METRIC VERSION

CALCULUS

METRIC VERSION | 9E

Early Transcendentals

James Stewart
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REFERENCE page 1

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Arithmetic Operations

$$a(b + c) = ab + ac \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{a + c}{b} = \frac{a}{b} + \frac{c}{b} \qquad \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

Exponents and Radicals

$$x^m x^n = x^{m+n} \qquad \frac{x^m}{x^n} = x^{m-n}$$

$$(x^m)^n = x^{mn} \qquad x^{-n} = \frac{1}{x^n}$$

$$(xy)^n = x^n y^n \qquad \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$x^{1/n} = \sqrt[n]{x} \qquad x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$$

$$\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y} \qquad \sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

Factoring Special Polynomials

$$x^2 - y^2 = (x + y)(x - y)$$

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Binomial Theorem

$$(x + y)^2 = x^2 + 2xy + y^2 \qquad (x - y)^2 = x^2 - 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + nxy^{n-1} + y^n$$

where $\displaystyle\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot k}$

Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Inequalities and Absolute Value

If $a < b$ and $b < c$, then $a < c$.

If $a < b$, then $a + c < b + c$.

If $a < b$ and $c > 0$, then $ca < cb$.

If $a < b$ and $c < 0$, then $ca > cb$.

If $a > 0$, then

$|x| = a$ means $x = a$ or $x = -a$

$|x| < a$ means $-a < x < a$

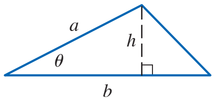
$|x| > a$ means $x > a$ or $x < -a$

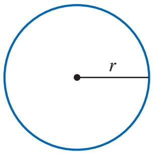
GEOMETRY

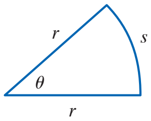
Geometric Formulas

Formulas for area A, circumference C, and volume V:

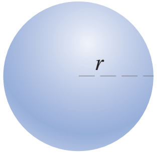
Triangle	Circle	Sector of Circle
$A = \frac{1}{2}bh$	$A = \pi r^2$	$A = \frac{1}{2}r^2\theta$
$= \frac{1}{2}ab\sin\theta$	$C = 2\pi r$	$s = r\theta$ (θ in radians)

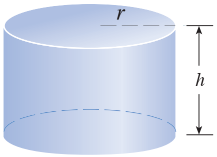


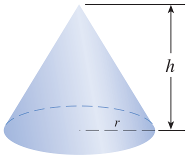




Sphere	Cylinder	Cone
$V = \frac{4}{3}\pi r^3$	$V = \pi r^2 h$	$V = \frac{1}{3}\pi r^2 h$
$A = 4\pi r^2$		$A = \pi r\sqrt{r^2 + h^2}$







Distance and Midpoint Formulas

Distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint of $\overline{P_1P_2}$: $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Lines

Slope of line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-slope equation of line through $P_1(x_1, y_1)$ with slope m:

$$y - y_1 = m(x - x_1)$$

Slope-intercept equation of line with slope m and y-intercept b:

$$y = mx + b$$

Circles

Equation of the circle with center (h, k) and radius r:

$$(x - h)^2 + (y - k)^2 = r^2$$

REFERENCE page 2

TRIGONOMETRY

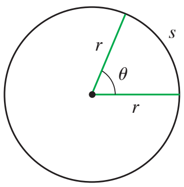
Angle Measurement

π radians $= 180°$

$1° = \frac{\pi}{180}$ rad $\qquad$ 1 rad $= \frac{180°}{\pi}$

$s = r\theta$

(θ in radians)

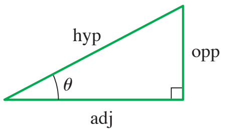


Right Angle Trigonometry

$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$

$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$

$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$

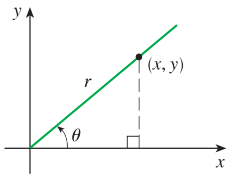


Trigonometric Functions

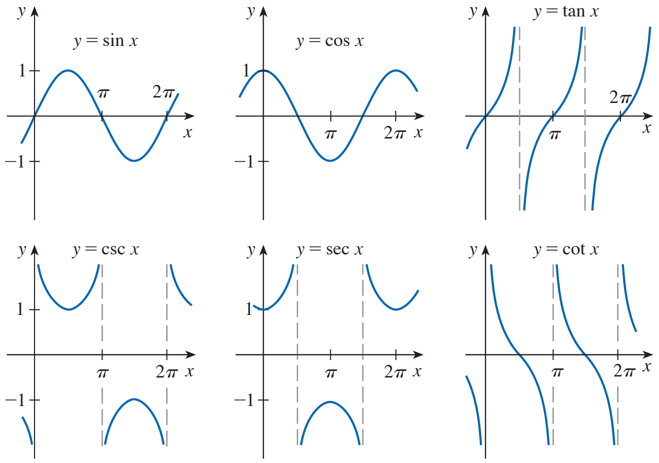
$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$

$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$

$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$



Graphs of Trigonometric Functions



Trigonometric Functions of Important Angles

θ	radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90°	$\pi/2$	1	0	—

Fundamental Identities

$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$

$\cot\theta = \frac{1}{\tan\theta} \qquad \sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$

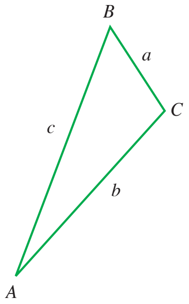
$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta \qquad \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \qquad \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$

The Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



The Law of Cosines

$a^2 = b^2 + c^2 - 2bc\cos A$

$b^2 = a^2 + c^2 - 2ac\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$

Addition and Subtraction Formulas

$\sin(x + y) = \sin x\cos y + \cos x\sin y$

$\sin(x - y) = \sin x\cos y - \cos x\sin y$

$\cos(x + y) = \cos x\cos y - \sin x\sin y$

$\cos(x - y) = \cos x\cos y + \sin x\sin y$

$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}$

$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x\tan y}$

Double-Angle Formulas

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$

Half-Angle Formulas

$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$

CALCULUS

EARLY TRANSCENDENTALS

NINTH EDITION

Metric Version

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Contents

Preface x

A Tribute to James Stewart xxii

About the Authors xxiii

Technology in the Ninth Edition xxiv

To the Student xxv

Diagnostic Tests xxvi

A Preview of Calculus 1

1 Functions and Models 7

1.1 Four Ways to Represent a Function 8
1.2 Mathematical Models: A Catalog of Essential Functions 21
1.3 New Functions from Old Functions 36
1.4 Exponential Functions 45
1.5 Inverse Functions and Logarithms 54
Review 67

Principles of Problem Solving 70

2 Limits and Derivatives 77

2.1 The Tangent and Velocity Problems 78
2.2 The Limit of a Function 83
2.3 Calculating Limits Using the Limit Laws 94
2.4 The Precise Definition of a Limit 105
2.5 Continuity 115
2.6 Limits at Infinity; Horizontal Asymptotes 127
2.7 Derivatives and Rates of Change 140
WRITING PROJECT • Early Methods for Finding Tangents 152
2.8 The Derivative as a Function 153
Review 166

Problems Plus 171

3 Differentiation Rules 173

3.1 Derivatives of Polynomials and Exponential Functions 174
APPLIED PROJECT • Building a Better Roller Coaster 184
3.2 The Product and Quotient Rules 185
3.3 Derivatives of Trigonometric Functions 191
3.4 The Chain Rule 199
APPLIED PROJECT • Where Should a Pilot Start Descent? 209
3.5 Implicit Differentiation 209
DISCOVERY PROJECT • Families of Implicit Curves 217
3.6 Derivatives of Logarithmic and Inverse Trigonometric Functions 217
3.7 Rates of Change in the Natural and Social Sciences 225
3.8 Exponential Growth and Decay 239
APPLIED PROJECT • Controlling Red Blood Cell Loss During Surgery 247
3.9 Related Rates 247
3.10 Linear Approximations and Differentials 254
DISCOVERY PROJECT • Polynomial Approximations 260
3.11 Hyperbolic Functions 261
Review 269

Problems Plus 274

4 Applications of Differentiation 279

4.1 Maximum and Minimum Values 280
APPLIED PROJECT • The Calculus of Rainbows 289
4.2 The Mean Value Theorem 290
4.3 What Derivatives Tell Us about the Shape of a Graph 296
4.4 Indeterminate Forms and l'Hospital's Rule 309
WRITING PROJECT • The Origins of l'Hospital's Rule 319
4.5 Summary of Curve Sketching 320
4.6 Graphing with Calculus and Technology 329
4.7 Optimization Problems 336
APPLIED PROJECT • The Shape of a Can 349
APPLIED PROJECT • Planes and Birds: Minimizing Energy 350
4.8 Newton's Method 351
4.9 Antiderivatives 356
Review 364

Problems Plus 369

5 Integrals 371

5.1 The Area and Distance Problems 372

5.2 The Definite Integral 384

DISCOVERY PROJECT • Area Functions 398

5.3 The Fundamental Theorem of Calculus 399

5.4 Indefinite Integrals and the Net Change Theorem 409

WRITING PROJECT • Newton, Leibniz, and the Invention of Calculus 418

5.5 The Substitution Rule 419

Review 428

Problems Plus 432

6 Applications of Integration 435

6.1 Areas Between Curves 436

APPLIED PROJECT • The Gini Index 445

6.2 Volumes 446

6.3 Volumes by Cylindrical Shells 460

6.4 Work 467

6.5 Average Value of a Function 473

APPLIED PROJECT • Calculus and Baseball 476

APPLIED PROJECT • Where to Sit at the Movies 478

Review 478

Problems Plus 481

7 Techniques of Integration 485

7.1 Integration by Parts 486

7.2 Trigonometric Integrals 493

7.3 Trigonometric Substitution 500

7.4 Integration of Rational Functions by Partial Fractions 507

7.5 Strategy for Integration 517

7.6 Integration Using Tables and Technology 523

DISCOVERY PROJECT • Patterns in Integrals 528

7.7 Approximate Integration 529

7.8 Improper Integrals 542

Review 552

Problems Plus 556

8 Further Applications of Integration 559

8.1 Arc Length 560

DISCOVERY PROJECT • Arc Length Contest 567

8.2 Area of a Surface of Revolution 567

DISCOVERY PROJECT • Rotating on a Slant 575

8.3 Applications to Physics and Engineering 576

DISCOVERY PROJECT • Complementary Coffee Cups 587

8.4 Applications to Economics and Biology 587

8.5 Probability 592

Review 600

Problems Plus 602

9 Differential Equations 605

9.1 Modeling with Differential Equations 606

9.2 Direction Fields and Euler's Method 612

9.3 Separable Equations 621

APPLIED PROJECT • How Fast Does a Tank Drain? 630

9.4 Models for Population Growth 631

9.5 Linear Equations 641

APPLIED PROJECT • Which Is Faster, Going Up or Coming Down? 648

9.6 Predator-Prey Systems 649

Review 656

Problems Plus 659

10 Parametric Equations and Polar Coordinates 661

10.1 Curves Defined by Parametric Equations 662

DISCOVERY PROJECT • Running Circles Around Circles 672

10.2 Calculus with Parametric Curves 673

DISCOVERY PROJECT • Bézier Curves 684

10.3 Polar Coordinates 684

DISCOVERY PROJECT • Families of Polar Curves 694

10.4 Calculus in Polar Coordinates 694

10.5 Conic Sections 702

10.6 Conic Sections in Polar Coordinates 711
Review 719

Problems Plus 722

11 Sequences, Series, and Power Series 723

11.1 Sequences 724
DISCOVERY PROJECT • Logistic Sequences 738
11.2 Series 738
11.3 The Integral Test and Estimates of Sums 751
11.4 The Comparison Tests 760
11.5 Alternating Series and Absolute Convergence 765
11.6 The Ratio and Root Tests 774
11.7 Strategy for Testing Series 779
11.8 Power Series 781
11.9 Representations of Functions as Power Series 787
11.10 Taylor and Maclaurin Series 795
DISCOVERY PROJECT • An Elusive Limit 810
WRITING PROJECT • How Newton Discovered the Binomial Series 811
11.11 Applications of Taylor Polynomials 811
APPLIED PROJECT • Radiation from the Stars 820
Review 821

Problems Plus 825

12 Vectors and the Geometry of Space 829

12.1 Three-Dimensional Coordinate Systems 830
12.2 Vectors 836
DISCOVERY PROJECT • The Shape of a Hanging Chain 846
12.3 The Dot Product 847
12.4 The Cross Product 855
DISCOVERY PROJECT • The Geometry of a Tetrahedron 864
12.5 Equations of Lines and Planes 864
DISCOVERY PROJECT • Putting 3D in Perspective 874
12.6 Cylinders and Quadric Surfaces 875
Review 883

Problems Plus 887

13 Vector Functions 889

13.1 Vector Functions and Space Curves 890
13.2 Derivatives and Integrals of Vector Functions 898
13.3 Arc Length and Curvature 904
13.4 Motion in Space: Velocity and Acceleration 916
APPLIED PROJECT • Kepler's Laws 925
Review 927

Problems Plus 930

14 Partial Derivatives 933

14.1 Functions of Several Variables 934
14.2 Limits and Continuity 951
14.3 Partial Derivatives 961
DISCOVERY PROJECT • Deriving the Cobb-Douglas Production Function 973
14.4 Tangent Planes and Linear Approximations 974
APPLIED PROJECT • The Speedo LZR Racer 984
14.5 The Chain Rule 985
14.6 Directional Derivatives and the Gradient Vector 994
14.7 Maximum and Minimum Values 1008
DISCOVERY PROJECT • Quadratic Approximations and Critical Points 1019
14.8 Lagrange Multipliers 1020
APPLIED PROJECT • Rocket Science 1028
APPLIED PROJECT • Hydro-Turbine Optimization 1030
Review 1031

Problems Plus 1035

15 Multiple Integrals 1037

15.1 Double Integrals over Rectangles 1038
15.2 Double Integrals over General Regions 1051
15.3 Double Integrals in Polar Coordinates 1062
15.4 Applications of Double Integrals 1069
15.5 Surface Area 1079
15.6 Triple Integrals 1082
DISCOVERY PROJECT • Volumes of Hyperspheres 1095
15.7 Triple Integrals in Cylindrical Coordinates 1095
DISCOVERY PROJECT • The Intersection of Three Cylinders 1101

15.8 Triple Integrals in Spherical Coordinates 1102
APPLIED PROJECT • Roller Derby 1108
15.9 Change of Variables in Multiple Integrals 1109
Review 1117

Problems Plus 1121

16 Vector Calculus 1123

16.1 Vector Fields 1124
16.2 Line Integrals 1131
16.3 The Fundamental Theorem for Line Integrals 1144
16.4 Green's Theorem 1154
16.5 Curl and Divergence 1161
16.6 Parametric Surfaces and Their Areas 1170
16.7 Surface Integrals 1182
16.8 Stokes' Theorem 1195
16.9 The Divergence Theorem 1201
16.10 Summary 1208
Review 1209

Problems Plus 1213

Appendixes A1

A Numbers, Inequalities, and Absolute Values A2
B Coordinate Geometry and Lines A10
C Graphs of Second-Degree Equations A16
D Trigonometry A24
E Sigma Notation A36
F Proofs of Theorems A41
G The Logarithm Defined as an Integral A53
H Answers to Odd-Numbered Exercises A61

Index A143

Preface

> *A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery.*
>
> GEORGE POLYA

The art of teaching, Mark Van Doren said, is the art of assisting discovery. In this Ninth Edition, Metric Version, as in all of the preceding editions, we continue the tradition of writing a book that, we hope, assists students in discovering calculus—both for its practical power and its surprising beauty. We aim to convey to the student a sense of the utility of calculus as well as to promote development of technical ability. At the same time, we strive to give some appreciation for the intrinsic beauty of the subject. Newton undoubtedly experienced a sense of triumph when he made his great discoveries. We want students to share some of that excitement.

The emphasis is on understanding concepts. Nearly all calculus instructors agree that conceptual understanding should be the ultimate goal of calculus instruction; to implement this goal we present fundamental topics graphically, numerically, algebraically, and verbally, with an emphasis on the relationships between these different representations. Visualization, numerical and graphical experimentation, and verbal descriptions can greatly facilitate conceptual understanding. Moreover, conceptual understanding and technical skill can go hand in hand, each reinforcing the other.

We are keenly aware that good teaching comes in different forms and that there are different approaches to teaching and learning calculus, so the exposition and exercises are designed to accommodate different teaching and learning styles. The features (including projects, extended exercises, principles of problem solving, and historical insights) provide a variety of enhancements to a central core of fundamental concepts and skills. Our aim is to provide instructors and their students with the tools they need to chart their own paths to discovering calculus.

Alternate Versions

The Stewart *Calculus* series includes several other calculus textbooks that might be preferable for some instructors. Most of them also come in single variable and multivariable versions.

- *Calculus*, Ninth Edition, Metric Version, is similar to the present textbook except that the exponential, logarithmic, and inverse trigonometric functions are covered after the chapter on integration.

- *Essential Calculus*, Second Edition, is a much briefer book (840 pages), though it contains almost all of the topics in *Calculus*, Ninth Edition. The relative brevity is achieved through briefer exposition of some topics and putting some features on the website.

- *Essential Calculus: Early Transcendentals*, Second Edition, resembles *Essential Calculus*, but the exponential, logarithmic, and inverse trigonometric functions are covered in Chapter 3.
- *Calculus: Concepts and Contexts*, Fourth Edition, emphasizes conceptual understanding even more strongly than this book. The coverage of topics is not encyclopedic and the material on transcendental functions and on parametric equations is woven throughout the book instead of being treated in separate chapters.
- *Brief Applied Calculus* is intended for students in business, the social sciences, and the life sciences.
- *Biocalculus: Calculus for the Life Sciences* is intended to show students in the life sciences how calculus relates to biology.
- *Biocalculus: Calculus, Probability, and Statistics for the Life Sciences* contains all the content of *Biocalculus: Calculus for the Life Sciences* as well as three additional chapters covering probability and statistics.

What's New in the Ninth Edition, Metric Version?

The overall structure of the text remains largely the same, but we have made many improvements that are intended to make the Ninth Edition, Metric Version even more usable as a teaching tool for instructors and as a learning tool for students. The changes are a result of conversations with our colleagues and students, suggestions from users and reviewers, insights gained from our own experiences teaching from the book, and from the copious notes that James Stewart entrusted to us about changes that he wanted us to consider for the new edition. In all the changes, both small and large, we have retained the features and tone that have contributed to the success of this book.

- More than 20% of the exercises are new:

  Basic exercises have been added, where appropriate, near the beginning of exercise sets. These exercises are intended to build student confidence and reinforce understanding of the fundamental concepts of a section. (See, for instance, Exercises 7.3.1–4, 9.1.1–5, 11.4.3–6.)

  Some new exercises include graphs intended to encourage students to understand how a graph facilitates the solution of a problem; these exercises complement subsequent exercises in which students need to supply their own graph. (See Exercises 6.2.1–4, Exercises 10.4.43–46 as well as 53–54, 15.5.1–2, 15.6.9–12, 16.7.15 and 24, 16.8.9 and 13.)

  Some exercises have been structured in two stages, where part (a) asks for the setup and part (b) is the evaluation. This allows students to check their answer to part (a) before completing the problem. (See Exercises 6.1.1–4, 6.3.3–4, 15.2.7–10.)

  Some challenging and extended exercises have been added toward the end of selected exercise sets (such as Exercises 6.2.87, 9.3.56, 11.2.79–81, and 11.9.47).

  Titles have been added to selected exercises when the exercise extends a concept discussed in the section. (See, for example, Exercises 2.6.66, 10.1.55–57, 15.2.80–81.)

  Some of our favorite new exercises are 1.3.71, 3.4.99, 3.5.65, 4.5.55–58, 6.2.79, 6.5.18, 10.5.69, 15.1.38, and 15.4.3–4. In addition, Problem 14 in the Problems Plus following Chapter 6 and Problem 4 in the Problems Plus following Chapter 15 are interesting and challenging.

- New examples have been added, and additional steps have been added to the solutions of some existing examples. (See, for instance, Example 2.7.5, Example 6.3.5, Example 10.1.5, Examples 14.8.1 and 14.8.4, and Example 16.3.4.)
- Several sections have been restructured and new subheads added to focus the organization around key concepts. (Good illustrations of this are Sections 2.3, 11.1, 11.2, and 14.2.)
- Many new graphs and illustrations have been added, and existing ones updated, to provide additional graphical insights into key concepts.
- A few new topics have been added and others expanded (within a section or in extended exercises) that were requested by reviewers. (Examples include a subsection on torsion in Section 13.3, symmetric difference quotients in Exercise 2.7.60, and improper integrals of more than one type in Exercises 7.8.65–68.)
- New projects have been added and some existing projects have been updated. (For instance, see the Discovery Project following Section 12.2, *The Shape of a Hanging Chain*.)
- Derivatives of logarithmic functions and inverse trigonometric functions are now covered in one section (3.6) that emphasizes the concept of the derivative of an inverse function.
- Alternating series and absolute convergence are now covered in one section (11.5).
- The chapter on Second-Order Differential Equations, as well as the associated appendix section on complex numbers, has been moved to the website.

Features

Each feature is designed to complement different teaching and learning practices. Throughout the text there are historical insights, extended exercises, projects, problem-solving principles, and many opportunities to experiment with concepts by using technology. We are mindful that there is rarely enough time in a semester to utilize all of these features, but their availability in the book gives the instructor the option to assign some and perhaps simply draw attention to others in order to emphasize the rich ideas of calculus and its crucial importance in the real world.

Conceptual Exercises

The most important way to foster conceptual understanding is through the problems that the instructor assigns. To that end we have included various types of problems. Some exercise sets begin with requests to explain the meanings of the basic concepts of the section (see, for instance, the first few exercises in Sections 2.2, 2.5, 11.2, 14.2, and 14.3) and most exercise sets contain exercises designed to reinforce basic understanding (such as Exercises 2.5.3–10, 5.5.1–8, 6.1.1–4, 7.3.1–4, 9.1.1–5, and 11.4.3–6). Other exercises test conceptual understanding through graphs or tables (see Exercises 2.7.17, 2.8.36–38, 2.8.47–52, 9.1.23–25, 10.1.30–33, 13.2.1–2, 13.3.37–43, 14.1.41–44, 14.3.2, 14.3.4–6, 14.6.1–2, 14.7.3–4, 15.1.6–8, 16.1.13–22, 16.2.19–20, and 16.3.1–2).

Many exercises provide a graph to aid in visualization (see for instance Exercises 6.2.1–4, 10.4.43–46, 15.5.1–2, 15.6.9–12, and 16.7.24). Another type of exercise uses verbal descriptions to gauge conceptual understanding (see Exercises 2.5.12, 2.8.66, 4.3.79–80, and 7.8.79). In addition, all the review sections begin with a Concept Check and a True-False Quiz.

We particularly value problems that combine and compare graphical, numerical, and algebraic approaches (see Exercises 2.6.45–46, 3.7.29, and 9.4.4).

Graded Exercise Sets

Each exercise set is carefully graded, progressing from basic conceptual exercises, to skill-development and graphical exercises, and then to more challenging exercises that often extend the concepts of the section, draw on concepts from previous sections, or involve applications or proofs.

Real-World Data

Real-world data provide a tangible way to introduce, motivate, or illustrate the concepts of calculus. As a result, many of the examples and exercises deal with functions defined by such numerical data or graphs. These real-world data have been obtained by contacting companies and government agencies as well as researching on the Internet and in libraries. See, for instance, Figure 1 in Section 1.1 (seismograms from the Northridge earthquake), Exercise 2.8.36 (number of cosmetic surgeries), Exercise 5.1.12 (velocity of the space shuttle *Endeavour*), Exercise 5.4.83 (power consumption in the New England states), Example 3 in Section 14.4 (the heat index), Figure 1 in Section 14.6 (temperature contour map), Example 9 in Section 15.1 (snowfall in Colorado), and Figure 1 in Section 16.1 (velocity vector fields of wind in San Francisco Bay).

Projects

One way of involving students and making them active learners is to have them work (perhaps in groups) on extended projects that give a feeling of substantial accomplishment when completed. There are three kinds of projects in the text.

Applied Projects involve applications that are designed to appeal to the imagination of students. The project after Section 9.5 asks whether a ball thrown upward takes longer to reach its maximum height or to fall back to its original height (the answer might surprise you). The project after Section 14.8 uses Lagrange multipliers to determine the masses of the three stages of a rocket so as to minimize the total mass while enabling the rocket to reach a desired velocity.

Discovery Projects anticipate results to be discussed later or encourage discovery through pattern recognition (see the project following Section 7.6, which explores patterns in integrals). Other discovery projects explore aspects of geometry: tetrahedra (after Section 12.4), hyperspheres (after Section 15.6), and intersections of three cylinders (after Section 15.7). Additionally, the project following Section 12.2 uses the geometric definition of the derivative to find a formula for the shape of a hanging chain. Some projects make substantial use of technology; the one following Section 10.2 shows how to use Bézier curves to design shapes that represent letters for a laser printer.

Writing Projects ask students to compare present-day methods with those of the founders of calculus—Fermat's method for finding tangents, for instance, following Section 2.7. Suggested references are supplied.

More projects can be found in the *Instructor's Guide*. There are also extended exercises that can serve as smaller projects. (See Exercise 4.7.53 on the geometry of beehive cells, Exercise 6.2.87 on scaling solids of revolution, or Exercise 9.3.56 on the formation of sea ice.)

Problem Solving

Students usually have difficulties with problems that have no single well-defined procedure for obtaining the answer. As a student of George Polya, James Stewart experienced first-hand Polya's delightful and penetrating insights into the process of problem solving. Accordingly, a modified version of Polya's four-stage problem-solving strategy is presented following Chapter 1 in Principles of Problem Solving. These principles are applied, both explicitly and implicitly, throughout the book. Each of the other chapters is followed by a section called *Problems Plus*, which features examples of how to tackle challenging calculus problems. In selecting the Problems Plus problems we have kept in mind the following advice from David Hilbert: "A mathematical problem should be difficult in order to entice us, yet not inaccessible lest it mock our efforts." We have used these problems to great effect in our own calculus classes; it is gratifying to see how students respond to a challenge. James Stewart said, "When I put these challenging problems on assignments and tests I grade them in a different way . . . I reward a student significantly for ideas toward a solution and for recognizing which problem-solving principles are relevant."

Technology

When using technology, it is particularly important to clearly understand the concepts that underlie the images on the screen or the results of a calculation. When properly used, graphing calculators and computers are powerful tools for discovering and understanding those concepts. This textbook can be used either with or without technology—we use two special symbols to indicate clearly when a particular type of assistance from technology is required. The icon 📈 indicates an exercise that definitely requires the use of graphing software or a graphing calculator to aid in sketching a graph. (That is not to say that the technology can't be used on the other exercises as well.) The symbol T means that the assistance of software or a graphing calculator is needed beyond just graphing to complete the exercise. Freely available websites such as WolframAlpha.com or Symbolab.com are often suitable. In cases where the full resources of a computer algebra system, such as Maple or Mathematica, are needed, we state this in the exercise. Of course, technology doesn't make pencil and paper obsolete. Hand calculation and sketches are often preferable to technology for illustrating and reinforcing some concepts. Both instructors and students need to develop the ability to decide where using technology is appropriate and where more insight is gained by working out an exercise by hand.

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Stewart Website

Visit StewartCalculus.com for these additional materials:

- Homework Hints
- Solutions to the Concept Checks (from the review section of each chapter)
- Algebra and Analytic Geometry Review
- Lies My Calculator and Computer Told Me
- History of Mathematics, with links to recommended historical websites

- Additional Topics (complete with exercise sets): Fourier Series, Rotation of Axes, Formulas for the Remainder Theorem in Taylor Series
- Additional chapter on second-order differential equations, including the method of series solutions, and an appendix section reviewing complex numbers and complex exponential functions
- Instructor Area that includes archived problems (drill exercises that appeared in previous editions, together with their solutions)
- Challenge Problems (some from the Problems Plus sections from prior editions)
- Links, for particular topics, to outside Web resources

Content

Diagnostic Tests

The book begins with four diagnostic tests, in Basic Algebra, Analytic Geometry, Functions, and Trigonometry.

A Preview of Calculus

This is an overview of the subject and includes a list of questions to motivate the study of calculus.

1 Functions and Models

From the beginning, multiple representations of functions are stressed: verbal, numerical, visual, and algebraic. A discussion of mathematical models leads to a review of the standard functions, including exponential and logarithmic functions, from these four points of view.

2 Limits and Derivatives

The material on limits is motivated by a prior discussion of the tangent and velocity problems. Limits are treated from descriptive, graphical, numerical, and algebraic points of view. Section 2.4, on the precise definition of a limit, is an optional section. Sections 2.7 and 2.8 deal with derivatives (including derivatives for functions defined graphically and numerically) before the differentiation rules are covered in Chapter 3. Here the examples and exercises explore the meaning of derivatives in various contexts. Higher derivatives are introduced in Section 2.8.

3 Differentiation Rules

All the basic functions, including exponential, logarithmic, and inverse trigonometric functions, are differentiated here. The latter two classes of functions are now covered in one section that focuses on the derivative of an inverse function. When derivatives are computed in applied situations, students are asked to explain their meanings. Exponential growth and decay are included in this chapter.

4 Applications of Differentiation

The basic facts concerning extreme values and shapes of curves are deduced from the Mean Value Theorem. Graphing with technology emphasizes the interaction between calculus and machines and the analysis of families of curves. Some substantial optimization problems are provided, including an explanation of why you need to raise your head 42° to see the top of a rainbow.

5 Integrals

The area problem and the distance problem serve to motivate the definite integral, with sigma notation introduced as needed. (Full coverage of sigma notation is provided in Appendix E.) Emphasis is placed on explaining the meanings of integrals in various contexts and on estimating their values from graphs and tables.

6 Applications of Integration

This chapter presents the applications of integration—area, volume, work, average value—that can reasonably be done without specialized techniques of integration. General methods are emphasized. The goal is for students to be able to divide a quantity into small pieces, estimate with Riemann sums, and recognize the limit as an integral.

7 Techniques of Integration

All the standard methods are covered but, of course, the real challenge is to be able to recognize which technique is best used in a given situation. Accordingly, a strategy for evaluating integrals is explained in Section 7.5. The use of mathematical software is discussed in Section 7.6.

8 Further Applications of Integration

This chapter contains the applications of integration—arc length and surface area—for which it is useful to have available all the techniques of integration, as well as applications to biology, economics, and physics (hydrostatic force and centers of mass). A section on probability is included. There are more applications here than can realistically be covered in a given course. Instructors may select applications suitable for their students and for which they themselves have enthusiasm.

9 Differential Equations

Modeling is the theme that unifies this introductory treatment of differential equations. Direction fields and Euler's method are studied before separable and linear equations are solved explicitly, so that qualitative, numerical, and analytic approaches are given equal consideration. These methods are applied to the exponential, logistic, and other models for population growth. The first four or five sections of this chapter serve as a good introduction to first-order differential equations. An optional final section uses predator-prey models to illustrate systems of differential equations.

10 Parametric Equations and Polar Coordinates

This chapter introduces parametric and polar curves and applies the methods of calculus to them. Parametric curves are well suited to projects that require graphing with technology; the two presented here involve families of curves and Bézier curves. A brief treatment of conic sections in polar coordinates prepares the way for Kepler's Laws in Chapter 13.

11 Sequences, Series, and Power Series

The convergence tests have intuitive justifications (see Section 11.3) as well as formal proofs. Numerical estimates of sums of series are based on which test was used to prove convergence. The emphasis is on Taylor series and polynomials and their applications to physics.

12 Vectors and the Geometry of Space

The material on three-dimensional analytic geometry and vectors is covered in this and the next chapter. Here we deal with vectors, the dot and cross products, lines, planes, and surfaces.

13 Vector Functions

This chapter covers vector-valued functions, their derivatives and integrals, the length and curvature of space curves, and velocity and acceleration along space curves, culminating in Kepler's laws.

14 Partial Derivatives

Functions of two or more variables are studied from verbal, numerical, visual, and algebraic points of view. In particular, partial derivatives are introduced by looking at a specific column in a table of values of the heat index (perceived air temperature) as a function of the actual temperature and the relative humidity.

15 Multiple Integrals

Contour maps and the Midpoint Rule are used to estimate the average snowfall and average temperature in given regions. Double and triple integrals are used to compute volumes, surface areas, and (in projects) volumes of hyperspheres and volumes of intersections of three cylinders. Cylindrical and spherical coordinates are introduced in the context of evaluating triple integrals. Several applications are considered, including computing mass, charge, and probabilities.

16 Vector Calculus

Vector fields are introduced through pictures of velocity fields showing San Francisco Bay wind patterns. The similarities among the Fundamental Theorem for line integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem are emphasized.

17 Second-Order Differential Equations

Since first-order differential equations are covered in Chapter 9, this online chapter deals with second-order linear differential equations, their application to vibrating springs and electric circuits, and series solutions.

Ancillaries

Calculus, Early Transcendentals, Ninth Edition, Metric Version, is supported by a complete set of ancillaries. Each piece has been designed to enhance student understanding and to facilitate creative instruction.

Ancillaries for Instructors

Instructor's Guide

by Douglas Shaw

Each section of the text is discussed from several viewpoints. Available online at the Instructor's Companion Site, the Instructor's Guide *contains suggested time to allot, points to stress, text discussion topics, core materials for lecture, workshop/discussion suggestions, group work exercises in a form suitable for handout, and suggested homework assignments.*

Complete Solutions Manual

Single Variable Calculus: Early Transcendentals, Ninth Edition, Metric Version
Chapters 1–11
By Joshua Babbin, Scott Barnett, and Jeffery A. Cole, with metrication by Anthony Tan and Michael Verwer, both from McMaster University.

Multivariable Calculus, Ninth Edition, Metric Version
Chapters 10–16
By Joshua Babbin and Gina Sanders, with metrication by Anthony Tan and Michael Verwer, both from McMaster University.

Includes worked-out solutions to all exercises in the text. Both volumes of the Complete Solutions Manual *are available online at the Instructor's Companion Site.*

Test Bank

Contains text-specific multiple-choice and free response test items and is available online at the Instructor's Companion Site.

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One of the main factors aiding in the preparation of this edition is the cogent advice from a large number of reviewers, all of whom have extensive experience teaching calculus. We greatly appreciate their suggestions and the time they spent to understand the approach taken in this book. We have learned something from each of them.

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We are deeply indebted to our friend and colleague Lothar Redlin who began working with us on this revision shortly before his untimely death in 2018. Lothar's deep insights into mathematics and its pedagogy, and his lightning fast problem-solving skills, were invaluable assets.

We especially thank Kathi Townes of TECHarts, our production service and copyeditor (for this as well as the past several editions). Her extraordinary ability to recall any detail of the manuscript as needed, her facility in simultaneously handling different editing tasks, and her comprehensive familiarity with the book were key factors in its accuracy and timely production. We also thank Lori Heckelman for the elegant and precise rendering of the new illustrations.

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This textbook has benefited greatly over the past three decades from the advice and guidance of some of the best mathematics editors: Ron Munro, Harry Campbell, Craig Barth, Jeremy Hayhurst, Gary Ostedt, Bob Pirtle, Richard Stratton, Liz Covello, Neha Taleja, and now Gary Whalen. They have all contributed significantly to the success of this book. Prominently, Gary Whalen's broad knowledge of current issues in the teaching of mathematics and his continual research into creating better ways of using technology as a teaching and learning tool were invaluable resources in the creation of this edition.

JAMES STEWART
DANIEL CLEGG
SALEEM WATSON

A Tribute to James Stewart

JAMES STEWART had a singular gift for teaching mathematics. The large lecture halls where he taught his calculus classes were always packed to capacity with students, whom he held engaged with interest and anticipation as he led them to discover a new concept or the solution to a stimulating problem. Stewart presented calculus the way he viewed it—as a rich subject with intuitive concepts, wonderful problems, powerful applications, and a fascinating history. As a testament to his success in teaching and lecturing, many of his students went on to become mathematicians, scientists, and engineers—and more than a few are now university professors themselves. It was his students who first suggested that he write a calculus textbook of his own. Over the years, former students, by then working scientists and engineers, would call him to discuss mathematical problems that they encountered in their work; some of these discussions resulted in new exercises or projects in the book.

We each met James Stewart—or Jim as he liked us to call him—through his teaching and lecturing, resulting in his inviting us to coauthor mathematics textbooks with him. In the years we have known him, he was in turn our teacher, mentor, and friend.

Jim had several special talents whose combination perhaps uniquely qualified him to write such a beautiful calculus textbook—a textbook with a narrative that speaks to students and that combines the fundamentals of calculus with conceptual insights on how to think about them. Jim always listened carefully to his students in order to find out precisely where they may have had difficulty with a concept. Crucially, Jim really enjoyed hard work—a necessary trait for completing the immense task of writing a calculus book. As his coauthors, we enjoyed his contagious enthusiasm and optimism, making the time we spent with him always fun and productive, never stressful.

Most would agree that writing a calculus textbook is a major enough feat for one lifetime, but amazingly, Jim had many other interests and accomplishments: he played violin professionally in the Hamilton and McMaster Philharmonic Orchestras for many years, he had an enduring passion for architecture, he was a patron of the arts and cared deeply about many social and humanitarian causes. He was also a world traveler, an eclectic art collector, and even a gourmet cook.

James Stewart was an extraordinary person, mathematician, and teacher. It has been our honor and privilege to be his coauthors and friends.

DANIEL CLEGG
SALEEM WATSON

About the Authors

For more than two decades, Daniel Clegg and Saleem Watson have worked with James Stewart on writing mathematics textbooks. The close working relationship between them was particularly productive because they shared a common viewpoint on teaching mathematics and on writing mathematics. In a 2014 interview James Stewart remarked on their collaborations: “We discovered that we could think in the same way . . . we agreed on almost everything, which is kind of rare.”

Daniel Clegg and Saleem Watson met James Stewart in different ways, yet in each case their initial encounter turned out to be the beginning of a long association. Stewart spotted Daniel’s talent for teaching during a chance meeting at a mathematics conference and asked him to review the manuscript for an upcoming edition of *Calculus* and to author the multivariable solutions manual. Since that time Daniel has played an ever-increasing role in the making of several editions of the Stewart calculus books. He and Stewart have also coauthored an applied calculus textbook. Stewart first met Saleem when Saleem was a student in his graduate mathematics class. Later Stewart spent a sabbatical leave doing research with Saleem at Penn State University, where Saleem was an instructor at the time. Stewart asked Saleem and Lothar Redlin (also a student of Stewart’s) to join him in writing a series of precalculus textbooks; their many years of collaboration resulted in several editions of these books.

JAMES STEWART was professor of mathematics at McMaster University and the University of Toronto for many years. James did graduate studies at Stanford University and the University of Toronto, and subsequently did research at the University of London. His research field was Harmonic Analysis and he also studied the connections between mathematics and music.

DANIEL CLEGG is professor of mathematics at Palomar College in Southern California. He did undergraduate studies at California State University, Fullerton and graduate studies at the University of California, Los Angeles (UCLA). Daniel is a consummate teacher; he has been teaching mathematics ever since he was a graduate student at UCLA.

SALEEM WATSON is professor emeritus of mathematics at California State University, Long Beach. He did undergraduate studies at Andrews University in Michigan and graduate studies at Dalhousie University and McMaster University. After completing a research fellowship at the University of Warsaw, he taught for several years at Penn State before joining the mathematics department at California State University, Long Beach.

Stewart and Clegg have published *Brief Applied Calculus.*

Stewart, Redlin, and Watson have published *Precalculus: Mathematics for Calculus, College Algebra, Trigonometry, Algebra and Trigonometry,* and (with Phyllis Panman) *College Algebra: Concepts and Contexts.*

Technology in the Ninth Edition

Graphing and computing devices are valuable tools for learning and exploring calculus, and some have become well established in calculus instruction. Graphing calculators are useful for drawing graphs and performing some numerical calculations, like approximating solutions to equations or numerically evaluating derivatives (Chapter 3) or definite integrals (Chapter 5). Mathematical software packages called computer algebra systems (CAS, for short) are more powerful tools. Despite the name, algebra represents only a small subset of the capabilities of a CAS. In particular, a CAS can do mathematics symbolically rather than just numerically. It can find exact solutions to equations and exact formulas for derivatives and integrals.

We now have access to a wider variety of tools of varying capabilities than ever before. These include Web-based resources (some of which are free of charge) and apps for smartphones and tablets. Many of these resources include at least some CAS functionality, so some exercises that may have typically required a CAS can now be completed using these alternate tools.

In this edition, rather than refer to a specific type of device (a graphing calculator, for instance) or software package (such as a CAS), we indicate the type of capability that is needed to work an exercise.

Graphing Icon

The appearance of this icon beside an exercise indicates that you are expected to use a machine or software to help you draw the graph. In many cases, a graphing calculator will suffice. Websites such as Desmos.com provide similar capability. If the graph is in 3D (see Chapters 12–16), WolframAlpha.com is a good resource. There are also many graphing software applications for computers, smartphones, and tablets. If an exercise asks for a graph but no graphing icon is shown, then you are expected to draw the graph by hand. In Chapter 1 we review graphs of basic functions and discuss how to use transformations to graph modified versions of these basic functions.

Technology Icon

This icon is used to indicate that software or a device with abilities beyond just graphing is needed to complete the exercise. Many graphing calculators and software resources can provide numerical approximations when needed. For working with mathematics symbolically, websites like WolframAlpha.com or Symbolab.com are helpful, as are more advanced graphing calculators such as the Texas Instrument TI-89 or TI-Nspire CAS. If the full power of a CAS is needed, this will be stated in the exercise, and access to software packages such as Mathematica, Maple, MATLAB, or SageMath may be required. If an exercise does not include a technology icon, then you are expected to evaluate limits, derivatives, and integrals, or solve equations by hand, arriving at exact answers. No technology is needed for these exercises beyond perhaps a basic scientific calculator.

To the Student

Reading a calculus textbook is different from reading a story or a news article. Don't be discouraged if you have to read a passage more than once in order to understand it. You should have pencil and paper and calculator at hand to sketch a diagram or make a calculation.

Some students start by trying their homework problems and read the text only if they get stuck on an exercise. We suggest that a far better plan is to read and understand a section of the text before attempting the exercises. In particular, you should look at the definitions to see the exact meanings of the terms. And before you read each example, we suggest that you cover up the solution and try solving the problem yourself.

Part of the aim of this course is to train you to think logically. Learn to write the solutions of the exercises in a connected, step-by-step fashion with explanatory sentences—not just a string of disconnected equations or formulas.

The answers to the odd-numbered exercises appear at the back of the book, in Appendix H. Some exercises ask for a verbal explanation or interpretation or description. In such cases there is no single correct way of expressing the answer, so don't worry that you haven't found the definitive answer. In addition, there are often several different forms in which to express a numerical or algebraic answer, so if your answer differs from the given one, don't immediately assume you're wrong. For example, if the answer given in the back of the book is $\sqrt{2} - 1$ and you obtain $1/(1 + \sqrt{2})$, then you're correct and rationalizing the denominator will show that the answers are equivalent.

The icon 📈 indicates an exercise that definitely requires the use of either a graphing calculator or a computer with graphing software to help you sketch the graph. But that doesn't mean that graphing devices can't be used to check your work on the other exercises as well. The symbol T indicates that technological assistance beyond just graphing is needed to complete the exercise. (See Technology in the Ninth Edition for more details.)

You will also encounter the symbol ⊘, which warns you against committing an error. This symbol is placed in the margin in situations where many students tend to make the same mistake.

Homework Hints are available for many exercises. These hints can be found on StewartCalculus.com as well as in WebAssign. The homework hints ask you questions that allow you to make progress toward a solution without actually giving you the answer. If a particular hint doesn't enable you to solve the problem, you can click to reveal the next hint.

We recommend that you keep this book for reference purposes after you finish the course. Because you will likely forget some of the specific details of calculus, the book will serve as a useful reminder when you need to use calculus in subsequent courses. And, because this book contains more material than can be covered in any one course, it can also serve as a valuable resource for a working scientist or engineer.

Calculus is an exciting subject, justly considered to be one of the greatest achievements of the human intellect. We hope you will discover that it is not only useful but also intrinsically beautiful.

Diagnostic Tests

Success in calculus depends to a large extent on knowledge of the mathematics that precedes calculus: algebra, analytic geometry, functions, and trigonometry. The following tests are intended to diagnose weaknesses that you might have in these areas. After taking each test you can check your answers against the given answers and, if necessary, refresh your skills by referring to the review materials that are provided.

A Diagnostic Test: Algebra

1. Evaluate each expression without using a calculator.

(a) $(-3)^4$ (b) -3^4 (c) 3^{-4}

(d) $\dfrac{5^{23}}{5^{21}}$ (e) $\left(\dfrac{2}{3}\right)^{-2}$ (f) $16^{-3/4}$

2. Simplify each expression. Write your answer without negative exponents.

(a) $\sqrt{200} - \sqrt{32}$

(b) $(3a^3b^3)(4ab^2)^2$

(c) $\left(\dfrac{3x^{3/2}y^3}{x^2y^{-1/2}}\right)^{-2}$

3. Expand and simplify.

(a) $3(x + 6) + 4(2x - 5)$ (b) $(x + 3)(4x - 5)$

(c) $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$ (d) $(2x + 3)^2$

(e) $(x + 2)^3$

4. Factor each expression.

(a) $4x^2 - 25$ (b) $2x^2 + 5x - 12$

(c) $x^3 - 3x^2 - 4x + 12$ (d) $x^4 + 27x$

(e) $3x^{3/2} - 9x^{1/2} + 6x^{-1/2}$ (f) $x^3y - 4xy$

5. Simplify the rational expression.

(a) $\dfrac{x^2 + 3x + 2}{x^2 - x - 2}$ (b) $\dfrac{2x^2 - x - 1}{x^2 - 9} \cdot \dfrac{x + 3}{2x + 1}$

(c) $\dfrac{x^2}{x^2 - 4} - \dfrac{x + 1}{x + 2}$ (d) $\dfrac{\dfrac{y}{x} - \dfrac{x}{y}}{\dfrac{1}{y} - \dfrac{1}{x}}$

6. Rationalize the expression and simplify.

(a) $\dfrac{\sqrt{10}}{\sqrt{5}-2}$ (b) $\dfrac{\sqrt{4+h}-2}{h}$

7. Rewrite by completing the square.

(a) $x^2 + x + 1$ (b) $2x^2 - 12x + 11$

8. Solve the equation. (Find only the real solutions.)

(a) $x + 5 = 14 - \frac{1}{2}x$ (b) $\dfrac{2x}{x+1} = \dfrac{2x-1}{x}$

(c) $x^2 - x - 12 = 0$ (d) $2x^2 + 4x + 1 = 0$

(e) $x^4 - 3x^2 + 2 = 0$ (f) $3|x - 4| = 10$

(g) $2x(4 - x)^{-1/2} - 3\sqrt{4 - x} = 0$

9. Solve each inequality. Write your answer using interval notation.

(a) $-4 < 5 - 3x \leqslant 17$ (b) $x^2 < 2x + 8$

(c) $x(x - 1)(x + 2) > 0$ (d) $|x - 4| < 3$

(e) $\dfrac{2x-3}{x+1} \leqslant 1$

10. State whether each equation is true or false.

(a) $(p + q)^2 = p^2 + q^2$ (b) $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$

(c) $\sqrt{a^2 + b^2} = a + b$ (d) $\dfrac{1 + TC}{C} = 1 + T$

(e) $\dfrac{1}{x-y} = \dfrac{1}{x} - \dfrac{1}{y}$ (f) $\dfrac{1/x}{a/x - b/x} = \dfrac{1}{a-b}$

ANSWERS TO DIAGNOSTIC TEST A: ALGEBRA

1. (a) 81 (b) -81 (c) $\frac{1}{81}$
(d) 25 (e) $\frac{9}{4}$ (f) $\frac{1}{8}$

2. (a) $6\sqrt{2}$ (b) $48a^5b^7$ (c) $\dfrac{x}{9y^7}$

3. (a) $11x - 2$ (b) $4x^2 + 7x - 15$
(c) $a - b$ (d) $4x^2 + 12x + 9$
(e) $x^3 + 6x^2 + 12x + 8$

4. (a) $(2x - 5)(2x + 5)$ (b) $(2x - 3)(x + 4)$
(c) $(x - 3)(x - 2)(x + 2)$ (d) $x(x + 3)(x^2 - 3x + 9)$
(e) $3x^{-1/2}(x - 1)(x - 2)$ (f) $xy(x - 2)(x + 2)$

5. (a) $\dfrac{x+2}{x-2}$ (b) $\dfrac{x-1}{x-3}$
(c) $\dfrac{1}{x-2}$ (d) $-(x + y)$

6. (a) $5\sqrt{2} + 2\sqrt{10}$ (b) $\dfrac{1}{\sqrt{4+h}+2}$

7. (a) $\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$ (b) $2(x - 3)^2 - 7$

8. (a) 6 (b) 1 (c) $-3, 4$
(d) $-1 \pm \frac{1}{2}\sqrt{2}$ (e) $\pm 1, \pm\sqrt{2}$ (f) $\frac{2}{3}, \frac{22}{3}$
(g) $\frac{12}{5}$

9. (a) $[-4, 3)$ (b) $(-2, 4)$
(c) $(-2, 0) \cup (1, \infty)$ (d) $(1, 7)$
(e) $(-1, 4]$

10. (a) False (b) True (c) False
(d) False (e) False (f) True

If you had difficulty with these problems, you may wish to consult the Review of Algebra on the website **StewartCalculus.com.**

B | Diagnostic Test: Analytic Geometry

1. Find an equation for the line that passes through the point $(2, -5)$ and
   (a) has slope -3
   (b) is parallel to the x-axis
   (c) is parallel to the y-axis
   (d) is parallel to the line $2x - 4y = 3$

2. Find an equation for the circle that has center $(-1, 4)$ and passes through the point $(3, -2)$.

3. Find the center and radius of the circle with equation $x^2 + y^2 - 6x + 10y + 9 = 0$.

4. Let $A(-7, 4)$ and $B(5, -12)$ be points in the plane.
   (a) Find the slope of the line that contains A and B.
   (b) Find an equation of the line that passes through A and B. What are the intercepts?
   (c) Find the midpoint of the segment AB.
   (d) Find the length of the segment AB.
   (e) Find an equation of the perpendicular bisector of AB.
   (f) Find an equation of the circle for which AB is a diameter.

5. Sketch the region in the xy-plane defined by the equation or inequalities.
   (a) $-1 \leqslant y \leqslant 3$
   (b) $|x| < 4$ and $|y| < 2$
   (c) $y < 1 - \frac{1}{2}x$
   (d) $y \geqslant x^2 - 1$
   (e) $x^2 + y^2 < 4$
   (f) $9x^2 + 16y^2 = 144$

ANSWERS TO DIAGNOSTIC TEST B: ANALYTIC GEOMETRY

1. (a) $y = -3x + 1$
   (b) $y = -5$
   (c) $x = 2$
   (d) $y = \frac{1}{2}x - 6$

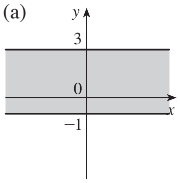
2. $(x + 1)^2 + (y - 4)^2 = 52$

3. Center $(3, -5)$, radius 5

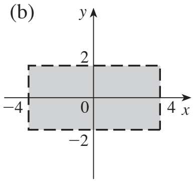
4. (a) $-\frac{4}{3}$
   (b) $4x + 3y + 16 = 0$; x-intercept -4, y-intercept $-\frac{16}{3}$
   (c) $(-1, -4)$
   (d) 20
   (e) $3x - 4y = 13$
   (f) $(x + 1)^2 + (y + 4)^2 = 100$

5.

(a)



(b)



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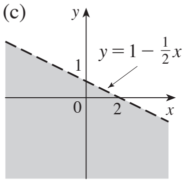

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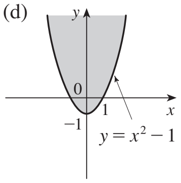

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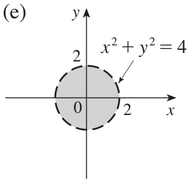
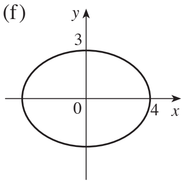


(f)



If you had difficulty with these problems, you may wish to consult
the review of analytic geometry in Appendixes B and C.

C Diagnostic Test: Functions

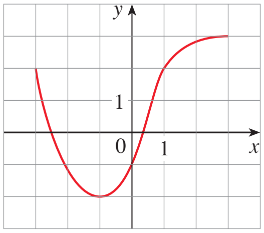


FIGURE FOR PROBLEM 1

1. The graph of a function f is given at the left.
   (a) State the value of $f(-1)$.
   (b) Estimate the value of $f(2)$.
   (c) For what values of x is $f(x) = 2$?
   (d) Estimate the values of x such that $f(x) = 0$.
   (e) State the domain and range of f.

2. If $f(x) = x^3$, evaluate the difference quotient $\dfrac{f(2+h) - f(2)}{h}$ and simplify your answer.

3. Find the domain of the function.
   (a) $f(x) = \dfrac{2x+1}{x^2+x-2}$ (b) $g(x) = \dfrac{\sqrt[3]{x}}{x^2+1}$ (c) $h(x) = \sqrt{4-x} + \sqrt{x^2-1}$

4. How are graphs of the functions obtained from the graph of f?
   (a) $y = -f(x)$ (b) $y = 2f(x) - 1$ (c) $y = f(x-3) + 2$

5. Without using a calculator, make a rough sketch of the graph.
   (a) $y = x^3$ (b) $y = (x+1)^3$ (c) $y = (x-2)^3 + 3$
   (d) $y = 4 - x^2$ (e) $y = \sqrt{x}$ (f) $y = 2\sqrt{x}$
   (g) $y = -2^x$ (h) $y = 1 + x^{-1}$

6. Let $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leqslant 0 \\ 2x + 1 & \text{if } x > 0 \end{cases}$
   (a) Evaluate $f(-2)$ and $f(1)$. (b) Sketch the graph of f.

7. If $f(x) = x^2 + 2x - 1$ and $g(x) = 2x - 3$, find each of the following functions.
   (a) $f \circ g$ (b) $g \circ f$ (c) $g \circ g \circ g$

ANSWERS TO DIAGNOSTIC TEST C: FUNCTIONS

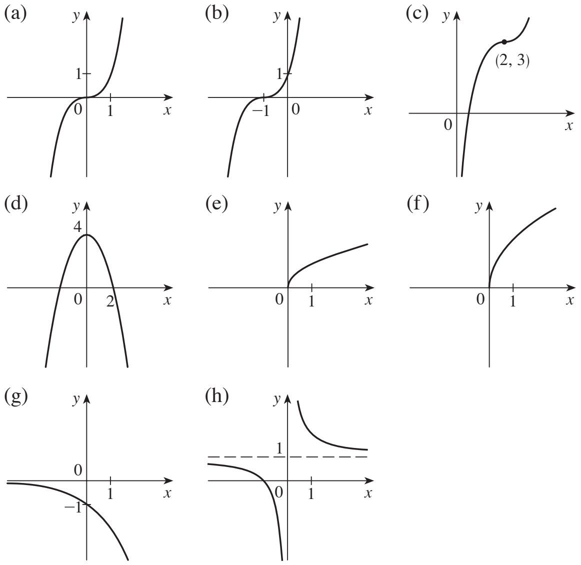
1. (a) -2 (b) 2.8
   (c) $-3, 1$ (d) $-2.5, 0.3$
   (e) $[-3, 3], [-2, 3]$

2. $12 + 6h + h^2$

3. (a) $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$
   (b) $(-\infty, \infty)$
   (c) $(-\infty, -1] \cup [1, 4]$

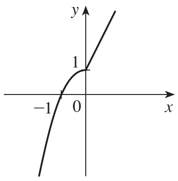
4. (a) Reflect about the x-axis
   (b) Stretch vertically by a factor of 2, then shift 1 unit downward
   (c) Shift 3 units to the right and 2 units upward

5.



6. (a) $-3, 3$

(b)



7. (a) $(f \circ g)(x) = 4x^2 - 8x + 2$

(b) $(g \circ f)(x) = 2x^2 + 4x - 5$

(c) $(g \circ g \circ g)(x) = 8x - 21$

If you had difficulty with these problems, you should look at sections 1.1–1.3 of this book.

D Diagnostic Test: Trigonometry

1. Convert from degrees to radians.

(a) $300°$ (b) $-18°$

2. Convert from radians to degrees.

(a) $5\pi/6$ (b) 2

3. Find the length of an arc of a circle with radius 12 cm if the arc subtends a central angle of 30°.

4. Find the exact values.

(a) $\tan(\pi/3)$ (b) $\sin(7\pi/6)$ (c) $\sec(5\pi/3)$

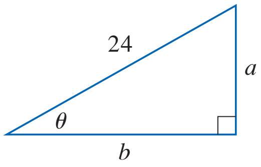


FIGURE FOR PROBLEM 5

5. Express the lengths a and b in the figure in terms of θ.

6. If $\sin x = \frac{1}{3}$ and $\sec y = \frac{5}{4}$, where x and y lie between 0 and $\pi/2$, evaluate $\sin(x + y)$.

7. Prove the identities.

(a) $\tan\theta \sin\theta + \cos\theta = \sec\theta$ (b) $\dfrac{2\tan x}{1 + \tan^2 x} = \sin 2x$

8. Find all values of x such that $\sin 2x = \sin x$ and $0 \leqslant x \leqslant 2\pi$.

9. Sketch the graph of the function $y = 1 + \sin 2x$ without using a calculator.

ANSWERS TO DIAGNOSTIC TEST D: TRIGONOMETRY

1. (a) $5\pi/3$ (b) $-\pi/10$

2. (a) $150°$ (b) $360°/\pi \approx 114.6°$

3. 2π cm

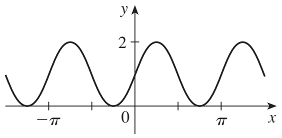
4. (a) $\sqrt{3}$ (b) $-\frac{1}{2}$ (c) 2

5. $a = 24\sin\theta$, $b = 24\cos\theta$

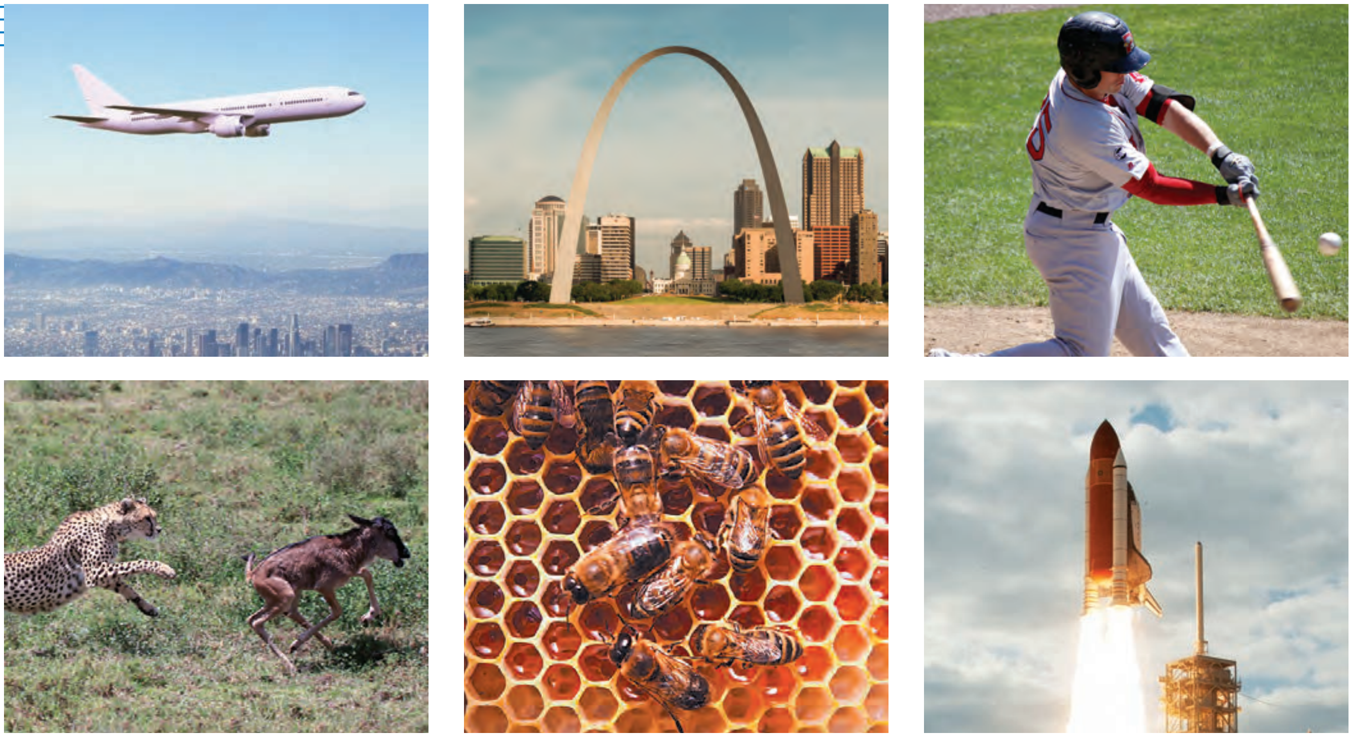
6. $\frac{1}{15}(4 + 6\sqrt{2})$

8. $0, \pi/3, \pi, 5\pi/3, 2\pi$

9.



If you had difficulty with these problems, you should look at Appendix D of this book.



By the time you finish this course, you will be able to determine where a pilot should start descent for a smooth landing, find the length of the curve used to design the Gateway Arch in St. Louis, compute the force on a baseball bat when it strikes the ball, predict the population sizes for competing predator-prey species, show that bees form the cells of a beehive in a way that uses the least amount of wax, and estimate the amount of fuel needed to propel a rocket into orbit.

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A Preview of Calculus

CALCULUS IS FUNDAMENTALLY DIFFERENT from the mathematics that you have studied previously: calculus is less static and more dynamic. It is concerned with change and motion; it deals with quantities that approach other quantities. For that reason it may be useful to have an overview of calculus before beginning your study of the subject. Here we give a preview of some of the main ideas of calculus and show how their foundations are built upon the concept of a *limit*.

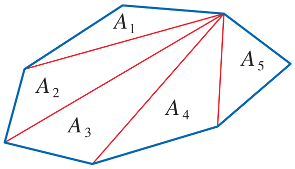
What Is Calculus?

The world around us is continually changing—populations increase, a cup of coffee cools, a stone falls, chemicals react with one another, currency values fluctuate, and so on. We would like to be able to analyze quantities or processes that are undergoing continuous change. For example, if a stone falls 10 feet each second we could easily tell how fast it is falling at any time, but this is *not* what happens—the stone falls faster and faster, its speed changing at each instant. In studying calculus, we will learn how to model (or describe) such instantaneously changing processes and how to find the cumulative effect of these changes.

Calculus builds on what you have learned in algebra and analytic geometry but advances these ideas spectacularly. Its uses extend to nearly every field of human activity. You will encounter numerous applications of calculus throughout this book.

At its core, calculus revolves around two key problems involving the graphs of functions—the *area problem* and the *tangent problem*—and an unexpected relationship between them. Solving these problems is useful because the area under the graph of a function and the tangent to the graph of a function have many important interpretations in a variety of contexts.

The Area Problem



$A = A_1 + A_2 + A_3 + A_4 + A_5$

FIGURE 1

The origins of calculus go back at least 2500 years to the ancient Greeks, who found areas using the "method of exhaustion." They knew how to find the area A of any polygon by dividing it into triangles, as in Figure 1, and adding the areas of these triangles.

It is a much more difficult problem to find the area of a curved figure. The Greek method of exhaustion was to inscribe polygons in the figure and circumscribe polygons about the figure, and then let the number of sides of the polygons increase. Figure 2 illustrates this process for the special case of a circle with inscribed regular polygons.

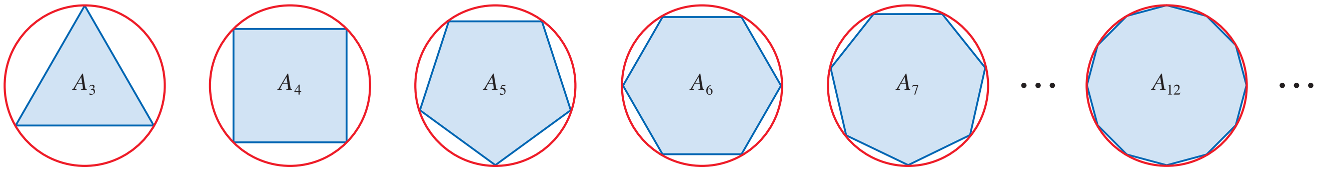


FIGURE 2

Let A_n be the area of the inscribed regular polygon of n sides. As n increases, it appears that A_n gets closer and closer to the area of the circle. We say that the area A of the circle is the *limit* of the areas of the inscribed polygons, and we write

$$A = \lim_{n \to \infty} A_n$$

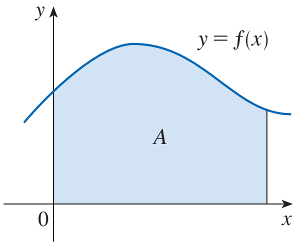


FIGURE 3
The area A of the region under the graph of f

The Greeks themselves did not use limits explicitly. However, by indirect reasoning, Eudoxus (fifth century BC) used exhaustion to prove the familiar formula for the area of a circle: $A = \pi r^2$.

We will use a similar idea in Chapter 5 to find areas of regions of the type shown in Figure 3. We approximate such an area by areas of rectangles as shown in Figure 4. If we approximate the area A of the region under the graph of f by using n rectangles $R_1, R_2, \ldots, R_n$, then the approximate area is

$$A_n = R_1 + R_2 + \cdots + R_n$$

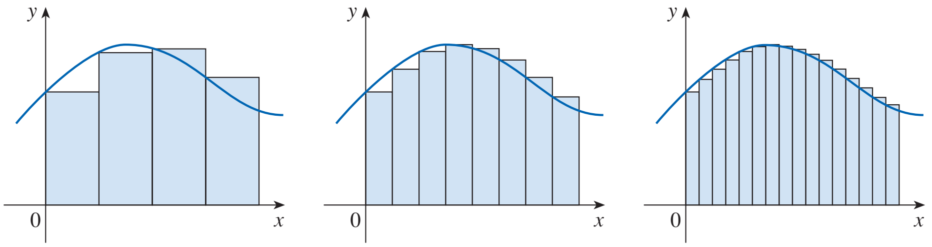


FIGURE 4 Approximating the area A using rectangles

Now imagine that we increase the number of rectangles (as the width of each one decreases) and calculate A as the limit of these sums of areas of rectangles:

$$A = \lim_{n \to \infty} A_n$$

In Chapter 5 we will learn how to calculate such limits.

The area problem is the central problem in the branch of calculus called *integral calculus*; it is important because the area under the graph of a function has different interpretations depending on what the function represents. In fact, the techniques that we develop for finding areas will also enable us to compute the volume of a solid, the length of a curve, the force of water against a dam, the mass and center of mass of a rod, the work done in pumping water out of a tank, and the amount of fuel needed to send a rocket into orbit.

The Tangent Problem

Consider the problem of trying to find an equation of the tangent line ℓ to a curve with equation $y = f(x)$ at a given point P. (We will give a precise definition of a tangent line in Chapter 2; for now you can think of it as the line that touches the curve at P and follows the direction of the curve at P, as in Figure 5.) Because the point P lies on the tangent line, we can find the equation of ℓ if we know its slope m. The problem is that we need two points to compute the slope and we know only one point, P, on ℓ. To get around the problem we first find an approximation to m by taking a nearby point Q on the curve and computing the slope m_{PQ} of the secant line PQ.

Now imagine that Q moves along the curve toward P as in Figure 6. You can see that the secant line PQ rotates and approaches the tangent line ℓ as its limiting position. This

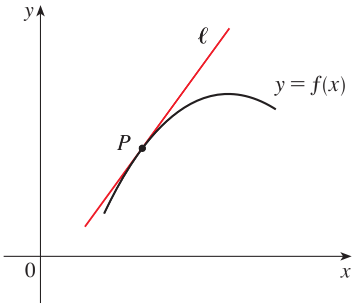


FIGURE 5
The tangent line at P

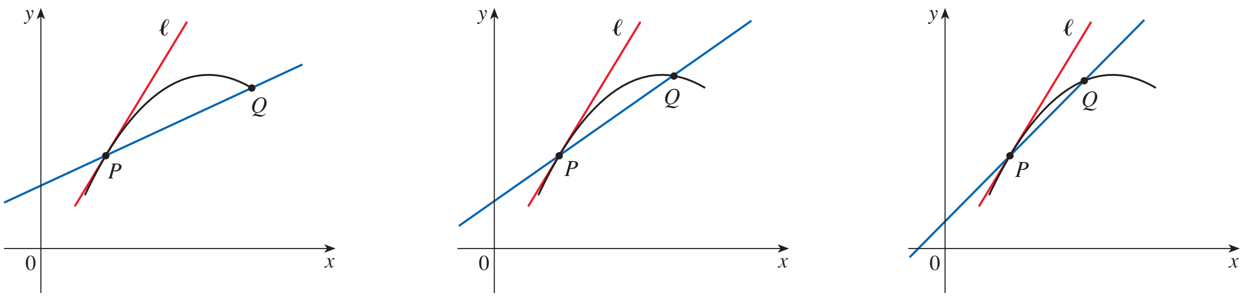


FIGURE 6 The secant lines approach the tangent line as Q approaches P.

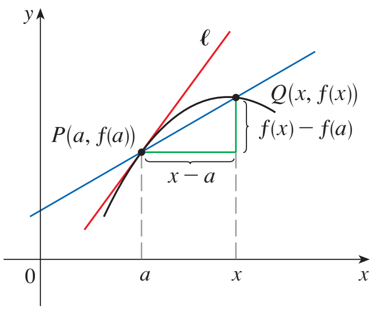


FIGURE 7
The secant line PQ

means that the slope m_{PQ} of the secant line becomes closer and closer to the slope m of the tangent line. We write

$$m = \lim_{Q \to P} m_{PQ}$$

and say that m is the limit of m_{PQ} as Q approaches P along the curve.

Notice from Figure 7 that if P is the point $(a, f(a))$ and Q is the point $(x, f(x))$, then

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Because x approaches a as Q approaches P, an equivalent expression for the slope of the tangent line is

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

In Chapter 3 we will learn rules for calculating such limits.

The tangent problem has given rise to the branch of calculus called *differential calculus*; it is important because the slope of a tangent to the graph of a function can have different interpretations depending on the context. For instance, solving the tangent problem allows us to find the instantaneous speed of a falling stone, the rate of change of a chemical reaction, or the direction of the forces on a hanging chain.

A Relationship between the Area and Tangent Problems

The area and tangent problems seem to be very different problems but, surprisingly, the problems are closely related—in fact, they are so closely related that solving one of them leads to a solution of the other. The relationship between these two problems is introduced in Chapter 5; it is the central discovery in calculus and is appropriately named the Fundamental Theorem of Calculus. Perhaps most importantly, the Fundamental Theorem vastly simplifies the solution of the area problem, making it possible to find areas without having to approximate by rectangles and evaluate the associated limits.

Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716) are credited with the invention of calculus because they were the first to recognize the importance of the Fundamental Theorem of Calculus and to utilize it as a tool for solving real-world problems. In studying calculus you will discover these powerful results for yourself.

Summary

We have seen that the concept of a limit arises in finding the area of a region and in finding the slope of a tangent line to a curve. It is this basic idea of a limit that sets calculus apart from other areas of mathematics. In fact, we could define calculus as the part of mathematics that deals with limits. We have mentioned that areas under curves and slopes of tangent lines to curves have many different interpretations in a variety of contexts. Finally, we have discussed that the area and tangent problems are closely related.

After Isaac Newton invented his version of calculus, he used it to explain the motion of the planets around the sun, giving a definitive answer to a centuries-long quest for a description of our solar system. Today calculus is applied in a great variety of contexts, such as determining the orbits of satellites and spacecraft, predicting population sizes, forecasting weather, measuring cardiac output, and gauging the efficiency of an economic market.

In order to convey a sense of the power and versatility of calculus, we conclude with a list of some of the questions that you will be able to answer using calculus.

1. How can we design a roller coaster for a safe and smooth ride?
   (See the Applied Project following Section 3.1.)
2. How far away from an airport should a pilot start descent?
   (See the Applied Project following Section 3.4.)
3. How can we explain the fact that the angle of elevation from an observer up to the highest point in a rainbow is always 42°?
   (See the Applied Project following Section 4.1.)
4. How can we estimate the amount of work that was required to build the Great Pyramid of Khufu in ancient Egypt?
   (See Exercise 36 in Section 6.4.)
5. With what speed must a projectile be launched with so that it escapes the earth's gravitation pull?
   (See Exercise 77 in Section 7.8.)
6. How can we explain the changes in the thickness of sea ice over time and why cracks in the ice tend to "heal"?
   (See Exercise 56 in Section 9.3.)
7. Does a ball thrown upward take longer to reach its maximum height or to fall back down to its original height?
   (See the Applied Project following Section 9.5.)
8. How can we fit curves together to design shapes to represent letters on a laser printer?
   (See the Applied Project following Section 10.2.)
9. How can we explain the fact that planets and satellites move in elliptical orbits?
   (See the Applied Project following Section 13.4.)
10. How can we distribute water flow among turbines at a hydroelectric station so as to maximize the total energy production?
    (See the Applied Project following Section 14.8.)

The electrical power produced by a wind turbine can be estimated by a mathematical function that incorporates several factors. We will explore this function in Exercise 1.2.25 and determine the expected power output of a particular turbine for various wind speeds.

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1

Functions and Models

THE FUNDAMENTAL OBJECTS THAT WE deal with in calculus are functions. This chapter prepares the way for calculus by discussing the basic ideas concerning functions, their graphs, and ways of transforming and combining them. We stress that a function can be represented in different ways: by an equation, in a table, by a graph, or in words. We look at the main types of functions that occur in calculus and describe the process of using these functions as mathematical models of real-world phenomena.

1.1 Four Ways to Represent a Function

Functions

Functions arise whenever one quantity depends on another. Consider the following four situations.

A. The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A, and we say that A is a *function* of r.

B. The human population of the world P depends on the time t. Table 1 gives estimates of the world population P at time t, for certain years. For instance,

$$P \approx 2{,}560{,}000{,}000 \qquad \text{when } t = 1950$$

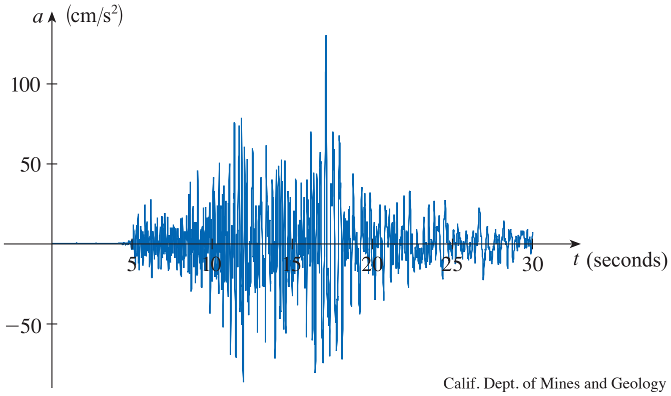
For each value of the time t there is a corresponding value of P, and we say that P is a function of t.

Table 1 World Population

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080
2010	6870

C. The cost C of mailing an envelope depends on its weight w. Although there is no simple formula that connects w and C, the post office has a rule for determining C when w is known.

D. The vertical acceleration a of the ground as measured by a seismograph during an earthquake is a function of the elapsed time t. Figure 1 shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of t, the graph provides a corresponding value of a.



Calif. Dept. of Mines and Geology

FIGURE 1
Vertical ground acceleration during the Northridge earthquake

Each of these examples describes a rule whereby, given a number (r in Example A), another number (A) is assigned. In each case we say that the second number is a function of the first number. If f represents the rule that connects A to r in Example A, then we express this in **function notation** as $A = f(r)$.

A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

We usually consider functions for which the sets D and E are sets of real numbers. The set D is called the **domain** of the function. The number $f(x)$ is the **value of f at x** and is read "f of x." The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain. A symbol that represents an arbitrary number in the *domain* of a function f is called an **independent variable**. A symbol that represents a number in the *range* of f is called a **dependent variable**. In Example A, for instance, r is the independent variable and A is the dependent variable.

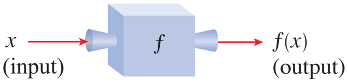


FIGURE 2
Machine diagram for a function f

It's helpful to think of a function as a **machine** (see Figure 2). If x is in the domain of the function f, then when x enters the machine, it's accepted as an **input** and the machine produces an **output** $f(x)$ according to the rule of the function. So we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs. The preprogrammed functions in a calculator are good examples of a function as a machine. For example, if you input a number and press the squaring key, the calculator displays the output, the square of the input.

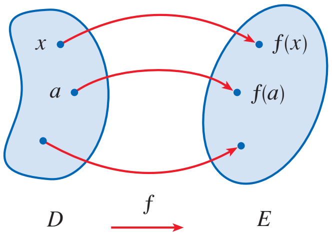


FIGURE 3
Arrow diagram for f

Another way to picture a function is by an **arrow diagram** as in Figure 3. Each arrow connects an element of D to an element of E. The arrow indicates that $f(x)$ is associated with x, $f(a)$ is associated with a, and so on.

Perhaps the most useful method for visualizing a function is its graph. If f is a function with domain D, then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}$$

(Notice that these are input-output pairs.) In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.

The graph of a function f gives us a useful picture of the behavior or "life history" of a function. Since the y-coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x. (See Figure 4.) The graph of f also allows us to picture the domain of f on the x-axis and its range on the y-axis as in Figure 5.

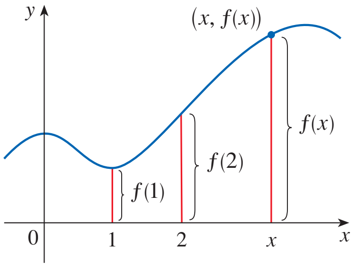


FIGURE 4

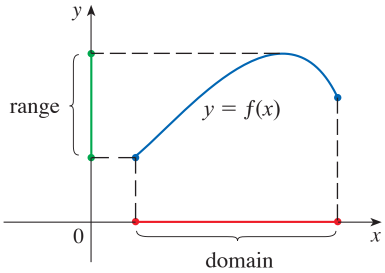


FIGURE 5

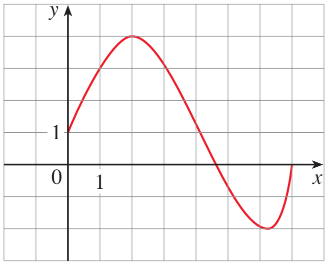


FIGURE 6

The notation for intervals is given in Appendix A.

EXAMPLE 1 The graph of a function f is shown in Figure 6.
(a) Find the values of $f(1)$ and $f(5)$.
(b) What are the domain and range of f?

SOLUTION
(a) We see from Figure 6 that the point $(1, 3)$ lies on the graph of f, so the value of f at 1 is $f(1) = 3$. (In other words, the point on the graph that lies above $x = 1$ is 3 units above the x-axis.)

When $x = 5$, the graph lies about 0.7 units below the x-axis, so we estimate that $f(5) \approx -0.7$.

(b) We see that $f(x)$ is defined when $0 \leqslant x \leqslant 7$, so the domain of f is the closed interval $[0, 7]$. Notice that f takes on all values from -2 to 4, so the range of f is

$$\{y \mid -2 \leqslant y \leqslant 4\} = [-2, 4]$$ ■

In calculus, the most common method of defining a function is by an algebraic equation. For example, the equation $y = 2x - 1$ defines y as a function of x. We can express this in function notation as $f(x) = 2x - 1$.

EXAMPLE 2 Sketch the graph and find the domain and range of each function.
(a) $f(x) = 2x - 1$ (b) $g(x) = x^2$

SOLUTION
(a) The equation of the graph is $y = 2x - 1$, and we recognize this as being the equation of a line with slope 2 and y-intercept -1. (Recall the slope-intercept form of the equation of a line: $y = mx + b$. See Appendix B.) This enables us to sketch a portion of the graph of f in Figure 7. The expression $2x - 1$ is defined for all real numbers, so the domain of f is the set of all real numbers, which we denote by $\mathbb{R}$. The graph shows that the range is also $\mathbb{R}$.

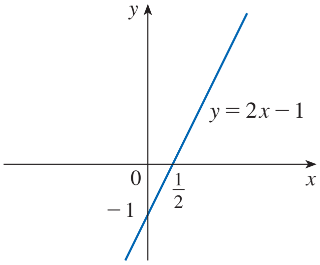


FIGURE 7

(b) Since $g(2) = 2^2 = 4$ and $g(-1) = (-1)^2 = 1$, we could plot the points $(2, 4)$ and $(-1, 1)$, together with a few other points on the graph, and join them to produce the graph (Figure 8). The equation of the graph is $y = x^2$, which represents a parabola (see Appendix C). The domain of g is $\mathbb{R}$. The range of g consists of all values of $g(x)$, that is, all numbers of the form x^2. But $x^2 \geqslant 0$ for all numbers x and any positive number y is a square. So the range of g is $\{y \mid y \geqslant 0\} = [0, \infty)$. This can also be seen from Figure 8. ■

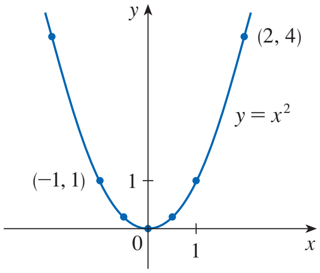


FIGURE 8

EXAMPLE 3 If $f(x) = 2x^2 - 5x + 1$ and $h \neq 0$, evaluate $\dfrac{f(a+h) - f(a)}{h}$.

SOLUTION We first evaluate $f(a + h)$ by replacing x by $a + h$ in the expression for $f(x)$:

$$\begin{aligned} f(a+h) &= 2(a+h)^2 - 5(a+h) + 1 \\ &= 2(a^2 + 2ah + h^2) - 5(a+h) + 1 \\ &= 2a^2 + 4ah + 2h^2 - 5a - 5h + 1 \end{aligned}$$

Then we substitute into the given expression and simplify:

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{(2a^2 + 4ah + 2h^2 - 5a - 5h + 1) - (2a^2 - 5a + 1)}{h} \\ &= \frac{2a^2 + 4ah + 2h^2 - 5a - 5h + 1 - 2a^2 + 5a - 1}{h} \\ &= \frac{4ah + 2h^2 - 5h}{h} = 4a + 2h - 5 \end{aligned}$$ ■

The expression

$$\frac{f(a+h) - f(a)}{h}$$

in Example 3 is called a **difference quotient** and occurs frequently in calculus. As we will see in Chapter 2, it represents the average rate of change of $f(x)$ between $x = a$ and $x = a + h$.

Representations of Functions

We consider four different ways to represent a function:

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

If a single function can be represented in all four ways, it's often useful to go from one representation to another to gain additional insight into the function. (In Example 2, for instance, we started with algebraic formulas and then obtained graphs.) But certain functions are described more naturally by one method than by another. With this in mind, let's reexamine the four situations that we considered at the beginning of this section.

Table 2 World Population

t (years since 1900)	Population (millions)
0	1650
10	1750
20	1860
30	2070
40	2300
50	2560
60	3040
70	3710
80	4450
90	5280
100	6080
110	6870

A. The most useful representation of the area of a circle as a function of its radius is probably the algebraic formula $A = \pi r^2$ or, in function notation, $A(r) = \pi r^2$. It is also possible to compile a table of values or sketch a graph (half a parabola). Because a circle has to have a positive radius, the domain is $\{r \mid r > 0\} = (0, \infty)$ and the range is also $(0, \infty)$.

B. We are given a description of the function in words: $P(t)$ is the human population of the world at time t. Let's measure t so that $t = 0$ corresponds to the year 1900. Table 2 provides a convenient representation of this function. If we plot the ordered pairs in the table, we get the graph (called a *scatter plot*) in Figure 9. It too is a useful representation; the graph allows us to absorb all the data at once. What about a formula? Of course, it's impossible to devise an explicit formula that gives the exact human population $P(t)$ at any time t. But it is possible to find an expression for a function that *approximates* $P(t)$. In fact, using methods explained in Section 1.4, we obtain an approximation for the population P:

$$P(t) \approx f(t) = (1.43653 \times 10^9) \cdot (1.01395)^t$$

Figure 10 shows that it is a reasonably good "fit." The function f is called a *mathematical model* for population growth. In other words, it is a function with an explicit formula that approximates the behavior of our given function. We will see, however, that the ideas of calculus can be applied to a table of values; an explicit formula is not necessary.

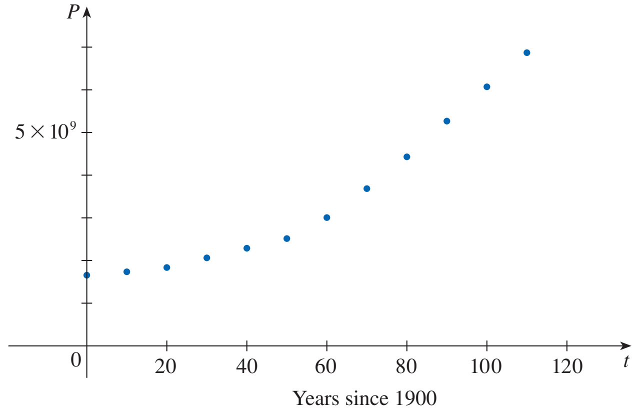


FIGURE 9

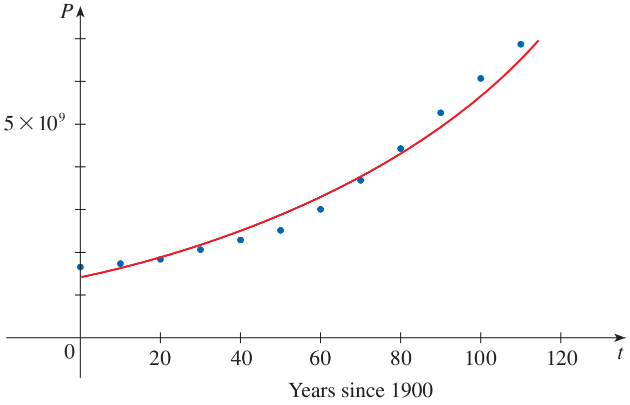


FIGURE 10

A function defined by a table of values is called a *tabular* function.

Table 3

w (grams)	$C(w)$ (dollars)
$0 < w < 25$	1.00
$25 < w < 50$	1.15
$50 < w < 75$	1.30
$75 < w < 100$	1.45
$100 < w < 125$	1.60
$\vdots$	$\vdots$

The function P is typical of the functions that arise whenever we attempt to apply calculus to the real world. We start with a verbal description of a function. Then we may be able to construct a table of values of the function, perhaps from instrument readings in a scientific experiment. Even though we don't have complete knowledge of the values of the function, we will see throughout the book that it is still possible to perform the operations of calculus on such a function.

C. Again, the function is described in words: Let $C(w)$ be the cost of mailing a large envelope with weight w. The rule that the US Postal Service used as of 2019 is as follows: The cost is 1 dollar for up to 25 g plus 15 cents for each additional gram (or less) up to 350 g. A table of values is the most convenient representation for this function (see Table 3), though it is possible to sketch a graph (see Example 10).

D. The graph shown in Figure 1 is the most natural representation of the vertical acceleration function $a(t)$. It's true that a table of values could be compiled, and it is even possible to devise an approximate formula. But everything a geologist needs to know—amplitudes and patterns—can be seen easily from the graph. (The same is true for the patterns seen in electrocardiograms of heart patients and polygraphs for lie-detection.)

In the next example we sketch the graph of a function that is defined verbally.

EXAMPLE 4 When you turn on a hot-water faucet that is connected to a hot-water tank, the temperature T of the water depends on how long the water has been running. Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.

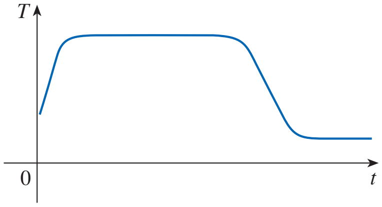


FIGURE 11

SOLUTION The initial temperature of the running water is close to room temperature because the water has been sitting in the pipes. When the water from the hot-water tank starts flowing from the faucet, T increases quickly. In the next phase, T is constant at the temperature of the heated water in the tank. When the tank is drained, T decreases to the temperature of the water supply. This enables us to make the rough sketch of T as a function of t shown in Figure 11. ■

In the following example we start with a verbal description of a function in a physical situation and obtain an explicit algebraic formula. The ability to do this is a useful skill in solving calculus problems that ask for the maximum or minimum values of quantities.

EXAMPLE 5 A rectangular storage container with an open top has a volume of 10 m^3. The length of its base is twice its width. Material for the base costs \$10 per square meter; material for the sides costs \$6 per square meter. Express the cost of materials as a function of the width of the base.

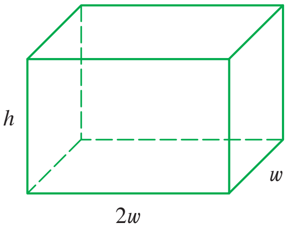


FIGURE 12

SOLUTION We draw a diagram as in Figure 12 and introduce notation by letting w and $2w$ be the width and length of the base, respectively, and h be the height.

The area of the base is $(2w)w = 2w^2$, so the cost, in dollars, of the material for the base is $10(2w^2)$. Two of the sides have area wh and the other two have area $2wh$, so the cost of the material for the sides is $6[2(wh) + 2(2wh)]$. The total cost is therefore

$$C = 10(2w^2) + 6[2(wh) + 2(2wh)] = 20w^2 + 36wh$$

To express C as a function of w alone, we need to eliminate h and we do so by using the fact that the volume is 10 m^3. Thus

$$w(2w)h = 10$$

which gives

$$h = \frac{10}{2w^2} = \frac{5}{w^2}$$

Substituting this into the expression for C, we have

$$C = 20w^2 + 36w\left(\frac{5}{w^2}\right) = 20w^2 + \frac{180}{w}$$

Therefore the equation

$$C(w) = 20w^2 + \frac{180}{w} \qquad w > 0$$

expresses C as a function of w. ■

PS In setting up applied functions as in Example 5, it may be useful to review the principles of problem solving at the end of this chapter, particularly *Step 1: Understand the Problem.*

In the next example we find the domain of a function that is defined algebraically. If a function is given by a formula and the domain is not stated explicitly, we use the following **domain convention**: the domain of the function is the set of all inputs for which the formula makes sense and gives a real-number output.

EXAMPLE 6 Find the domain of each function.

(a) $f(x) = \sqrt{x + 2}$ (b) $g(x) = \dfrac{1}{x^2 - x}$

SOLUTION

(a) Because the square root of a negative number is not defined (as a real number), the domain of f consists of all values of x such that $x + 2 \geqslant 0$. This is equivalent to $x \geqslant -2$, so the domain is the interval $[-2, \infty)$.

(b) Since

$$g(x) = \frac{1}{x^2 - x} = \frac{1}{x(x - 1)}$$

and division by 0 is not allowed, we see that $g(x)$ is not defined when $x = 0$ or $x = 1$. So the domain of g is

$$\{x \mid x \neq 0, x \neq 1\}$$

which could also be written in interval notation as

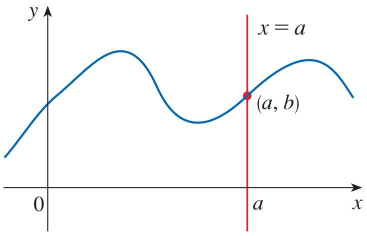
$$(-\infty, 0) \cup (0, 1) \cup (1, \infty)$$

■

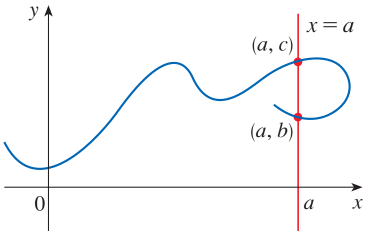
Which Rules Define Functions?

Not every equation defines a function. The equation $y = x^2$ defines y as a function of x because the equation determines exactly one value of y for each value of x. However, the equation $y^2 = x$ does *not* define y as a function of x because some input values x correspond to more than one output y; for instance, for the input $x = 4$ the equation gives the outputs $y = 2$ and $y = -2$.

Similarly, not every table defines a function. Table 3 defined C as a function of w—each package weight w corresponds to exactly one mailing cost. On the other hand, Table 4 does *not* define y as a function of x because some input values x in the table correspond to more than one output y; for instance, the input $x = 5$ gives the outputs $y = 7$ and $y = 8$.



(a) This curve represents a function.



(b) This curve doesn't represent a function.

FIGURE 13

Table 4

x	2	4	5	5	6
y	3	6	7	8	9

What about curves drawn in the xy-plane? Which curves are graphs of functions? The following test gives an answer.

The Vertical Line Test A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

The reason for the truth of the Vertical Line Test can be seen in Figure 13. If each vertical line $x = a$ intersects a curve only once, at (a, b), then exactly one function value is defined by $f(a) = b$. But if a line $x = a$ intersects the curve twice, at (a, b) and (a, c), then the curve can't represent a function because a function can't assign two different values to a.

For example, the parabola $x = y^2 - 2$ shown in Figure 14(a) is not the graph of a function of x because, as you can see, there are vertical lines that intersect the parabola twice. The parabola, however, does contain the graphs of *two* functions of x. Notice that the equation $x = y^2 - 2$ implies $y^2 = x + 2$, so $y = \pm\sqrt{x + 2}$. Thus the upper and lower halves of the parabola are the graphs of the functions $f(x) = \sqrt{x + 2}$ [from Example 6(a)] and $g(x) = -\sqrt{x + 2}$. [See Figures 14(b) and (c).]

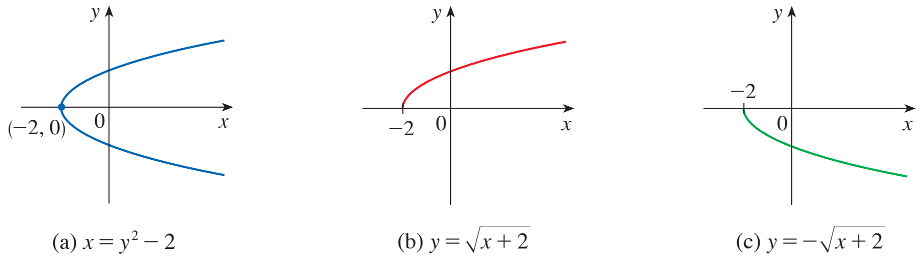


FIGURE 14

(a) $x = y^2 - 2$ (b) $y = \sqrt{x + 2}$ (c) $y = -\sqrt{x + 2}$

We observe that if we reverse the roles of x and y, then the equation $x = h(y) = y^2 - 2$ *does* define x as a function of y (with y as the independent variable and x as the dependent variable). The graph of the function h is the parabola in Figure 14(a).

■ Piecewise Defined Functions

The functions in the following four examples are defined by different formulas in different parts of their domains. Such functions are called **piecewise defined functions**.

EXAMPLE 7 A function f is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leqslant -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph.

SOLUTION Remember that a function is a rule. For this particular function the rule is the following: First look at the value of the input x. If it happens that $x \leqslant -1$, then the value of $f(x)$ is $1 - x$. On the other hand, if $x > -1$, then the value of $f(x)$ is x^2. Note that even though two different formulas are used, f is *one* function, not two.

$$\text{Since } -2 \leqslant -1, \text{ we have } f(-2) = 1 - (-2) = 3.$$

$$\text{Since } -1 \leqslant -1, \text{ we have } f(-1) = 1 - (-1) = 2.$$

$$\text{Since } 0 > -1, \text{ we have } f(0) = 0^2 = 0.$$

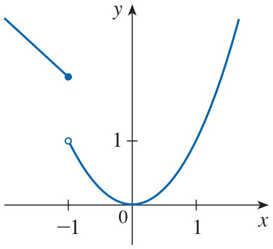


FIGURE 15

How do we draw the graph of f? We observe that if $x \leqslant -1$, then $f(x) = 1 - x$, so the part of the graph of f that lies to the left of the vertical line $x = -1$ must coincide with the line $y = 1 - x$, which has slope -1 and y-intercept 1. If $x > -1$, then $f(x) = x^2$, so the part of the graph of f that lies to the right of the line $x = -1$ must coincide with the graph of $y = x^2$, which is a parabola. This enables us to sketch the graph in Figure 15. The solid dot indicates that the point $(-1, 2)$ is included on the graph; the open dot indicates that the point $(-1, 1)$ is excluded from the graph. ■

The next example of a piecewise defined function is the absolute value function. Recall that the **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line. Distances are always positive or 0, so we have

For a more extensive review of absolute values, see Appendix A.

$$|a| \geqslant 0 \qquad \text{for every number } a$$

For example,

$$|3| = 3 \qquad |-3| = 3 \qquad |0| = 0 \qquad |\sqrt{2} - 1| = \sqrt{2} - 1 \qquad |3 - \pi| = \pi - 3$$

In general, we have

$$\boxed{\begin{aligned} |a| &= a && \text{if } a \geqslant 0 \\ |a| &= -a && \text{if } a < 0 \end{aligned}}$$

(Remember that if a is negative, then $-a$ is positive.)

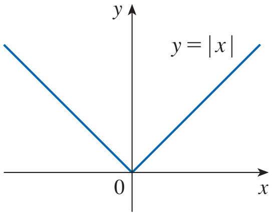


FIGURE 16

EXAMPLE 8 Sketch the graph of the absolute value function $f(x) = |x|$.

SOLUTION From the preceding discussion we know that

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

Using the same method as in Example 7, we see that the graph of f coincides with the line $y = x$ to the right of the y-axis and coincides with the line $y = -x$ to the left of the y-axis (see Figure 16). ■

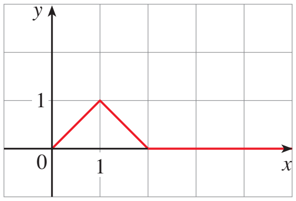


FIGURE 17

The point-slope form of the equation of a line is $y - y_1 = m(x - x_1)$. See Appendix B.

EXAMPLE 9 Find a formula for the function f graphed in Figure 17.

SOLUTION The line through (0, 0) and (1, 1) has slope $m = 1$ and y-intercept $b = 0$, so its equation is $y = x$. Thus, for the part of the graph of f that joins (0, 0) to (1, 1), we have

$$f(x) = x \qquad \text{if } 0 \leqslant x \leqslant 1$$

The line through (1, 1) and (2, 0) has slope $m = -1$, so its point-slope form is

$$y - 0 = (-1)(x - 2) \qquad \text{or} \qquad y = 2 - x$$

So we have

$$f(x) = 2 - x \qquad \text{if } 1 < x \leqslant 2$$

We also see that the graph of f coincides with the x-axis for $x > 2$. Putting this information together, we have the following three-piece formula for f:

$$f(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant 1 \\ 2 - x & \text{if } 1 < x \leqslant 2 \\ 0 & \text{if } x > 2 \end{cases}$$

■

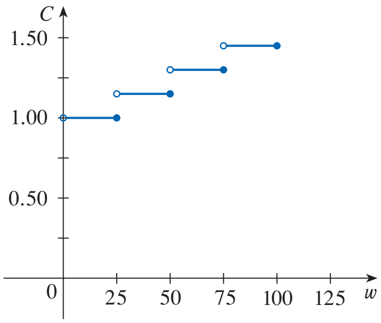


FIGURE 18

EXAMPLE 10 In Example C at the beginning of this section we considered the cost $C(w)$ of mailing a large envelope with weight w. In effect, this is a piecewise defined function because, from Table 3, we have

$$C(w) = \begin{cases} 1.00 & \text{if } 0 < w < 25 \\ 1.15 & \text{if } 25 < w < 50 \\ 1.30 & \text{if } 50 < w < 75 \\ 1.45 & \text{if } 75 < w < 100 \\ \quad\vdots & \end{cases}$$

The graph is shown in Figure 18. ■

Looking at Figure 18, you can see why a function like the one in Example 10 is called a **step function**.

Even and Odd Functions

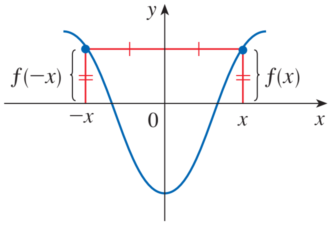


FIGURE 19
An even function

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

The geometric significance of an even function is that its graph is symmetric with respect to the y-axis (see Figure 19). This means that if we have plotted the graph of f for $x \geqslant 0$, we obtain the entire graph simply by reflecting this portion about the y-axis.

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

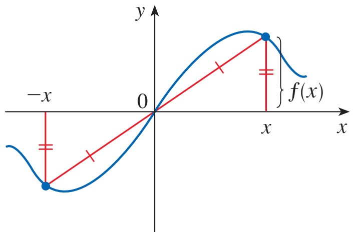


FIGURE 20
An odd function

The graph of an odd function is symmetric about the origin (see Figure 20). If we already have the graph of f for $x \geqslant 0$, we can obtain the entire graph by rotating this portion through 180° about the origin.

EXAMPLE 11 Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

SOLUTION

(a)
$$\begin{aligned} f(-x) &= (-x)^5 + (-x) = (-1)^5 x^5 + (-x) \\ &= -x^5 - x = -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore f is an odd function.

(b)
$$g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

So g is even.

(c)
$$h(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd. ■

The graphs of the functions in Example 11 are shown in Figure 21. Notice that the graph of h is symmetric neither about the y-axis nor about the origin.

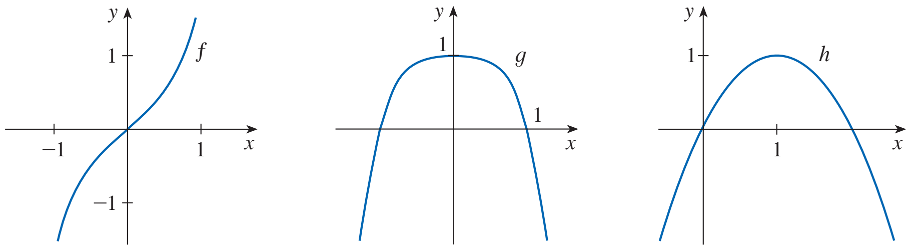


FIGURE 21 (a) (b) (c)

Increasing and Decreasing Functions

The graph shown in Figure 22 rises from A to B, falls from B to C, and rises again from C to D. The function f is said to be increasing on the interval $[a, b]$, decreasing on $[b, c]$, and increasing again on $[c, d]$. Notice that if x_1 and x_2 are any two numbers between

a and b with $x_1 < x_2$, then $f(x_1) < f(x_2)$. We use this as the defining property of an increasing function.

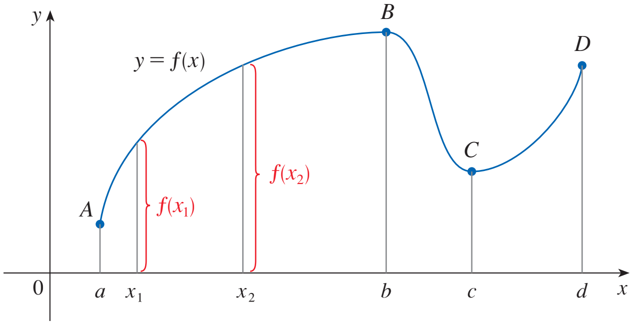


FIGURE 22

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

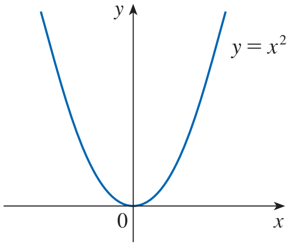


FIGURE 23

In the definition of an increasing function it is important to realize that the inequality $f(x_1) < f(x_2)$ must be satisfied for *every* pair of numbers x_1 and x_2 in I with $x_1 < x_2$.

You can see from Figure 23 that the function $f(x) = x^2$ is decreasing on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$.

1.1 Exercises

1. If $f(x) = x + \sqrt{2 - x}$ and $g(u) = u + \sqrt{2 - u}$, is it true that $f = g$?

2. If

$$f(x) = \frac{x^2 - x}{x - 1} \qquad \text{and} \qquad g(x) = x$$

is it true that $f = g$?

3. The graph of a function g is given.

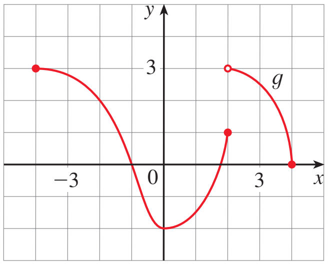
(a) State the values of $g(-2)$, $g(0)$, $g(2)$, and $g(3)$.

(b) For what value(s) of x is $g(x) = 3$?

(c) For what value(s) of x is $g(x) \leqslant 3$?

(d) State the domain and range of g.

(e) On what interval(s) is g increasing?



4. The graphs of f and g are given.

(a) State the values of $f(-4)$ and $g(3)$.

(b) Which is larger, $f(-3)$ or $g(-3)$?

(c) For what values of x is $f(x) = g(x)$?

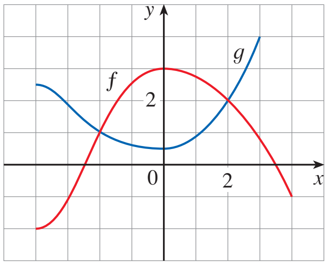
(d) On what interval(s) is $f(x) \leqslant g(x)$?

(e) State the solution of the equation $f(x) = -1$.

(f) On what interval(s) is g decreasing?

(g) State the domain and range of f.

(h) State the domain and range of g.



5. Figure 1 was recorded by an instrument operated by the California Department of Mines and Geology at the

University Hospital of the University of Southern California in Los Angeles. Use it to estimate the range of the vertical ground acceleration function at USC during the Northridge earthquake.

6. In this section we discussed examples of ordinary, everyday functions: population is a function of time, postage cost is a function of package weight, water temperature is a function of time. Give three other examples of functions from everyday life that are described verbally. What can you say about the domain and range of each of your functions? If possible, sketch a rough graph of each function.

7–14 Determine whether the equation or table defines y as a function of x.

7. $3x - 5y = 7$

8. $3x^2 - 2y = 5$

9. $x^2 + (y - 3)^2 = 5$

10. $2xy + 5y^2 = 4$

11. $(y + 3)^3 + 1 = 2x$

12. $2x - |y| = 0$

13.

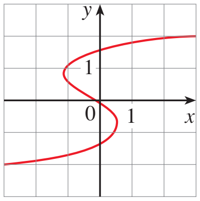
x Height (cm)	y Shoe size
180	12
150	8
150	7
160	9
175	10

14.

x Year	y Tuition cost ($)
2016	10,900
2017	11,000
2018	11,200
2019	11,200
2020	11,300

15–18 Determine whether the curve is the graph of a function of x. If it is, state the domain and range of the function.

15.



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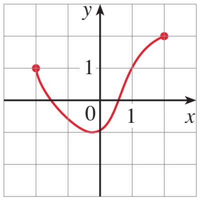

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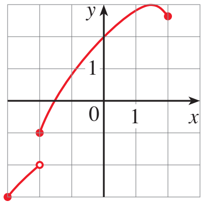
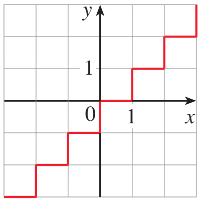


18.



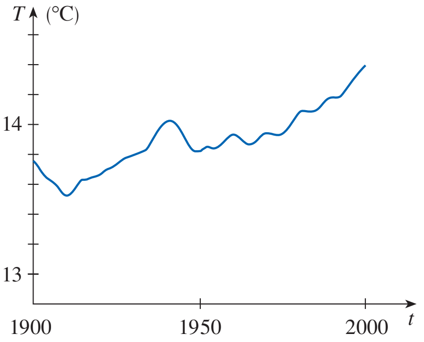
19. Shown is a graph of the global average temperature T during the 20th century. Estimate the following.

(a) The global average temperature in 1950

(b) The year when the average temperature was 14.2°C

(c) The years when the temperature was smallest and largest

(d) The range of T

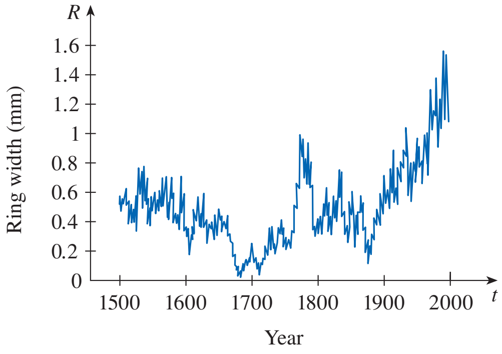


Source: Adapted from *Globe and Mail* [Toronto], 5 Dec. 2009. Print.

20. Trees grow faster and form wider rings in warm years and grow more slowly and form narrower rings in cooler years. The figure shows ring widths of a Siberian pine from 1500 to 2000.

(a) What is the range of the ring width function?

(b) What does the graph tend to say about the temperature of the earth? Does the graph reflect the volcanic eruptions of the mid-19th century?



Source: Adapted from G. Jacoby et al., "Mongolian Tree Rings and 20th-Century Warming," *Science* 273 (1996): 771–73.

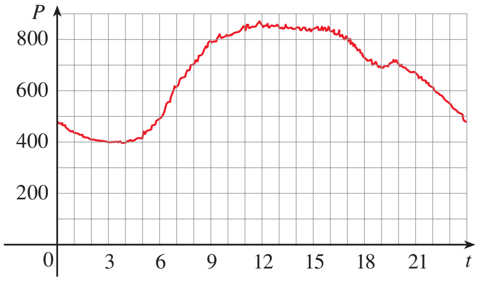
21. You put some ice cubes in a glass, fill the glass with cold water, and then let the glass sit on a table. Describe how the temperature of the water changes as time passes. Then sketch a rough graph of the temperature of the water as a function of the elapsed time.

22. You place a frozen pie in an oven and bake it for an hour. Then you take it out and let it cool. Describe how the temperature of the pie changes as time passes. Then sketch a rough graph of the temperature of the pie as a function of time.

23. The graph shows the power consumption for a day in September in San Francisco. (P is measured in megawatts; t is measured in hours starting at midnight.)

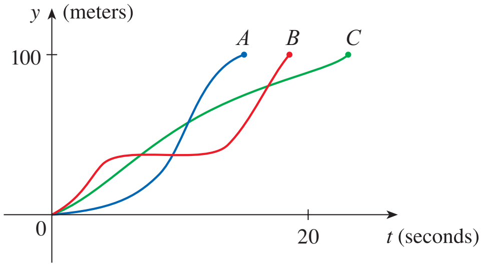
(a) What was the power consumption at 6 AM? At 6 PM?

(b) When was the power consumption the lowest? When was it the highest? Do these times seem reasonable?



Pacific Gas & Electric

24. Three runners compete in a 100-meter race. The graph depicts the distance run as a function of time for each runner. Describe in words what the graph tells you about this race. Who won the race? Did each runner finish the race?



25. Sketch a rough graph of the outdoor temperature as a function of time during a typical spring day.

26. Sketch a rough graph of the number of hours of daylight as a function of the time of year.

27. Sketch a rough graph of the amount of a particular brand of coffee sold by a store as a function of the price of the coffee.

28. Sketch a rough graph of the market value of a new car as a function of time for a period of 20 years. Assume the car is well maintained.

29. A homeowner mows the lawn every Wednesday afternoon. Sketch a rough graph of the height of the grass as a function of time over the course of a four-week period.

30. An airplane takes off from an airport and lands an hour later at another airport, 650 kilometers away. If t represents the time in minutes since the plane has left the terminal building, let $x(t)$ be the horizontal distance traveled and $y(t)$ be the altitude of the plane.
(a) Sketch a possible graph of $x(t)$.
(b) Sketch a possible graph of $y(t)$.
(c) Sketch a possible graph of the ground speed.
(d) Sketch a possible graph of the vertical velocity.

31. Temperature readings T (in °C) were recorded every two hours from midnight to 2:00 PM in Atlanta on a day in June. The time t was measured in hours from midnight.

t	0	2	4	6	8	10	12	14
T	23	21	20	19	21	26	28	30

(a) Use the readings to sketch a rough graph of T as a function of t.
(b) Use your graph to estimate the temperature at 9:00 AM.

32. Researchers measured the blood alcohol concentration (BAC) of eight adult male subjects after rapid consumption of 30 mL of ethanol (corresponding to two standard alcoholic drinks). The table shows the data they obtained by averaging the BAC (in g/dL) of the eight men.
(a) Use the readings to sketch a graph of the BAC as a function of t.
(b) Use your graph to describe how the effect of alcohol varies with time.

t (hours)	BAC	t (hours)	BAC
0	0	1.75	0.022
0.2	0.025	2.0	0.018
0.5	0.041	2.25	0.015
0.75	0.040	2.5	0.012
1.0	0.033	3.0	0.007
1.25	0.029	3.5	0.003
1.5	0.024	4.0	0.001

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

33. If $f(x) = 3x^2 - x + 2$, find $f(2)$, $f(-2)$, $f(a)$, $f(-a)$, $f(a + 1)$, $2f(a)$, $f(2a)$, $f(a^2)$, $[f(a)]^2$, and $f(a + h)$.

34. If $g(x) = \dfrac{x}{\sqrt{x + 1}}$, find $g(0)$, $g(3)$, $5g(a)$, $\frac{1}{2}g(4a)$, $g(a^2)$, $[g(a)]^2$, $g(a + h)$, and $g(x - a)$.

35–38 Evaluate the difference quotient for the given function. Simplify your answer.

35. $f(x) = 4 + 3x - x^2$, $\quad \dfrac{f(3 + h) - f(3)}{h}$

36. $f(x) = x^3$, $\quad \dfrac{f(a + h) - f(a)}{h}$

37. $f(x) = \dfrac{1}{x}$, $\quad \dfrac{f(x) - f(a)}{x - a}$

38. $f(x) = \sqrt{x + 2}$, $\quad \dfrac{f(x) - f(1)}{x - 1}$

39–46 Find the domain of the function.

39. $f(x) = \dfrac{x + 4}{x^2 - 9}$

40. $f(x) = \dfrac{x^2 + 1}{x^2 + 4x - 21}$

41. $f(t) = \sqrt[3]{2t - 1}$

42. $g(t) = \sqrt{3 - t} - \sqrt{2 + t}$

43. $h(x) = \dfrac{1}{\sqrt[4]{x^2 - 5x}}$

44. $f(u) = \dfrac{u + 1}{1 + \dfrac{1}{u + 1}}$

45. $F(p) = \sqrt{2 - \sqrt{p}}$

46. $h(x) = \sqrt{x^2 - 4x - 5}$

47. Find the domain and range and sketch the graph of the function $h(x) = \sqrt{4 - x^2}$.

48. Find the domain and sketch the graph of the function

$$f(x) = \frac{x^2 - 4}{x - 2}$$

49–52 Evaluate $f(-3)$, $f(0)$, and $f(2)$ for the piecewise defined function. Then sketch the graph of the function.

49. $f(x) = \begin{cases} x^2 + 2 & \text{if } x < 0 \\ x & \text{if } x \geqslant 0 \end{cases}$

50. $f(x) = \begin{cases} 5 & \text{if } x < 2 \\ \frac{1}{2}x - 3 & \text{if } x \geqslant 2 \end{cases}$

51. $f(x) = \begin{cases} x + 1 & \text{if } x \leqslant -1 \\ x^2 & \text{if } x > -1 \end{cases}$

52. $f(x) = \begin{cases} -1 & \text{if } x \leqslant 1 \\ 7 - 2x & \text{if } x > 1 \end{cases}$

53–58 Sketch the graph of the function.

53. $f(x) = x + |x|$

54. $f(x) = |x + 2|$

55. $g(t) = |1 - 3t|$

56. $f(x) = \dfrac{|x|}{x}$

57. $f(x) = \begin{cases} |x| & \text{if } |x| \leqslant 1 \\ 1 & \text{if } |x| > 1 \end{cases}$

58. $g(x) = \big||x| - 1\big|$

59–64 Find a formula for the function whose graph is the given curve.

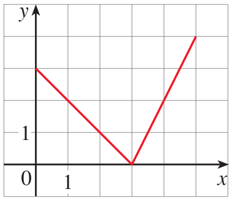
59. The line segment joining the points $(1, -3)$ and $(5, 7)$

60. The line segment joining the points $(-5, 10)$ and $(7, -10)$

61. The bottom half of the parabola $x + (y - 1)^2 = 0$

62. The top half of the circle $x^2 + (y - 2)^2 = 4$

63.



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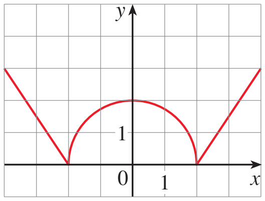


65–70 Find a formula for the described function and state its domain.

65. A rectangle has perimeter 20 m. Express the area of the rectangle as a function of the length of one of its sides.

66. A rectangle has area 16 m^2. Express the perimeter of the rectangle as a function of the length of one of its sides.

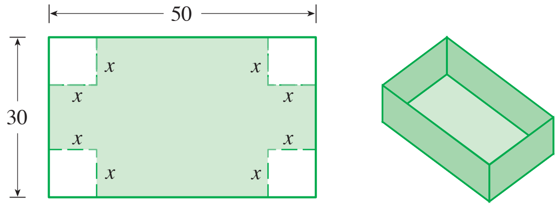
67. Express the area of an equilateral triangle as a function of the length of a side.

68. A closed rectangular box with volume 0.25 m^3 has length twice the width. Express the height of the box as a function of the width.

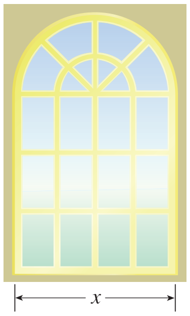
69. An open rectangular box with volume 2 m^3 has a square base. Express the surface area of the box as a function of the length of a side of the base.

70. A right circular cylinder has volume 400 cm^3. Express the radius of the cylinder as a function of the height.

71. A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 30 cm by 50 cm by cutting out equal squares of side x at each corner and then folding up the sides as in the figure. Express the volume V of the box as a function of x.



72. A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 10 m, express the area A of the window as a function of the width x of the window.



73. In a certain state the maximum speed permitted on freeways is 100 km/h and the minimum speed is 60 km/h. The fine for violating these limits is \$15 for every kilometer per hour above the maximum speed or below the minimum speed. Express the amount of the fine F as a function of the driving speed x and graph $F(x)$ for $0 \leqslant x \leqslant 150$.

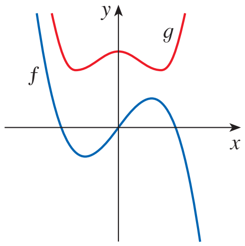
74. An electricity company charges its customers a base rate of \$10 a month, plus 6 cents per kilowatt-hour (kWh) for the first 1200 kWh and 7 cents per kWh for all usage over 1200 kWh. Express the monthly cost E as a function of the amount x of electricity used. Then graph the function E for $0 \leqslant x \leqslant 2000$.

75. In a certain country, income tax is assessed as follows. There is no tax on income up to \$10,000. Any income over \$10,000 is taxed at a rate of 10%, up to an income of \$20,000. Any income over \$20,000 is taxed at 15%.
(a) Sketch the graph of the tax rate R as a function of the income I.
(b) How much tax is assessed on an income of \$14,000? On \$26,000?
(c) Sketch the graph of the total assessed tax T as a function of the income I.

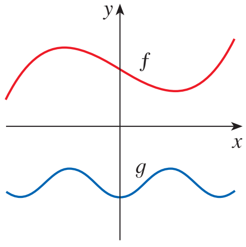
76. (a) If the point (5, 3) is on the graph of an even function, what other point must also be on the graph?
(b) If the point (5, 3) is on the graph of an odd function, what other point must also be on the graph?

77–78 Graphs of f and g are shown. Decide whether each function is even, odd, or neither. Explain your reasoning.

77.

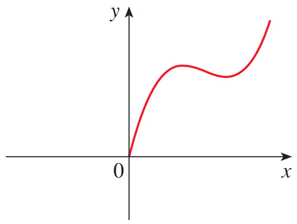


78.



79–80 The graph of a function defined for $x \geqslant 0$ is given. Complete the graph for $x < 0$ to make (a) an even function and (b) an odd function.

79.



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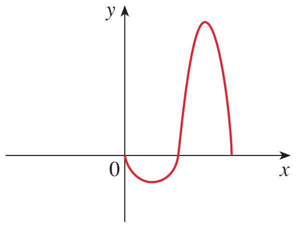


81–86 Determine whether f is even, odd, or neither. You may wish to use a graphing calculator or computer to check your answer visually.

81. $f(x) = \dfrac{x}{x^2 + 1}$

82. $f(x) = \dfrac{x^2}{x^4 + 1}$

83. $f(x) = \dfrac{x}{x + 1}$

84. $f(x) = x|x|$

85. $f(x) = 1 + 3x^2 - x^4$

86. $f(x) = 1 + 3x^3 - x^5$

87. If f and g are both even functions, is $f + g$ even? If f and g are both odd functions, is $f + g$ odd? What if f is even and g is odd? Justify your answers.

88. If f and g are both even functions, is the product fg even? If f and g are both odd functions, is fg odd? What if f is even and g is odd? Justify your answers.

1.2 Mathematical Models: A Catalog of Essential Functions

A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon such as the size of a population, the demand for a product, the speed of a falling object, the concentration of a product in a chemical reaction, the life expectancy of a person at birth, or the cost of emissions reductions. The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

Given a real-world problem, our first task in the mathematical modeling process is to formulate a mathematical model by identifying and naming the independent and dependent variables and making assumptions that simplify the phenomenon enough to make it mathematically tractable. We use our knowledge of the physical situation and our mathematical skills to obtain equations that relate the variables. In situations where there is no physical law to guide us, we may need to collect data (either from the Internet or a library or by conducting our own experiments) and examine the data in the form of a table in order to discern patterns. From this numerical representation of a function we may wish to obtain a graphical representation by plotting the data. The graph might even suggest a suitable algebraic formula in some cases.

The second stage is to apply the mathematics that we know (such as the calculus that will be developed throughout this book) to the mathematical model that we have formulated in order to derive mathematical conclusions. Then, in the third stage, we take those mathematical conclusions and interpret them as information about the original real-world phenomenon by way of offering explanations or making predictions. The final step is to test our predictions by checking against new real data. If the predictions don't compare well with reality, we need to refine our model or formulate a new model and start the cycle again. Figure 1 illustrates the process of mathematical modeling.

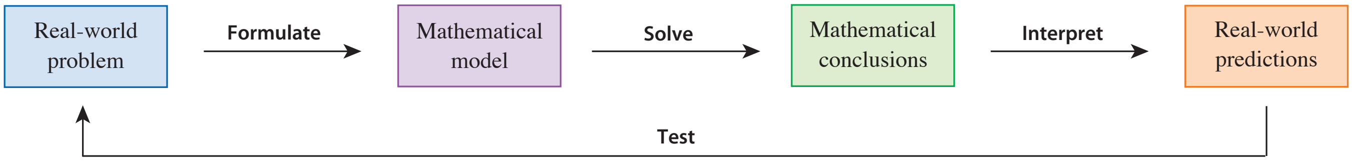


FIGURE 1
The modeling process

A mathematical model is never a completely accurate representation of a physical situation—it is an *idealization*. A good model simplifies reality enough to permit mathematical calculations but is accurate enough to provide valuable conclusions. It is important to realize the limitations of a model.

There are many different types of functions that can be used to model relationships observed in the real world. In what follows, we discuss the behavior and graphs of some of these functions and give examples of situations appropriately modeled by such functions.

Linear Models

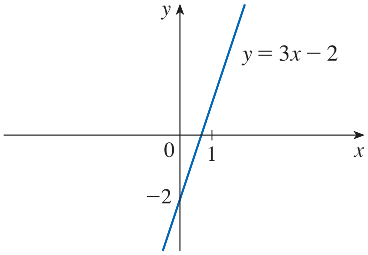
The coordinate geometry of lines is reviewed in Appendix B.

When we say that y is a **linear function** of x, we mean that the graph of the function is a line, so we can use the slope-intercept form of the equation of a line to write a formula for the function as

$$y = f(x) = mx + b$$

where m is the slope of the line and b is the y-intercept.

A characteristic feature of linear functions is that they change at a constant rate. For instance, Figure 2 shows a graph of the linear function $f(x) = 3x - 2$ and a table of sample values. Notice that whenever x increases by 0.1, the value of $f(x)$ increases by 0.3. So $f(x)$ increases three times as fast as x. This means that the slope of the graph of $y = 3x - 2$, namely 3, can be interpreted as the rate of change of y with respect to x.



x	$f(x) = 3x - 2$
1.0	1.0
1.1	1.3
1.2	1.6
1.3	1.9
1.4	2.2
1.5	2.5

FIGURE 2

EXAMPLE 1

(a) As dry air moves upward, it expands and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C, express the temperature T (in °C) as a function of the height h (in kilometers), assuming that a linear model is appropriate.
(b) Draw the graph of the function in part (a). What does the slope represent?
(c) What is the temperature at a height of 2.5 km?

SOLUTION

(a) Because we are assuming that T is a linear function of h, we can write

$$T = mh + b$$

We are given that $T = 20$ when $h = 0$, so

$$20 = m \cdot 0 + b = b$$

In other words, the y-intercept is $b = 20$.

We are also given that $T = 10$ when $h = 1$, so

$$10 = m \cdot 1 + 20$$

The slope of the line is therefore $m = 10 - 20 = -10$ and the required linear function is

$$T = -10h + 20$$

(b) The graph is sketched in Figure 3. The slope is $m = -10$°C/km, and this represents the rate of change of temperature with respect to height.

(c) At a height of $h = 2.5$ km, the temperature is

$$T = -10(2.5) + 20 = -5°\text{C}$$ ■

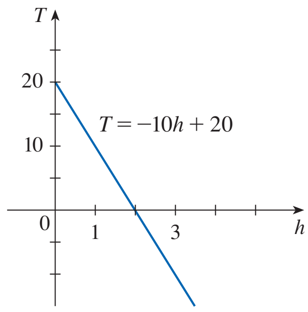


FIGURE 3

If there is no physical law or principle to help us formulate a model, we construct an **empirical model**, which is based entirely on collected data. We seek a curve that "fits" the data in the sense that it captures the basic trend of the data points.

EXAMPLE 2 Table 1 lists the average carbon dioxide level in the atmosphere, measured in parts per million at Mauna Loa Observatory from 1980 to 2016. Use the data in Table 1 to find a model for the carbon dioxide level.

Table 1

Year	CO_2 level (in ppm)	Year	CO_2 level (in ppm)
1980	338.7	2000	369.4
1984	344.4	2004	377.5
1988	351.5	2008	385.6
1992	356.3	2012	393.8
1996	362.4	2016	404.2

SOLUTION We use the data in Table 1 to make the scatter plot in Figure 4, where t represents time (in years) and C represents the CO_2 level (in parts per million, ppm).

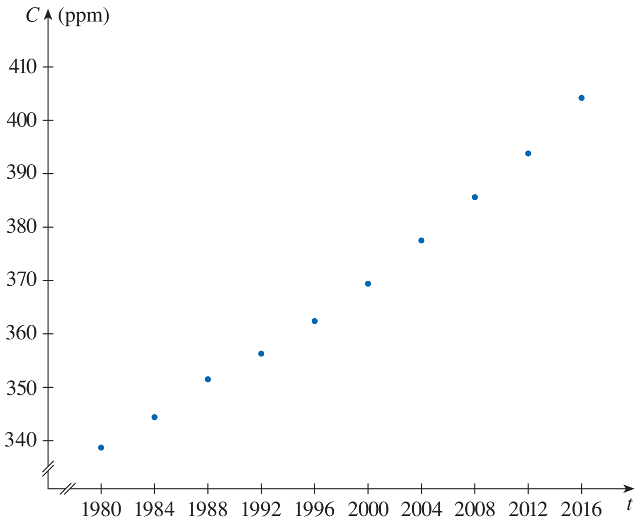


FIGURE 4
Scatter plot for the average CO_2 level

Notice that the data points appear to lie close to a straight line, so it's natural to choose a linear model in this case. But there are many possible lines that approximate these data points, so which one should we use? One possibility is the line that passes through the first and last data points. The slope of this line is

$$\frac{404.2 - 338.7}{2016 - 1980} = \frac{65.5}{36} \approx 1.819$$

We write its equation as

$$C - 338.7 = 1.819(t - 1980)$$

or

$$\boxed{1} \qquad C = 1.819t - 3262.92$$

A computer or graphing calculator finds the regression line by the **method of least squares**, which is to minimize the sum of the squares of the vertical distances between the data points and the line. The details are explained in Exercise 14.7.61.

Equation 1 gives one possible linear model for the carbon dioxide level; it is graphed in Figure 5. Notice that our model gives values higher than most of the actual CO_2 levels. A better linear model is obtained by a procedure from statistics called *linear regression*. Many graphing calculators and computer software applications can determine the regression line for a set of data. One such calculator gives the slope and y-intercept of the regression line for the data from Table 1 as

$$m = 1.78242 \qquad b = -3192.90$$

So our least squares model for the CO_2 level is

$$\boxed{2} \qquad C = 1.78242t - 3192.90$$

In Figure 6 we graph the regression line as well as the data points. Comparing with Figure 5, we see that the regression line gives a better fit.

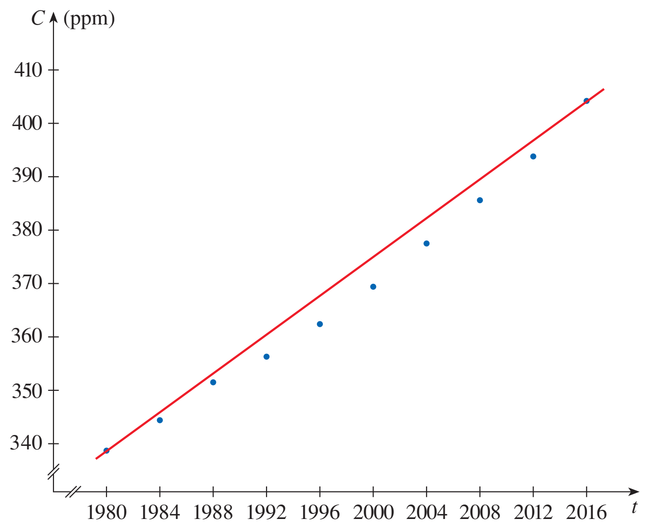


FIGURE 5
Linear model through first and last data points

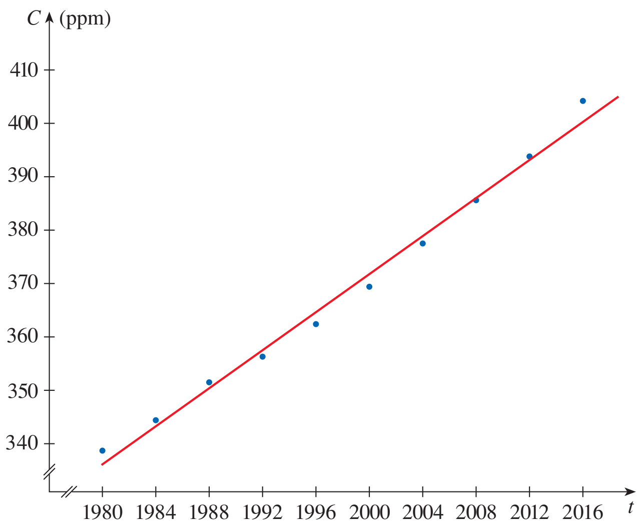


FIGURE 6
The regression line ■

EXAMPLE 3 Use the linear model given by Equation 2 to estimate the average CO_2 level for 1987 and to predict the level for the year 2025. According to this model, when will the CO_2 level exceed 440 parts per million?

SOLUTION Using Equation 2 with $t = 1987$, we estimate that the average CO_2 level in 1987 was

$$C(1987) = 1.78242(1987) - 3192.90 \approx 348.77$$

This is an example of *interpolation* because we have estimated a value *between* observed values. (In fact, the Mauna Loa Observatory reported that the average CO_2 level in 1987 was 348.93 ppm, so our estimate is quite accurate.)

With $t = 2025$, we get

$$C(2025) = 1.78242(2025) - 3192.90 \approx 416.50$$

So we predict that the average CO_2 level in the year 2025 will be 416.5 ppm. This is an example of *extrapolation* because we have predicted a value *outside* the time frame of observations. Consequently, we are far less certain about the accuracy of our prediction.

Using Equation 2, we see that the CO_2 level exceeds 440 ppm when

$$1.78242t - 3192.90 > 440$$

Solving this inequality, we get

$$t > \frac{3632.9}{1.78242} \approx 2038.18$$

We therefore predict that the CO_2 level will exceed 440 ppm by the year 2038. This prediction is risky because it involves a time quite remote from our observations. In fact, we see from Figure 6 that the trend has been for CO_2 levels to increase rather more rapidly in recent years, so the level might exceed 440 ppm well before 2038. ■

Polynomials

A function P is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the **coefficients** of the polynomial. The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the **leading coefficient** $a_n \neq 0$, then the **degree** of the polynomial is n. For example, the function

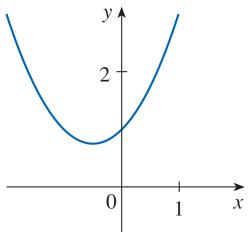
$$P(x) = 2x^6 - x^4 + \tfrac{2}{5}x^3 + \sqrt{2}$$

is a polynomial of degree 6.

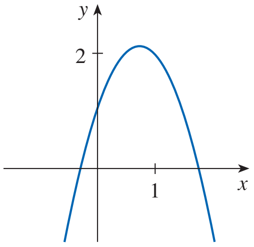
A polynomial of degree 1 is of the form $P(x) = mx + b$ and so it is a linear function. A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic function**. Its graph is always a parabola obtained by shifting the parabola $y = ax^2$, as we will see in Section 1.3. The parabola opens upward if $a > 0$ and downward if $a < 0$. (See Figure 7.)

A polynomial of degree 3 is of the form

$$P(x) = ax^3 + bx^2 + cx + d \qquad a \neq 0$$



(a) $y = x^2 + x + 1$



(b) $y = -2x^2 + 3x + 1$

FIGURE 7
The graphs of quadratic functions are parabolas.

and is called a **cubic function**. Figure 8 shows the graph of a cubic function in part (a) and graphs of polynomials of degrees 4 and 5 in parts (b) and (c). We will see later why the graphs have these shapes.

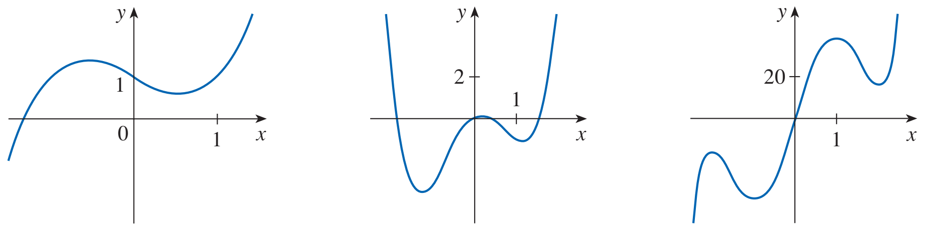


FIGURE 8

(a) $y = x^3 - x + 1$ (b) $y = x^4 - 3x^2 + x$ (c) $y = 3x^5 - 25x^3 + 60x$

Polynomials are commonly used to model various quantities that occur in the natural and social sciences. For instance, in Section 3.7 we will explain why economists often use a polynomial $P(x)$ to represent the cost of producing x units of a commodity. In the following example we use a quadratic function to model the fall of a ball.

Table 2

Time (seconds)	Height (meters)
0	450
1	445
2	431
3	408
4	375
5	332
6	279
7	216
8	143
9	61

EXAMPLE 4 A ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground, and its height h above the ground is recorded at 1-second intervals in Table 2. Find a model to fit the data and use the model to predict the time at which the ball hits the ground.

SOLUTION We draw a scatter plot of the data in Figure 9 and observe that a linear model is inappropriate. But it looks as if the data points might lie on a parabola, so we try a quadratic model instead. Using a graphing calculator or computer algebra system (which uses the least squares method), we obtain the following quadratic model:

$$h = 449.36 + 0.96t - 4.90t^2 \tag{3}$$

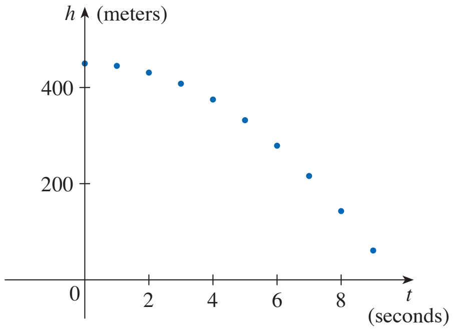


FIGURE 9
Scatter plot for a falling ball

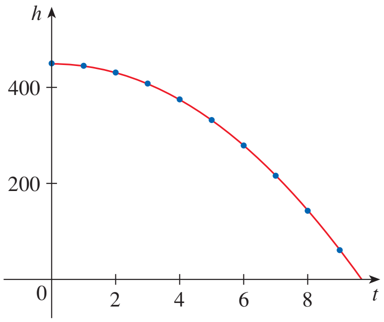


FIGURE 10
Quadratic model for a falling ball

In Figure 10 we plot the graph of Equation 3 together with the data points and see that the quadratic model gives a very good fit.

The ball hits the ground when $h = 0$, so we solve the quadratic equation

$$-4.90t^2 + 0.96t + 449.36 = 0$$

The quadratic formula gives

$$t = \frac{-0.96 \pm \sqrt{(0.96)^2 - 4(-4.90)(449.36)}}{2(-4.90)}$$

The positive root is $t \approx 9.67$, so we predict that the ball will hit the ground after falling about 9.7 seconds. ■

Power Functions

A function of the form $f(x) = x^a$, where a is a constant, is called a **power function**. We consider several cases.

(i) $a = n$, where n is a positive integer

The graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4$, and 5 are shown in Figure 11. (These are polynomials with only one term.) We already know the shape of the graphs of $y = x$ (a line through the origin with slope 1) and $y = x^2$ [a parabola, see Example 1.1.2(b)].

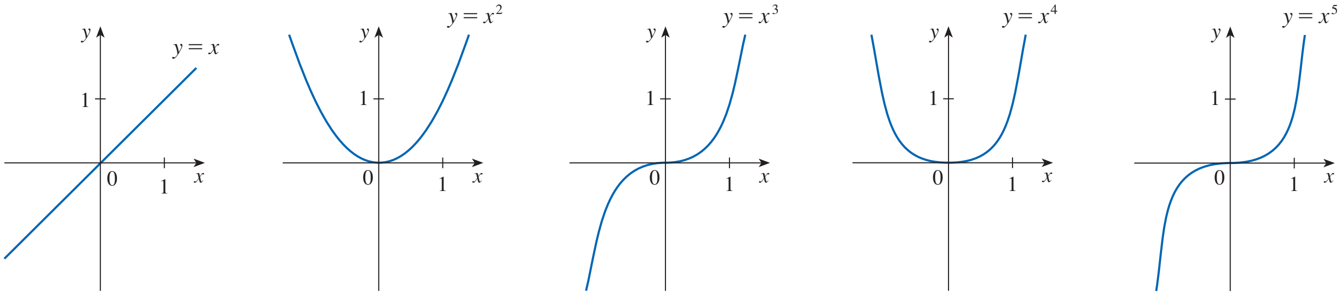


FIGURE 11 Graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4, 5$

The general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd. If n is even, then $f(x) = x^n$ is an even function and its graph is similar to the parabola $y = x^2$. If n is odd, then $f(x) = x^n$ is an odd function and its graph is similar to that of $y = x^3$. Notice from Figure 12, however, that as n increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $|x| \geqslant 1$. (If x is small, then x^2 is smaller, x^3 is even smaller, x^4 is smaller still, and so on.)

A **family of functions** is a collection of functions whose equations are related. Figure 12 shows two families of power functions, one with even powers and one with odd powers.

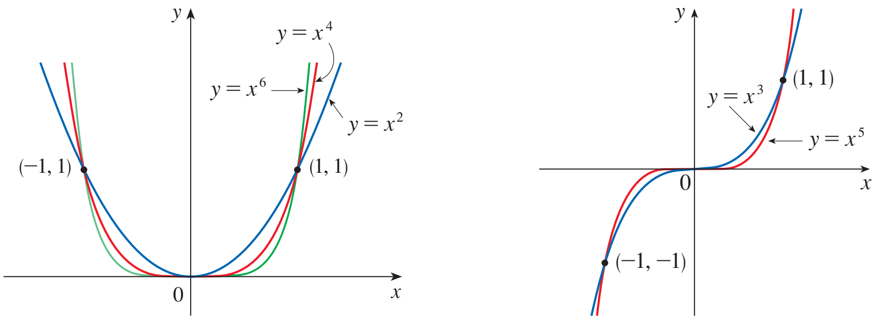
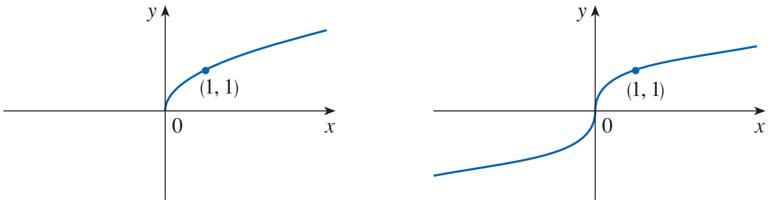


FIGURE 12

(ii) $a = 1/n$, where n is a positive integer

The function $f(x) = x^{1/n} = \sqrt[n]{x}$ is a **root function**. For $n = 2$ it is the square root function $f(x) = \sqrt{x}$, whose domain is $[0, \infty)$ and whose graph is the upper half of the parabola $x = y^2$. [See Figure 13(a).] For other even values of n, the graph of $y = \sqrt[n]{x}$ is similar to that of $y = \sqrt{x}$. For $n = 3$ we have the cube root function $f(x) = \sqrt[3]{x}$ whose domain is $\mathbb{R}$ (recall that every real number has a cube root) and whose graph is shown in Figure 13(b). The graph of $y = \sqrt[n]{x}$ for n odd ($n > 3$) is similar to that of $y = \sqrt[3]{x}$.



(a) $f(x) = \sqrt{x}$

(b) $f(x) = \sqrt[3]{x}$

FIGURE 13
Graphs of root functions

(iii) $a = -1$

The graph of the **reciprocal function** $f(x) = x^{-1} = 1/x$ is shown in Figure 14. Its graph has the equation $y = 1/x$, or $xy = 1$, and is a hyperbola with the coordinate axes as its asymptotes. This function arises in physics and chemistry in connection with Boyle's Law, which says that when the temperature is constant, the volume V of a gas is inversely proportional to the pressure P:

$$V = \frac{C}{P}$$

where C is a constant. Thus the graph of V as a function of P (see Figure 15) has the same general shape as the right half of Figure 14.

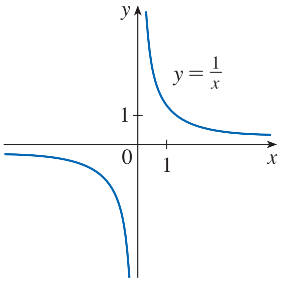


FIGURE 14
The reciprocal function

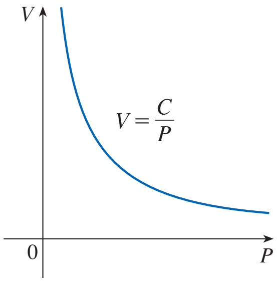


FIGURE 15
Volume as a function of pressure at constant temperature

(iv) $a = -2$

Among the remaining negative powers for the power function $f(x) = x^a$, by far the most important is that of $a = -2$. Many natural laws state that one quantity is inversely proportional to the square of another quantity. In other words, the first quantity is modeled by a function of the form $f(x) = C/x^2$ and we refer to this as an **inverse square law**. For instance, the illumination I of an object by a light source is inversely proportional to the square of the distance x from the source:

$$I = \frac{C}{x^2}$$

where C is a constant. Thus the graph of I as a function of x (see Figure 17) has the same general shape as the right half of Figure 16.

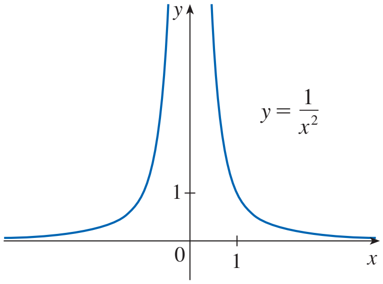


FIGURE 16
The reciprocal of the squaring function

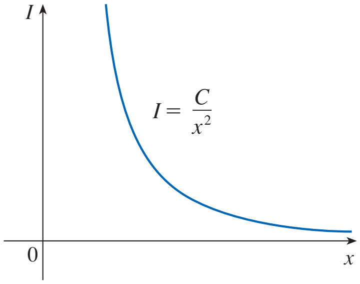


FIGURE 17
Illumination from a light source as a function of distance from the source

Inverse square laws model gravitational force, loudness of sound, and electrostatic force between two charged particles. See Exercise 37 for a geometric reason why inverse square laws often occur in nature.

Power functions are also used to model species-area relationships (Exercises 35–36) and the period of revolution of a planet as a function of its distance from the sun (see Exercise 34).

Rational Functions

A **rational function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain consists of all values of x such that $Q(x) \neq 0$. A simple example of a rational function is the function $f(x) = 1/x$, whose domain is $\{x \mid x \neq 0\}$; this is the reciprocal function graphed in Figure 14. The function

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

is a rational function with domain $\{x \mid x \neq \pm 2\}$. Its graph is shown in Figure 18.

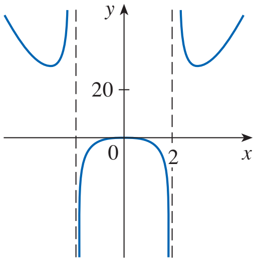


FIGURE 18
$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$

Algebraic Functions

A function f is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials. Any rational function is automatically an algebraic function. Here are two more examples:

$$f(x) = \sqrt{x^2 + 1} \qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

In Chapter 4 we will sketch a variety of algebraic functions, and we will see that their graphs can assume many different shapes.

An example of an algebraic function occurs in the theory of relativity. The mass of a particle with velocity v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

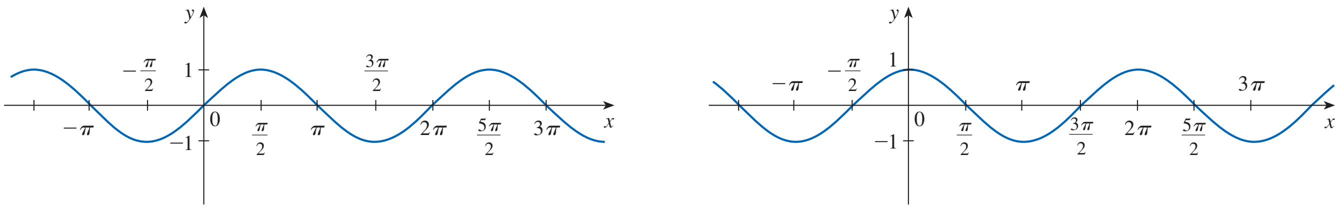
where m_0 is the rest mass of the particle and $c = 3.0 \times 10^5$ km/s is the speed of light in a vacuum.

Functions that are not algebraic are called **transcendental**; these include the trigonometric, exponential, and logarithmic functions.

Trigonometric Functions

The Reference Pages are located at the front and back of the book.

Trigonometry and the trigonometric functions are reviewed on Reference Page 2 and also in Appendix D. In calculus the convention is that *radian measure* is always used (except when otherwise indicated). For example, when we use the function $f(x) = \sin x$, it is understood that $\sin x$ means the sine of the angle whose radian measure is x. Thus the graphs of the sine and cosine functions are as shown in Figure 19.



(a) $f(x) = \sin x$

(b) $g(x) = \cos x$

FIGURE 19

Notice that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$. Thus, for all values of x, we have

$$\boxed{-1 \leqslant \sin x \leqslant 1 \qquad -1 \leqslant \cos x \leqslant 1}$$

or, in terms of absolute values,

$$|\sin x| \leqslant 1 \qquad |\cos x| \leqslant 1$$

An important property of the sine and cosine functions is that they are periodic functions and have period 2π. This means that, for all values of x,

$$\boxed{\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x}$$

The periodic nature of these functions makes them suitable for modeling repetitive phenomena such as tides, vibrating springs, and sound waves. For instance, in Example 1.3.4 we will see that a reasonable model for the number of hours of daylight in Philadelphia t days after January 1 is given by the function

$$L(t) = 12 + 2.8 \sin\left[\frac{2\pi}{365}(t - 80)\right]$$

EXAMPLE 5 Find the domain of the function $f(x) = \dfrac{1}{1 - 2\cos x}$.

SOLUTION This function is defined for all values of x except for those that make the denominator 0. But

$$1 - 2\cos x = 0 \iff \cos x = \frac{1}{2} \iff x = \frac{\pi}{3} + 2n\pi \quad \text{or} \quad x = \frac{5\pi}{3} + 2n\pi$$

where n is any integer (because the cosine function has period 2π). So the domain of f is the set of all real numbers except for the ones noted above. ■

The tangent function is related to the sine and cosine functions by the equation

$$\tan x = \frac{\sin x}{\cos x}$$

and its graph is shown in Figure 20. It is undefined whenever $\cos x = 0$, that is, when $x = \pm\pi/2, \pm 3\pi/2, \ldots$. Its range is $(-\infty, \infty)$. Notice that the tangent function has period π:

$$\tan(x + \pi) = \tan x \qquad \text{for all } x$$

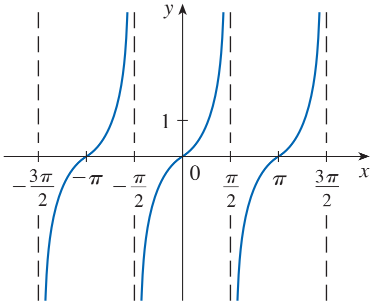
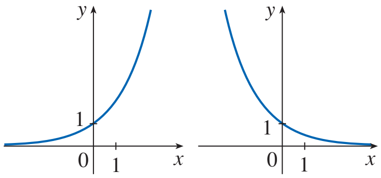


FIGURE 20
$y = \tan x$

The remaining three trigonometric functions (cosecant, secant, and cotangent) are the reciprocals of the sine, cosine, and tangent functions. Their graphs are shown in Appendix D.

Exponential Functions

The **exponential functions** are the functions of the form $f(x) = b^x$, where the base b is a positive constant. The graphs of $y = 2^x$ and $y = (0.5)^x$ are shown in Figure 21. In both cases the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.



(a) $y = 2^x$ (b) $y = (0.5)^x$

FIGURE 21

Exponential functions will be studied in detail in Section 1.4, and we will see that they are useful for modeling many natural phenomena, such as when populations grow (if $b > 1$) or decline (if $b < 1$).

Logarithmic Functions

The **logarithmic functions** $f(x) = \log_b x$, where the base b is a positive constant, are the inverse functions of the exponential functions. They will be studied in Section 1.5. Figure 22 shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the function increases slowly when $x > 1$.

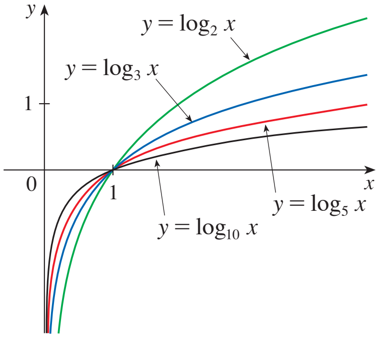


FIGURE 22

EXAMPLE 6 Classify the following functions as one of the types of functions that we have discussed.

(a) $f(x) = 5^x$ (b) $g(x) = x^5$ (c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$ (d) $u(t) = 1 - t + 5t^4$

SOLUTION

(a) $f(x) = 5^x$ is an exponential function. (The variable x is the exponent.)

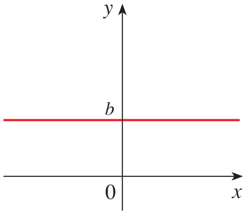
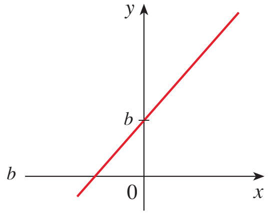
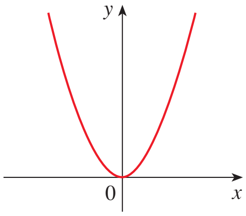
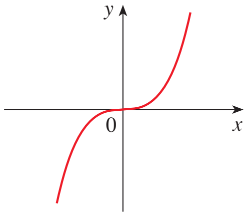
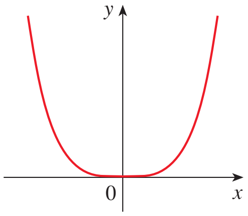
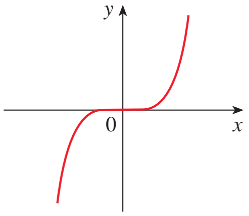
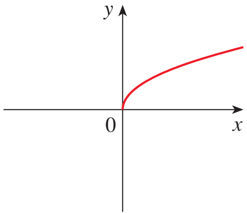
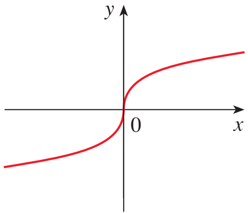
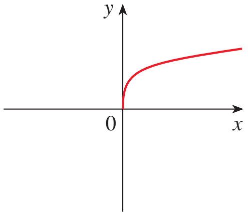
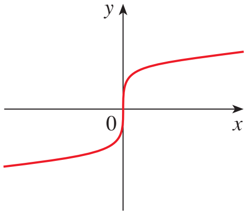
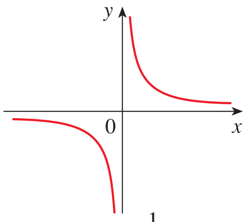
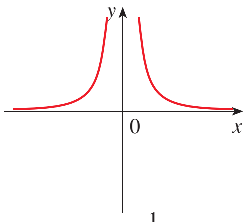
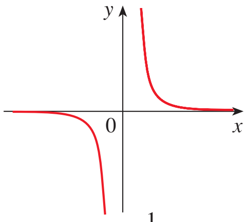
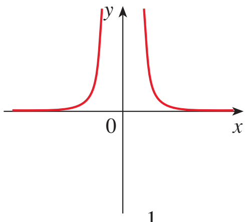
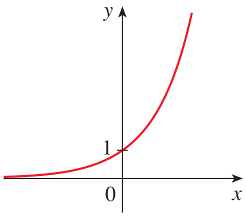
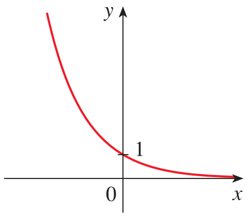
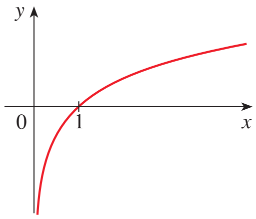
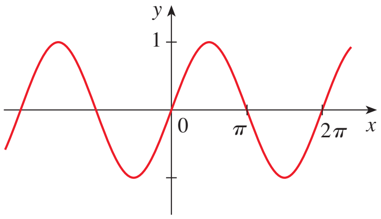
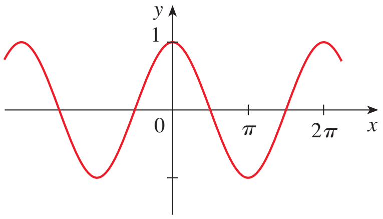
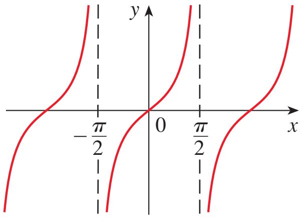
(b) $g(x) = x^5$ is a power function. (The variable x is the base.) We could also consider it to be a polynomial of degree 5.

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$ is an algebraic function. (It is not a rational function because the denominator is not a polynomial.)

(d) $u(t) = 1 - t + 5t^4$ is a polynomial of degree 4. ■

Table 3 (on the following page) shows a summary of graphs of some families of essential functions that will be used frequently throughout the book.

Table 3 Families of Essential Functions and Their Graphs

Linear Functions $f(x) = mx + b$	 $f(x) = b$  $f(x) = mx + b$
Power Functions $f(x) = x^n$	 $f(x) = x^2$  $f(x) = x^3$  $f(x) = x^4$  $f(x) = x^5$
Root Functions $f(x) = \sqrt[n]{x}$	 $f(x) = \sqrt{x}$  $f(x) = \sqrt[3]{x}$  $f(x) = \sqrt[4]{x}$  $f(x) = \sqrt[5]{x}$
Reciprocal Functions $f(x) = \dfrac{1}{x^n}$	 $f(x) = \dfrac{1}{x}$  $f(x) = \dfrac{1}{x^2}$  $f(x) = \dfrac{1}{x^3}$  $f(x) = \dfrac{1}{x^4}$
Exponential and Logarithmic Functions $f(x) = b^x$ $f(x) = \log_b x$	 $f(x) = b^x \ (b > 1)$  $f(x) = b^x \ (b < 1)$  $f(x) = \log_b x \ (b > 1)$
Trigonometric Functions $f(x) = \sin x$ $f(x) = \cos x$ $f(x) = \tan x$	 $f(x) = \sin x$  $f(x) = \cos x$  $f(x) = \tan x$

1.2 Exercises

1–2 Classify each function as a power function, root function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function.

1. (a) $f(x) = x^3 + 3x^2$ (b) $g(t) = \cos^2 t - \sin t$

(c) $r(t) = t^{\sqrt{3}}$ (d) $v(t) = 8^t$

(e) $y = \dfrac{\sqrt{x}}{x^2 + 1}$ (f) $g(u) = \log_{10} u$

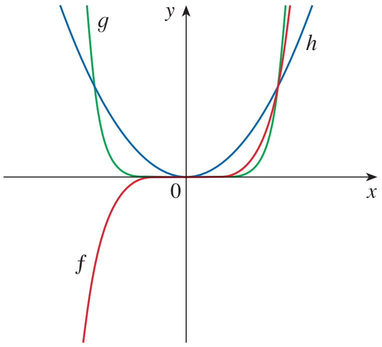
2. (a) $f(t) = \dfrac{3t^2 + 2}{t}$ (b) $h(r) = 2.3^r$

(c) $s(t) = \sqrt{t + 4}$ (d) $y = x^4 + 5$

(e) $g(x) = \sqrt[3]{x}$ (f) $y = \dfrac{1}{x^2}$

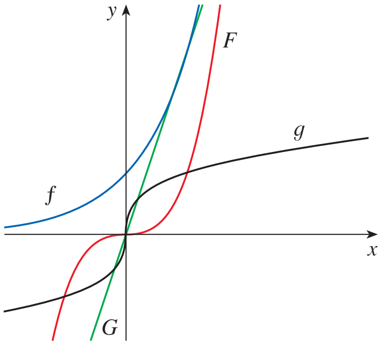
3–4 Match each equation with its graph. Explain your choices. (Don't use a computer or graphing calculator.)

3. (a) $y = x^2$ (b) $y = x^5$ (c) $y = x^8$



4. (a) $y = 3x$ (b) $y = 3^x$

(c) $y = x^3$ (d) $y = \sqrt[3]{x}$



5–6 Find the domain of the function.

5. $f(x) = \dfrac{\cos x}{1 - \sin x}$ **6.** $g(x) = \dfrac{1}{1 - \tan x}$

---

7. (a) Find an equation for the family of linear functions with slope 2 and sketch several members of the family.

(b) Find an equation for the family of linear functions such that $f(2) = 1$. Sketch several members of the family.

(c) Which function belongs to both families?

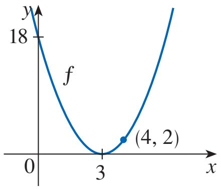
8. What do all members of the family of linear functions $f(x) = 1 + m(x + 3)$ have in common? Sketch several members of the family.

9. What do all members of the family of linear functions $f(x) = c - x$ have in common? Sketch several members of the family.

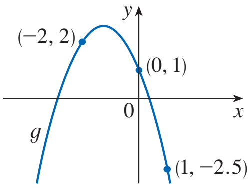
10. Sketch several members of the family of polynomials $P(x) = x^3 - cx^2$. How does the graph change when c changes?

11–12 Find a formula for the quadratic function whose graph is shown.

11.



12.



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13. Find a formula for a cubic function f if $f(1) = 6$ and $f(-1) = f(0) = f(2) = 0$.

14. Recent studies indicate that the average surface temperature of the earth has been rising steadily. Some scientists have modeled the temperature by the linear function $T = 0.02t + 8.50$, where T is temperature in °C and t represents years since 1900.

(a) What do the slope and T-intercept represent?

(b) Use the equation to predict the earth's average surface temperature in 2100.

15. If the recommended adult dosage for a drug is D (in mg), then to determine the appropriate dosage c for a child of age a, pharmacists use the equation $c = 0.0417D(a + 1)$. Suppose the dosage for an adult is 200 mg.

(a) Find the slope of the graph of c. What does it represent?

(b) What is the dosage for a newborn?

16. The manager of a weekend flea market knows from past experience that if he charges x dollars for a rental space at the market, then the number y of spaces that will be rented is given by the equation $y = 200 - 4x$.

(a) Sketch a graph of this linear function. (Remember that the rental charge per space and the number of spaces rented can't be negative quantities.)

(b) What do the slope, the y-intercept, and the x-intercept of the graph represent?

17. The relationship between the Fahrenheit (F) and Celsius (C) temperature scales is given by the linear function $F = \frac{9}{5}C + 32$.

(a) Sketch a graph of this function.

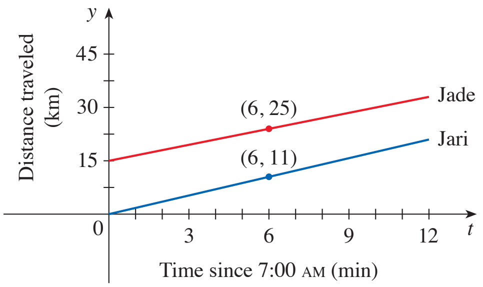
(b) What is the slope of the graph and what does it represent? What is the F-intercept and what does it represent?

18. Jade and her roommate Jari commute to work each morning, traveling west on I-10. One morning Jade left for work at 6:50 AM, but Jari left 10 minutes later. Both drove at a constant speed. The graphs show the distance (in kilometers) each of them has traveled on I-10, t minutes after 7:00 AM.

(a) Use the graph to decide which driver is traveling faster.

(b) Find the speed (in km/h) at which each of them is driving.

(c) Find linear functions f and g that model the distances traveled by Jade and Jari as functions of t (in minutes).



19. The manager of a furniture factory finds that it costs \$2200 to manufacture 100 chairs in one day and \$4800 to produce 300 chairs in one day.

(a) Express the cost as a function of the number of chairs produced, assuming that it is linear. Then sketch the graph.

(b) What is the slope of the graph and what does it represent?

(c) What is the y-intercept of the graph and what does it represent?

20. The monthly cost of driving a car depends on the number of kilometers driven. Lynn found that in May it cost her \$380 to drive 770 km and in June it cost her \$460 to drive 1290 km.

(a) Express the monthly cost C as a function of the distance driven d, assuming that a linear relationship gives a suitable model.

(b) Use part (a) to predict the cost of driving 2400 km per month.

(c) Draw the graph of the linear function. What does the slope represent?

(d) What does the C-intercept represent?

(e) Why does a linear function give a suitable model in this situation?

21. At the surface of the ocean, the water pressure is the same as the air pressure above the water, 1.05 kg/cm^2. Below the surface, the water pressure increases by 0.3 kg/cm^2 for every 3 m of descent.

(a) Express the water pressure as a function of the depth below the ocean surface.

(b) At what depth is the pressure 7 kg/cm^2?

22. The resistance R of a wire of fixed length is related to its diameter x by an inverse square law, that is, by a function of the form $R(x) = kx^{-2}$.

(a) A wire of fixed length and 0.005 meters in diameter has a resistance of 140 ohms. Find the value of k.

(b) Find the resistance of a wire made of the same material and of the same length as the wire in part (a) but with a diameter of 0.008 meters.

23. The illumination of an object by a light source is related to the distance from the source by an inverse square law. Suppose that after dark you are sitting in a room with just one lamp, trying to read a book. The light is too dim, so you move your chair halfway to the lamp. How much brighter is the light?

24. The pressure P of a sample of oxygen gas that is compressed at a constant temperature is related to the volume V of gas by a reciprocal function of the form $P = k/V$.

(a) A sample of oxygen gas that occupies 0.671 m^3 exerts a pressure of 39 kPa at a temperature of 293 K (absolute temperature measured on the Kelvin scale). Find the value of k in the given model.

(b) If the sample expands to a volume of 0.916 m^3, find the new pressure.

25. The power output of a wind turbine depends on many factors. It can be shown using physical principles that the power P generated by a wind turbine is modeled by

$$P = kAv^3$$

where v is the wind speed, A is the area swept out by the blades, and k is a constant that depends on air density, efficiency of the turbine, and the design of the wind turbine blades.

(a) If only wind speed is doubled, by what factor is the power output increased?

(b) If only the length of the blades is doubled, by what factor is the power output increased?

(c) For a particular wind turbine, the length of the blades is 30 m and $k = 0.214 \text{ kg/m}^3$. Find the power output (in watts, $\text{W} = \text{m}^2 \cdot \text{kg/s}^3$) when the wind speed is 10 m/s, 15 m/s, and 25 m/s.

26. Astronomers infer the radiant exitance (radiant flux emitted per unit area) of stars using the Stefan Boltzmann Law:

$$E(T) = (5.67 \times 10^{-8})T^4$$

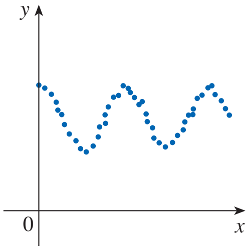
where E is the energy radiated per unit of surface area measured in watts (W) and T is the absolute temperature measured in kelvins (K).

(a) Graph the function E for temperatures T between 100 K and 300 K.

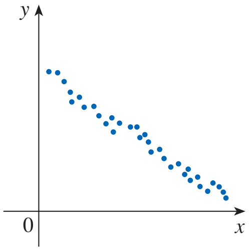
(b) Use the graph to describe the change in energy E as the temperature T increases.

27–28 For each scatter plot, decide what type of function you might choose as a model for the data. Explain your choices.

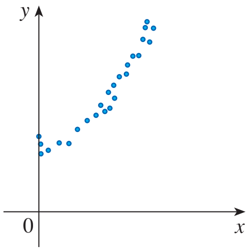
27. (a)



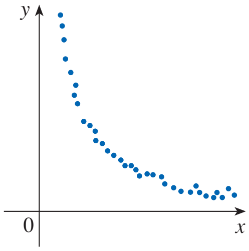
(b)



28. (a)



(b)



T **29.** The table shows (lifetime) peptic ulcer rates (per 100 population) for various family incomes as reported by the National Health Interview Survey.

(a) Make a scatter plot of these data and decide whether a linear model is appropriate.

(b) Find and graph a linear model using the first and last data points.

(c) Find and graph the regression line.

(d) Use the linear model in part (c) to estimate the ulcer rate for people with an income of \$25,000.

(e) According to the model, how likely is someone with an income of \$80,000 to suffer from peptic ulcers?

(f) Do you think it would be reasonable to apply the model to someone with an income of \$200,000?

Income	Ulcer rate (per 100 population)
\$4,000	14.1
\$6,000	13.0
\$8,000	13.4
\$12,000	12.5
\$16,000	12.0
\$20,000	12.4
\$30,000	10.5
\$45,000	9.4
\$60,000	8.2

T **30.** When laboratory rats are exposed to asbestos fibers, some of them develop lung tumors. The table lists the results of several experiments by different scientists.

(a) Find the regression line for the data.

(b) Make a scatter plot and graph the regression line. Does the regression line appear to be a suitable model for the data?

(c) What does the y-intercept of the regression line represent?

Asbestos exposure (fibers/mL)	Percent of mice that develop lung tumors	Asbestos exposure (fibers/mL)	Percent of mice that develop lung tumors
50	2	1600	42
400	6	1800	37
500	5	2000	38
900	10	3000	50
1100	26		

T **31.** Anthropologists use a linear model that relates human femur (thighbone) length to height. The model allows an anthropologist to determine the height of an individual when only a partial skeleton (including the femur) is found. Here we find the model by analyzing the data on femur length and height for the eight males given in the table.

(a) Make a scatter plot of the data.

(b) Find and graph the regression line that models the data.

(c) An anthropologist finds a human femur of length 53 cm. How tall was the person?

Femur length (cm)	Height (cm)	Femur length (cm)	Height (cm)
50.1	178.5	44.5	168.3
48.3	173.6	42.7	165.0
45.2	164.8	39.5	155.4
44.7	163.7	38.0	155.8

T **32.** The table shows average US retail residential prices of electricity from 2000 to 2016, measured in cents per kilowatt hour.

(a) Make a scatter plot. Is a linear model appropriate?

(b) Find and graph the regression line.

(c) Use your linear model from part (b) to estimate the average retail price of electricity in 2005 and 2017.

Years since 2000	Cents/kWh	Years since 2000	Cents/kWh
0	8.24	10	11.54
2	8.44	12	11.88
4	8.95	14	12.52
6	10.40	16	12.90
8	11.26		

Source: US Energy Information Administration

T **33.** The table shows world average daily oil consumption from 1985 to 2015, measured in thousands of barrels per day.

(a) Make a scatter plot and decide whether a linear model is appropriate.
(b) Find and graph the regression line.
(c) Use the linear model to estimate the oil consumption in 2002 and 2017.

Years since 1985	Thousands of barrels of oil per day
0	60,083
5	66,533
10	70,099
15	76,784
20	84,077
25	87,302
30	94,071

Source: US Energy Information Administration

T **34.** The table shows the mean (average) distances d of the planets from the sun (taking the unit of measurement to be the distance from the earth to the sun) and their periods T (time of revolution in years).

(a) Fit a power model to the data.
(b) Kepler's Third Law of Planetary Motion states that

> "The square of the period of revolution of a planet is proportional to the cube of its mean distance from the sun."

Does your model corroborate Kepler's Third Law?

Planet	d	T
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.523	1.881
Jupiter	5.203	11.861
Saturn	9.541	29.457
Uranus	19.190	84.008
Neptune	30.086	164.784

35. It makes sense that the larger the area of a region, the larger the number of species that inhabit the region. Many ecologists have modeled the species-area relation with a power function. In particular, the number of species S of bats living in caves in central Mexico has been related to the surface area A of the caves by the equation $S = 0.7A^{0.3}$.

(a) The cave called *Misión Imposible* near Puebla, Mexico, has a surface area of $A = 60 \text{ m}^2$. How many species of bats would you expect to find in that cave?
(b) If you discover that four species of bats live in a cave, estimate the area of the cave.

T **36.** The table shows the number N of species of reptiles and amphibians inhabiting Caribbean islands and the area A of the island in square kilometers.

(a) Use a power function to model N as a function of A.
(b) The Caribbean island of Dominica has area 753 km^2. How many species of reptiles and amphibians would you expect to find on Dominica?

Island	A	N
Saba	10	5
Monserrat	103	9
Puerto Rico	8,959	40
Jamaica	11,424	39
Hispaniola	79,192	84
Cuba	114,524	76

37. Suppose that a force or energy originates from a point source and spreads its influence equally in all directions, such as the light from a lightbulb or the gravitational force of a planet. So at a distance r from the source, the intensity I of the force or energy is equal to the source strength S divided by the surface area of a sphere of radius r. Show that I satisfies the inverse square law $I = k/r^2$, where k is a positive constant.

1.3 New Functions from Old Functions

In this section we start with the basic functions we discussed in Section 1.2 and obtain new functions by shifting, stretching, and reflecting their graphs. We also show how to combine pairs of functions by the standard arithmetic operations and by composition.

Transformations of Functions

By applying certain transformations to the graph of a given function we can obtain the graphs of related functions. This will give us the ability to sketch the graphs of many functions quickly by hand. It will also enable us to write equations for given graphs.

Let's first consider **translations** of graphs. If c is a positive number, then the graph of $y = f(x) + c$ is just the graph of $y = f(x)$ shifted upward a distance of c units (because each y-coordinate is increased by the same number c). Likewise, if $g(x) = f(x - c)$, where $c > 0$, then the value of g at x is the same as the value of f at $x - c$ (c units to the left of x). Therefore the graph of $y = f(x - c)$ is just the graph of $y = f(x)$ shifted c units to the right (see Figure 1).

Vertical and Horizontal Shifts Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward

$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward

$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right

$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

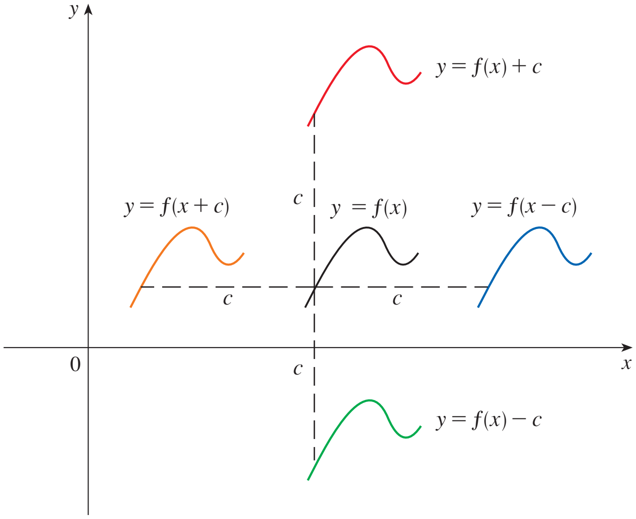


FIGURE 1 Translating the graph of f

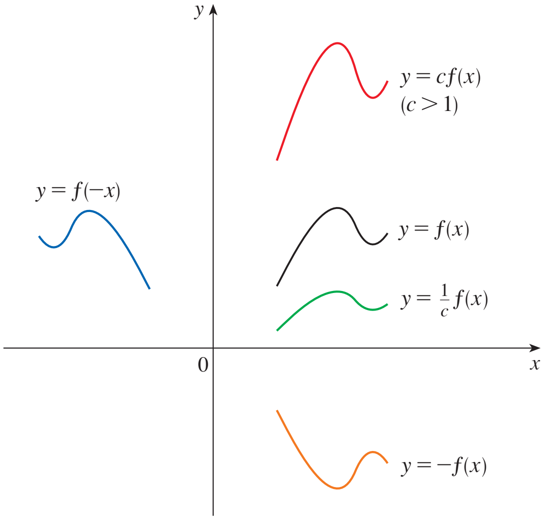


FIGURE 2 Stretching and reflecting the graph of f

Now let's consider the **stretching** and **reflecting** transformations. If $c > 1$, then the graph of $y = cf(x)$ is the graph of $y = f(x)$ stretched by a factor of c in the vertical direction (because each y-coordinate is multiplied by the same number c). The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x-axis because the point (x, y) is replaced by the point $(x, -y)$. (See Figure 2 and the following chart, where the results of other stretching, shrinking, and reflecting transformations are also given.)

Vertical and Horizontal Stretching and Reflecting Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c

$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c

$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c

$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c

$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis

$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

Figure 3 illustrates these stretching transformations when applied to the cosine function with $c = 2$. For instance, in order to get the graph of $y = 2\cos x$ we multiply the y-coordinate of each point on the graph of $y = \cos x$ by 2. This means that the graph of $y = \cos x$ gets stretched vertically by a factor of 2.

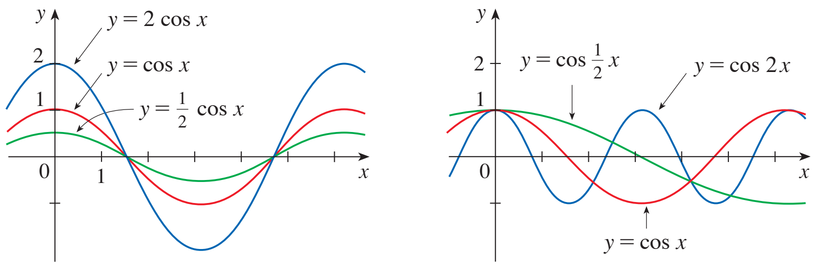
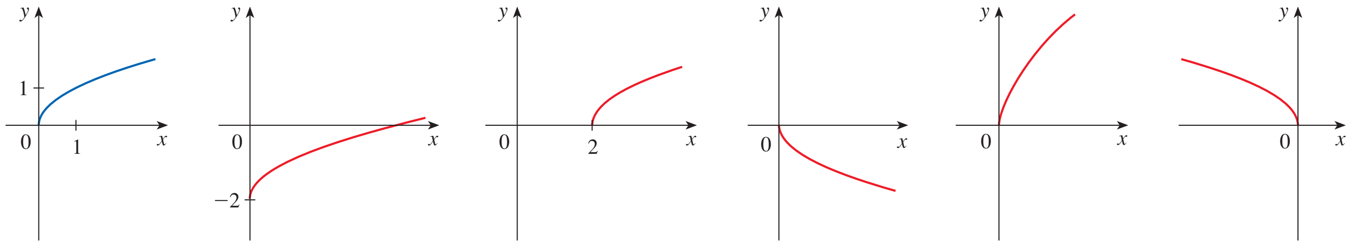


FIGURE 3

EXAMPLE 1 Given the graph of $y = \sqrt{x}$, use transformations to graph $y = \sqrt{x} - 2$, $y = \sqrt{x - 2}$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, and $y = \sqrt{-x}$.

SOLUTION The graph of the square root function $y = \sqrt{x}$, obtained from Figure 1.2.13(a), is shown in Figure 4(a). In the other parts of the figure we sketch $y = \sqrt{x} - 2$ by shifting 2 units downward, $y = \sqrt{x - 2}$ by shifting 2 units to the right, $y = -\sqrt{x}$ by reflecting about the x-axis, $y = 2\sqrt{x}$ by stretching vertically by a factor of 2, and $y = \sqrt{-x}$ by reflecting about the y-axis.



(a) $y = \sqrt{x}$ (b) $y = \sqrt{x} - 2$ (c) $y = \sqrt{x - 2}$ (d) $y = -\sqrt{x}$ (e) $y = 2\sqrt{x}$ (f) $y = \sqrt{-x}$

FIGURE 4

■

EXAMPLE 2 Sketch the graph of the function $f(x) = x^2 + 6x + 10$.

SOLUTION Completing the square, we write the equation of the graph as

$$y = x^2 + 6x + 10 = (x + 3)^2 + 1$$

This means we obtain the desired graph by starting with the parabola $y = x^2$ and shifting 3 units to the left and then 1 unit upward (see Figure 5).

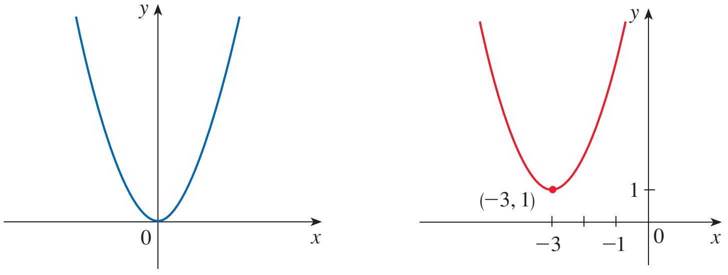


FIGURE 5

(a) $y = x^2$ (b) $y = (x + 3)^2 + 1$

■

EXAMPLE 3 Sketch the graph of each function.

(a) $y = \sin 2x$ (b) $y = 1 - \sin x$

SOLUTION

(a) We obtain the graph of $y = \sin 2x$ from that of $y = \sin x$ by compressing horizontally by a factor of 2. (See Figures 6 and 7.) Because the period of $y = \sin x$ is 2π, the period of $y = \sin 2x$ is $2\pi/2 = \pi$.

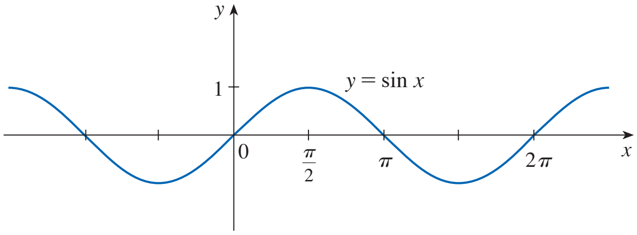


FIGURE 6

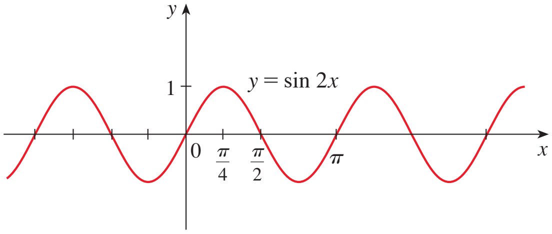


FIGURE 7

(b) To obtain the graph of $y = 1 - \sin x$, we again start with $y = \sin x$. We reflect about the x-axis to get the graph of $y = -\sin x$ and then we shift 1 unit upward to get $y = 1 - \sin x$. (See Figure 8.)

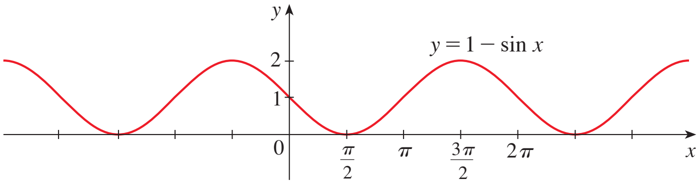


FIGURE 8

■

EXAMPLE 4 Figure 9 shows graphs of the number of hours of daylight as functions of the time of the year at several latitudes. Given that Philadelphia is located at approximately 40°N latitude, find a function that models the length of daylight at Philadelphia.

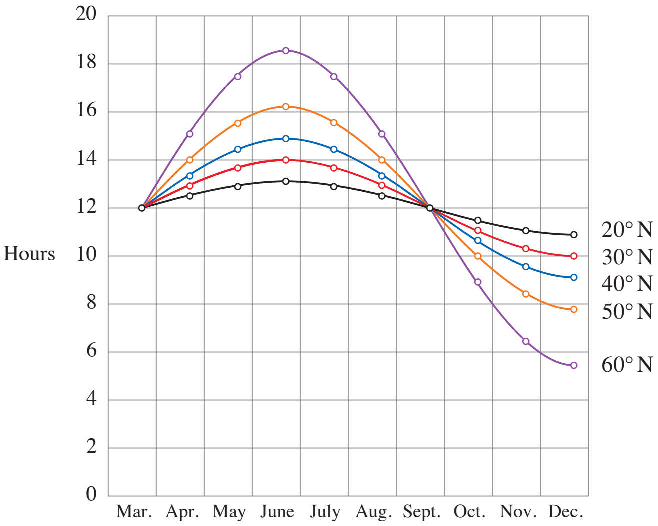


FIGURE 9
Graph of the length of daylight from March 21 through December 21 at various latitudes

Source: Adapted from L. Harrison, *Daylight, Twilight, Darkness and Time* (New York: Silver, Burdett, 1935), 40.

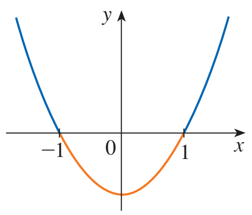
SOLUTION Notice that each curve resembles a shifted and stretched sine function. By looking at the blue curve we see that, at the latitude of Philadelphia, daylight lasts about 14.8 hours on June 21 and 9.2 hours on December 21, so the amplitude of the curve (the factor by which we have to stretch the sine curve vertically) is $\frac{1}{2}(14.8 - 9.2) = 2.8$.

By what factor do we need to stretch the sine curve horizontally if we measure the time t in days? Because there are about 365 days in a year, the period of our model should be 365. But the period of $y = \sin t$ is 2π, so the horizontal stretching factor is $2\pi/365$.

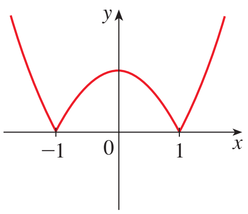
We also notice that the curve begins its cycle on March 21, the 80th day of the year, so we have to shift the curve 80 units to the right. In addition, we shift it 12 units upward. Therefore we model the length of daylight in Philadelphia on the tth day of the year by the function

$$L(t) = 12 + 2.8 \sin\left[\frac{2\pi}{365}(t - 80)\right]$$ ■

Another transformation of some interest is taking the *absolute value* of a function. If $y = |f(x)|$, then according to the definition of absolute value, $y = f(x)$ when $f(x) \geqslant 0$ and $y = -f(x)$ when $f(x) < 0$. This tells us how to get the graph of $y = |f(x)|$ from the graph of $y = f(x)$: the part of the graph that lies above the x-axis remains the same, and the part that lies below the x-axis is reflected about the x-axis.



(a) $y = x^2 - 1$



(b) $y = |x^2 - 1|$

FIGURE 10

EXAMPLE 5 Sketch the graph of the function $y = |x^2 - 1|$.

SOLUTION We first graph the parabola $y = x^2 - 1$ in Figure 10(a) by shifting the parabola $y = x^2$ downward 1 unit. We see that the graph lies below the x-axis when $-1 < x < 1$, so we reflect that part of the graph about the x-axis to obtain the graph of $y = |x^2 - 1|$ in Figure 10(b). ■

Combinations of Functions

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers.

Definitio Given two functions f and g, the **sum**, **difference**, **product**, and **quotient** functions are defined by

$$(f + g)(x) = f(x) + g(x) \qquad (f - g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)\,g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

If the domain of f is A and the domain of g is B, then the domain of $f + g$ (and the domain of $f - g$) is the intersection $A \cap B$ because both $f(x)$ and $g(x)$ have to be defined. For example, the domain of $f(x) = \sqrt{x}$ is $A = [0, \infty)$ and the domain of $g(x) = \sqrt{2 - x}$ is $B = (-\infty, 2]$, so the domain of $(f + g)(x) = \sqrt{x} + \sqrt{2 - x}$ is $A \cap B = [0, 2]$.

The domain of fg is also $A \cap B$. Because we can't divide by 0, the domain of f/g is $\{x \in A \cap B \mid g(x) \neq 0\}$. For instance, if $f(x) = x^2$ and $g(x) = x - 1$, then the domain of the rational function $(f/g)(x) = x^2/(x - 1)$ is $\{x \mid x \neq 1\}$, or $(-\infty, 1) \cup (1, \infty)$.

There is another way of combining two functions to obtain a new function. For example, suppose that $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$. Since y is a function of u and u is, in turn, a function of x, it follows that y is ultimately a function of x. We compute this by substitution:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The procedure is called *composition* because the new function is *composed* of the two given functions f and g.

In general, given any two functions f and g, we start with a number x in the domain of g and calculate $g(x)$. If this number $g(x)$ is in the domain of f, then we can calculate the value of $f(g(x))$. Notice that the output of one function is used as the input to the next function. The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f. It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ ("f circle g").

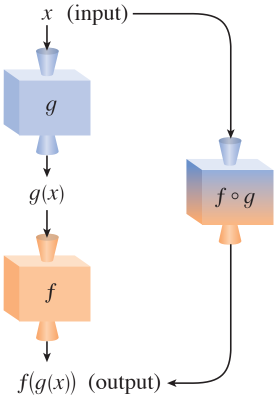


FIGURE 11
The $f \circ g$ machine is composed of the g machine (first) and then the f machine.

Definition Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined. Figure 11 shows how to picture $f \circ g$ in terms of machines.

EXAMPLE 6 If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

SOLUTION We have

$$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2$$
$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$$

■

NOTE You can see from Example 6 that, in general, $f \circ g \neq g \circ f$. Remember, the notation $f \circ g$ means that the function g is applied first and then f is applied second. In Example 6, $f \circ g$ is the function that *first* subtracts 3 and *then* squares; $g \circ f$ is the function that *first* squares and *then* subtracts 3.

EXAMPLE 7 If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, find each function and its domain.

(a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$

SOLUTION

(a) $$(f \circ g)(x) = f(g(x)) = f\left(\sqrt{2 - x}\right) = \sqrt{\sqrt{2 - x}} = \sqrt[4]{2 - x}$$

The domain of $f \circ g$ is $\{x \mid 2 - x \geqslant 0\} = \{x \mid x \leqslant 2\} = (-\infty, 2]$.

(b) $$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x}\right) = \sqrt{2 - \sqrt{x}}$$

For $\sqrt{x}$ to be defined we must have $x \geqslant 0$. For $\sqrt{2 - \sqrt{x}}$ to be defined we must have $2 - \sqrt{x} \geqslant 0$, that is, $\sqrt{x} \leqslant 2$, or $x \leqslant 4$. Thus we have $0 \leqslant x \leqslant 4$, so the domain of $g \circ f$ is the closed interval $[0, 4]$.

If $0 \leqslant a \leqslant b$, then $a^2 \leqslant b^2$.

(c) $$(f \circ f)(x) = f(f(x)) = f\left(\sqrt{x}\right) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$$

The domain of $f \circ f$ is $[0, \infty)$.

(d) $$(g \circ g)(x) = g(g(x)) = g\left(\sqrt{2-x}\right) = \sqrt{2-\sqrt{2-x}}$$

This expression is defined when both $2 - x \geqslant 0$ and $2 - \sqrt{2-x} \geqslant 0$. The first inequality means $x \leqslant 2$, and the second is equivalent to $\sqrt{2-x} \leqslant 2$, or $2 - x \leqslant 4$, or $x \geqslant -2$. Thus $-2 \leqslant x \leqslant 2$, so the domain of $g \circ g$ is the closed interval $[-2, 2]$. ■

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

EXAMPLE 8 Find $f \circ g \circ h$ if $f(x) = x/(x+1)$, $g(x) = x^{10}$, and $h(x) = x + 3$.

SOLUTION

$$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x+3)) \\ &= f((x+3)^{10}) = \frac{(x+3)^{10}}{(x+3)^{10}+1}\end{aligned}$$ ■

So far we have used composition to build complicated functions from simpler ones. But in calculus it is often useful to be able to *decompose* a complicated function into simpler ones, as in the following example.

EXAMPLE 9 Given $F(x) = \cos^2(x+9)$, find functions f, g, and h such that $F = f \circ g \circ h$.

SOLUTION Since $F(x) = [\cos(x+9)]^2$, the formula for F says: first add 9, then take the cosine of the result, and finally square. So we let

$$h(x) = x + 9 \qquad g(x) = \cos x \qquad f(x) = x^2$$

Then $$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x+9)) = f(\cos(x+9)) \\ &= [\cos(x+9)]^2 = F(x)\end{aligned}$$ ■

1.3 Exercises

1. Suppose the graph of f is given. Write equations for the graphs that are obtained from the graph of f as follows.

(a) Shift 3 units upward.

(b) Shift 3 units downward.

(c) Shift 3 units to the right.

(d) Shift 3 units to the left.

(e) Reflect about the x-axis.

(f) Reflect about the y-axis.

(g) Stretch vertically by a factor of 3.

(h) Shrink vertically by a factor of 3.

2. Explain how each graph is obtained from the graph of $y = f(x)$.

(a) $y = f(x) + 8$ (b) $y = f(x+8)$

(c) $y = 8f(x)$ (d) $y = f(8x)$

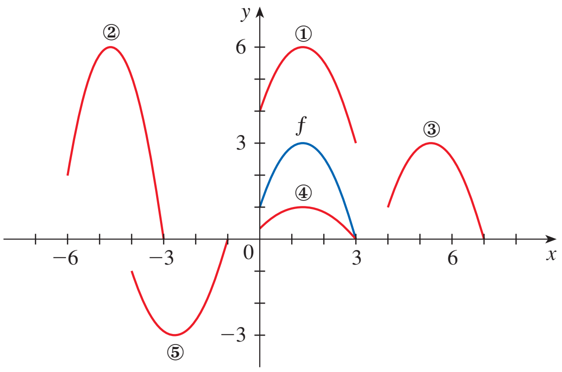
(e) $y = -f(x) - 1$ (f) $y = 8f\left(\frac{1}{8}x\right)$

3. The graph of $y = f(x)$ is given. Match each equation with its graph and give reasons for your choices.

(a) $y = f(x-4)$ (b) $y = f(x) + 3$

(c) $y = \frac{1}{3}f(x)$ (d) $y = -f(x+4)$

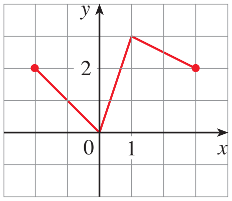
(e) $y = 2f(x+6)$



4. The graph of f is given. Draw the graphs of the following functions.

(a) $y = f(x) - 3$ (b) $y = f(x + 1)$

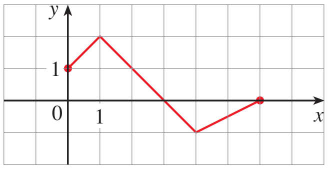
(c) $y = \frac{1}{2}f(x)$ (d) $y = -f(x)$



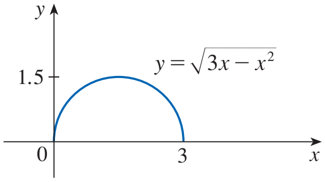
5. The graph of f is given. Use it to graph the following functions.

(a) $y = f(2x)$ (b) $y = f\left(\frac{1}{2}x\right)$

(c) $y = f(-x)$ (d) $y = -f(-x)$



6–7 The graph of $y = \sqrt{3x - x^2}$ is given. Use transformations to create a function whose graph is as shown.



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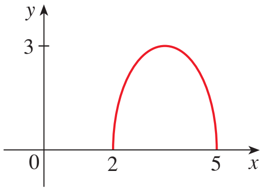

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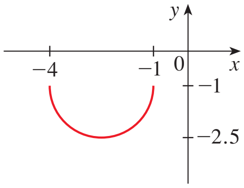


8. (a) How is the graph of $y = 1 + \sqrt{x}$ related to the graph of $y = \sqrt{x}$? Use your answer and Figure 4(a) to sketch the graph of $y = 1 + \sqrt{x}$.

(b) How is the graph of $y = 5 \sin \pi x$ related to the graph of $y = \sin x$? Use your answer and Figure 6 to sketch the graph of $y = 5 \sin \pi x$.

9–26 Graph the function by hand, not by plotting points, but by starting with the graph of one of the standard functions given in Table 1.2.3, and then applying the appropriate transformations.

9. $y = 1 + x^2$ **10.** $y = (x + 1)^2$

11. $y = |x + 2|$ **12.** $y = 1 - x^3$

13. $y = \dfrac{1}{x} + 2$ **14.** $y = -\sqrt{x} - 1$

15. $y = \sin 4x$ **16.** $y = 1 + \dfrac{1}{x^2}$

17. $y = 2 + \sqrt{x + 1}$ **18.** $y = -(x - 1)^2 + 3$

19. $y = x^2 - 2x + 5$ **20.** $y = (x + 1)^3 + 2$

21. $y = 2 - |x|$ **22.** $y = 2 - 2 \cos x$

23. $y = 3 \sin \frac{1}{2}x + 1$ **24.** $y = \dfrac{1}{4}\tan\left(x - \dfrac{\pi}{4}\right)$

25. $y = |\cos \pi x|$ **26.** $y = |\sqrt{x} - 1|$

27. The city of New Orleans is located at latitude 30°N. Use Figure 9 to find a function that models the number of hours of daylight at New Orleans as a function of the time of year. To check the accuracy of your model, use the fact that on March 31 the sun rises at 5:51 AM and sets at 6:18 PM in New Orleans.

28. A variable star is one whose brightness alternately increases and decreases. For the most visible variable star, Delta Cephei, the time between periods of maximum brightness is 5.4 days, the average brightness (or magnitude) of the star is 4.0, and its brightness varies by ±0.35 magnitude. Find a function that models the brightness of Delta Cephei as a function of time.

29. Some of the highest tides in the world occur in the Bay of Fundy on the Atlantic Coast of Canada. At Hopewell Cape the water depth at low tide is about 2.0 m and at high tide it is about 12.0 m. The natural period of oscillation is about 12 hours and on a particular day, high tide occurred at 6:45 AM. Find a function involving the cosine function that models the water depth $D(t)$ (in meters) as a function of time t (in hours after midnight) on that day.

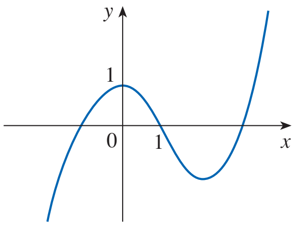
30. In a normal respiratory cycle the volume of air that moves into and out of the lungs is about 500 mL. The reserve and residue volumes of air that remain in the lungs occupy about 2000 mL and a single respiratory cycle for an average human takes about 4 seconds. Find a model for the total volume of air $V(t)$ in the lungs as a function of time.

31. (a) How is the graph of $y = f(|x|)$ related to the graph of f?

(b) Sketch the graph of $y = \sin |x|$.

(c) Sketch the graph of $y = \sqrt{|x|}$.

32. Use the given graph of f to sketch the graph of $y = 1/f(x)$. Which features of f are the most important in sketching $y = 1/f(x)$? Explain how they are used.



33–34 Find (a) $f + g$, (b) $f - g$, (c) fg, and (d) f/g and state their domains.

33. $f(x) = \sqrt{25 - x^2}, \quad g(x) = \sqrt{x + 1}$

34. $f(x) = \dfrac{1}{x - 1}, \quad g(x) = \dfrac{1}{x} - 2$

35–40 Find the functions (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, and (d) $g \circ g$ and their domains.

35. $f(x) = x^3 + 5, \quad g(x) = \sqrt[3]{x}$

36. $f(x) = \dfrac{1}{x}, \quad g(x) = 2x + 1$

37. $f(x) = \dfrac{1}{\sqrt{x}}, \quad g(x) = x + 1$

38. $f(x) = \dfrac{x}{x + 1}, \quad g(x) = 2x - 1$

39. $f(x) = \dfrac{2}{x}, \quad g(x) = \sin x$

40. $f(x) = \sqrt{5 - x}, \quad g(x) = \sqrt{x - 1}$

41–44 Find $f \circ g \circ h$.

41. $f(x) = 3x - 2, \quad g(x) = \sin x, \quad h(x) = x^2$

42. $f(x) = |x - 4|, \quad g(x) = 2^x, \quad h(x) = \sqrt{x}$

43. $f(x) = \sqrt{x - 3}, \quad g(x) = x^2, \quad h(x) = x^3 + 2$

44. $f(x) = \tan x, \quad g(x) = \dfrac{x}{x - 1}, \quad h(x) = \sqrt[3]{x}$

45–50 Express the function in the form $f \circ g$.

45. $F(x) = (2x + x^2)^4$

46. $F(x) = \cos^2 x$

47. $F(x) = \dfrac{\sqrt[3]{x}}{1 + \sqrt[3]{x}}$

48. $G(x) = \sqrt[3]{\dfrac{x}{1 + x}}$

49. $v(t) = \sec(t^2)\tan(t^2)$

50. $H(x) = \sqrt{1 + \sqrt{x}}$

51–54 Express the function in the form $f \circ g \circ h$.

51. $R(x) = \sqrt{\sqrt{x} - 1}$

52. $H(x) = \sqrt[8]{2 + |x|}$

53. $S(t) = \sin^2(\cos t)$

54. $H(t) = \cos\left(\sqrt{\tan t} + 1\right)$

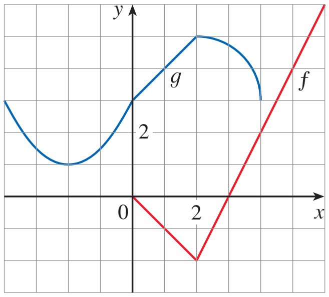
55–56 Use the table to evaluate each expression.

x	1	2	3	4	5	6
$f(x)$	3	1	5	6	2	4
$g(x)$	5	3	4	1	3	2

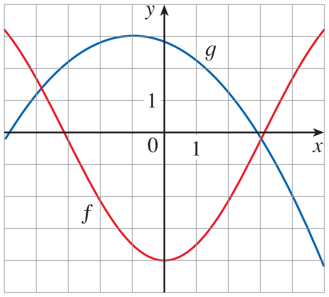
55. (a) $f(g(3))$ (b) $g(f(2))$
(c) $(f \circ g)(5)$ (d) $(g \circ f)(5)$

56. (a) $g(g(g(2)))$ (b) $(f \circ f \circ f)(1)$
(c) $(f \circ f \circ g)(1)$ (d) $(g \circ f \circ g)(3)$

57. Use the given graphs of f and g to evaluate each expression, or explain why it is undefined.
(a) $f(g(2))$ (b) $g(f(0))$ (c) $(f \circ g)(0)$
(d) $(g \circ f)(6)$ (e) $(g \circ g)(-2)$ (f) $(f \circ f)(4)$



58. Use the given graphs of f and g to estimate the value of $f(g(x))$ for $x = -5, -4, -3, \ldots, 5$. Use these estimates to sketch a rough graph of $f \circ g$.



59. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.
(a) Express the radius r of this circle as a function of the time t (in seconds).
(b) If A is the area of this circle as a function of the radius, find $A \circ r$ and interpret it.

60. A spherical balloon is being inflated and the radius of the balloon is increasing at a rate of 2 cm/s.
(a) Express the radius r of the balloon as a function of the time t (in seconds).
(b) If V is the volume of the balloon as a function of the radius, find $V \circ r$ and interpret it.

61. A ship is moving at a speed of 30 km/h parallel to a straight shoreline. The ship is 6 km from shore and it passes a lighthouse at noon.
(a) Express the distance s between the lighthouse and the ship as a function of d, the distance the ship has traveled since noon; that is, find f so that $s = f(d)$.
(b) Express d as a function of t, the time elapsed since noon; that is, find g so that $d = g(t)$.
(c) Find $f \circ g$. What does this function represent?

62. An airplane is flying at a speed of 560 km/h at an altitude of two kilometers and passes directly over a radar station at time $t = 0$.
(a) Express the horizontal distance d (in kilometers) that the plane has flown as a function of t.
(b) Express the distance s between the plane and the radar station as a function of d.
(c) Use composition to express s as a function of t.

63. The Heaviside Function The *Heaviside function H* is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

It is used in the study of electric circuits to represent the sudden surge of electric current, or voltage, when a switch is instantaneously turned on.
(a) Sketch the graph of the Heaviside function.
(b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and 120 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$.
(c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 5$ seconds and 240 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$. (Note that starting at $t = 5$ corresponds to a translation.)

64. The Ramp Function The Heaviside function defined in Exercise 63 can also be used to define the *ramp function* $y = ctH(t)$, which represents a gradual increase in voltage or current in a circuit.
(a) Sketch the graph of the ramp function $y = tH(t)$.
(b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and the voltage is gradually increased to 120 volts over a 60-second time interval. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leqslant 60$.
(c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 7$ seconds and the voltage is gradually increased to 100 volts over a period of 25 seconds. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leqslant 32$.

65. Let f and g be linear functions with equations $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$. Is $f \circ g$ also a linear function? If so, what is the slope of its graph?

66. If you invest x dollars at 4% interest compounded annually, then the amount $A(x)$ of the investment after one year is $A(x) = 1.04x$. Find $A \circ A$, $A \circ A \circ A$, and $A \circ A \circ A \circ A$. What do these compositions represent? Find a formula for the composition of n copies of A.

67. (a) If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$. (Think about what operations you would have to perform on the formula for g to end up with the formula for h.)
(b) If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.

68. If $f(x) = x + 4$ and $h(x) = 4x - 1$, find a function g such that $g \circ f = h$.

69. Suppose g is an even function and let $h = f \circ g$. Is h always an even function?

70. Suppose g is an odd function and let $h = f \circ g$. Is h always an odd function? What if f is odd? What if f is even?

71. Let $f(x)$ be a function with domain $\mathbb{R}$.
(a) Show that $E(x) = f(x) + f(-x)$ is an even function.
(b) Show that $O(x) = f(x) - f(-x)$ is an odd function.
(c) Prove that every function $f(x)$ can be written as a sum of an even function and an odd function.
(d) Express the function $f(x) = 2^x + (x - 3)^2$ as a sum of an even function and an odd function.

1.4 Exponential Functions

The function $f(x) = 2^x$ is called an *exponential function* because the variable, x, is the exponent. It should not be confused with the power function $g(x) = x^2$, in which the variable is the base.

Exponential Functions and Their Graphs

In general, an **exponential function** is a function of the form

$$f(x) = b^x$$

In Appendix G we present an alternative approach to the exponential and logarithmic functions using integral calculus.

where b is a positive constant. Let's recall what this means.

If $x = n$, a positive integer, then

$$b^n = \underbrace{b \cdot b \cdot \cdots \cdot b}_{n \text{ factors}}$$

If $x = 0$, then $b^0 = 1$, and if $x = -n$, where n is a positive integer, then

$$b^{-n} = \frac{1}{b^n}$$

If x is a rational number, $x = p/q$, where p and q are integers and $q > 0$, then

$$b^x = b^{p/q} = \sqrt[q]{b^p} = \left(\sqrt[q]{b}\right)^p$$

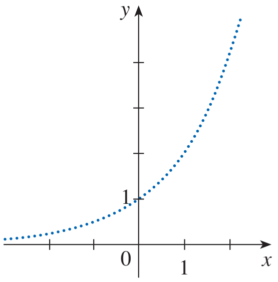


FIGURE 1
Representation of $y = 2^x$, x rational

But what is the meaning of b^x if x is an irrational number? For instance, what is meant by $2^{\sqrt{3}}$ or 5^{π}?

To help us answer this question we first look at the graph of the function $y = 2^x$, where x is rational. A representation of this graph is shown in Figure 1. We want to enlarge the domain of $y = 2^x$ to include both rational and irrational numbers.

There are holes in the graph in Figure 1 corresponding to irrational values of x. We want to fill in the holes by defining $f(x) = 2^x$, where $x \in \mathbb{R}$, so that f is an increasing function. In particular, since the irrational number $\sqrt{3}$ satisfies

$$1.7 < \sqrt{3} < 1.8$$

we must have
$$2^{1.7} < 2^{\sqrt{3}} < 2^{1.8}$$

and we know what $2^{1.7}$ and $2^{1.8}$ mean because 1.7 and 1.8 are rational numbers. Similarly, if we use better approximations for $\sqrt{3}$, we obtain better approximations for $2^{\sqrt{3}}$:

$$\begin{array}{rcl} 1.73 < \sqrt{3} < 1.74 & \Rightarrow & 2^{1.73} < 2^{\sqrt{3}} < 2^{1.74} \\ 1.732 < \sqrt{3} < 1.733 & \Rightarrow & 2^{1.732} < 2^{\sqrt{3}} < 2^{1.733} \\ 1.7320 < \sqrt{3} < 1.7321 & \Rightarrow & 2^{1.7320} < 2^{\sqrt{3}} < 2^{1.7321} \\ 1.73205 < \sqrt{3} < 1.73206 & \Rightarrow & 2^{1.73205} < 2^{\sqrt{3}} < 2^{1.73206} \\ \vdots \quad \vdots & & \vdots \quad \vdots \end{array}$$

A proof of this fact is given in J. Marsden and A. Weinstein, *Calculus Unlimited* (Menlo Park, CA: Benjamin/Cummings, 1981).

It can be shown that there is exactly one number that is greater than all of the numbers

$$2^{1.7},\quad 2^{1.73},\quad 2^{1.732},\quad 2^{1.7320},\quad 2^{1.73205},\quad \ldots$$

and less than all of the numbers

$$2^{1.8},\quad 2^{1.74},\quad 2^{1.733},\quad 2^{1.7321},\quad 2^{1.73206},\quad \ldots$$

We define $2^{\sqrt{3}}$ to be this number. Using the preceding approximation process we can compute it correct to six decimal places:

$$2^{\sqrt{3}} \approx 3.321997$$

Similarly, we can define 2^x (or b^x, if $b > 0$) where x is any irrational number. Figure 2 shows how all the holes in Figure 1 have been filled to complete the graph of the function $f(x) = 2^x$, $x \in \mathbb{R}$.

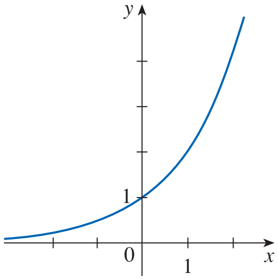


FIGURE 2
$y = 2^x$, x real

The graphs of members of the family of functions $y = b^x$ are shown in Figure 3 for various values of the base b. Notice that all of these graphs pass through the same point $(0, 1)$ because $b^0 = 1$ for $b \neq 0$. Notice also that as the base b gets larger, the exponential function grows more rapidly (for $x > 0$).

If $0 < b < 1$, then b^x approaches 0 as x becomes large. If $b > 1$, then b^x approaches 0 as x decreases through negative values. In both cases the x-axis is a horizontal asymptote. These matters are discussed in Section 2.6.

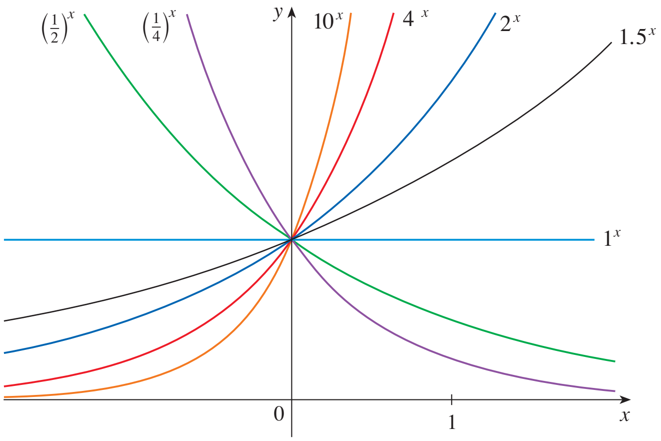


FIGURE 3

You can see from Figure 3 that there are basically three kinds of exponential functions $y = b^x$. If $0 < b < 1$, the exponential function decreases; if $b = 1$, it is a constant; and if $b > 1$, it increases. These three cases are illustrated in Figure 4. Observe that if $b \neq 1$, then the exponential function $y = b^x$ has domain $\mathbb{R}$ and range $(0, \infty)$. Notice also that, since $(1/b)^x = 1/b^x = b^{-x}$, the graph of $y = (1/b)^x$ is just the reflection of the graph of $y = b^x$ about the y-axis.

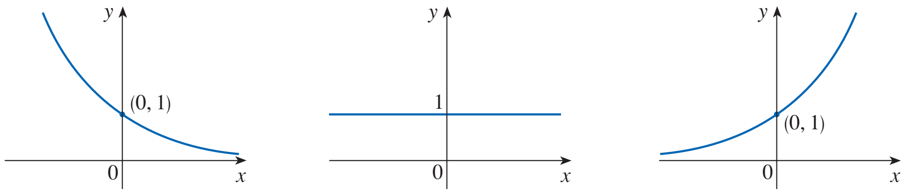


FIGURE 4

(a) $y = b^x$, $0 < b < 1$

(b) $y = 1^x$

(c) $y = b^x$, $b > 1$

One reason for the importance of the exponential function lies in the following properties. If x and y are rational numbers, then these laws are well known from elementary algebra. It can be proved that they remain true for arbitrary real numbers x and y.

www.StewartCalculus.com
For review and practice using the Laws of Exponents, click on *Review of Algebra*.

Laws of Exponents If a and b are positive numbers and x and y are any real numbers, then

1. $b^{x+y} = b^x b^y$ **2.** $b^{x-y} = \dfrac{b^x}{b^y}$ **3.** $(b^x)^y = b^{xy}$ **4.** $(ab)^x = a^x b^x$

EXAMPLE 1 Sketch the graph of the function $y = 3 - 2^x$ and determine its domain and range.

For a review of reflecting and shifting graphs, see Section 1.3.

SOLUTION First we reflect the graph of $y = 2^x$ [shown in Figures 2 and 5(a)] about the x-axis to get the graph of $y = -2^x$ in Figure 5(b). Then we shift the graph of $y = -2^x$

upward 3 units to obtain the graph of $y = 3 - 2^x$ in Figure 5(c). The domain is $\mathbb{R}$ and the range is $(-\infty, 3)$.

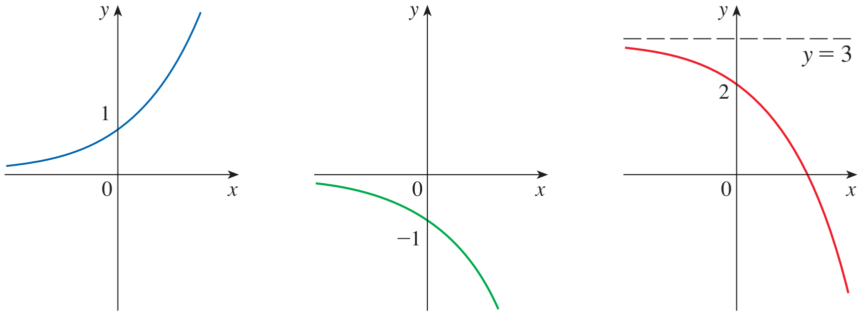


FIGURE 5

(a) $y = 2^x$ (b) $y = -2^x$ (c) $y = 3 - 2^x$

EXAMPLE 2 Use a graphing calculator or computer to compare the exponential function $f(x) = 2^x$ and the power function $g(x) = x^2$. Which function grows more quickly when x is large?

SOLUTION Figure 6 shows both functions graphed in the viewing rectangle $[-2, 6]$ by $[0, 40]$. We see that the graphs intersect three times, but for $x > 4$ the graph of $f(x) = 2^x$ stays above the graph of $g(x) = x^2$. Figure 7 gives a more global view and shows that for large values of x, the exponential function $f(x) = 2^x$ grows far more rapidly than the power function $g(x) = x^2$.

Example 2 shows that $y = 2^x$ increases more quickly than $y = x^2$. To demonstrate just how quickly $f(x) = 2^x$ increases, let's perform the following thought experiment. Suppose we start with a piece of paper 25 micrometers thick and we fold it in half 50 times. Each time we fold the paper in half, the thickness of the paper doubles, so the thickness of the resulting paper would be $2^{50}/2500$ centimeters. How thick do you think that is? It works out to be more than 17 million miles!

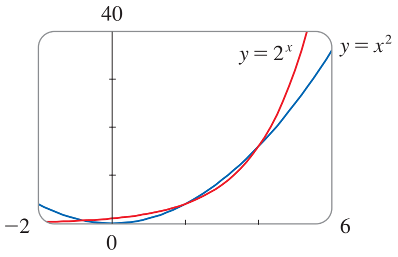


FIGURE 6

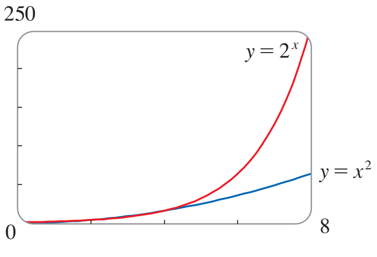


FIGURE 7

Applications of Exponential Functions

The exponential function occurs very frequently in mathematical models of nature and society. Here we indicate briefly how it arises in the description of increasing population or decreasing viral loads. In later chapters we will pursue these and other applications in greater detail.

First we consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals, it is determined that the population doubles every hour. If the number of bacteria at time t is $p(t)$, where t is measured in hours, and the initial population is $p(0) = 1000$, then we have

$$p(1) = 2p(0) = 2 \times 1000$$

$$p(2) = 2p(1) = 2^2 \times 1000$$

$$p(3) = 2p(2) = 2^3 \times 1000$$

It seems from this pattern that, in general,

$$p(t) = 2^t \times 1000 = (1000)2^t$$

This population function is a constant multiple of the exponential function $y = 2^t$, so it exhibits the rapid growth that we observed in Figure 7. Under ideal conditions (unlimited space and nutrition and absence of disease) this exponential growth is typical of what actually occurs in nature.

Table 1 World Population

t (years since 1900)	Population P (millions)
0	1650
10	1750
20	1860
30	2070
40	2300
50	2560
60	3040
70	3710
80	4450
90	5280
100	6080
110	6870

EXAMPLE 3 Table 1 shows data for the population of the world in the 20th century and Figure 8 shows the corresponding scatter plot.

The pattern of the data points in Figure 8 suggests exponential growth, so we use a graphing calculator (or computer) with exponential regression capability to apply the method of least squares and obtain the exponential model

$$P(t) = (1.43653 \times 10^9) \cdot (1.01395)^t$$

where $t = 0$ corresponds to 1900. Figure 9 shows the graph of this exponential function together with the original data points. We see that the exponential curve fits the data reasonably well. The period of relatively slow population growth is explained by the two world wars and the Great Depression of the 1930s.

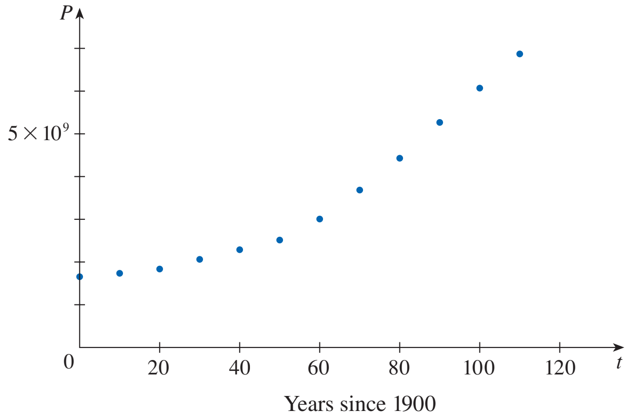


FIGURE 8 Scatter plot for world population growth

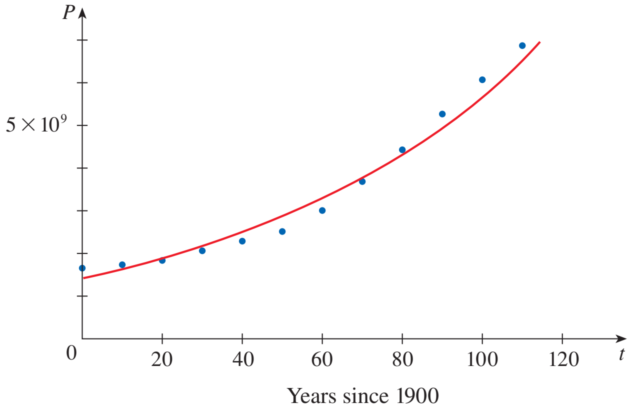


FIGURE 9 Exponential model for world population growth ■

Table 2

t (days)	$V(t)$
1	76.0
4	53.0
8	18.0
11	9.4
15	5.2
22	3.6

EXAMPLE 4 In 1995 a research article was published that detailed the effect of the protease inhibitor ABT-538 on the human immunodeficiency virus HIV-1.[1] Table 2 shows values of the plasma viral load $V(t)$ of patient 303, measured in RNA copies per mL, t days after ABT-538 treatment was begun. The corresponding scatter plot is shown in Figure 10 (on the following page).

The rather dramatic decline of the viral load that we see in Figure 10 reminds us of the graphs of the exponential function $y = b^x$ in Figures 3 and 4(a) for the case where the base b is less than 1. So let's model the function $V(t)$ by an exponential function. Using a graphing calculator or computer to fit the data in Table 2 with an exponential function of the form $y = a \cdot b^t$, we obtain the model

---

[1] D. Ho et al., "Rapid Turnover of Plasma Virions and CD4 Lymphocytes in HIV-1 Infection," *Nature* 373 (1995): 123–26.

$$V = 96.39785 \cdot (0.818656)^t$$

In Figure 11 we graph this exponential function with the data points and observe that the model represents the viral load reasonably well for the first month of treatment.

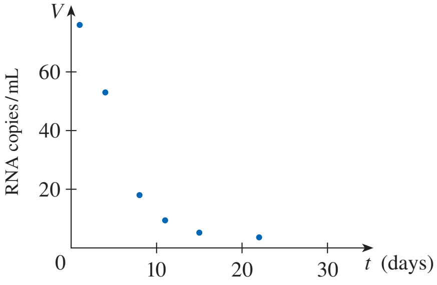


FIGURE 10
Plasma viral load in patient 303

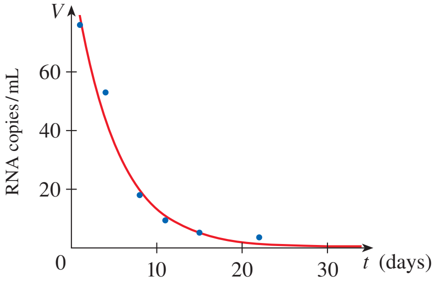


FIGURE 11
Exponential model for viral load ■

In Example 3 we used an exponential function of the form $y = a \cdot b^t$, $b > 1$, to model an increasing population and in Example 4 we used $y = a \cdot b^t$, $b < 1$, to model a decreasing viral load. In Section 3.8 we will explore additional examples of quantities that grow or decline exponentially, including the value of an investment account with compounding interest and the amount of radioactive material that remains as the material decays.

■ The Number *e*

Of all possible bases for an exponential function, there is one that is most convenient for the purposes of calculus. The choice of a base b is influenced by the way the graph of $y = b^x$ crosses the y-axis. Figures 12 and 13 show the tangent lines to the graphs of $y = 2^x$ and $y = 3^x$ at the point $(0, 1)$. (Tangent lines will be defined precisely in Section 2.7. For present purposes, you can think of the tangent line to an exponential graph at a point as the line that touches the graph only at that point.) If we measure the slopes of these tangent lines at $(0, 1)$, we find that $m \approx 0.7$ for $y = 2^x$ and $m \approx 1.1$ for $y = 3^x$.

It turns out, as we will see in Chapter 3, that some of the formulas of calculus will be greatly simplified if we choose the base b so that the slope of the tangent line to $y = b^x$ at $(0, 1)$ is *exactly* 1. (See Figure 14.) In fact, there *is* such a number and it is denoted by the letter e. (This notation was chosen by the Swiss mathematician Leonhard Euler in 1727, probably because it is the first letter of the word *exponential*.) In view of

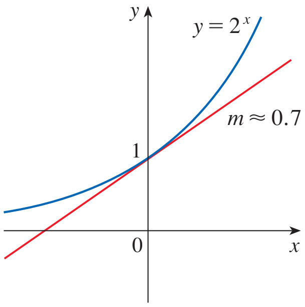


FIGURE 12

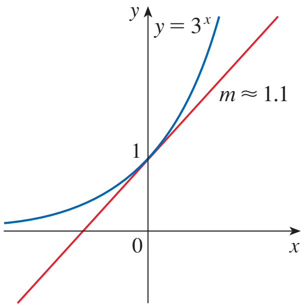


FIGURE 13

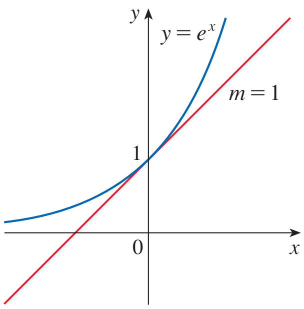


FIGURE 14

Figures 12 and 13, it comes as no surprise that the number e lies between 2 and 3 and the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$. (See Figure 15.) In Chapter 3 we will see that the value of e, correct to five decimal places, is

$$e \approx 2.71828$$

We call the function $f(x) = e^x$ the **natural exponential function**.

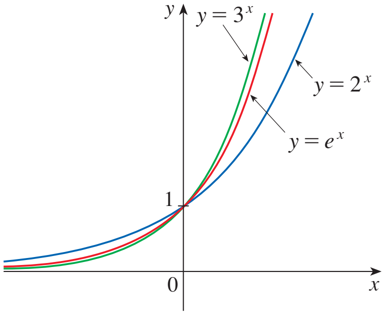
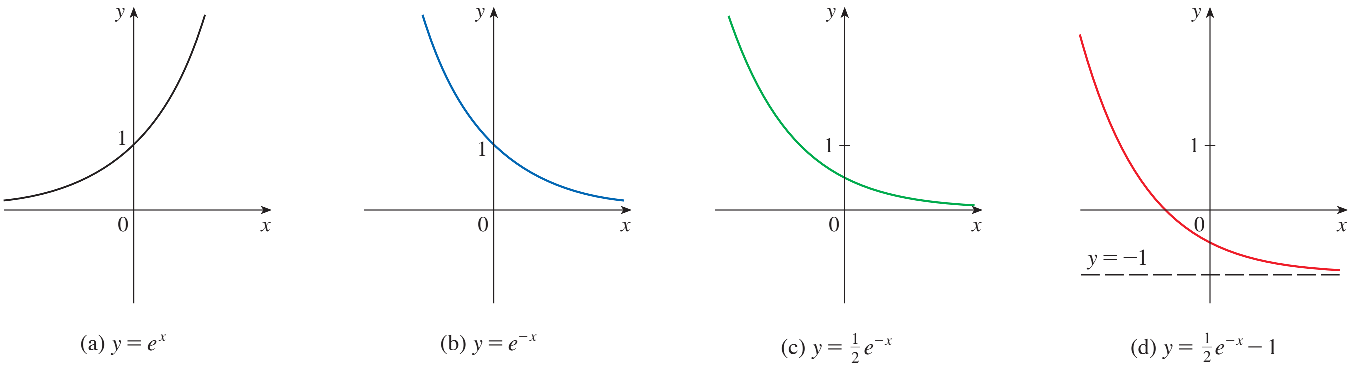


FIGURE 15
The graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$.

EXAMPLE 5 Graph the function $y = \frac{1}{2}e^{-x} - 1$ and state the domain and range.

SOLUTION We start with the graph of $y = e^x$ from Figures 14 and 16(a) and reflect about the y-axis to get the graph of $y = e^{-x}$ in Figure 16(b). (Notice that the tangent line to the graph at the y-intercept has slope -1.) Then we compress the graph vertically by a factor of 2 to obtain the graph of $y = \frac{1}{2}e^{-x}$ in Figure 16(c). Finally, we shift the graph downward one unit to get the desired graph in Figure 16(d). The domain is $\mathbb{R}$ and the range is $(-1, \infty)$.



(a) $y = e^x$ (b) $y = e^{-x}$ (c) $y = \frac{1}{2}e^{-x}$ (d) $y = \frac{1}{2}e^{-x} - 1$

FIGURE 16 ■

How far to the right do you think we would have to go for the height of the graph of $y = e^x$ to exceed a million? The next example demonstrates the rapid growth of this function by providing an answer that might surprise you.

EXAMPLE 6 Use a graphing calculator or computer to find the values of x for which $e^x > 1{,}000{,}000$.

SOLUTION In Figure 17 we graph both the function $y = e^x$ and the horizontal line $y = 1{,}000{,}000$. We see that these curves intersect when $x \approx 13.8$. So $e^x > 10^6$ when $x > 13.8$. It is perhaps surprising that the values of the exponential function have already surpassed a million when x is only 14. ■

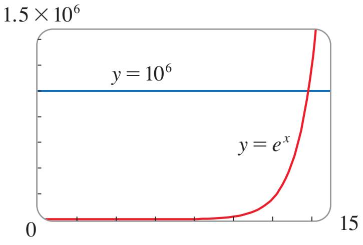


FIGURE 17

1.4 Exercises

1–2 Use the Laws of Exponents to rewrite and simplify each expression.

1. (a) $\dfrac{-2^6}{4^3}$ (b) $\dfrac{(-3)^6}{9^6}$ (c) $\dfrac{1}{\sqrt[4]{x^5}}$

(d) $\dfrac{x^3 \cdot x^n}{x^{n+1}}$ (e) $b^3(3b^{-1})^{-2}$ (f) $\dfrac{2x^2y}{(3x^{-2}y)^2}$

2. (a) $\dfrac{\sqrt[3]{4}}{\sqrt[3]{108}}$ (b) $27^{2/3}$ (c) $2x^2(3x^5)^2$

(d) $(2x^{-2})^{-3}x^{-3}$ (e) $\dfrac{3a^{3/2} \cdot a^{1/2}}{a^{-1}}$ (f) $\dfrac{\sqrt{a\sqrt{b}}}{\sqrt[3]{ab}}$

---

3. (a) Write an equation that defines the exponential function with base $b > 0$.
(b) What is the domain of this function?
(c) If $b \neq 1$, what is the range of this function?
(d) Sketch the general shape of the graph of the exponential function for each of the following cases.
(i) $b > 1$
(ii) $b = 1$
(iii) $0 < b < 1$

4. (a) How is the number e defined?
(b) What is an approximate value for e?
(c) What is the natural exponential function?

5–8 Graph the given functions on a common screen. How are these graphs related?

5. $y = 2^x$, $y = e^x$, $y = 5^x$, $y = 20^x$

6. $y = e^x$, $y = e^{-x}$, $y = 8^x$, $y = 8^{-x}$

7. $y = 3^x$, $y = 10^x$, $y = \left(\frac{1}{3}\right)^x$, $y = \left(\frac{1}{10}\right)^x$

8. $y = 0.9^x$, $y = 0.6^x$, $y = 0.3^x$, $y = 0.1^x$

9–14 Make a rough sketch by hand of the graph of the function. Use the graphs given in Figures 3 and 15 and, if necessary, the transformations of Section 1.3.

9. $g(x) = 3^x + 1$

10. $h(x) = 2\left(\frac{1}{2}\right)^x - 3$

11. $y = -e^{-x}$

12. $y = 4^{x+2}$

13. $y = 1 - \frac{1}{2}e^{-x}$

14. $y = e^{|x|}$

---

15. Starting with the graph of $y = e^x$, write the equation of the graph that results from
(a) shifting 2 units downward.
(b) shifting 2 units to the right.
(c) reflecting about the x-axis.
(d) reflecting about the y-axis.
(e) reflecting about the x-axis and then about the y-axis.

16. Starting with the graph of $y = e^x$, find the equation of the graph that results from
(a) reflecting about the line $y = 4$.
(b) reflecting about the line $x = 2$.

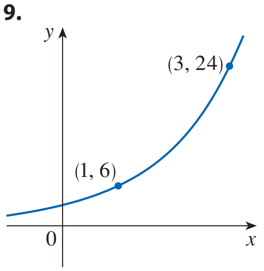
17–18 Find the domain of each function.

17. (a) $f(x) = \dfrac{1 - e^{x^2}}{1 - e^{1-x^2}}$ (b) $f(x) = \dfrac{1 + x}{e^{\cos x}}$

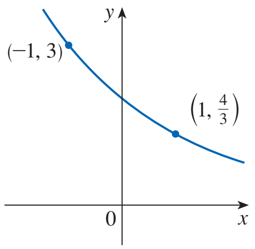
18. (a) $g(t) = \sqrt{10^t - 100}$ (b) $g(t) = \sin(e^t - 1)$

19–20 Find the exponential function $f(x) = Cb^x$ whose graph is given.

19.



20.



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21. If $f(x) = 5^x$, show that

$$\frac{f(x + h) - f(x)}{h} = 5^x\left(\frac{5^h - 1}{h}\right)$$

22. Suppose you are offered a job that lasts one month. Which of the following methods of payment do you prefer?
I. One million dollars at the end of the month.
II. One cent on the first day of the month, two cents on the second day, four cents on the third day, and, in general, 2^{n-1} cents on the nth day.

23. Suppose the graphs of $f(x) = x^2$ and $g(x) = 2^x$ are drawn on a coordinate grid where the unit of measurement is 3 centimeters. Show that at a distance 1 m to the right of the origin, the height of the graph of f is 15 m but the height of the graph of g is about 419 km.

24. Compare the functions $f(x) = x^5$ and $g(x) = 5^x$ by graphing both functions in several viewing rectangles. Find all points of intersection of the graphs correct to one decimal place. Which function grows more rapidly when x is large?

25. Compare the functions $f(x) = x^{10}$ and $g(x) = e^x$ by graphing both functions in several viewing rectangles. When does the graph of g finally surpass the graph of f?

26. Use a graph to estimate the values of x such that $e^x > 1{,}000{,}000{,}000$.

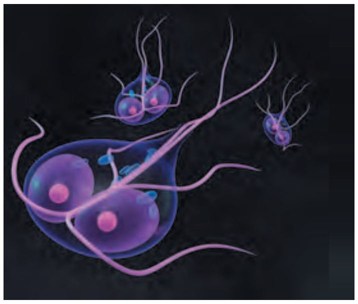
27. A researcher is trying to determine the doubling time for a population of the bacterium *Giardia lamblia*. He starts a culture in a nutrient solution and estimates the bacteria count every four hours. His data are shown in the table.

Time (hours)	0	4	8	12	16	20	24
Bacteria count (CFU/mL)	37	47	63	78	105	130	173

(a) Make a scatter plot of the data.

(b) Use a calculator or computer to find an exponential curve $f(t) = a \cdot b^t$ that models the bacteria population t hours later.

(c) Graph the model from part (b) together with the scatter plot in part (a). Use the graph to estimate how long it takes for the bacteria count to double.



Sebastian Kaulitzki / Shutterstock.com

G. lamblia

28. The table gives the population of the United States, in millions, for the years 1900–2010. Use a graphing calculator (or computer) with exponential regression capability to model the US population since 1900. Use the model to estimate the population in 1925 and to predict the population in the year 2020.

Year	Population
1900	76
1910	92
1920	106
1930	123
1940	131
1950	150
1960	179
1970	203
1980	227
1990	250
2000	281
2010	310

29. A bacteria culture starts with 500 bacteria and doubles in size every half hour.

(a) How many bacteria are there after 3 hours?

(b) How many bacteria are there after t hours?

(c) How many bacteria are there after 40 minutes?

(d) Graph the population function and estimate the time for the population to reach 100,000.

30. A gray squirrel population was introduced in a certain region 18 years ago. Biologists observe that the population doubles every six years, and now the population is 600.

(a) What was the initial squirrel population?

(b) What is the expected squirrel population t years after introduction?

(c) Estimate the expected squirrel population 10 years from now.

31. In Example 4, the patient's viral load V was 76.0 RNA copies per mL after one day of treatment. Use the graph of V in Figure 11 to estimate the additional time required for the viral load to decrease to half that amount.

32. After alcohol is fully absorbed into the body, it is metabolized. Suppose that after consuming several alcoholic drinks earlier in the evening, your blood alcohol concentration (BAC) at midnight is 0.14 g/dL. After 1.5 hours your BAC is half this amount.

(a) Find an exponential model for your BAC t hours after midnight.

(b) Graph your BAC and use the graph to determine when your BAC reaches the legal limit of 0.08 g/dL.

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

33. If you graph the function

$$f(x) = \frac{1 - e^{1/x}}{1 + e^{1/x}}$$

you'll see that f appears to be an odd function. Prove it.

34. Graph several members of the family of functions

$$f(x) = \frac{1}{1 + ae^{bx}}$$

where $a > 0$. How does the graph change when b changes? How does it change when a changes?

35. Graph several members of the family of functions

$$f(x) = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$$

where $a > 0$. How does the graph change as a increases?

1.5 Inverse Functions and Logarithms

Inverse Functions

Table 1 gives data from an experiment in which a biologist started a bacteria culture with 100 bacteria in a limited nutrient medium; the size of the bacteria population was recorded at hourly intervals. The number of bacteria N is a function of the time t: $N = f(t)$.

Suppose, however, that the biologist changes her point of view and becomes interested in the time required for the population to reach various levels. In other words, she is thinking of t as a function of N. This function is called the *inverse function* of f, denoted by f^{-1}, and read "f inverse." Here $t = f^{-1}(N)$ is the time required for the population level to reach N. The values of f^{-1} can be found by reading Table 1 from right to left or by consulting Table 2. For instance, $f^{-1}(550) = 6$ because $f(6) = 550$.

Table 1 N as a function of t

t (hours)	$N = f(t)$ = population at time t
0	100
1	168
2	259
3	358
4	445
5	509
6	550
7	573
8	586

Table 2 t as a function of N

N	$t = f^{-1}(N)$ = time to reach N bacteria
100	0
168	1
259	2
358	3
445	4
509	5
550	6
573	7
586	8

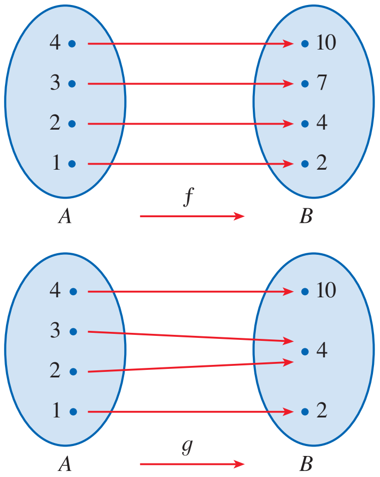


FIGURE 1
f is one-to-one; g is not.

Not all functions possess inverses. Let's compare the functions f and g whose arrow diagrams are shown in Figure 1. Note that f never takes on the same value twice (any two inputs in A have different outputs), whereas g does take on the same value twice (both 2 and 3 have the same output, 4). In symbols,

$$g(2) = g(3)$$

but
$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

Functions that share this property with f are called *one-to-one functions*.

In the language of inputs and outputs, Definition 1 says that f is one-to-one if each output corresponds to only one input.

1 Definition A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

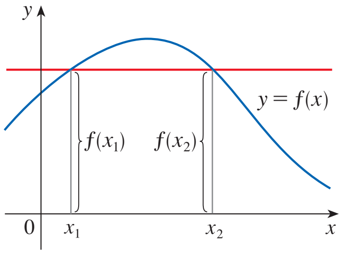


FIGURE 2
This function is not one-to-one because $f(x_1) = f(x_2)$.

If a horizontal line intersects the graph of f in more than one point, then we see from Figure 2 that there are numbers x_1 and x_2 such that $f(x_1) = f(x_2)$. This means that f is not one-to-one.

Therefore we have the following geometric method for determining whether a function is one-to-one.

Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.

EXAMPLE 1 Is the function $f(x) = x^3$ one-to-one?

SOLUTION 1 If $x_1 \neq x_2$, then $x_1^3 \neq x_2^3$ (two different numbers can't have the same cube). Therefore, by Definition 1, $f(x) = x^3$ is one-to-one.

SOLUTION 2 From Figure 3 we see that no horizontal line intersects the graph of $f(x) = x^3$ more than once. Therefore, by the Horizontal Line Test, f is one-to-one. ■

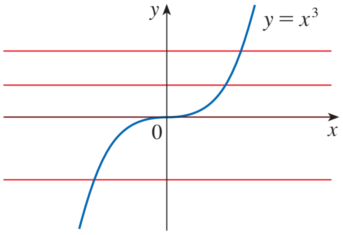


FIGURE 3
$f(x) = x^3$ is one-to-one.

EXAMPLE 2 Is the function $g(x) = x^2$ one-to-one?

SOLUTION 1 This function is not one-to-one because, for instance,

$$g(1) = 1 = g(-1)$$

and so 1 and -1 have the same output.

SOLUTION 2 From Figure 4 we see that there are horizontal lines that intersect the graph of g more than once. Therefore, by the Horizontal Line Test, g is not one-to-one. ■

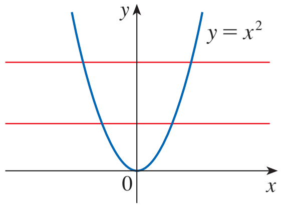


FIGURE 4
$g(x) = x^2$ is not one-to-one.

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

2 Definition Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

This definition says that if f maps x into y, then f^{-1} maps y back into x. (If f were not one-to-one, then f^{-1} would not be uniquely defined.) The arrow diagram in Figure 5 indicates that f^{-1} reverses the effect of f. Note that

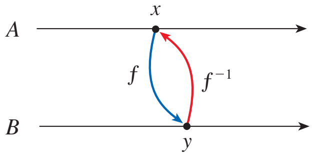


FIGURE 5

$$\boxed{\begin{aligned} \text{domain of } f^{-1} &= \text{range of } f \\ \text{range of } f^{-1} &= \text{domain of } f \end{aligned}}$$

For example, the inverse function of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$ because if $y = x^3$, then

$$f^{-1}(y) = f^{-1}(x^3) = (x^3)^{1/3} = x$$

CAUTION Do not mistake the -1 in f^{-1} for an exponent. Thus

$$f^{-1}(x) \quad \text{does } \textit{not} \text{ mean} \quad \frac{1}{f(x)}$$

The reciprocal $1/f(x)$ could be written as $[f(x)]^{-1}$.

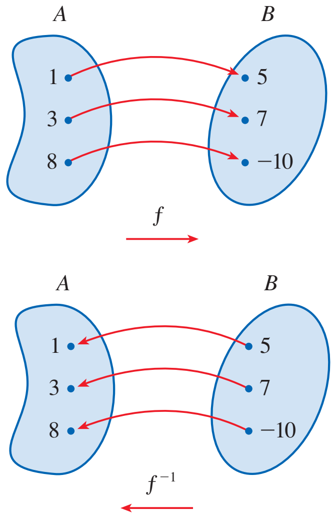


FIGURE 6
The inverse function reverses inputs and outputs.

EXAMPLE 3 If f is a one-to-one function and $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$, find $f^{-1}(7)$, $f^{-1}(5)$, and $f^{-1}(-10)$.

SOLUTION From the definition of f^{-1} we have

$$\begin{aligned} f^{-1}(7) &= 3 && \text{because} && f(3) = 7 \\ f^{-1}(5) &= 1 && \text{because} && f(1) = 5 \\ f^{-1}(-10) &= 8 && \text{because} && f(8) = -10 \end{aligned}$$

The diagram in Figure 6 makes it clear how f^{-1} reverses the effect of f in this case. ■

The letter x is traditionally used as the independent variable, so when we concentrate on f^{-1} rather than on f, we usually reverse the roles of x and y in Definition 2 and write

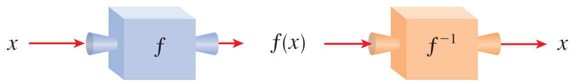
$$f^{-1}(x) = y \iff f(y) = x \tag{3}$$

By substituting for y in Definition 2 and substituting for x in (3), we get the following **cancellation equations**:

$$\begin{aligned} f^{-1}(f(x)) &= x && \text{for every } x \text{ in } A \\ f(f^{-1}(x)) &= x && \text{for every } x \text{ in } B \end{aligned} \tag{4}$$

The first cancellation equation says that if we start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started (see the machine diagram in Figure 7). Thus f^{-1} undoes what f does. The second equation says that f undoes what f^{-1} does.

FIGURE 7



For example, if $f(x) = x^3$, then $f^{-1}(x) = x^{1/3}$ and so the cancellation equations become

$$f^{-1}(f(x)) = (x^3)^{1/3} = x$$

$$f(f^{-1}(x)) = (x^{1/3})^3 = x$$

These equations simply say that the cube function and the cube root function cancel each other when applied in succession.

Now let's see how to compute inverse functions. If we have a function $y = f(x)$ and are able to solve this equation for x in terms of y, then according to Definition 2 we must have $x = f^{-1}(y)$. If we want to call the independent variable x, we then interchange x and y and arrive at the equation $y = f^{-1}(x)$.

5 How to Find the Inverse Function of a One-to-One Function f

STEP 1 Write $y = f(x)$.

STEP 2 Solve this equation for x in terms of y (if possible).

STEP 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

EXAMPLE 4 Find the inverse function of $f(x) = x^3 + 2$.

SOLUTION According to (5) we first write

$$y = x^3 + 2$$

Then we solve this equation for x:

$$x^3 = y - 2$$
$$x = \sqrt[3]{y - 2}$$

Finally, we interchange x and y:

$$y = \sqrt[3]{x - 2}$$

Therefore the inverse function is $f^{-1}(x) = \sqrt[3]{x - 2}$. ■

In Example 4, notice how f^{-1} reverses the effect of f. The function f is the rule "Cube, then add 2"; f^{-1} is the rule "Subtract 2, then take the cube root."

The principle of interchanging x and y to find the inverse function also gives us the method for obtaining the graph of f^{-1} from the graph of f. Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from (a, b) by reflecting about the line $y = x$. (See Figure 8.)

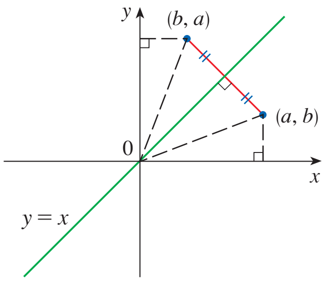


FIGURE 8

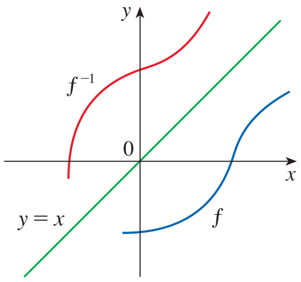


FIGURE 9

Therefore, as illustrated by Figure 9:

> The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

EXAMPLE 5 Sketch the graphs of $f(x) = \sqrt{-1 - x}$ and its inverse function using the same coordinate axes.

SOLUTION First we sketch the curve $y = \sqrt{-1 - x}$ (the top half of the parabola $y^2 = -1 - x$, or $x = -y^2 - 1$) and then we reflect about the line $y = x$ to get the graph of f^{-1}. (See Figure 10.) As a check on our graph, notice that the expression for f^{-1} is $f^{-1}(x) = -x^2 - 1$, $x \geqslant 0$. So the graph of f^{-1} is the right half of the parabola $y = -x^2 - 1$ and this seems reasonable from Figure 10. ■

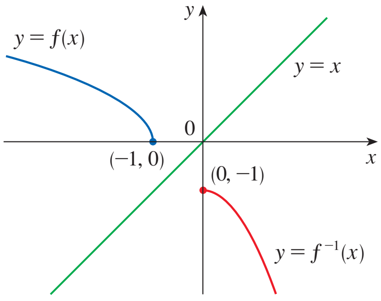


FIGURE 10

Logarithmic Functions

If $b > 0$ and $b \neq 1$, the exponential function $f(x) = b^x$ is either increasing or decreasing and so it is one-to-one by the Horizontal Line Test. It therefore has an inverse function f^{-1}, which is called the **logarithmic function with base *b*** and is denoted by $\log_b$. If we use the formulation of an inverse function given by (3),

$$f^{-1}(x) = y \iff f(y) = x$$

then we have

6
$$\log_b x = y \iff b^y = x$$

Thus, if $x > 0$, then $\log_b x$ is the exponent to which the base b must be raised to give x. For example, $\log_{10} 0.001 = -3$ because $10^{-3} = 0.001$.

The cancellation equations (4), when applied to the functions $f(x) = b^x$ and $f^{-1}(x) = \log_b x$, become

7
$$\begin{aligned} \log_b(b^x) &= x \quad \text{for every } x \in \mathbb{R} \\ b^{\log_b x} &= x \quad \text{for every } x > 0 \end{aligned}$$

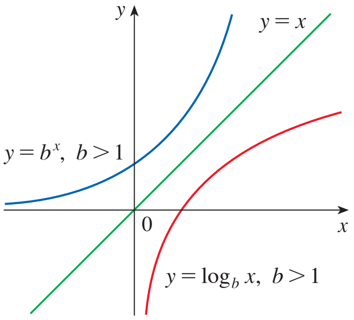


FIGURE 11

The logarithmic function $\log_b$ has domain $(0, \infty)$ and range $\mathbb{R}$. Its graph is the reflection of the graph of $y = b^x$ about the line $y = x$.

Figure 11 shows the case where $b > 1$. (The most important logarithmic functions have base $b > 1$.) The fact that $y = b^x$ is a very rapidly increasing function for $x > 0$ is reflected in the fact that $y = \log_b x$ is a very slowly increasing function for $x > 1$.

Figure 12 shows the graphs of $y = \log_b x$ with various values of the base $b > 1$. Because $\log_b 1 = 0$, the graphs of all logarithmic functions pass through the point $(1, 0)$.

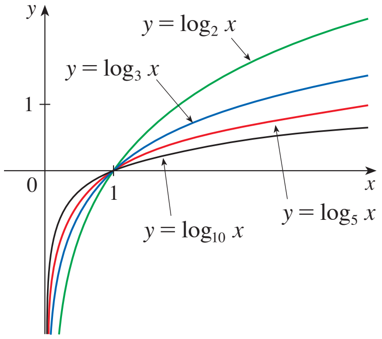


FIGURE 12

The following properties of logarithmic functions follow from the corresponding properties of exponential functions given in Section 1.4.

Laws of Logarithms If x and y are positive numbers, then

1. $\log_b(xy) = \log_b x + \log_b y$
2. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
3. $\log_b(x^r) = r \log_b x$ (where r is any real number)

EXAMPLE 6 Use the laws of logarithms to evaluate $\log_2 80 - \log_2 5$.

SOLUTION Using Law 2, we have

$$\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right) = \log_2 16 = 4$$

because $2^4 = 16$. ■

Natural Logarithms

Notation for Logarithms
Most textbooks in calculus and the sciences, as well as calculators, use the notation $\ln x$ for the natural logarithm and $\log x$ for the "common logarithm," $\log_{10} x$. In the more advanced mathematical and scientific literature and in computer languages, however, the notation $\log x$ usually denotes the natural logarithm.

Of all possible bases b for logarithms, we will see in Chapter 3 that the most convenient choice of a base is the number e, which was defined in Section 1.4. The logarithm with base e is called the **natural logarithm** and has a special notation:

$$\boxed{\log_e x = \ln x}$$

If we set $b = e$ and replace $\log_e$ with "ln" in (6) and (7), then the defining properties of the natural logarithm function become

8
$$\boxed{\ln x = y \iff e^y = x}$$

9
$$\boxed{\begin{array}{rl} \ln(e^x) = x & x \in \mathbb{R} \\ e^{\ln x} = x & x > 0 \end{array}}$$

In particular, if we set $x = 1$, we get

$$\boxed{\ln e = 1}$$

Combining Property 9 with Law 3 allows us to write

$$x^r = e^{\ln(x^r)} = e^{r \ln x} \qquad x > 0$$

Thus a power of x can be expressed in an equivalent exponential form; we will find this useful in the chapters to come.

10
$$\boxed{x^r = e^{r \ln x}}$$

EXAMPLE 7 Find x if $\ln x = 5$.

SOLUTION 1 From (8) we see that

$$\ln x = 5 \qquad \text{means} \qquad e^5 = x$$

Therefore $x = e^5$.

(If you have trouble working with the "ln" notation, just replace it by $\log_e$. Then the equation becomes $\log_e x = 5$; so, by the definition of logarithm, $e^5 = x$.)

SOLUTION 2 Start with the equation

$$\ln x = 5$$

and apply the exponential function to both sides of the equation:

$$e^{\ln x} = e^5$$

But the second cancellation equation in (9) says that $e^{\ln x} = x$. Therefore $x = e^5$. ■

EXAMPLE 8 Solve the equation $e^{5-3x} = 10$.

SOLUTION We take natural logarithms of both sides of the equation and use (9):

$$\begin{aligned} \ln(e^{5-3x}) &= \ln 10 \\ 5 - 3x &= \ln 10 \\ 3x &= 5 - \ln 10 \\ x &= \tfrac{1}{3}(5 - \ln 10) \end{aligned}$$

Using a calculator, we can approximate the solution: to four decimal places, $x \approx 0.8991$. ■

The laws of logarithms allow us to expand logarithms of products and quotients as sums and differences of logarithms. These same laws also allow us to combine sums and differences of logarithms into a single logarithmic expression. These processes are illustrated in Examples 9 and 10.

EXAMPLE 9 Use the laws of logarithms to expand $\ln \dfrac{x^2\sqrt{x^2+2}}{3x+1}$.

SOLUTION Using Laws 1, 2, and 3 of logarithms, we have

$$\begin{aligned} \ln \frac{x^2\sqrt{x^2+2}}{3x+1} &= \ln x^2 + \ln \sqrt{x^2+2} - \ln(3x+1) \\ &= 2\ln x + \tfrac{1}{2}\ln(x^2+2) - \ln(3x+1) \end{aligned}$$ ■

EXAMPLE 10 Express $\ln a + \frac{1}{2}\ln b$ as a single logarithm.

SOLUTION Using Laws 3 and 1 of logarithms, we have

$$\begin{aligned} \ln a + \tfrac{1}{2}\ln b &= \ln a + \ln b^{1/2} \\ &= \ln a + \ln\sqrt{b} \\ &= \ln\left(a\sqrt{b}\,\right) \end{aligned}$$ ■

The following formula shows that logarithms with any base can be expressed in terms of the natural logarithm.

11 **Change of Base Formula** For any positive number b ($b \neq 1$), we have

$$\log_b x = \frac{\ln x}{\ln b}$$

PROOF Let $y = \log_b x$. Then, from (6), we have $b^y = x$. Taking natural logarithms of both sides of this equation, we get $y \ln b = \ln x$. Therefore

$$y = \frac{\ln x}{\ln b}$$ ■

Formula 11 enables us to use a calculator to compute a logarithm with any base (as shown in the following example). Similarly, Formula 11 allows us to graph any logarithmic function on a graphing calculator or computer (see Exercises 49 and 50).

EXAMPLE 11 Evaluate $\log_8 5$ correct to six decimal places.

SOLUTION Formula 11 gives

$$\log_8 5 = \frac{\ln 5}{\ln 8} \approx 0.773976$$ ■

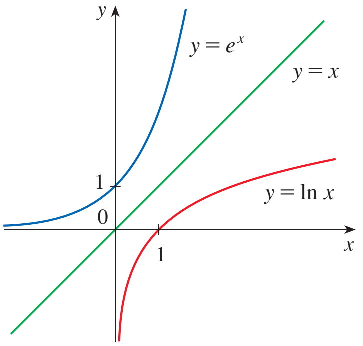


FIGURE 13
The graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ about the line $y = x$.

Graph and Growth of the Natural Logarithm

The graphs of the exponential function $y = e^x$ and its inverse function, the natural logarithm function, are shown in Figure 13. In common with all other logarithmic functions with base greater than 1, the natural logarithm is an increasing function defined on $(0, \infty)$ and the y-axis is a vertical asymptote. (This means that the values of $\ln x$ become very large negative as x approaches 0.)

EXAMPLE 12 Sketch the graph of the function $y = \ln(x - 2) - 1$.

SOLUTION We start with the graph of $y = \ln x$ as given in Figure 13. Using the transformations of Section 1.3, we shift it 2 units to the right to get the graph of $y = \ln(x - 2)$ and then we shift it 1 unit downward to get the graph of $y = \ln(x - 2) - 1$. (See Figure 14.)

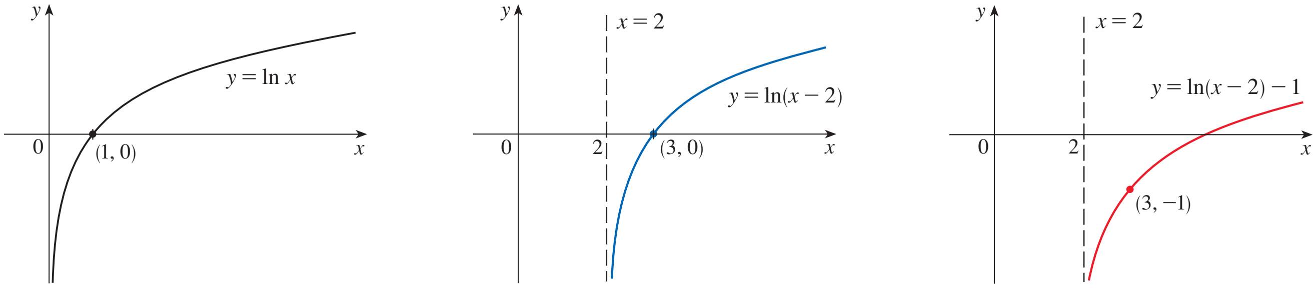


FIGURE 14 ■

Although $\ln x$ is an increasing function, it grows *very* slowly when $x > 1$. In fact, $\ln x$ grows more slowly than any positive power of x. To illustrate this fact, we graph $y = \ln x$ and $y = x^{1/2} = \sqrt{x}$ in Figures 15 and 16. You can see that the graphs initially grow at comparable rates, but eventually the root function far surpasses the logarithm.

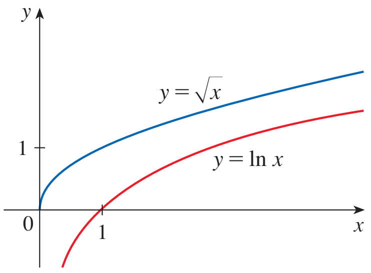


FIGURE 15

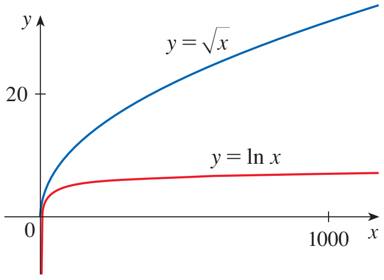


FIGURE 16

Inverse Trigonometric Functions

When we try to find the inverse trigonometric functions, we have a slight difficulty: because the trigonometric functions are not one-to-one, they don't have inverse functions. The difficulty is overcome by restricting the domains of these functions so that they become one-to-one.

You can see from Figure 17 that the sine function $y = \sin x$ is not one-to-one (use the Horizontal Line Test). However, if we restrict the domain to the interval $[-\pi/2, \pi/2]$, then the function *is* one-to-one and all values in the range of $y = \sin x$ are attained (see Figure 18). The inverse function of this restricted sine function f exists and is denoted by $\sin^{-1}$ or arcsin. It is called the **inverse sine function** or the **arcsine function**.

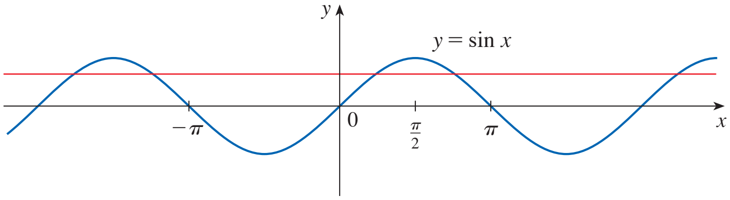


FIGURE 17

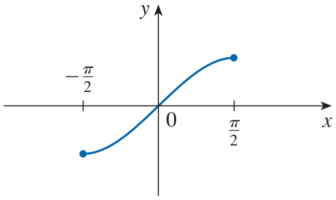


FIGURE 18
$y = \sin x,\ -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

Since the definition of an inverse function says that

$$f^{-1}(x) = y \iff f(y) = x$$

we have

$$\boxed{\sin^{-1}x = y \iff \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}}$$

$\sin^{-1}x \neq \dfrac{1}{\sin x}$

So, if $-1 \leqslant x \leqslant 1$, $\sin^{-1}x$ is the number between $-\pi/2$ and $\pi/2$ whose sine is x.

EXAMPLE 13 Evaluate (a) $\sin^{-1}\left(\frac{1}{2}\right)$ and (b) $\tan\left(\arcsin \frac{1}{3}\right)$.

SOLUTION
(a) We have

$$\sin^{-1}\left(\tfrac{1}{2}\right) = \frac{\pi}{6}$$

because $\sin(\pi/6) = \frac{1}{2}$ and $\pi/6$ lies between $-\pi/2$ and $\pi/2$.

(b) Let $\theta = \arcsin \frac{1}{3}$, so $\sin\theta = \frac{1}{3}$. Then we can draw a right triangle with angle θ as in Figure 19 and deduce from the Pythagorean Theorem that the third side has length $\sqrt{9 - 1} = 2\sqrt{2}$. This enables us to read from the triangle that

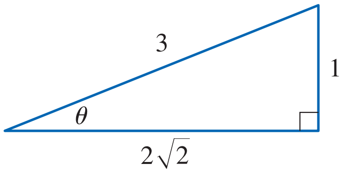


FIGURE 19

$$\tan\left(\arcsin \tfrac{1}{3}\right) = \tan u = \frac{1}{2\sqrt{2}}$$ ■

The cancellation equations for inverse functions become, in this case,

$$\boxed{\begin{aligned} \sin^{-1}(\sin x) &= x \quad \text{for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \\ \sin(\sin^{-1}x) &= x \quad \text{for } -1 \leqslant x \leqslant 1 \end{aligned}}$$

The inverse sine function, $\sin^{-1}$, has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$, and its graph, shown in Figure 20, is obtained from that of the restricted sine function (Figure 18) by reflection about the line $y = x$.

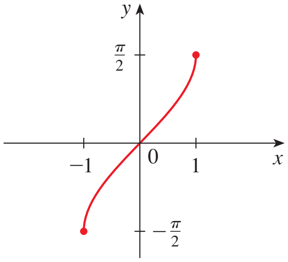


FIGURE 20
$y = \sin^{-1}x = \arcsin x$

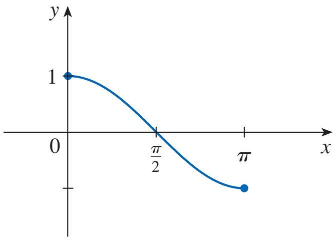


FIGURE 21
$y = \cos x,\ 0 \leqslant x \leqslant \pi$

The **inverse cosine function** is handled similarly. The restricted cosine function $f(x) = \cos x$, $0 \leqslant x \leqslant \pi$, is one-to-one (see Figure 21) and so it has an inverse function denoted by $\cos^{-1}$ or arccos.

$$\boxed{\cos^{-1}x = y \iff \cos y = x \quad \text{and} \quad 0 \leqslant y \leqslant \pi}$$

The cancellation equations are

$$\boxed{\begin{aligned} \cos^{-1}(\cos x) &= x \quad \text{for } 0 \leqslant x \leqslant \pi \\ \cos(\cos^{-1}x) &= x \quad \text{for } -1 \leqslant x \leqslant 1 \end{aligned}}$$

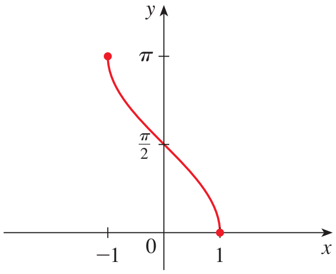


FIGURE 22
$y = \cos^{-1}x = \arccos x$

The inverse cosine function, $\cos^{-1}$, has domain $[-1, 1]$ and range $[0, \pi]$. Its graph is shown in Figure 22.

The tangent function can be made one-to-one by restricting it to the interval $(-\pi/2, \pi/2)$. Thus the **inverse tangent function** is defined as the inverse of the function $f(x) = \tan x$, $-\pi/2 < x < \pi/2$. (See Figure 23.) It is denoted by $\tan^{-1}$ or arctan.

$$\boxed{\tan^{-1}x = y \iff \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}}$$

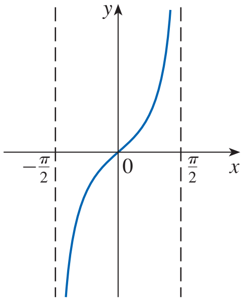


FIGURE 23
$y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

EXAMPLE 14 Simplify the expression $\cos(\tan^{-1}x)$.

SOLUTION 1 Let $y = \tan^{-1}x$. Then $\tan y = x$ and $-\pi/2 < y < \pi/2$. We want to find $\cos y$ but, since $\tan y$ is known, it is easier to find $\sec y$ first:

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$\sec y = \sqrt{1 + x^2} \qquad \text{(since } \sec y > 0 \text{ for } -\pi/2 < y < \pi/2\text{)}$$

Thus $$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1 + x^2}}$$

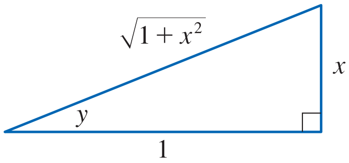


FIGURE 24

SOLUTION 2 Instead of using trigonometric identities as in Solution 1, it is perhaps easier to use a diagram. If $y = \tan^{-1}x$, then $\tan y = x$, and we can read from Figure 24 (which illustrates the case $y > 0$) that

$$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sqrt{1 + x^2}}$$ ■

The inverse tangent function, $\tan^{-1} = \arctan$, has domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$. Its graph is shown in Figure 25.

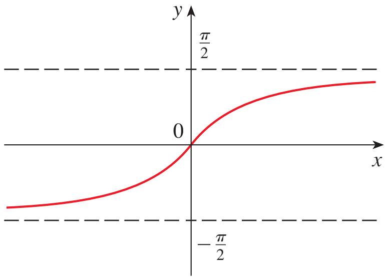


FIGURE 25
$y = \tan^{-1}x = \arctan x$

We know that the lines $x = \pm\pi/2$ are vertical asymptotes of the graph of tan. Since the graph of $\tan^{-1}$ is obtained by reflecting the graph of the restricted tangent function about the line $y = x$, it follows that the lines $y = \pi/2$ and $y = -\pi/2$ are horizontal asymptotes of the graph of $\tan^{-1}$.

The remaining inverse trigonometric functions are not used as frequently and are summarized here.

$$\boxed{12}\quad \begin{array}{llll} y = \csc^{-1}x \;\left(|x| \geqslant 1\right) & \iff & \csc y = x \quad\text{and} & y \in (0, \pi/2] \cup (\pi, 3\pi/2] \\ y = \sec^{-1}x \;\left(|x| \geqslant 1\right) & \iff & \sec y = x \quad\text{and} & y \in [0, \pi/2) \cup [\pi, 3\pi/2) \\ y = \cot^{-1}x \;\;(x \in \mathbb{R}) & \iff & \cot y = x \quad\text{and} & y \in (0, \pi) \end{array}$$

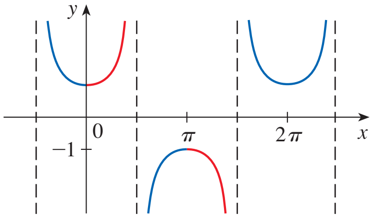


FIGURE 26
$y = \sec x$

The choice of intervals for y in the definitions of $\csc^{-1}$ and $\sec^{-1}$ is not universally agreed upon. For instance, some authors use $y \in [0, \pi/2) \cup (\pi/2, \pi]$ in the definition of $\sec^{-1}$. [You can see from the graph of the secant function in Figure 26 that both this choice and the one in (12) will work.]

1.5 Exercises

1. (a) What is a one-to-one function?
(b) How can you tell from the graph of a function whether it is one-to-one?

2. (a) Suppose f is a one-to-one function with domain A and range B. How is the inverse function f^{-1} defined? What is the domain of f^{-1}? What is the range of f^{-1}?
(b) If you are given a formula for f, how do you find a formula for f^{-1}?
(c) If you are given the graph of f, how do you find the graph of f^{-1}?

3–16 A function is given by a table of values, a graph, a formula, or a verbal description. Determine whether it is one-to-one.

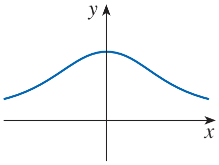
3.

x	1	2	3	4	5	6
$f(x)$	1.5	2.0	3.6	5.3	2.8	2.0

4.

x	1	2	3	4	5	6
$f(x)$	1.0	1.9	2.8	3.5	3.1	2.9

5.



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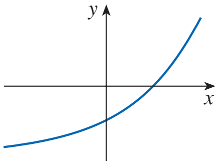
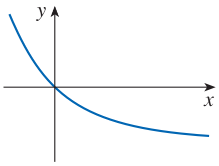


7.



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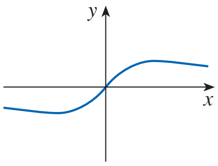


9. $f(x) = 2x - 3$

10. $f(x) = x^4 - 16$

11. $r(t) = t^3 + 4$

12. $g(x) = \sqrt[3]{x}$

13. $g(x) = 1 - \sin x$

14. $f(x) = x^4 - 1, \quad 0 \leqslant x \leqslant 10$

15. $f(t)$ is the height of a football t seconds after kickoff.

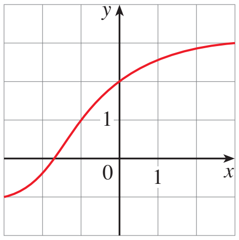
16. $f(t)$ is your height at age t.

17. Assume that f is a one-to-one function.
(a) If $f(6) = 17$, what is $f^{-1}(17)$?
(b) If $f^{-1}(3) = 2$, what is $f(2)$?

18. If $f(x) = x^5 + x^3 + x$, find $f^{-1}(3)$ and $f(f^{-1}(2))$.

19. If $g(x) = 3 + x + e^x$, find $g^{-1}(4)$.

20. The graph of f is given.
(a) Why is f one-to-one?
(b) What are the domain and range of f^{-1}?
(c) What is the value of $f^{-1}(2)$?
(d) Estimate the value of $f^{-1}(0)$.



21. The formula $C = \frac{5}{9}(F - 32)$, where $F \geqslant -459.67$, expresses the Celsius temperature C as a function of the Fahrenheit temperature F. Find a formula for the inverse function and interpret it. What is the domain of the inverse function?

22. In the theory of relativity, the mass of a particle with speed v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and c is the speed of light in a vacuum. Find the inverse function of f and explain its meaning.

23–30 Find a formula for the inverse of the function.

23. $f(x) = 1 - x^2, \quad x \geqslant 0$

24. $g(x) = x^2 - 2x, \quad x \geqslant 1$

25. $g(x) = 2 + \sqrt{x + 1}$

26. $h(x) = \dfrac{6 - 3x}{5x + 7}$

27. $y = e^{1-x}$

28. $y = 3\ln(x - 2)$

29. $y = \left(2 + \sqrt[3]{x}\,\right)^5$

30. $y = \dfrac{1 - e^{-x}}{1 + e^{-x}}$

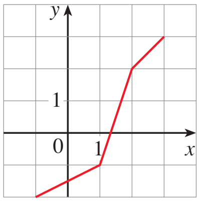
31–32 Find an explicit formula for f^{-1} and use it to graph f^{-1}, f, and the line $y = x$ on the same screen. To check your work, see whether the graphs of f and f^{-1} are reflections about the line.

31. $f(x) = \sqrt{4x + 3}$

32. $f(x) = 1 + e^{-x}$

33–34 Use the given graph of f to sketch the graph of f^{-1}.

33.



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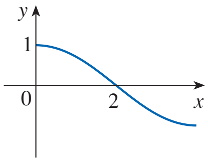


35. Let $f(x) = \sqrt{1 - x^2}$, $0 \leqslant x \leqslant 1$.
(a) Find f^{-1}. How is it related to f?
(b) Identify the graph of f and explain your answer to part (a).

36. Let $g(x) = \sqrt[3]{1 - x^3}$.
(a) Find g^{-1}. How is it related to g?
(b) Graph g. How do you explain your answer to part (a)?

37. (a) How is the logarithmic function $y = \log_b x$ defined?
(b) What is the domain of this function?
(c) What is the range of this function?
(d) Sketch the general shape of the graph of the function $y = \log_b x$ if $b > 1$.

38. (a) What is the natural logarithm?
(b) What is the common logarithm?
(c) Sketch the graphs of the natural logarithm function and the natural exponential function with a common set of axes.

39–42 Find the exact value of each expression.

39. (a) $\log_3 81$ (b) $\log_3\left(\frac{1}{81}\right)$ (c) $\log_9 3$

40. (a) $\ln \dfrac{1}{e^2}$ (b) $\ln \sqrt{e}$ (c) $\ln\left(\ln e^{e^{50}}\right)$

41. (a) $\log_2 30 - \log_2 15$
(b) $\log_3 10 - \log_3 5 - \log_3 18$
(c) $2\log_5 100 - 4\log_5 50$

42. (a) $e^{3\ln 2}$ (b) $e^{-2\ln 5}$ (c) $e^{\ln(\ln e^3)}$

43–44 Use the laws of logarithms to expand each expression.

43. (a) $\log_{10}(x^2y^3z)$ (b) $\ln\left(\dfrac{x^4}{\sqrt{x^2 - 4}}\right)$

44. (a) $\ln\sqrt{\dfrac{3x}{x - 3}}$ (b) $\log_2\left[(x^3 + 1)\sqrt[3]{(x - 3)^2}\,\right]$

45–46 Express as a single logarithm.

45. (a) $\log_{10} 20 - \frac{1}{3}\log_{10} 1000$ (b) $\ln a - 2\ln b + 3\ln c$

46. (a) $3\ln(x - 2) - \ln(x^2 - 5x + 6) + 2\ln(x - 3)$
(b) $c\log_a x - d\log_a y + \log_a z$

47–48 Use Formula 11 to evaluate each logarithm correct to six decimal places.

47. (a) $\log_5 10$ (b) $\log_{15} 12$

48. (a) $\log_3 12$ (b) $\log_{12} 6$

49–50 Use Formula 11 to graph the given functions on a common screen. How are these graphs related?

49. $y = \log_{1.5} x, \quad y = \ln x, \quad y = \log_{10} x, \quad y = \log_{50} x$

50. $y = \ln x, \quad y = \log_8 x, \quad y = e^x, \quad y = 8^x$

51. Suppose that the graph of $y = \log_2 x$ is drawn on a coordinate grid where the unit of measurement is a centimeter. How many kilometers to the right of the origin do we have to move before the height of the curve reaches 25 cm?

52. Compare the functions $f(x) = x^{0.1}$ and $g(x) = \ln x$ by graphing both functions in several viewing rectangles. When does the graph of f finally surpass the graph of g?

53–54 Make a rough sketch by hand of the graph of each function. Use the graphs given in Figures 12 and 13 and, if necessary, the transformations of Section 1.3.

53. (a) $y = \log_{10}(x + 5)$ (b) $y = -\ln x$

54. (a) $y = \ln(-x)$ (b) $y = \ln|x|$

55–56
(a) What are the domain and range of f?
(b) What is the x-intercept of the graph of f?
(c) Sketch the graph of f.

55. $f(x) = \ln x + 2$ **56.** $f(x) = \ln(x - 1) - 1$

57–60 Solve each equation for x. Give both an exact value and a decimal approximation, correct to three decimal places.

57. (a) $\ln(4x + 2) = 3$ (b) $e^{2x-3} = 12$

58. (a) $\log_2(x^2 - x - 1) = 2$ (b) $1 + e^{4x+1} = 20$

59. (a) $\ln x + \ln(x - 1) = 0$ (b) $5^{1-2x} = 9$

60. (a) $\ln(\ln x) = 0$ (b) $\dfrac{60}{1 + e^{-x}} = 4$

61–62 Solve each inequality for x.

61. (a) $\ln x < 0$ (b) $e^x > 5$

62. (a) $1 < e^{3x-1} < 2$ (b) $1 - 2\ln x < 3$

63. (a) Find the domain of $f(x) = \ln(e^x - 3)$.
(b) Find f^{-1} and its domain.

64. (a) What are the values of $e^{\ln 300}$ and $\ln(e^{300})$?
(b) Use your calculator to evaluate $e^{\ln 300}$ and $\ln(e^{300})$. What do you notice? Can you explain why the calculator has trouble?

T **65.** Graph the function $f(x) = \sqrt{x^3 + x^2 + x + 1}$ and explain why it is one-to-one. Then use a computer algebra system to find an explicit expression for $f^{-1}(x)$. (Your CAS will produce three possible expressions. Explain why two of them are irrelevant in this context.)

T **66.** (a) If $g(x) = x^6 + x^4$, $x \geqslant 0$, use a computer algebra system to find an expression for $g^{-1}(x)$.
(b) Use the expression in part (a) to graph $y = g(x)$, $y = x$, and $y = g^{-1}(x)$ on the same screen.

67. If a bacteria population starts with 100 bacteria and doubles every three hours, then the number of bacteria after t hours is $n = f(t) = 100 \cdot 2^{t/3}$.
(a) Find the inverse of this function and explain its meaning.
(b) When will the population reach 50,000?

68. The National Ignition Facility at the Lawrence Livermore National Laboratory maintains the world's largest laser facility. The lasers, which are used to start a nuclear fusion reaction, are powered by a capacitor bank that stores a total of about 400 megajoules of energy. When the lasers are fired the capacitors discharge completely and then immediately begin recharging. The charge Q of the capacitors t seconds after the discharge is given by

$$Q(t) = Q_0(1 - e^{-t/a})$$

(The maximum charge capacity is Q_0 and t is measured in seconds.)
(a) Find a formula for the inverse of this function and explain its meaning.
(b) How long does it take to recharge the capacitors to 90% of capacity if $a = 50$?

69–74 Find the exact value of each expression.

69. (a) $\cos^{-1}(-1)$ (b) $\sin^{-1}(0.5)$

70. (a) $\tan^{-1}\sqrt{3}$ (b) $\arctan(-1)$

71. (a) $\csc^{-1}\sqrt{2}$ (b) $\arcsin 1$

72. (a) $\sin^{-1}(-1/\sqrt{2})$ (b) $\cos^{-1}(\sqrt{3}/2)$

73. (a) $\cot^{-1}(-\sqrt{3})$ (b) $\sec^{-1} 2$

74. (a) $\arcsin(\sin(5\pi/4))$ (b) $\cos\left(2\sin^{-1}\left(\frac{5}{13}\right)\right)$

75. Prove that $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$.

76–78 Simplify the expression.

76. $\tan(\sin^{-1}x)$ **77.** $\sin(\tan^{-1}x)$ **78.** $\sin(2\arccos x)$

79-80 Graph the given functions on the same screen. How are these graphs related?

79. $y = \sin x$, $-\pi/2 \leqslant x \leqslant \pi/2$; $y = \sin^{-1}x$; $y = x$

80. $y = \tan x$, $-\pi/2 < x < \pi/2$; $y = \tan^{-1}x$; $y = x$

81. Find the domain and range of the function

$$g(x) = \sin^{-1}(3x + 1)$$

82. (a) Graph the function $f(x) = \sin(\sin^{-1}x)$ and explain the appearance of the graph.
(b) Graph the function $g(x) = \sin^{-1}(\sin x)$. How do you explain the appearance of this graph?

83. (a) If we shift a curve to the left, what happens to its reflection about the line $y = x$? In view of this geometric principle, find an expression for the inverse of $g(x) = f(x + c)$, where f is a one-to-one function.
(b) Find an expression for the inverse of $h(x) = f(cx)$, where $c \neq 0$.

1 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. (a) What is a function? What are its domain and range?
(b) What is the graph of a function?
(c) How can you tell whether a given curve is the graph of a function?

2. Discuss four ways of representing a function. Illustrate your discussion with examples.

3. (a) What is an even function? How can you tell if a function is even by looking at its graph? Give three examples of an even function.
(b) What is an odd function? How can you tell if a function is odd by looking at its graph? Give three examples of an odd function.

4. What is an increasing function?

5. What is a mathematical model?

6. Give an example of each type of function.
(a) Linear function (b) Power function
(c) Exponential function (d) Quadratic function
(e) Polynomial of degree 5 (f) Rational function

7. Sketch by hand, on the same axes, the graphs of the following functions.
(a) $f(x) = x$ (b) $g(x) = x^2$
(c) $h(x) = x^3$ (d) $j(x) = x^4$

8. Draw, by hand, a rough sketch of the graph of each function.
(a) $y = \sin x$ (b) $y = \tan x$ (c) $y = e^x$
(d) $y = \ln x$ (e) $y = 1/x$ (f) $y = |x|$
(g) $y = \sqrt{x}$ (h) $y = \tan^{-1}x$

9. Suppose that f has domain A and g has domain B.
(a) What is the domain of $f + g$?
(b) What is the domain of fg?
(c) What is the domain of f/g?

10. How is the composite function $f \circ g$ defined? What is its domain?

11. Suppose the graph of f is given. Write an equation for each of the graphs that are obtained from the graph of f as follows.
(a) Shift 2 units upward. (b) Shift 2 units downward.
(c) Shift 2 units to the right. (d) Shift 2 units to the left.
(e) Reflect about the x-axis.
(f) Reflect about the y-axis.
(g) Stretch vertically by a factor of 2.
(h) Shrink vertically by a factor of 2.
(i) Stretch horizontally by a factor of 2.
(j) Shrink horizontally by a factor of 2.

12. (a) What is a one-to-one function? How can you tell if a function is one-to-one by looking at its graph?
(b) If f is a one-to-one function, how is its inverse function f^{-1} defined? How do you obtain the graph of f^{-1} from the graph of f?

13. (a) How is the inverse sine function $f(x) = \sin^{-1}x$ defined? What are its domain and range?
(b) How is the inverse cosine function $f(x) = \cos^{-1}x$ defined? What are its domain and range?
(c) How is the inverse tangent function $f(x) = \tan^{-1}x$ defined? What are its domain and range?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If f is a function, then $f(s + t) = f(s) + f(t)$.

2. If $f(s) = f(t)$, then $s = t$.

3. If f is a function, then $f(3x) = 3f(x)$.

4. If the function f has an inverse and $f(2) = 3$, then $f^{-1}(3) = 2$.

5. A vertical line intersects the graph of a function at most once.

6. If f and g are functions, then $f \circ g = g \circ f$.

7. If f is one-to-one, then $f^{-1}(x) = \dfrac{1}{f(x)}$.

8. You can always divide by e^x.

9. If $0 < a < b$, then $\ln a < \ln b$.

10. If $x > 0$, then $(\ln x)^6 = 6 \ln x$.

11. If $x > 0$ and $a > 1$, then $\dfrac{\ln x}{\ln a} = \ln \dfrac{x}{a}$.

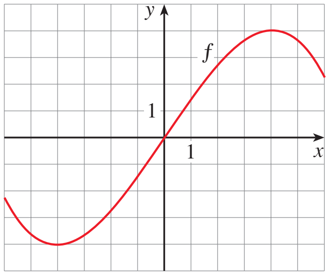
12. $\tan^{-1}(-1) = 3\pi/4$

13. $\tan^{-1}x = \dfrac{\sin^{-1}x}{\cos^{-1}x}$

14. If x is any real number, then $\sqrt{x^2} = x$.

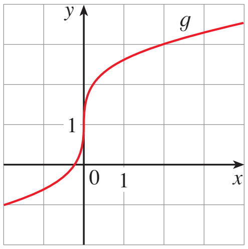
EXERCISES

1. Let f be the function whose graph is given.



(a) Estimate the value of $f(2)$.
(b) Estimate the values of x such that $f(x) = 3$.
(c) State the domain of f.
(d) State the range of f.
(e) On what interval is f increasing?
(f) Is f one-to-one? Explain.
(g) Is f even, odd, or neither even nor odd? Explain.

2. The graph of g is given.



(a) State the value of $g(2)$.
(b) Why is g one-to-one?
(c) Estimate the value of $g^{-1}(2)$.
(d) Estimate the domain of g^{-1}.
(e) Sketch the graph of g^{-1}.

3. If $f(x) = x^2 - 2x + 3$, evaluate the difference quotient

$$\frac{f(a+h) - f(a)}{h}$$

4. Sketch a rough graph of the yield of a crop as a function of the amount of fertilizer used.

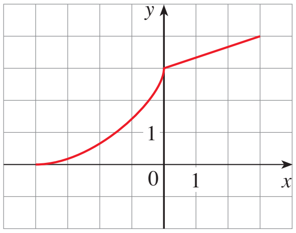
5–8 Find the domain and range of the function. Write your answer in interval notation.

5. $f(x) = 2/(3x - 1)$
6. $g(x) = \sqrt{16 - x^4}$
7. $h(x) = \ln(x + 6)$
8. $F(t) = 3 + \cos 2t$

---

9. Suppose that the graph of f is given. Describe how the graphs of the following functions can be obtained from the graph of f.
(a) $y = f(x) + 5$
(b) $y = f(x + 5)$
(c) $y = 1 + 2f(x)$
(d) $y = f(x - 2) - 2$
(e) $y = -f(x)$
(f) $y = f^{-1}(x)$

10. The graph of f is given. Draw the graphs of the following functions.
(a) $y = f(x - 8)$
(b) $y = -f(x)$
(c) $y = 2 - f(x)$
(d) $y = \frac{1}{2}f(x) - 1$
(e) $y = f^{-1}(x)$
(f) $y = f^{-1}(x + 3)$



11–18 Use transformations to sketch the graph of the function.

11. $f(x) = x^3 + 2$
12. $f(x) = (x - 3)^2$
13. $y = \sqrt{x + 2}$
14. $y = \ln(x + 5)$
15. $g(x) = 1 + \cos 2x$
16. $h(x) = -e^x + 2$
17. $s(x) = 1 + 0.5^x$
18. $f(x) = \begin{cases} -x & \text{if } x < 0 \\ e^x - 1 & \text{if } x \geqslant 0 \end{cases}$

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19. Determine whether f is even, odd, or neither even nor odd.
(a) $f(x) = 2x^5 - 3x^2 + 2$
(b) $f(x) = x^3 - x^7$
(c) $f(x) = e^{-x^2}$
(d) $f(x) = 1 + \sin x$
(e) $f(x) = 1 - \cos 2x$
(f) $f(x) = (x + 1)^2$

20. Find an expression for the function whose graph consists of the line segment from the point $(-2, 2)$ to the point $(-1, 0)$ together with the top half of the circle with center the origin and radius 1.

21. If $f(x) = \ln x$ and $g(x) = x^2 - 9$, find the functions (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, (d) $g \circ g$, and their domains.

22. Express the function $F(x) = 1/\sqrt{x + \sqrt{x}}$ as a composition of three functions.

23. Life expectancy has improved dramatically in recent decades. The table gives the life expectancy at birth (in years) of males born in the United States. Use a scatter plot to choose an appropriate type of model. Use your model to predict the life span of a male born in the year 2030.

Birth year	Life expectancy	Birth year	Life expectancy
1900	48.3	1960	66.6
1910	51.1	1970	67.1
1920	55.2	1980	70.0
1930	57.4	1990	71.8
1940	62.5	2000	73.0
1950	65.6	2010	76.2

24. A small-appliance manufacturer finds that it costs \$9000 to produce 1000 toaster ovens a week and \$12,000 to produce 1500 toaster ovens a week.
(a) Express the cost as a function of the number of toaster ovens produced, assuming that it is linear. Then sketch the graph.
(b) What is the slope of the graph and what does it represent?
(c) What is the y-intercept of the graph and what does it represent?

25. If $f(x) = 2x + 4^x$, find $f^{-1}(6)$.

26. Find the inverse function of $f(x) = \dfrac{2x + 3}{1 - 5x}$.

27. Use the laws of logarithms to expand each expression.

(a) $\ln x\sqrt{x + 1}$ (b) $\log_2 \sqrt{\dfrac{x^2 + 1}{x - 1}}$

28. Express as a single logarithm.
(a) $\frac{1}{2}\ln x - 2\ln(x^2 + 1)$
(b) $\ln(x - 3) + \ln(x + 3) - 2\ln(x^2 - 9)$

29–30 Find the exact value of each expression.

29. (a) $e^{2\ln 5}$ (b) $\log_6 4 + \log_6 54$ (c) $\tan\left(\arcsin \frac{4}{5}\right)$

30. (a) $\ln \dfrac{1}{e^3}$ (b) $\sin(\tan^{-1} 1)$ (c) $10^{-3\log 4}$

31–36 Solve the equation for x. Give both an exact value and a decimal approximation, correct to three decimal places.

31. $e^{2x} = 3$

32. $\ln x^2 = 5$

33. $e^{e^x} = 10$

34. $\cos^{-1}x = 2$

35. $\tan^{-1}(3x^2) = \dfrac{\pi}{4}$

36. $\ln x - 1 = \ln(5 + x) - 4$

37. The viral load for an HIV patient is 52.0 RNA copies/mL before treatment begins. Eight days later the viral load is half of the initial amount.
(a) Find the viral load after 24 days.
(b) Find the viral load $V(t)$ that remains after t days.
(c) Find a formula for the inverse of the function V and explain its meaning.
(d) After how many days will the viral load be reduced to 2.0 RNA copies/mL?

38. The population of a certain species in a limited environment with initial population 100 and carrying capacity 1000 is

$$P(t) = \frac{100{,}000}{100 + 900e^{-t}}$$

where t is measured in years.
(a) Graph this function and estimate how long it takes for the population to reach 900.
(b) Find the inverse of this function and explain its meaning.
(c) Use the inverse function to find the time required for the population to reach 900. Compare with the result of part (a).

Principles of Problem Solving

There are no hard and fast rules that will ensure success in solving problems. However, it is possible to outline some general steps in the problem-solving process and to give some principles that may be useful in the solution of certain problems. These steps and principles are just common sense made explicit. They have been adapted from George Polya's book *How To Solve It*.

1 UNDERSTAND THE PROBLEM

The first step is to read the problem and make sure that you understand it clearly. Ask yourself the following questions:

What is the unknown?

What are the given quantities?

What are the given conditions?

For many problems it is useful to

draw a diagram

and identify the given and required quantities on the diagram.

Usually it is necessary to

introduce suitable notation

In choosing symbols for the unknown quantities we often use letters such as a, b, c, m, n, x, and y, but in some cases it helps to use initials as suggestive symbols; for instance, V for volume or t for time.

2 THINK OF A PLAN

Find a connection between the given information and the unknown that will enable you to calculate the unknown. It often helps to ask yourself explicitly: "How can I relate the given to the unknown?" If you don't see a connection immediately, the following ideas may be helpful in devising a plan.

Try to Recognize Something Familiar Relate the given situation to previous knowledge. Look at the unknown and try to recall a more familiar problem that has a similar unknown.

Try to Recognize Patterns Some problems are solved by recognizing that some kind of pattern is occurring. The pattern could be geometric, or numerical, or algebraic. If you can see regularity or repetition in a problem, you might be able to guess what the continuing pattern is and then prove it.

Use Analogy Try to think of an analogous problem—that is, a similar problem, a related problem—but one that is easier than the original problem. If you can solve the similar, simpler problem, then it might give you the clues you need to solve the original, more difficult problem. For instance, if a problem involves very large numbers, you could first try a similar problem with smaller numbers. Or if the problem involves three-dimensional geometry, you could look for a similar problem in two-dimensional geometry. Or if the problem you start with is a general one, you could first try a special case.

Introduce Something Extra It may sometimes be necessary to introduce something new—an auxiliary aid—to help make the connection between the given and the unknown. For instance, in a problem where a diagram is useful the auxiliary aid could be a new line drawn in a diagram. In a more algebraic problem it could be a new unknown that is related to the original unknown.

Take Cases We may sometimes have to split a problem into several cases and give a different argument for each of the cases. For instance, we often have to use this strategy in dealing with absolute value.

Work Backward Sometimes it is useful to imagine that your problem is solved and then to work backward, step by step, until you arrive at the given data. At this point you may be able to reverse your steps and thereby construct a solution to the original problem. This procedure is commonly used in solving equations. For instance, in solving the equation $3x - 5 = 7$, we suppose that x is a number that satisfies $3x - 5 = 7$ and work backward. We add 5 to each side of the equation and then divide each side by 3 to get $x = 4$. Since each of these steps can be reversed, we have solved the problem.

Establish Subgoals In a complex problem it is often useful to set subgoals (in which the desired situation is only partially fulfilled). If we can first reach these subgoals, then we may be able to build on them to reach our final goal.

Indirect Reasoning Sometimes it is appropriate to attack a problem indirectly. In using proof by contradiction to prove that P implies Q, we assume that P is true and Q is false and try to see why this can't happen. Somehow we have to use this information and arrive at a contradiction to what we absolutely know is true.

Mathematical Induction In proving statements that involve a positive integer n, it is frequently helpful to use the following principle.

Principle of Mathematical Induction Let S_n be a statement about the positive integer n. Suppose that

1. S_1 is true.
2. S_{k+1} is true whenever S_k is true.

Then S_n is true for all positive integers n.

This is reasonable because, since S_1 is true, it follows from condition 2 (with $k = 1$) that S_2 is true. Then, using condition 2 with $k = 2$, we see that S_3 is true. Again using condition 2, this time with $k = 3$, we have that S_4 is true. This procedure can be followed indefinitely.

3 CARRY OUT THE PLAN

In Step 2 a plan was devised. In carrying out that plan we have to check each stage of the plan and write the details that prove that each stage is correct.

4 LOOK BACK

Having completed our solution, it is wise to look back over it, partly to see if we have made errors in the solution and partly to see if we can think of an easier way to solve the problem. Another reason for looking back is that it will familiarize us with the method of solution and this may be useful for solving a future problem. Descartes said, "Every problem that I solved became a rule which served afterwards to solve other problems."

These principles of problem solving are illustrated in the following examples. Before you look at the solutions, try to solve these problems yourself, referring to these principles of problem solving if you get stuck. You may find it useful to refer to this section from time to time as you solve the exercises in the remaining chapters of this book.

EXAMPLE 1 Express the hypotenuse h of a right triangle with area 25 m^2 as a function of its perimeter P.

PS Understand the problem.

SOLUTION Let's first sort out the information by identifying the unknown quantity and the data:

Unknown: hypotenuse h

Given quantities: perimeter P, area 25 m^2

PS Draw a diagram.

It helps to draw a diagram and we do so in Figure 1.

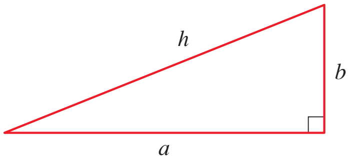


FIGURE 1

PS Connect the given with the unknown.

PS Introduce something extra.

In order to connect the given quantities to the unknown, we introduce two extra variables a and b, which are the lengths of the other two sides of the triangle. This enables us to express the given condition, which is that the triangle is right-angled, by the Pythagorean Theorem:

$$h^2 = a^2 + b^2$$

The other connections among the variables come by writing expressions for the area and perimeter:

$$25 = \tfrac{1}{2}ab \qquad\qquad P = a + b + h$$

Since P is given, notice that we now have three equations in the three unknowns a, b, and h:

$$\boxed{1} \qquad h^2 = a^2 + b^2$$

$$\boxed{2} \qquad 25 = \tfrac{1}{2}ab$$

$$\boxed{3} \qquad P = a + b + h$$

PS Relate to the familiar.

Although we have the correct number of equations, they are not easy to solve in a straightforward fashion. But if we use the problem-solving strategy of trying to recognize something familiar, then we can solve these equations by an easier method. Look at the right sides of Equations 1, 2, and 3. Do these expressions remind you of anything familiar? Notice that they contain the ingredients of a familiar formula:

$$(a + b)^2 = a^2 + 2ab + b^2$$

Using this idea, we express $(a + b)^2$ in two ways. From Equations 1 and 2 we have

$$(a + b)^2 = (a^2 + b^2) + 2ab = h^2 + 4(25) = h^2 + 100$$

From Equation 3 we have

$$(a + b)^2 = (P - h)^2 = P^2 - 2Ph + h^2$$

Thus

$$h^2 + 100 = P^2 - 2Ph + h^2$$

$$2Ph = P^2 - 100$$

$$h = \frac{P^2 - 100}{2P}$$

This is the required expression for h as a function of P. ■

As the next example illustrates, it is often necessary to use the problem-solving principle of *taking cases* when dealing with absolute values.

EXAMPLE 2 Solve the inequality $|x-3|+|x+2|<11$.

SOLUTION Recall the definition of absolute value:

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

It follows that

$$\begin{aligned} |x-3| &= \begin{cases} x-3 & \text{if } x-3 \geqslant 0 \\ -(x-3) & \text{if } x-3 < 0 \end{cases} \\ &= \begin{cases} x-3 & \text{if } x \geqslant 3 \\ -x+3 & \text{if } x < 3 \end{cases} \end{aligned}$$

Similarly

$$\begin{aligned} |x+2| &= \begin{cases} x+2 & \text{if } x+2 \geqslant 0 \\ -(x+2) & \text{if } x+2 < 0 \end{cases} \\ &= \begin{cases} x+2 & \text{if } x \geqslant -2 \\ -x-2 & \text{if } x < -2 \end{cases} \end{aligned}$$

PS Take cases.

These expressions show that we must consider three cases:

$$x < -2 \qquad -2 \leqslant x < 3 \qquad x \geqslant 3$$

CASE I If $x < -2$, we have

$$\begin{aligned} |x-3|+|x+2| &< 11 \\ -x+3-x-2 &< 11 \\ -2x &< 10 \\ x &> -5 \end{aligned}$$

CASE II If $-2 \leqslant x < 3$, the given inequality becomes

$$\begin{aligned} -x+3+x+2 &< 11 \\ 5 &< 11 \qquad \text{(always true)} \end{aligned}$$

CASE III If $x \geqslant 3$, the inequality becomes

$$\begin{aligned} x-3+x+2 &< 11 \\ 2x &< 12 \\ x &< 6 \end{aligned}$$

Combining cases I, II, and III, we see that the inequality is satisfied when $-5 < x < 6$. So the solution is the interval $(-5, 6)$. ■

In the following example we first guess the answer by looking at special cases and recognizing a pattern. Then we prove our conjecture by mathematical induction.

In using the Principle of Mathematical Induction, we follow three steps:

Step 1 Prove that S_n is true when $n = 1$.

Step 2 Assume that S_n is true when $n = k$ and deduce that S_n is true when $n = k + 1$.

Step 3 Conclude that S_n is true for all n by the Principle of Mathematical Induction.

EXAMPLE 3 If $f_0(x) = x/(x+1)$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \ldots$, find a formula for $f_n(x)$.

PS Analogy: Try a similar, simpler problem.

SOLUTION We start by finding formulas for $f_n(x)$ for the special cases $n = 1, 2$, and 3.

$$f_1(x) = (f_0 \circ f_0)(x) = f_0(f_0(x)) = f_0\left(\frac{x}{x+1}\right)$$

$$= \frac{\dfrac{x}{x+1}}{\dfrac{x}{x+1}+1} = \frac{\dfrac{x}{x+1}}{\dfrac{2x+1}{x+1}} = \frac{x}{2x+1}$$

$$f_2(x) = (f_0 \circ f_1)(x) = f_0(f_1(x)) = f_0\left(\frac{x}{2x+1}\right)$$

$$= \frac{\dfrac{x}{2x+1}}{\dfrac{x}{2x+1}+1} = \frac{\dfrac{x}{2x+1}}{\dfrac{3x+1}{2x+1}} = \frac{x}{3x+1}$$

$$f_3(x) = (f_0 \circ f_2)(x) = f_0(f_2(x)) = f_0\left(\frac{x}{3x+1}\right)$$

$$= \frac{\dfrac{x}{3x+1}}{\dfrac{x}{3x+1}+1} = \frac{\dfrac{x}{3x+1}}{\dfrac{4x+1}{3x+1}} = \frac{x}{4x+1}$$

PS Look for a pattern.

We notice a pattern: the coefficient of x in the denominator of $f_n(x)$ is $n+1$ in the three cases we have computed. So we make the guess that, in general,

$$\boxed{4} \qquad f_n(x) = \frac{x}{(n+1)x+1}$$

To prove this, we use the Principle of Mathematical Induction. We have already verified that (4) is true for $n = 1$. Assume that it is true for $n = k$, that is,

$$f_k(x) = \frac{x}{(k+1)x+1}$$

Then $$f_{k+1}(x) = (f_0 \circ f_k)(x) = f_0(f_k(x)) = f_0\left(\frac{x}{(k+1)x+1}\right)$$

$$= \frac{\dfrac{x}{(k+1)x+1}}{\dfrac{x}{(k+1)x+1}+1} = \frac{\dfrac{x}{(k+1)x+1}}{\dfrac{(k+2)x+1}{(k+1)x+1}} = \frac{x}{(k+2)x+1}$$

This expression shows that (4) is true for $n = k+1$. Therefore, by mathematical induction, it is true for all positive integers n. ■

Problems

1. One of the legs of a right triangle has length 4 cm. Express the length of the altitude perpendicular to the hypotenuse as a function of the length of the hypotenuse.

2. The altitude perpendicular to the hypotenuse of a right triangle is 12 cm. Express the length of the hypotenuse as a function of the perimeter.

3. Solve the equation $\left|4x - |x+1|\right| = 3$.

4. Solve the inequality $|x-1| - |x-3| \geqslant 5$.

5. Sketch the graph of the function $f(x) = \left|x^2 - 4|x| + 3\right|$.

6. Sketch the graph of the function $g(x) = |x^2 - 1| - |x^2 - 4|$.

7. Draw the graph of the equation $x + |x| = y + |y|$.

8. Sketch the region in the plane consisting of all points (x, y) such that

$$|x - y| + |x| - |y| \leqslant 2$$

9. The notation $\max\{a, b, \ldots\}$ means the largest of the numbers $a, b, \ldots$. Sketch the graph of each function.
   (a) $f(x) = \max\{x, 1/x\}$
   (b) $f(x) = \max\{\sin x, \cos x\}$
   (c) $f(x) = \max\{x^2, 2 + x, 2 - x\}$

10. Sketch the region in the plane defined by each of the following equations or inequalities.
    (a) $\max\{x, 2y\} = 1$
    (b) $-1 \leqslant \max\{x, 2y\} \leqslant 1$
    (c) $\max\{x, y^2\} = 1$

11. Show that if $x > 0$ and $x \neq 1$, then

$$\frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \frac{1}{\log_5 x} = \frac{1}{\log_{30} x}$$

12. Find the number of solutions of the equation $\sin x = \dfrac{x}{100}$.

13. Find the exact value of

$$\sin\frac{\pi}{100} + \sin\frac{2\pi}{100} + \sin\frac{3\pi}{100} + \cdots + \sin\frac{200\pi}{100}$$

14. (a) Show that the function $f(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$ is an odd function.
    (b) Find the inverse function of f.

15. Solve the inequality $\ln(x^2 - 2x - 2) \leqslant 0$.

16. Use indirect reasoning to prove that $\log_2 5$ is an irrational number.

17. A driver sets out on a journey. For the first half of the distance she drives at the leisurely pace of 50 km/h; she drives the second half at 100 km/h. What is her average speed on this trip?

18. Is it true that $f \circ (g + h) = f \circ g + f \circ h$?

19. Prove that if n is a positive integer, then $7^n - 1$ is divisible by 6.

20. Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

21. If $f_0(x) = x^2$ and $f_{n+1}(x) = f_0(f_n(x))$ for $n = 0, 1, 2, \ldots$, find a formula for $f_n(x)$.

22. (a) If $f_0(x) = \dfrac{1}{2 - x}$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \ldots$, find an expression for $f_n(x)$ and use mathematical induction to prove it.

(b) Graph f_0, f_1, f_2, f_3 on the same screen and describe the effects of repeated composition.

We know that when an object is dropped from a height it falls faster and faster. Galileo discovered that the distance the object has fallen is proportional to the square of the time elapsed. Calculus enables us to calculate the precise speed of the object at any time. In Exercise 2.7.11 you are asked to determine the speed at which a cliff diver plunges into the ocean.

Icealex / Shutterstock.com

2

Limits and Derivatives

IN *A PREVIEW OF CALCULUS* (immediately preceding Chapter 1) we saw how the idea of a limit underlies the various branches of calculus. It is therefore appropriate to begin our study of calculus by investigating limits and their properties. The special type of limit that is used to find tangents and velocities gives rise to the central idea in differential calculus, the derivative.

2.1 The Tangent and Velocity Problems

In this section we see how limits arise when we attempt to find the tangent to a curve or the velocity of an object.

The Tangent Problem

The word *tangent* is derived from the Latin word *tangens*, which means "touching." We can think of a tangent to a curve as a line that touches the curve and follows the same direction as the curve at the point of contact. How can this idea be made precise?

For a circle we could simply follow Euclid and say that a tangent is a line ℓ that intersects the circle once and only once, as in Figure 1(a). For more complicated curves this definition is inadequate. Figure l(b) shows a line ℓ that appears to be a tangent to the curve C at point P, but it intersects C twice.

To be specific, let's look at the problem of trying to find a tangent line ℓ to the parabola $y = x^2$ in the following example.

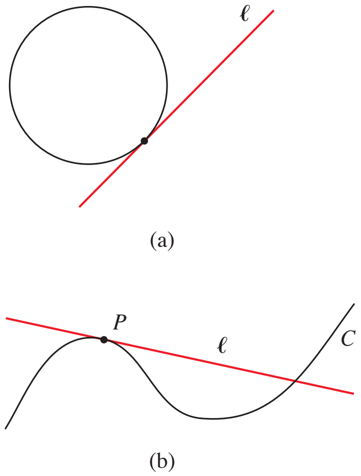


FIGURE 1

EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

SOLUTION We will be able to find an equation of the tangent line ℓ as soon as we know its slope m. The difficulty is that we know only one point, P, on ℓ, whereas we need two points to compute the slope. But observe that we can compute an approximation to m by choosing a nearby point $Q(x, x^2)$ on the parabola (as in Figure 2) and computing the slope m_{PQ} of the secant line PQ. (A **secant line**, from the Latin word *secans*, meaning cutting, is a line that cuts [intersects] a curve more than once.)

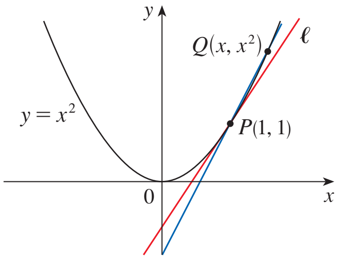


FIGURE 2

We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point $Q(1.5, 2.25)$ we have

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{0.5} = 2.5$$

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

The tables in the margin show the values of m_{PQ} for several values of x close to 1. The closer Q is to P, the closer x is to 1 and, it appears from the tables, the closer m_{PQ} is to 2. This suggests that the slope of the tangent line ℓ should be $m = 2$.

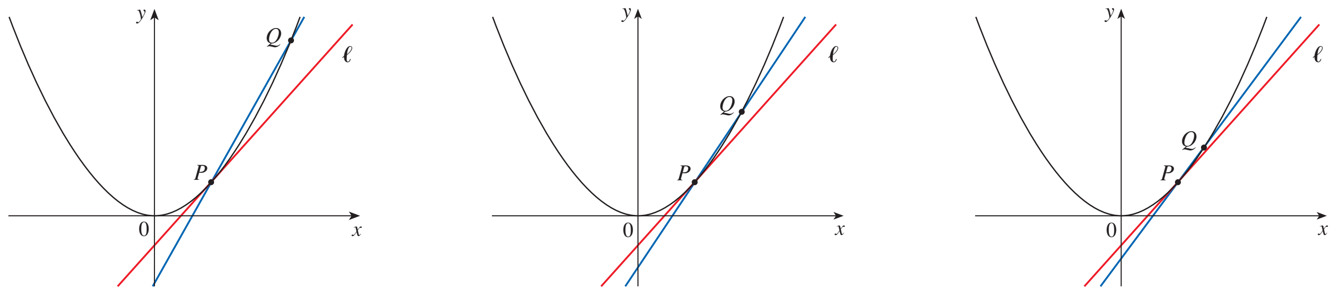
We say that the slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically by writing

$$\lim_{Q \to P} m_{PQ} = m \qquad \text{and} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

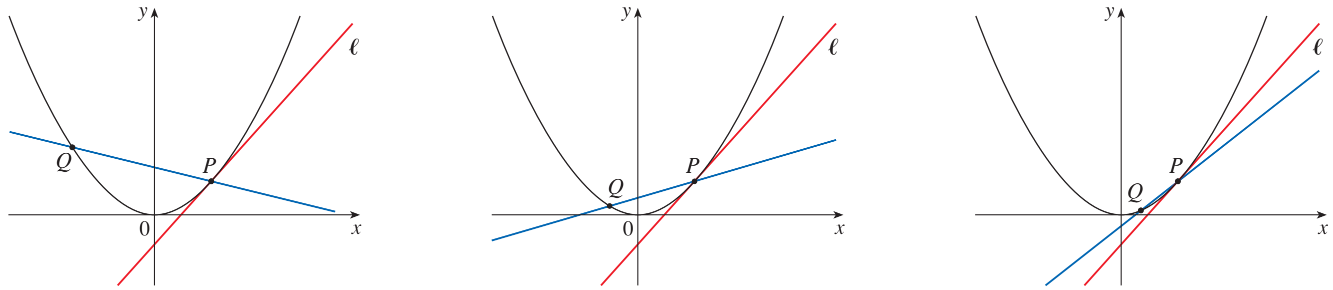
Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line [$y - y_1 = m(x - x_1)$, see Appendix B] to write the equation of the tangent line through $(1, 1)$ as

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1 \qquad ■$$

Figure 3 illustrates the limiting process that occurs in Example 1. As Q approaches P along the parabola, the corresponding secant lines rotate about P and approach the tangent line ℓ.



Q approaches P from the right



Q approaches P from the left

FIGURE 3

Many functions that occur in the sciences are not described by explicit equations; they are defined by experimental data. The next example shows how to estimate the slope of the tangent line to the graph of such a function.

t	Q
0	10
0.02	8.187
0.04	6.703
0.06	5.488
0.08	4.493
0.1	3.676

EXAMPLE 2 A pulse laser operates by storing charge on a capacitor and releasing it suddenly when the laser is fired. The data in the table describe the charge Q remaining on the capacitor (measured in coulombs) at time t (measured in seconds after the laser is fired). Use the data to draw the graph of this function and estimate the slope of the tangent line at the point where $t = 0.04$. (*Note:* The slope of the tangent line represents the electric current flowing from the capacitor to the laser [measured in amperes].)

SOLUTION In Figure 4 we plot the given data and use these points to sketch a curve that approximates the graph of the function.

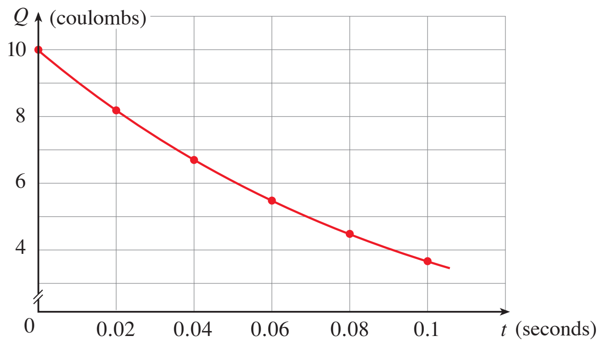


FIGURE 4

Given the points $P(0.04, 6.703)$ and $R(0, 10)$ on the graph, we find that the slope of the secant line PR is

$$m_{PR} = \frac{10 - 6.703}{0 - 0.04} = -82.425$$

R	m_{PR}
(0, 10)	−82.425
(0.02, 8.187)	−74.200
(0.06, 5.488)	−60.750
(0.08, 4.493)	−55.250
(0.1, 3.676)	−50.450

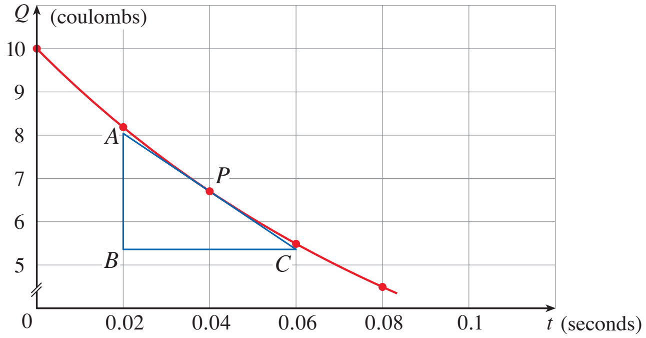
The table at the left shows the results of similar calculations for the slopes of other secant lines. From this table we would expect the slope of the tangent line at $t = 0.04$ to lie somewhere between -74.20 and -60.75. In fact, the average of the slopes of the two closest secant lines is

$$\tfrac{1}{2}(-74.20 - 60.75) = -67.475$$

So, by this method, we estimate the slope of the tangent line to be about -67.5.

Another method is to draw an approximation to the tangent line at P and measure the sides of the triangle ABC, as in Figure 5.

FIGURE 5



The physical meaning of the answer in Example 2 is that the electric current flowing from the capacitor to the laser after 0.04 seconds is about −65 amperes.

This gives an estimate of the slope of the tangent line as

$$-\frac{|AB|}{|BC|} \approx -\frac{8.0 - 5.4}{0.06 - 0.02} = -65.0$$ ■

■ The Velocity Problem

If you watch the speedometer of a car as you drive in city traffic, you see that the speed doesn't stay the same for very long; that is, the velocity of the car is not constant. We assume from watching the speedometer that the car has a definite velocity at each moment, but how is the "instantaneous" velocity defined?

Let's consider the *velocity problem:* Find the instantaneous velocity of an object moving along a straight path at a specific time if the position of the object at any time is known. In the next example, we investigate the velocity of a falling ball. Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.) If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then (at the earth's surface) Galileo's observation is expressed by the equation

$$s(t) = 4.9t^2$$

Steve Allen / Stockbyte / Getty Images

CN Tower in Toronto

XAMPLE 3 Suppose that a ball is dropped from the upper observation deck of he CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after seconds.

OLUTION The difficulty in finding the instantaneous velocity at 5 seconds is that e are dealing with a single instant of time ($t = 5$), so no time interval is involved. owever, we can approximate the desired quantity by computing the average velocity ver the brief time interval of a tenth of a second from $t = 5$ to $t = 5.1$:

$$\begin{aligned}\text{average velocity} &= \frac{\text{change in position}}{\text{time elapsed}}\\ &= \frac{s(5.1) - s(5)}{0.1}\\ &= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1} = 49.49 \text{ m/s}\end{aligned}$$

he following table shows the results of similar calculations of the average velocity over successively smaller time periods.

Time interval	Average velocity (m/s)
$5 \leq t \leq 5.1$	49.49
$5 \leq t \leq 5.05$	49.245
$5 \leq t \leq 5.01$	49.049
$5 \leq t \leq 5.001$	49.0049

It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s. The **instantaneous velocity** when $t = 5$ is defined to be the *limiting value* of these average velocities over shorter and shorter time periods that start at $t = 5$. Thus it appears that the (instantaneous) velocity after 5 seconds is 49 m/s. ■

You may have the feeling that the calculations used in solving this problem are very similar to those used earlier in this section to find tangents. In fact, there is a close connection between the tangent problem and the velocity problem. If we draw the graph of the distance function of the ball (as in Figure 6) and we consider the points $P(5, 4.9(5)^2)$ and $Q(5 + h, 4.9(5 + h)^2)$ on the graph, then the slope of the secant line PQ is

$$m_{PQ} = \frac{4.9(5 + h)^2 - 4.9(5)^2}{(5 + h) - 5}$$

which is the same as the average velocity over the time interval $[5, 5 + h]$. Therefore the velocity at time $t = 5$ (the limit of these average velocities as h approaches 0) must be equal to the slope of the tangent line at P (the limit of the slopes of the secant lines).

Examples 1 and 3 show that in order to solve tangent and velocity problems we must be able to find limits. After studying methods for computing limits in the next five sections, we will return to the problems of finding tangents and velocities in Section 2.7.

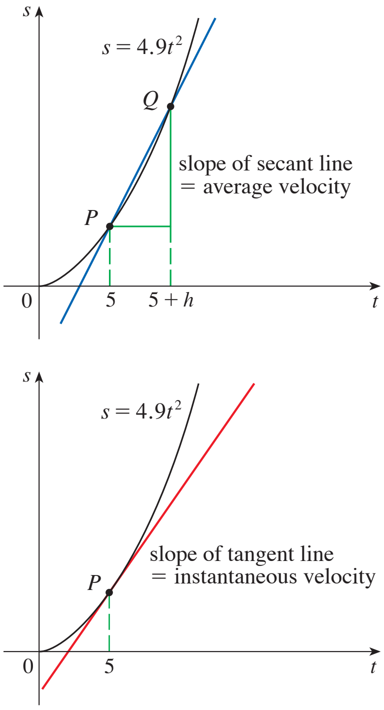

s
s = 4.9t²
Q
slope of secant line = average velocity
P
0
5
5 + h
t
s
s = 4.9t²
slope of tangent line = instantaneous velocity
P
0
5
t

FIGURE 6

2.1 Exercises

1. A tank holds 1000 liters of water, which drains from the bottom of the tank in half an hour. The values in the table show the volume V of water remaining in the tank (in liters) after t minutes.

t (min)	5	10	15	20	25	30
V (L)	694	444	250	111	28	0

(a) If P is the point (15, 250) on the graph of V, find the slopes of the secant lines PQ when Q is the point on the graph with $t = 5, 10, 20, 25$, and 30.

(b) Estimate the slope of the tangent line at P by averaging the slopes of two secant lines.

(c) Use a graph of V to estimate the slope of the tangent line at P. (This slope represents the rate at which the water is flowing from the tank after 15 minutes.)

2. A student bought a smartwatch that tracks the number of steps she walks throughout the day. The table shows the number of steps recorded t minutes after 3:00 PM on the first day she wore the watch.

t (min)	0	10	20	30	40
Steps	3438	4559	5622	6536	7398

(a) Find the slopes of the secant lines corresponding to the given intervals of t. What do these slopes represent?
(i) [0, 40] (ii) [10, 20] (iii) [20, 30]

(b) Estimate the student's walking pace, in steps per minute, at 3:20 PM by averaging the slopes of two secant lines.

3. The point $P(2, -1)$ lies on the curve $y = 1/(1 - x)$.

(a) If Q is the point $(x, 1/(1 - x))$, find the slope of the secant line PQ (correct to six decimal places) for the following values of x:
(i) 1.5 (ii) 1.9 (iii) 1.99 (iv) 1.999
(v) 2.5 (vi) 2.1 (vii) 2.01 (viii) 2.001

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(2, -1)$.

(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(2, -1)$.

4. The point $P(0.5, 0)$ lies on the curve $y = \cos \pi x$.

(a) If Q is the point $(x, \cos \pi x)$, find the slope of the secant line PQ (correct to six decimal places) for the following values of x:
(i) 0 (ii) 0.4 (iii) 0.49
(iv) 0.499 (v) 1 (vi) 0.6
(vii) 0.51 (viii) 0.501

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(0.5, 0)$.

(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(0.5, 0)$.

(d) Sketch the curve, two of the secant lines, and the tangent line.

5. The deck of a bridge is suspended 80 meters above a river. If a pebble falls off the side of the bridge, the height, in meters, of the pebble above the water surface after t seconds is given by $y = 80 - 4.9t^2$.

(a) Find the average velocity of the pebble for the time period beginning when $t = 4$ and lasting
(i) 0.1 seconds (ii) 0.05 seconds (iii) 0.01 seconds

(b) Estimate the instantaneous velocity of the pebble after 4 seconds.

6. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height in meters t seconds later is given by $y = 10t - 1.86t^2$.

(a) Find the average velocity over the given time intervals:
(i) [1, 2] (ii) [1, 1.5] (iii) [1, 1.1]
(iv) [1, 1.01] (v) [1, 1.001]

(b) Estimate the instantaneous velocity when $t = 1$.

7. The table shows the position of a motorcyclist after accelerating from rest.

t (seconds)	0	1	2	3	4	5	6
s (meters)	0	1.5	6.3	14.2	24.1	38.0	53.9

(a) Find the average velocity for each time period:
(i) [2, 4] (ii) [3, 4] (iii) [4, 5] (iv) [4, 6]

(b) Use the graph of s as a function of t to estimate the instantaneous velocity when $t = 3$.

8. The displacement (in centimeters) of a particle moving back and forth along a straight line is given by the equation of motion $s = 2 \sin \pi t + 3 \cos \pi t$, where t is measured in seconds.

(a) Find the average velocity during each time period:
(i) [1, 2] (ii) [1, 1.1]
(iii) [1, 1.01] (iv) [1, 1.001]

(b) Estimate the instantaneous velocity of the particle when $t = 1$.

9. The point $P(1, 0)$ lies on the curve $y = \sin(10\pi/x)$.

(a) If Q is the point $(x, \sin(10\pi/x))$, find the slope of the secant line PQ (correct to four decimal places) for $x = 2$, 1.5, 1.4, 1.3, 1.2, 1.1, 0.5, 0.6, 0.7, 0.8, and 0.9. Do the slopes appear to be approaching a limit?

(b) Use a graph of the curve to explain why the slopes of the secant lines in part (a) are not close to the slope of the tangent line at P.

(c) By choosing appropriate secant lines, estimate the slope of the tangent line at P.

2.2 The Limit of a Function

Having seen in the preceding section how limits arise when we want to find the tangent to a curve or the velocity of an object, we now turn our attention to limits in general and numerical and graphical methods for computing them.

Finding Limits Numerically and Graphically

Let's investigate the behavior of the function f defined by $f(x) = (x - 1)/(x^2 - 1)$ for values of x near 1. The following table gives values of $f(x)$ for values of x close to 1 but not equal to 1.

$x < 1$	$f(x)$	$x > 1$	$f(x)$
0.5	0.666667	1.5	0.400000
0.9	0.526316	1.1	0.476190
0.99	0.502513	1.01	0.497512
0.999	0.500250	1.001	0.499750
0.9999	0.500025	1.0001	0.499975
↓ 1	↓ 0.5	↓ 1	↓ 0.5

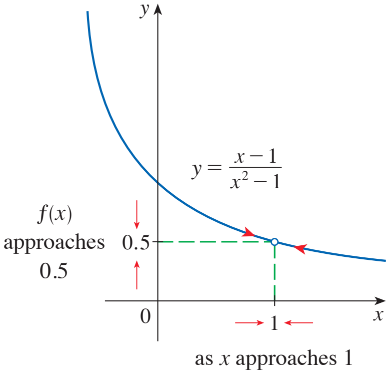


FIGURE 1

From the table and the graph of f shown in Figure 1 we see that the closer x is to 1 (on either side of 1), the closer $f(x)$ is to 0.5. In fact, it appears that we can make the values of $f(x)$ as close as we like to 0.5 by taking x sufficiently close to 1. We express this by saying "the limit of the function $f(x) = (x - 1)/(x^2 - 1)$ as x approaches 1 is equal to 0.5." The notation for this is

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 1} = 0.5$$

In general, we use the following notation.

1 Intuitive Definition of a Limit Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

Roughly speaking, this says that the values of $f(x)$ approach L as x approaches a. In other words, the values of $f(x)$ tend to get closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but $x \neq a$. (A more precise definition will be given in Section 2.4.)

An alternative notation for

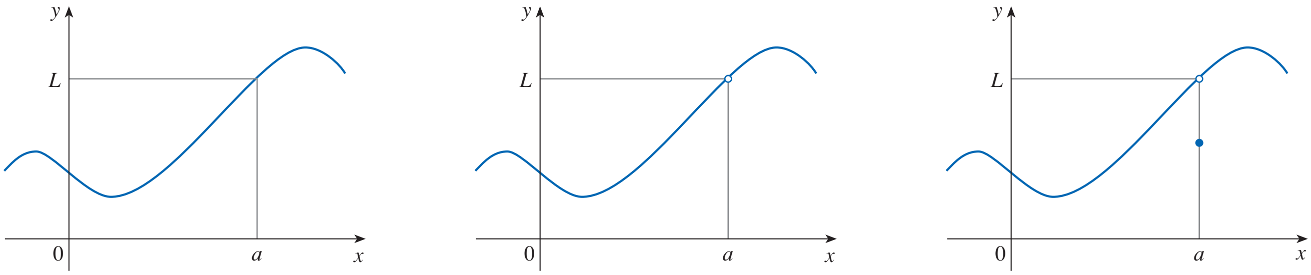
$$\lim_{x \to a} f(x) = L$$

is $$f(x) \to L \quad \text{as} \quad x \to a$$

which is usually read "$f(x)$ approaches L as x approaches a."

Notice the phrase "but x not equal to a" in the definition of limit. This means that in finding the limit of $f(x)$ as x approaches a, we never consider $x = a$. In fact, $f(x)$ need not even be defined when $x = a$. The only thing that matters is how f is defined *near* a.

Figure 2 shows the graphs of three functions. Note that in part (b), $f(a)$ is not defined and in part (c), $f(a) \neq L$. But in each case, regardless of what happens at a, it is true that $\lim_{x \to a} f(x) = L$.



(a) (b) (c)

FIGURE 2 $\lim_{x \to a} f(x) = L$ in all three cases

EXAMPLE 1 Estimate the value of $\displaystyle\lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$.

SOLUTION The table lists values of the function for several values of t near 0.

t	$\dfrac{\sqrt{t^2+9}-3}{t^2}$
± 1.0	$0.162277\ldots$
± 0.5	$0.165525\ldots$
± 0.1	$0.166620\ldots$
± 0.05	$0.166655\ldots$
± 0.01	$0.166666\ldots$

As t approaches 0, the values of the function seem to approach $0.1666666\ldots$ and so we guess that

$$\lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} = \frac{1}{6} \qquad ■$$

t	$\dfrac{\sqrt{t^2+9}-3}{t^2}$
± 0.001	0.166667
± 0.0001	0.166670
± 0.00001	0.167000
± 0.000001	0.000000

In Example 1 what would have happened if we had taken even smaller values of t? The table in the margin shows the results from one calculator; you can see that something strange seems to be happening.

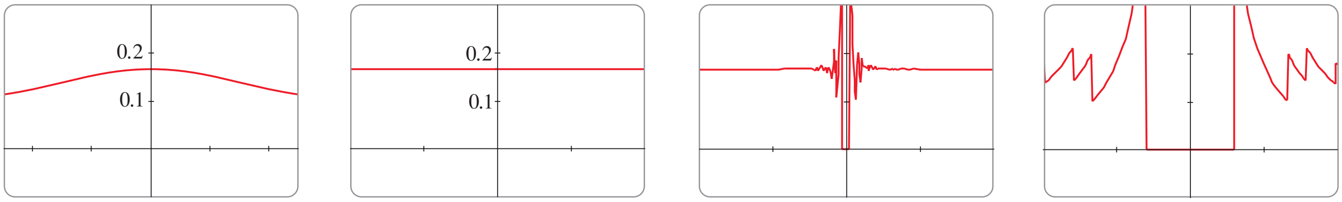
If you try these calculations on your own calculator you might get different values, but eventually you will get the value 0 if you make t sufficiently small. Does this mean that the answer is really 0 instead of $\frac{1}{6}$? No, the value of the limit is $\frac{1}{6}$, as we will show in the next section. The problem is that the calculator gave false values because $\sqrt{t^2 + 9}$ is very close to 3 when t is small. (In fact, when t is sufficiently small, a calculator's value for $\sqrt{t^2 + 9}$ is $3.000\ldots$ to as many digits as the calculator is capable of carrying.)

www.StewartCalculus.com
For a further explanation of why calculators sometimes give false values, click on *Lies My Calculator and Computer Told Me.* In particular, see the section called *The Perils of Subtraction.*

Something similar happens when we try to graph the function

$$f(t) = \frac{\sqrt{t^2 + 9} - 3}{t^2}$$

of Example 1 on a graphing calculator or computer. Parts (a) and (b) of Figure 3 show quite accurate graphs of f, and if we trace along the curve, we can estimate easily that the limit is about $\frac{1}{6}$. But if we zoom in too much, as in parts (c) and (d), then we get inaccurate graphs, again due to rounding errors within the calculations.



(a) $-5 \leqslant t \leqslant 5$ (b) $-0.1 \leqslant t \leqslant 0.1$ (c) $-10^{-6} \leqslant t \leqslant 10^{-6}$ (d) $-10^{-7} \leqslant t \leqslant 10^{-7}$

FIGURE 3

EXAMPLE 2 Guess the value of $\displaystyle\lim_{x \to 0} \frac{\sin x}{x}$.

x	$\dfrac{\sin x}{x}$
± 1.0	0.84147098
± 0.5	0.95885108
± 0.4	0.97354586
± 0.3	0.98506736
± 0.2	0.99334665
± 0.1	0.99833417
± 0.05	0.99958339
± 0.01	0.99998333
± 0.005	0.99999583
± 0.001	0.99999983

SOLUTION The function $f(x) = (\sin x)/x$ is not defined when $x = 0$. Using a calculator (and remembering that, if $x \in \mathbb{R}$, $\sin x$ means the sine of the angle whose *radian* measure is x), we construct a table of values correct to eight decimal places. From the table at the left and the graph in Figure 4 we guess that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

This guess is in fact correct, as will be proved in Chapter 3 using a geometric argument.

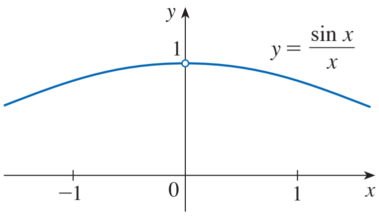


FIGURE 4

■

EXAMPLE 3 Find $\lim_{x \to 0} \left(x^3 + \dfrac{\cos 5x}{10{,}000} \right)$.

SOLUTION As before, we construct a table of values. From the first table it appears that the limit might be zero.

x	$x^3 + \dfrac{\cos 5x}{10{,}000}$
1	1.000028
0.5	0.124920
0.1	0.001088
0.05	0.000222
0.01	0.000101

x	$x^3 + \dfrac{\cos 5x}{10{,}000}$
0.005	0.00010009
0.001	0.00010000

But if we persevere with smaller values of x, the second table suggests that the limit is more likely to be 0.0001. In Section 2.5 we will be able to show that $\lim_{x \to 0} \cos 5x = 1$ and that it follows that

$$\lim_{x \to 0} \left(x^3 + \frac{\cos 5x}{10{,}000} \right) = \frac{1}{10{,}000} = 0.0001 \qquad \blacksquare$$

One-Sided Limits

The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

(This function is named after the electrical engineer Oliver Heaviside [1850–1925] and can be used to describe an electric current that is switched on at time $t = 0$.) Its graph is shown in Figure 5.

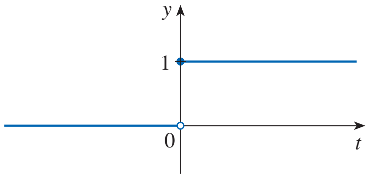


FIGURE 5
The Heaviside function

There is no single number that $H(t)$ approaches as t approaches 0, so $\lim_{t \to 0} H(t)$ does not exist. However, as t approaches 0 from the left, $H(t)$ approaches 0. As t approaches 0 from the right, $H(t)$ approaches 1. We indicate this situation symbolically by writing

$$\lim_{t \to 0^-} H(t) = 0 \qquad \text{and} \qquad \lim_{t \to 0^+} H(t) = 1$$

and we call these *one-sided limits*. The notation $t \to 0^-$ indicates that we consider only values of t that are less than 0. Likewise, $t \to 0^+$ indicates that we consider only values of t that are greater than 0.

2 **Intuitive Definition of One-Sided Limits** We write

$$\lim_{x \to a^-} f(x) = L$$

and say that the **left-hand limit** of $f(x)$ as x approaches a [or the limit of $f(x)$ as x approaches a *from the left*] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a with x *less than* a.

We write

$$\lim_{x \to a^+} f(x) = L$$

and say that the **right-hand limit** of $f(x)$ as x approaches a [or the limit of $f(x)$ as x approaches a *from the right*] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a with x *greater than* a.

For instance, the notation $x \to 5^-$ means that we consider only $x < 5$, and $x \to 5^+$ means that we consider only $x > 5$. Definition 2 is illustrated in Figure 6.

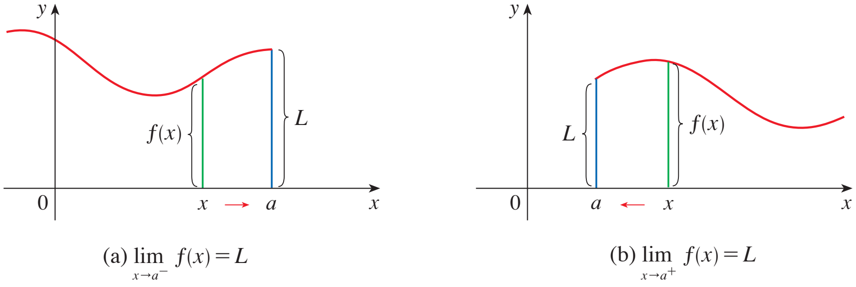


FIGURE 6

(a) $\lim_{x \to a^-} f(x) = L$ (b) $\lim_{x \to a^+} f(x) = L$

Notice that Definition 2 differs from Definition 1 only in that we require x to be less than (or greater than) a. By comparing these definitions, we see that the following is true.

3 $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

EXAMPLE 4 The graph of a function g is shown in Figure 7.

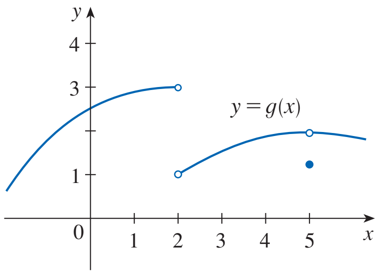


FIGURE 7

Use the graph to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$ (b) $\lim_{x \to 2^+} g(x)$ (c) $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x)$ (e) $\lim_{x \to 5^+} g(x)$ (f) $\lim_{x \to 5} g(x)$

SOLUTION Looking at the graph we see that the values of $g(x)$ approach 3 as x approaches 2 from the left, but they approach 1 as x approaches 2 from the right. Therefore

$$\text{(a)} \quad \lim_{x \to 2^-} g(x) = 3 \qquad \text{and} \qquad \text{(b)} \quad \lim_{x \to 2^+} g(x) = 1$$

(c) Since the left and right limits are different, we conclude from (3) that $\lim_{x \to 2} g(x)$ does not exist.

The graph also shows that

$$\text{(d)} \quad \lim_{x \to 5^-} g(x) = 2 \qquad \text{and} \qquad \text{(e)} \quad \lim_{x \to 5^+} g(x) = 2$$

(f) This time the left and right limits are the same and so, by (3), we have

$$\lim_{x \to 5} g(x) = 2$$

Despite this fact, notice that $g(5) \neq 2$. ■

How Can a Limit Fail to Exist?

We have seen that a limit fails to exist at a number a if the left- and right-hand limits are not equal (as in Example 4). The next two examples illustrate additional ways that a limit can fail to exist.

EXAMPLE 5 Investigate $\lim_{x \to 0} \sin \dfrac{\pi}{x}$.

Limits and Technology
Some software applications, including computer algebra systems (CAS), can compute limits. In order to avoid the types of pitfalls demonstrated in Examples 1, 3, and 5, such applications don't find limits by numerical experimentation. Instead, they use more sophisticated techniques such as computing infinite series. You are encouraged to use one of these resources to compute the limits in the examples of this section and check your answers to the exercises in this chapter.

SOLUTION Notice that the function $f(x) = \sin(\pi/x)$ is undefined at 0. Evaluating the function for some small values of x, we get

$$f(1) = \sin \pi = 0 \qquad f\left(\tfrac{1}{2}\right) = \sin 2\pi = 0$$

$$f\left(\tfrac{1}{3}\right) = \sin 3\pi = 0 \qquad f\left(\tfrac{1}{4}\right) = \sin 4\pi = 0$$

$$f(0.1) = \sin 10\pi = 0 \qquad f(0.01) = \sin 100\pi = 0$$

Similarly, $f(0.001) = f(0.0001) = 0$. On the basis of this information we might be tempted to guess that the limit is 0, but this time our guess is wrong. Note that although $f(1/n) = \sin n\pi = 0$ for any integer n, it is also true that $f(x) = 1$ for infinitely many values of x (such as 2/5 or 2/101) that approach 0. You can see this from the graph of f shown in Figure 8.

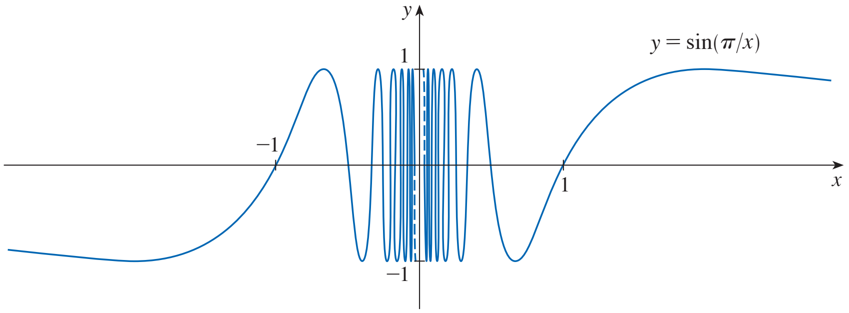


FIGURE 8

The dashed lines near the y-axis indicate that the values of $\sin(\pi/x)$ oscillate between 1 and -1 infinitely often as x approaches 0.

Since the values of $f(x)$ do not approach a fixed number as x approaches 0,

$$\lim_{x \to 0} \sin \frac{\pi}{x} \quad \text{does not exist}$$

■

Examples 3 and 5 illustrate some of the pitfalls in guessing the value of a limit. It is easy to guess the wrong value if we use inappropriate values of x, but it is difficult to know when to stop calculating values. And, as the discussion after Example 1 shows, sometimes calculators and computers give the wrong values. In the next section, however, we will develop foolproof methods for calculating limits.

Another way a limit at a number a can fail to exist is when the function values grow arbitrarily large (in absolute value) as x approaches a.

EXAMPLE 6 Find $\lim_{x \to 0} \dfrac{1}{x^2}$ if it exists.

SOLUTION As x becomes close to 0, x^2 also becomes close to 0, and $1/x^2$ becomes very large. (See the following table.) In fact, it appears from the graph of the function $f(x) = 1/x^2$ shown in Figure 9 that the values of $f(x)$ can be made arbitrarily large by taking x close enough to 0. Thus the values of $f(x)$ do not approach a number, so $\lim_{x \to 0} (1/x^2)$ does not exist.

x	$\dfrac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000

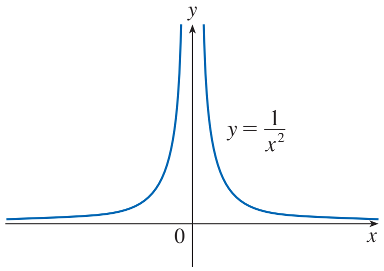


FIGURE 9

■

Infinite Limits; Vertical Asymptotes

To indicate the kind of behavior exhibited in Example 6, we use the notation

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

This does not mean that we are regarding ∞ as a number. Nor does it mean that the limit exists. It simply expresses the particular way in which the limit does not exist: $1/x^2$ can be made as large as we like by taking x close enough to 0.

In general, we write symbolically

$$\lim_{x \to a} f(x) = \infty$$

to indicate that the values of $f(x)$ tend to become larger and larger (or "increase without bound") as x becomes closer and closer to a.

4 Intuitive Definition of an Infinite Limit Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

Another notation for $\lim_{x \to a} f(x) = \infty$ is

$$f(x) \to \infty \qquad \text{as} \qquad x \to a$$

Again, the symbol ∞ is not a number, but the expression $\lim_{x \to a} f(x) = \infty$ is often read as

"the limit of $f(x)$, as x approaches a, is infinity"

or "$f(x)$ becomes infinite as x approaches a"

or "$f(x)$ increases without bound as x approaches a"

This definition is illustrated graphically in Figure 10.

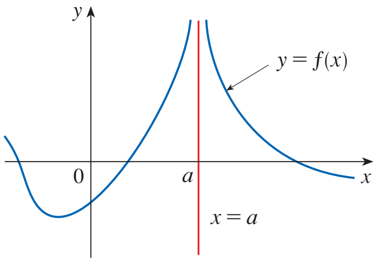


FIGURE 10
$\lim_{x \to a} f(x) = \infty$

When we say a number is "large negative," we mean that it is negative but its magnitude (absolute value) is large.

A similar sort of limit, for functions that become large negative as x gets close to a, is defined in Definition 5 and is illustrated in Figure 11.

5 Definition Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

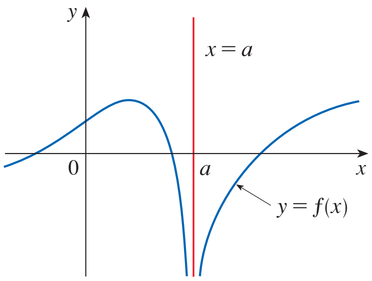


FIGURE 11

$\lim_{x \to a} f(x) = -\infty$

The symbol $\lim_{x \to a} f(x) = -\infty$ can be read as "the limit of $f(x)$, as x approaches a, is negative infinity" or "$f(x)$ decreases without bound as x approaches a." As an example we have

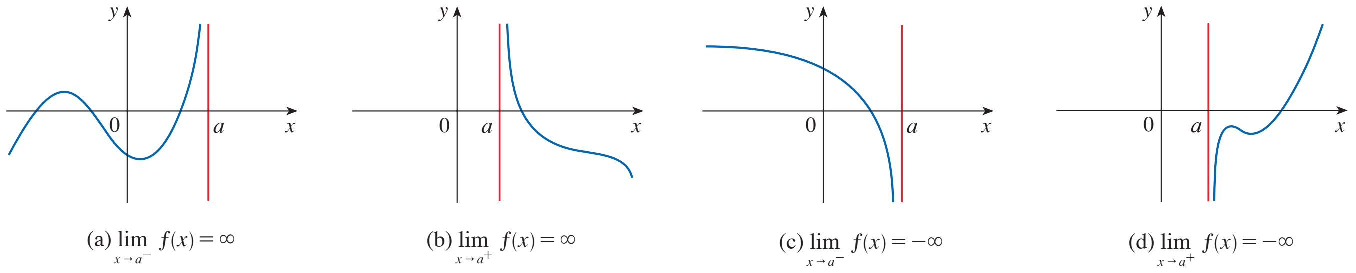
$$\lim_{x \to 0} \left(-\frac{1}{x^2} \right) = -\infty$$

Similar definitions can be given for the one-sided infinite limits

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

remembering that $x \to a^-$ means that we consider only values of x that are less than a, and similarly $x \to a^+$ means that we consider only $x > a$. Illustrations of these four cases are given in Figure 12.



(a) $\lim_{x \to a^-} f(x) = \infty$ (b) $\lim_{x \to a^+} f(x) = \infty$ (c) $\lim_{x \to a^-} f(x) = -\infty$ (d) $\lim_{x \to a^+} f(x) = -\infty$

FIGURE 12

6 Definition The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

For instance, the y-axis is a vertical asymptote of the curve $y = 1/x^2$ because $\lim_{x \to 0} (1/x^2) = \infty$. In Figure 12 the line $x = a$ is a vertical asymptote in each of the four cases shown. In general, knowledge of vertical asymptotes is very useful in sketching graphs.

EXAMPLE 7 Does the curve $y = \dfrac{2x}{x-3}$ have a vertical asymptote?

SOLUTION There is a potential vertical asymptote where the denominator is 0, that is, at $x = 3$, so we investigate the one-sided limits there.

If x is close to 3 but larger than 3, then the denominator $x - 3$ is a small positive number and $2x$ is close to 6. So the quotient $2x/(x-3)$ is a large *positive* number. [For instance, if $x = 3.01$ then $2x/(x-3) = 6.02/0.01 = 602$.] Thus, intuitively, we see that

$$\lim_{x \to 3^+} \frac{2x}{x-3} = \infty$$

Likewise, if x is close to 3 but smaller than 3, then $x - 3$ is a small negative number but $2x$ is still a positive number (close to 6). So $2x/(x-3)$ is a numerically large *negative* number. Thus

$$\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty$$

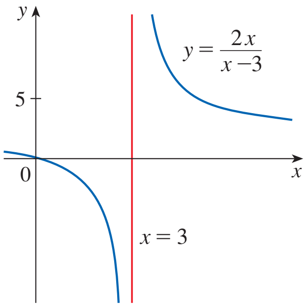


FIGURE 13

The graph of the curve $y = 2x/(x-3)$ is given in Figure 13. According to Definition 6, the line $x = 3$ is a vertical asymptote. ■

NOTE Neither of the limits in Examples 6 and 7 exist, but in Example 6 we can write $\lim_{x \to 0} (1/x^2) = \infty$ because $f(x) \to \infty$ as x approaches 0 from either the left or the right. In Example 7, $f(x) \to \infty$ as x approaches 3 from the right but $f(x) \to -\infty$ as x approaches 3 from the left, so we simply say that $\lim_{x \to 3} f(x)$ does not exist.

EXAMPLE 8 Find the vertical asymptotes of $f(x) = \tan x$.

SOLUTION Because

$$\tan x = \frac{\sin x}{\cos x}$$

there are potential vertical asymptotes where $\cos x = 0$. In fact, since $\cos x \to 0^+$ as $x \to (\pi/2)^-$ and $\cos x \to 0^-$ as $x \to (\pi/2)^+$, whereas $\sin x$ is positive (near 1) when x is near $\pi/2$, we have

$$\lim_{x \to (\pi/2)^-} \tan x = \infty \qquad \text{and} \qquad \lim_{x \to (\pi/2)^+} \tan x = -\infty$$

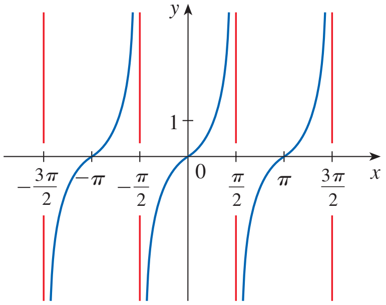


FIGURE 14
$y = \tan x$

This shows that the line $x = \pi/2$ is a vertical asymptote. Similar reasoning shows that the lines $x = \pi/2 + n\pi$, where n is an integer, are all vertical asymptotes of $f(x) = \tan x$. The graph in Figure 14 confirms this. ■

Another example of a function whose graph has a vertical asymptote is the natural logarithmic function $y = \ln x$. From Figure 15 we see that

$$\boxed{\lim_{x \to 0^+} \ln x = -\infty}$$

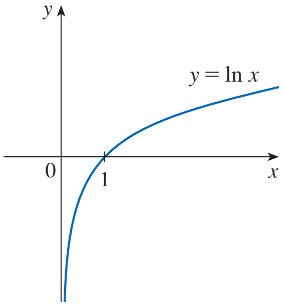


FIGURE 15
The y-axis is a vertical asymptote of the natural logarithmic function.

and so the line $x = 0$ (the y-axis) is a vertical asymptote. In fact, the same is true for $y = \log_b x$ provided that $b > 1$. (See Figures 1.5.11 and 1.5.12.)

2.2 Exercises

1. Explain in your own words what is meant by the equation

$$\lim_{x \to 2} f(x) = 5$$

Is it possible for this statement to be true and yet $f(2) = 3$? Explain.

2. Explain what it means to say that

$$\lim_{x \to 1^-} f(x) = 3 \qquad \text{and} \qquad \lim_{x \to 1^+} f(x) = 7$$

In this situation is it possible that $\lim_{x \to 1} f(x)$ exists? Explain.

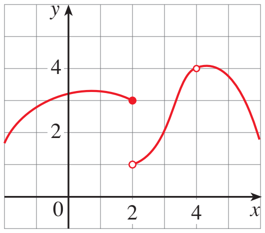
3. Explain the meaning of each of the following.

(a) $\lim_{x \to -3} f(x) = \infty$ (b) $\lim_{x \to 4^+} f(x) = -\infty$

4. Use the given graph of f to state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 2^-} f(x)$ (b) $\lim_{x \to 2^+} f(x)$ (c) $\lim_{x \to 2} f(x)$

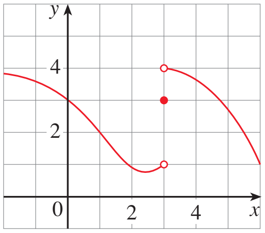
(d) $f(2)$ (e) $\lim_{x \to 4} f(x)$ (f) $f(4)$



5. For the function f whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 1} f(x)$ (b) $\lim_{x \to 3^-} f(x)$ (c) $\lim_{x \to 3^+} f(x)$

(d) $\lim_{x \to 3} f(x)$ (e) $f(3)$



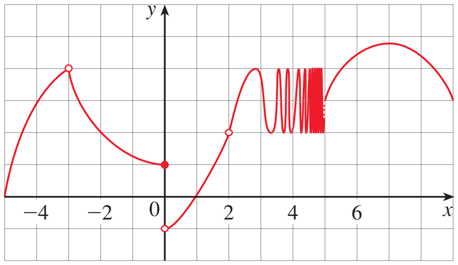
6. For the function h whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to -3^-} h(x)$ (b) $\lim_{x \to -3^+} h(x)$ (c) $\lim_{x \to -3} h(x)$

(d) $h(-3)$ (e) $\lim_{x \to 0^-} h(x)$ (f) $\lim_{x \to 0^+} h(x)$

(g) $\lim_{x \to 0} h(x)$ (h) $h(0)$ (i) $\lim_{x \to 2} h(x)$

(j) $h(2)$ (k) $\lim_{x \to 5^+} h(x)$ (l) $\lim_{x \to 5^-} h(x)$



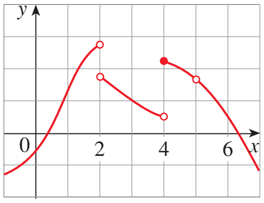
7. For the function g whose graph is shown, find a number a that satisfies the given description.

(a) $\lim_{x \to a} g(x)$ does not exist but $g(a)$ is defined.

(b) $\lim_{x \to a} g(x)$ exists but $g(a)$ is not defined.

(c) $\lim_{x \to a^-} g(x)$ and $\lim_{x \to a^+} g(x)$ both exist but $\lim_{x \to a} g(x)$ does not exist.

(d) $\lim_{x \to a^+} g(x) = g(a)$ but $\lim_{x \to a^-} g(x) \neq g(a)$.

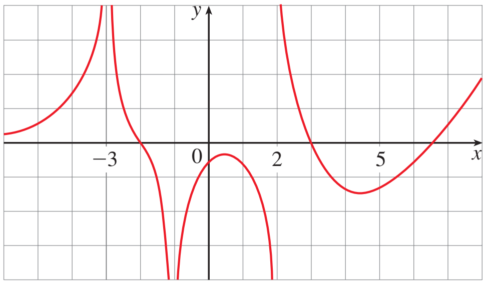


8. For the function A whose graph is shown, state the following.

(a) $\lim_{x \to -3} A(x)$ (b) $\lim_{x \to 2^-} A(x)$

(c) $\lim_{x \to 2^+} A(x)$ (d) $\lim_{x \to -1} A(x)$

(e) The equations of the vertical asymptotes

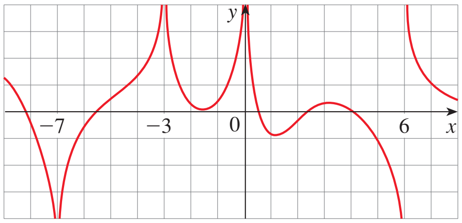


9. For the function f whose graph is shown, state the following.

(a) $\lim_{x \to -7} f(x)$ (b) $\lim_{x \to -3} f(x)$ (c) $\lim_{x \to 0} f(x)$

(d) $\lim_{x \to 6^-} f(x)$ (e) $\lim_{x \to 6^+} f(x)$

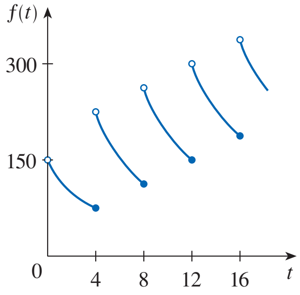
(f) The equations of the vertical asymptotes



10. A patient receives a 150-mg injection of a drug every 4 hours. The graph shows the amount $f(t)$ of the drug in the bloodstream after t hours. Find

$$\lim_{t\to 12^-} f(t) \qquad \text{and} \qquad \lim_{t\to 12^+} f(t)$$

and explain the significance of these one-sided limits.



11–12 Sketch the graph of the function and use it to determine the values of a for which $\lim_{x\to a} f(x)$ exists.

11. $f(x) = \begin{cases} e^x & \text{if } x \leqslant 0 \\ x - 1 & \text{if } 0 < x < 1 \\ \ln x & \text{if } x \geqslant 1 \end{cases}$

12. $f(x) = \begin{cases} \sqrt[3]{x} & \text{if } x \leqslant -1 \\ x & \text{if } -1 < x \leqslant 2 \\ (x-1)^2 & \text{if } x > 2 \end{cases}$

13–14 Use the graph of the function f to state the value of each limit, if it exists. If it does not exist, explain why.

(a) $\lim_{x\to 0^-} f(x)$ (b) $\lim_{x\to 0^+} f(x)$ (c) $\lim_{x\to 0} f(x)$

13. $f(x) = x\sqrt{1 + x^{-2}}$

14. $f(x) = \dfrac{e^{1/x} - 2}{e^{1/x} + 1}$

15–18 Sketch the graph of an example of a function f that satisfies all of the given conditions.

15. $\lim_{x\to 1^-} f(x) = 3$, $\lim_{x\to 1^+} f(x) = 0$, $f(1) = 2$

16. $\lim_{x\to 0} f(x) = 4$, $\lim_{x\to 8^-} f(x) = 1$, $\lim_{x\to 8^+} f(x) = -3$, $f(0) = 6$, $f(8) = -1$

17. $\lim_{x\to -1^-} f(x) = 0$, $\lim_{x\to -1^+} f(x) = 1$, $\lim_{x\to 2} f(x) = 3$, $f(-1) = 2$, $f(2) = 1$

18. $\lim_{x\to -3^-} f(x) = 3$, $\lim_{x\to -3^+} f(x) = 2$, $\lim_{x\to 3^-} f(x) = -1$, $\lim_{x\to 3^+} f(x) = 2$, $f(-3) = 2$, $f(3) = 0$

19–22 Guess the value of the limit (if it exists) by evaluating the function at the given numbers (correct to six decimal places).

19. $\lim_{x\to 3} \dfrac{x^2 - 3x}{x^2 - 9}$,
$x = 3.1, 3.05, 3.01, 3.001, 3.0001,$
$2.9, 2.95, 2.99, 2.999, 2.9999$

20. $\lim_{x\to -3} \dfrac{x^2 - 3x}{x^2 - 9}$,
$x = -2.5, -2.9, -2.95, -2.99, -2.999, -2.9999,$
$-3.5, -3.1, -3.05, -3.01, -3.001, -3.0001$

21. $\lim_{t\to 0} \dfrac{e^{5t} - 1}{t}$, $t = \pm 0.5, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$

22. $\lim_{h\to 0} \dfrac{(2 + h)^5 - 32}{h}$,
$h = \pm 0.5, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$

23–28 Use a table of values to estimate the value of the limit. If you have a graphing device, use it to confirm your result graphically.

23. $\lim_{x\to 4} \dfrac{\ln x - \ln 4}{x - 4}$

24. $\lim_{p\to -1} \dfrac{1 + p^9}{1 + p^{15}}$

25. $\lim_{\theta\to 0} \dfrac{\sin 3\theta}{\tan 2\theta}$

26. $\lim_{t\to 0} \dfrac{5^t - 1}{t}$

27. $\lim_{x\to 0^+} x^x$

28. $\lim_{x\to 0^+} x^2 \ln x$

29–40 Determine the infinite limit.

29. $\lim_{x\to 5^+} \dfrac{x + 1}{x - 5}$

30. $\lim_{x\to 5^-} \dfrac{x + 1}{x - 5}$

31. $\lim_{x\to 2} \dfrac{x^2}{(x - 2)^2}$

32. $\lim_{x\to 3^-} \dfrac{\sqrt{x}}{(x - 3)^5}$

33. $\lim_{x\to 1^+} \ln(\sqrt{x} - 1)$

34. $\lim_{x\to 0^+} \ln(\sin x)$

35. $\lim_{x\to (\pi/2)^+} \dfrac{1}{x} \sec x$

36. $\lim_{x\to \pi^-} x \cot x$

37. $\lim_{x\to 1} \dfrac{x^2 + 2x}{x^2 - 2x + 1}$

38. $\lim_{x\to 3^-} \dfrac{x^2 + 4x}{x^2 - 2x - 3}$

39. $\lim_{x\to 0} (\ln x^2 - x^{-2})$

40. $\lim_{x\to 0^+} \left(\dfrac{1}{x} - \ln x\right)$

41. Find the vertical asymptote of the function

$$f(x) = \frac{x - 1}{2x + 4}$$

42. (a) Find the vertical asymptotes of the function

$$y = \frac{x^2 + 1}{3x - 2x^2}$$

(b) Confirm your answer to part (a) by graphing the function.

43. Determine $\lim_{x \to 1^-} \frac{1}{x^3 - 1}$ and $\lim_{x \to 1^+} \frac{1}{x^3 - 1}$

(a) by evaluating $f(x) = 1/(x^3 - 1)$ for values of x that approach 1 from the left and from the right,

(b) by reasoning as in Example 7, and

(c) from a graph of f.

44. (a) By graphing the function

$$f(x) = \frac{\cos 2x - \cos x}{x^2}$$

and zooming in toward the point where the graph crosses the y-axis, estimate the value of $\lim_{x \to 0} f(x)$.

(b) Check your answer in part (a) by evaluating $f(x)$ for values of x that approach 0.

45. (a) Estimate the value of the limit $\lim_{x \to 0} (1 + x)^{1/x}$ to five decimal places. Does this number look familiar?

(b) Illustrate part (a) by graphing the function $y = (1 + x)^{1/x}$.

46. (a) Graph the function $f(x) = e^x + \ln|x - 4|$ for $0 \leqslant x \leqslant 5$. Do you think the graph is an accurate representation of f?

(b) How would you get a graph that represents f better?

47. (a) Evaluate the function $f(x) = x^2 - (2^x/1000)$ for $x = 1, 0.8, 0.6, 0.4, 0.2, 0.1$, and 0.05, and guess the value of

$$\lim_{x \to 0} \left(x^2 - \frac{2^x}{1000} \right)$$

(b) Evaluate $f(x)$ for $x = 0.04, 0.02, 0.01, 0.005, 0.003$, and 0.001. Guess again.

48. (a) Evaluate the function

$$h(x) = \frac{\tan x - x}{x^3}$$

for $x = 1, 0.5, 0.1, 0.05, 0.01$, and 0.005.

(b) Guess the value of $\lim_{x \to 0} \frac{\tan x - x}{x^3}$.

(c) Evaluate $h(x)$ for successively smaller values of x until you finally reach a value of 0 for $h(x)$. Are you still confident that your guess in part (b) is correct? Explain why you eventually obtained 0 values. (In Section 4.4 a method for evaluating this limit will be explained.)

(d) Graph the function h in the viewing rectangle $[-1, 1]$ by $[0, 1]$. Then zoom in toward the point where the graph crosses the y-axis to estimate the limit of $h(x)$ as x approaches 0. Continue to zoom in until you observe distortions in the graph of h. Compare with the results of part (c).

49. Use a graph to estimate the equations of all the vertical asymptotes of the curve

$$y = \tan(2 \sin x) \qquad -\pi \leqslant x \leqslant \pi$$

Then find the exact equations of these asymptotes.

50. Consider the function $f(x) = \tan \frac{1}{x}$.

(a) Show that $f(x) = 0$ for $x = \frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \ldots$

(b) Show that $f(x) = 1$ for $x = \frac{4}{\pi}, \frac{4}{5\pi}, \frac{4}{9\pi}, \ldots$

(c) What can you conclude about $\lim_{x \to 0^+} \tan \frac{1}{x}$?

51. In the theory of relativity, the mass of a particle with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of the particle at rest and c is the speed of light. What happens as $v \to c^-$?

2.3 Calculating Limits Using the Limit Laws

Properties of Limits

In Section 2.2 we used calculators and graphs to guess the values of limits, but we saw that such methods don't always lead to the correct answers. In this section we use the following properties of limits, called the *Limit Laws*, to calculate limits.

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

exist. Then

1. $\displaystyle \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\displaystyle \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\displaystyle \lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$
4. $\displaystyle \lim_{x \to a} [f(x)\,g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\displaystyle \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$

These five laws can be stated verbally as follows:

Sum Law

1. The limit of a sum is the sum of the limits.

Difference Law

2. The limit of a difference is the difference of the limits.

Constant Multiple Law

3. The limit of a constant times a function is the constant times the limit of the function.

Product Law

4. The limit of a product is the product of the limits.

Quotient Law

5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

It is easy to believe that these properties are true. For instance, if $f(x)$ is close to L and $g(x)$ is close to M, it is reasonable to conclude that $f(x) + g(x)$ is close to $L + M$. This gives us an intuitive basis for believing that Law 1 is true. In Section 2.4 we give a precise definition of a limit and use it to prove this law. The proofs of the remaining laws are given in Appendix F.

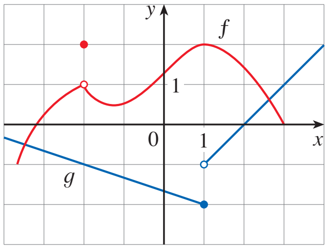


FIGURE 1

EXAMPLE 1 Use the Limit Laws and the graphs of f and g in Figure 1 to evaluate the following limits, if they exist.

(a) $\displaystyle \lim_{x \to -2} [f(x) + 5g(x)]$ (b) $\displaystyle \lim_{x \to 1} [f(x)g(x)]$ (c) $\displaystyle \lim_{x \to 2} \frac{f(x)}{g(x)}$

SOLUTION

(a) From the graphs of f and g we see that

$$\lim_{x \to -2} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to -2} g(x) = -1$$

Therefore we have

$$\begin{aligned}
\lim_{x \to -2} [f(x) + 5g(x)] &= \lim_{x \to -2} f(x) + \lim_{x \to -2} [5g(x)] && \text{(by Limit Law 1)} \\
&= \lim_{x \to -2} f(x) + 5 \lim_{x \to -2} g(x) && \text{(by Limit Law 3)} \\
&= 1 + 5(-1) = -4
\end{aligned}$$

(b) We see that $\lim_{x\to 1} f(x) = 2$. But $\lim_{x\to 1} g(x)$ does not exist because the left and right limits are different:

$$\lim_{x\to 1^-} g(x) = -2 \qquad \lim_{x\to 1^+} g(x) = -1$$

So we can't use Law 4 for the desired limit. But we can use Law 4 for the one-sided limits:

$$\lim_{x\to 1^-} [f(x)g(x)] = \lim_{x\to 1^-} f(x) \cdot \lim_{x\to 1^-} g(x) = 2 \cdot (-2) = -4$$

$$\lim_{x\to 1^+} [f(x)g(x)] = \lim_{x\to 1^+} f(x) \cdot \lim_{x\to 1^+} g(x) = 2 \cdot (-1) = -2$$

The left and right limits aren't equal, so $\lim_{x\to 1} [f(x)g(x)]$ does not exist.

(c) The graphs show that

$$\lim_{x\to 2} f(x) \approx 1.4 \qquad \text{and} \qquad \lim_{x\to 2} g(x) = 0$$

Because the limit of the denominator is 0, we can't use Law 5. The given limit does not exist because the denominator approaches 0 while the numerator approaches a nonzero number. ■

If we use the Product Law repeatedly with $g(x) = f(x)$, we obtain the following law.

Power Law

$$\textbf{6.}\quad \lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n \qquad \text{where } n \text{ is a positive integer}$$

A similar property, which you are asked to prove in Exercise 2.5.69, holds for roots:

Root Law

$$\textbf{7.}\quad \lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} \qquad \text{where } n \text{ is a positive integer}$$

$\left[\text{If } n \text{ is even, we assume that } \lim_{x\to a} f(x) > 0.\right]$

In applying these seven limit laws, we need to use two special limits:

$$\textbf{8.}\quad \lim_{x\to a} c = c \qquad\qquad \textbf{9.}\quad \lim_{x\to a} x = a$$

These limits are obvious from an intuitive point of view (state them in words or draw graphs of $y = c$ and $y = x$), but proofs based on the precise definition are requested in Exercises 2.4.23–24.

If we now put $f(x) = x$ in Law 6 and use Law 9, we get a useful special limit for power functions.

$$\textbf{10.}\quad \lim_{x\to a} x^n = a^n \qquad \text{where } n \text{ is a positive integer}$$

If we put $f(x) = x$ in Law 7 and use Law 9, we get a similar special limit for roots. (For square roots the proof is outlined in Exercise 2.4.37.)

11. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

EXAMPLE 2 Evaluate the following limits and justify each step.

(a) $\lim_{x\to 5} (2x^2 - 3x + 4)$ (b) $\lim_{x\to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

SOLUTION

(a)
$$\begin{aligned}
\lim_{x\to 5} (2x^2 - 3x + 4) &= \lim_{x\to 5} (2x^2) - \lim_{x\to 5} (3x) + \lim_{x\to 5} 4 && \text{(by Laws 2 and 1)}\\
&= 2 \lim_{x\to 5} x^2 - 3 \lim_{x\to 5} x + \lim_{x\to 5} 4 && \text{(by 3)}\\
&= 2(5^2) - 3(5) + 4 && \text{(by 10, 9, and 8)}\\
&= 39
\end{aligned}$$

(b) We start by using Law 5, but its use is fully justified only at the final stage when we see that the limits of the numerator and denominator exist and the limit of the denominator is not 0.

$$\begin{aligned}
\lim_{x\to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} &= \frac{\lim_{x\to -2} (x^3 + 2x^2 - 1)}{\lim_{x\to -2} (5 - 3x)} && \text{(by Law 5)}\\
&= \frac{\lim_{x\to -2} x^3 + 2 \lim_{x\to -2} x^2 - \lim_{x\to -2} 1}{\lim_{x\to -2} 5 - 3 \lim_{x\to -2} x} && \text{(by 1, 2, and 3)}\\
&= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} && \text{(by 10, 9, and 8)}\\
&= -\frac{1}{11}
\end{aligned}$$

■

■ Evaluating Limits by Direct Substitution

In Example 2(a) we determined that $\lim_{x\to 5} f(x) = 39$, where $f(x) = 2x^2 - 3x + 4$. Notice that $f(5) = 39$; in other words, we would have gotten the correct result simply by substituting 5 for x. Similarly, direct substitution provides the correct answer in part (b). The functions in Example 2 are a polynomial and a rational function, respectively, and similar use of the Limit Laws proves that direct substitution always works for such functions (see Exercises 59 and 60). We state this fact as follows.

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

Newton and Limits

Isaac Newton was born on Christmas Day in 1642, the year of Galileo's death. When he entered Cambridge University in 1661 Newton didn't know much mathematics, but he learned quickly by reading Euclid and Descartes and by attending the lectures of Isaac Barrow. Cambridge was closed because of the plague from 1665 to 1666, and Newton returned home to reflect on what he had learned. Those two years were amazingly productive for at that time he made four of his major discoveries: (1) his representation of functions as sums of infinite series, including the binomial theorem; (2) his work on differential and integral calculus; (3) his laws of motion and law of universal gravitation; and (4) his prism experiments on the nature of light and color. Because of a fear of controversy and criticism, he was reluctant to publish his discoveries and it wasn't until 1687, at the urging of the astronomer Halley, that Newton published *Principia Mathematica*. In this work, the greatest scientific treatise ever written, Newton set forth his version of calculus and used it to investigate mechanics, fluid dynamics, and wave motion, and to explain the motion of planets and comets.

The beginnings of calculus are found in the calculations of areas and volumes by ancient Greek scholars such as Eudoxus and Archimedes. Although aspects of the idea of a limit are implicit in their "method of exhaustion," Eudoxus and Archimedes never explicitly formulated the concept of a limit. Likewise, mathematicians such as Cavalieri, Fermat, and Barrow, the immediate precursors of Newton in the development of calculus, did not actually use limits. It was Isaac Newton who was the first to talk explicitly about limits. He explained that the main idea behind limits is that quantities "approach nearer than by any given difference." Newton stated that the limit was the basic concept in calculus, but it was left to later mathematicians like Cauchy to clarify his ideas about limits.

Functions that have the Direct Substitution Property are called *continuous at a* and will be studied in Section 2.5. However, not all limits can be evaluated initially by direct substitution, as the following examples show.

EXAMPLE 3 Find $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$.

SOLUTION Let $f(x) = (x^2 - 1)/(x - 1)$. We can't find the limit by substituting $x = 1$ because $f(1)$ isn't defined. Nor can we apply the Quotient Law, because the limit of the denominator is 0. Instead, we need to do some preliminary algebra. We factor the numerator as a difference of squares:

$$\frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1}$$

Notice that in Example 3 we do not have an infinite limit even though the denominator approaches 0 as $x \to 1$. When both numerator and denominator approach 0, the limit may be infinite or it may be some finite value.

The numerator and denominator have a common factor of $x - 1$. When we take the limit as x approaches 1, we have $x \neq 1$ and so $x - 1 \neq 0$. Therefore we can cancel the common factor, $x - 1$, and then compute the limit by direct substitution as follows:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$
$$= \lim_{x \to 1} (x + 1) = 1 + 1 = 2$$

The limit in this example arose in Example 2.1.1 in finding the tangent to the parabola $y = x^2$ at the point $(1, 1)$. ■

NOTE In Example 3 we were able to compute the limit by replacing the given function $f(x) = (x^2 - 1)/(x - 1)$ by a simpler function, $g(x) = x + 1$, that has the same limit. This is valid because $f(x) = g(x)$ except when $x = 1$, and in computing a limit as x approaches 1 we don't consider what happens when x is actually *equal* to 1. In general, we have the following useful fact.

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided the limits exist.

EXAMPLE 4 Find $\lim_{x \to 1} g(x)$ where

$$g(x) = \begin{cases} x + 1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1 \end{cases}$$

SOLUTION Here g is defined at $x = 1$ and $g(1) = \pi$, but the value of a limit as x *approaches* 1 does not depend on the value of the function *at* 1. Since $g(x) = x + 1$ for $x \neq 1$, we have

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} (x + 1) = 2$$ ■

Note that the values of the functions in Examples 3 and 4 are identical except when $x = 1$ (see Figure 2) and so they have the same limit as x approaches 1.

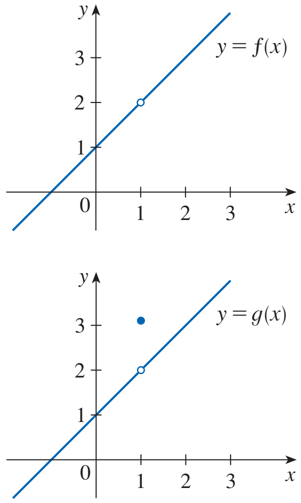


FIGURE 2
The graphs of the functions f (from Example 3) and g (from Example 4)

EXAMPLE 5 Evaluate $\lim_{h \to 0} \dfrac{(3+h)^2 - 9}{h}$.

SOLUTION If we define

$$F(h) = \frac{(3+h)^2 - 9}{h}$$

then, as in Example 3, we can't compute $\lim_{h \to 0} F(h)$ by letting $h = 0$ because $F(0)$ is undefined. But if we simplify $F(h)$ algebraically, we find that

$$\begin{aligned} F(h) &= \frac{(9 + 6h + h^2) - 9}{h} = \frac{6h + h^2}{h} \\ &= \frac{h(6+h)}{h} = 6 + h \end{aligned}$$

(Recall that we consider only $h \neq 0$ when letting h approach 0.) Thus

$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h} = \lim_{h \to 0} (6 + h) = 6$$

■

EXAMPLE 6 Find $\lim_{t \to 0} \dfrac{\sqrt{t^2 + 9} - 3}{t^2}$.

SOLUTION We can't apply the Quotient Law immediately because the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing the numerator:

$$\begin{aligned} \lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2} &= \lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2} \cdot \frac{\sqrt{t^2+9} + 3}{\sqrt{t^2+9} + 3} \\ &= \lim_{t \to 0} \frac{(t^2 + 9) - 9}{t^2\left(\sqrt{t^2+9} + 3\right)} \\ &= \lim_{t \to 0} \frac{t^2}{t^2\left(\sqrt{t^2+9} + 3\right)} \\ &= \lim_{t \to 0} \frac{1}{\sqrt{t^2+9} + 3} \\ &= \frac{1}{\sqrt{\lim_{t \to 0} (t^2 + 9)} + 3} \\ &= \frac{1}{3 + 3} = \frac{1}{6} \end{aligned}$$

(Here we use several properties of limits: 5, 1, 7, 8, 10.)

This calculation confirms the guess that we made in Example 2.2.1. ■

Using One-Sided Limits

Some limits are best calculated by first finding the left- and right-hand limits. The following theorem is a reminder of what we discovered in Section 2.2. It says that a two-sided limit exists if and only if both of the one-sided limits exist and are equal.

1 Theorem $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$

When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

EXAMPLE 7 Show that $\lim_{x\to 0} |x| = 0$.

SOLUTION Recall that

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

Since $|x| = x$ for $x > 0$, we have

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ and so

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0$$

Therefore, by Theorem 1,

$$\lim_{x\to 0} |x| = 0$$ ■

The result of Example 7 looks plausible from Figure 3.

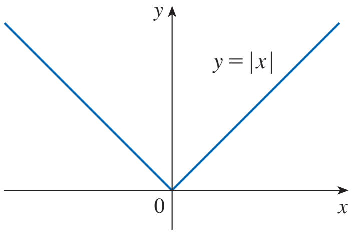


FIGURE 3

EXAMPLE 8 Prove that $\lim_{x\to 0} \dfrac{|x|}{x}$ does not exist.

SOLUTION Using the facts that $|x| = x$ when $x > 0$ and $|x| = -x$ when $x < 0$, we have

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$$

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} (-1) = -1$$

Since the right- and left-hand limits are different, it follows from Theorem 1 that $\lim_{x\to 0} |x|/x$ does not exist. The graph of the function $f(x) = |x|/x$ is shown in Figure 4 and supports the one-sided limits that we found. ■

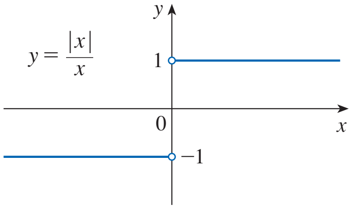


FIGURE 4

EXAMPLE 9 If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8 - 2x & \text{if } x < 4 \end{cases}$$

determine whether $\lim_{x\to 4} f(x)$ exists.

It is shown in Example 2.4.4 that $\lim_{x \to 0^+} \sqrt{x} = 0$.

SOLUTION Since $f(x) = \sqrt{x-4}$ for $x > 4$, we have

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x-4} = \sqrt{4-4} = 0$$

Since $f(x) = 8 - 2x$ for $x < 4$, we have

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (8 - 2x) = 8 - 2 \cdot 4 = 0$$

The right- and left-hand limits are equal. Thus the limit exists and

$$\lim_{x \to 4} f(x) = 0$$

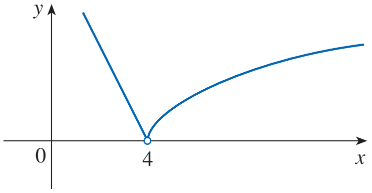


FIGURE 5

The graph of f is shown in Figure 5. ■

EXAMPLE 10 The **greatest integer function** is defined by $[\![x]\!]$ = the largest integer that is less than or equal to x. (For instance, $[\![4]\!] = 4$, $[\![4.8]\!] = 4$, $[\![\pi]\!] = 3$, $[\![\sqrt{2}]\!] = 1$, $[\![-\frac{1}{2}]\!] = -1$.) Show that $\lim_{x \to 3} [\![x]\!]$ does not exist.

Other notations for $[\![x]\!]$ are $[x]$ and $\lfloor x \rfloor$. The greatest integer function is sometimes called the *floor function*.

SOLUTION The graph of the greatest integer function is shown in Figure 6. Since $[\![x]\!] = 3$ for $3 \leqslant x < 4$, we have

$$\lim_{x \to 3^+} [\![x]\!] = \lim_{x \to 3^+} 3 = 3$$

Since $[\![x]\!] = 2$ for $2 \leqslant x < 3$, we have

$$\lim_{x \to 3^-} [\![x]\!] = \lim_{x \to 3^-} 2 = 2$$

Because these one-sided limits are not equal, $\lim_{x \to 3} [\![x]\!]$ does not exist by Theorem 1. ■

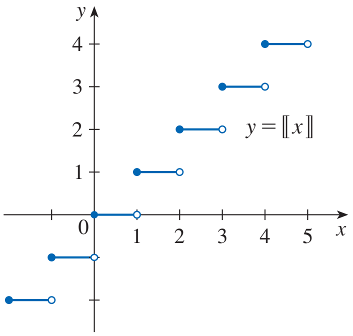


FIGURE 6
Greatest integer function

The Squeeze Theorem

The following two theorems describe how the limits of functions are related when the values of one function are greater than (or equal to) those of another. Their proofs can be found in Appendix F.

2 Theorem If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then
$$\lim_{x \to a} g(x) = L$$

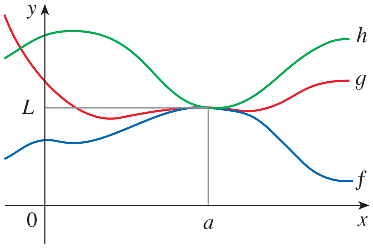


FIGURE 7

The Squeeze Theorem, which is sometimes called the Sandwich Theorem or the Pinching Theorem, is illustrated by Figure 7. It says that if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a, and if f and h have the same limit L at a, then g is forced to have the same limit L at a.

EXAMPLE 11 Show that $\displaystyle\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

SOLUTION First note that we **cannot** rewrite the limit as the product of the limits $\lim_{x\to 0} x^2$ and $\lim_{x\to 0} \sin(1/x)$ because $\lim_{x\to 0} \sin(1/x)$ does not exist (see Example 2.2.5).

We *can* find the limit by using the Squeeze Theorem. To apply the Squeeze Theorem we need to find a function f smaller than $g(x) = x^2 \sin(1/x)$ and a function h bigger than g such that both $f(x)$ and $h(x)$ approach 0 as $x \to 0$. To do this we use our knowledge of the sine function. Because the sine of any number lies between -1 and 1, we can write

$$\boxed{4} \qquad -1 \leqslant \sin\frac{1}{x} \leqslant 1$$

Any inequality remains true when multiplied by a positive number. We know that $x^2 \geqslant 0$ for all x and so, multiplying each side of the inequalities in (4) by x^2, we get

$$-x^2 \leqslant x^2 \sin\frac{1}{x} \leqslant x^2$$

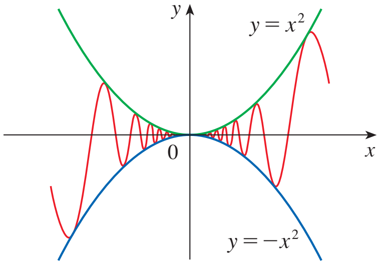


FIGURE 8
$y = x^2 \sin(1/x)$

as illustrated by Figure 8. We know that

$$\lim_{x\to 0} x^2 = 0 \qquad \text{and} \qquad \lim_{x\to 0} (-x^2) = 0$$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0 \qquad ■$$

2.3 Exercises

1. Given that

$$\lim_{x\to 2} f(x) = 4 \qquad \lim_{x\to 2} g(x) = -2 \qquad \lim_{x\to 2} h(x) = 0$$

find the limits that exist. If the limit does not exist, explain why.

(a) $\displaystyle\lim_{x\to 2} [f(x) + 5g(x)]$ (b) $\displaystyle\lim_{x\to 2} [g(x)]^3$

(c) $\displaystyle\lim_{x\to 2} \sqrt{f(x)}$ (d) $\displaystyle\lim_{x\to 2} \frac{3f(x)}{g(x)}$

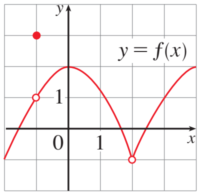
(e) $\displaystyle\lim_{x\to 2} \frac{g(x)}{h(x)}$ (f) $\displaystyle\lim_{x\to 2} \frac{g(x)h(x)}{f(x)}$

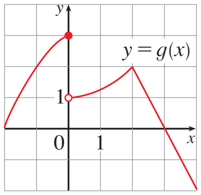
2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.

(a) $\displaystyle\lim_{x\to 2} [f(x) + g(x)]$ (b) $\displaystyle\lim_{x\to 0} [f(x) - g(x)]$

(c) $\displaystyle\lim_{x\to -1} [f(x)g(x)]$ (d) $\displaystyle\lim_{x\to 3} \frac{f(x)}{g(x)}$

(e) $\displaystyle\lim_{x\to 2} [x^2 f(x)]$ (f) $\displaystyle f(-1) + \lim_{x\to -1} g(x)$





3–9 Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\displaystyle\lim_{x\to 5} (4x^2 - 5x)$

4. $\displaystyle\lim_{x\to -3} (2x^3 + 6x^2 - 9)$

5. $\displaystyle\lim_{v\to 2} (v^2 + 2v)(2v^3 - 5)$

6. $\displaystyle\lim_{t\to 7} \frac{3t^2 + 1}{t^2 - 5t + 2}$

7. $\displaystyle\lim_{u\to -2} \sqrt{9 - u^3 + 2u^2}$

8. $\displaystyle\lim_{x\to 3} \sqrt[3]{x + 5}\,(2x^2 - 3x)$

9. $\displaystyle\lim_{t\to -1} \left(\frac{2t^5 - t^4}{5t^2 + 4}\right)^3$

10. (a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

(b) In view of part (a), explain why the equation

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2} (x + 3)$$

is correct.

11–34 Evaluate the limit, if it exists.

11. $\lim_{x \to -2} (3x - 7)$

12. $\lim_{x \to 6} \left(8 - \frac{1}{2}x\right)$

13. $\lim_{t \to 4} \dfrac{t^2 - 2t - 8}{t - 4}$

14. $\lim_{x \to -3} \dfrac{x^2 + 3x}{x^2 - x - 12}$

15. $\lim_{x \to 2} \dfrac{x^2 + 5x + 4}{x - 2}$

16. $\lim_{x \to 4} \dfrac{x^2 + 3x}{x^2 - x - 12}$

17. $\lim_{x \to -2} \dfrac{x^2 - x - 6}{3x^2 + 5x - 2}$

18. $\lim_{x \to -5} \dfrac{2x^2 + 9x - 5}{x^2 - 25}$

19. $\lim_{t \to 3} \dfrac{t^3 - 27}{t^2 - 9}$

20. $\lim_{u \to -1} \dfrac{u + 1}{u^3 + 1}$

21. $\lim_{h \to 0} \dfrac{(h - 3)^2 - 9}{h}$

22. $\lim_{x \to 9} \dfrac{9 - x}{3 - \sqrt{x}}$

23. $\lim_{h \to 0} \dfrac{\sqrt{9 + h} - 3}{h}$

24. $\lim_{x \to 2} \dfrac{2 - x}{\sqrt{x + 2} - 2}$

25. $\lim_{x \to 3} \dfrac{\dfrac{1}{x} - \dfrac{1}{3}}{x - 3}$

26. $\lim_{h \to 0} \dfrac{(-2 + h)^{-1} + 2^{-1}}{h}$

27. $\lim_{t \to 0} \dfrac{\sqrt{1 + t} - \sqrt{1 - t}}{t}$

28. $\lim_{t \to 0} \left(\dfrac{1}{t} - \dfrac{1}{t^2 + t}\right)$

29. $\lim_{x \to 16} \dfrac{4 - \sqrt{x}}{16x - x^2}$

30. $\lim_{x \to 2} \dfrac{x^2 - 4x + 4}{x^4 - 3x^2 - 4}$

31. $\lim_{t \to 0} \left(\dfrac{1}{t\sqrt{1 + t}} - \dfrac{1}{t}\right)$

32. $\lim_{x \to -4} \dfrac{\sqrt{x^2 + 9} - 5}{x + 4}$

33. $\lim_{h \to 0} \dfrac{(x + h)^3 - x^3}{h}$

34. $\lim_{h \to 0} \dfrac{\dfrac{1}{(x + h)^2} - \dfrac{1}{x^2}}{h}$

---

35. (a) Estimate the value of

$$\lim_{x \to 0} \frac{x}{\sqrt{1 + 3x} - 1}$$

by graphing the function $f(x) = x/\left(\sqrt{1 + 3x} - 1\right)$.

(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

36. (a) Use a graph of

$$f(x) = \frac{\sqrt{3 + x} - \sqrt{3}}{x}$$

to estimate the value of $\lim_{x \to 0} f(x)$ to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Use the Limit Laws to find the exact value of the limit.

37. Use the Squeeze Theorem to show that

$$\lim_{x \to 0} x^2 \cos 20\pi x = 0$$

Illustrate by graphing the functions $f(x) = -x^2$, $g(x) = x^2 \cos 20\pi x$, and $h(x) = x^2$ on the same screen.

38. Use the Squeeze Theorem to show that

$$\lim_{x \to 0} \sqrt{x^3 + x^2} \sin \frac{\pi}{x} = 0$$

Illustrate by graphing the functions f, g, and h (in the notation of the Squeeze Theorem) on the same screen.

39. If $4x - 9 \leqslant f(x) \leqslant x^2 - 4x + 7$ for $x \geqslant 0$, find $\lim_{x \to 4} f(x)$.

40. If $2x \leqslant g(x) \leqslant x^4 - x^2 + 2$ for all x, evaluate $\lim_{x \to 1} g(x)$.

41. Prove that $\lim_{x \to 0} x^4 \cos \dfrac{2}{x} = 0$.

42. Prove that $\lim_{x \to 0^+} \sqrt{x}\, e^{\sin(\pi/x)} = 0$.

43–48 Find the limit, if it exists. If the limit does not exist, explain why.

43. $\lim_{x \to -4} \left(|x + 4| - 2x\right)$

44. $\lim_{x \to -4} \dfrac{|x + 4|}{2x + 8}$

45. $\lim_{x \to 0.5^-} \dfrac{2x - 1}{|2x^3 - x^2|}$

46. $\lim_{x \to -2} \dfrac{2 - |x|}{2 + x}$

47. $\lim_{x \to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{|x|}\right)$

48. $\lim_{x \to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{|x|}\right)$

---

49. The Signum Function The *signum* (or sign) *function*, denoted by sgn, is defined by

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

(a) Sketch the graph of this function.

(b) Find each of the following limits or explain why it does not exist.

(i) $\lim_{x \to 0^+} \operatorname{sgn} x$

(ii) $\lim_{x \to 0^-} \operatorname{sgn} x$

(iii) $\lim_{x \to 0} \operatorname{sgn} x$

(iv) $\lim_{x \to 0} |\operatorname{sgn} x|$

50. Let $g(x) = \operatorname{sgn}(\sin x)$.

(a) Find each of the following limits or explain why it does not exist.

(i) $\lim_{x\to 0^+} g(x)$ (ii) $\lim_{x\to 0^-} g(x)$ (iii) $\lim_{x\to 0} g(x)$

(iv) $\lim_{x\to \pi^+} g(x)$ (v) $\lim_{x\to \pi^-} g(x)$ (vi) $\lim_{x\to \pi} g(x)$

(b) For which values of a does $\lim_{x\to a} g(x)$ not exist?

(c) Sketch a graph of g.

51. Let $g(x) = \dfrac{x^2 + x - 6}{|x - 2|}$.

(a) Find

(i) $\lim_{x\to 2^+} g(x)$ (ii) $\lim_{x\to 2^-} g(x)$

(b) Does $\lim_{x\to 2} g(x)$ exist?

(c) Sketch the graph of g.

52. Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ (x - 2)^2 & \text{if } x \geqslant 1 \end{cases}$$

(a) Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

(b) Does $\lim_{x\to 1} f(x)$ exist?

(c) Sketch the graph of f.

53. Let

$$B(t) = \begin{cases} 4 - \frac{1}{2}t & \text{if } t < 2 \\ \sqrt{t + c} & \text{if } t \geqslant 2 \end{cases}$$

Find the value of c so that $\lim_{t\to 2} B(t)$ exists.

54. Let

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leqslant 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

(a) Evaluate each of the following, if it exists.

(i) $\lim_{x\to 1^-} g(x)$ (ii) $\lim_{x\to 1} g(x)$ (iii) $g(1)$

(iv) $\lim_{x\to 2^-} g(x)$ (v) $\lim_{x\to 2^+} g(x)$ (vi) $\lim_{x\to 2} g(x)$

(b) Sketch the graph of g.

55. (a) If the symbol $\llbracket\ \rrbracket$ denotes the greatest integer function defined in Example 10, evaluate

(i) $\lim_{x\to -2^+} \llbracket x \rrbracket$ (ii) $\lim_{x\to -2} \llbracket x \rrbracket$ (iii) $\lim_{x\to -2.4} \llbracket x \rrbracket$

(b) If n is an integer, evaluate

(i) $\lim_{x\to n^-} \llbracket x \rrbracket$ (ii) $\lim_{x\to n^+} \llbracket x \rrbracket$

(c) For what values of a does $\lim_{x\to a} \llbracket x \rrbracket$ exist?

56. Let $f(x) = \llbracket \cos x \rrbracket$, $-\pi \leqslant x \leqslant \pi$.

(a) Sketch the graph of f.

(b) Evaluate each limit, if it exists.

(i) $\lim_{x\to 0} f(x)$ (ii) $\lim_{x\to (\pi/2)^-} f(x)$

(iii) $\lim_{x\to (\pi/2)^+} f(x)$ (iv) $\lim_{x\to \pi/2} f(x)$

(c) For what values of a does $\lim_{x\to a} f(x)$ exist?

57. If $f(x) = \llbracket x \rrbracket + \llbracket -x \rrbracket$, show that $\lim_{x\to 2} f(x)$ exists but is not equal to $f(2)$.

58. In the theory of relativity, the Lorentz contraction formula

$$L = L_0\sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v\to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

59. If p is a polynomial, show that $\lim_{x\to a} p(x) = p(a)$.

60. If r is a rational function, use Exercise 59 to show that $\lim_{x\to a} r(x) = r(a)$ for every number a in the domain of r.

61. If $\lim_{x\to 1} \dfrac{f(x) - 8}{x - 1} = 10$, find $\lim_{x\to 1} f(x)$.

62. If $\lim_{x\to 0} \dfrac{f(x)}{x^2} = 5$, find the following limits.

(a) $\lim_{x\to 0} f(x)$ (b) $\lim_{x\to 0} \dfrac{f(x)}{x}$

63. If

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x\to 0} f(x) = 0$.

64. Show by means of an example that $\lim_{x\to a} [f(x) + g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

65. Show by means of an example that $\lim_{x\to a} [f(x)\,g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

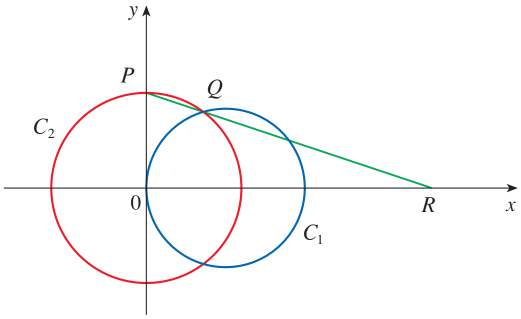
66. Evaluate $\lim_{x\to 2} \dfrac{\sqrt{6 - x} - 2}{\sqrt{3 - x} - 1}$.

67. Is there a number a such that

$$\lim_{x\to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

68. The figure shows a fixed circle C_1 with equation $(x - 1)^2 + y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \to 0^+$?



2.4 The Precise Definition of a Limit

The intuitive definition of a limit given in Section 2.2 is inadequate for some purposes because such phrases as "x is close to 2" and "$f(x)$ gets closer and closer to L" are vague. In order to be able to prove conclusively that

$$\lim_{x \to 0} \left(x^3 + \frac{\cos 5x}{10{,}000} \right) = 0.0001 \qquad \text{or} \qquad \lim_{x \to 0} \frac{\sin x}{x} = 1$$

we must make the definition of a limit precise.

The Precise Definition of a Limit

To motivate the precise definition of a limit, let's consider the function

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

Intuitively, it is clear that when x is close to 3 but $x \neq 3$, then $f(x)$ is close to 5, and so $\lim_{x \to 3} f(x) = 5$.

To obtain more detailed information about how $f(x)$ varies when x is close to 3, we ask the following question:

How close to 3 does x have to be so that $f(x)$ differs from 5 by less than 0.1?

The distance from x to 3 is $|x - 3|$ and the distance from $f(x)$ to 5 is $|f(x) - 5|$, so our problem is to find a number δ (the Greek letter delta) such that

$$|f(x) - 5| < 0.1 \qquad \text{if} \qquad |x - 3| < \delta \quad \text{but } x \neq 3$$

If $|x - 3| > 0$, then $x \neq 3$, so an equivalent formulation of our problem is to find a number δ such that

$$|f(x) - 5| < 0.1 \qquad \text{if} \qquad 0 < |x - 3| < \delta$$

Notice that if $0 < |x - 3| < (0.1)/2 = 0.05$, then

$$|f(x) - 5| = |(2x - 1) - 5| = |2x - 6| = 2|x - 3| < 2(0.05) = 0.1$$

that is, $$|f(x) - 5| < 0.1 \qquad \text{if} \qquad 0 < |x - 3| < 0.05$$

Thus an answer to the problem is given by $\delta = 0.05$; that is, if x is within a distance of 0.05 from 3, then $f(x)$ will be within a distance of 0.1 from 5.

If we change the number 0.1 in our problem to the smaller number 0.01, then by using the same method we find that $f(x)$ will differ from 5 by less than 0.01 provided that x differs from 3 by less than $(0.01)/2 = 0.005$:

$$|f(x) - 5| < 0.01 \qquad \text{if} \qquad 0 < |x - 3| < 0.005$$

Similarly,

$$|f(x) - 5| < 0.001 \qquad \text{if} \qquad 0 < |x - 3| < 0.0005$$

The numbers 0.1, 0.01, and 0.001 that we have considered are *error tolerances* that we might allow. For 5 to be the precise limit of $f(x)$ as x approaches 3, we must not only be able to bring the difference between $f(x)$ and 5 below each of these three numbers; we must be able to bring it below *any* positive number. And, by the same reasoning, we can! If we write ε (the Greek letter epsilon) for an arbitrary positive number, then we find as before that

$$\boxed{1} \qquad |f(x) - 5| < \varepsilon \quad \text{if} \quad 0 < |x - 3| < \delta = \frac{\varepsilon}{2}$$

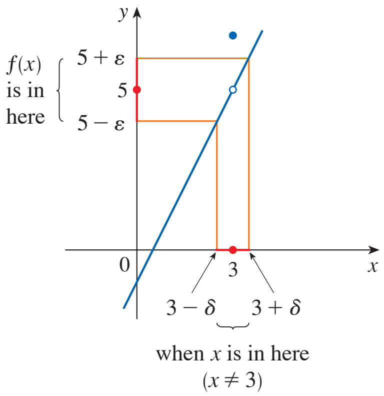


FIGURE 1

This is a precise way of saying that $f(x)$ is close to 5 when x is close to 3 because (1) says that we can make the values of $f(x)$ within an arbitrary distance ε from 5 by restricting the values of x to be within a distance $\varepsilon/2$ from 3 (but $x \neq 3$).

Note that (1) can be rewritten as follows:

$$\text{if} \quad 3 - \delta < x < 3 + \delta \quad (x \neq 3) \qquad \text{then} \qquad 5 - \varepsilon < f(x) < 5 + \varepsilon$$

and this is illustrated in Figure 1. By taking the values of x ($\neq 3$) to lie in the interval $(3 - \delta, 3 + \delta)$ we can make the values of $f(x)$ lie in the interval $(5 - \varepsilon, 5 + \varepsilon)$.

Using (1) as a model, we give a precise definition of a limit.

It is traditional to use the Greek letters ε and δ in the precise definition of a limit.

2 Precise Definition of a Limit Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

Since $|x - a|$ is the distance from x to a and $|f(x) - L|$ is the distance from $f(x)$ to L, and since ε can be arbitrarily small, the definition of a limit can be expressed in words as follows:

> $\lim_{x \to a} f(x) = L$ means that the distance between $f(x)$ and L can be made arbitrarily small by requiring that the distance from x to a be sufficiently small (but not 0).

Alternatively,

> $\lim_{x \to a} f(x) = L$ means that the values of $f(x)$ can be made as close as we please to L by requiring x to be close enough to a (but not equal to a).

We can also reformulate Definition 2 in terms of intervals by observing that the inequality $|x - a| < \delta$ is equivalent to $-\delta < x - a < \delta$, which in turn can be written as $a - \delta < x < a + \delta$. Also $0 < |x - a|$ is true if and only if $x - a \neq 0$, that is, $x \neq a$. Similarly, the inequality $|f(x) - L| < \varepsilon$ is equivalent to the pair of inequalities $L - \varepsilon < f(x) < L + \varepsilon$. Therefore, in terms of intervals, Definition 2 can be stated as follows:

> $\lim_{x \to a} f(x) = L$ means that for every $\varepsilon > 0$ (no matter how small ε is) we can find $\delta > 0$ such that if x lies in the open interval $(a - \delta, a + \delta)$ and $x \neq a$, then $f(x)$ lies in the open interval $(L - \varepsilon, L + \varepsilon)$.

We interpret this statement geometrically by representing a function by an arrow diagram as in Figure 2, where f maps a subset of $\mathbb{R}$ onto another subset of $\mathbb{R}$.

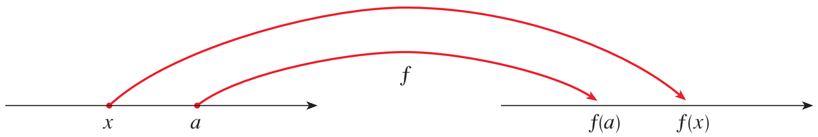


FIGURE 2

The definition of limit says that if any small interval $(L - \varepsilon, L + \varepsilon)$ is given around L, then we can find an interval $(a - \delta, a + \delta)$ around a such that f maps all the points in $(a - \delta, a + \delta)$ (except possibly a) into the interval $(L - \varepsilon, L + \varepsilon)$. (See Figure 3.)

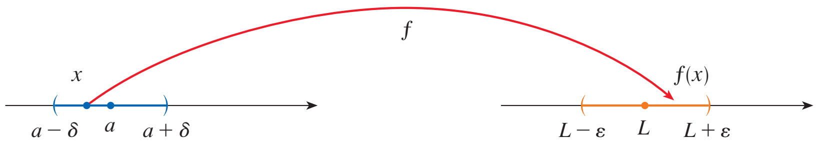


FIGURE 3

Another geometric interpretation of limits can be given in terms of the graph of a function. If $\varepsilon > 0$ is given, then we draw the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ and the graph of f. (See Figure 4.) If $\lim_{x \to a} f(x) = L$, then we can find a number $\delta > 0$ such that if we restrict x to lie in the interval $(a - \delta, a + \delta)$ and take $x \neq a$, then the curve $y = f(x)$ lies between the lines $y = L - \varepsilon$ and $y = L + \varepsilon$. (See Figure 5.) You can see that if such a δ has been found, then any smaller δ will also work.

It is important to realize that the process illustrated in Figures 4 and 5 must work for *every* positive number ε, no matter how small it is chosen. Figure 6 shows that if a smaller ε is chosen, then a smaller δ may be required.

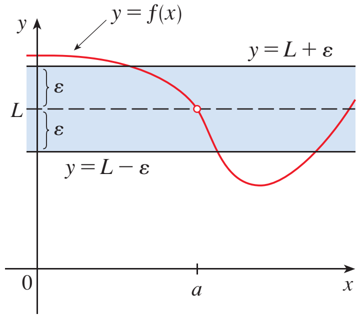


FIGURE 4

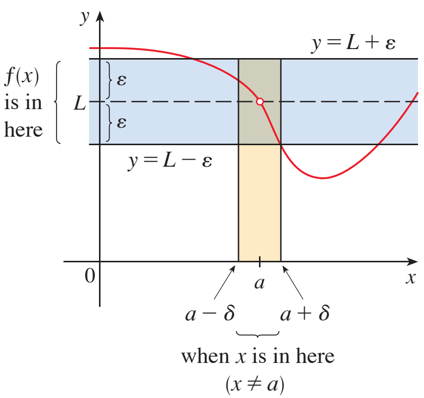


FIGURE 5

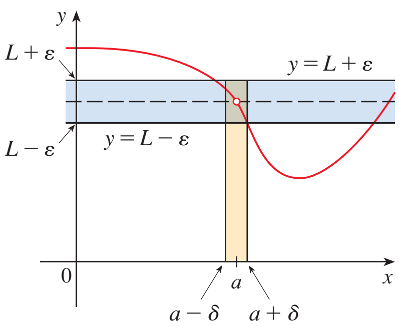


FIGURE 6

EXAMPLE 1 Since $f(x) = x^3 - 5x + 6$ is a polynomial, we know from the Direct Substitution Property that $\lim_{x \to 1} f(x) = f(1) = 1^3 - 5(1) + 6 = 2$. Use a graph to find a number δ such that if x is within δ of 1, then y is within 0.2 of 2, that is,

$$\text{if} \qquad |x - 1| < \delta \qquad \text{then} \qquad |(x^3 - 5x + 6) - 2| < 0.2$$

In other words, find a number δ that corresponds to $\varepsilon = 0.2$ in the definition of a limit for the function $f(x) = x^3 - 5x + 6$ with $a = 1$ and $L = 2$.

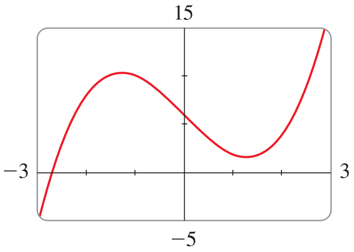


FIGURE 7

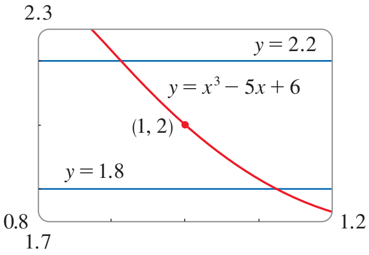


FIGURE 8

SOLUTION A graph of f is shown in Figure 7; we are interested in the region near the point $(1, 2)$. Notice that we can rewrite the inequality

$$|(x^3 - 5x + 6) - 2| < 0.2$$

as

$$-0.2 < (x^3 - 5x + 6) - 2 < 0.2$$

or equivalently

$$1.8 < x^3 - 5x + 6 < 2.2$$

So we need to determine the values of x for which the curve $y = x^3 - 5x + 6$ lies between the horizontal lines $y = 1.8$ and $y = 2.2$. Therefore we graph the curves $y = x^3 - 5x + 6$, $y = 1.8$, and $y = 2.2$ near the point $(1, 2)$ in Figure 8. We estimate that the x-coordinate of the point of intersection of the line $y = 2.2$ and the curve $y = x^3 - 5x + 6$ is about 0.911. Similarly, $y = x^3 - 5x + 6$ intersects the line $y = 1.8$ when $x \approx 1.124$. So, rounding toward 1 to be safe, we can say that

$$\text{if} \qquad 0.92 < x < 1.12 \qquad \text{then} \qquad 1.8 < x^3 - 5x + 6 < 2.2$$

This interval $(0.92, 1.12)$ is not symmetric about $x = 1$. The distance from $x = 1$ to the left endpoint is $1 - 0.92 = 0.08$ and the distance to the right endpoint is 0.12. We can choose δ to be the smaller of these numbers, that is, $\delta = 0.08$. Then we can rewrite our inequalities in terms of distances as follows:

$$\text{if} \qquad |x - 1| < 0.08 \qquad \text{then} \qquad |(x^3 - 5x + 6) - 2| < 0.2$$

This just says that by keeping x within 0.08 of 1, we are able to keep $f(x)$ within 0.2 of 2.

Although we chose $\delta = 0.08$, any smaller positive value of δ would also have worked. ■

The graphical procedure used in Example 1 gives an illustration of the definition for $\varepsilon = 0.2$, but it does not *prove* that the limit is equal to 2. A proof has to provide a δ for *every* ε.

In proving limit statements it may be helpful to think of the definition of limit as a challenge. First it challenges you with a number ε. Then you must be able to produce a suitable δ. You have to be able to do this for every $\varepsilon > 0$, not just a particular ε.

Imagine a contest between two people, A and B, and imagine yourself to be B. Person A stipulates that the fixed number L should be approximated by the values of $f(x)$ to within a degree of accuracy ε (say, 0.01). Person B then responds by finding a number δ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. Then A may become more exacting and challenge B with a smaller value of ε (say, 0.0001). Again B has to respond by finding a corresponding δ. Usually the smaller the value of ε, the smaller the corresponding value of δ must be. If B always wins, no matter how small A makes ε, then $\lim_{x \to a} f(x) = L$.

EXAMPLE 2 Prove that $\lim\limits_{x \to 3} (4x - 5) = 7$.

SOLUTION

1. *Preliminary analysis of the problem (guessing a value for δ).* Let ε be a given positive number. We want to find a number δ such that

$$\text{if} \qquad 0 < |x - 3| < \delta \qquad \text{then} \qquad |(4x - 5) - 7| < \varepsilon$$

But $|(4x-5)-7| = |4x-12| = |4(x-3)| = 4|x-3|$. Therefore we want δ such that

$$\text{if} \quad 0 < |x-3| < \delta \quad \text{then} \quad 4|x-3| < \varepsilon$$

that is,

$$\text{if} \quad 0 < |x-3| < \delta \quad \text{then} \quad |x-3| < \frac{\varepsilon}{4}$$

This suggests that we should choose $\delta = \varepsilon/4$.

2. *Proof (showing that this δ works).* Given $\varepsilon > 0$, choose $\delta = \varepsilon/4$. If $0 < |x-3| < \delta$, then

$$|(4x-5)-7| = |4x-12| = 4|x-3| < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon$$

Thus

$$\text{if} \quad 0 < |x-3| < \delta \quad \text{then} \quad |(4x-5)-7| < \varepsilon$$

Therefore, by the definition of a limit,

$$\lim_{x\to 3}(4x-5) = 7$$

This example is illustrated by Figure 9.

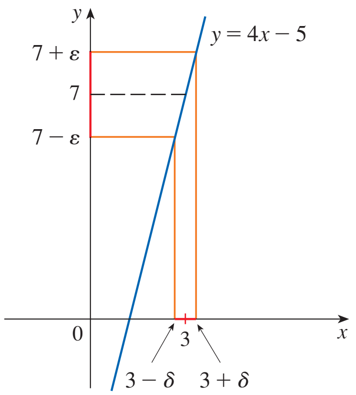


FIGURE 9

Note that in the solution of Example 2 there were two stages—guessing and proving. We made a preliminary analysis that enabled us to guess a value for δ. But then in the second stage we had to go back and prove in a careful, logical fashion that we had made a correct guess. This procedure is typical of much of mathematics. Sometimes it is necessary to first make an intelligent guess about the answer to a problem and then prove that the guess is correct.

EXAMPLE 3 Prove that $\lim_{x\to 3} x^2 = 9$.

SOLUTION

1. *Guessing a value for δ.* Let $\varepsilon > 0$ be given. We have to find a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x-3| < \delta \quad \text{then} \quad |x^2-9| < \varepsilon$$

To connect $|x^2-9|$ with $|x-3|$ we write $|x^2-9| = |(x+3)(x-3)|$. Then we want

$$\text{if} \quad 0 < |x-3| < \delta \quad \text{then} \quad |x+3||x-3| < \varepsilon$$

Cauchy and Limits

After the invention of calculus in the 17th century, there followed a period of free development of the subject in the 18th century. Mathematicians like the Bernoulli brothers and Euler were eager to exploit the power of calculus and boldly explored the consequences of this new and wonderful mathematical theory without worrying too much about whether their proofs were completely correct.

The 19th century, by contrast, was the Age of Rigor in mathematics. There was a movement to go back to the foundations of the subject—to provide careful definitions and rigorous proofs. At the forefront of this movement was the French mathematician Augustin-Louis Cauchy (1789–1857), who started out as a military engineer before becoming a mathematics professor in Paris. Cauchy took Newton's idea of a limit, which was kept alive in the 18th century by the French mathematician Jean d'Alembert, and made it more precise. His definition of a limit reads as follows: "When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last is called the *limit* of all the others." But when Cauchy used this definition in examples and proofs, he often employed delta-epsilon inequalities similar to the ones in this section. A typical Cauchy proof starts with: "Designate by δ and ε two very small numbers; . . ." He used ε because of the correspondence between epsilon and the French word *erreur* and δ because delta corresponds to *différence*. Later, the German mathematician Karl Weierstrass (1815–1897) stated the definition of a limit exactly as in our Definition 2

Notice that if we can find a positive constant C such that $|x+3| < C$, then

$$|x+3||x-3| < C|x-3|$$

and we can make $C|x-3| < \varepsilon$ by taking $|x-3| < \varepsilon/C$, so we could choose $\delta = \varepsilon/C$.

We can find such a number C if we restrict x to lie in some interval centered at 3. In fact, since we are interested only in values of x that are close to 3, it is reasonable to assume that x is within a distance 1 from 3, that is, $|x-3| < 1$. Then $2 < x < 4$, so $5 < x+3 < 7$. Thus we have $|x+3| < 7$, and so $C = 7$ is a suitable choice for the constant.

But now there are two restrictions on $|x-3|$, namely

$$|x-3| < 1 \qquad \text{and} \qquad |x-3| < \frac{\varepsilon}{C} = \frac{\varepsilon}{7}$$

To make sure that both of these inequalities are satisfied, we take δ to be the smaller of the two numbers 1 and $\varepsilon/7$. The notation for this is $\delta = \min\{1, \varepsilon/7\}$.

2. *Showing that this δ works.* Given $\varepsilon > 0$, let $\delta = \min\{1, \varepsilon/7\}$. If $0 < |x-3| < \delta$, then $|x-3| < 1 \Rightarrow 2 < x < 4 \Rightarrow |x+3| < 7$ (as in part l). We also have $|x-3| < \varepsilon/7$, so

$$|x^2-9| = |x+3||x-3| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon$$

This shows that $\lim_{x\to 3} x^2 = 9$. ■

One-Sided Limits

The intuitive definitions of one-sided limits that were given in Section 2.2 can be precisely reformulated as follows.

3 Precise Definition of Left-Hand Limit

$$\lim_{x\to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \qquad a-\delta < x < a \qquad \text{then} \qquad |f(x)-L| < \varepsilon$$

4 Precise Definition of Right-Hand Limit

$$\lim_{x\to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \qquad a < x < a+\delta \qquad \text{then} \qquad |f(x)-L| < \varepsilon$$

Notice that Definition 3 is the same as Definition 2 except that x is restricted to lie in the *left* half $(a-\delta, a)$ of the interval $(a-\delta, a+\delta)$. In Definition 4, x is restricted to lie in the *right* half $(a, a+\delta)$ of the interval $(a-\delta, a+\delta)$.

EXAMPLE 4 Use Definition 4 to prove that $\lim_{x\to 0^+} \sqrt{x} = 0$.

SOLUTION

1. *Guessing a value for* δ. Let ε be a given positive number. Here $a = 0$ and $L = 0$, so we want to find a number δ such that

$$\text{if} \quad 0 < x < \delta \quad \text{then} \quad |\sqrt{x} - 0| < \varepsilon$$

that is,

$$\text{if} \quad 0 < x < \delta \quad \text{then} \quad \sqrt{x} < \varepsilon$$

or, squaring both sides of the inequality $\sqrt{x} < \varepsilon$, we get

$$\text{if} \quad 0 < x < \delta \quad \text{then} \quad x < \varepsilon^2$$

This suggests that we should choose $\delta = \varepsilon^2$.

2. *Showing that this* δ *works.* Given $\varepsilon > 0$, let $\delta = \varepsilon^2$. If $0 < x < \delta$, then

$$\sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$$

so

$$|\sqrt{x} - 0| < \varepsilon$$

According to Definition 4, this shows that $\lim_{x\to 0^+} \sqrt{x} = 0$. ■

■ The Limit Laws

As the preceding examples show, it is not always easy to prove that limit statements are true using the ε, δ definition. In fact, if we had been given a more complicated function such as $f(x) = (6x^2 - 8x + 9)/(2x^2 - 1)$, a proof would require a great deal of ingenuity. Fortunately this is unnecessary because the Limit Laws stated in Section 2.3 can be proved using Definition 2, and then the limits of complicated functions can be found rigorously from the Limit Laws without resorting to the definition directly.

For instance, we prove the Sum Law: If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ both exist, then

$$\lim_{x\to a} [f(x) + g(x)] = L + M$$

The remaining laws are proved in the exercises and in Appendix F.

PROOF OF THE SUM LAW Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) + g(x) - (L + M)| < \varepsilon$$

Using the Triangle Inequality we can write

Triangle Inequality:

$$|a + b| \leqslant |a| + |b|$$

(See Appendix A.)

$$\boxed{5} \qquad \begin{aligned} |f(x) + g(x) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leqslant |f(x) - L| + |g(x) - M| \end{aligned}$$

We make $|f(x) + g(x) - (L + M)|$ less than ε by making each of the terms $|f(x) - L|$ and $|g(x) - M|$ less than $\varepsilon/2$.

Since $\varepsilon/2 > 0$ and $\lim_{x\to a} f(x) = L$, there exists a number $\delta_1 > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta_1 \quad \text{then} \quad |f(x) - L| < \frac{\varepsilon}{2}$$

Similarly, since $\lim_{x\to a} g(x) = M$, there exists a number $\delta_2 > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta_2 \quad \text{then} \quad |g(x) - M| < \frac{\varepsilon}{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$, the smaller of the numbers δ_1 and δ_2. Notice that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad 0 < |x - a| < \delta_1 \quad \text{and} \quad 0 < |x - a| < \delta_2$$

and so
$$|f(x) - L| < \frac{\varepsilon}{2} \qquad \text{and} \qquad |g(x) - M| < \frac{\varepsilon}{2}$$

Therefore, by (5),

$$\begin{aligned} |f(x) + g(x) - (L + M)| &\leqslant |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

To summarize,

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) + g(x) - (L + M)| < \varepsilon$$

Thus, by the definition of a limit,

$$\lim_{x \to a} [f(x) + g(x)] = L + M$$

■

Infinite Limits

Infinite limits can also be defined in a precise way. The following is a precise version of Definition 2.2.4.

6 Precise Definition of an Infinite Limit Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) > M$$

This says that the values of $f(x)$ can be made arbitrarily large (larger than any given number M) by requiring x to be close enough to a (within a distance δ, where δ depends on M, but with $x \neq a$). A geometric illustration is shown in Figure 10.

Given any horizontal line $y = M$, we can find a number $\delta > 0$ such that if we restrict x to lie in the interval $(a - \delta, a + \delta)$ but $x \neq a$, then the curve $y = f(x)$ lies above the line $y = M$. You can see that if a larger M is chosen, then a smaller δ may be required.

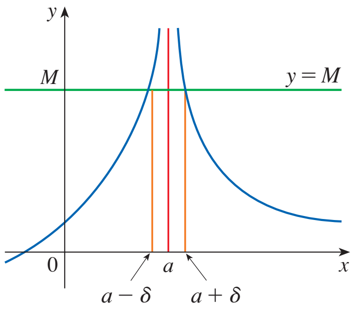


FIGURE 10

EXAMPLE 5 Use Definition 6 to prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

SOLUTION Let M be a given positive number. We want to find a number δ such that

$$\text{if} \quad 0 < |x| < \delta \quad \text{then} \quad 1/x^2 > M$$

But
$$\frac{1}{x^2} > M \iff x^2 < \frac{1}{M} \iff \sqrt{x^2} < \sqrt{\frac{1}{M}} \iff |x| < \frac{1}{\sqrt{M}}$$

So if we choose $\delta = 1/\sqrt{M}$ and $0 < |x| < \delta = 1/\sqrt{M}$, then $1/x^2 > M$. This shows that $1/x^2 \to \infty$ as $x \to 0$. ■

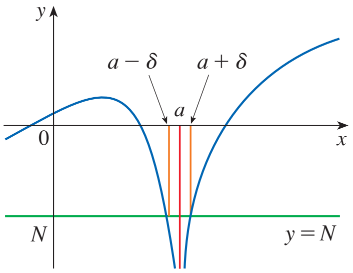


FIGURE 11

Similarly, the following is a precise version of Definition 2.2.5. It is illustrated by Figure 11.

7 Definition Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

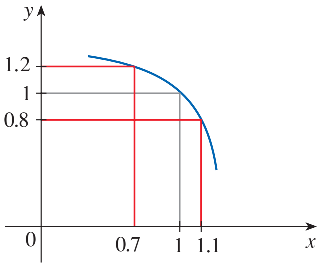
means that for every negative number N there is a positive number δ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) < N$$

2.4 Exercises

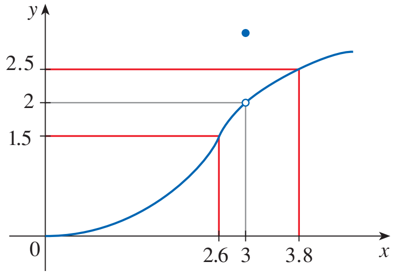
1. Use the given graph of f to find a number δ such that

$$\text{if} \quad |x - 1| < \delta \quad \text{then} \quad |f(x) - 1| < 0.2$$



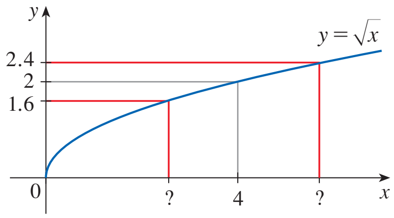
2. Use the given graph of f to find a number δ such that

$$\text{if} \quad 0 < |x - 3| < d \quad \text{then} \quad |f(x) - 2| < 0.5$$



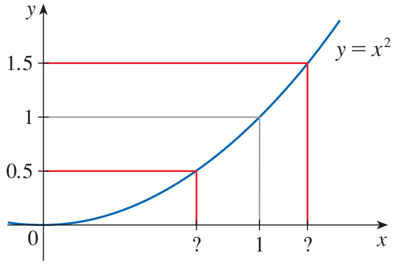
3. Use the given graph of $f(x) = \sqrt{x}$ to find a number δ such that

$$\text{if} \quad |x - 4| < \delta \quad \text{then} \quad |\sqrt{x} - 2| < 0.4$$



4. Use the given graph of $f(x) = x^2$ to find a number δ such that

$$\text{if} \quad |x - 1| < \delta \quad \text{then} \quad |x^2 - 1| < \tfrac{1}{2}$$



5. Use a graph to find a number δ such that

$$\text{if} \quad |x - 2| < \delta \quad \text{then} \quad \left|\sqrt{x^2 + 5} - 3\right| < 0.3$$

6. Use a graph to find a number δ such that

$$\text{if} \quad \left|x - \frac{\pi}{6}\right| < \delta \quad \text{then} \quad \left|\cos^2 x - \frac{3}{4}\right| < 0.1$$

7. For the limit

$$\lim_{x \to 2} (x^3 - 3x + 4) = 6$$

illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 0.2$ and $\varepsilon = 0.1$.

8. For the limit

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x} = 2$$

illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

9. (a) Use a graph to find a number δ such that

$$\text{if} \quad 2 < x < 2 + \delta \quad \text{then} \quad \frac{1}{\ln(x - 1)} > 100$$

(b) What limit does part (a) suggest is true?

10. Given that $\lim_{x\to\pi} \csc^2 x = \infty$, illustrate Definition 6 by finding values of δ that correspond to (a) $M = 500$ and (b) $M = 1000$.

11. A machinist is required to manufacture a circular metal disk with area 1000 cm^2.
(a) What radius produces such a disk?
(b) If the machinist is allowed an error tolerance of ±5 cm^2 in the area of the disk, how close to the ideal radius in part (a) must the machinist control the radius?
(c) In terms of the ε, δ definition of $\lim_{x\to a} f(x) = L$, what is x? What is $f(x)$? What is a? What is L? What value of ε is given? What is the corresponding value of δ?

12. Crystal growth furnaces are used in research to determine how best to manufacture crystals used in electronic components. For proper growth of a crystal, the temperature must be controlled accurately by adjusting the input power. Suppose the relationship is given by

$$T(w) = 0.1w^2 + 2.155w + 20$$

where T is the temperature in degrees Celsius and w is the power input in watts.
(a) How much power is needed to maintain the temperature at 200°C?
(b) If the temperature is allowed to vary from 200°C by up to ±1°C, what range of wattage is allowed for the input power?
(c) In terms of the ε, δ definition of $\lim_{x\to a} f(x) = L$, what is x? What is $f(x)$? What is a? What is L? What value of ε is given? What is the corresponding value of δ?

13. (a) Find a number δ such that if $|x - 2| < \delta$, then $|4x - 8| < \varepsilon$, where $\varepsilon = 0.1$.
(b) Repeat part (a) with $\varepsilon = 0.01$.

14. Given that $\lim_{x\to 2}(5x - 7) = 3$, illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 0.1$, $\varepsilon = 0.05$, and $\varepsilon = 0.01$.

15–18 Prove the statement using the ε, δ definition of a limit and illustrate with a diagram like Figure 9.

15. $\lim_{x\to 4}\left(\frac{1}{2}x - 1\right) = 1$

16. $\lim_{x\to 2}(2 - 3x) = -4$

17. $\lim_{x\to -2}(-2x + 1) = 5$

18. $\lim_{x\to 1}(2x - 5) = -3$

19–32 Prove the statement using the ε, δ definition of a limit.

19. $\lim_{x\to 9}\left(1 - \frac{1}{3}x\right) = -2$

20. $\lim_{x\to 5}\left(\frac{3}{2}x - \frac{1}{2}\right) = 7$

21. $\lim_{x\to 4}\dfrac{x^2 - 2x - 8}{x - 4} = 6$

22. $\lim_{x\to -1.5}\dfrac{9 - 4x^2}{3 + 2x} = 6$

23. $\lim_{x\to a} x = a$

24. $\lim_{x\to a} c = c$

25. $\lim_{x\to 0} x^2 = 0$

26. $\lim_{x\to 0} x^3 = 0$

27. $\lim_{x\to 0} |x| = 0$

28. $\lim_{x\to -6^+} \sqrt[8]{6 + x} = 0$

29. $\lim_{x\to 2}(x^2 - 4x + 5) = 1$

30. $\lim_{x\to 2}(x^2 + 2x - 7) = 1$

31. $\lim_{x\to -2}(x^2 - 1) = 3$

32. $\lim_{x\to 2} x^3 = 8$

33. Verify that another possible choice of δ for showing that $\lim_{x\to 3} x^2 = 9$ in Example 3 is $\delta = \min\{2, \varepsilon/8\}$.

34. Verify, by a geometric argument, that the largest possible choice of δ for showing that $\lim_{x\to 3} x^2 = 9$ is $\delta = \sqrt{9 + \varepsilon} - 3$.

35. (a) For the limit $\lim_{x\to 1}(x^3 + x + 1) = 3$, use a graph to find a value of δ that corresponds to $\varepsilon = 0.4$.
(b) By solving the cubic equation $x^3 + x + 1 = 3 + \varepsilon$, find the largest possible value of δ that works for any given $\varepsilon > 0$.
(c) Put $\varepsilon = 0.4$ in your answer to part (b) and compare with your answer to part (a).

36. Prove that $\lim_{x\to 2}\dfrac{1}{x} = \dfrac{1}{2}$.

37. Prove that $\lim_{x\to a}\sqrt{x} = \sqrt{a}$ if $a > 0$.

$$\left[\textit{Hint}\text{: Use } \left|\sqrt{x} - \sqrt{a}\right| = \frac{|x - a|}{\sqrt{x} + \sqrt{a}}.\right]$$

38. If H is the Heaviside function defined in Section 2.2, prove, using Definition 2, that $\lim_{t\to 0} H(t)$ does not exist. [*Hint:* Use an indirect proof as follows. Suppose that the limit is L. Take $\varepsilon = \frac{1}{2}$ in the definition of a limit and try to arrive at a contradiction.]

39. If the function f is defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x\to 0} f(x)$ does not exist.

40. By comparing Definitions 2, 3, and 4, prove Theorem 2.3.1:

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

41. How close to -3 do we have to take x so that

$$\frac{1}{(x + 3)^4} > 10{,}000$$

42. Prove, using Definition 6, that $\lim_{x\to -3}\dfrac{1}{(x + 3)^4} = \infty$.

43. Prove that $\lim_{x\to 0^+} \ln x = -\infty$.

44. Suppose that $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = c$, where c is a real number. Prove each statement.
(a) $\lim_{x\to a}[f(x) + g(x)] = \infty$
(b) $\lim_{x\to a}[f(x)g(x)] = \infty \quad \text{if } c > 0$
(c) $\lim_{x\to a}[f(x)g(x)] = -\infty \quad \text{if } c < 0$

2.5 Continuity

Continuity of a Function

We noticed in Section 2.3 that the limit of a function as x approaches a can often be found simply by calculating the value of the function at a. Functions having this property are called *continuous at a*. We will see that the mathematical definition of continuity corresponds closely with the meaning of the word *continuity* in everyday language. (A continuous process is one that takes place without interruption.)

1 Definition A function f is **continuous at a number *a*** if

$$\lim_{x\to a} f(x) = f(a)$$

Notice that Definition 1 implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

As illustrated in Figure 1, if f is continuous, then the points $(x, f(x))$ on the graph of f approach the point $(a, f(a))$ on the graph. So there is no gap in the curve.

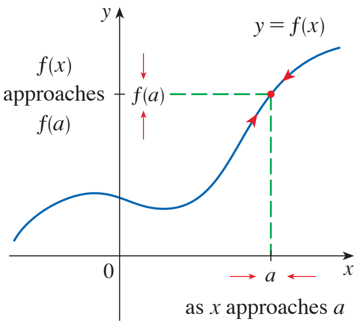


FIGURE 1

The definition says that f is continuous at a if $f(x)$ approaches $f(a)$ as x approaches a. Thus a continuous function f has the property that a small change in x produces only a small change in $f(x)$. In fact, the change in $f(x)$ can be kept as small as we please by keeping the change in x sufficiently small.

If f is defined near a (in other words, f is defined on an open interval containing a, except perhaps at a), we say that f is **discontinuous at *a*** (or f has a **discontinuity** at a) if f is not continuous at a.

Physical phenomena are usually continuous. For instance, the displacement or velocity of a moving vehicle varies continuously with time, as does a person's height. But discontinuities do occur in such situations as electric currents. [The Heaviside function, introduced in Section 2.2, is discontinuous at 0 because $\lim_{t\to 0} H(t)$ does not exist.]

Geometrically, you can think of a function that is continuous at every number in an interval as a function whose graph has no break in it: the graph can be drawn without removing your pen from the paper.

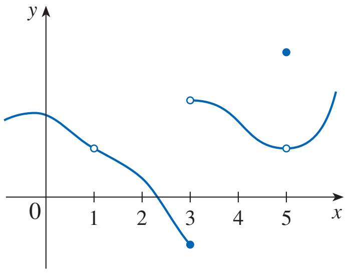


FIGURE 2

EXAMPLE 1 Figure 2 shows the graph of a function f. At which numbers is f discontinuous? Why?

SOLUTION It looks as if there is a discontinuity when $a = 1$ because the graph has a break there. The official reason that f is discontinuous at 1 is that $f(1)$ is not defined.

The graph also has a break when $a = 3$, but the reason for the discontinuity is different. Here, $f(3)$ is defined, but $\lim_{x\to 3} f(x)$ does not exist (because the left and right limits are different). So f is discontinuous at 3.

What about $a = 5$? Here, $f(5)$ is defined and $\lim_{x\to 5} f(x)$ exists (because the left and right limits are the same). But

$$\lim_{x\to 5} f(x) \neq f(5)$$

So f is discontinuous at 5. ■

Now let's see how to detect discontinuities when a function is defined by a formula.

EXAMPLE 2 Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(d) $f(x) = [\![x]\!]$

SOLUTION

(a) Notice that $f(2)$ is not defined, so f is discontinuous at 2. Later we'll see why f is continuous at all other numbers.

(b) Here $f(2) = 1$ is defined and

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 1)}{x - 2} = \lim_{x \to 2} (x + 1) = 3$$

exists. But

$$\lim_{x \to 2} f(x) \neq f(2)$$

so f is not continuous at 2.

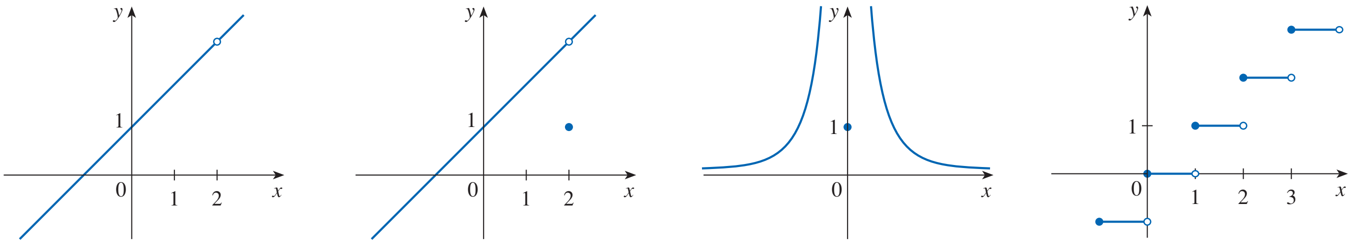
(c) Here $f(0) = 1$ is defined but

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2}$$

does not exist. (See Example 2.2.6.) So f is discontinuous at 0.

(d) The greatest integer function $f(x) = [\![x]\!]$ has discontinuities at all of the integers because $\lim_{x \to n} [\![x]\!]$ does not exist if n is an integer. (See Example 2.3.10 and Exercise 2.3.55.) ■

Figure 3 shows the graphs of the functions in Example 2. In each case the graph can't be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph. The kind of discontinuity illustrated in parts (a) and (b) is called **removable** because we could remove the discontinuity by redefining f at just the single number 2. [If we redefine f to be 3 at $x = 2$, then f is equivalent to the function $g(x) = x + 1$, which is continuous.] The discontinuity in part (c) is called an **infinite discontinuity**. The discontinuities in part (d) are called **jump discontinuities** because the function "jumps" from one value to another.



(a) A removable discontinuity (b) A removable discontinuity (c) An infinite discontinuity (d) Jump discontinuities

FIGURE 3
Graphs of the functions in Example 2

2 Definition A function f is **continuous from the right at a number a** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at a** if

$$\lim_{x \to a^-} f(x) = f(a)$$

EXAMPLE 3 At each integer n, the function $f(x) = [\![x]\!]$ [see Figure 3(d)] is continuous from the right but discontinuous from the left because

$$\lim_{x \to n^+} f(x) = \lim_{x \to n^+} [\![x]\!] = n = f(n)$$

but
$$\lim_{x \to n^-} f(x) = \lim_{x \to n^-} [\![x]\!] = n - 1 \neq f(n)$$ ■

3 Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left.*)

EXAMPLE 4 Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

SOLUTION If $-1 < a < 1$, then using the Limit Laws from Section 2.3, we have

$$\begin{aligned}\lim_{x \to a} f(x) &= \lim_{x \to a} \left(1 - \sqrt{1 - x^2}\right)\\ &= 1 - \lim_{x \to a} \sqrt{1 - x^2} && \text{(by Laws 2 and 8)}\\ &= 1 - \sqrt{\lim_{x \to a} (1 - x^2)} && \text{(by 7)}\\ &= 1 - \sqrt{1 - a^2} && \text{(by 2, 8, and 10)}\\ &= f(a)\end{aligned}$$

Thus, by Definition 1, f is continuous at a if $-1 < a < 1$. Similar calculations show that

$$\lim_{x \to -1^+} f(x) = 1 = f(-1) \qquad \text{and} \qquad \lim_{x \to 1^-} f(x) = 1 = f(1)$$

so f is continuous from the right at -1 and continuous from the left at 1. Therefore, according to Definition 3, f is continuous on $[-1, 1]$.

The graph of f is sketched in Figure 4. It is the lower half of the circle

$$x^2 + (y - 1)^2 = 1$$ ■

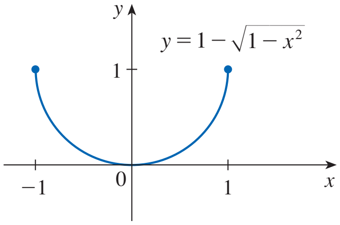


FIGURE 4

Properties of Continuous Functions

Instead of always using Definitions 1, 2, and 3 to verify the continuity of a function as we did in Example 4, it is often convenient to use the next theorem, which shows how to build up complicated continuous functions from simple ones.

4 Theorem If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$ **2.** $f - g$ **3.** cf

4. fg **5.** $\dfrac{f}{g}$ if $g(a) \neq 0$

PROOF Each of the five parts of this theorem follows from the corresponding Limit Law in Section 2.3. For instance, we give the proof of part 1. Since f and g are continuous at a, we have

$$\lim_{x \to a} f(x) = f(a) \qquad \text{and} \qquad \lim_{x \to a} g(x) = g(a)$$

Therefore

$$\begin{aligned} \lim_{x \to a} (f + g)(x) &= \lim_{x \to a} [f(x) + g(x)] \\ &= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \qquad \text{(by Law 1)} \\ &= f(a) + g(a) \\ &= (f + g)(a) \end{aligned}$$

This shows that $f + g$ is continuous at a. ■

It follows from Theorem 4 and Definition 3 that if f and g are continuous on an interval, then so are the functions $f + g$, $f - g$, cf, fg, and (if g is never 0) f/g. The following theorem was stated in Section 2.3 as the Direct Substitution Property.

5 Theorem

(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

PROOF

(a) A polynomial is a function of the form

$$P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

where $c_0, c_1, \ldots, c_n$ are constants. We know that

$$\lim_{x \to a} c_0 = c_0 \qquad \text{(by Law 8)}$$

and

$$\lim_{x \to a} x^m = a^m \qquad m = 1, 2, \ldots, n \qquad \text{(by 10)}$$

This equation is precisely the statement that the function $f(x) = x^m$ is a continuous function. Thus, by part 3 of Theorem 4, the function $g(x) = cx^m$ is continuous. Since P is a sum of functions of this form and a constant function, it follows from part 1 of Theorem 4 that P is continuous.

(b) A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain of f is $D = \{x \in \mathbb{R} \mid Q(x) \neq 0\}$. We know from part (a) that P and Q are continuous everywhere. Thus, by part 5 of Theorem 4, f is continuous at every number in D. ■

As an illustration of Theorem 5, observe that the volume of a sphere varies continuously with its radius because the formula $V(r) = \frac{4}{3}\pi r^3$ shows that V is a polynomial function of r. Likewise, if a ball is thrown vertically into the air with an initial velocity of 15 m/s, then the height of the ball in meters t seconds later is given by the formula $h = 15t - 4.9t^2$. Again this is a polynomial function, so the height is a continuous function of the elapsed time, as we might expect.

Knowledge of which functions are continuous enables us to evaluate some limits very quickly, as the following example shows. Compare it with Example 2.3.2(b).

EXAMPLE 5 Find $\lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$.

SOLUTION The function

$$f(x) = \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

is rational, so by Theorem 5 it is continuous on its domain, which is $\left\{x \mid x \neq \frac{5}{3}\right\}$. Therefore

$$\begin{aligned}\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} &= \lim_{x \to -2} f(x) = f(-2) \\ &= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = -\frac{1}{11}\end{aligned}$$

■

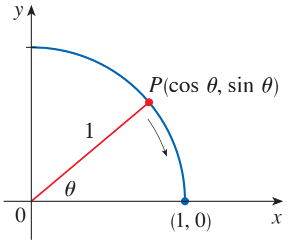


FIGURE 5

It turns out that most of the familiar functions are continuous at every number in their domains. For instance, Limit Law 11 in Section 2.3 is exactly the statement that root functions are continuous.

From the appearance of the graphs of the sine and cosine functions (Figure 1.2.19), we would certainly guess that they are continuous. We know from the definitions of $\sin\theta$ and $\cos\theta$ that the coordinates of the point P in Figure 5 are $(\cos\theta, \sin\theta)$. As $\theta \to 0$, we see that P approaches the point $(1, 0)$ and so $\cos\theta \to 1$ and $\sin\theta \to 0$. Thus

6 $$\lim_{\theta \to 0} \cos\theta = 1 \qquad \lim_{\theta \to 0} \sin\theta = 0$$

Another way to establish the limits in (6) is to use the Squeeze Theorem with the inequality $\sin\theta < \theta$ (for $\theta > 0$), which is proved in Section 3.3.

Since $\cos 0 = 1$ and $\sin 0 = 0$, the equations in (6) assert that the cosine and sine functions are continuous at 0. The addition formulas for cosine and sine can then be used to deduce that these functions are continuous everywhere (see Exercises 66 and 67).

It follows from part 5 of Theorem 4 that

$$\tan x = \frac{\sin x}{\cos x}$$

is continuous except where $\cos x = 0$. This happens when x is an odd integer multiple of $\pi/2$, so $y = \tan x$ has infinite discontinuities when $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2$, and so on (see Figure 6).

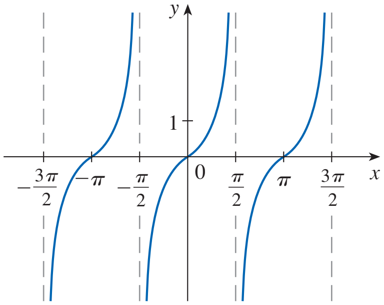


FIGURE 6
$y = \tan x$

The inverse function of any continuous one-to-one function is also continuous. (This fact is proved in Appendix F, but our geometric intuition makes it seem plausible: the graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$. So if the graph of f has no break in it, neither does the graph of f^{-1}.) Thus the inverse trigonometric functions are continuous.

In Section 1.4 we defined the exponential function $y = b^x$ so as to fill in the holes in the graph of $y = b^x$ where x is rational. In other words, the very definition of $y = b^x$ makes it a continuous function on $\mathbb{R}$. Therefore its inverse function $y = \log_b x$ is continuous on $(0, \infty)$.

The inverse trigonometric functions are reviewed in Section 1.5.

7 Theorem The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

EXAMPLE 6 Where is the function $f(x) = \dfrac{\ln x + \tan^{-1}x}{x^2 - 1}$ continuous?

SOLUTION We know from Theorem 7 that the function $y = \ln x$ is continuous for $x > 0$ and $y = \tan^{-1}x$ is continuous on $\mathbb{R}$. Thus, by part 1 of Theorem 4, $y = \ln x + \tan^{-1}x$ is continuous on $(0, \infty)$. The denominator, $y = x^2 - 1$, is a polynomial, so it is continuous everywhere. Therefore, by part 5 of Theorem 4, f is continuous at all positive numbers x except where $x^2 - 1 = 0 \iff x = \pm 1$. So f is continuous on the intervals $(0, 1)$ and $(1, \infty)$. ■

EXAMPLE 7 Evaluate $\displaystyle\lim_{x \to \pi} \frac{\sin x}{2 + \cos x}$.

SOLUTION Theorem 7 tells us that $y = \sin x$ is continuous. The function in the denominator, $y = 2 + \cos x$, is the sum of two continuous functions and is therefore continuous. Notice that this function is never 0 because $\cos x \geqslant -1$ for all x and so $2 + \cos x > 0$ everywhere. Thus the ratio

$$f(x) = \frac{\sin x}{2 + \cos x}$$

is continuous everywhere. Hence, by the definition of a continuous function,

$$\lim_{x \to \pi} \frac{\sin x}{2 + \cos x} = \lim_{x \to \pi} f(x) = f(\pi) = \frac{\sin \pi}{2 + \cos \pi} = \frac{0}{2 - 1} = 0$$ ■

Another way of combining continuous functions f and g to get a new continuous function is to form the composite function $f \circ g$. This fact is a consequence of the following theorem.

This theorem says that a limit symbol can be moved through a function symbol if the function is continuous and the limit exists. In other words, the order of these two symbols can be reversed.

8 Theorem If f is continuous at b and $\displaystyle\lim_{x \to a} g(x) = b$, then $\displaystyle\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Intuitively, Theorem 8 is reasonable because if x is close to a, then $g(x)$ is close to b, and since f is continuous at b, if $g(x)$ is close to b, then $f(g(x))$ is close to $f(b)$. A proof of Theorem 8 is given in Appendix F.

EXAMPLE 8 Evaluate $\lim_{x \to 1} \arcsin\left(\dfrac{1 - \sqrt{x}}{1 - x}\right)$.

SOLUTION Because arcsin is a continuous function, we can apply Theorem 8:

$$\begin{aligned}\lim_{x \to 1} \arcsin\left(\frac{1 - \sqrt{x}}{1 - x}\right) &= \arcsin\left(\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x}\right) \\ &= \arcsin\left(\lim_{x \to 1} \frac{1 - \sqrt{x}}{(1 - \sqrt{x})(1 + \sqrt{x})}\right) \\ &= \arcsin\left(\lim_{x \to 1} \frac{1}{1 + \sqrt{x}}\right) \\ &= \arcsin\frac{1}{2} = \frac{\pi}{6}\end{aligned}$$

■

Let's now apply Theorem 8 in the special case where $f(x) = \sqrt[n]{x}$, with n being a positive integer. Then

$$f(g(x)) = \sqrt[n]{g(x)}$$

and

$$f\left(\lim_{x \to a} g(x)\right) = \sqrt[n]{\lim_{x \to a} g(x)}$$

If we put these expressions into Theorem 8, we get

$$\lim_{x \to a} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x \to a} g(x)}$$

and so Limit Law 7 has now been proved. (We assume that the roots exist.)

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

This theorem is often expressed informally by saying "a continuous function of a continuous function is a continuous function."

PROOF Since g is continuous at a, we have

$$\lim_{x \to a} g(x) = g(a)$$

Since f is continuous at $b = g(a)$, we can apply Theorem 8 to obtain

$$\lim_{x \to a} f(g(x)) = f(g(a))$$

which is precisely the statement that the function $h(x) = f(g(x))$ is continuous at a; that is, $f \circ g$ is continuous at a. ■

EXAMPLE 9 Where are the following functions continuous?

(a) $h(x) = \sin(x^2)$ (b) $F(x) = \ln(1 + \cos x)$

SOLUTION

(a) We have $h(x) = f(g(x))$, where

$$g(x) = x^2 \qquad \text{and} \qquad f(x) = \sin x$$

We know that g is continuous on $\mathbb{R}$ since it is a polynomial, and f is also continuous everywhere. Thus $h = f \circ g$ is continuous on $\mathbb{R}$ by Theorem 9.

(b) We know from Theorem 7 that $f(x) = \ln x$ is continuous and $g(x) = 1 + \cos x$ is continuous (because both $y = 1$ and $y = \cos x$ are continuous). Therefore, by Theorem 9, $F(x) = f(g(x))$ is continuous wherever it is defined. The expression $\ln(1 + \cos x)$ is defined when $1 + \cos x > 0$, so it is undefined when $\cos x = -1$, and this happens when $x = \pm\pi, \pm 3\pi, \ldots$. Thus F has discontinuities when x is an odd multiple of π and is continuous on the intervals between these values (see Figure 7). ■

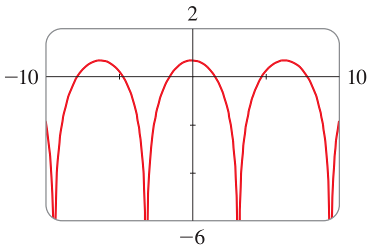


FIGURE 7
$y = \ln(1 + \cos x)$

■ The Intermediate Value Theorem

An important property of continuous functions is expressed by the following theorem, whose proof is found in more advanced books on calculus.

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. It is illustrated by Figure 8. Note that the value N can be taken on once [as in part (a)] or more than once [as in part (b)].

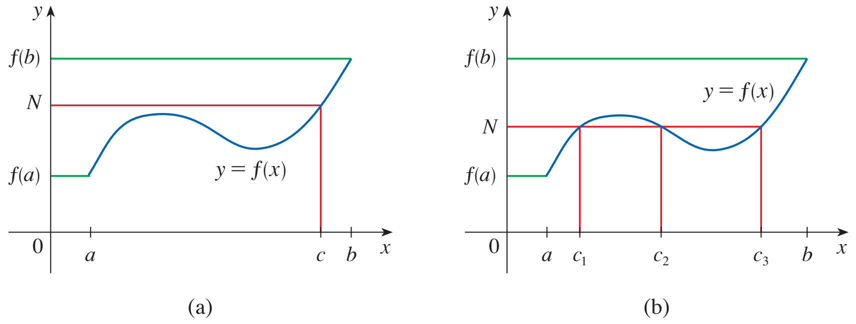


FIGURE 8

If we think of a continuous function as a function whose graph has no hole or break, then it is easy to believe that the Intermediate Value Theorem is true. In geometric terms it says that if any horizontal line $y = N$ is given between $y = f(a)$ and $y = f(b)$ as in Figure 9, then the graph of f can't jump over the line. It must intersect $y = N$ somewhere.

It is important that the function f in Theorem 10 be continuous. The Intermediate Value Theorem is not true in general for discontinuous functions (see Exercise 52).

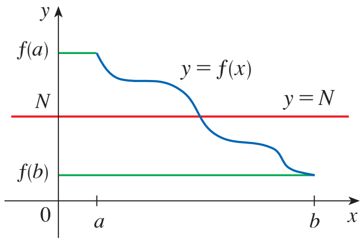


FIGURE 9

One use of the Intermediate Value Theorem is in locating solutions of equations as in the following example.

EXAMPLE 10 Show that there is a solution of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

SOLUTION Let $f(x) = 4x^3 - 6x^2 + 3x - 2$. We are looking for a solution of the given equation, that is, a number c between 1 and 2 such that $f(c) = 0$. Therefore we take $a = 1$, $b = 2$, and $N = 0$ in Theorem 10. We have

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$

and

$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

Thus $f(1) < 0 < f(2)$; that is, $N = 0$ is a number between $f(1)$ and $f(2)$. The function f is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number c between 1 and 2 such that $f(c) = 0$. In other words, the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one solution c in the interval $(1, 2)$.

In fact, we can locate a solution more precisely by using the Intermediate Value Theorem again. Since

$$f(1.2) = -0.128 < 0 \qquad \text{and} \qquad f(1.3) = 0.548 > 0$$

a solution must lie between 1.2 and 1.3. A calculator gives, by trial and error,

$$f(1.22) = -0.007008 < 0 \qquad \text{and} \qquad f(1.23) = 0.056068 > 0$$

so a solution lies in the interval $(1.22, 1.23)$. ■

We can use a graphing calculator or computer to illustrate the use of the Intermediate Value Theorem in Example 10. Figure 10 shows the graph of f in the viewing rectangle $[-1, 3]$ by $[-3, 3]$ and you can see that the graph crosses the x-axis between 1 and 2. Figure 11 shows the result of zooming in to the viewing rectangle $[1.2, 1.3]$ by $[-0.2, 0.2]$.

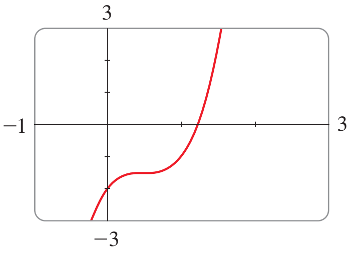


FIGURE 10

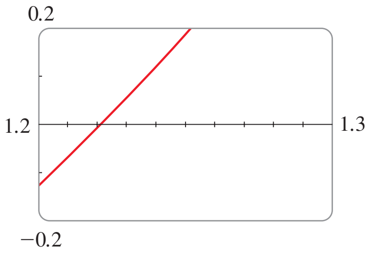


FIGURE 11

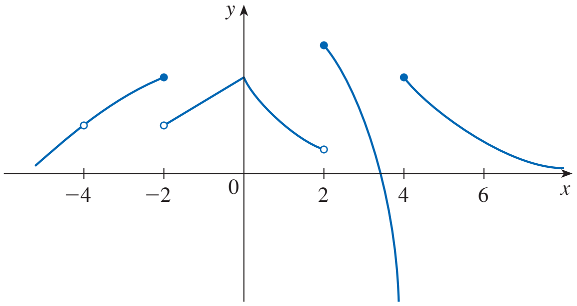
In fact, the Intermediate Value Theorem plays a role in the very way these graphing devices work. A computer calculates a finite number of points on the graph and turns on the pixels that contain these calculated points. It assumes that the function is continuous and takes on all the intermediate values between two consecutive points. The computer therefore “connects the dots” by turning on the intermediate pixels.

2.5 Exercises

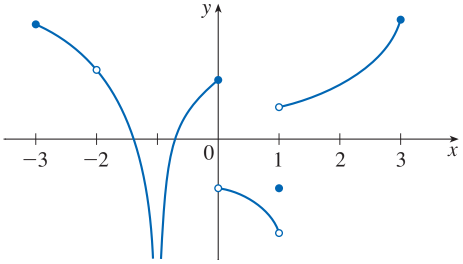
1. Write an equation that expresses the fact that a function f is continuous at the number 4.

2. If f is continuous on $(-\infty, \infty)$, what can you say about its graph?

3. (a) From the given graph of f, state the numbers at which f is discontinuous and explain why.
(b) For each of the numbers stated in part (a), determine whether f is continuous from the right, or from the left, or neither.



4. From the given graph of g, state the numbers at which g is discontinuous and explain why.



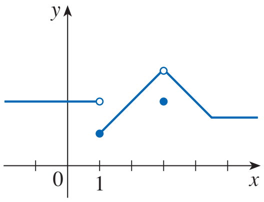
5–6 The graph of a function f is given.

(a) At what numbers a does $\lim_{x\to a} f(x)$ not exist?

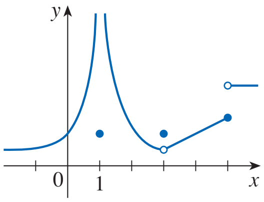
(b) At what numbers a is f not continuous?

(c) At what numbers a does $\lim_{x\to a} f(x)$ exist but f is not continuous at a?

5.



6.



7–10 Sketch the graph of a function f that is defined on $\mathbb{R}$ and continuous except for the stated discontinuities.

7. Removable discontinuity at -2, infinite discontinuity at 2

8. Jump discontinuity at -3, removable discontinuity at 4

9. Discontinuities at 0 and 3, but continuous from the right at 0 and from the left at 3

10. Continuous only from the left at -1, not continuous from the left or right at 3

11. The toll T charged for driving on a certain stretch of a toll road is \$5 except during rush hours (between 7 AM and 10 AM and between 4 PM and 7 PM) when the toll is \$7.
(a) Sketch a graph of T as a function of the time t, measured in hours past midnight.
(b) Discuss the discontinuities of this function and their significance to someone who uses the road.

12. Explain why each function is continuous or discontinuous.
(a) The temperature at a specific location as a function of time
(b) The temperature at a specific time as a function of the distance due west from New York City
(c) The altitude above sea level as a function of the distance due west from New York City
(d) The cost of a taxi ride as a function of the distance traveled
(e) The current in the circuit for the lights in a room as a function of time

13–16 Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

13. $f(x) = 3x^2 + (x + 2)^5, \quad a = -1$

14. $g(t) = \dfrac{t^2 + 5t}{2t + 1}, \quad a = 2$

15. $p(v) = 2\sqrt{3v^2 + 1}, \quad a = 1$

16. $f(r) = \sqrt[3]{4r^2 - 2r + 7}, \quad a = -2$

17–18 Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

17. $f(x) = x + \sqrt{x - 4}, \quad [4, \infty)$

18. $g(x) = \dfrac{x - 1}{3x + 6}, \quad (-\infty, -2)$

19–24 Explain why the function is discontinuous at the given number a. Sketch the graph of the function.

19. $f(x) = \dfrac{1}{x + 2} \qquad a = -2$

20. $f(x) = \begin{cases} \dfrac{1}{x + 2} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases} \qquad a = -2$

21. $f(x) = \begin{cases} x + 3 & \text{if } x \leqslant -1 \\ 2^x & \text{if } x > -1 \end{cases}$ $\qquad a = -1$

22. $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$ $\qquad a = 1$

23. $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases}$ $\qquad a = 0$

24. $f(x) = \begin{cases} \dfrac{2x^2 - 5x - 3}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$ $\qquad a = 3$

25–26

(a) Show that f has a removable discontinuity at $x = 3$.

(b) Redefine $f(3)$ so that f is continuous at $x = 3$ (and thus the discontinuity is "removed").

25. $f(x) = \dfrac{x - 3}{x^2 - 9}$ $\qquad$ **26.** $f(x) = \dfrac{x^2 - 7x + 12}{x - 3}$

27–34 Explain, using Theorems 4, 5, 7, and 9, why the function is continuous at every number in its domain. State the domain.

27. $f(x) = \dfrac{x^2}{\sqrt{x^4 + 2}}$ $\qquad$ **28.** $g(v) = \dfrac{3v - 1}{v^2 + 2v - 15}$

29. $h(t) = \dfrac{\cos(t^2)}{1 - e^t}$

30. $B(u) = \sqrt{3u - 2} + \sqrt[3]{2u - 3}$

31. $L(v) = v \ln(1 - v^2)$ $\qquad$ **32.** $f(t) = e^{-t^2} \ln(1 + t^2)$

33. $M(x) = \sqrt{1 + \dfrac{1}{x}}$ $\qquad$ **34.** $g(t) = \cos^{-1}(e^t - 1)$

35–38 Use continuity to evaluate the limit.

35. $\lim\limits_{x \to 2} x\sqrt{20 - x^2}$ $\qquad$ **36.** $\lim\limits_{\theta \to \pi/2} \sin(\tan(\cos \theta))$

37. $\lim\limits_{x \to 1} \ln\left(\dfrac{5 - x^2}{1 + x}\right)$ $\qquad$ **38.** $\lim\limits_{x \to 4} 3^{\sqrt{x^2 - 2x - 4}}$

39–40 Locate the discontinuities of the function and illustrate by graphing.

39. $f(x) = \dfrac{1}{\sqrt{1 - \sin x}}$ $\qquad$ **40.** $y = \arctan \dfrac{1}{x}$

41–42 Show that f is continuous on $(-\infty, \infty)$.

41. $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leqslant 1 \\ \ln x & \text{if } x > 1 \end{cases}$

42. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geqslant \pi/4 \end{cases}$

43–45 Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f.

43. $f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ x & \text{if } -1 \leqslant x < 1 \\ 1/x & \text{if } x \geqslant 1 \end{cases}$

44. $f(x) = \begin{cases} 2^x & \text{if } x \leqslant 1 \\ 3 - x & \text{if } 1 < x \leqslant 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$

45. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ e^x & \text{if } 0 \leqslant x \leqslant 1 \\ 2 - x & \text{if } x > 1 \end{cases}$

46. The gravitational force exerted by the planet Earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \dfrac{GMr}{R^3} & \text{if } r < R \\ \dfrac{GM}{r^2} & \text{if } r \geqslant R \end{cases}$$

where M is the mass of Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r?

47. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geqslant 2 \end{cases}$$

48. Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leqslant x < 3 \\ 2x - a + b & \text{if } x \geqslant 3 \end{cases}$$

49. Suppose f and g are continuous functions such that $g(2) = 6$ and $\lim_{x \to 2} [3f(x) + f(x)g(x)] = 36$. Find $f(2)$.

50. Let $f(x) = 1/x$ and $g(x) = 1/x^2$.

(a) Find $(f \circ g)(x)$.

(b) Is $f \circ g$ continuous everywhere? Explain.

51. Which of the following functions f has a removable discontinuity at a? If the discontinuity is removable, find a function g that agrees with f for $x \neq a$ and is continuous at a.

(a) $f(x) = \dfrac{x^4 - 1}{x - 1}, \quad a = 1$

(b) $f(x) = \dfrac{x^3 - x^2 - 2x}{x - 2}, \quad a = 2$

(c) $f(x) = [\![\sin x]\!], \quad a = \pi$

52. Suppose that a function f is continuous on $[0, 1]$ except at 0.25 and that $f(0) = 1$ and $f(1) = 3$. Let $N = 2$. Sketch two possible graphs of f, one showing that f might not satisfy the conclusion of the Intermediate Value Theorem and one showing that f might still satisfy the conclusion of the Intermediate Value Theorem (even though it doesn't satisfy the hypothesis).

53. If $f(x) = x^2 + 10 \sin x$, show that there is a number c such that $f(c) = 1000$.

54. Suppose f is continuous on $[1, 5]$ and the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. If $f(2) = 8$, explain why $f(3) > 6$.

55–58 Use the Intermediate Value Theorem to show that there is a solution of the given equation in the specified interval.

55. $-x^3 + 4x + 1 = 0, \quad (-1, 0)$

56. $\ln x = x - \sqrt{x}, \quad (2, 3)$

57. $e^x = 3 - 2x, \quad (0, 1)$

58. $\sin x = x^2 - x, \quad (1, 2)$

59–60

(a) Prove that the equation has at least one real solution.

(b) Use a calculator to find an interval of length 0.01 that contains a solution.

59. $\cos x = x^3$

60. $\ln x = 3 - 2x$

61–62

(a) Prove that the equation has at least one real solution.

(b) Find the solution correct to three decimal places, by graphing.

61. $100e^{-x/100} = 0.01x^2$

62. $\arctan x = 1 - x$

63–64 Prove, without graphing, that the graph of the function has at least two x-intercepts in the specified interval.

63. $y = \sin x^3, \quad (1, 2)$

64. $y = x^2 - 3 + 1/x, \quad (0, 2)$

65. Prove that f is continuous at a if and only if

$$\lim_{h \to 0} f(a + h) = f(a)$$

66. To prove that sine is continuous, we need to show that $\lim_{x \to a} \sin x = \sin a$ for every real number a. By Exercise 65 an equivalent statement is that

$$\lim_{h \to 0} \sin(a + h) = \sin a$$

Use (6) to show that this is true.

67. Prove that cosine is a continuous function.

68. (a) Prove Theorem 4, part 3.
(b) Prove Theorem 4, part 5.

69. Use Theorem 8 to prove Limit Laws 6 and 7 from Section 2.3.

70. Is there a number that is exactly 1 more than its cube?

71. For what values of x is f continuous?

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

72. For what values of x is g continuous?

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$$

73. Show that the function

$$f(x) = \begin{cases} x^4 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous on $(-\infty, \infty)$.

74. If a and b are positive numbers, prove that the equation

$$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0$$

has at least one solution in the interval $(-1, 1)$.

75. A woman leaves her house at 7:00 AM and takes her usual path to the top of a mountain, arriving at 7:00 PM. The following morning, she starts at 7:00 AM at the top and takes the same path back, arriving at her home at 7:00 PM. Use the Intermediate Value Theorem to show that there is a point on the path that the woman will cross at exactly the same time of day on both days.

76. Absolute Value and Continuity

(a) Show that the absolute value function $F(x) = |x|$ is continuous everywhere.

(b) Prove that if f is a continuous function on an interval, then so is $|f|$.

(c) Is the converse of the statement in part (b) also true? In other words, if $|f|$ is continuous, does it follow that f is continuous? If so, prove it. If not, find a counterexample.

2.6 Limits at Infinity; Horizontal Asymptotes

In Sections 2.2 and 2.4 we investigated infinite limits and vertical asymptotes of a curve $y = f(x)$. There we let x approach a number and the result was that the values of y became arbitrarily large (positive or negative). In this section we let x become arbitrarily large (positive or negative) and see what happens to y.

Limits at Infinity and Horizontal Asymptotes

Let's begin by investigating the behavior of the function f defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

as x becomes large. The table at the left gives values of this function correct to six decimal places, and the graph of f has been drawn by a computer in Figure 1.

x	$f(x)$
0	−1
±1	0
±2	0.600000
±3	0.800000
±4	0.882353
±5	0.923077
±10	0.980198
±50	0.999200
±100	0.999800
±1000	0.999998

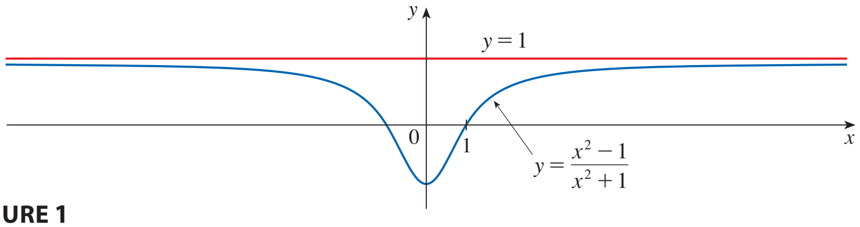


FIGURE 1

You can see that as x grows larger and larger, the values of $f(x)$ get closer and closer to 1. (The graph of f approaches the horizontal line $y = 1$ as we look to the right.) In fact, it seems that we can make the values of $f(x)$ as close as we like to 1 by taking x sufficiently large. This situation is expressed symbolically by writing

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

In general, we use the notation

$$\lim_{x \to \infty} f(x) = L$$

to indicate that the values of $f(x)$ approach L as x becomes larger and larger.

1 Intuitive Definition of a Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Another notation for $\lim_{x \to \infty} f(x) = L$ is

$$f(x) \to L \quad \text{as} \quad x \to \infty$$

The symbol ∞ does not represent a number. Nonetheless, the expression $\lim_{x\to\infty} f(x) = L$ is often read as

"the limit of $f(x)$, as x approaches infinity, is L"

or "the limit of $f(x)$, as x becomes infinite, is L"

or "the limit of $f(x)$, as x increases without bound, is L"

The meaning of such phrases is given by Definition 1. A more precise definition, similar to the ε, δ definition of Section 2.4, is given at the end of this section.

Geometric illustrations of Definition 1 are shown in Figure 2. Notice that there are many ways for the graph of f to approach the line $y = L$ (which is called a *horizontal asymptote*) as we look to the far right of each graph.

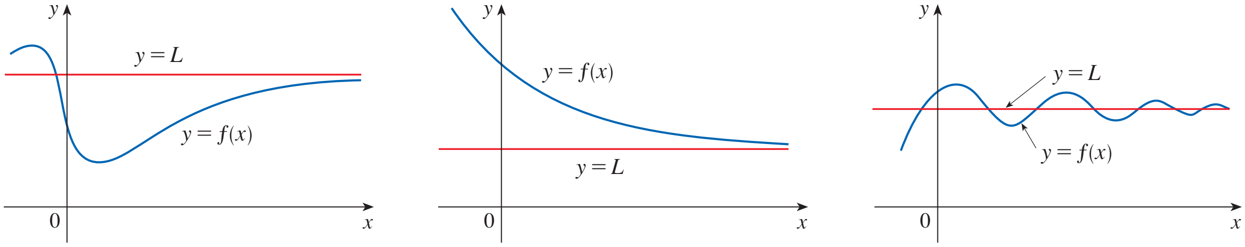


FIGURE 2 Examples illustrating $\lim_{x\to\infty} f(x) = L$

Referring back to Figure 1, we see that for numerically large negative values of x, the values of $f(x)$ are close to 1. By letting x decrease through negative values without bound, we can make $f(x)$ as close to 1 as we like. This is expressed by writing

$$\lim_{x\to-\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

The general definition is as follows.

2 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

Again, the symbol $-\infty$ does not represent a number, but the expression $\lim_{x\to-\infty} f(x) = L$ is often read as

"the limit of $f(x)$, as x approaches negative infinity, is L"

Definition 2 is illustrated in Figure 3. Notice that the graph approaches the line $y = L$ as we look to the far left of each graph.

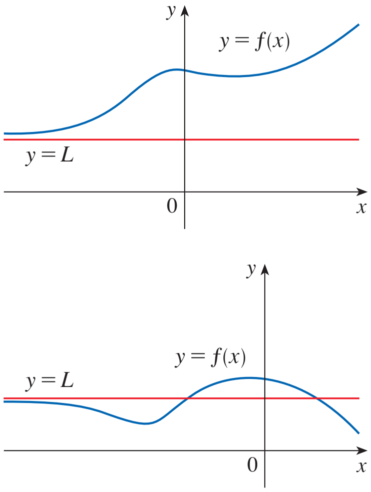


FIGURE 3
Examples illustrating $\lim_{x\to-\infty} f(x) = L$

3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L$$

For instance, the curve illustrated in Figure 1 has the line $y = 1$ as a horizontal asymptote because

$$\lim_{x\to\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

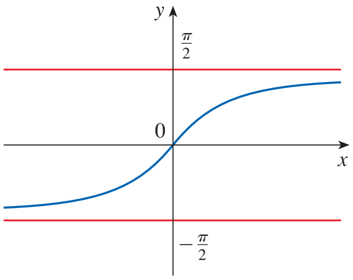


FIGURE 4

$y = \tan^{-1}x$

An example of a curve with two horizontal asymptotes is $y = \tan^{-1}x$. (See Figure 4.) In fact,

$$\boxed{\lim_{x\to-\infty} \tan^{-1}x = -\frac{\pi}{2} \qquad \lim_{x\to\infty} \tan^{-1}x = \frac{\pi}{2}} \tag{4}$$

so both of the lines $y = -\pi/2$ and $y = \pi/2$ are horizontal asymptotes. (This follows from the fact that the lines $x = \pm\pi/2$ are vertical asymptotes of the graph of the tangent function.)

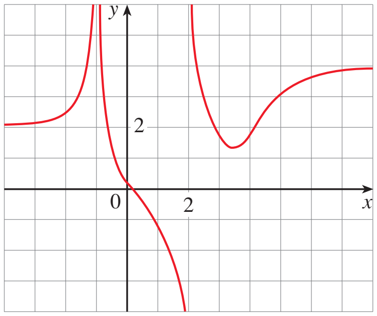


FIGURE 5

EXAMPLE 1 Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown in Figure 5.

SOLUTION We see that the values of $f(x)$ become large as $x \to -1$ from both sides, so

$$\lim_{x\to-1} f(x) = \infty$$

Notice that $f(x)$ becomes large negative as x approaches 2 from the left, but large positive as x approaches 2 from the right. So

$$\lim_{x\to2^-} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to2^+} f(x) = \infty$$

Thus both of the lines $x = -1$ and $x = 2$ are vertical asymptotes.

As x becomes large, it appears that $f(x)$ approaches 4. But as x decreases through negative values, $f(x)$ approaches 2. So

$$\lim_{x\to\infty} f(x) = 4 \qquad \text{and} \qquad \lim_{x\to-\infty} f(x) = 2$$

This means that both $y = 4$ and $y = 2$ are horizontal asymptotes. ■

EXAMPLE 2 Find $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$.

SOLUTION Observe that when x is large, $1/x$ is small. For instance,

$$\frac{1}{100} = 0.01 \qquad \frac{1}{10{,}000} = 0.0001 \qquad \frac{1}{1{,}000{,}000} = 0.000001$$

In fact, by taking x large enough, we can make $1/x$ as close to 0 as we please. Therefore, according to Definition 1, we have

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Similar reasoning shows that when x is large negative, $1/x$ is small negative, so we also have

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$

It follows that the line $y = 0$ (the x-axis) is a horizontal asymptote of the curve $y = 1/x$. (This is a hyperbola; see Figure 6.) ■

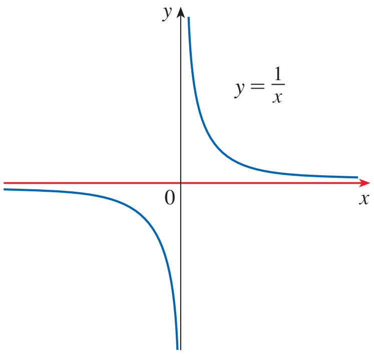


FIGURE 6

$\lim_{x\to\infty} \frac{1}{x} = 0, \quad \lim_{x\to-\infty} \frac{1}{x} = 0$

Evaluating Limits at Infinity

Most of the Limit Laws that were given in Section 2.3 also hold for limits at infinity. It can be proved that *the Limit Laws listed in Section 2.3 (with the exception of Laws 10 and 11) are also valid if "$x \to a$" is replaced by "$x \to \infty$" or "$x \to -\infty$."* In particular, if we combine Laws 6 and 7 with the results of Example 2, we obtain the following important rule for calculating limits.

5 Theorem If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$

EXAMPLE 3 Evaluate the following limit and indicate which properties of limits are used at each stage.

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

SOLUTION As x becomes large, both numerator and denominator become large, so it isn't obvious what happens to their ratio. We need to do some preliminary algebra.

To evaluate the limit at infinity of any rational function, we first divide both the numerator and denominator by the highest power of x that occurs in the denominator. (We may assume that $x \neq 0$, since we are interested only in large values of x.) In this case the highest power of x in the denominator is x^2, so we have

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{\dfrac{3x^2 - x - 2}{x^2}}{\dfrac{5x^2 + 4x + 1}{x^2}} = \lim_{x\to\infty} \frac{3 - \dfrac{1}{x} - \dfrac{2}{x^2}}{5 + \dfrac{4}{x} + \dfrac{1}{x^2}}$$

$$= \frac{\lim\limits_{x\to\infty}\left(3 - \dfrac{1}{x} - \dfrac{2}{x^2}\right)}{\lim\limits_{x\to\infty}\left(5 + \dfrac{4}{x} + \dfrac{1}{x^2}\right)} \qquad \text{(by Limit Law 5)}$$

$$= \frac{\lim\limits_{x\to\infty} 3 - \lim\limits_{x\to\infty} \dfrac{1}{x} - 2\lim\limits_{x\to\infty} \dfrac{1}{x^2}}{\lim\limits_{x\to\infty} 5 + 4\lim\limits_{x\to\infty} \dfrac{1}{x} + \lim\limits_{x\to\infty} \dfrac{1}{x^2}} \qquad \text{(by 1, 2, and 3)}$$

$$= \frac{3 - 0 - 0}{5 + 0 + 0} \qquad \text{(by 8 and Theorem 5)}$$

$$= \frac{3}{5}$$

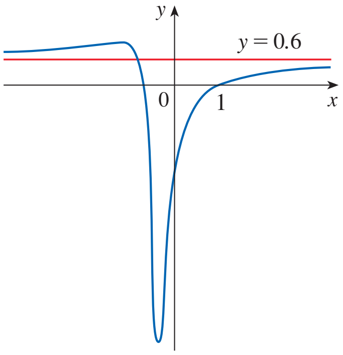


FIGURE 7

$y = \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$

A similar calculation shows that the limit as $x \to -\infty$ is also $\frac{3}{5}$. Figure 7 illustrates the results of these calculations by showing how the graph of the given rational function approaches the horizontal asymptote $y = \frac{3}{5} = 0.6$. ■

EXAMPLE 4 Find the horizontal asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

SOLUTION Dividing both numerator and denominator by x (which is the highest power of x in the denominator) and using the properties of limits, we have

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\dfrac{\sqrt{2x^2+1}}{x}}{\dfrac{3x-5}{x}} = \lim_{x\to\infty} \frac{\sqrt{\dfrac{2x^2+1}{x^2}}}{\dfrac{3x-5}{x}} \qquad \left(\text{since } \sqrt{x^2} = x \text{ for } x > 0\right)$$

$$= \frac{\lim\limits_{x\to\infty} \sqrt{2 + \dfrac{1}{x^2}}}{\lim\limits_{x\to\infty} \left(3 - \dfrac{5}{x}\right)} = \frac{\sqrt{\lim\limits_{x\to\infty} 2 + \lim\limits_{x\to\infty} \dfrac{1}{x^2}}}{\lim\limits_{x\to\infty} 3 - 5 \lim\limits_{x\to\infty} \dfrac{1}{x}} = \frac{\sqrt{2+0}}{3 - 5 \cdot 0} = \frac{\sqrt{2}}{3}$$

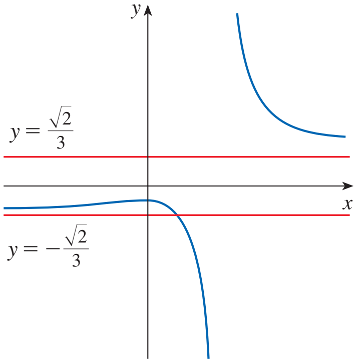


FIGURE 8

$y = \dfrac{\sqrt{2x^2+1}}{3x-5}$

Therefore the line $y = \sqrt{2}/3$ is a horizontal asymptote of the graph of f.

In computing the limit as $x \to -\infty$, we must remember that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$. So when we divide the numerator by x, for $x < 0$ we get

$$\frac{\sqrt{2x^2+1}}{x} = \frac{\sqrt{2x^2+1}}{-\sqrt{x^2}} = -\sqrt{\frac{2x^2+1}{x^2}} = -\sqrt{2 + \frac{1}{x^2}}$$

Therefore

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{-\sqrt{2 + \dfrac{1}{x^2}}}{3 - \dfrac{5}{x}} = \frac{-\sqrt{2 + \lim\limits_{x\to-\infty} \dfrac{1}{x^2}}}{3 - 5 \lim\limits_{x\to-\infty} \dfrac{1}{x}} = -\frac{\sqrt{2}}{3}$$

Thus the line $y = -\sqrt{2}/3$ is also a horizontal asymptote. See Figure 8. ■

EXAMPLE 5 Compute $\lim\limits_{x\to\infty} \left(\sqrt{x^2+1} - x\right)$.

We can think of the given function as having a denominator of 1.

SOLUTION Because both $\sqrt{x^2+1}$ and x are large when x is large, it's difficult to see what happens to their difference, so we use algebra to rewrite the function. We first multiply numerator and denominator by the conjugate radical:

$$\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) = \lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) \cdot \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x}$$

$$= \lim_{x\to\infty} \frac{(x^2+1) - x^2}{\sqrt{x^2+1} + x} = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1} + x}$$

Notice that the denominator of this last expression $\left(\sqrt{x^2+1} + x\right)$ becomes large as $x \to \infty$ (it's bigger than x). So

$$\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1} + x} = 0$$

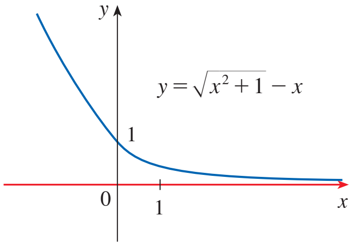


FIGURE 9

Figure 9 illustrates this result. ■

EXAMPLE 6 Evaluate $\displaystyle\lim_{x\to 2^+} \arctan\left(\frac{1}{x-2}\right)$.

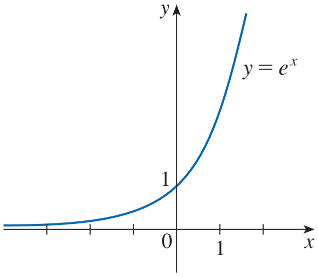
SOLUTION If we let $t = 1/(x-2)$, we know that $t \to \infty$ as $x \to 2^+$. Therefore, by the second equation in (4), we have

$$\lim_{x\to 2^+} \arctan\left(\frac{1}{x-2}\right) = \lim_{t\to\infty} \arctan t = \frac{\pi}{2}$$ ■

The graph of the natural exponential function $y = e^x$ has the line $y = 0$ (the x-axis) as a horizontal asymptote. (The same is true of any exponential function with base $b > 1$.) In fact, from the graph in Figure 10 and the corresponding table of values, we see that

6
$$\boxed{\lim_{x\to -\infty} e^x = 0}$$

Notice that the values of e^x approach 0 very rapidly.



x	e^x
0	1.00000
−1	0.36788
−2	0.13534
−3	0.04979
−5	0.00674
−8	0.00034
−10	0.00005

FIGURE 10

EXAMPLE 7 Evaluate $\displaystyle\lim_{x\to 0^-} e^{1/x}$.

SOLUTION If we let $t = 1/x$, we know that $t \to -\infty$ as $x \to 0^-$. Therefore, by (6),

$$\lim_{x\to 0^-} e^{1/x} = \lim_{t\to -\infty} e^t = 0$$

(See Exercise 81.) ■

PS The problem-solving strategy for Examples 6 and 7 is *introducing something extra* (see Principles of Problem Solving following Chapter 1). Here, the something extra, the auxiliary aid, is the new variable t.

EXAMPLE 8 Evaluate $\displaystyle\lim_{x\to\infty} \sin x$.

SOLUTION As x increases, the values of $\sin x$ oscillate between 1 and -1 infinitely often and so they don't approach any definite number. Thus $\lim_{x\to\infty} \sin x$ does not exist. ■

■ Infinite Limits at Infinity

The notation

$$\lim_{x\to\infty} f(x) = \infty$$

is used to indicate that the values of $f(x)$ become large as x becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x\to -\infty} f(x) = \infty \qquad \lim_{x\to\infty} f(x) = -\infty \qquad \lim_{x\to -\infty} f(x) = -\infty$$

EXAMPLE 9 Find $\lim_{x\to\infty} x^3$ and $\lim_{x\to-\infty} x^3$.

SOLUTION When x becomes large, x^3 also becomes large. For instance,

$$10^3 = 1000 \qquad 100^3 = 1{,}000{,}000 \qquad 1000^3 = 1{,}000{,}000{,}000$$

In fact, we can make x^3 as big as we like by requiring x to be large enough. Therefore we can write

$$\lim_{x\to\infty} x^3 = \infty$$

Similarly, when x is large negative, so is x^3. Thus

$$\lim_{x\to-\infty} x^3 = -\infty$$

These limit statements can also be seen from the graph of $y = x^3$ in Figure 11. ■

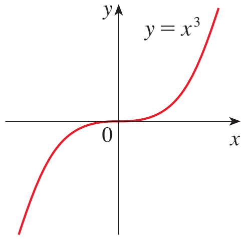


FIGURE 11
$\lim_{x\to\infty} x^3 = \infty$, $\lim_{x\to-\infty} x^3 = -\infty$

Looking at Figure 10 we see that

$$\lim_{x\to\infty} e^x = \infty$$

but, as Figure 12 demonstrates, $y = e^x$ becomes large as $x \to \infty$ at a much faster rate than $y = x^3$.

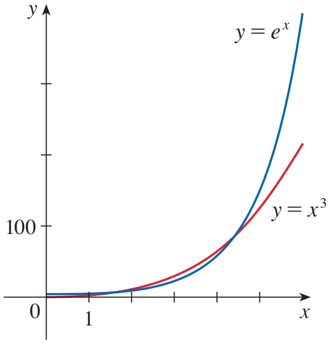


FIGURE 12
e^x is much larger than x^3 when x is large.

EXAMPLE 10 Find $\lim_{x\to\infty} (x^2 - x)$.

SOLUTION Limit Law 2 says that the limit of a difference is the difference of the limits, provided that these limits exist. We cannot use Law 2 here because

$$\lim_{x\to\infty} x^2 = \infty \qquad \text{and} \qquad \lim_{x\to\infty} x = \infty$$

In general, the Limit Laws can't be applied to infinite limits because ∞ is not a number ($\infty - \infty$ can't be defined). However, we *can* write

$$\lim_{x\to\infty} (x^2 - x) = \lim_{x\to\infty} x(x - 1) = \infty$$

because both x and $x - 1$ become arbitrarily large and so their product does too. ■

EXAMPLE 11 Find $\lim_{x\to\infty} \dfrac{x^2 + x}{3 - x}$.

SOLUTION As in Example 3, we divide the numerator and denominator by the highest power of x in the denominator, which is simply x:

$$\lim_{x\to\infty} \frac{x^2 + x}{3 - x} = \lim_{x\to\infty} \frac{x + 1}{\dfrac{3}{x} - 1} = -\infty$$

because $x + 1 \to \infty$ and $3/x - 1 \to 0 - 1 = -1$ as $x \to \infty$. ■

The next example shows that by using infinite limits at infinity, together with intercepts, we can get a rough idea of the graph of a polynomial without having to plot a large number of points.

EXAMPLE 12 Sketch the graph of $y = (x - 2)^4(x + 1)^3(x - 1)$ by finding its intercepts and its limits as $x \to \infty$ and as $x \to -\infty$.

SOLUTION The y-intercept is $f(0) = (-2)^4(1)^3(-1) = -16$ and the x-intercepts are found by setting $y = 0$: $x = 2, -1, 1$. Notice that since $(x - 2)^4$ is never negative, the function doesn't change sign at 2; thus the graph doesn't cross the x-axis at 2. The graph crosses the axis at -1 and 1.

When x is large positive, all three factors are large, so

$$\lim_{x \to \infty} (x - 2)^4(x + 1)^3(x - 1) = \infty$$

When x is large negative, the first factor is large positive and the second and third factors are both large negative, so

$$\lim_{x \to -\infty} (x - 2)^4(x + 1)^3(x - 1) = \infty$$

Combining this information, we give a rough sketch of the graph in Figure 13. ■

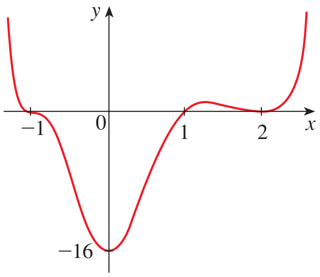


FIGURE 13
$y = (x - 2)^4(x + 1)^3(x - 1)$

Precise Definitions

Definition 1 can be stated precisely as follows.

$\boxed{7}$ **Precise Definition of a Limit at Infinity** Let f be a function defined onsome interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \qquad x > N \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

In words, this says that the values of $f(x)$ can be made arbitrarily close to L (within a distance ε, where ε is any positive number) by requiring x to be sufficiently large (larger than N, where N depends on ε). Graphically, it says that by keeping x large enough (larger than some number N) we can make the graph of f lie between the given horizontal lines $y = L - \varepsilon$ and $y = L + \varepsilon$ as in Figure 14. This must be true no matter how small we choose ε.

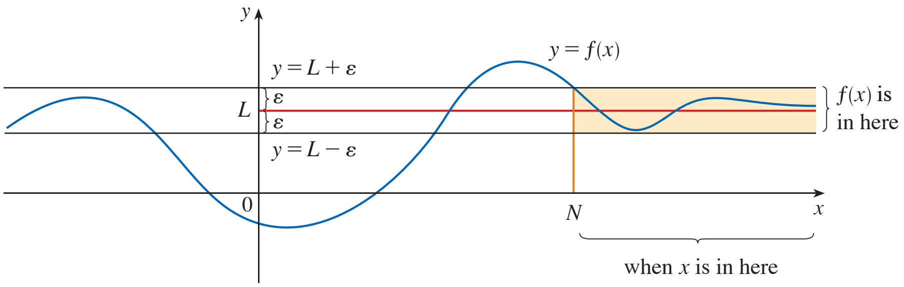


FIGURE 14
$\lim_{x \to \infty} f(x) = L$

Figure 15 shows that if a smaller value of ε is chosen, then a larger value of N may be required.

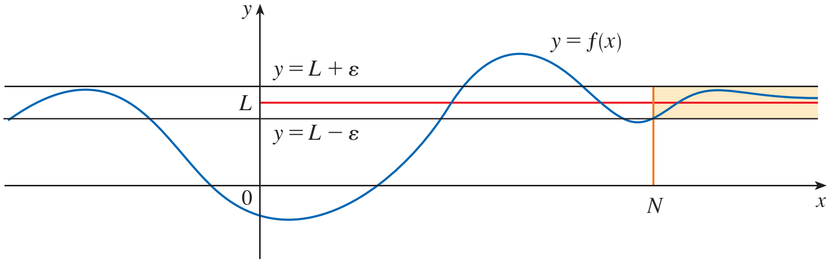


FIGURE 15
$\lim_{x \to \infty} f(x) = L$

Similarly, a precise version of Definition 2 is given by Definition 8, which is illustrated in Figure 16.

8 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x < N \quad \text{then} \quad |f(x) - L| < \varepsilon$$

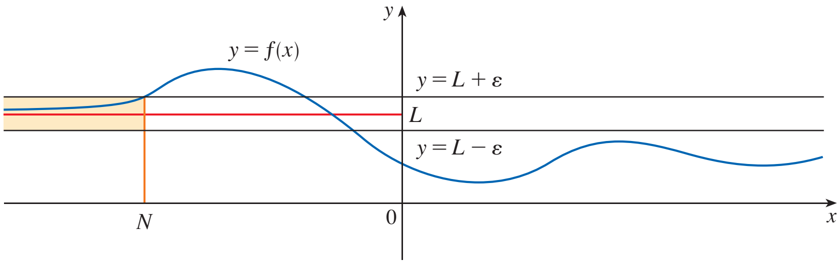


FIGURE 16
$\lim_{x \to -\infty} f(x) = L$

In Example 3 we calculated that

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{3}{5}$$

In the next example we use a calculator (or computer) to relate this statement to Definition 7 with $L = \frac{3}{5} = 0.6$ and $\varepsilon = 0.1$.

EXAMPLE 13 Use a graph to find a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad \left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1$$

SOLUTION We rewrite the given inequality as

$$0.5 < \frac{3x^2 - x - 2}{5x^2 + 4x + 1} < 0.7$$

We need to determine the values of x for which the given curve lies between the horizontal lines $y = 0.5$ and $y = 0.7$. So we graph the curve and these lines in Figure 17. Then we use the graph to estimate that the curve crosses the line $y = 0.5$ when $x \approx 6.7$. To the right of this number it seems that the curve stays between the lines $y = 0.5$ and $y = 0.7$. Rounding up to be safe, we can say that

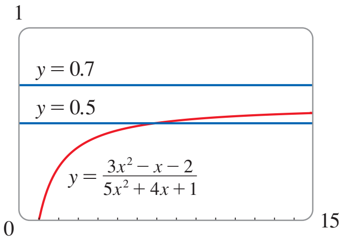


FIGURE 17

$$\text{if} \quad x > 7 \quad \text{then} \quad \left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1$$

In other words, for $\varepsilon = 0.1$ we can choose $N = 7$ (or any larger number) in Definition 7. ■

EXAMPLE 14 Use Definition 7 to prove that $\lim_{x \to \infty} \frac{1}{x} = 0$.

SOLUTION Given $\varepsilon > 0$, we want to find N such that

$$\text{if} \quad x > N \quad \text{then} \quad \left| \frac{1}{x} - 0 \right| < \varepsilon$$

In computing the limit we may assume that $x > 0$. Then $1/x < \varepsilon \iff x > 1/\varepsilon$. Let's choose $N = 1/\varepsilon$. So

$$\text{if} \quad x > N = \frac{1}{\varepsilon} \quad \text{then} \quad \left| \frac{1}{x} - 0 \right| = \frac{1}{x} < \varepsilon$$

Therefore, by Definition 7,

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Figure 18 illustrates the proof by showing some values of ε and the corresponding values of N.

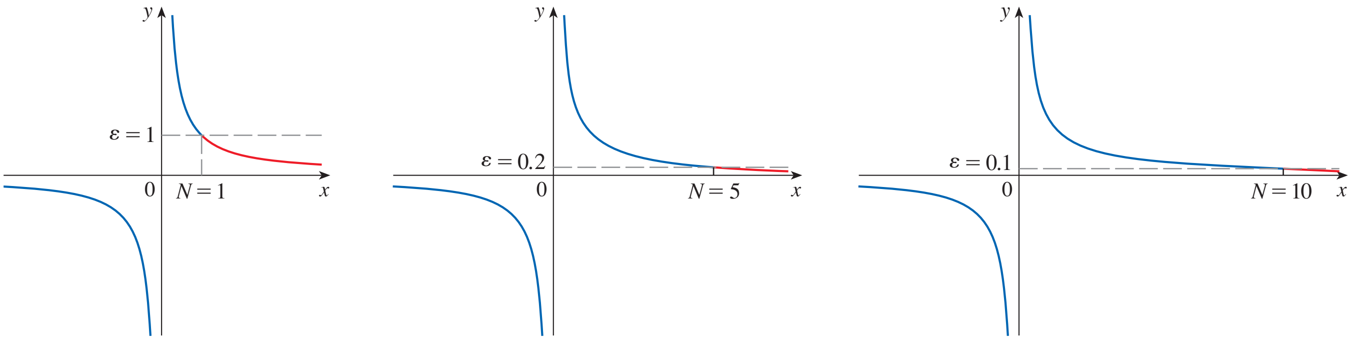


FIGURE 18 ■

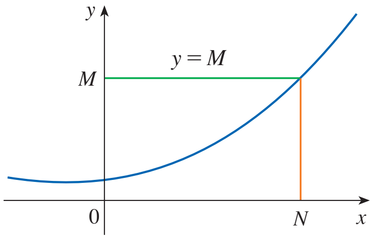


FIGURE 19
$\lim_{x\to\infty} f(x) = \infty$

Finally we note that an infinite limit at infinity can be defined as follows. The geometric illustration is given in Figure 19.

9 **Precise Definition of an Infinite Limit at Infinity** Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \qquad \text{then} \qquad f(x) > M$$

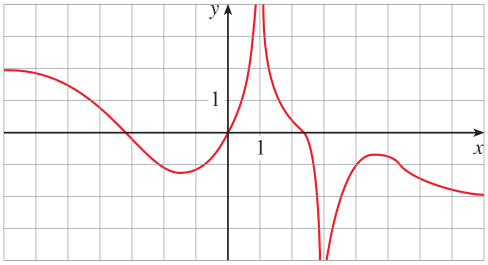
Similar definitions apply when the symbol ∞ is replaced by $-\infty$. (See Exercise 80.)

2.6 Exercises

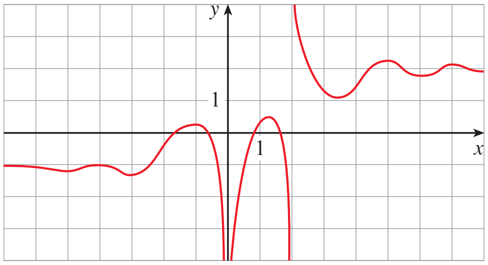
1. Explain in your own words the meaning of each of the following.
(a) $\lim_{x\to\infty} f(x) = 5$ (b) $\lim_{x\to-\infty} f(x) = 3$

2. (a) Can the graph of $y = f(x)$ intersect a vertical asymptote? Can it intersect a horizontal asymptote? Illustrate by sketching graphs.
(b) How many horizontal asymptotes can the graph of $y = f(x)$ have? Sketch graphs to illustrate the possibilities.

3. For the function f whose graph is given, state the following.
(a) $\lim_{x\to\infty} f(x)$ (b) $\lim_{x\to-\infty} f(x)$
(c) $\lim_{x\to 1} f(x)$ (d) $\lim_{x\to 3} f(x)$
(e) The equations of the asymptotes



4. For the function g whose graph is given, state the following.
(a) $\lim_{x\to\infty} g(x)$ (b) $\lim_{x\to-\infty} g(x)$
(c) $\lim_{x\to 0} g(x)$ (d) $\lim_{x\to 2^-} g(x)$
(e) $\lim_{x\to 2^+} g(x)$ (f) The equations of the asymptotes



5–10 Sketch the graph of an example of a function f that satisfies all of the given conditions.

5. $f(2) = 4, \quad f(-2) = -4, \quad \lim_{x\to-\infty} f(x) = 0, \quad \lim_{x\to\infty} f(x) = 2$

6. $f(0) = 0, \quad \lim_{x\to 1^-} f(x) = \infty, \quad \lim_{x\to 1^+} f(x) = -\infty,$
$\lim_{x\to-\infty} f(x) = -2, \quad \lim_{x\to\infty} f(x) = -2$

7. $\lim_{x\to 0} f(x) = \infty, \quad \lim_{x\to 3^-} f(x) = -\infty, \quad \lim_{x\to 3^+} f(x) = \infty,$
$\lim_{x\to-\infty} f(x) = 1, \quad \lim_{x\to\infty} f(x) = -1$

8. $\lim_{x\to-\infty} f(x) = -\infty, \quad \lim_{x\to-2^-} f(x) = \infty, \quad \lim_{x\to-2^+} f(x) = -\infty,$
$\lim_{x\to 2} f(x) = \infty, \quad \lim_{x\to\infty} f(x) = \infty$

9. $f(0) = 0, \quad \lim_{x\to 1} f(x) = -\infty, \quad \lim_{x\to\infty} f(x) = -\infty, \quad f$ is odd

10. $\lim_{x\to-\infty} f(x) = -1, \quad \lim_{x\to 0^-} f(x) = \infty, \quad \lim_{x\to 0^+} f(x) = -\infty,$
$\lim_{x\to 3^-} f(x) = 1, \quad f(3) = 4, \quad \lim_{x\to 3^+} f(x) = 4, \quad \lim_{x\to\infty} f(x) = 1$

11. Guess the value of the limit

$$\lim_{x\to\infty}\frac{x^2}{2^x}$$

by evaluating the function $f(x) = x^2/2^x$ for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50$, and 100. Then use a graph of f to support your guess.

12. (a) Use a graph of

$$f(x) = \left(1 - \frac{2}{x}\right)^x$$

to estimate the value of $\lim_{x\to\infty} f(x)$ correct to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

13–14 Evaluate the limit and justify each step by indicating the appropriate properties of limits.

13. $\displaystyle\lim_{x\to\infty}\frac{2x^2 - 7}{5x^2 + x - 3}$

14. $\displaystyle\lim_{x\to\infty}\sqrt{\frac{9x^3 + 8x - 4}{3 - 5x + x^3}}$

15–42 Find the limit or show that it does not exist.

15. $\displaystyle\lim_{x\to\infty}\frac{4x + 3}{5x - 1}$

16. $\displaystyle\lim_{x\to\infty}\frac{-2}{3x + 7}$

17. $\displaystyle\lim_{t\to-\infty}\frac{3t^2 + t}{t^3 - 4t + 1}$

18. $\displaystyle\lim_{t\to-\infty}\frac{6t^2 + t - 5}{9 - 2t^2}$

19. $\displaystyle\lim_{r\to\infty}\frac{r - r^3}{2 - r^2 + 3r^3}$

20. $\displaystyle\lim_{x\to\infty}\frac{3x^3 - 8x + 2}{4x^3 - 5x^2 - 2}$

21. $\displaystyle\lim_{x\to\infty}\frac{4 - \sqrt{x}}{2 + \sqrt{x}}$

22. $\displaystyle\lim_{u\to-\infty}\frac{(u^2 + 1)(2u^2 - 1)}{(u^2 + 2)^2}$

23. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x + 3x^2}}{4x - 1}$

24. $\displaystyle\lim_{t\to\infty}\frac{t + 3}{\sqrt{2t^2 - 1}}$

25. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{1 + 4x^6}}{2 - x^3}$

26. $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{1 + 4x^6}}{2 - x^3}$

27. $\displaystyle\lim_{x\to-\infty}\frac{2x^5 - x}{x^4 + 3}$

28. $\displaystyle\lim_{q\to\infty}\frac{q^3 + 6q - 4}{4q^2 - 3q + 3}$

29. $\displaystyle\lim_{t\to\infty}\left(\sqrt{25t^2 + 2} - 5t\right)$

30. $\displaystyle\lim_{x\to-\infty}\left(\sqrt{4x^2 + 3x} + 2x\right)$

31. $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^2 + ax} - \sqrt{x^2 + bx}\right)$

32. $\displaystyle\lim_{x\to\infty}\left(x - \sqrt{x}\right)$

33. $\displaystyle\lim_{x\to-\infty}(x^2 + 2x^7)$

34. $\displaystyle\lim_{x\to\infty}(e^{-x} + 2\cos 3x)$

35. $\displaystyle\lim_{x\to\infty}(e^{-2x}\cos x)$

36. $\displaystyle\lim_{x\to\infty}\frac{\sin^2 x}{x^2 + 1}$

37. $\displaystyle\lim_{x\to\infty}\frac{1 - e^x}{1 + 2e^x}$

38. $\displaystyle\lim_{x\to\infty}\frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

39. $\displaystyle\lim_{x\to(\pi/2)^+} e^{\sec x}$

40. $\displaystyle\lim_{x\to 0^+}\tan^{-1}(\ln x)$

41. $\displaystyle\lim_{x\to\infty}\left[\ln(1 + x^2) - \ln(1 + x)\right]$

42. $\displaystyle\lim_{x\to\infty}\left[\ln(2 + x) - \ln(1 + x)\right]$

43. (a) For $f(x) = \dfrac{x}{\ln x}$ find each of the following limits.

(i) $\displaystyle\lim_{x\to 0^+} f(x)$ (ii) $\displaystyle\lim_{x\to 1^-} f(x)$ (iii) $\displaystyle\lim_{x\to 1^+} f(x)$

(b) Use a table of values to estimate $\displaystyle\lim_{x\to\infty} f(x)$.

(c) Use the information from parts (a) and (b) to make a rough sketch of the graph of f.

44. (a) For $f(x) = \dfrac{2}{x} - \dfrac{1}{\ln x}$ find each of the following limits.

(i) $\displaystyle\lim_{x\to\infty} f(x)$ (ii) $\displaystyle\lim_{x\to 0^+} f(x)$

(iii) $\displaystyle\lim_{x\to 1^-} f(x)$ (iv) $\displaystyle\lim_{x\to 1^+} f(x)$

(b) Use the information from part (a) to make a rough sketch of the graph of f.

45. (a) Estimate the value of

$$\lim_{x\to-\infty}\left(\sqrt{x^2 + x + 1} + x\right)$$

by graphing the function $f(x) = \sqrt{x^2 + x + 1} + x$.

(b) Use a table of values of $f(x)$ to guess the value of the limit.

(c) Prove that your guess is correct.

46. (a) Use a graph of

$$f(x) = \sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}$$

to estimate the value of $\lim_{x\to\infty} f(x)$ to one decimal place.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Find the exact value of the limit.

47–52 Find the horizontal and vertical asymptotes of each curve. You may want to use a graphing calculator (or computer) to check your work by graphing the curve and estimating the asymptotes.

47. $y = \dfrac{5 + 4x}{x + 3}$

48. $y = \dfrac{2x^2 + 1}{3x^2 + 2x - 1}$

49. $y = \dfrac{2x^2 + x - 1}{x^2 + x - 2}$

50. $y = \dfrac{1 + x^4}{x^2 - x^4}$

51. $y = \dfrac{x^3 - x}{x^2 - 6x + 5}$

52. $y = \dfrac{2e^x}{e^x - 5}$

53. Estimate the horizontal asymptote of the function

$$f(x) = \frac{3x^3 + 500x^2}{x^3 + 500x^2 + 100x + 2000}$$

by graphing f for $-10 \leqslant x \leqslant 10$. Then calculate the equation of the asymptote by evaluating the limit. How do you explain the discrepancy?

54. (a) Graph the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

How many horizontal and vertical asymptotes do you observe? Use the graph to estimate the values of the limits

$$\lim_{x \to \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} \qquad \text{and} \qquad \lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

(b) By calculating values of $f(x)$, give numerical estimates of the limits in part (a).

(c) Calculate the exact values of the limits in part (a). Did you get the same value or different values for these two limits? [In view of your answer to part (a), you might have to check your calculation for the second limit.]

55. Let P and Q be polynomials. Find

$$\lim_{x \to \infty} \frac{P(x)}{Q(x)}$$

if the degree of P is (a) less than the degree of Q and (b) greater than the degree of Q.

56. Make a rough sketch of the curve $y = x^n$ (n an integer) for the following five cases:

(i) $n = 0$ (ii) $n > 0$, n odd
(iii) $n > 0$, n even (iv) $n < 0$, n odd
(v) $n < 0$, n even

Then use these sketches to find the following limits.

(a) $\lim_{x \to 0^+} x^n$ (b) $\lim_{x \to 0^-} x^n$
(c) $\lim_{x \to \infty} x^n$ (d) $\lim_{x \to -\infty} x^n$

57. Find a formula for a function f that satisfies the following conditions:

$$\lim_{x \to \pm\infty} f(x) = 0, \quad \lim_{x \to 0} f(x) = -\infty, \quad f(2) = 0,$$
$$\lim_{x \to 3^-} f(x) = \infty, \quad \lim_{x \to 3^+} f(x) = -\infty$$

58. Find a formula for a function that has vertical asymptotes $x = 1$ and $x = 3$ and horizontal asymptote $y = 1$.

59. A function f is a ratio of quadratic functions and has a vertical asymptote $x = 4$ and just one x-intercept, $x = 1$. It is known that f has a removable discontinuity at $x = -1$ and $\lim_{x \to -1} f(x) = 2$. Evaluate

(a) $f(0)$ (b) $\lim_{x \to \infty} f(x)$

60–64 Find the limits as $x \to \infty$ and as $x \to -\infty$. Use this information, together with intercepts, to give a rough sketch of the graph as in Example 12.

60. $y = 2x^3 - x^4$ **61.** $y = x^4 - x^6$

62. $y = x^3(x + 2)^2(x - 1)$

63. $y = (3 - x)(1 + x)^2(1 - x)^4$

64. $y = x^2(x^2 - 1)^2(x + 2)$

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65. (a) Use the Squeeze Theorem to evaluate $\lim_{x \to \infty} \dfrac{\sin x}{x}$.

(b) Graph $f(x) = (\sin x)/x$. How many times does the graph cross the asymptote?

66. End Behavior of a Function By the *end behavior* of a function we mean the behavior of its values as $x \to \infty$ and as $x \to -\infty$.

(a) Describe and compare the end behavior of the functions

$$P(x) = 3x^5 - 5x^3 + 2x \qquad Q(x) = 3x^5$$

by graphing both functions in the viewing rectangles $[-2, 2]$ by $[-2, 2]$ and $[-10, 10]$ by $[-10{,}000, 10{,}000]$.

(b) Two functions are said to have the *same end behavior* if their ratio approaches 1 as $x \to \infty$. Show that P and Q have the same end behavior.

67. Find $\lim_{x \to \infty} f(x)$ if, for all $x > 1$,

$$\frac{10e^x - 21}{2e^x} < f(x) < \frac{5\sqrt{x}}{\sqrt{x - 1}}$$

68. (a) A tank contains 5000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min. Show that the concentration of salt after t minutes (in grams per liter) is

$$C(t) = \frac{30t}{200 + t}$$

(b) What happens to the concentration as $t \to \infty$?

69. In Chapter 9 we will be able to show, under certain assumptions, that the velocity $v(t)$ of a falling raindrop at time t is

$$v(t) = v^*(1 - e^{-gt/v^*})$$

where g is the acceleration due to gravity and v^* is the *terminal velocity* of the raindrop.

(a) Find $\lim_{t \to \infty} v(t)$.

(b) Graph $v(t)$ if $v^* = 1$ m/s and $g = 9.8$ m/s^2. How long does it take for the velocity of the raindrop to reach 99% of its terminal velocity?

70. (a) By graphing $y = e^{-x/10}$ and $y = 0.1$ on a common screen, discover how large you need to make x so that $e^{-x/10} < 0.1$.

(b) Can you solve part (a) without using a graph?

71. Use a graph to find a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad \left| \frac{3x^2 + 1}{2x^2 + x + 1} - 1.5 \right| < 0.05$$

72. For the limit

$$\lim_{x \to \infty} \frac{1 - 3x}{\sqrt{x^2 + 1}} = -3$$

illustrate Definition 7 by finding values of N that correspond to $\varepsilon = 0.1$ and $\varepsilon = 0.05$.

73. For the limit

$$\lim_{x \to -\infty} \frac{1 - 3x}{\sqrt{x^2 + 1}} = 3$$

illustrate Definition 8 by finding values of N that correspond to $\varepsilon = 0.1$ and $\varepsilon = 0.05$.

74. For the limit

$$\lim_{x \to \infty} \sqrt{x \ln x} = \infty$$

illustrate Definition 9 by finding a value of N that corresponds to $M = 100$.

75. (a) How large do we have to take x so that $1/x^2 < 0.0001$?

(b) Taking $r = 2$ in Theorem 5, we have the statement

$$\lim_{x \to \infty} \frac{1}{x^2} = 0$$

Prove this directly using Definition 7.

76. (a) How large do we have to take x so that $1/\sqrt{x} < 0.0001$?

(b) Taking $r = \frac{1}{2}$ in Theorem 5, we have the statement

$$\lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0$$

Prove this directly using Definition 7.

77. Use Definition 8 to prove that $\displaystyle\lim_{x \to -\infty} \frac{1}{x} = 0$.

78. Prove, using Definition 9, that $\displaystyle\lim_{x \to \infty} x^3 = \infty$.

79. Use Definition 9 to prove that $\displaystyle\lim_{x \to \infty} e^x = \infty$.

80. Formulate a precise definition of

$$\lim_{x \to -\infty} f(x) = -\infty$$

Then use your definition to prove that

$$\lim_{x \to -\infty} (1 + x^3) = -\infty$$

81. (a) Prove that

$$\lim_{x \to \infty} f(x) = \lim_{t \to 0^+} f(1/t)$$

and

$$\lim_{x \to -\infty} f(x) = \lim_{t \to 0^-} f(1/t)$$

assuming that these limits exist.

(b) Use part (a) and Exercise 65 to find

$$\lim_{x \to 0^+} x \sin \frac{1}{x}$$

2.7 Derivatives and Rates of Change

Now that we have defined limits and have learned techniques for computing them, we revisit the problems of finding tangent lines and velocities from Section 2.1. The special type of limit that occurs in both of these problems is called a *derivative* and we will see that it can be interpreted as a rate of change in any of the natural or social sciences or engineering.

Tangents

If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider (as we did in Section 2.1) a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Then we let Q approach P along the curve C by letting x approach a. If m_{PQ} approaches a number m, then we define the *tangent line* ℓ to be the line through P with slope m. (This amounts to saying that the tangent line is the limiting position of the secant line PQ as Q approaches P. See Figure 1.)

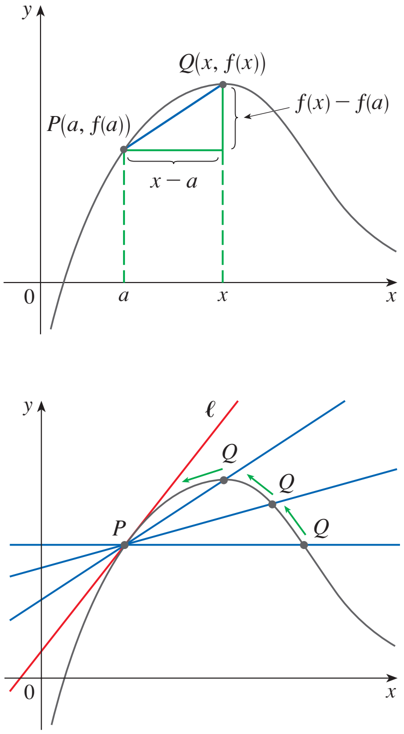


FIGURE 1

1 Definition The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

In our first example we confirm the guess we made in Example 2.1.1.

EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

SOLUTION Here we have $a = 1$ and $f(x) = x^2$, so the slope is

$$\begin{aligned} m &= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1} \\ &= \lim_{x \to 1} (x + 1) = 1 + 1 = 2 \end{aligned}$$

Point-slope form for a line through the point (x_1, y_1) with slope m:

$$y - y_1 = m(x - x_1)$$

Using the point-slope form of the equation of a line, we find that an equation of the tangent line at $(1, 1)$ is

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$ ■

We sometimes refer to the slope of the tangent line to a curve at a point as the **slope of the curve** at the point. The idea is that if we zoom in far enough toward the point, the curve looks almost like a straight line. Figure 2 illustrates this procedure for the curve $y = x^2$ in Example 1. The more we zoom in, the more the parabola looks like a line. In other words, the curve becomes almost indistinguishable from its tangent line.

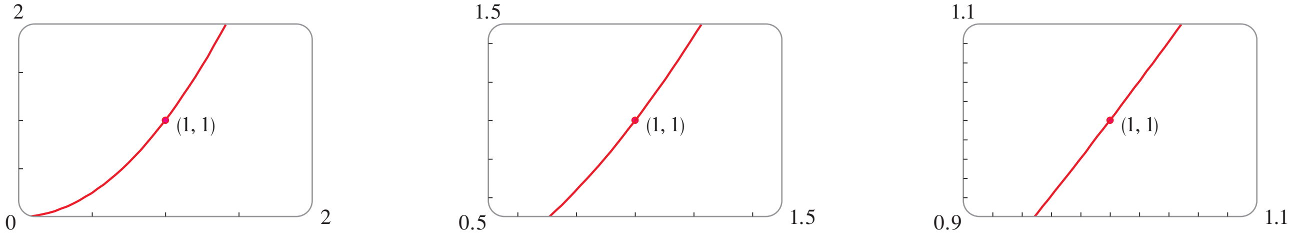


FIGURE 2
Zooming in toward the point $(1, 1)$ on the parabola $y = x^2$

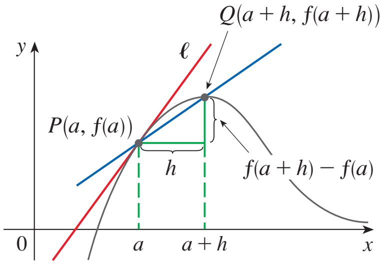


FIGURE 3

There is another expression for the slope of a tangent line that is sometimes easier to use. If $h = x - a$, then $x = a + h$ and so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$

(See Figure 3 where the case $h > 0$ is illustrated and Q is located to the right of P. If it happened that $h < 0$, however, Q would be to the left of P.)

Notice that as x approaches a, h approaches 0 (because $h = x - a$) and so the expression for the slope of the tangent line in Definition 1 becomes

$$\boxed{2} \qquad \boxed{m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

EXAMPLE 2 Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point (3, 1).

SOLUTION Let $f(x) = 3/x$. Then, by Equation 2, the slope of the tangent at (3, 1) is

$$\begin{aligned} m &= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h \to 0} \frac{\dfrac{3}{3+h} - 1}{h} = \lim_{h \to 0} \frac{\dfrac{3 - (3+h)}{3+h}}{h} \\ &= \lim_{h \to 0} \frac{-h}{h(3+h)} = \lim_{h \to 0} -\frac{1}{3+h} = -\frac{1}{3} \end{aligned}$$

Therefore an equation of the tangent at the point (3, 1) is

$$y - 1 = -\tfrac{1}{3}(x - 3)$$

which simplifies to

$$x + 3y - 6 = 0$$

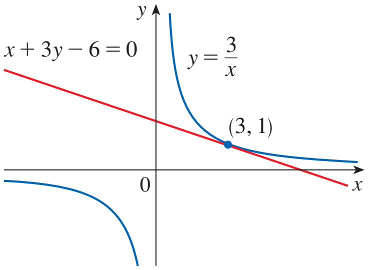


FIGURE 4

The hyperbola and its tangent are shown in Figure 4. ■

■ Velocities

In Section 2.1 we investigated the motion of a ball dropped from the CN Tower and defined its velocity to be the limiting value of average velocities over shorter and shorter time periods.

In general, suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t. The function f that describes the motion is called the **position function** of the object. In the time interval from $t = a$ to $t = a + h$, the change in position is $f(a+h) - f(a)$. (See Figure 5.)

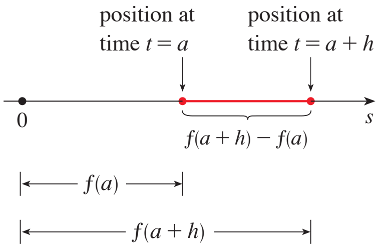
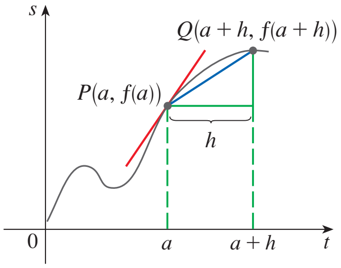


FIGURE 5



$m_{PQ} = \dfrac{f(a+h) - f(a)}{h}$

= average velocity

FIGURE 6

The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

which is the same as the slope of the secant line PQ in Figure 6.

Now suppose we compute the average velocities over shorter and shorter time intervals $[a, a+h]$. In other words, we let h approach 0. As in the example of the falling ball, we define the **velocity** (or **instantaneous velocity**) $v(a)$ at time $t = a$ to be the limit of these average velocities.

3 Definition The **instantaneous velocity** of an object with position function $f(t)$ at time $t = a$ is

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that this limit exists.

This means that the velocity at time $t = a$ is equal to the slope of the tangent line at P (compare Equation 2 and the expression in Definition 3).

Now that we know how to compute limits, let's reconsider the problem of the falling ball from Example 2.1.3.

EXAMPLE 3 Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground.

(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball traveling when it hits the ground?

Recall from Section 2.1: The distance (in meters) fallen after t seconds is $4.9t^2$.

SOLUTION Since two different velocities are requested, it's efficient to start by finding the velocity at a general time $t = a$. Using the equation of motion $s = f(t) = 4.9t^2$, we have

$$\begin{aligned} v(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{4.9(a+h)^2 - 4.9a^2}{h} \\ &= \lim_{h \to 0} \frac{4.9(a^2 + 2ah + h^2 - a^2)}{h} = \lim_{h \to 0} \frac{4.9(2ah + h^2)}{h} \\ &= \lim_{h \to 0} \frac{4.9h(2a + h)}{h} = \lim_{h \to 0} 4.9(2a + h) = 9.8a \end{aligned}$$

(a) The velocity after 5 seconds is $v(5) = (9.8)(5) = 49$ m/s.

(b) Since the observation deck is 450 m above the ground, the ball will hit the ground at the time t when $s(t) = 450$, that is,

$$4.9t^2 = 450$$

This gives

$$t^2 = \frac{450}{4.9} \qquad \text{and} \qquad t = \sqrt{\frac{450}{4.9}} \approx 9.6 \text{ s}$$

The velocity of the ball as it hits the ground is therefore

$$v\left(\sqrt{\frac{450}{4.9}}\right) = 9.8\sqrt{\frac{450}{4.9}} \approx 94 \text{ m/s}$$ ■

Derivatives

We have seen that the same type of limit arises in finding the slope of a tangent line (Equation 2) or the velocity of an object (Definition 3). In fact, limits of the form

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

arise whenever we calculate a rate of change in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics. Since this type of limit occurs so widely, it is given a special name and notation.

$f'(a)$ is read "f prime of a."

4 Definition The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

If we write $x = a + h$, then we have $h = x - a$ and h approaches 0 if and only if x approaches a. Therefore an equivalent way of stating the definition of the derivative, as we saw in finding tangent lines (see Definition 1), is

5
$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

EXAMPLE 4 Use Definition 4 to find the derivative of the function $f(x) = x^2 - 8x + 9$ at the numbers (a) 2 and (b) a.

SOLUTION

Definitions 4 and 5 are equivalent, so we can use either one to compute the derivative. In practice, Definition 4 often leads to simpler computations.

(a) From Definition 4 we have

$$\begin{aligned} f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0} \frac{(2+h)^2 - 8(2+h) + 9 - (-3)}{h} \\ &= \lim_{h \to 0} \frac{4 + 4h + h^2 - 16 - 8h + 9 + 3}{h} \\ &= \lim_{h \to 0} \frac{h^2 - 4h}{h} = \lim_{h \to 0} \frac{h(h-4)}{h} = \lim_{h \to 0} (h - 4) = -4 \end{aligned}$$

(b)

$$\begin{aligned} f'(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \\ &= \lim_{h\to 0}\frac{[(a+h)^2-8(a+h)+9]-[a^2-8a+9]}{h} \\ &= \lim_{h\to 0}\frac{a^2+2ah+h^2-8a-8h+9-a^2+8a-9}{h} \\ &= \lim_{h\to 0}\frac{2ah+h^2-8h}{h} = \lim_{h\to 0}(2a+h-8) = 2a-8 \end{aligned}$$

As a check on our work in part (a), notice that if we let $a = 2$, then $f'(2) = 2(2) - 8 = -4$. ■

EXAMPLE 5 Use Equation 5 to find the derivative of the function $f(x) = 1/\sqrt{x}$ at the number a $(a > 0)$.

SOLUTION From Equation 5 we get

$$\begin{aligned} f'(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \\ &= \lim_{x\to a}\frac{\dfrac{1}{\sqrt{x}}-\dfrac{1}{\sqrt{a}}}{x-a} = \lim_{x\to a}\frac{\dfrac{1}{\sqrt{x}}-\dfrac{1}{\sqrt{a}}}{x-a}\cdot\frac{\sqrt{x}\,\sqrt{a}}{\sqrt{x}\,\sqrt{a}} \\ &= \lim_{x\to a}\frac{\sqrt{a}-\sqrt{x}}{\sqrt{ax}\,(x-a)} = \lim_{x\to a}\frac{\sqrt{a}-\sqrt{x}}{\sqrt{ax}\,(x-a)}\cdot\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}+\sqrt{x}} \\ &= \lim_{x\to a}\frac{-(x-a)}{\sqrt{ax}\,(x-a)(\sqrt{a}+\sqrt{x})} = \lim_{x\to a}\frac{-1}{\sqrt{ax}(\sqrt{a}+\sqrt{x})} \\ &= \frac{-1}{\sqrt{a^2}(\sqrt{a}+\sqrt{a})} = \frac{-1}{a\cdot 2\sqrt{a}} = -\frac{1}{2a^{3/2}} \end{aligned}$$

You can verify that using Definition 4 gives the same result. ■

We defined the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ to be the line that passes through P and has slope m given by Equation 1 or 2. Since, by Definition 4 (and Equation 5), this is the same as the derivative $f'(a)$, we can now say the following.

The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

If we use the point-slope form of the equation of a line, we can write an equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$:

$$y - f(a) = f'(a)(x - a)$$

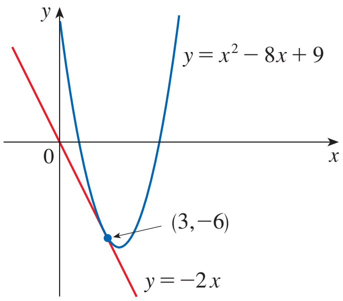


FIGURE 7

EXAMPLE 6 Find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.

SOLUTION From Example 4(b) we know that the derivative of $f(x) = x^2 - 8x + 9$ at the number a is $f'(a) = 2a - 8$. Therefore the slope of the tangent line at $(3, -6)$ is $f'(3) = 2(3) - 8 = -2$. Thus an equation of the tangent line, shown in Figure 7, is

$$y - (-6) = (-2)(x - 3) \qquad \text{or} \qquad y = -2x$$ ■

■ Rates of Change

Suppose y is a quantity that depends on another quantity x. Thus y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the **increment** of x) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of y with respect to x** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line PQ in Figure 8.

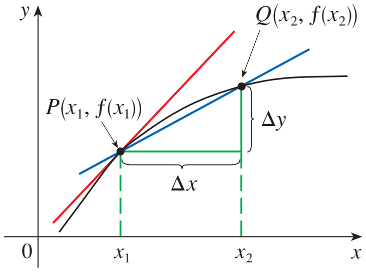


FIGURE 8

average rate of change $= m_{PQ}$

instantaneous rate of change = slope of tangent at P

By analogy with velocity, we consider the average rate of change over smaller and smaller intervals by letting x_2 approach x_1 and therefore letting Δx approach 0. The limit of these average rates of change is called the **(instantaneous) rate of change of y with respect to x** at $x = x_1$, which (as in the case of velocity) is interpreted as the slope of the tangent to the curve $y = f(x)$ at $P(x_1, f(x_1))$:

$$\boxed{6} \qquad \text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

We recognize this limit as being the derivative $f'(x_1)$.

We know that one interpretation of the derivative $f'(a)$ is as the slope of the tangent line to the curve $y = f(x)$ when $x = a$. We now have a second interpretation:

> The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

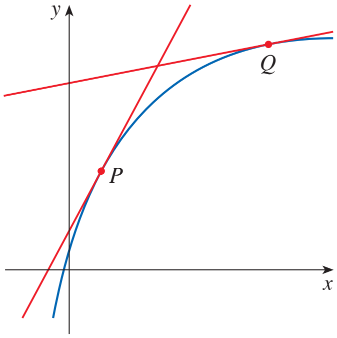


FIGURE 9
The y-values are changing rapidly at P and slowly at Q.

The connection with the first interpretation is that if we sketch the curve $y = f(x)$, then the instantaneous rate of change is the slope of the tangent to this curve at the point where $x = a$. This means that when the derivative is large (and therefore the curve is steep, as at the point P in Figure 9), the y-values change rapidly. When the derivative is small, the curve is relatively flat (as at point Q) and the y-values change slowly.

In particular, if $s = f(t)$ is the position function of a particle that moves along a straight line, then $f'(a)$ is the rate of change of the displacement s with respect to the time t. In other words, *$f'(a)$ is the velocity of the particle at time $t = a$*. The **speed** of the particle is the absolute value of the velocity, that is, $|f'(a)|$.

In the next example we discuss the meaning of the derivative of a function that is defined verbally.

EXAMPLE 7 A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x meters of this fabric is $C = f(x)$ dollars.
(a) What is the meaning of the derivative $f'(x)$? What are its units?
(b) In practical terms, what does it mean to say that $f'(1000) = 9$?
(c) Which do you think is greater, $f'(50)$ or $f'(500)$? What about $f'(5000)$?

SOLUTION
(a) The derivative $f'(x)$ is the instantaneous rate of change of C with respect to x; that is, $f'(x)$ means the rate of change of the production cost with respect to the number of meters produced. (Economists call this rate of change the *marginal cost*. This idea is discussed in more detail in Sections 3.7 and 4.7.)

Because

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x}$$

the units for $f'(x)$ are the same as the units for the difference quotient $\Delta C/\Delta x$. Since ΔC is measured in dollars and Δx in meters, it follows that the units for $f'(x)$ are dollars per meter.

(b) The statement that $f'(1000) = 9$ means that, after 1000 meters of fabric have been manufactured, the rate at which the production cost is increasing is \$9/meter. (When $x = 1000$, C is increasing 9 times as fast as x.)

Here we are assuming that the cost function is well behaved; in other words, $C(x)$ doesn't oscillate rapidly near $x = 1000$.

Since $\Delta x = 1$ is small compared with $x = 1000$, we could use the approximation

$$f'(1000) \approx \frac{\Delta C}{\Delta x} = \frac{\Delta C}{1} = \Delta C$$

and say that the cost of manufacturing the 1000th meter (or the 1001st) is about \$9.

(c) The rate at which the production cost is increasing (per meter) is probably lower when $x = 500$ than when $x = 50$ (the cost of making the 500th meter is less than the cost of the 50th meter) because of economies of scale. (The manufacturer makes more efficient use of the fixed costs of production.) So

$$f'(50) > f'(500)$$

But, as production expands, the resulting large-scale operation might become inefficient and there might be overtime costs. Thus it is possible that the rate of increase of costs will eventually start to rise. So it may happen that

$$f'(5000) > f'(500)$$ ■

In the following example we estimate the rate of change of the national debt with respect to time. Here the function is defined not by a formula but by a table of values.

EXAMPLE 8 Let $D(t)$ be the US national debt at time t. The table in the margin gives approximate values of this function by providing end of year estimates, in billions of dollars, from 2000 to 2016. Interpret and estimate the value of $D'(2008)$.

t	$D(t)$
2000	5662.2
2004	7596.1
2008	10,699.8
2012	16,432.7
2016	19,976.8

Source: US Dept. of the Treasury

SOLUTION The derivative $D'(2008)$ means the rate of change of D with respect to t when $t = 2008$, that is, the rate of increase of the national debt in 2008.

According to Equation 5,

$$D'(2008) = \lim_{t \to 2008} \frac{D(t) - D(2008)}{t - 2008}$$

One way we can estimate this value is to compare average rates of change over different time intervals by computing difference quotients, as compiled in the following table.

t	Time interval	Average rate of change $= \dfrac{D(t) - D(2008)}{t - 2008}$
2000	[2000, 2008]	629.7
2004	[2004, 2008]	775.93
2012	[2008, 2012]	1433.23
2016	[2008, 2016]	1159.63

From this table we see that $D'(2008)$ lies somewhere between 775.93 and 1433.23 billion dollars per year. [Here we are making the reasonable assumption that the debt didn't fluctuate wildly between 2004 and 2012.] A good estimate for the rate of increase of the US national debt in 2008 would be the average of these two numbers, namely

$$D'(2008) \approx 1105 \text{ billion dollars per year}$$

Another method would be to plot the debt function and estimate the slope of the tangent line when $t = 2008$. ■

A Note On Units

The units for the average rate of change $\Delta D/\Delta t$ are the units for ΔD divided by the units for Δt, namely billions of dollars per year. The instantaneous rate of change is the limit of the average rates of change, so it is measured in the same units: billions of dollars per year.

In Examples 3, 7, and 8 we saw three specific examples of rates of change: the velocity of an object is the rate of change of displacement with respect to time; marginal cost is the rate of change of production cost with respect to the number of items produced; the rate of change of the debt with respect to time is of interest in economics. Here is a small sample of other rates of change: In physics, the rate of change of work with respect to time is called *power*. Chemists who study a chemical reaction are interested in the rate of change in the concentration of a reactant with respect to time (called the *rate of reaction*). A biologist is interested in the rate of change of the population of a colony of bacteria with respect to time. In fact, the computation of rates of change is important in all of the natural sciences, in engineering, and even in the social sciences. Further examples will be given in Section 3.7.

All these rates of change are derivatives and can therefore be interpreted as slopes of tangents. This gives added significance to the solution of the tangent problem. Whenever we solve a problem involving tangent lines, we are not just solving a problem in geometry. We are also implicitly solving a great variety of problems involving rates of change in science and engineering.

2.7 Exercises

1. A curve has equation $y = f(x)$.
- (a) Write an expression for the slope of the secant line through the points $P(3, f(3))$ and $Q(x, f(x))$.
- (b) Write an expression for the slope of the tangent line at P.

2. Graph the curve $y = e^x$ in the viewing rectangles $[-1, 1]$ by $[0, 2]$, $[-0.5, 0.5]$ by $[0.5, 1.5]$, and $[-0.1, 0.1]$ by $[0.9, 1.1]$. What do you notice about the curve as you zoom in toward the point $(0, 1)$?

3. (a) Find the slope of the tangent line to the parabola $y = x^2 + 3x$ at the point $(-1, -2)$
- (i) using Definition 1 (ii) using Equation 2

(b) Find an equation of the tangent line in part (a).

(c) Graph the parabola and the tangent line. As a check on your work, zoom in toward the point $(-1, -2)$ until the parabola and the tangent line are indistinguishable.

4. (a) Find the slope of the tangent line to the curve $y = x^3 + 1$ at the point $(1, 2)$
- (i) using Definition 1 (ii) using Equation 2

(b) Find an equation of the tangent line in part (a).

(c) Graph the curve and the tangent line in successively smaller viewing rectangles centered at $(1, 2)$ until the curve and the line appear to coincide.

5–8 Find an equation of the tangent line to the curve at the given point.

5. $y = 2x^2 - 5x + 1$, $(3, 4)$

6. $y = x^2 - 2x^3$, $(1, -1)$

7. $y = \dfrac{x+2}{x-3}$, $(2, -4)$ **8.** $y = \sqrt{1 - 3x}$, $(-1, 2)$

---

9. (a) Find the slope of the tangent to the curve $y = 3 + 4x^2 - 2x^3$ at the point where $x = a$.

(b) Find equations of the tangent lines at the points $(1, 5)$ and $(2, 3)$.

(c) Graph the curve and both tangents on a common screen.

10. (a) Find the slope of the tangent to the curve $y = 2\sqrt{x}$ at the point where $x = a$.

(b) Find equations of the tangent lines at the points $(1, 2)$ and $(9, 6)$.

(c) Graph the curve and both tangents on a common screen.

11. A cliff diver plunges from a height of 30 m above the water surface. The distance the diver falls in t seconds is given by the function $d(t) = 4.9t^2$ m.
- (a) After how many seconds will the diver hit the water?
- (b) With what velocity does the diver hit the water?

12. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height (in meters) after t seconds is given by $H = 10t - 1.86t^2$.
- (a) Find the velocity of the rock after one second.
- (b) Find the velocity of the rock when $t = a$.
- (c) When will the rock hit the surface?
- (d) With what velocity will the rock hit the surface?

13. The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $s = 1/t^2$, where t is measured in seconds. Find the velocity of the particle at times $t = a$, $t = 1$, $t = 2$, and $t = 3$.

14. The displacement (in meters) of a particle moving in a straight line is given by $s = \frac{1}{2}t^2 - 6t + 23$, where t is measured in seconds.

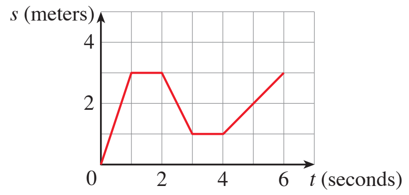
(a) Find the average velocity over each time interval:
- (i) $[4, 8]$ (ii) $[6, 8]$
- (iii) $[8, 10]$ (iv) $[8, 12]$

(b) Find the instantaneous velocity when $t = 8$.

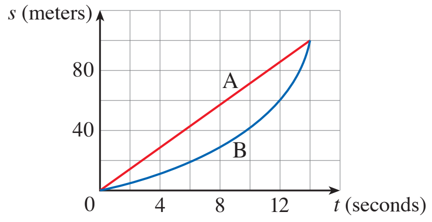
(c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities in part (a). Then draw the tangent line whose slope is the instantaneous velocity in part (b).

15. (a) A particle starts by moving to the right along a horizontal line; the graph of its position function is shown in the figure. When is the particle moving to the right? Moving to the left? Standing still?

(b) Draw a graph of the velocity function.

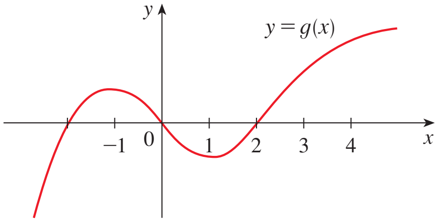


16. Shown are graphs of the position functions of two runners, A and B, who run a 100-meter race and finish in a tie.
- (a) Describe and compare how the runners run the race.
- (b) At what time is the distance between the runners the greatest?
- (c) At what time do they have the same velocity?



17. For the function g whose graph is given, arrange the following numbers in increasing order and explain your reasoning:

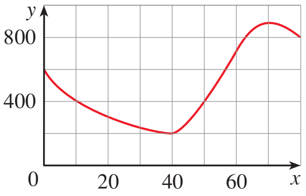
$$0 \qquad g'(-2) \qquad g'(0) \qquad g'(2) \qquad g'(4)$$



18. The graph of a function f is shown.
(a) Find the average rate of change of f on the interval $[20, 60]$.
(b) Identify an interval on which the average rate of change of f is 0.
(c) Compute

$$\frac{f(40) - f(10)}{40 - 10}$$

What does this value represent geometrically?
(d) Estimate the value of $f'(50)$.
(e) Is $f'(10) > f'(30)$?
(f) Is $f'(60) > \dfrac{f(80) - f(40)}{80 - 40}$? Explain.



19–20 Use Definition 4 to find $f'(a)$ at the given number a.

19. $f(x) = \sqrt{4x + 1}, \quad a = 6$

20. $f(x) = 5x^4, \quad a = -1$

21–22 Use Equation 5 to find $f'(a)$ at the given number a.

21. $f(x) = \dfrac{x^2}{x + 6}, \quad a = 3$ **22.** $f(x) = \dfrac{1}{\sqrt{2x + 2}}, \quad a = 1$

23–26 Find $f'(a)$.

23. $f(x) = 2x^2 - 5x + 3$ **24.** $f(t) = t^3 - 3t$

25. $f(t) = \dfrac{1}{t^2 + 1}$ **26.** $f(x) = \dfrac{x}{1 - 4x}$

---

27. Find an equation of the tangent line to the graph of $y = B(x)$ at $x = 6$ if $B(6) = 0$ and $B'(6) = -\frac{1}{2}$.

28. Find an equation of the tangent line to the graph of $y = g(x)$ at $x = 5$ if $g(5) = -3$ and $g'(5) = 4$.

29. If $f(x) = 3x^2 - x^3$, find $f'(1)$ and use it to find an equation of the tangent line to the curve $y = 3x^2 - x^3$ at the point $(1, 2)$.

30. If $g(x) = x^4 - 2$, find $g'(1)$ and use it to find an equation of the tangent line to the curve $y = x^4 - 2$ at the point $(1, -1)$.

31. (a) If $F(x) = 5x/(1 + x^2)$, find $F'(2)$ and use it to find an equation of the tangent line to the curve $y = 5x/(1 + x^2)$ at the point $(2, 2)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

32. (a) If $G(x) = 4x^2 - x^3$, find $G'(a)$ and use it to find equations of the tangent lines to the curve $y = 4x^2 - x^3$ at the points $(2, 8)$ and $(3, 9)$.
(b) Illustrate part (a) by graphing the curve and the tangent lines on the same screen.

33. If an equation of the tangent line to the curve $y = f(x)$ at the point where $a = 2$ is $y = 4x - 5$, find $f(2)$ and $f'(2)$.

34. If the tangent line to $y = f(x)$ at $(4, 3)$ passes through the point $(0, 2)$, find $f(4)$ and $f'(4)$.

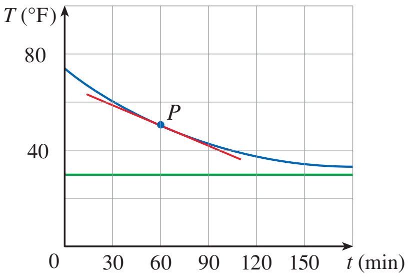
35–36 A particle moves along a straight line with equation of motion $s = f(t)$, where s is measured in meters and t in seconds. Find the velocity and the speed when $t = 4$.

35. $f(t) = 80t - 6t^2$ **36.** $f(t) = 10 + \dfrac{45}{t + 1}$

---

37. A warm can of soda is placed in a cold refrigerator. Sketch the graph of the temperature of the soda as a function of time. Is the initial rate of change of temperature greater or less than the rate of change after an hour?

38. A roast turkey is taken from an oven when its temperature has reached 85°C and is placed on a table in a room where the temperature is 24°C. The graph shows how the temperature of the turkey decreases and eventually approaches room temperature. By measuring the slope of the tangent, estimate the rate of change of the temperature after an hour.



39. Sketch the graph of a function f for which $f(0) = 0$, $f'(0) = 3$, $f'(1) = 0$, and $f'(2) = -1$.

40. Sketch the graph of a function g for which $g(0) = g(2) = g(4) = 0$, $g'(1) = g'(3) = 0$, $g'(0) = g'(4) = 1$, $g'(2) = -1$, $\lim_{x\to\infty} g(x) = \infty$, and $\lim_{x\to-\infty} g(x) = -\infty$.

41. Sketch the graph of a function g that is continuous on its domain $(-5, 5)$ and where $g(0) = 1$, $g'(0) = 1$, $g'(-2) = 0$, $\lim_{x\to-5^+} g(x) = \infty$, and $\lim_{x\to5^-} g(x) = 3$.

42. Sketch the graph of a function f where the domain is $(-2, 2)$, $f'(0) = -2$, $\lim_{x\to2^-} f(x) = \infty$, f is continuous at all numbers in its domain except ±1, and f is odd.

43–48 Each limit represents the derivative of some function f at some number a. State such an f and a in each case.

43. $\displaystyle\lim_{h\to0} \frac{\sqrt{9+h}-3}{h}$

44. $\displaystyle\lim_{h\to0} \frac{e^{-2+h}-e^{-2}}{h}$

45. $\displaystyle\lim_{x\to2} \frac{x^6-64}{x-2}$

46. $\displaystyle\lim_{x\to1/4} \frac{\frac{1}{x}-4}{x-\frac{1}{4}}$

47. $\displaystyle\lim_{h\to0} \frac{\tan\left(\frac{\pi}{4}+h\right)-1}{h}$

48. $\displaystyle\lim_{\theta\to\pi/6} \frac{\sin\theta-\frac{1}{2}}{\theta-\frac{\pi}{6}}$

---

49. The cost (in dollars) of producing x units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.

(a) Find the average rate of change of C with respect to x when the production level is changed

(i) from $x = 100$ to $x = 105$

(ii) from $x = 100$ to $x = 101$

(b) Find the instantaneous rate of change of C with respect to x when $x = 100$. (This is called the *marginal cost*. Its significance will be explained in Section 3.7.)

50. Let $H(t)$ be the daily cost (in dollars) to heat an office building when the outside temperature is t degrees Celsius.

(a) What is the meaning of $H'(14)$? What are its units?

(b) Would you expect $H'(14)$ to be positive or negative? Explain.

51. The cost of producing x kilograms of gold from a new gold mine is $C = f(x)$ dollars.

(a) What is the meaning of the derivative $f'(x)$? What are its units?

(b) What does the statement $f'(22) = 17$ mean?

(c) Do you think the values of $f'(x)$ will increase or decrease in the short term? What about the long term? Explain.

52. The quantity (in kilograms) of a gourmet ground coffee that is sold by a coffee company at a price of p dollars per kilogram is $Q = f(p)$.

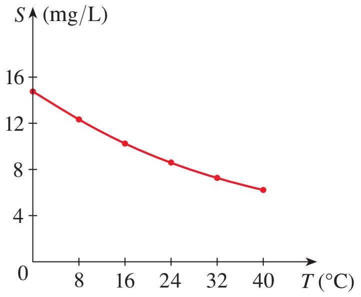
(a) What is the meaning of the derivative $f'(8)$? What are its units?

(b) Is $f'(8)$ positive or negative? Explain.

53. The quantity of oxygen that can dissolve in water depends on the temperature of the water. (So thermal pollution influences the oxygen content of water.) The graph shows how oxygen solubility S varies as a function of the water temperature T.

(a) What is the meaning of the derivative $S'(T)$? What are its units?

(b) Estimate the value of $S'(16)$ and interpret it.

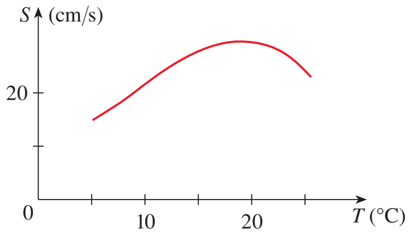


Source: C. Kupchella et al., *Environmental Science: Living Within the System of Nature*, 2d ed. (Boston: Allyn and Bacon, 1989).

54. The graph shows the influence of the temperature T on the maximum sustainable swimming speed S of Coho salmon.

(a) What is the meaning of the derivative $S'(T)$? What are its units?

(b) Estimate the values of $S'(15)$ and $S'(25)$ and interpret them.



55. Researchers measured the average blood alcohol concentration $C(t)$ of eight men starting one hour after consumption of 30 mL of ethanol (corresponding to two alcoholic drinks).

t (hours)	1.0	1.5	2.0	2.5	3.0
$C(t)$ (g/dL)	0.033	0.024	0.018	0.012	0.007

(a) Find the average rate of change of C with respect to t over each time interval:

(i) [1.0, 2.0] (ii) [1.5, 2.0]

(iii) [2.0, 2.5] (iv) [2.0, 3.0]

In each case, include the units.

(b) Estimate the instantaneous rate of change at $t = 2$ and interpret your result. What are the units?

Source: Adapted from P. Wilkinson et al., “Pharmacokinetics of Ethanol after Oral Administration in the Fasting State,” *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

56. The number N of locations of a popular coffeehouse chain is given in the table. (The numbers of locations as of October 1 are given.)

Year	N
2008	16,680
2010	16,858
2012	18,066
2014	21,366
2016	25,085

(a) Find the average rate of growth
  (i) from 2008 to 2010
  (ii) from 2010 to 2012
  In each case, include the units. What can you conclude?
(b) Estimate the instantaneous rate of growth in 2010 by taking the average of two average rates of change. What are its units?
(c) Estimate the instantaneous rate of growth in 2010 by measuring the slope of a tangent.

57–58 Determine whether $f'(0)$ exists.

57. $f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

58. $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

59. (a) Graph the function $f(x) = \sin x - \frac{1}{1000}\sin(1000x)$ in the viewing rectangle $[-2\pi, 2\pi]$ by $[-4, 4]$. What slope does the graph appear to have at the origin?
(b) Zoom in to the viewing window $[-0.4, 0.4]$ by $[-0.25, 0.25]$ and estimate the value of $f'(0)$. Does this agree with your answer from part (a)?
(c) Now zoom in to the viewing window $[-0.008, 0.008]$ by $[-0.005, 0.005]$. Do you wish to revise your estimate for $f'(0)$?

60. Symmetric Difference Quotients In Example 8 we approximated an instantaneous rate of change by averaging two average rates of change. An alternative method is to use a single average rate of change over an interval *centered* at the desired value. We define the *symmetric difference quotient* of a function f at $x = a$ on the interval $[a - d, a + d]$ as

$$\frac{f(a + d) - f(a - d)}{(a + d) - (a - d)} = \frac{f(a + d) - f(a - d)}{2d}$$

(a) Compute the symmetric difference quotient for the function D in Example 8 on the interval $[2004, 2012]$ and verify that your result agrees with the estimate for $D'(2008)$ computed in the example.
(b) Show that the symmetric difference quotient of a function f at $x = a$ is equivalent to averaging the average rates of change of f over the intervals $[a - d, a]$ and $[a, a + d]$.
(c) Use a symmetric difference quotient to estimate $f'(1)$ for $f(x) = x^3 - 2x^2 + 2$ with $d = 0.4$. Draw a graph of f along with secant lines corresponding to average rates of change over the intervals $[1 - d, 1]$, $[1, 1 + d]$, and $[1 - d, 1 + d]$. Which of these secant lines appears to have slope closest to that of the tangent line at $x = 1$?

WRITING PROJECT | EARLY METHODS FOR FINDING TANGENTS

The first person to explicitly formulate the ideas of limits and derivatives was Sir Isaac Newton in the 1660s. But Newton acknowledged that "If I have seen further than other men, it is because I have stood on the shoulders of giants." Two of those giants were Pierre Fermat (1601–1665) and Newton's mentor at Cambridge, Isaac Barrow (1630–1677). Newton was familiar with the methods that these men used to find tangent lines, and their methods played a role in Newton's eventual formulation of calculus.

Learn about these methods by researching on the Internet or reading one of the references listed here. Write an essay comparing the methods of either Fermat or Barrow to modern methods. In particular, use the method of Section 2.7 to find an equation of the tangent line to the curve $y = x^3 + 2x$ at the point $(1, 3)$ and show how either Fermat or Barrow would have solved the same problem. Although you used derivatives and they did not, point out similarities between the methods.

1. C. H. Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979), pp. 124, 132.

2. Howard Eves, *An Introduction to the History of Mathematics*, 6th ed. (New York: Saunders, 1990), pp. 391, 395.

3. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), pp. 344, 346.

4. Uta Merzbach and Carl Boyer, *A History of Mathematics*, 3rd ed. (Hoboken, NJ: Wiley, 2011), pp. 323, 356.

2.8 The Derivative as a Function

The Derivative Function

In the preceding section we considered the derivative of a function f at a fixed number a:

$$\boxed{1} \qquad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Here we change our point of view and let the number a vary. If we replace a in Equation 1 by a variable x, we obtain

$$\boxed{2} \qquad \boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Given any number x for which this limit exists, we assign to x the number $f'(x)$. So we can regard f' as a new function, called the **derivative of f** and defined by Equation 2. We know that the value of f' at x, $f'(x)$, can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.

The function f' is called the derivative of f because it has been "derived" from f by the limiting operation in Equation 2. The domain of f' is the set $\{x \mid f'(x) \text{ exists}\}$ and may be smaller than the domain of f.

EXAMPLE 1 The graph of a function f is given in Figure 1. Use it to sketch the graph of the derivative f'.

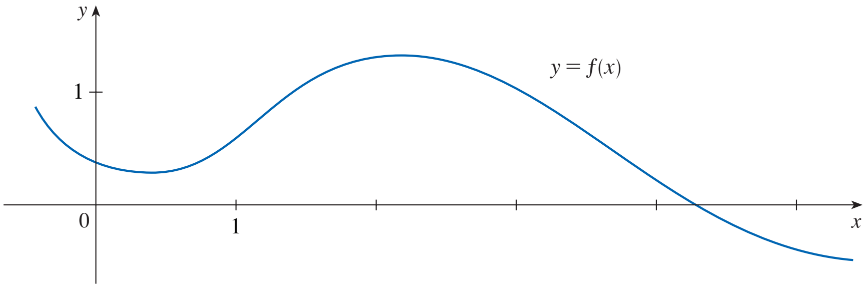


FIGURE 1

SOLUTION We can estimate the value of the derivative at any value of x by drawing the tangent at the point $(x, f(x))$ and estimating its slope. For instance, for $x = 3$ we draw a tangent at P in Figure 2 and estimate its slope to be about $-\frac{2}{3}$. (We have drawn a triangle to help estimate the slope.) Thus $f'(3) \approx -\frac{2}{3} \approx -0.67$ and this allows us to plot the point $P'(3, -0.67)$ on the graph of f' directly beneath P. (The slope of the graph of f becomes the y-value on the graph of f'.)

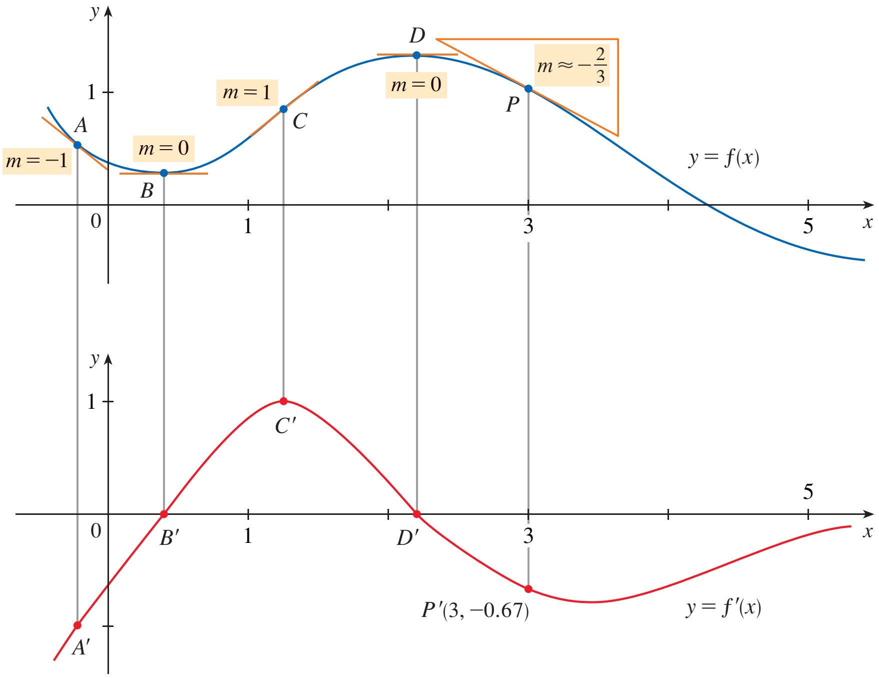


FIGURE 2

The slope of the tangent drawn at A appears to be about -1, so we plot the point A' with a y-value of -1 on the graph of f' (directly beneath A). The tangents at B and D are horizontal, so the derivative is 0 there and the graph of f' crosses the x-axis (where $y = 0$) at the points B' and D', directly beneath B and D. Between B and D, the graph of f is steepest at C and the tangent line there appears to have slope 1, so the largest value of $f'(x)$ between B' and D' is 1 (at C').

Notice that between B and D the tangents have positive slope, so $f'(x)$ is positive there. (The graph of f' is above the x-axis.) But to the right of D the tangents have negative slope, so $f'(x)$ is negative there. (The graph of f' is below the x-axis.) ■

EXAMPLE 2

(a) If $f(x) = x^3 - x$, find a formula for $f'(x)$.
(b) Illustrate this formula by comparing the graphs of f and f'.

SOLUTION

(a) When using Equation 2 to compute a derivative, we must remember that the variable

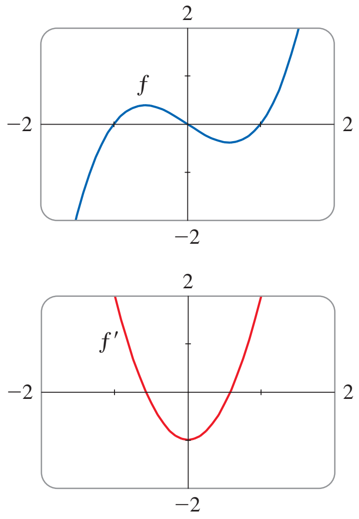


FIGURE 3

is h and that x is temporarily regarded as a constant during the calculation of the limit.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{[(x+h)^3-(x+h)]-[x^3-x]}{h} \\ &= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x-h-x^3+x}{h} \\ &= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-h}{h} \\ &= \lim_{h\to 0}(3x^2+3xh+h^2-1) = 3x^2-1 \end{aligned}$$

(b) We use a calculator to graph f and f' in Figure 3. Notice that $f'(x)=0$ when f has horizontal tangents and $f'(x)$ is positive when the tangents have positive slope. So these graphs serve as a check on our work in part (a). ■

EXAMPLE 3 If $f(x)=\sqrt{x}$, find the derivative of f. State the domain of f'.

SOLUTION

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right) \qquad \text{(Rationalize the numerator.)} \\ &= \lim_{h\to 0}\frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\ &= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

We see that $f'(x)$ exists if $x>0$, so the domain of f' is $(0,\infty)$. This is slightly smaller than the domain of f, which is $[0,\infty)$. ■

Let's check to see that the result of Example 3 is reasonable by looking at the graphs of f and f' in Figure 4. When x is close to 0, $\sqrt{x}$ is also close to 0, so $f'(x)=1/\left(2\sqrt{x}\right)$ is very large and this corresponds to the steep tangent lines near $(0, 0)$ in Figure 4(a) and the large values of $f'(x)$ just to the right of 0 in Figure 4(b). When x is large, $f'(x)$ is very small and this corresponds to the flatter tangent lines at the far right of the graph of f and the horizontal asymptote of the graph of f'.

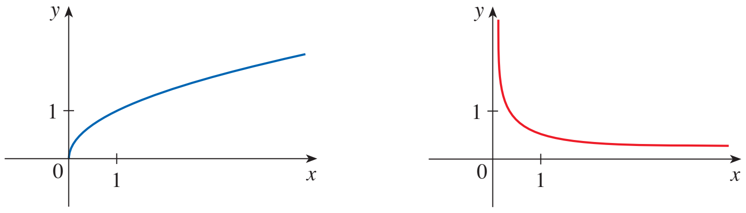


FIGURE 4

(a) $f(x)=\sqrt{x}$

(b) $f'(x)=\dfrac{1}{2\sqrt{x}}$

EXAMPLE 4 Find f' if $f(x) = \dfrac{1 - x}{2 + x}$.

SOLUTION

$$\frac{\dfrac{a}{b} - \dfrac{c}{d}}{e} = \frac{ad - bc}{bd} \cdot \frac{1}{e}$$

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\dfrac{1 - (x + h)}{2 + (x + h)} - \dfrac{1 - x}{2 + x}}{h} \\
&= \lim_{h \to 0} \frac{(1 - x - h)(2 + x) - (1 - x)(2 + x + h)}{h(2 + x + h)(2 + x)} \\
&= \lim_{h \to 0} \frac{(2 - x - 2h - x^2 - xh) - (2 - x + h - x^2 - xh)}{h(2 + x + h)(2 + x)} \\
&= \lim_{h \to 0} \frac{-3h}{h(2 + x + h)(2 + x)} \\
&= \lim_{h \to 0} \frac{-3}{(2 + x + h)(2 + x)} = -\frac{3}{(2 + x)^2}
\end{aligned}$$

■

Leibniz

Gottfried Wilhelm Leibniz was born in Leipzig in 1646 and studied law, theology, philosophy, and mathematics at the university there, graduating with a bachelor's degree at age 17. After earning his doctorate in law at age 20, Leibniz entered the diplomatic service and spent most of his life traveling to the capitals of Europe on political missions. In particular, he worked to avert a French military threat against Germany and attempted to reconcile the Catholic and Protestant churches.

His serious study of mathematics did not begin until 1672 while he was on a diplomatic mission in Paris. There he built a calculating machine and met scientists, like Huygens, who directed his attention to the latest developments in mathematics and science. Leibniz sought to develop a symbolic logic and system of notation that would simplify logical reasoning. In particular, the version of calculus that he published in 1684 established the notation and the rules for finding derivatives that we use today.

Unfortunately, a dreadful priority dispute arose in the 1690s between the followers of Newton and those of Leibniz as to who had invented calculus first. Leibniz was even accused of plagiarism by members of the Royal Society in England. The truth is that each man invented calculus independently. Newton arrived at his version of calculus first but, because of his fear of controversy, did not publish it immediately. So Leibniz's 1684 account of calculus was the first to be published.

Other Notations

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

The symbols D and d/dx are called **differentiation operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative.

The symbol dy/dx, which was introduced by Leibniz, should not be regarded as a ratio (for the time being); it is simply a synonym for $f'(x)$. Nonetheless, it is a very useful and suggestive notation, especially when used in conjunction with increment notation. Referring to Equation 2.7.6, we can rewrite the definition of derivative in Leibniz notation in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

If we want to indicate the value of a derivative dy/dx in Leibniz notation at a specific number a, we use the notation

$$\left.\frac{dy}{dx}\right|_{x=a} \qquad \text{or} \qquad \left.\frac{dy}{dx}\right]_{x=a}$$

which is a synonym for $f'(a)$. The vertical bar means "evaluate at."

$\boxed{3}$ **Definition** A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

EXAMPLE 5 Where is the function $f(x) = |x|$ differentiable?

SOLUTION If $x > 0$, then $|x| = x$ and we can choose h small enough that $x + h > 0$ and hence $|x + h| = x + h$. Therefore, for $x > 0$, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{|x + h| - |x|}{h} = \lim_{h\to 0} \frac{(x + h) - x}{h} \\ &= \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1 \end{aligned}$$

and so f is differentiable for any $x > 0$.

Similarly, for $x < 0$ we have $|x| = -x$ and h can be chosen small enough that $x + h < 0$ and so $|x + h| = -(x + h)$. Therefore, for $x < 0$,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{|x + h| - |x|}{h} = \lim_{h\to 0} \frac{-(x + h) - (-x)}{h} \\ &= \lim_{h\to 0} \frac{-h}{h} = \lim_{h\to 0} (-1) = -1 \end{aligned}$$

and so f is differentiable for any $x < 0$.

For $x = 0$ we have to investigate

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0 + h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{|0 + h| - |0|}{h} = \lim_{h\to 0} \frac{|h|}{h} \qquad \text{(if it exists)} \end{aligned}$$

Let's compute the left and right limits separately:

$$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1$$

and

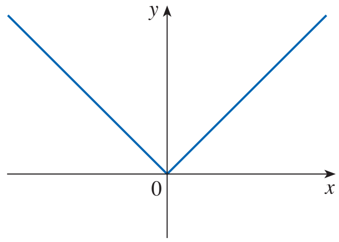
$$\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} (-1) = -1$$

Since these limits are different, $f'(0)$ does not exist. Thus f is differentiable at all x except 0.

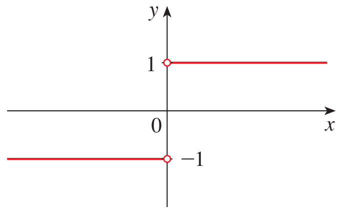
A formula for f' is given by

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

and its graph is shown in Figure 5(b). The fact that $f'(0)$ does not exist is reflected geometrically in the fact that the curve $y = |x|$ does not have a tangent line at $(0, 0)$. [See Figure 5(a).] ■



(a) $y = f(x) = |x|$



(b) $y = f'(x)$

FIGURE 5

Both continuity and differentiability are desirable properties for a function to have. The following theorem shows how these properties are related.

4 Theorem If f is differentiable at a, then f is continuous at a.

PROOF To prove that f is continuous at a, we have to show that $\lim_{x\to a} f(x) = f(a)$. We will do this by showing that the difference $f(x) - f(a)$ approaches 0.

The given information is that f is differentiable at a, that is,

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

PS An important aspect of problem solving is trying to find a connection between the given and the unknown. See Step 2 (Think of a Plan) in Principles of Problem Solving following Chapter 1.

exists (see Equation 2.7.5). To connect the given and the unknown, we divide and multiply $f(x) - f(a)$ by $x - a$ (which we can do when $x \neq a$):

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a)$$

Thus, using Limit Law 4, we can write

$$\begin{aligned} \lim_{x\to a} [f(x) - f(a)] &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a}(x - a) \\ &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x\to a} (x - a) \\ &= f'(a) \cdot 0 = 0 \end{aligned}$$

To use what we have just proved, we start with $f(x)$ and add and subtract $f(a)$:

$$\begin{aligned} \lim_{x\to a} f(x) &= \lim_{x\to a} [f(a) + (f(x) - f(a))] \\ &= \lim_{x\to a} f(a) + \lim_{x\to a} [f(x) - f(a)] \\ &= f(a) + 0 = f(a) \end{aligned}$$

Therefore f is continuous at a. ■

⊘ **NOTE** The converse of Theorem 4 is false; that is, there are functions that are continuous but not differentiable. For instance, the function $f(x) = |x|$ is continuous at 0 because

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} |x| = 0 = f(0)$$

(See Example 2.3.7.) But in Example 5 we showed that f is not differentiable at 0.

How Can a Function Fail To Be Differentiable?

We saw that the function $y = |x|$ in Example 5 is not differentiable at 0 and Figure 5(a) shows that its graph changes direction abruptly when $x = 0$. In general, if the graph of a function f has a "corner" or "kink" in it, then the graph of f has no tangent at this point and f is not differentiable there. [In trying to compute $f'(a)$, we find that the left and right limits are different.]

Theorem 4 gives another way for a function not to have a derivative. It says that if f is not continuous at a, then f is not differentiable at a. So at any discontinuity (for instance, a jump discontinuity) f fails to be differentiable.

A third possibility is that the curve has a **vertical tangent line** when $x = a$; that is, f is continuous at a and

$$\lim_{x \to a} |f'(x)| = \infty$$

This means that the tangent lines become steeper and steeper as $x \to a$. Figure 6 shows one way that this can happen; Figure 7(c) shows another. Figure 7 illustrates the three possibilities that we have discussed.

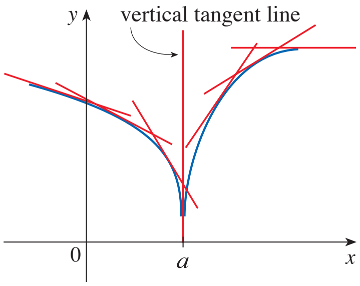
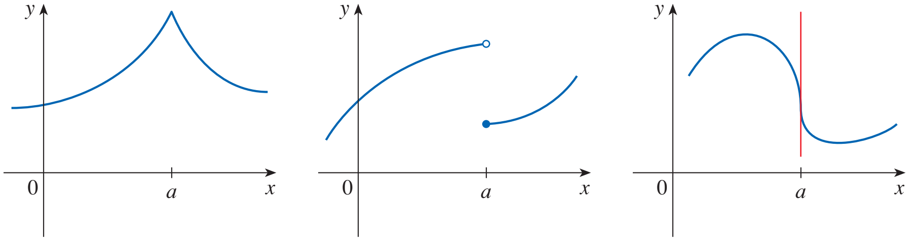


FIGURE 6



(a) A corner (b) A discontinuity (c) A vertical tangent

FIGURE 7
Three ways for f not to be differentiable at a

A graphing calculator or computer provides another way of looking at differentiability. If f is differentiable at a, then when we zoom in toward the point $(a, f(a))$ the graph straightens out and appears more and more like a line. (See Figure 8. We saw a specific example of this in Figure 2.7.2.) But no matter how much we zoom in toward a point like the ones in Figures 6 and 7(a), we can't eliminate the sharp point or corner (see Figure 9).

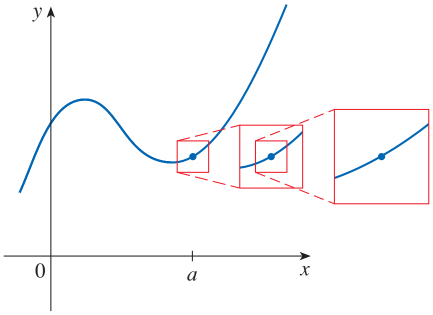


FIGURE 8
f is differentiable at a.

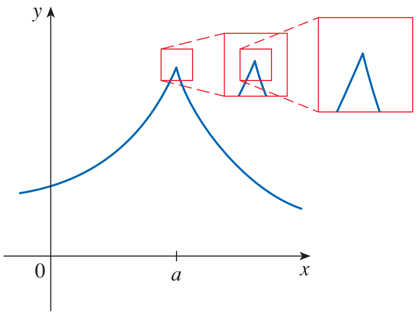


FIGURE 9
f is not differentiable at a.

Higher Derivatives

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the **second derivative** of f because it is the derivative of the derivative of f. Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$$\underbrace{\frac{d}{dx}}_{\text{derivative of}} \underbrace{\left(\frac{dy}{dx}\right)}_{\text{first derivative}} = \underbrace{\frac{d^2y}{dx^2}}_{\text{second derivative}}$$

EXAMPLE 6 If $f(x) = x^3 - x$, find and interpret $f''(x)$.

SOLUTION In Example 2 we found that the first derivative is $f'(x) = 3x^2 - 1$. So the second derivative is

$$\begin{aligned} f''(x) = (f')'(x) &= \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} \\ &= \lim_{h \to 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h} \\ &= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 1 - 3x^2 + 1}{h} \\ &= \lim_{h \to 0} (6x + 3h) = 6x \end{aligned}$$

The graphs of f, f', and f'' are shown in Figure 10.

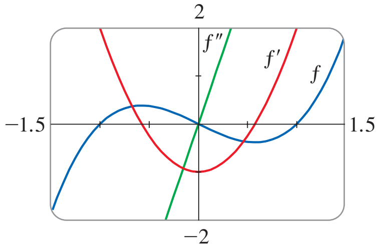


FIGURE 10

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$. In other words, it is the rate of change of the slope of the original curve $y = f(x)$.

Notice from Figure 10 that $f''(x)$ is negative when $y = f'(x)$ has negative slope and positive when $y = f'(x)$ has positive slope. So the graphs serve as a check on our calculations. ■

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is *acceleration*, which we define as follows.

If $s = s(t)$ is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity $v(t)$ of the object as a function of time:

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity with respect to time is called the **acceleration** $a(t)$ of the object. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t)$$

or, in Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Acceleration is the change in velocity you feel when speeding up or slowing down in a car.

The **third derivative** f''' is the derivative of the second derivative: $f''' = (f'')'$. So $f'''(x)$ can be interpreted as the slope of the curve $y = f''(x)$ or as the rate of change of $f''(x)$. If $y = f(x)$, then alternative notations for the third derivative are

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

We can also interpret the third derivative physically in the case where the function is the position function $s = s(t)$ of an object that moves along a straight line. Because $s''' = (s'')' = a'$, the third derivative of the position function is the derivative of the acceleration function and is called the **jerk**:

$$j = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

Thus the jerk j is the rate of change of acceleration. It is aptly named because a large jerk means a sudden change in acceleration, which causes an abrupt movement.

The differentiation process can be continued. The fourth derivative f'''' is usually denoted by $f^{(4)}$. In general, the nth derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times. If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

EXAMPLE 7 If $f(x) = x^3 - x$, find $f'''(x)$ and $f^{(4)}(x)$.

SOLUTION In Example 6 we found that $f''(x) = 6x$. The graph of the second derivative has equation $y = 6x$ and so it is a straight line with slope 6. Since the derivative $f'''(x)$ is the slope of $f''(x)$, we have

$$f'''(x) = 6$$

for all values of x. So f''' is a constant function and its graph is a horizontal line. Therefore, for all values of x,

$$f^{(4)}(x) = 0$$

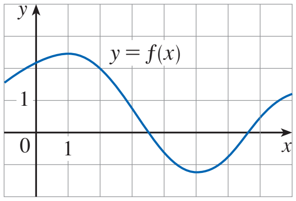
■

We have seen that one application of second and third derivatives occurs in analyzing the motion of objects using acceleration and jerk. We will investigate another application of second derivatives in Section 4.3, where we show how knowledge of f'' gives us information about the shape of the graph of f. In Chapter 11 we will see how second and higher derivatives enable us to represent functions as sums of infinite series.

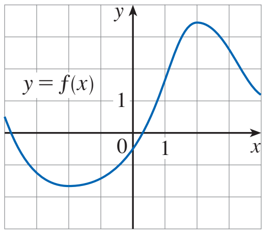
2.8 Exercises

1–2 Use the given graph to estimate the value of each derivative. Then sketch the graph of f'.

1. (a) $f'(0)$ (b) $f'(1)$ (c) $f'(2)$ (d) $f'(3)$
(e) $f'(4)$ (f) $f'(5)$ (g) $f'(6)$ (h) $f'(7)$

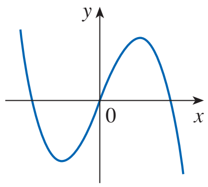


2. (a) $f'(-3)$ (b) $f'(-2)$ (c) $f'(-1)$
(d) $f'(0)$ (e) $f'(1)$ (f) $f'(2)$
(g) $f'(3)$

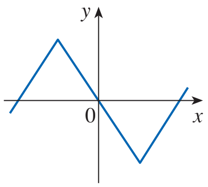


3. Match the graph of each function in (a)–(d) with the graph of its derivative in I–IV. Give reasons for your choices.

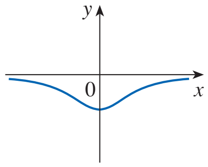
(a)



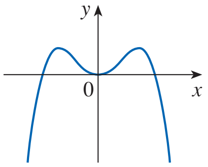
(b)



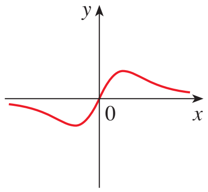
(c)



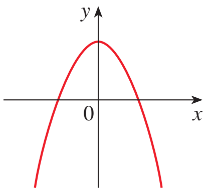
(d)



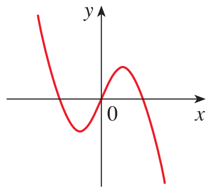
I



II



III



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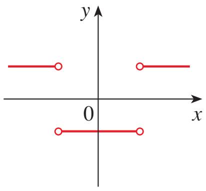
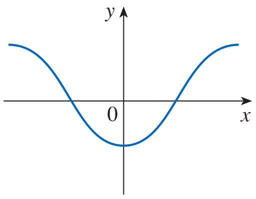


4–11 Trace or copy the graph of the given function f. (Assume that the axes have equal scales.) Then use the method of Example 1 to sketch the graph of f' below it.

4.



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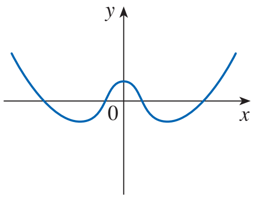

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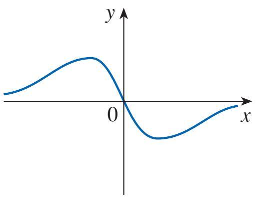

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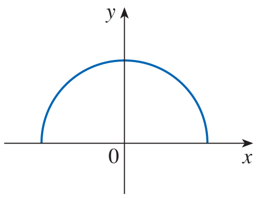

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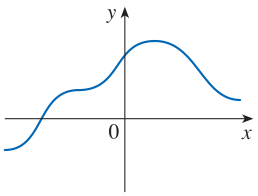

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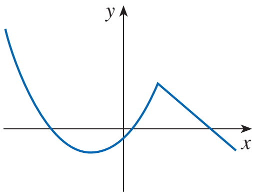

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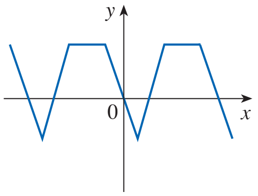

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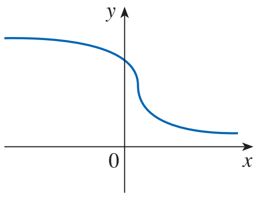
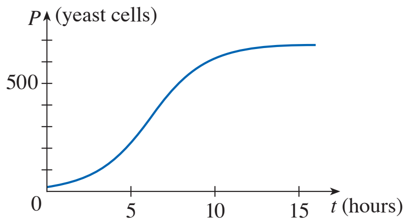


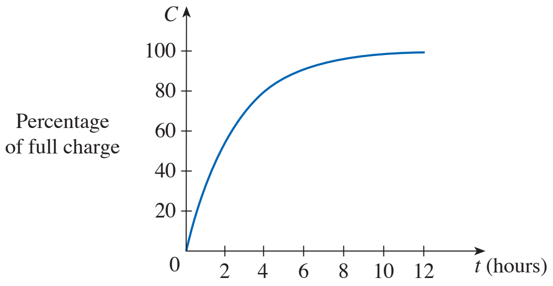
12. Shown is the graph of the population function $P(t)$ for yeast cells in a laboratory culture. Use the method of Example 1 to graph the derivative $P'(t)$. What does the graph of P' tell us about the yeast population?



13. A rechargeable battery is plugged into a charger. The graph shows $C(t)$, the percentage of full capacity that the battery reaches as a function of time t elapsed (in hours).

(a) What is the meaning of the derivative $C'(t)$?

(b) Sketch the graph of $C'(t)$. What does the graph tell you?

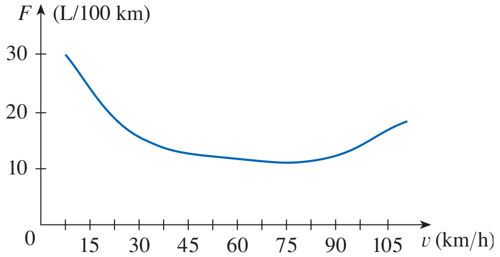


14. The graph (from the US Department of Energy) shows how driving speed affects gas mileage. Fuel economy F is measured in liters per 100 km and speed v is measured in kilometers per hour.

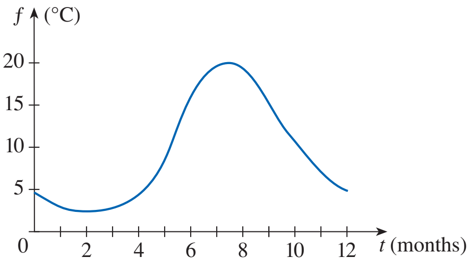
(a) What is the meaning of the derivative $F'(v)$?

(b) Sketch the graph of $F'(v)$.

(c) At what speed should you drive if you want to save on gas?



15. The graph shows how the average surface water temperature f of Lake Michigan varies over the course of a year (where t is measured in months with $t = 0$ corresponding to January 1). The average was calculated from data obtained over a 20-year period ending in 2011. Sketch the graph of the derivative function f'. When is $f'(t)$ largest?



16–18 Make a careful sketch of the graph of f and below it sketch the graph of f' in the same manner as in Exercises 4–11. Can you guess a formula for $f'(x)$ from its graph?

16. $f(x) = \sin x$ **17.** $f(x) = e^x$ **18.** $f(x) = \ln x$

---

19. Let $f(x) = x^2$.

(a) Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, and $f'(2)$ by zooming in on the graph of f.

(b) Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, and $f'(-2)$.

(c) Use the results from parts (a) and (b) to guess a formula for $f'(x)$.

(d) Use the definition of derivative to prove that your guess in part (c) is correct.

20. Let $f(x) = x^3$.

(a) Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, $f'(2)$, and $f'(3)$ by zooming in on the graph of f.

(b) Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, $f'(-2)$, and $f'(-3)$.

(c) Use the values from parts (a) and (b) to graph f'.

(d) Guess a formula for $f'(x)$.

(e) Use the definition of derivative to prove that your guess in part (d) is correct.

21–32 Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

21. $f(x) = 3x - 8$

22. $f(x) = mx + b$

23. $f(t) = 2.5t^2 + 6t$

24. $f(x) = 4 + 8x - 5x^2$

25. $A(p) = 4p^3 + 3p$

26. $F(t) = t^3 - 5t + 1$

27. $f(x) = \dfrac{1}{x^2 - 4}$

28. $F(v) = \dfrac{v}{v + 2}$

29. $g(u) = \dfrac{u + 1}{4u - 1}$

30. $f(x) = x^4$

31. $f(x) = \dfrac{1}{\sqrt{1 + x}}$

32. $g(x) = \dfrac{1}{1 + \sqrt{x}}$

---

33. (a) Sketch the graph of $f(x) = 1 + \sqrt{x + 3}$ by starting with the graph of $y = \sqrt{x}$ and using the transformations of Section 1.3.

(b) Use the graph from part (a) to sketch the graph of f'.

(c) Use the definition of a derivative to find $f'(x)$. What are the domains of f and f'?

(d) Graph f' and compare with your sketch in part (b).

34. (a) If $f(x) = x + 1/x$, find $f'(x)$.

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

35. (a) If $f(x) = x^4 + 2x$, find $f'(x)$.

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

36. The table gives the number $N(t)$, measured in thousands, of minimally invasive cosmetic surgery procedures performed in the United States for various years t.

t	$N(t)$ (thousands)
2000	5,500
2002	4,897
2004	7,470
2006	9,138
2008	10,897
2010	11,561
2012	13,035
2014	13,945

Source: American Society of Plastic Surgeons

(a) What is the meaning of $N'(t)$? What are its units?

(b) Construct a table of estimated values for $N'(t)$.

(c) Graph N and N'.

(d) How would it be possible to get more accurate values for $N'(t)$?

37. The table gives the height as time passes of a typical pine tree grown for lumber at a managed site.

Tree age (years)	14	21	28	35	42	49
Height (meters)	12	16	19	22	24	25

Source: Arkansas Forestry Commission

If $H(t)$ is the height of the tree after t years, construct a table of estimated values for H' and sketch its graph.

38. Water temperature affects the growth rate of brook trout. The table shows the amount of weight gained by brook trout after 24 days in various water temperatures.

Temperature (°C)	15.5	17.7	20.0	22.4	24.4
Weight gained (g)	37.2	31.0	19.8	9.7	−9.8

If $W(x)$ is the weight gain at temperature x, construct a table of estimated values for W' and sketch its graph. What are the units for $W'(x)$?

Source: Adapted from J. Chadwick Jr., "Temperature Effects on Growth and Stress Physiology of Brook Trout: Implications for Climate Change Impacts on an Iconic Cold-Water Fish." *Masters Theses*. Paper 897. 2012. scholarworks.umass.edu/theses/897.

39. Let P represent the percentage of a city's electrical power that is produced by solar panels t years after January 1, 2020.

(a) What does dP/dt represent in this context?

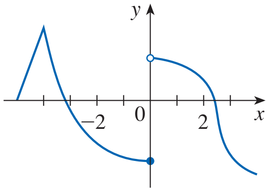
(b) Interpret the statement

$$\left.\frac{dP}{dt}\right|_{t=2} = 3.5$$

40. Suppose N is the number of people in the United States who travel by car to another state for a vacation in a year when the average price of gasoline is p dollars per liter. Do you expect dN/dp to be positive or negative? Explain.

41–44 The graph of f is given. State, with reasons, the numbers at which f is *not* differentiable.

41.



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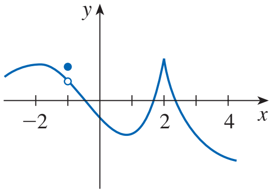

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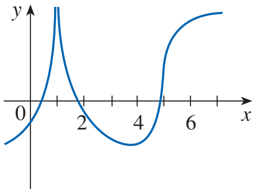

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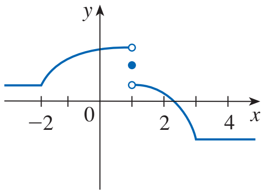

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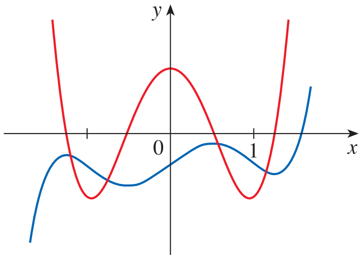
45. Graph the function $f(x) = x + \sqrt{|x|}$. Zoom in repeatedly, first toward the point $(-1, 0)$ and then toward the origin. What is different about the behavior of f in the vicinity of these two points? What do you conclude about the differentiability of f?

46. Zoom in toward the points $(1, 0)$, $(0, 1)$, and $(-1, 0)$ on the graph of the function $g(x) = (x^2 - 1)^{2/3}$. What do you notice? Account for what you see in terms of the differentiability of g.

47–48 The graphs of a function f and its derivative f' are shown. Which is bigger, $f'(-1)$ or $f''(1)$?

47.



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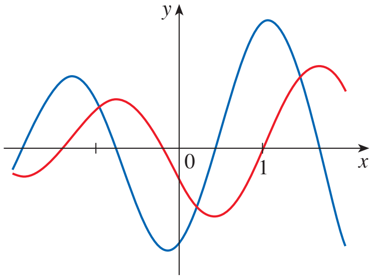

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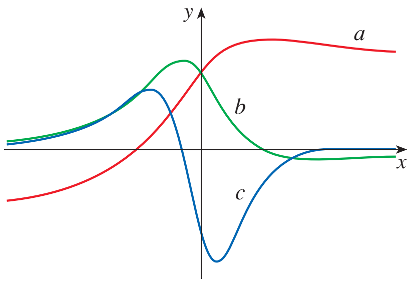
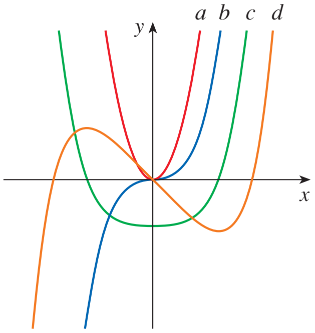
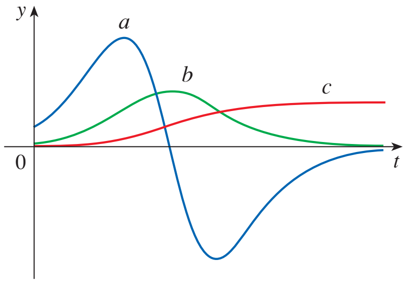
49. The figure shows the graphs of f, f', and f''. Identify each curve, and explain your choices.



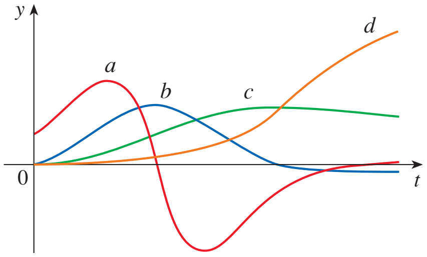
50. The figure shows graphs of f, f', f'', and f'''. Identify each curve, and explain your choices.



51. The figure shows the graphs of three functions. One is the position function of a car, one is the velocity of the car, and one is its acceleration. Identify each curve, and explain your choices.



52. The figure shows the graphs of four functions. One is the position function of a car, one is the velocity of the car, one is its acceleration, and one is its jerk. Identify each curve, and explain your choices.



53–54 Use the definition of a derivative to find $f'(x)$ and $f''(x)$. Then graph f, f', and f'' on a common screen and check to see if your answers are reasonable.

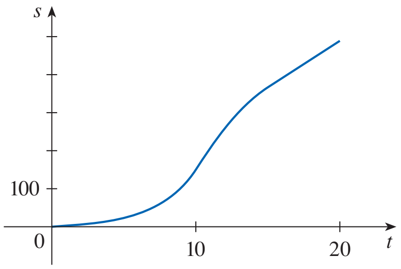
53. $f(x) = 3x^2 + 2x + 1$

54. $f(x) = x^3 - 3x$

---

55. If $f(x) = 2x^2 - x^3$, find $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$. Graph f, f', f'', and f''' on a common screen. Are the graphs consistent with the geometric interpretations of these derivatives?

56. (a) The graph of a position function of a car is shown, where s is measured in meters and t in seconds. Use it to graph the velocity and acceleration of the car. What is the acceleration at $t = 10$ seconds?



(b) Use the acceleration curve from part (a) to estimate the jerk at $t = 10$ seconds. What are the units for jerk?

57. Let $f(x) = \sqrt[3]{x}$.
(a) If $a \neq 0$, use Equation 2.7.5 to find $f'(a)$.
(b) Show that $f'(0)$ does not exist.
(c) Show that $y = \sqrt[3]{x}$ has a vertical tangent line at $(0, 0)$. (Recall the shape of the graph of f. See Figure 1.2.13.)

58. (a) If $g(x) = x^{2/3}$, show that $g'(0)$ does not exist.
(b) If $a \neq 0$, find $g'(a)$.
(c) Show that $y = x^{2/3}$ has a vertical tangent line at $(0, 0)$.
(d) Illustrate part (c) by graphing $y = x^{2/3}$.

59. Show that the function $f(x) = |x - 6|$ is not differentiable at 6. Find a formula for f' and sketch its graph.

60. Where is the greatest integer function $f(x) = [\![x]\!]$ not differentiable? Find a formula for f' and sketch its graph.

61. (a) Sketch the graph of the function $f(x) = x|x|$.
(b) For what values of x is f differentiable?
(c) Find a formula for f'.

62. (a) Sketch the graph of the function $g(x) = x + |x|$.
(b) For what values of x is g differentiable?
(c) Find a formula for g'.

63. Derivatives of Even and Odd Functions Recall that a function f is called *even* if $f(-x) = f(x)$ for all x in its domain and *odd* if $f(-x) = -f(x)$ for all such x. Prove each of the following.
(a) The derivative of an even function is an odd function.
(b) The derivative of an odd function is an even function.

64–65 Left- and Right-Hand Derivatives The *left-hand* and *right-hand derivatives* of f at a are defined by

$$f'_-(a) = \lim_{h \to 0^-} \frac{f(a + h) - f(a)}{h}$$

and

$$f'_+(a) = \lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h}$$

if these limits exist. Then $f'(a)$ exists if and only if these one-sided derivatives exist and are equal.

64. Find $f'_-(0)$ and $f'_+(0)$ for the given function f. Is f differentiable at 0?

(a) $$f(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ x & \text{if } x > 0 \end{cases}$$

(b) $$f(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ x^2 & \text{if } x > 0 \end{cases}$$

65. Let

$$f(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ 5 - x & \text{if } 0 < x < 4 \\ \dfrac{1}{5 - x} & \text{if } x \geqslant 4 \end{cases}$$

(a) Find $f'_-(4)$ and $f'_+(4)$.
(b) Sketch the graph of f.
(c) Where is f discontinuous?
(d) Where is f not differentiable?

66. When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running. In Example 1.1.4 we sketched a possible graph of T as a function of the time t that has elapsed since the faucet was turned on.
(a) Describe how the rate of change of T with respect to t varies as t increases.
(b) Sketch a graph of the derivative of T.

67. Nick starts jogging and runs faster and faster for 3 mintues, then he walks for 5 minutes. He stops at an intersection for 2 minutes, runs fairly quickly for 5 minutes, then walks for 4 minutes.
(a) Sketch a possible graph of the distance s Nick has covered after t minutes.
(b) Sketch a graph of ds/dt.

68. Let ℓ be the tangent line to the parabola $y = x^2$ at the point (1, 1). The *angle of inclination* of ℓ is the angle ϕ that ℓ makes with the positive direction of the x-axis. Calculate ϕ correct to the nearest degree.

2 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. Explain what each of the following means and illustrate with a sketch.
(a) $\lim_{x \to a} f(x) = L$ (b) $\lim_{x \to a^+} f(x) = L$ (c) $\lim_{x \to a^-} f(x) = L$
(d) $\lim_{x \to a} f(x) = \infty$ (e) $\lim_{x \to \infty} f(x) = L$

2. Describe several ways in which a limit can fail to exist. Illustrate with sketches.

3. State the following Limit Laws.
(a) Sum Law
(b) Difference Law
(c) Constant Multiple Law
(d) Product Law
(e) Quotient Law
(f) Power Law
(g) Root Law

4. What does the Squeeze Theorem say?

5. (a) What does it mean to say that the line $x = a$ is a vertical asymptote of the curve $y = f(x)$? Draw curves to illustrate the various possibilities.
(b) What does it mean to say that the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$? Draw curves to illustrate the various possibilities.

6. Which of the following curves have vertical asymptotes? Which have horizontal asymptotes?
(a) $y = x^4$ (b) $y = \sin x$ (c) $y = \tan x$
(d) $y = \tan^{-1}x$ (e) $y = e^x$ (f) $y = \ln x$
(g) $y = 1/x$ (h) $y = \sqrt{x}$

7. (a) What does it mean for f to be continuous at a?
(b) What does it mean for f to be continuous on the interval $(-\infty, \infty)$? What can you say about the graph of such a function?

8. (a) Give examples of functions that are continuous on $[-1, 1]$.
(b) Give an example of a function that is not continuous on $[0, 1]$.

9. What does the Intermediate Value Theorem say?

10. Write an expression for the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.

11. Suppose an object moves along a straight line with position $f(t)$ at time t. Write an expression for the instantaneous velocity of the object at time $t = a$. How can you interpret this velocity in terms of the graph of f?

12. If $y = f(x)$ and x changes from x_1 to x_2, write expressions for the following.

(a) The average rate of change of y with respect to x over the interval $[x_1, x_2]$

(b) The instantaneous rate of change of y with respect to x at $x = x_1$

13. Define the derivative $f'(a)$. Discuss two ways of interpreting this number.

14. Define the second derivative of f. If $f(t)$ is the position function of a particle, how can you interpret the second derivative?

15. (a) What does it mean for f to be differentiable at a?
(b) What is the relation between the differentiability and continuity of a function?
(c) Sketch the graph of a function that is continuous but not differentiable at $a = 2$.

16. Describe several ways in which a function can fail to be differentiable. Illustrate with sketches.

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. $\displaystyle \lim_{x\to 4}\left(\frac{2x}{x-4} - \frac{8}{x-4}\right) = \lim_{x\to 4}\frac{2x}{x-4} - \lim_{x\to 4}\frac{8}{x-4}$

2. $\displaystyle \lim_{x\to 1}\frac{x^2+6x-7}{x^2+5x-6} = \frac{\lim_{x\to 1}(x^2+6x-7)}{\lim_{x\to 1}(x^2+5x-6)}$

3. $\displaystyle \lim_{x\to 1}\frac{x-3}{x^2+2x-4} = \frac{\lim_{x\to 1}(x-3)}{\lim_{x\to 1}(x^2+2x-4)}$

4. $\displaystyle \frac{x^2-9}{x-3} = x+3$

5. $\displaystyle \lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3}(x+3)$

6. If $\lim_{x\to 5} f(x) = 2$ and $\lim_{x\to 5} g(x) = 0$, then $\lim_{x\to 5}[f(x)/g(x)]$ does not exist.

7. If $\lim_{x\to 5} f(x) = 0$ and $\lim_{x\to 5} g(x) = 0$, then $\lim_{x\to 5}[f(x)/g(x)]$ does not exist.

8. If neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists, then $\lim_{x\to a}[f(x) + g(x)]$ does not exist.

9. If $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a}[f(x) + g(x)]$ does not exist.

10. If p is a polynomial, then $\lim_{x\to b} p(x) = p(b)$.

11. If $\lim_{x\to 0} f(x) = \infty$ and $\lim_{x\to 0} g(x) = \infty$, then $\lim_{x\to 0}[f(x) - g(x)] = 0$.

12. A function can have two different horizontal asymptotes.

13. If f has domain $[0, \infty)$ and has no horizontal asymptote, then $\lim_{x\to\infty} f(x) = \infty$ or $\lim_{x\to\infty} f(x) = -\infty$.

14. If the line $x = 1$ is a vertical asymptote of $y = f(x)$, then f is not defined at 1.

15. If $f(1) > 0$ and $f(3) < 0$, then there exists a number c between 1 and 3 such that $f(c) = 0$.

16. If f is continuous at 5 and $f(5) = 2$ and $f(4) = 3$, then $\lim_{x\to 2} f(4x^2 - 11) = 2$.

17. If f is continuous on $[-1, 1]$ and $f(-1) = 4$ and $f(1) = 3$, then there exists a number r such that $|r| < 1$ and $f(r) = \pi$.

18. Let f be a function such that $\lim_{x\to 0} f(x) = 6$. Then there exists a positive number δ such that if $0 < |x| < \delta$, then $|f(x) - 6| < 1$.

19. If $f(x) > 1$ for all x and $\lim_{x\to 0} f(x)$ exists, then $\lim_{x\to 0} f(x) > 1$.

20. If f is continuous at a, then f is differentiable at a.

21. If $f'(r)$ exists, then $\lim_{x\to r} f(x) = f(r)$.

22. $\displaystyle \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

23. The equation $x^{10} - 10x^2 + 5 = 0$ has a solution in the interval $(0, 2)$.

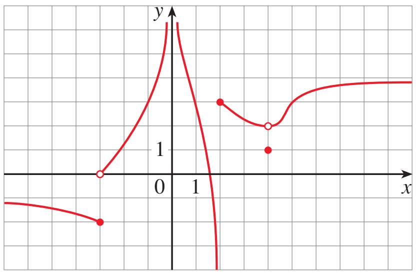
24. If f is continuous at a, so is $|f|$.

25. If $|f|$ is continuous at a, so is f.

26. If f is differentiable at a, so is $|f|$.

EXERCISES

1. The graph of f is given.



(a) Find each limit, or explain why it does not exist.

(i) $\lim_{x\to 2^+} f(x)$ (ii) $\lim_{x\to -3^+} f(x)$ (iii) $\lim_{x\to -3} f(x)$

(iv) $\lim_{x\to 4} f(x)$ (v) $\lim_{x\to 0} f(x)$ (vi) $\lim_{x\to 2^-} f(x)$

(vii) $\lim_{x\to \infty} f(x)$ (viii) $\lim_{x\to -\infty} f(x)$

(b) State the equations of the horizontal asymptotes.

(c) State the equations of the vertical asymptotes.

(d) At what numbers is f discontinuous? Explain.

2. Sketch the graph of a function f that satisfies all of the following conditions:

$$\lim_{x\to -\infty} f(x) = -2, \quad \lim_{x\to \infty} f(x) = 0, \quad \lim_{x\to -3} f(x) = \infty,$$

$$\lim_{x\to 3^-} f(x) = -\infty, \quad \lim_{x\to 3^+} f(x) = 2,$$

f is continuous from the right at 3

3–20 Find the limit.

3. $\displaystyle\lim_{x\to 0} \cos(x^3 + 3x)$

4. $\displaystyle\lim_{x\to 3} \frac{x^2 - 9}{x^2 + 2x - 3}$

5. $\displaystyle\lim_{x\to -3} \frac{x^2 - 9}{x^2 + 2x - 3}$

6. $\displaystyle\lim_{x\to 1^+} \frac{x^2 - 9}{x^2 + 2x - 3}$

7. $\displaystyle\lim_{h\to 0} \frac{(h - 1)^3 + 1}{h}$

8. $\displaystyle\lim_{t\to 2} \frac{t^2 - 4}{t^3 - 8}$

9. $\displaystyle\lim_{r\to 9} \frac{\sqrt{r}}{(r - 9)^4}$

10. $\displaystyle\lim_{v\to 4^+} \frac{4 - v}{|4 - v|}$

11. $\displaystyle\lim_{r\to -1} \frac{r^2 - 3r - 4}{4r^2 + r - 3}$

12. $\displaystyle\lim_{t\to 5} \frac{3 - \sqrt{t + 4}}{t - 5}$

13. $\displaystyle\lim_{x\to \infty} \frac{\sqrt{x^2 - 9}}{2x - 6}$

14. $\displaystyle\lim_{x\to -\infty} \frac{\sqrt{x^2 - 9}}{2x - 6}$

15. $\displaystyle\lim_{x\to \pi^-} \ln(\sin x)$

16. $\displaystyle\lim_{x\to -\infty} \frac{1 - 2x^2 - x^4}{5 + x - 3x^4}$

17. $\displaystyle\lim_{x\to \infty} \left(\sqrt{x^2 + 4x + 1} - x\right)$

18. $\displaystyle\lim_{x\to \infty} e^{x - x^2}$

19. $\displaystyle\lim_{x\to 0^+} \tan^{-1}(1/x)$

20. $\displaystyle\lim_{x\to 1} \left(\frac{1}{x - 1} + \frac{1}{x^2 - 3x + 2}\right)$

21–22 Use graphs to discover the asymptotes of the curve. Then prove what you have discovered.

21. $\displaystyle y = \frac{\cos^2 x}{x^2}$

22. $y = \sqrt{x^2 + x + 1} - \sqrt{x^2 - x}$

23. If $2x - 1 \leqslant f(x) \leqslant x^2$ for $0 < x < 3$, find $\lim_{x\to 1} f(x)$.

24. Prove that $\lim_{x\to 0} x^2 \cos(1/x^2) = 0$.

25–28 Prove the statement using the precise definition of a limit.

25. $\displaystyle\lim_{x\to 2} (14 - 5x) = 4$

26. $\displaystyle\lim_{x\to 0} \sqrt[3]{x} = 0$

27. $\displaystyle\lim_{x\to 2} (x^2 - 3x) = -2$

28. $\displaystyle\lim_{x\to 4^+} \frac{2}{\sqrt{x - 4}} = \infty$

29. Let

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ 3 - x & \text{if } 0 \leqslant x < 3 \\ (x - 3)^2 & \text{if } x > 3 \end{cases}$$

(a) Evaluate each limit, if it exists.

(i) $\lim_{x\to 0^+} f(x)$ (ii) $\lim_{x\to 0^-} f(x)$ (iii) $\lim_{x\to 0} f(x)$

(iv) $\lim_{x\to 3^-} f(x)$ (v) $\lim_{x\to 3^+} f(x)$ (vi) $\lim_{x\to 3} f(x)$

(b) Where is f discontinuous?

(c) Sketch the graph of f.

30. Let

$$g(x) = \begin{cases} 2x - x^2 & \text{if } 0 \leqslant x \leqslant 2 \\ 2 - x & \text{if } 2 < x \leqslant 3 \\ x - 4 & \text{if } 3 < x < 4 \\ \pi & \text{if } x \geqslant 4 \end{cases}$$

(a) For each of the numbers 2, 3, and 4, determine whether g is continuous from the left, continuous from the right, or continuous at the number.

(b) Sketch the graph of g.

31–32 Show that the function is continuous on its domain. State the domain.

31. $h(x) = xe^{\sin x}$

32. $\displaystyle g(x) = \frac{\sqrt{x^2 - 9}}{x^2 - 2}$

33–34 Use the Intermediate Value Theorem to show that there is a solution of the equation in the given interval.

33. $x^5 - x^3 + 3x - 5 = 0, \quad (1, 2)$

34. $\cos\sqrt{x} = e^x - 2, \quad (0, 1)$

35. (a) Find the slope of the tangent line to the curve $y = 9 - 2x^2$ at the point (2, 1).
(b) Find an equation of this tangent line.

36. Find equations of the tangent lines to the curve

$$y = \frac{2}{1 - 3x}$$

at the points with x-coordinates 0 and -1.

37. The displacement (in meters) of an object moving in a straight line is given by $s = 1 + 2t + \frac{1}{4}t^2$, where t is measured in seconds.
(a) Find the average velocity over each time period.
(i) $[1, 3]$ (ii) $[1, 2]$ (iii) $[1, 1.5]$ (iv) $[1, 1.1]$
(b) Find the instantaneous velocity when $t = 1$.

38. According to Boyle's Law, if the temperature of a confined gas is held fixed, then the product of the pressure P and the volume V is a constant. Suppose that, for a certain gas, $PV = 4000$, where P is measured in pascals and V is measured in liters.
(a) Find the average rate of change of P as V increases from 3 L to 4 L.
(b) Express V as a function of P and show that the instantaneous rate of change of V with respect to P is inversely proportional to the square of P.

39. (a) Use the definition of a derivative to find $f'(2)$, where $f(x) = x^3 - 2x$.
(b) Find an equation of the tangent line to the curve $y = x^3 - 2x$ at the point (2, 4).
(c) Illustrate part (b) by graphing the curve and the tangent line on the same screen.

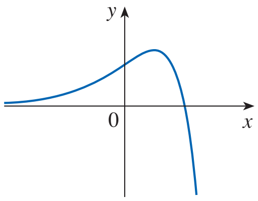
40. Find a function f and a number a such that

$$\lim_{h \to 0} \frac{(2 + h)^6 - 64}{h} = f'(a)$$

41. The total cost of repaying a student loan at an interest rate of r% per year is $C = f(r)$.
(a) What is the meaning of the derivative $f'(r)$? What are its units?
(b) What does the statement $f'(10) = 1200$ mean?
(c) Is $f'(r)$ always positive or does it change sign?

42–44 Trace or copy the graph of the function. Then sketch a graph of its derivative directly beneath.

42.



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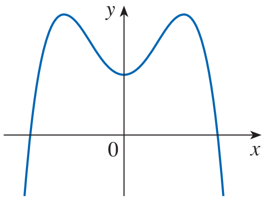

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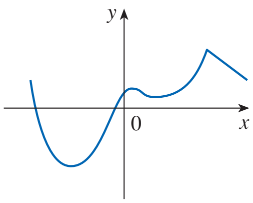


45–46 Find the derivative of f using the definition of a derivative. What is the domain of f'?

45. $f(x) = \dfrac{2}{x^2}$

46. $f(t) = \dfrac{1}{\sqrt{t + 1}}$

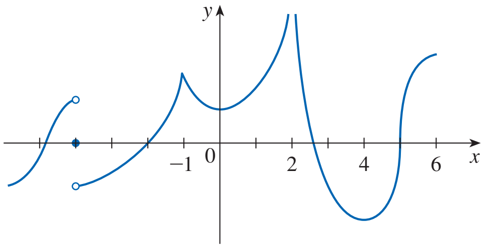
47. (a) If $f(x) = \sqrt{3 - 5x}$, use the definition of a derivative to find $f'(x)$.
(b) Find the domains of f and f'.
(c) Graph f and f' on a common screen. Compare the graphs to see whether your answer to part (a) is reasonable.

48. (a) Find the asymptotes of the graph of

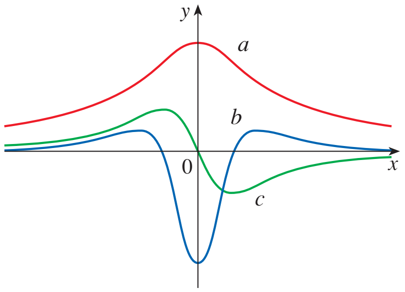
$$f(x) = \frac{4 - x}{3 + x}$$

and use them to sketch the graph.
(b) Use your graph from part (a) to sketch the graph of f'.
(c) Use the definition of a derivative to find $f'(x)$.
(d) Graph f' and compare with your sketch in part (b).

49. The graph of f is shown. State, with reasons, the numbers at which f is not differentiable.



50. The figure shows the graphs of f, f', and f''. Identify each curve, and explain your choices.



51. Sketch the graph of a function f that satisfies all of the following conditions:

The domain of f is all real numbers except 0,

$\lim_{x\to 0^-} f(x) = 1, \quad \lim_{x\to 0^+} f(x) = 0,$

$f'(x) > 0$ for all x in the domain of f,

$\lim_{x\to -\infty} f'(x) = 0, \quad \lim_{x\to \infty} f'(x) = 1$

52. Let $P(t)$ be the percentage of Americans under the age of 18 at time t. The table gives values of this function in census years from 1950 to 2010.

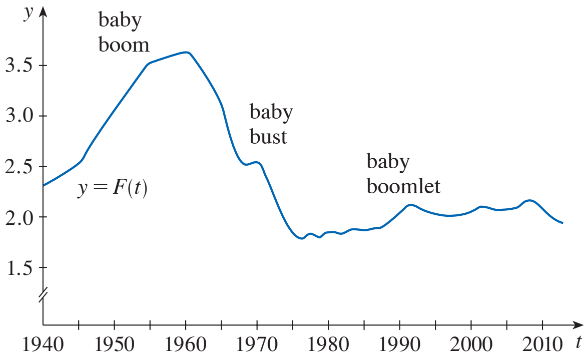
t	$P(t)$	t	$P(t)$
1950	31.1	1990	25.7
1960	35.7	2000	25.7
1970	34.0	2010	24.0
1980	28.0		

(a) What is the meaning of $P'(t)$? What are its units?
(b) Construct a table of estimated values for $P'(t)$.
(c) Graph P and P'.
(d) How would it be possible to get more accurate values for $P'(t)$?

53. Let $B(t)$ be the number of US \$20 bills in circulation at time t. The table gives values of this function from 1995 to 2015, as of December 31, in billions. Interpret and estimate the value of $B'(2010)$.

t	1995	2000	2005	2010	2015
$B(t)$	4.21	4.93	5.77	6.53	8.57

54. The *total fertility rate* at time t, denoted by $F(t)$, is an estimate of the average number of children born to each woman (assuming that current birth rates remain constant). The graph of the total fertility rate in the United States shows the fluctuations from 1940 to 2010.
(a) Estimate the values of $F'(1950)$, $F'(1965)$, and $F'(1987)$.
(b) What are the meanings of these derivatives?
(c) Can you suggest reasons for the values of these derivatives?



55. Suppose that $|f(x)| \leqslant g(x)$ for all x, where $\lim_{x\to a} g(x) = 0$. Find $\lim_{x\to a} f(x)$.

56. Let $f(x) = [\![x]\!] + [\![-x]\!]$.
(a) For what values of a does $\lim_{x\to a} f(x)$ exist?
(b) At what numbers is f discontinuous?

Problems Plus

In the Principles of Problem Solving following Chapter 1 we considered the problem-solving strategy of *introducing something extra*. In the following example we show how this principle is sometimes useful when we evaluate limits. The idea is to change the variable—to introduce a new variable that is related to the original variable—in such a way as to make the problem simpler. Later, in Section 5.5, we will make more extensive use of this general idea.

EXAMPLE Evaluate $\lim_{x \to 0} \dfrac{\sqrt[3]{1 + cx} - 1}{x}$, where c is a constant.

SOLUTION As it stands, this limit looks challenging. In Section 2.3 we evaluated limits in which both numerator and denominator approached 0. There our strategy was to perform some sort of algebraic manipulation that led to a simplifying cancellation, but here it's not clear what kind of algebra is necessary.

So we introduce a new variable t by the equation

$$t = \sqrt[3]{1 + cx}$$

We also need to express x in terms of t, so we solve this equation:

$$t^3 = 1 + cx \qquad x = \frac{t^3 - 1}{c} \quad (\text{if } c \neq 0)$$

Notice that $x \to 0$ is equivalent to $t \to 1$. This allows us to convert the given limit into one involving the variable t:

$$\begin{aligned} \lim_{x \to 0} \frac{\sqrt[3]{1 + cx} - 1}{x} &= \lim_{t \to 1} \frac{t - 1}{(t^3 - 1)/c} \\ &= \lim_{t \to 1} \frac{c(t - 1)}{t^3 - 1} \end{aligned}$$

The change of variable allowed us to replace a relatively complicated limit by a simpler one of a type that we have seen before. Factoring the denominator as a difference of cubes, we get

$$\begin{aligned} \lim_{t \to 1} \frac{c(t - 1)}{t^3 - 1} &= \lim_{t \to 1} \frac{c(t - 1)}{(t - 1)(t^2 + t + 1)} \\ &= \lim_{t \to 1} \frac{c}{t^2 + t + 1} = \frac{c}{3} \end{aligned}$$

In making the change of variable we had to rule out the case $c = 0$. But if $c = 0$, the function is 0 for all nonzero x and so its limit is 0. Therefore, in all cases, the limit is $c/3$. ■

The following problems are meant to test and challenge your problem-solving skills. Some of them require a considerable amount of time to think through, so don't be discouraged if you can't solve them right away. If you get stuck, you might find it helpful to refer to the discussion of the principles of problem solving following Chapter 1.

Problems

1. Evaluate $\lim_{x \to 1} \dfrac{\sqrt[3]{x} - 1}{\sqrt{x} - 1}$.
2. Find numbers a and b such that $\lim_{x \to 0} \dfrac{\sqrt{ax + b} - 2}{x} = 1$.
3. Evaluate $\lim_{x \to 0} \dfrac{|2x - 1| - |2x + 1|}{x}$.

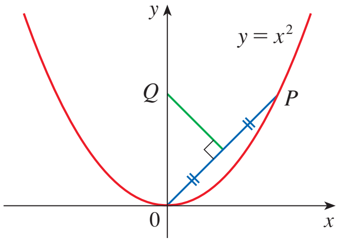


FIGURE FOR PROBLEM 4

4. The figure shows a point P on the parabola $y = x^2$ and the point Q where the perpendicular bisector of OP intersects the y-axis. As P approaches the origin along the parabola, what happens to Q? Does it have a limiting position? If so, find it.

5. Evaluate the following limits, if they exist, where $[\![x]\!]$ denotes the greatest integer function.

(a) $\displaystyle\lim_{x \to 0} \frac{[\![x]\!]}{x}$ (b) $\displaystyle\lim_{x \to 0} x\,[\![1/x]\!]$

6. Sketch the region in the plane defined by each of the following equations.

(a) $[\![x]\!]^2 + [\![y]\!]^2 = 1$ (b) $[\![x]\!]^2 - [\![y]\!]^2 = 3$

(c) $[\![x + y]\!]^2 = 1$ (d) $[\![x]\!] + [\![y]\!] = 1$

7. Let $f(x) = x/[\![x]\!]$.

(a) Find the domain and range of f. (b) Evaluate $\lim_{x \to \infty} f(x)$.

8. A **fixed point** of a function f is a number c in its domain such that $f(c) = c$. (The function doesn't move c; it stays fixed.)

(a) Sketch the graph of a continuous function with domain $[0, 1]$ whose range also lies in $[0, 1]$. Locate a fixed point of f.

(b) Try to draw the graph of a continuous function with domain $[0, 1]$ and range in $[0, 1]$ that does *not* have a fixed point. What is the obstacle?

(c) Use the Intermediate Value Theorem to prove that any continuous function with domain $[0, 1]$ and range in $[0, 1]$ must have a fixed point.

9. If $\lim_{x \to a} [f(x) + g(x)] = 2$ and $\lim_{x \to a} [f(x) - g(x)] = 1$, find $\lim_{x \to a} [f(x)\,g(x)]$.

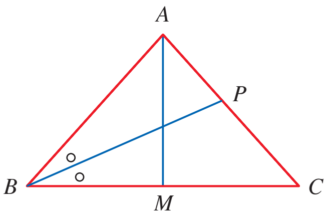


FIGURE FOR PROBLEM 10

10. (a) The figure shows an isosceles triangle ABC with $\angle B = \angle C$. The bisector of angle B intersects the side AC at the point P. Suppose that the base BC remains fixed but the altitude $|AM|$ of the triangle approaches 0, so A approaches the midpoint M of BC. What happens to P during this process? Does it have a limiting position? If so, find it.

(b) Try to sketch the path traced out by P during this process. Then find an equation of this curve and use this equation to sketch the curve.

11. (a) If we start from 0° latitude and proceed in a westerly direction, we can let $T(x)$ denote the temperature at the point x at any given time. Assuming that T is a continuous function of x, show that at any fixed time there are at least two diametrically opposite points on the equator that have exactly the same temperature.

(b) Does the result in part (a) hold for points lying on any circle on the earth's surface?

(c) Does the result in part (a) hold for barometric pressure and for altitude above sea level?

12. If f is a differentiable function and $g(x) = x f(x)$, use the definition of a derivative to show that $g'(x) = x f'(x) + f(x)$.

13. Suppose f is a function that satisfies the equation

$$f(x + y) = f(x) + f(y) + x^2 y + x y^2$$

for all real numbers x and y. Suppose also that

$$\lim_{x \to 0} \frac{f(x)}{x} = 1$$

(a) Find $f(0)$. (b) Find $f'(0)$. (c) Find $f'(x)$.

14. Suppose f is a function with the property that $|f(x)| \leqslant x^2$ for all x. Show that $f(0) = 0$. Then show that $f'(0) = 0$.

In the project following Section 3.4 you will calculate the distance from an airport runway at which a pilot should start descent for a smooth landing.

Who is Danny / Shutterstock.com

3

Differentiation Rules

WE HAVE SEEN HOW TO INTERPRET derivatives as slopes and rates of change. We have used the definition of a derivative to calculate the derivatives of functions defined by formulas. But it would be tedious if we always had to use the definition, so in this chapter we develop rules for finding derivatives without having to use the definition directly. These differentiation rules enable us to calculate with relative ease the derivatives of polynomials, rational functions, algebraic functions, exponential and logarithmic functions, and trigonometric and inverse trigonometric functions. We then use these rules to solve problems involving rates of change and the approximation of functions.

3.1 Derivatives of Polynomials and Exponential Functions

In this section we learn how to differentiate constant functions, power functions, polynomials, and exponential functions.

Constant Functions

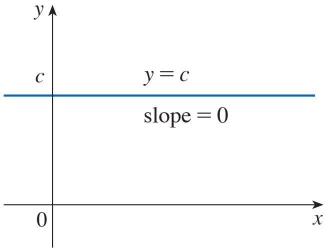


FIGURE 1
The graph of $f(x) = c$ is the line $y = c$, so $f'(x) = 0$.

Let's start with the simplest of all functions, the constant function $f(x) = c$. The graph of this function is the horizontal line $y = c$, which has slope 0, so we must have $f'(x) = 0$. (See Figure 1.) A formal proof, from the definition of a derivative, is also easy:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0$$

In Leibniz notation, we write this rule as follows.

Derivative of a Constant Function

$$\frac{d}{dx}(c) = 0$$

Power Functions

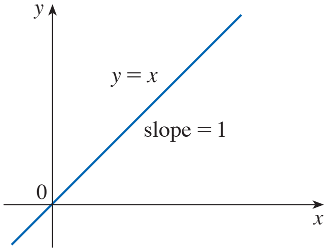


FIGURE 2
The graph of $f(x) = x$ is the line $y = x$, so $f'(x) = 1$.

We next look at the functions $f(x) = x^n$, where n is a positive integer. If $n = 1$, the graph of $f(x) = x$ is the line $y = x$, which has slope 1. (See Figure 2.) So

$$\boxed{1} \qquad \boxed{\frac{d}{dx}(x) = 1}$$

(You can also verify Equation 1 from the definition of a derivative.) We have already investigated the cases $n = 2$ and $n = 3$. In fact, in Section 2.8 (Exercises 19 and 20) we found that

$$\boxed{2} \qquad \frac{d}{dx}(x^2) = 2x \qquad \frac{d}{dx}(x^3) = 3x^2$$

For $n = 4$ we find the derivative of $f(x) = x^4$ as follows:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^4 - x^4}{h} \\ &= \lim_{h \to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} \\ &= \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \\ &= \lim_{h \to 0} (4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3 \end{aligned}$$

Thus

$$\boxed{3} \qquad \frac{d}{dx}(x^4) = 4x^3$$

Comparing the equations in (1), (2), and (3), we see a pattern emerging. It seems to be a reasonable guess that, when n is a positive integer, $(d/dx)(x^n) = nx^{n-1}$. This turns out to be true. We prove it in two ways; the second proof uses the Binomial Theorem.

The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

FIRST PROOF The formula

$$x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})$$

can be verified simply by multiplying out the right-hand side (or by summing the second factor as a geometric series). If $f(x) = x^n$, we can use Equation 2.7.5 for $f'(a)$ and the equation above to write

$$\begin{aligned} f'(a) &= \lim_{x\to a}\frac{f(x) - f(a)}{x - a} = \lim_{x\to a}\frac{x^n - a^n}{x - a} \\ &= \lim_{x\to a}(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}) \\ &= a^{n-1} + a^{n-2}a + \cdots + aa^{n-2} + a^{n-1} \\ &= na^{n-1} \end{aligned}$$

SECOND PROOF

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

The Binomial Theorem is given on Reference Page 1.

In finding the derivative of x^4 we had to expand $(x + h)^4$. Here we need to expand $(x + h)^n$ and we use the Binomial Theorem to do so:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{\left[x^n + nx^{n-1}h + \dfrac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h} \\ &= \lim_{h\to 0}\frac{nx^{n-1}h + \dfrac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\ &= \lim_{h\to 0}\left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right] \\ &= nx^{n-1} \end{aligned}$$

because every term except the first has h as a factor and therefore approaches 0. ■

We illustrate the Power Rule using various notations in Example 1.

EXAMPLE 1

(a) If $f(x) = x^6$, then $f'(x) = 6x^5$. (b) If $y = x^{1000}$, then $y' = 1000x^{999}$.

(c) If $y = t^4$, then $\dfrac{dy}{dt} = 4t^3$. (d) $\dfrac{d}{dr}(r^3) = 3r^2$ ■

What about power functions with negative integer exponents? In Exercise 69 we ask you to verify from the definition of a derivative that

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

We can rewrite this equation as

$$\frac{d}{dx}(x^{-1}) = (-1)x^{-2}$$

and so the Power Rule is true when $n = -1$. In fact, we will show in the next section [Exercise 3.2.66(c)] that it holds for all negative integers.

What if the exponent is a fraction? In Example 2.8.3 we found that

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

which can be written as

$$\frac{d}{dx}(x^{1/2}) = \tfrac{1}{2}x^{-1/2}$$

This shows that the Power Rule is true even when the exponent is $\frac{1}{2}$. In fact, we will show in Section 3.6 that it is true for all real exponents n.

The Power Rule (General Version) If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

EXAMPLE 2 Differentiate:

(a) $f(x) = \dfrac{1}{x^2}$ (b) $y = \sqrt[3]{x^2}$

SOLUTION In each case we rewrite the function as a power of x.

(a) Since $f(x) = x^{-2}$, we use the Power Rule with $n = -2$:

$$f'(x) = \frac{d}{dx}(x^{-2}) = -2x^{-2-1} = -2x^{-3} = -\frac{2}{x^3}$$

(b)

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt[3]{x^2}\right) = \frac{d}{dx}(x^{2/3}) = \tfrac{2}{3}x^{(2/3)-1} = \tfrac{2}{3}x^{-1/3}$$ ■

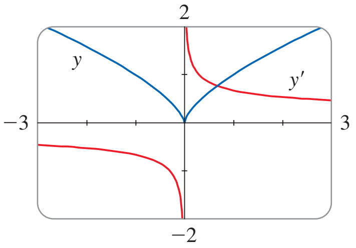


FIGURE 3
$y = \sqrt[3]{x^2}$

Figure 3 shows the function y in Example 2(b) and its derivative y'. Notice that y is not differentiable at 0 (y' is not defined there). Also observe that the function y is increasing when y' is positive and is decreasing when y' is negative. In Chapter 4 we will prove that, in general, *a function increases when its derivative is positive and decreases when its derivative is negative.*

The Power Rule enables us to find tangent lines without having to resort to the definition of a derivative. It also enables us to find *normal lines*. The **normal line** to a curve C at a point P is the line through P that is perpendicular to the tangent line at P. (In the study of optics, the Law of Reflection involves the angle between a light ray and the normal line to a lens.)

EXAMPLE 3 Find equations of the tangent line and normal line to the curve $y = x\sqrt{x}$ at the point $(1, 1)$. Illustrate by graphing the curve and these lines.

SOLUTION The derivative of $f(x) = x\sqrt{x} = xx^{1/2} = x^{3/2}$ is

$$f'(x) = \tfrac{3}{2}x^{(3/2)-1} = \tfrac{3}{2}x^{1/2} = \tfrac{3}{2}\sqrt{x}$$

So the slope of the tangent line at $(1, 1)$ is $f'(1) = \frac{3}{2}$. Therefore an equation of the tangent line is

$$y - 1 = \tfrac{3}{2}(x - 1) \qquad \text{or} \qquad y = \tfrac{3}{2}x - \tfrac{1}{2}$$

The normal line is perpendicular to the tangent line, so its slope is the negative reciprocal of $\frac{3}{2}$, that is, $-\frac{2}{3}$. Thus an equation of the normal line is

$$y - 1 = -\tfrac{2}{3}(x - 1) \qquad \text{or} \qquad y = -\tfrac{2}{3}x + \tfrac{5}{3}$$

We graph the curve and its tangent line and normal line in Figure 4. ■

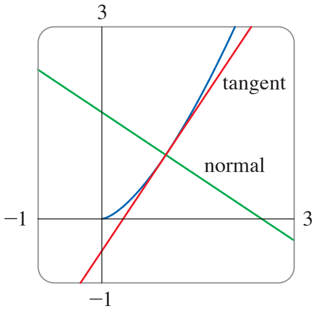
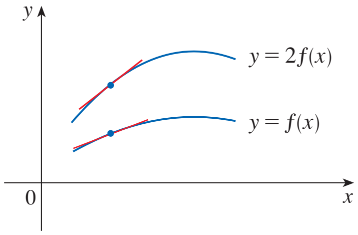


FIGURE 4
$y = x\sqrt{x}$

New Derivatives from Old

When new functions are formed from old functions by addition, subtraction, or multiplication by a constant, their derivatives can be calculated in terms of derivatives of the old functions. In particular, the following formula says that *the derivative of a constant times a function is the constant times the derivative of the function.*

Geometric Interpretation of the Constant Multiple Rule



Multiplying by $c = 2$ stretches the graph vertically by a factor of 2. All the rises have been doubled but the runs stay the same. So the slopes are also doubled.

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

PROOF Let $g(x) = cf(x)$. Then

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h) - cf(x)}{h} \\ &= \lim_{h\to 0} c\left[\frac{f(x+h) - f(x)}{h}\right] \\ &= c\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \qquad \text{(by Limit Law 3)} \\ &= cf'(x) \end{aligned}$$

■

EXAMPLE 4

(a) $\dfrac{d}{dx}(3x^4) = 3\dfrac{d}{dx}(x^4) = 3(4x^3) = 12x^3$

(b) $\dfrac{d}{dx}(-x) = \dfrac{d}{dx}[(-1)x] = (-1)\dfrac{d}{dx}(x) = -1(1) = -1$ ■

The next rule tells us that *the derivative of a sum (or difference) of functions is the sum (or difference) of the derivatives*.

Using prime notation, we can write the Sum and Difference Rules as

$$(f+g)' = f' + g'$$

$$(f-g)' = f' - g'$$

The Sum and Difference Rules If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

PROOF To prove the Sum Rule, we let $F(x) = f(x) + g(x)$. Then

$$\begin{aligned}
F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} \\
&= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\
&= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right] \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} && \text{(by Limit Law 1)} \\
&= f'(x) + g'(x)
\end{aligned}$$

To prove the Difference Rule, we write $f - g$ as $f + (-1)g$ and apply the Sum Rule and the Constant Multiple Rule. ■

The Sum Rule can be extended to the sum of any number of functions. For instance, using this theorem twice, we get

$$(f+g+h)' = [(f+g)+h]' = (f+g)' + h' = f' + g' + h'$$

The Constant Multiple Rule, the Sum Rule, and the Difference Rule can be combined with the Power Rule to differentiate any polynomial, as the following three examples demonstrate.

EXAMPLE 5

$$\begin{aligned}
&\frac{d}{dx}(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5) \\
&\qquad = \frac{d}{dx}(x^8) + 12\frac{d}{dx}(x^5) - 4\frac{d}{dx}(x^4) + 10\frac{d}{dx}(x^3) - 6\frac{d}{dx}(x) + \frac{d}{dx}(5) \\
&\qquad = 8x^7 + 12(5x^4) - 4(4x^3) + 10(3x^2) - 6(1) + 0 \\
&\qquad = 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6
\end{aligned}$$

■

EXAMPLE 6 Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

SOLUTION Horizontal tangents occur where the derivative is zero. We have

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^4) - 6\frac{d}{dx}(x^2) + \frac{d}{dx}(4)\\ &= 4x^3 - 12x + 0 = 4x(x^2 - 3)\end{aligned}$$

Thus $dy/dx = 0$ if $x = 0$ or $x^2 - 3 = 0$, that is, $x = \pm\sqrt{3}$. So the given curve has horizontal tangents when $x = 0$, $\sqrt{3}$, and $-\sqrt{3}$. The corresponding points are $(0, 4)$, $(\sqrt{3}, -5)$, and $(-\sqrt{3}, -5)$. (See Figure 5.) ■

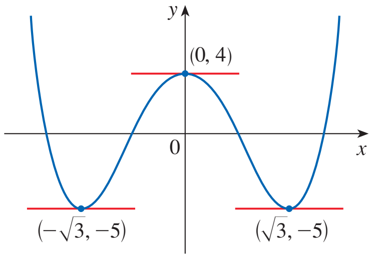


FIGURE 5
The curve $y = x^4 - 6x^2 + 4$ and its horizontal tangents

EXAMPLE 7 The equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 4$, where s is measured in centimeters and t in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?

SOLUTION The velocity and acceleration are

$$v(t) = \frac{ds}{dt} = 6t^2 - 10t + 3$$

$$a(t) = \frac{dv}{dt} = 12t - 10$$

The acceleration after 2 seconds is $a(2) = 14\ \text{cm/s}^2$. ■

Exponential Functions

Let's try to compute the derivative of the exponential function $f(x) = b^x$ using the definition of a derivative:

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{b^{x+h} - b^x}{h}\\ &= \lim_{h\to 0}\frac{b^x b^h - b^x}{h} = \lim_{h\to 0}\frac{b^x(b^h - 1)}{h}\end{aligned}$$

The factor b^x doesn't depend on h, so we can take it in front of the limit:

$$f'(x) = b^x \lim_{h\to 0}\frac{b^h - 1}{h}$$

Notice that the limit is the value of the derivative of f at 0, that is,

$$\lim_{h\to 0}\frac{b^h - 1}{h} = f'(0)$$

Therefore we have shown that if the exponential function $f(x) = b^x$ is differentiable at 0, then it is differentiable everywhere and

$$\boxed{4} \qquad f'(x) = f'(0)\,b^x$$

This equation says that *the rate of change of any exponential function is proportional to the function itself*. (The slope is proportional to the height.)

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$
0.1	0.71773	1.16123
0.01	0.69556	1.10467
0.001	0.69339	1.09922
0.0001	0.69317	1.09867
0.00001	0.69315	1.09862

Numerical evidence for the existence of $f'(0)$ is given in the table at the left for the cases $b = 2$ and $b = 3$. (Values are stated correct to four decimal places.) It can be proved that the limits exist and

$$\text{for } b = 2, \quad f'(0) = \lim_{h \to 0} \frac{2^h - 1}{h} \approx 0.693$$

$$\text{for } b = 3, \quad f'(0) = \lim_{h \to 0} \frac{3^h - 1}{h} \approx 1.099$$

Thus, from Equation 4 we have

$$\boxed{5} \qquad \frac{d}{dx}(2^x) \approx (0.693)2^x \qquad \frac{d}{dx}(3^x) \approx (1.099)3^x$$

Of all possible choices for the base b in Equation 4, the simplest differentiation formula occurs when $f'(0) = 1$. In view of the estimates of $f'(0)$ for $b = 2$ and $b = 3$, it seems reasonable that there is a number b between 2 and 3 for which $f'(0) = 1$. It is traditional to denote this value by the letter e. (In fact, that is how we introduced e in Section 1.4.) Thus we have the following definition.

In Exercise 1 we will see that e lies between 2.7 and 2.8. Later we will be able to show that, correct to five decimal places,

$$e \approx 2.71828$$

Definition of the Number e

$$e \text{ is the number such that } \quad \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

Geometrically, this means that of all possible exponential functions $y = b^x$, the function $f(x) = e^x$ is the one whose tangent line at (0, 1) has a slope $f'(0)$ that is exactly 1. (See Figures 6 and 7.)

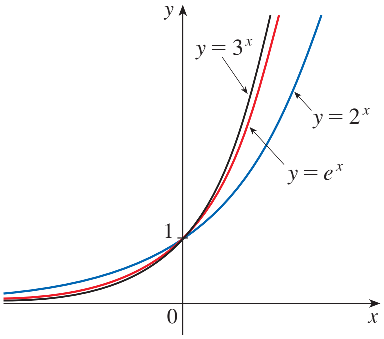


FIGURE 6

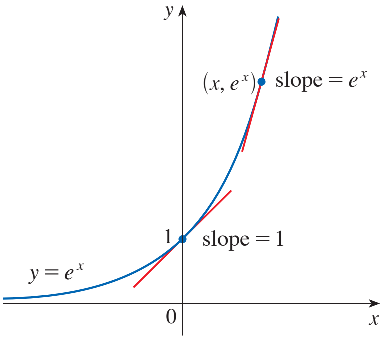


FIGURE 7

If we put $b = e$ and, therefore, $f'(0) = 1$ in Equation 4, it becomes the following important differentiation formula.

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

Thus the exponential function $f(x) = e^x$ has the property that it is its own derivative. The geometrical significance of this fact is that the slope of a tangent line to the curve $y = e^x$ at a point (x, e^x) is equal to the y-coordinate of the point (see Figure 7).

EXAMPLE 8 If $f(x) = e^x - x$, find f' and f''. Compare the graphs of f and f'.

SOLUTION Using the Difference Rule, we have

$$f'(x) = \frac{d}{dx}(e^x - x) = \frac{d}{dx}(e^x) - \frac{d}{dx}(x) = e^x - 1$$

In Section 2.8 we defined the second derivative as the derivative of f', so

$$f''(x) = \frac{d}{dx}(e^x - 1) = \frac{d}{dx}(e^x) - \frac{d}{dx}(1) = e^x$$

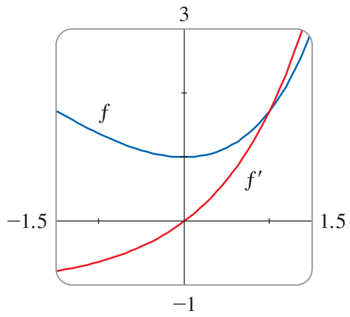


FIGURE 8

The function f and its derivative f' are graphed in Figure 8. Notice that f has a horizontal tangent when $x = 0$; this corresponds to the fact that $f'(0) = 0$. Notice also that, for $x > 0$, $f'(x)$ is positive and f is increasing. When $x < 0$, $f'(x)$ is negative and f is decreasing. ■

EXAMPLE 9 At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

SOLUTION Since $y = e^x$, we have $y' = e^x$. Let the x-coordinate of the point in question be a. Then the slope of the tangent line at that point is e^a. This tangent line will be parallel to the line $y = 2x$ if it has the same slope, that is, 2. Equating slopes, we get

$$e^a = 2 \qquad a = \ln 2$$

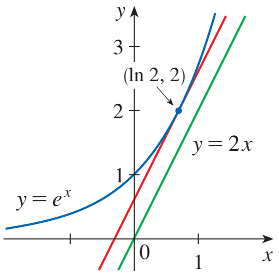


FIGURE 9

Therefore the required point is $(a, e^a) = (\ln 2, 2)$. (See Figure 9.) ■

3.1 Exercises

1. (a) How is the number e defined?

(b) Use a calculator to estimate the values of the limits

$$\lim_{h \to 0} \frac{2.7^h - 1}{h} \qquad \text{and} \qquad \lim_{h \to 0} \frac{2.8^h - 1}{h}$$

correct to two decimal places. What can you conclude about the value of e?

2. (a) Sketch, by hand, the graph of the function $f(x) = e^x$, paying particular attention to how the graph crosses the y-axis. What is the slope of the tangent line at that point?

(b) What types of functions are $f(x) = e^x$ and $g(x) = x^e$? Compare the differentiation formulas for f and g.

(c) Which of the two functions in part (b) grows more rapidly when x is large?

3–34 Differentiate the function.

3. $g(x) = 4x + 7$

4. $g(t) = 5t + 4t^2$

5. $f(x) = x^{75} - x + 3$

6. $g(x) = \frac{7}{4}x^2 - 3x + 12$

7. $f(t) = -2e^t$

8. $F(t) = t^3 + e^3$

9. $W(v) = 1.8v^{-3}$

10. $r(z) = z^{-5} - z^{1/2}$

11. $f(x) = x^{3/2} + x^{-3}$

12. $V(t) = t^{-3/5} + t^4$

13. $s(t) = \dfrac{1}{t} + \dfrac{1}{t^2}$

14. $r(t) = \dfrac{a}{t^2} + \dfrac{b}{t^4}$

15. $y = 2x + \sqrt{x}$

16. $h(w) = \sqrt{2}\,w - \sqrt{2}$

17. $g(x) = \dfrac{1}{\sqrt{x}} + \sqrt[4]{x}$

18. $W(t) = \sqrt{t} - 2e^t$

19. $f(x) = x^3(x + 3)$

20. $F(t) = (2t - 3)^2$

21. $y = 3e^x + \dfrac{4}{\sqrt[3]{x}}$

22. $S(R) = 4\pi R^2$

23. $f(x) = \dfrac{3x^2 + x^3}{x}$

24. $y = \dfrac{\sqrt{x} + x}{x^2}$

25. $G(r) = \dfrac{3r^{3/2} + r^{5/2}}{r}$

26. $G(t) = \sqrt{5t} + \dfrac{\sqrt{7}}{t}$

27. $j(x) = x^{2.4} + e^{2.4}$

28. $k(r) = e^r + r^e$

29. $F(z) = \dfrac{A + Bz + Cz^2}{z^2}$

30. $G(q) = (1 + q^{-1})^2$

31. $D(t) = \dfrac{1 + 16t^2}{(4t)^3}$

32. $f(v) = \dfrac{\sqrt[3]{v} - 2ve^v}{v}$

33. $P(w) = \dfrac{2w^2 - w + 4}{\sqrt{w}}$

34. $y = e^{x+1} + 1$

35–36 Find dy/dx and dy/dt.

35. $y = tx^2 + t^3x$

36. $y = \dfrac{t}{x^2} + \dfrac{x}{t}$

37–40 Find an equation of the tangent line to the curve at the given point.

37. $y = 2x^3 - x^2 + 2, \quad (1, 3)$

38. $y = 2e^x + x, \quad (0, 2)$

39. $y = x + \dfrac{2}{x}, \quad (2, 3)$

40. $y = \sqrt[4]{x} - x, \quad (1, 0)$

41–42 Find equations of the tangent line and normal line to the curve at the given point.

41. $y = x^4 + 2e^x, \quad (0, 2)$

42. $y = x^{3/2}, \quad (1, 1)$

43–44 Find an equation of the tangent line to the curve at the given point. Illustrate by graphing the curve and the tangent line on the same screen.

43. $y = 3x^2 - x^3, \quad (1, 2)$

44. $y = x - \sqrt{x}, \quad (1, 0)$

45–46 Find $f'(x)$. Compare the graphs of f and f' and use them to explain why your answer is reasonable.

45. $f(x) = x^4 - 2x^3 + x^2$

46. $f(x) = x^5 - 2x^3 + x - 1$

47. (a) Graph the function

$$f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30$$

in the viewing rectangle $[-3, 5]$ by $[-10, 50]$.

(b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of f'. (See Example 2.8.1.)

(c) Calculate $f'(x)$ and use this expression to graph f'. Compare with your sketch in part (b).

48. (a) Graph the function $g(x) = e^x - 3x^2$ in the viewing rectangle $[-1, 4]$ by $[-8, 8]$.

(b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of g'. (See Example 2.8.1.)

(c) Calculate $g'(x)$ and use this expression to graph g'. Compare with your sketch in part (b).

49–50 Find the first and second derivatives of the function.

49. $f(x) = 0.001x^5 - 0.02x^3$

50. $G(r) = \sqrt{r} + \sqrt[3]{r}$

51–52 Find the first and second derivatives of the function. Check to see that your answers are reasonable by comparing the graphs of f, f', and f''.

51. $f(x) = 2x - 5x^{3/4}$

52. $f(x) = e^x - x^3$

53. The equation of motion of a particle is $s = t^3 - 3t$, where s is in meters and t is in seconds. Find

(a) the velocity and acceleration as functions of t,

(b) the acceleration after 2 s, and

(c) the acceleration when the velocity is 0.

54. The equation of motion of a particle is $s = t^4 - 2t^3 + t^2 - t$, where s is in meters and t is in seconds.

(a) Find the velocity and acceleration as functions of t.

(b) Find the acceleration after 1 s.

(c) Graph the position, velocity, and acceleration functions on the same screen.

55. Biologists have proposed a cubic polynomial to model the length L of Alaskan rockfish at age A:

$$L = 0.0390A^3 - 0.945A^2 + 10.03A + 3.07$$

where L is measured in centimeters and A in years. Calculate

$$\left.\frac{dL}{dA}\right|_{A=12}$$

and interpret your answer.

56. The number of tree species S in a given area A in a forest reserve has been modeled by the power function

$$S(A) = 0.882A^{0.842}$$

where A is measured in square meters. Find $S'(100)$ and interpret your answer.

57. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P of the gas is inversely proportional to the volume V of the gas.
(a) Suppose that the pressure of a sample of air that occupies $0.106\ \text{m}^3$ at 25°C is 50 kPa. Write V as a function of P.
(b) Calculate dV/dP when $P = 50$ kPa. What is the meaning of the derivative? What are its units?

58. Car tires need to be inflated properly because overinflation or underinflation can cause premature tread wear. The data in the table show tire life L (in thousands of kilometers) for a certain type of tire at various pressures P (in kPa).

P	179	193	214	242	262	290	311
L	80	106	126	130	119	113	95

(a) Use a calculator or computer to model tire life with a quadratic function of the pressure.
(b) Use the model to estimate dL/dP when $P = 200$ and when $P = 300$. What is the meaning of the derivative? What are the units? What is the significance of the signs of the derivatives?

59. Find the points on the curve $y = x^3 + 3x^2 - 9x + 10$ where the tangent is horizontal.

60. For what value of x does the graph of $f(x) = e^x - 2x$ have a horizontal tangent?

61. Show that the curve $y = 2e^x + 3x + 5x^3$ has no tangent line with slope 2.

62. Find an equation of the line that is both tangent to the curve $y = x^4 + 1$ and parallel to the line $32x - y = 15$.

63. Find equations for two lines that are both tangent to the curve $y = x^3 - 3x^2 + 3x - 3$ and parallel to the line $3x - y = 15$.

64. At what point on the curve $y = 1 + 2e^x - 3x$ is the tangent line parallel to the line $3x - y = 5$? Illustrate by graphing the curve and both lines.

65. Find an equation of the normal line to the curve $y = \sqrt{x}$ that is parallel to the line $2x + y = 1$.

66. Where does the normal line to the parabola $y = x^2 - 1$ at the point $(-1, 0)$ intersect the parabola a second time? Illustrate with a sketch.

67. Draw a diagram to show that there are two tangent lines to the parabola $y = x^2$ that pass through the point $(0, -4)$. Find the coordinates of the points where these tangent lines intersect the parabola.

68. (a) Find equations of both lines through the point $(2, -3)$ that are tangent to the parabola $y = x^2 + x$.
(b) Show that there is no line through the point $(2, 7)$ that is tangent to the parabola. Then draw a diagram to see why.

69. Use the definition of a derivative to show that if $f(x) = 1/x$, then $f'(x) = -1/x^2$. (This proves the Power Rule for the case $n = -1$.)

70. Find the nth derivative of each function by calculating the first few derivatives and observing the pattern that occurs.
(a) $f(x) = x^n$ (b) $f(x) = 1/x$

71. Find a second-degree polynomial P such that $P(2) = 5$, $P'(2) = 3$, and $P''(2) = 2$.

72. The equation $y'' + y' - 2y = x^2$ is called a **differential equation** because it involves an unknown function y and its derivatives y' and y''. Find constants A, B, and C such that the function $y = Ax^2 + Bx + C$ satisfies this equation. (Differential equations will be studied in detail in Chapter 9.)

73. Find a cubic function $y = ax^3 + bx^2 + cx + d$ whose graph has horizontal tangents at the points $(-2, 6)$ and $(2, 0)$.

74. Find a parabola with equation $y = ax^2 + bx + c$ that has slope 4 at $x = 1$, slope -8 at $x = -1$, and passes through the point $(2, 15)$.

75. Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ x + 1 & \text{if } x \geqslant 1 \end{cases}$$

Is f differentiable at 1? Sketch the graphs of f and f'.

76. At what numbers is the following function g differentiable?

$$g(x) = \begin{cases} 2x & \text{if } x \leqslant 0 \\ 2x - x^2 & \text{if } 0 < x < 2 \\ 2 - x & \text{if } x \geqslant 2 \end{cases}$$

Give a formula for g' and sketch the graphs of g and g'.

77. (a) For what values of x is the function $f(x) = |x^2 - 9|$ differentiable? Find a formula for f'.
(b) Sketch the graphs of f and f'.

78. Where is the function $h(x) = |x - 1| + |x + 2|$ differentiable? Give a formula for h' and sketch the graphs of h and h'.

79. Find the parabola with equation $y = ax^2 + bx$ whose tangent line at $(1, 1)$ has equation $y = 3x - 2$.

80. Suppose the curve $y = x^4 + ax^3 + bx^2 + cx + d$ has a tangent line when $x = 0$ with equation $y = 2x + 1$ and a tangent line when $x = 1$ with equation $y = 2 - 3x$. Find the values of a, b, c, and d.

81. For what values of a and b is the line $2x + y = b$ tangent to the parabola $y = ax^2$ when $x = 2$?

82. Find the value of c such that the line $y = \frac{3}{2}x + 6$ is tangent to the curve $y = c\sqrt{x}$.

83. What is the value of c such that the line $y = 2x + 3$ is tangent to the parabola $y = cx^2$?

84. The graph of any quadratic function $f(x) = ax^2 + bx + c$ is a parabola. Prove that the average of the slopes of the tangent lines to the parabola at the endpoints of any interval $[p, q]$ equals the slope of the tangent line at the midpoint of the interval.

85. Let

$$f(x) = \begin{cases} x^2 & \text{if } x \leqslant 2 \\ mx + b & \text{if } x > 2 \end{cases}$$

Find the values of m and b that make f differentiable everywhere.

86. Find numbers a and b such that the given function g is differentiable at 1.

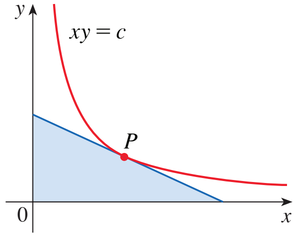
$$g(x) = \begin{cases} ax^3 - 3x & \text{if } x \leqslant 1 \\ bx^2 + 2 & \text{if } x > 1 \end{cases}$$

87. Evaluate $\lim\limits_{x \to 1} \dfrac{x^{1000} - 1}{x - 1}$.

88. A tangent line is drawn to the hyperbola $xy = c$ at a point P as shown in the figure.

(a) Show that the midpoint of the line segment cut from this tangent line by the coordinate axes is P.

(b) Show that the triangle formed by the tangent line and the coordinate axes always has the same area, no matter where P is located on the hyperbola.

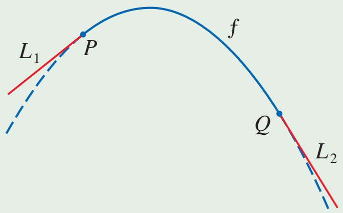


89. Draw a diagram showing two perpendicular lines that intersect on the y-axis and are both tangent to the parabola $y = x^2$. Where do these lines intersect?

90. Sketch the parabolas $y = x^2$ and $y = x^2 - 2x + 2$. Do you think there is a line that is tangent to both curves? If so, find its equation. If not, why not?

91. If $c > \frac{1}{2}$, how many lines through the point $(0, c)$ are normal lines to the parabola $y = x^2$? What if $c \leqslant \frac{1}{2}$?

APPLIED PROJECT | BUILDING A BETTER ROLLER COASTER



Suppose you are asked to design the first ascent and drop for a new roller coaster. By studying photographs of your favorite coasters, you decide to make the slope of the ascent 0.8 and the slope of the drop -1.6. You decide to connect these two straight stretches $y = L_1(x)$ and $y = L_2(x)$ with part of a parabola $y = f(x) = ax^2 + bx + c$, where x and $f(x)$ are measured in meters. For the track to be smooth there can't be abrupt changes in direction, so you want the linear segments L_1 and L_2 to be tangent to the parabola at the transition points P and Q. (See the figure.) To simplify the equations, you decide to place the origin at P.

1. (a) Suppose the horizontal distance between P and Q is 30 m. Write equations in a, b, and c that will ensure that the track is smooth at the transition points.

(b) Solve the equations in part (a) for a, b, and c to find a formula for $f(x)$.

(c) Plot L_1, f, and L_2 to verify graphically that the transitions are smooth.

(d) Find the difference in elevation between P and Q.

Susana Ortega / Shutterstock.com

2. The solution in Problem 1 might *look* smooth, but it might not *feel* smooth because the piecewise defined function [consisting of $L_1(x)$ for $x < 0$, $f(x)$ for $0 \leqslant x \leqslant 30$, and $L_2(x)$ for $x > 30$] doesn't have a continuous second derivative. So you decide to improve the design by using a quadratic function $q(x) = ax^2 + bx + c$ only on the interval $3 \leqslant x \leqslant 27$ and connecting it to the linear functions by means of two cubic functions:

$$g(x) = kx^3 + lx^2 + mx + n \qquad 0 \leqslant x < 3$$

$$h(x) = px^3 + qx^2 + rx + s \qquad 27 < x \leqslant 30$$

(a) Write a system of equations in 11 unknowns that ensures that the functions and their first two derivatives agree at the transition points.

(b) Solve the system of equations in part (a) to find formulas for $q(x)$, $g(x)$, and $h(x)$.

(c) Plot L_1, g, q, h, and L_2, and compare with the plot in Problem 1(c).

3.2 The Product and Quotient Rules

The formulas of this section enable us to differentiate new functions that are formed from old functions by using multiplication or division.

The Product Rule

By analogy with the Sum and Difference Rules, we might be tempted to guess, as Leibniz did three centuries ago, that the derivative of a product is the product of the derivatives. We can see, however, that this guess is wrong by looking at a particular example. Let $f(x) = x$ and $g(x) = x^2$. Then the Power Rule gives $f'(x) = 1$ and $g'(x) = 2x$. But $(fg)(x) = x^3$, so $(fg)'(x) = 3x^2$. Thus $(fg)' \neq f'g'$. The correct formula was discovered by Leibniz (soon after his false start) and is called the Product Rule.

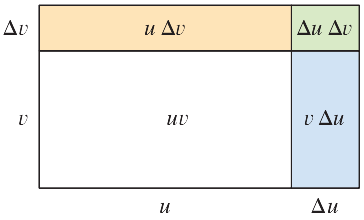


FIGURE 1

The geometry of the Product Rule

Before stating the Product Rule, let's see how we might discover it. We start by assuming that $u = f(x)$ and $v = g(x)$ are both positive differentiable functions. Then we can interpret the product uv as an area of a rectangle (see Figure 1). If x changes by an amount Δx, then the corresponding changes in u and v are

$$\Delta u = f(x + \Delta x) - f(x) \qquad \Delta v = g(x + \Delta x) - g(x)$$

and the new value of the product, $(u + \Delta u)(v + \Delta v)$, can be interpreted as the area of the large rectangle in Figure 1 (provided that Δu and Δv happen to be positive).

The change in the area of the rectangle is

$$\boxed{1} \qquad \begin{aligned}\Delta(uv) &= (u + \Delta u)(v + \Delta v) - uv = u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v \\ &= \text{the sum of the three shaded areas}\end{aligned}$$

If we divide by Δx, we get

$$\frac{\Delta(uv)}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}$$

If we now let $\Delta x \to 0$, we get the derivative of uv:

Recall that in Leibniz notation the definition of a derivative can be written as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\begin{aligned}\frac{d}{dx}(uv) &= \lim_{\Delta x \to 0} \frac{\Delta(uv)}{\Delta x} = \lim_{\Delta x \to 0}\left(u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}\right) \\ &= u \lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x} + v \lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} + \left(\lim_{\Delta x \to 0} \Delta u\right)\left(\lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x}\right) \\ &= u\frac{dv}{dx} + v\frac{du}{dx} + 0 \cdot \frac{dv}{dx}\end{aligned}$$

$$\boxed{2} \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

(Notice that $\Delta u \to 0$ as $\Delta x \to 0$ since f is differentiable and therefore continuous.)

Although we started by assuming (for the geometric interpretation) that all the quantities are positive, we notice that Equation 1 is always true. (The algebra is valid whether u, v, Δu, and Δv are positive or negative.) So we have proved Equation 2, known as the Product Rule, for all differentiable functions u and v.

In prime notation the Product Rule is written as

$$(fg)' = fg' + gf'$$

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

In words, the Product Rule says that *the derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.*

EXAMPLE 1

(a) If $f(x) = xe^x$, find $f'(x)$.

(b) Find the nth derivative, $f^{(n)}(x)$.

Figure 2 shows the graphs of the function f of Example 1 and its derivative f'. Notice that $f'(x)$ is positive when f is increasing and negative when f is decreasing.

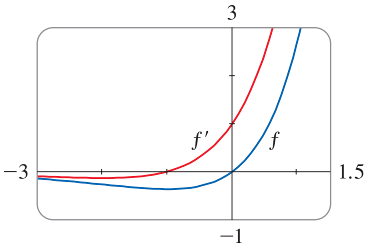


FIGURE 2

SOLUTION

(a) By the Product Rule, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}(xe^x) \\ &= x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x) \\ &= xe^x + e^x \cdot 1 = (x+1)e^x \end{aligned}$$

(b) Using the Product Rule a second time, we get

$$\begin{aligned} f''(x) &= \frac{d}{dx}[(x+1)e^x] \\ &= (x+1)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x+1) \\ &= (x+1)e^x + e^x \cdot 1 = (x+2)e^x \end{aligned}$$

Further applications of the Product Rule give

$$f'''(x) = (x+3)e^x \qquad f^{(4)}(x) = (x+4)e^x$$

In fact, each successive differentiation adds another term e^x, so

$$f^{(n)}(x) = (x+n)e^x$$

■

In Example 2, a and b are constants. It is customary in mathematics to use letters near the beginning of the alphabet to represent constants and letters near the end of the alphabet to represent variables.

EXAMPLE 2 Differentiate the function $f(t) = \sqrt{t}\,(a + bt)$.

SOLUTION 1 Using the Product Rule, we have

$$\begin{aligned} f'(t) &= \sqrt{t}\,\frac{d}{dt}(a+bt) + (a+bt)\frac{d}{dt}\left(\sqrt{t}\,\right) \\ &= \sqrt{t} \cdot b + (a+bt) \cdot \tfrac{1}{2}t^{-1/2} \\ &= b\sqrt{t} + \frac{a+bt}{2\sqrt{t}} = \frac{a+3bt}{2\sqrt{t}} \end{aligned}$$

SOLUTION 2 If we first use the laws of exponents to rewrite $f(t)$, then we can proceed directly without using the Product Rule.

$$f(t) = a\sqrt{t} + bt\sqrt{t} = at^{1/2} + bt^{3/2}$$

$$f'(t) = \tfrac{1}{2}at^{-1/2} + \tfrac{3}{2}bt^{1/2}$$

which is equivalent to the answer given in Solution 1. ■

Example 2 shows that it is sometimes easier to simplify a product of functions before differentiating than to use the Product Rule. In Example 1, however, the Product Rule is the only possible method.

EXAMPLE 3 If $f(x) = \sqrt{x}\, g(x)$, where $g(4) = 2$ and $g'(4) = 3$, find $f'(4)$.

SOLUTION Applying the Product Rule, we get

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\sqrt{x}\, g(x)\right] = \sqrt{x}\,\frac{d}{dx}[g(x)] + g(x)\,\frac{d}{dx}\left[\sqrt{x}\right] \\ &= \sqrt{x}\, g'(x) + g(x) \cdot \tfrac{1}{2}x^{-1/2} \\ &= \sqrt{x}\, g'(x) + \frac{g(x)}{2\sqrt{x}} \end{aligned}$$

So

$$f'(4) = \sqrt{4}\, g'(4) + \frac{g(4)}{2\sqrt{4}} = 2 \cdot 3 + \frac{2}{2 \cdot 2} = 6.5$$

■

The Quotient Rule

We find a rule for differentiating the quotient of two differentiable functions $u = f(x)$ and $v = g(x)$ in much the same way that we found the Product Rule. If x, u, and v change by amounts Δx, Δu, and Δv, then the corresponding change in the quotient u/v is

$$\begin{aligned} \Delta\left(\frac{u}{v}\right) &= \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{(u + \Delta u)v - u(v + \Delta v)}{v(v + \Delta v)} \\ &= \frac{v\Delta u - u\,\Delta v}{v(v + \Delta v)} \end{aligned}$$

so

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{\Delta x \to 0} \frac{\Delta(u/v)}{\Delta x} = \lim_{\Delta x \to 0} \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v(v + \Delta v)}$$

As $\Delta x \to 0$, $\Delta v \to 0$ also, because $v = g(x)$ is differentiable and therefore continuous. Thus, using the Limit Laws, we get

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\displaystyle\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} - u\lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x}}{v\displaystyle\lim_{\Delta x \to 0}(v + \Delta v)} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

In prime notation the Quotient Rule is written as

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2}$$

In words, the Quotient Rule says that the *derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

The Quotient Rule and the other differentiation formulas enable us to compute the derivative of any rational function, as the next example illustrates.

Figure 3 shows the graphs of the function of Example 4 and its derivative. Notice that when y grows rapidly (near $-\sqrt[3]{6} \approx -1.8$), y' is large. And when y grows slowly, y' is near 0.

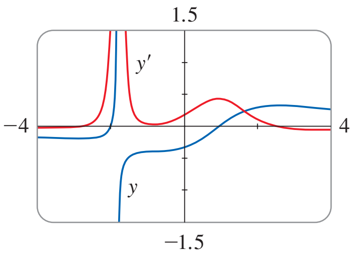


FIGURE 3

EXAMPLE 4 Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$. Then

$$y' = \frac{(x^3 + 6)\dfrac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2)\dfrac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2}$$

$$= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2}$$

$$= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}$$ ■

EXAMPLE 5 Find an equation of the tangent line to the curve $y = e^x/(1 + x^2)$ at the point $\left(1, \frac{1}{2}e\right)$.

SOLUTION According to the Quotient Rule, we have

$$\frac{dy}{dx} = \frac{(1 + x^2)\dfrac{d}{dx}(e^x) - e^x\dfrac{d}{dx}(1 + x^2)}{(1 + x^2)^2}$$

$$= \frac{(1 + x^2)e^x - e^x(2x)}{(1 + x^2)^2} = \frac{e^x(1 - 2x + x^2)}{(1 + x^2)^2}$$

$$= \frac{e^x(1 - x)^2}{(1 + x^2)^2}$$

So the slope of the tangent line at $\left(1, \frac{1}{2}e\right)$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = 0$$

This means that the tangent line at $\left(1, \frac{1}{2}e\right)$ is horizontal and its equation is $y = \frac{1}{2}e$. (See Figure 4.) ■

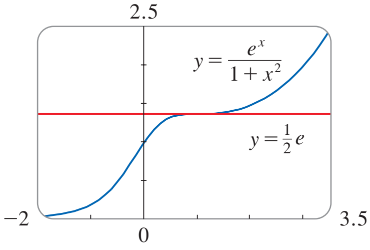


FIGURE 4

NOTE Don't use the Quotient Rule *every* time you see a quotient. Sometimes it's easier to rewrite a quotient first to put it in a form that is simpler for the purpose of differentiation. For instance, although it is possible to differentiate the function

$$F(x) = \frac{3x^2 + 2\sqrt{x}}{x}$$

using the Quotient Rule, it is much easier to perform the division first and write the function as

$$F(x) = 3x + 2x^{-1/2}$$

before differentiating.

We summarize the differentiation formulas we have learned so far as follows.

Table of Differentiation Formulas

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1} \qquad \frac{d}{dx}(e^x) = e^x$$

$$(cf)' = cf' \qquad (f+g)' = f' + g' \qquad (f-g)' = f' - g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

3.2 Exercises

1. Find the derivative of $f(x) = (1 + 2x^2)(x - x^2)$ in two ways: by using the Product Rule and by performing the multiplication first. Do your answers agree?

2. Find the derivative of the function

$$F(x) = \frac{x^4 - 5x^3 + \sqrt{x}}{x^2}$$

in two ways: by using the Quotient Rule and by simplifying first. Show that your answers are equivalent. Which method do you prefer?

3–30 Differentiate.

3. $y = (4x^2 + 3)(2x + 5)$

4. $y = (10x^2 + 7x - 2)(2 - x^2)$

5. $y = x^3 e^x$

6. $y = (e^x + 2)(2e^x - 1)$

7. $f(x) = (3x^2 - 5x)e^x$

8. $g(x) = \left(x + 2\sqrt{x}\right)e^x$

9. $y = \dfrac{x}{e^x}$

10. $y = \dfrac{e^x}{1 - e^x}$

11. $g(t) = \dfrac{3 - 2t}{5t + 1}$

12. $G(u) = \dfrac{6u^4 - 5u}{u + 1}$

13. $f(t) = \dfrac{5t}{t^3 - t - 1}$

14. $F(x) = \dfrac{1}{2x^3 - 6x^2 + 5}$

15. $y = \dfrac{s - \sqrt{s}}{s^2}$

16. $y = \dfrac{\sqrt{x}}{\sqrt{x} + 1}$

17. $J(u) = \left(\dfrac{1}{u} + \dfrac{1}{u^2}\right)\left(u + \dfrac{1}{u}\right)$

18. $h(w) = (w^2 + 3w)(w^{-1} - w^{-4})$

19. $H(u) = \left(u - \sqrt{u}\right)\left(u + \sqrt{u}\right)$

20. $f(z) = (1 - e^z)(z + e^z)$

21. $V(t) = (t + 2e^t)\sqrt{t}$

22. $W(t) = e^t(1 + te^t)$

23. $y = e^p\left(p + p\sqrt{p}\right)$

24. $h(r) = \dfrac{ae^r}{b + e^r}$

25. $f(t) = \dfrac{\sqrt[3]{t}}{t - 3}$

26. $y = (z^2 + e^z)\sqrt{z}$

27. $f(x) = \dfrac{x^2 e^x}{x^2 + e^x}$

28. $F(t) = \dfrac{At}{Bt^2 + Ct^3}$

29. $f(x) = \dfrac{x}{x + \dfrac{c}{x}}$

30. $f(x) = \dfrac{ax + b}{cx + d}$

31–34 Find $f'(x)$ and $f''(x)$.

31. $f(x) = x^2 e^x$

32. $f(x) = \sqrt{x}\,e^x$

33. $f(x) = \dfrac{x}{x^2 - 1}$

34. $f(x) = \dfrac{x}{1 + \sqrt{x}}$

35–36 Find an equation of the tangent line to the given curve at the specified point.

35. $y = \dfrac{x^2}{1 + x}, \quad \left(1, \tfrac{1}{2}\right)$

36. $y = \dfrac{1 + x}{1 + e^x}, \quad \left(0, \tfrac{1}{2}\right)$

37–38 Find equations of the tangent line and normal line to the given curve at the specified point.

37. $y = \dfrac{3x}{1 + 5x^2}, \quad \left(1, \tfrac{1}{2}\right)$

38. $y = x + xe^x, \quad (0, 0)$

39. (a) The curve $y = 1/(1 + x^2)$ is called a **witch of Maria Agnesi**. Find an equation of the tangent line to this curve at the point $\left(-1, \frac{1}{2}\right)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

40. (a) The curve $y = x/(1 + x^2)$ is called a **serpentine**. Find an equation of the tangent line to this curve at the point (3, 0.3).
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

41. (a) If $f(x) = (x^3 - x)e^x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

42. (a) If $f(x) = (x^2 - 1)/(x^2 + 1)$, find $f'(x)$ and $f''(x)$.
(b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f, f', and f''.

43. If $f(x) = x^2/(1 + x)$, find $f''(1)$.

44. If $g(x) = x/e^x$, find $g^{(n)}(x)$.

45. Suppose that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$. Find the following values.
(a) $(fg)'(5)$ (b) $(f/g)'(5)$ (c) $(g/f)'(5)$

46. Suppose that $f(4) = 2$, $g(4) = 5$, $f'(4) = 6$, and $g'(4) = -3$. Find $h'(4)$.
(a) $h(x) = 3f(x) + 8g(x)$ (b) $h(x) = f(x)g(x)$
(c) $h(x) = \dfrac{f(x)}{g(x)}$ (d) $h(x) = \dfrac{g(x)}{f(x) + g(x)}$

47. If $f(x) = e^x g(x)$, where $g(0) = 2$ and $g'(0) = 5$, find $f'(0)$.

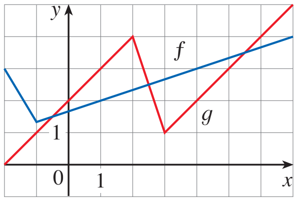
48. If $h(2) = 4$ and $h'(2) = -3$, find

$$\frac{d}{dx}\left(\frac{h(x)}{x}\right)\bigg|_{x=2}$$

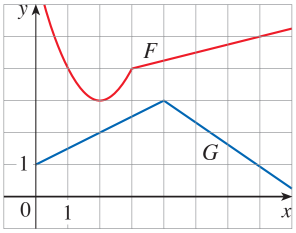
49. If $g(x) = xf(x)$, where $f(3) = 4$ and $f'(3) = -2$, find an equation of the tangent line to the graph of g at the point where $x = 3$.

50. If $f(2) = 10$ and $f'(x) = x^2 f(x)$ for all x, find $f''(2)$.

51. If f and g are the functions whose graphs are shown, let $u(x) = f(x)g(x)$ and $v(x) = f(x)/g(x)$.
(a) Find $u'(1)$. (b) Find $v'(4)$.



52. Let $P(x) = F(x)G(x)$ and $Q(x) = F(x)/G(x)$, where F and G are the functions whose graphs are shown.
(a) Find $P'(2)$. (b) Find $Q'(7)$.



53. If g is a differentiable function, find an expression for the derivative of each of the following functions.
(a) $y = xg(x)$ (b) $y = \dfrac{x}{g(x)}$
(c) $y = \dfrac{g(x)}{x}$

54. If f is a differentiable function, find an expression for the derivative of each of the following functions.
(a) $y = x^2 f(x)$ (b) $y = \dfrac{f(x)}{x^2}$
(c) $y = \dfrac{x^2}{f(x)}$ (d) $y = \dfrac{1 + xf(x)}{\sqrt{x}}$

55. How many tangent lines to the curve $y = x/(x + 1)$ pass through the point (1, 2)? At which points do these tangent lines touch the curve?

56. Find equations of the tangent lines to the curve

$$y = \frac{x - 1}{x + 1}$$

that are parallel to the line $x - 2y = 2$.

57. Find $R'(0)$, where

$$R(x) = \frac{x - 3x^3 + 5x^5}{1 + 3x^3 + 6x^6 + 9x^9}$$

Hint: Instead of finding $R'(x)$ first, let $f(x)$ be the numerator and $g(x)$ be the denominator of $R(x)$, and compute $R'(0)$ from $f(0)$, $f'(0)$, $g(0)$, and $g'(0)$.

58. Use the method of Exercise 57 to compute $Q'(0)$, where

$$Q(x) = \frac{1 + x + x^2 + xe^x}{1 - x + x^2 - xe^x}$$

59. In this exercise we estimate the rate at which the total personal income is rising in Boulder, Colorado. In 2015, the population of this city was 107,350 and the population was increasing by roughly 1960 people per year. The average annual income was \$60,220 per capita, and this average was increasing at about \$2250 per year (a little above the national average of about \$1810 yearly). Use the Product Rule and these figures to estimate the rate at which total personal income was rising in Boulder in 2015. Explain the meaning of each term in the Product Rule.

60. A manufacturer produces bolts of a fabric with a fixed width. The quantity q of this fabric (measured in meters) that is sold is a function of the selling price p (in dollars per meter), so we can write $q = f(p)$. Then the total revenue earned with selling price p is $R(p) = pf(p)$.
(a) What does it mean to say that $f(20) = 10{,}000$ and $f'(20) = -350$?
(b) Assuming the values in part (a), find $R'(20)$ and interpret your answer.

61. The Michaelis-Menten equation for the enzyme chymotrypsin is

$$v = \frac{0.14[\mathrm{S}]}{0.015 + [\mathrm{S}]}$$

where v is the rate of an enzymatic reaction and [S] is the concentration of a substrate S. Calculate $dv/d[\mathrm{S}]$ and interpret it.

62. The *biomass* $B(t)$ of a fish population is the total mass of the members of the population at time t. It is the product of the number of individuals $N(t)$ in the population and the average mass $M(t)$ of a fish at time t. In the case of guppies, breeding occurs continually. Suppose that at time $t = 4$ weeks the population is 820 guppies and is growing at a rate of 50 guppies per week, while the average mass is 1.2 g and is increasing at a rate of 0.14 g/week. At what rate is the biomass increasing when $t = 4$?

63. Extended Product Rule The Product Rule can be extended to the product of three functions.
(a) Use the Product Rule twice to prove that if f, g, and h are differentiable, then $(fgh)' = f'gh + fg'h + fgh'$.
(b) Taking $f = g = h$ in part (a), show that

$$\frac{d}{dx}[f(x)]^3 = 3[f(x)]^2 f'(x)$$

(c) Use part (b) to differentiate $y = e^{3x}$.

64. (a) If $F(x) = f(x)\,g(x)$, where f and g have derivatives of all orders, show that $F'' = f''g + 2f'g' + fg''$.
(b) Find similar formulas for F''' and $F^{(4)}$.
(c) Guess a formula for $F^{(n)}$.

65. Find expressions for the first five derivatives of $f(x) = x^2e^x$. Do you see a pattern in these expressions? Guess a formula for $f^{(n)}(x)$ and prove it using mathematical induction.

66. Reciprocal Rule If g is differentiable, the *Reciprocal Rule* says that

$$\frac{d}{dx}\left[\frac{1}{g(x)}\right] = -\frac{g'(x)}{[g(x)]^2}$$

(a) Use the Quotient Rule to prove the Reciprocal Rule.
(b) Use the Reciprocal Rule to differentiate the function in Exercise 14.
(c) Use the Reciprocal Rule to verify that the Power Rule is valid for negative integers, that is,

$$\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$$

for all positive integers n.

3.3 Derivatives of Trigonometric Functions

A review of the trigonometric functions is given in Appendix D.

Before starting this section, you might need to review the trigonometric functions. In particular, it is important to remember that when we talk about the function f defined for all real numbers x by

$$f(x) = \sin x$$

it is understood that $\sin x$ means the sine of the angle whose *radian* measure is x. A similar convention holds for the other trigonometric functions cos, tan, csc, sec, and cot. Recall from Section 2.5 that all of the trigonometric functions are continuous at every number in their domains.

Derivatives of the Trigonometric Functions

If we sketch the graph of the function $f(x) = \sin x$ and use the interpretation of $f'(x)$ as the slope of the tangent to the sine curve in order to sketch the graph of f' (see

Exercise 2.8.16), then it looks as if the graph of f' may be the same as the cosine curve (see Figure 1).

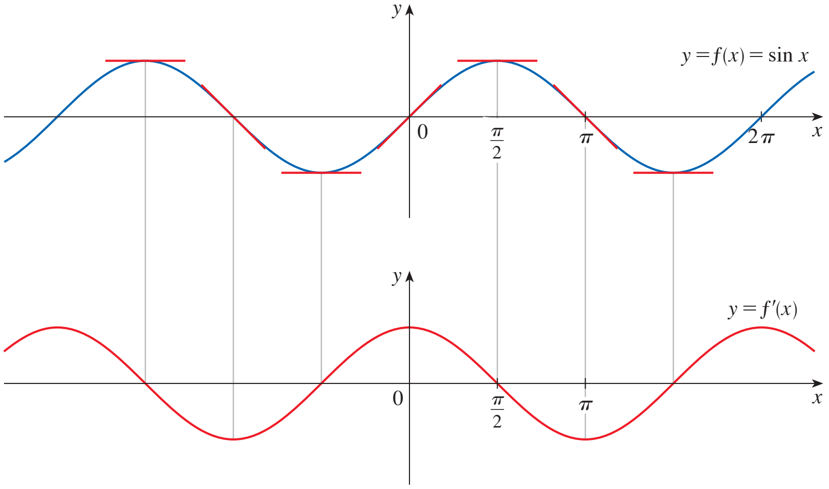


FIGURE 1

Let's try to confirm our guess that if $f(x) = \sin x$, then $f'(x) = \cos x$. From the definition of a derivative, we have

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \qquad \text{(using the addition formula for sine; see Appendix D)} \\
&= \lim_{h\to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right] \\
&= \lim_{h\to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right] \\
\boxed{1} \qquad &= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}
\end{aligned}
$$

Two of these four limits are easy to evaluate. Because we regard x as a constant when computing a limit as $h \to 0$, we have

$$\lim_{h\to 0} \sin x = \sin x \qquad \text{and} \qquad \lim_{h\to 0} \cos x = \cos x$$

Later in this section we will prove that

$$\lim_{h\to 0} \frac{\sin h}{h} = 1 \qquad \text{and} \qquad \lim_{h\to 0} \frac{\cos h - 1}{h} = 0$$

Putting these limits into (1), we get

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h} \\ &= (\sin x)\cdot 0 + (\cos x)\cdot 1 = \cos x \end{aligned}$$

So we have proved the formula for the derivative of the sine function:

2
$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

EXAMPLE 1 Differentiate $y = x^2 \sin x$.

Figure 2 shows the graphs of the function of Example 1 and its derivative. Notice that $y' = 0$ whenever y has a horizontal tangent.

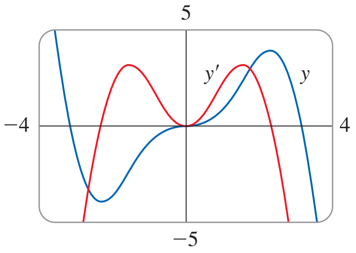


FIGURE 2

SOLUTION Using the Product Rule and Formula 2, we have

$$\begin{aligned} \frac{dy}{dx} &= x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2) \\ &= x^2 \cos x + 2x \sin x \end{aligned}$$ ■

Using the same methods as in the proof of Formula 2, we can prove (see Exercise 26) that

3
$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

The tangent function can also be differentiated by using the definition of a derivative, but it is easier to use the Quotient Rule together with Formulas 2 and 3:

$$\begin{aligned} \frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\ &= \frac{\cos x \dfrac{d}{dx}(\sin x) - \sin x \dfrac{d}{dx}(\cos x)}{\cos^2 x} \\ &= \frac{\cos x \cdot \cos x - \sin x\,(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} = \sec^2 x \qquad (\cos^2 x + \sin^2 x = 1) \end{aligned}$$

4
$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

The derivatives of the remaining trigonometric functions, csc, sec, and cot, can also be found easily using the Quotient Rule (see Exercises 23–25). We collect all the differentiation formulas for trigonometric functions in the following table. Remember that they are valid only when x is measured in radians.

When you memorize this table, it is helpful to notice that the minus signs go with the derivatives of the "cofunctions," that is, cosine, cosecant, and cotangent.

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

EXAMPLE 2 Differentiate $f(x) = \dfrac{\sec x}{1 + \tan x}$. For what values of x does the graph of f have a horizontal tangent?

SOLUTION The Quotient Rule gives

$$\begin{aligned} f'(x) &= \frac{(1 + \tan x)\dfrac{d}{dx}(\sec x) - \sec x \dfrac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2} \\ &= \frac{(1 + \tan x)\sec x \tan x - \sec x \cdot \sec^2 x}{(1 + \tan x)^2} \\ &= \frac{\sec x\,(\tan x + \tan^2 x - \sec^2 x)}{(1 + \tan x)^2} \\ &= \frac{\sec x\,(\tan x - 1)}{(1 + \tan x)^2} \qquad (\sec^2 x = \tan^2 x + 1) \end{aligned}$$

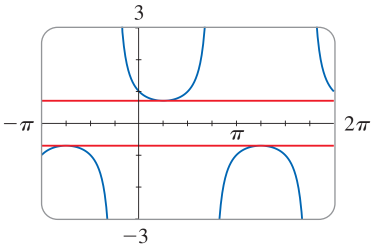


FIGURE 3
The horizontal tangents in Example 2

Because $\sec x$ is never 0, we see that $f'(x) = 0$ when $\tan x = 1$, and this occurs when $x = \pi/4 + n\pi$, where n is an integer (see Figure 3). ■

Trigonometric functions are often used in modeling real-world phenomena. In particular, vibrations, waves, elastic motions, and other quantities that vary in a periodic manner can be described using trigonometric functions. In the following example we discuss an instance of simple harmonic motion.

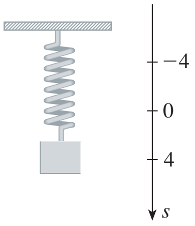


FIGURE 4

EXAMPLE 3 An object fastened to the end of a vertical spring is stretched 4 cm beyond its rest position and released at time $t = 0$. (See Figure 4 and note that the downward direction is positive.) Its position at time t is

$$s = f(t) = 4 \cos t$$

Find the velocity and acceleration at time t and use them to analyze the motion of the object.

SOLUTION The velocity and acceleration are

$$v = \frac{ds}{dt} = \frac{d}{dt}(4 \cos t) = 4\frac{d}{dt}(\cos t) = -4 \sin t$$

$$a = \frac{dv}{dt} = \frac{d}{dt}(-4 \sin t) = -4\frac{d}{dt}(\sin t) = -4 \cos t$$

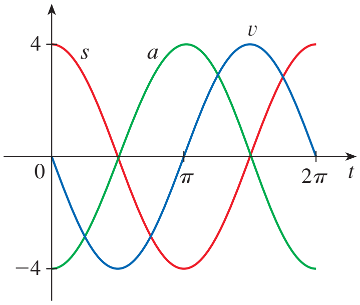


FIGURE 5

The object oscillates from the lowest point ($s = 4$ cm) to the highest point ($s = -4$ cm). The period of the oscillation is 2π, the period of $\cos t$.

The speed is $|v| = 4|\sin t|$, which is greatest when $|\sin t| = 1$, that is, when $\cos t = 0$. So the object moves fastest as it passes through its equilibrium position ($s = 0$). Its speed is 0 when $\sin t = 0$, that is, at the highest and lowest points.

The acceleration $a = -4\cos t = 0$ when $s = 0$. It has greatest magnitude at the highest and lowest points. See the graphs in Figure 5. ■

EXAMPLE 4 Find the 27th derivative of $\cos x$.

PS Look for a pattern.

SOLUTION The first few derivatives of $f(x) = \cos x$ are as follows:

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

$$f^{(4)}(x) = \cos x$$

$$f^{(5)}(x) = -\sin x$$

We see that the successive derivatives occur in a cycle of length 4 and, in particular, $f^{(n)}(x) = \cos x$ whenever n is a multiple of 4. Therefore

$$f^{(24)}(x) = \cos x$$

and, differentiating three more times, we have

$$f^{(27)}(x) = \sin x$$ ■

Two Special Trigonometric Limits

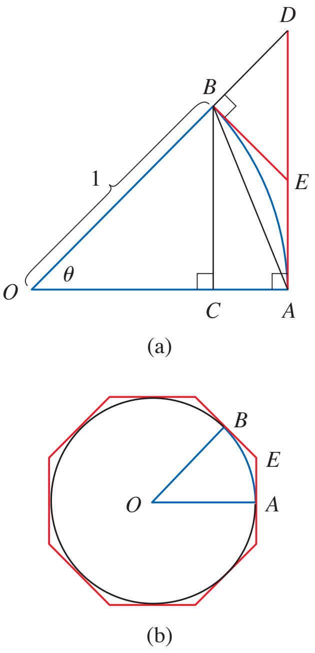
In proving the formula for the derivative of sine we used two special limits, which we now prove.

5
$$\boxed{\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1}$$

PROOF Assume first that θ lies between 0 and $\pi/2$. Figure 6(a) shows a sector of a circle with center O, central angle θ, and radius 1. BC is drawn perpendicular to OA. By the definition of radian measure, we have arc $AB = \theta$. Also $|BC| = |OB| \sin \theta = \sin \theta$. From the diagram we see that

$$|BC| < |AB| < \text{arc } AB$$

Therefore $\qquad \sin \theta < \theta \qquad$ so $\qquad \dfrac{\sin \theta}{\theta} < 1$



(a)

(b)

FIGURE 6

Let the tangent lines at A and B intersect at E. You can see from Figure 6(b) that the circumference of a circle is smaller than the length of a circumscribed polygon, and so arc $AB < |AE| + |EB|$. Thus

$$\begin{aligned} \theta = \text{arc } AB &< |AE| + |EB| \\ &< |AE| + |ED| \\ &= |AD| = |OA| \tan \theta \\ &= \tan \theta \end{aligned}$$

(In Appendix F the inequality $\theta \leqslant \tan\theta$ is proved directly from the definition of the length of an arc without resorting to geometric intuition as we did here.) Therefore we have

$$\theta < \frac{\sin\theta}{\cos\theta}$$

so
$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

We know that $\lim_{\theta\to 0} 1 = 1$ and $\lim_{\theta\to 0}\cos\theta = 1$, so by the Squeeze Theorem, we have

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1 \qquad (0 < \theta < \pi/2)$$

But the function $(\sin\theta)/\theta$ is an even function, so its right and left limits must be equal. Hence, we have

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

so we have proved Equation 5. ■

The first special limit we considered concerned the sine function. The following special limit involves cosine.

6
$$\boxed{\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0}$$

PROOF We multiply numerator and denominator by $\cos\theta + 1$ in order to put the function in a form in which we can use limits that we know.

$$\begin{aligned}
\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} &= \lim_{\theta\to 0}\left(\frac{\cos\theta - 1}{\theta}\cdot\frac{\cos\theta + 1}{\cos\theta + 1}\right) = \lim_{\theta\to 0}\frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)}\\
&= \lim_{\theta\to 0}\frac{-\sin^2\theta}{\theta(\cos\theta + 1)} = -\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{\cos\theta + 1}\right)\\
&= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\cos\theta + 1}\\
&= -1\cdot\left(\frac{0}{1 + 1}\right) = 0 \qquad \text{(by Equation 5)}
\end{aligned}$$
■

EXAMPLE 5 Find $\displaystyle\lim_{x\to 0}\frac{\sin 7x}{4x}$.

SOLUTION In order to apply Equation 5, we first rewrite the function by multiplying and dividing by 7:

Note that $\sin 7x \neq 7\sin x$.

$$\frac{\sin 7x}{4x} = \frac{7}{4}\left(\frac{\sin 7x}{7x}\right)$$

If we let $\theta = 7x$, then $\theta \to 0$ as $x \to 0$, so by Equation 5 we have

$$\lim_{x\to 0} \frac{\sin 7x}{4x} = \frac{7}{4} \lim_{x\to 0} \left(\frac{\sin 7x}{7x} \right)$$

$$= \frac{7}{4} \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = \frac{7}{4} \cdot 1 = \frac{7}{4}$$ ■

EXAMPLE 6 Calculate $\lim_{x\to 0} x \cot x$.

SOLUTION Here we divide numerator and denominator by x:

$$\lim_{x\to 0} x \cot x = \lim_{x\to 0} \frac{x\cos x}{\sin x}$$

$$= \lim_{x\to 0} \frac{\cos x}{\dfrac{\sin x}{x}} = \frac{\lim_{x\to 0} \cos x}{\lim_{x\to 0} \dfrac{\sin x}{x}}$$

$$= \frac{\cos 0}{1} \qquad \text{(by the continuity of cosine and Equation 5)}$$

$$= 1$$ ■

EXAMPLE 7 Find $\lim_{\theta\to 0} \dfrac{\cos\theta - 1}{\sin\theta}$.

SOLUTION In order to use Equations 5 and 6 we divide numerator and denominator by θ:

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\sin\theta} = \lim_{\theta\to 0} \frac{\dfrac{\cos\theta - 1}{\theta}}{\dfrac{\sin\theta}{\theta}}$$

$$= \frac{\lim_{\theta\to 0} \dfrac{\cos\theta - 1}{\theta}}{\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta}} = \frac{0}{1} = 0$$ ■

3.3 Exercises

1–22 Differentiate.

1. $f(x) = 3\sin x - 2\cos x$

2. $f(x) = \tan x - 4\sin x$

3. $y = x^2 + \cot x$

4. $y = 2\sec x - \csc x$

5. $h(\theta) = \theta^2 \sin\theta$

6. $g(x) = 3x + x^2\cos x$

7. $y = \sec\theta \tan\theta$

8. $y = \sin\theta \cos\theta$

9. $f(\theta) = (\theta - \cos\theta)\sin\theta$

10. $g(\theta) = e^\theta(\tan\theta - \theta)$

11. $H(t) = \cos^2 t$

12. $f(x) = e^x \sin x + \cos x$

13. $f(\theta) = \dfrac{\sin\theta}{1 + \cos\theta}$

14. $y = \dfrac{\cos x}{1 - \sin x}$

15. $y = \dfrac{x}{2 - \tan x}$

16. $f(t) = \dfrac{\cot t}{e^t}$

17. $f(w) = \dfrac{1 + \sec w}{1 - \sec w}$

18. $y = \dfrac{\sin t}{1 + \tan t}$

19. $y = \dfrac{t \sin t}{1 + t}$

20. $g(z) = \dfrac{z}{\sec z + \tan z}$

21. $f(\theta) = \theta\cos\theta \sin\theta$

22. $f(t) = te^t \cot t$

23. Show that $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$.

24. Show that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$.

25. Show that $\dfrac{d}{dx}(\cot x) = -\csc^2 x$.

26. Prove, using the definition of a derivative, that if $f(x) = \cos x$, then $f'(x) = -\sin x$.

27–30 Find an equation of the tangent line to the curve at the given point.

27. $y = \sin x + \cos x, \quad (0, 1)$

28. $y = x + \sin x, \quad (\pi, \pi)$

29. $y = e^x \cos x + \sin x, \quad (0, 1)$

30. $y = \dfrac{1 + \sin x}{\cos x}, \quad (\pi, -1)$

---

31. (a) Find an equation of the tangent line to the curve $y = 2x \sin x$ at the point $(\pi/2, \pi)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

32. (a) Find an equation of the tangent line to the curve $y = 3x + 6 \cos x$ at the point $(\pi/3, \pi + 3)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

33. (a) If $f(x) = \sec x - x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by graphing both f and f' for $|x| < \pi/2$.

34. (a) If $f(x) = e^x \cos x$, find $f'(x)$ and $f''(x)$.
(b) Check to see that your answers to part (a) are reasonable by graphing f, f', and f''.

35. If $g(\theta) = \dfrac{\sin \theta}{\theta}$, find $g'(\theta)$ and $g''(\theta)$.

36. If $f(t) = \sec t$, find $f''(\pi/4)$.

37. (a) Use the Quotient Rule to differentiate the function

$$f(x) = \frac{\tan x - 1}{\sec x}$$

(b) Simplify the expression for $f(x)$ by writing it in terms of $\sin x$ and $\cos x$, and then find $f'(x)$.
(c) Show that your answers to parts (a) and (b) are equivalent.

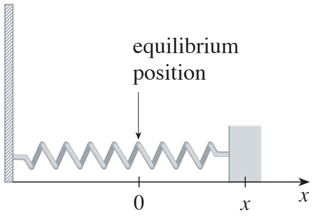
38. Suppose $f(\pi/3) = 4$ and $f'(\pi/3) = -2$, and let $g(x) = f(x) \sin x$ and $h(x) = (\cos x)/f(x)$. Find
(a) $g'(\pi/3)$ (b) $h'(\pi/3)$

39–40 For what values of x does the graph of f have a horizontal tangent?

39. $f(x) = x + 2 \sin x$ **40.** $f(x) = e^x \cos x$

---

41. A mass on a spring vibrates horizontally on a smooth level surface (see the figure). Its equation of motion is $x(t) = 8 \sin t$, where t is in seconds and x in centimeters.
(a) Find the velocity and acceleration at time t.
(b) Find the position, velocity, and acceleration of the mass at time $t = 2\pi/3$. In what direction is it moving at that time?



42. An elastic band is hung on a hook and a mass is hung on the lower end of the band. When the mass is pulled downward and then released, it vibrates vertically. The equation of motion is $s = 2 \cos t + 3 \sin t$, $t \geqslant 0$, where s is measured in centimeters and t in seconds. (Take the positive direction to be downward.)
(a) Find the velocity and acceleration at time t.
(b) Graph the velocity and acceleration functions.
(c) When does the mass pass through the equilibrium position for the first time?
(d) How far from its equilibrium position does the mass travel?
(e) When is the speed the greatest?

43. A ladder 6 m long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does x change with respect to θ when $\theta = \pi/3$?

44. An object with mass m is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu m g}{\mu \sin \theta + \cos \theta}$$

where μ is a constant called the *coefficient of friction*.
(a) Find the rate of change of F with respect to θ.
(b) When is this rate of change equal to 0?
(c) If $m = 20$ kg and $\mu = 0.6$, draw the graph of F as a function of θ and use it to locate the value of θ for which $dF/d\theta = 0$. Is the value consistent with your answer to part (b)?

45–60 Find the limit.

45. $\displaystyle\lim_{x \to 0} \frac{\sin 5x}{3x}$ **46.** $\displaystyle\lim_{x \to 0} \frac{\sin x}{\sin \pi x}$

47. $\displaystyle\lim_{t \to 0} \frac{\sin 3t}{\sin t}$ **48.** $\displaystyle\lim_{x \to 0} \frac{\sin^2 3x}{x}$

49. $\displaystyle\lim_{x\to 0}\frac{\sin x - \sin x\cos x}{x^2}$

50. $\displaystyle\lim_{x\to 0}\frac{1-\sec x}{2x}$

51. $\displaystyle\lim_{x\to 0}\frac{\tan 2x}{x}$

52. $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\tan 7\theta}$

53. $\displaystyle\lim_{x\to 0}\frac{\sin 3x}{5x^3-4x}$

54. $\displaystyle\lim_{x\to 0}\frac{\sin 3x\sin 5x}{x^2}$

55. $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta+\tan\theta}$

56. $\displaystyle\lim_{x\to 0}\csc x\ \sin(\sin x)$

57. $\displaystyle\lim_{\theta\to 0}\frac{\cos\theta-1}{2\theta^2}$

58. $\displaystyle\lim_{x\to 0}\frac{\sin(x^2)}{x}$

59. $\displaystyle\lim_{x\to\pi/4}\frac{1-\tan x}{\sin x-\cos x}$

60. $\displaystyle\lim_{x\to 1}\frac{\sin(x-1)}{x^2+x-2}$

61–62 Find the given derivative by finding the first few derivatives and observing the pattern that occurs.

61. $\displaystyle\frac{d^{99}}{dx^{99}}(\sin x)$

62. $\displaystyle\frac{d^{35}}{dx^{35}}(x\sin x)$

---

63. Find constants A and B such that the function $y = A\sin x + B\cos x$ satisfies the differential equation $y'' + y' - 2y = \sin x$.

64. (a) Evaluate $\displaystyle\lim_{x\to\infty} x\sin\frac{1}{x}$.

(b) Evaluate $\displaystyle\lim_{x\to 0} x\sin\frac{1}{x}$.

(c) Illustrate parts (a) and (b) by graphing $y = x\sin(1/x)$.

65. Differentiate each trigonometric identity to obtain a new (or familiar) identity.

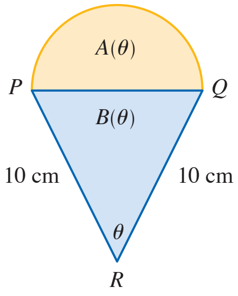
(a) $\tan x = \dfrac{\sin x}{\cos x}$

(b) $\sec x = \dfrac{1}{\cos x}$

(c) $\sin x + \cos x = \dfrac{1+\cot x}{\csc x}$

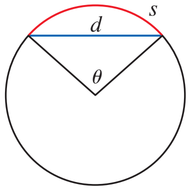
66. A semicircle with diameter PQ sits on an isosceles triangle PQR to form a region shaped like a two-dimensional ice-cream cone, as shown in the figure. If $A(\theta)$ is the area of the semicircle and $B(\theta)$ is the area of the triangle, find

$$\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)}$$



67. The figure shows a circular arc of length s and a chord of length d, both subtended by a central angle θ. Find

$$\lim_{\theta\to 0^+}\frac{s}{d}$$



68. Let $f(x) = \dfrac{x}{\sqrt{1-\cos 2x}}$.

(a) Graph f. What type of discontinuity does it appear to have at 0?

(b) Calculate the left and right limits of f at 0. Do these values confirm your answer to part (a)?

3.4 The Chain Rule

Suppose you are asked to differentiate the function

$$F(x) = \sqrt{x^2+1}$$

The differentiation formulas you learned in the previous sections of this chapter do not enable you to calculate $F'(x)$.

See Section 1.3 for a review of composite functions.

Observe that F is a composite function. In fact, if we let $y = f(u) = \sqrt{u}$ and let $u = g(x) = x^2 + 1$, then we can write $y = F(x) = f(g(x))$, that is, $F = f\circ g$. We know how to differentiate both f and g, so it would be useful to have a rule that tells us how to find the derivative of $F = f\circ g$ in terms of the derivatives of f and g.

The Chain Rule

It turns out that the derivative of the composite function $f \circ g$ is the product of the derivatives of f and g. This fact is one of the most important of the differentiation rules and is called the *Chain Rule*. It seems plausible if we interpret derivatives as rates of change. Regard du/dx as the rate of change of u with respect to x, dy/du as the rate of change of y with respect to u, and dy/dx as the rate of change of y with respect to x. If u changes twice as fast as x and y changes three times as fast as u, then it seems reasonable that y changes six times as fast as x, and so we expect that dy/dx is the product of dy/du and du/dx.

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x) \tag{1}$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} \tag{2}$$

Formula 2 is easy to remember because if we think of dy/du and du/dx as quotients, then we could cancel du; however, du has not been defined and du/dx should not be considered as an actual quotient.

James Gregory

The first person to formulate the Chain Rule was the Scottish mathematician James Gregory (1638–1675), who also designed the first practical reflecting telescope. Gregory discovered the basic ideas of calculus at about the same time as Newton. He became the first professor of Mathematics at the University of St. Andrews and later held the same position at the University of Edinburgh. But one year after accepting that position, he died at the age of 36.

COMMENTS ON THE PROOF OF THE CHAIN RULE Let Δu be the change in u corresponding to a change of Δx in x, that is,

$$\Delta u = g(x + \Delta x) - g(x)$$

Then the corresponding change in y is

$$\Delta y = f(u + \Delta u) - f(u)$$

It is tempting to write

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \qquad (3) \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \qquad \text{(Note that } \Delta u \to 0 \text{ as } \Delta x \to 0 \text{ because } g \text{ is continuous.)} \\
&= \frac{dy}{du}\,\frac{du}{dx}
\end{aligned}$$

The only flaw in this reasoning is that in (3) it might happen that $\Delta u = 0$ (even when $\Delta x \neq 0$) and, of course, we can't divide by 0. Nonetheless, this reasoning does at least *suggest* that the Chain Rule is true. A full proof of the Chain Rule is given at the end of this section. ■

EXAMPLE 1 Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$.

SOLUTION 1 (using Formula 1): At the beginning of this section we expressed F as $F(x) = (f \circ g)(x) = f(g(x))$ where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$. Since

$$f'(u) = \tfrac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} \qquad \text{and} \qquad g'(x) = 2x$$

we have

$$\begin{aligned} F'(x) &= f'(g(x)) \cdot g'(x) \\ &= \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}} \end{aligned}$$

SOLUTION 2 (using Formula 2): If we let $u = x^2 + 1$ and $y = \sqrt{u}$, then

$$F'(x) = \frac{dy}{du}\,\frac{du}{dx} = \frac{1}{2\sqrt{u}}(2x) = \frac{1}{2\sqrt{x^2+1}}(2x) = \frac{x}{\sqrt{x^2+1}}$$ ■

When using Formula 2 we should bear in mind that dy/dx refers to the derivative of y when y is considered as a function of x (the derivative of y with respect to x), whereas dy/du refers to the derivative of y when considered as a function of u (the derivative of y with respect to u). For instance, in Example 1, y can be considered as a function of x $\left(y = \sqrt{x^2+1}\right)$ and also as a function of u $\left(y = \sqrt{u}\right)$. Note that

$$\frac{dy}{dx} = F'(x) = \frac{x}{\sqrt{x^2+1}} \qquad \text{whereas} \qquad \frac{dy}{du} = f'(u) = \frac{1}{2\sqrt{u}}$$

NOTE In using the Chain Rule we work from the outside to the inside. Formula 1 says that *we differentiate the outer function* f [*at the inner function* $g(x)$] *and then we multiply by the derivative of the inner function.*

$$\frac{d}{dx}\;\underbrace{f}_{\substack{\text{outer}\\\text{function}}}\;\underbrace{(g(x))}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} \;=\; \underbrace{f'}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}}\;\underbrace{(g(x))}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} \;\cdot\; \underbrace{g'(x)}_{\substack{\text{derivative}\\\text{of inner}\\\text{function}}}$$

EXAMPLE 2 Differentiate (a) $y = \sin(x^2)$ and (b) $y = \sin^2 x$.

SOLUTION

(a) If $y = \sin(x^2)$, then the outer function is the sine function and the inner function is the squaring function, so the Chain Rule gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\;\underbrace{\sin}_{\substack{\text{outer}\\\text{function}}}\;\underbrace{(x^2)}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} \;=\; \underbrace{\cos}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}}\;\underbrace{(x^2)}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} \;\cdot\; \underbrace{2x}_{\substack{\text{derivative}\\\text{of inner}\\\text{function}}} \\ &= 2x\cos(x^2) \end{aligned}$$

(b) Note that $\sin^2 x = (\sin x)^2$. Here the outer function is the squaring function and the inner function is the sine function. So

$$\frac{dy}{dx} = \frac{d}{dx}\underbrace{(\sin x)}_{\substack{\text{inner}\\\text{function}}}{}^2 \;=\; \underbrace{2}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}} \cdot \underbrace{(\sin x)}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} \cdot \underbrace{\cos x}_{\substack{\text{derivative}\\\text{of inner}\\\text{function}}}$$

The answer can be left as $2 \sin x \cos x$ or written as $\sin 2x$ (by a trigonometric identity known as the double-angle formula). ■

See Reference Page 2 or Appendix D.

In Example 2(a) we combined the Chain Rule with the rule for differentiating the sine function. In general, if $y = \sin u$, where u is a differentiable function of x, then, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \cos u \frac{du}{dx}$$

Thus
$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$$

In a similar fashion, all of the formulas for differentiating trigonometric functions can be combined with the Chain Rule.

Let's make explicit the special case of the Chain Rule where the outer function f is a power function. If $y = [g(x)]^n$, then we can write $y = f(u) = u^n$ where $u = g(x)$. By using the Chain Rule and then the Power Rule, we get

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = nu^{n-1}\frac{du}{dx} = n[g(x)]^{n-1}g'(x)$$

4 The Power Rule Combined with the Chain Rule If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively,
$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Notice that the derivative we found in Example 1 could be calculated by taking $n = \frac{1}{2}$ in Rule 4.

EXAMPLE 3 Differentiate $y = (x^3 - 1)^{100}$.

SOLUTION Taking $u = g(x) = x^3 - 1$ and $n = 100$ in (4), we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^3 - 1)^{100} = 100(x^3 - 1)^{99}\frac{d}{dx}(x^3 - 1)\\ &= 100(x^3 - 1)^{99} \cdot 3x^2 = 300x^2(x^3 - 1)^{99}\end{aligned}$$ ■

EXAMPLE 4 Find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$.

SOLUTION First rewrite f:
$$f(x) = (x^2 + x + 1)^{-1/3}$$

Thus
$$\begin{aligned}f'(x) &= -\tfrac{1}{3}(x^2 + x + 1)^{-4/3}\frac{d}{dx}(x^2 + x + 1)\\ &= -\tfrac{1}{3}(x^2 + x + 1)^{-4/3}(2x + 1)\end{aligned}$$ ■

EXAMPLE 5 Find the derivative of the function

$$g(t) = \left(\frac{t-2}{2t+1}\right)^9$$

SOLUTION Combining the Power Rule, Chain Rule, and Quotient Rule, we get

$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^8 \frac{d}{dt}\left(\frac{t-2}{2t+1}\right)$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{(2t+1)\cdot 1 - 2(t-2)}{(2t+1)^2} = \frac{45(t-2)^8}{(2t+1)^{10}} \qquad ■$$

EXAMPLE 6 Differentiate $y = (2x+1)^5(x^3 - x + 1)^4$.

SOLUTION In this example we must use the Product Rule before using the Chain Rule:

$$\frac{dy}{dx} = (2x+1)^5 \frac{d}{dx}(x^3 - x + 1)^4 + (x^3 - x + 1)^4 \frac{d}{dx}(2x+1)^5$$

$$= (2x+1)^5 \cdot 4(x^3 - x + 1)^3 \frac{d}{dx}(x^3 - x + 1)$$

$$+ (x^3 - x + 1)^4 \cdot 5(2x+1)^4 \frac{d}{dx}(2x+1)$$

$$= 4(2x+1)^5(x^3 - x + 1)^3(3x^2 - 1) + 5(x^3 - x + 1)^4(2x+1)^4 \cdot 2$$

Noticing that each term has the common factor $2(2x+1)^4(x^3 - x + 1)^3$, we could factor it out and write the answer as

$$\frac{dy}{dx} = 2(2x+1)^4(x^3 - x + 1)^3(17x^3 + 6x^2 - 9x + 3) \qquad ■$$

The graphs of the functions y and y' in Example 6 are shown in Figure 1. Notice that y' is large when y increases rapidly and $y' = 0$ when y has a horizontal tangent. So our answer appears to be reasonable.

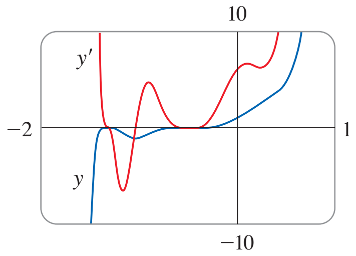


FIGURE 1

EXAMPLE 7 Differentiate $y = e^{\sin x}$.

SOLUTION Here the inner function is $g(x) = \sin x$ and the outer function is the exponential function $f(x) = e^x$. So, by the Chain Rule,

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\sin x}) = e^{\sin x}\frac{d}{dx}(\sin x) = e^{\sin x}\cos x \qquad ■$$

More generally, the Chain Rule gives

$$\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$$

The reason for the name "Chain Rule" becomes clear when we make a longer chain by adding another link. Suppose that $y = f(u)$, $u = g(x)$, and $x = h(t)$ where f, g, and h are differentiable functions. Then, to compute the derivative of y with respect to t, we use the Chain Rule twice:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{du}\frac{du}{dx}\frac{dx}{dt}$$

EXAMPLE 8 If $f(x) = \sin(\cos(\tan x))$, then

$$\begin{aligned} f'(x) &= \cos(\cos(\tan x)) \frac{d}{dx} \cos(\tan x) \\ &= \cos(\cos(\tan x)) [-\sin(\tan x)] \frac{d}{dx} (\tan x) \\ &= -\cos(\cos(\tan x)) \sin(\tan x) \sec^2 x \end{aligned}$$

Notice that we used the Chain Rule twice. ■

EXAMPLE 9 Differentiate $y = e^{\sec 3\theta}$.

SOLUTION The outer function is the exponential function, the middle function is the secant function, and the inner function is the tripling function. So we have

$$\begin{aligned} \frac{dy}{d\theta} &= e^{\sec 3\theta} \frac{d}{d\theta} (\sec 3\theta) \\ &= e^{\sec 3\theta} \sec 3\theta \tan 3\theta \frac{d}{d\theta} (3\theta) \\ &= 3e^{\sec 3\theta} \sec 3\theta \tan 3\theta \end{aligned}$$ ■

Derivatives of General Exponential Functions

We can use the Chain Rule to differentiate an exponential function with any base $b > 0$. Recall from Equation 1.5.10 that we can write

$$b^x = e^{(\ln b)x}$$

and then the Chain Rule gives

$$\begin{aligned} \frac{d}{dx} (b^x) = \frac{d}{dx} (e^{(\ln b)x}) &= e^{(\ln b)x} \frac{d}{dx} [(\ln b)x] \\ &= e^{(\ln b)x} (\ln b) = b^x \ln b \end{aligned}$$

because $\ln b$ is a constant. So we have the formula

Don't confuse Formula 5 (where x is the *exponent*) with the Power Rule (where x is the *base*):

$$\frac{d}{dx} (x^n) = nx^{n-1}$$

5
$$\boxed{\frac{d}{dx} (b^x) = b^x \ln b}$$

EXAMPLE 10 Find the derivative of each of the functions.

(a) $g(x) = 2^x$ (b) $h(x) = 5^{x^2}$

SOLUTION

(a) We use Formula 5 with $b = 2$:

$$g'(x) = \frac{d}{dx} (2^x) = 2^x \ln 2$$

This is consistent with the estimate

$$\frac{d}{dx} (2^x) \approx (0.693)2^x$$

we gave in Section 3.1 because $\ln 2 \approx 0.693147$.

(b) The outer function is an exponential function and the inner function is the squaring function, so we use Formula 5 and the Chain Rule to get

$$h'(x) = \frac{d}{dx}\left(5^{x^2}\right) = 5^{x^2} \ln 5 \cdot \frac{d}{dx}(x^2) = 2x \cdot 5^{x^2} \ln 5$$ ■

■ How to Prove the Chain Rule

Recall that if $y = f(x)$ and x changes from a to $a + \Delta x$, we define the increment of y as

$$\Delta y = f(a + \Delta x) - f(a)$$

According to the definition of a derivative, we have

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(a)$$

So if we denote by ε the difference between $\Delta y/\Delta x$ and $f'(a)$, we obtain

$$\lim_{\Delta x \to 0} \varepsilon = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x} - f'(a)\right) = f'(a) - f'(a) = 0$$

But $$\varepsilon = \frac{\Delta y}{\Delta x} - f'(a) \quad \Rightarrow \quad \Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x$$

If we define ε to be 0 when $\Delta x = 0$, then ε becomes a continuous function of Δx. Thus, for a differentiable function f, we can write

$$\boxed{6} \qquad \Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x \qquad \text{where} \quad \varepsilon \to 0 \text{ as } \Delta x \to 0$$

and ε is a continuous function of Δx. This property of differentiable functions is what enables us to prove the Chain Rule.

PROOF OF THE CHAIN RULE Suppose $u = g(x)$ is differentiable at a and $y = f(u)$ is differentiable at $b = g(a)$. If Δx is an increment in x and Δu and Δy are the corresponding increments in u and y, then we can use Equation 6 to write

$$\boxed{7} \qquad \Delta u = g'(a)\,\Delta x + \varepsilon_1\,\Delta x = [g'(a) + \varepsilon_1]\,\Delta x$$

where $\varepsilon_1 \to 0$ as $\Delta x \to 0$. Similarly

$$\boxed{8} \qquad \Delta y = f'(b)\,\Delta u + \varepsilon_2\,\Delta u = [f'(b) + \varepsilon_2]\,\Delta u$$

where $\varepsilon_2 \to 0$ as $\Delta u \to 0$. If we now substitute the expression for Δu from Equation 7 into Equation 8, we get

$$\Delta y = [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]\,\Delta x$$

so $$\frac{\Delta y}{\Delta x} = [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]$$

As $\Delta x \to 0$, Equation 7 shows that $\Delta u \to 0$. Taking the limit as $\Delta x \to 0$, we get

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]$$

$$= f'(b)\,g'(a) = f'(g(a))\,g'(a)$$

This proves the Chain Rule. ■

3.4 Exercises

1–6 Write the composite function in the form $f(g(x))$. [Identify the inner function $u = g(x)$ and the outer function $y = f(u)$.] Then find the derivative dy/dx.

1. $y = (5 - x^4)^3$

2. $y = \sqrt{x^3 + 2}$

3. $y = \sin(\cos x)$

4. $y = \tan(x^2)$

5. $y = e^{\sqrt{x}}$

6. $y = \sqrt[3]{e^x + 1}$

7–52 Find the derivative of the function.

7. $f(x) = (2x^3 - 5x^2 + 4)^5$

8. $f(x) = (x^5 + 3x^2 - x)^{50}$

9. $f(x) = \sqrt{5x + 1}$

10. $f(x) = \dfrac{1}{\sqrt[3]{x^2 - 1}}$

11. $g(t) = \dfrac{1}{(2t + 1)^2}$

12. $F(t) = \left(\dfrac{1}{2t + 1}\right)^4$

13. $f(\theta) = \cos(\theta^2)$

14. $g(\theta) = \cos^2\theta$

15. $g(x) = e^{x^2 - x}$

16. $y = 5^{\sqrt{x}}$

17. $y = x^2e^{-3x}$

18. $f(t) = t \sin \pi t$

19. $f(t) = e^{at} \sin bt$

20. $A(r) = \sqrt{r} \cdot e^{r^2 + 1}$

21. $F(x) = (4x + 5)^3(x^2 - 2x + 5)^4$

22. $G(z) = (1 - 4z)^2\sqrt{z^2 + 1}$

23. $y = \sqrt{\dfrac{x}{x + 1}}$

24. $y = \left(x + \dfrac{1}{x}\right)^5$

25. $y = e^{\tan \theta}$

26. $f(t) = 2^{t^3}$

27. $g(u) = \left(\dfrac{u^3 - 1}{u^3 + 1}\right)^8$

28. $s(t) = \sqrt{\dfrac{1 + \sin t}{1 + \cos t}}$

29. $r(t) = 10^{2\sqrt{t}}$

30. $f(z) = e^{z/(z - 1)}$

31. $H(r) = \dfrac{(r^2 - 1)^3}{(2r + 1)^5}$

32. $J(\theta) = \tan^2(n\theta)$

33. $F(t) = e^{t \sin 2t}$

34. $F(t) = \dfrac{t^2}{\sqrt{t^3 + 1}}$

35. $G(x) = 4^{C/x}$

36. $U(y) = \left(\dfrac{y^4 + 1}{y^2 + 1}\right)^5$

37. $f(x) = \sin x \cos(1 - x^2)$

38. $g(x) = e^{-x}\cos(x^2)$

39. $F(t) = \tan\sqrt{1 + t^2}$

40. $G(z) = (1 + \cos^2 z)^3$

41. $y = \sin^2(x^2 + 1)$

42. $y = e^{\sin 2x} + \sin(e^{2x})$

43. $g(x) = \sin\left(\dfrac{e^x}{1 + e^x}\right)$

44. $f(t) = e^{1/t}\sqrt{t^2 - 1}$

45. $f(t) = \tan(\sec(\cos t))$

46. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

47. $f(x) = e^{\sin^2(x^2)}$

48. $y = 2^{3^{4^x}}$

49. $y = (3^{\cos(x^2)} - 1)^4$

50. $y = \sin(\theta + \tan(\theta + \cos\theta))$

51. $y = \cos\sqrt{\sin(\tan \pi x)}$

52. $y = \sin^3(\cos(x^2))$

53–56 Find y' and y''.

53. $y = \cos(\sin 3\theta)$

54. $y = \left(1 + \sqrt{x}\right)^3$

55. $y = \sqrt{\cos x}$

56. $y = e^{e^x}$

57–60 Find an equation of the tangent line to the curve at the given point.

57. $y = 2^x$, $(0, 1)$

58. $y = \sqrt{1 + x^3}$, $(2, 3)$

59. $y = \sin(\sin x)$, $(\pi, 0)$

60. $y = xe^{-x^2}$, $(0, 0)$

61. (a) Find an equation of the tangent line to the curve $y = 2/(1 + e^{-x})$ at the point $(0, 1)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

62. (a) The curve $y = |x|/\sqrt{2 - x^2}$ is called a *bullet-nose curve*. Find an equation of the tangent line to this curve at the point $(1, 1)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

63. (a) If $f(x) = x\sqrt{2 - x^2}$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

64. The function $f(x) = \sin(x + \sin 2x)$, $0 \leqslant x \leqslant \pi$, arises in applications to frequency modulation (FM) synthesis.
(a) Use a graph of f produced by a calculator or computer to make a rough sketch of the graph of f'.
(b) Calculate $f'(x)$ and use this expression, with a calculator or computer, to graph f'. Compare with your sketch in part (a).

65. Find all points on the graph of the function $f(x) = 2\sin x + \sin^2 x$ at which the tangent line is horizontal.

66. At what point on the curve $y = \sqrt{1 + 2x}$ is the tangent line perpendicular to the line $6x + 2y = 1$?

67. If $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$, find $F'(5)$.

68. If $h(x) = \sqrt{4 + 3f(x)}$, where $f(1) = 7$ and $f'(1) = 4$, find $h'(1)$.

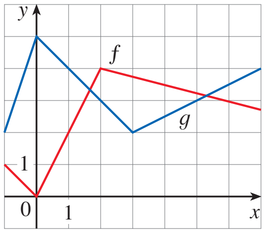
69. A table of values for f, g, f', and g' is given.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

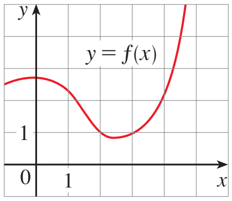
(a) If $h(x) = f(g(x))$, find $h'(1)$.
(b) If $H(x) = g(f(x))$, find $H'(1)$.

70. Let f and g be the functions in Exercise 69.
(a) If $F(x) = f(f(x))$, find $F'(2)$.
(b) If $G(x) = g(g(x))$, find $G'(3)$.

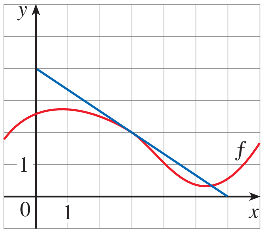
71. If f and g are the functions whose graphs are shown, let $u(x) = f(g(x))$, $v(x) = g(f(x))$, and $w(x) = g(g(x))$. Find each derivative, if it exists. If it does not exist, explain why.
(a) $u'(1)$ (b) $v'(1)$ (c) $w'(1)$



72. If f is the function whose graph is shown, let $h(x) = f(f(x))$ and $g(x) = f(x^2)$. Use the graph of f to estimate the value of each derivative.
(a) $h'(2)$ (b) $g'(2)$



73. If $g(x) = \sqrt{f(x)}$, where the graph of f is shown, evaluate $g'(3)$.



74. Suppose f is differentiable on $\mathbb{R}$ and α is a real number. Let $F(x) = f(x^\alpha)$ and $G(x) = [f(x)]^\alpha$. Find expressions for (a) $F'(x)$ and (b) $G'(x)$.

75. Suppose f is differentiable on $\mathbb{R}$. Let $F(x) = f(e^x)$ and $G(x) = e^{f(x)}$. Find expressions for (a) $F'(x)$ and (b) $G'(x)$.

76. Let $g(x) = e^{cx} + f(x)$ and $h(x) = e^{kx}f(x)$, where $f(0) = 3$, $f'(0) = 5$, and $f''(0) = -2$.
(a) Find $g'(0)$ and $g''(0)$ in terms of c.
(b) In terms of k, find an equation of the tangent line to the graph of h at the point where $x = 0$.

77. Let $r(x) = f(g(h(x)))$, where $h(1) = 2$, $g(2) = 3$, $h'(1) = 4$, $g'(2) = 5$, and $f'(3) = 6$. Find $r'(1)$.

78. If g is a twice differentiable function and $f(x) = xg(x^2)$, find f'' in terms of g, g', and g''.

79. If $F(x) = f(3f(4f(x)))$, where $f(0) = 0$ and $f'(0) = 2$, find $F'(0)$.

80. If $F(x) = f(xf(xf(x)))$, where $f(1) = 2$, $f(2) = 3$, $f'(1) = 4$, $f'(2) = 5$, and $f'(3) = 6$, find $F'(1)$.

81. Show that the function $y = e^{2x}(A \cos 3x + B \sin 3x)$ satisfies the differential equation $y'' - 4y' + 13y = 0$.

82. For what values of r does the function $y = e^{rx}$ satisfy the differential equation $y'' - 4y' + y = 0$?

83. Find the 50th derivative of $y = \cos 2x$.

84. Find the 1000th derivative of $f(x) = xe^{-x}$.

85. The displacement of a particle on a vibrating string is given by the equation

$$s(t) = 10 + \tfrac{1}{4}\sin(10\pi t)$$

where s is measured in centimeters and t in seconds. Find the velocity of the particle after t seconds.

86. If the equation of motion of a particle is given by $s = A\cos(\omega t + \delta)$, the particle is said to undergo *simple harmonic motion*.
(a) Find the velocity of the particle at time t.
(b) When is the velocity 0?

87. A Cepheid variable star is a star whose brightness alternately increases and decreases. The most easily visible such star is Delta Cephei, for which the interval between times of maximum brightness is 5.4 days. The average brightness of this star is 4.0 and its brightness changes by ±0.35. In view of these data, the brightness of Delta Cephei at time t, where t is measured in days, has been modeled by the function

$$B(t) = 4.0 + 0.35 \sin\left(\frac{2\pi t}{5.4}\right)$$

(a) Find the rate of change of the brightness after t days.
(b) Find, correct to two decimal places, the rate of increase after one day.

88. In Example 1.3.4 we arrived at a model for the length of daylight (in hours) in Philadelphia on the tth day of the year:

$$L(t) = 12 + 2.8 \sin\left[\frac{2\pi}{365}(t - 80)\right]$$

Use this model to compare how the number of hours of daylight is increasing in Philadelphia on March 21 ($t = 80$) and May 21 ($t = 141$).

89. The motion of a spring that is subject to a frictional force or a damping force (such as a shock absorber in a car) is often modeled by the product of an exponential function and a sine or cosine function. Suppose the equation of motion of a point on such a spring is

$$s(t) = 2e^{-1.5t} \sin 2\pi t$$

where s is measured in centimeters and t in seconds. Find the velocity after t seconds and graph both the position and velocity functions for $0 \leqslant t \leqslant 2$.

90. Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that has heard the rumor at time t and a and k are positive constants. [In Section 9.4 we will see that this is a reasonable equation for $p(t)$.]

(a) Find $\lim_{t \to \infty} p(t)$ and interpret your answer.
(b) Find the rate of spread of the rumor.
(c) Graph p for the case $a = 10$, $k = 0.5$ with t measured in hours. Use the graph to estimate how long it will take for 80% of the population to have heard the rumor.

91. The average blood alcohol concentration (BAC) of eight male subjects was measured after consumption of 15 mL of ethanol (corresponding to one alcoholic drink). The resulting data were modeled by the concentration function

$$C(t) = 0.00225te^{-0.0467t}$$

where t is measured in minutes after consumption and C is measured in g/dL.

(a) How rapidly was the BAC increasing after 10 minutes?
(b) How rapidly was it decreasing half an hour later?

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

92. Air is being pumped into a spherical weather balloon. At any time t, the volume of the balloon is $V(t)$ and its radius is $r(t)$.

(a) What do the derivatives dV/dr and dV/dt represent?
(b) Express dV/dt in terms of dr/dt.

93. A particle moves along a straight line with displacement $s(t)$, velocity $v(t)$, and acceleration $a(t)$. Show that

$$a(t) = v(t)\frac{dv}{ds}$$

Explain the difference between the meanings of the derivatives dv/dt and dv/ds.

94. The table gives the US population from 1790 to 1860.

Year	Population	Year	Population
1790	3,929,000	1830	12,861,000
1800	5,308,000	1840	17,063,000
1810	7,240,000	1850	23,192,000
1820	9,639,000	1860	31,443,000

(a) Fit an exponential function to the data. Graph the data points and the exponential model. How good is the fit?
(b) Estimate the rates of population growth in 1800 and 1850 by averaging slopes of secant lines.
(c) Use the exponential model in part (a) to estimate the rates of growth in 1800 and 1850. Compare these estimates with the ones in part (b).
(d) Use the exponential model to predict the population in 1870. Compare with the actual population of 38,558,000. Can you explain the discrepancy?

95. Use the Chain Rule to prove the following.

(a) The derivative of an even function is an odd function.
(b) The derivative of an odd function is an even function.

96. Use the Chain Rule and the Product Rule to give an alternative proof of the Quotient Rule.
[*Hint:* Write $f(x)/g(x) = f(x)[g(x)]^{-1}$.]

97. Use the Chain Rule to show that if θ is measured in degrees, then

$$\frac{d}{d\theta}(\sin\theta) = \frac{\pi}{180}\cos\theta$$

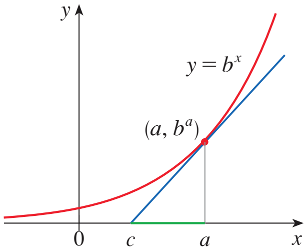
(This gives one reason for the convention that radian measure is always used when dealing with trigonometric functions in calculus: the differentiation formulas would not be as simple if we used degree measure.)

98. (a) Write $|x| = \sqrt{x^2}$ and use the Chain Rule to show that

$$\frac{d}{dx}|x| = \frac{x}{|x|}$$

(b) If $f(x) = |\sin x|$, find $f'(x)$ and sketch the graphs of f and f'. Where is f not differentiable?
(c) If $g(x) = \sin|x|$, find $g'(x)$ and sketch the graphs of g and g'. Where is g not differentiable?

99. Let c be the x-intercept of the tangent line to the curve $y = b^x$ ($b > 0$, $b \neq 1$) at the point (a, b^a). Show that the distance between the points $(a, 0)$ and $(c, 0)$ is the same for all values of a.



100. On every exponential curve $y = b^x$ ($b > 0$, $b \neq 1$), there is exactly one point (x_0, y_0) at which the tangent line to the curve passes through the origin. Show that in every case, $y_0 = e$. [*Hint:* You may wish to use Formula 1.5.10.]

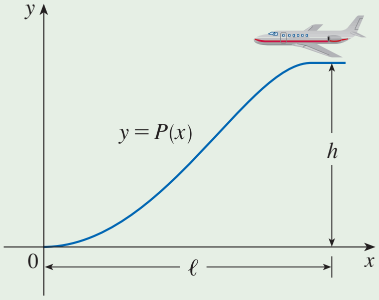
101. If $F = f \circ g \circ h$, where f, g, and h are differentiable functions, use the Chain Rule to show that

$$F'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

102. If $F = f \circ g$, where f and g are twice differentiable functions, use the Chain Rule and the Product Rule to show that the second derivative of F is given by

$$F''(x) = f''(g(x)) \cdot [g'(x)]^2 + f'(g(x)) \cdot g''(x)$$

APPLIED PROJECT | WHERE SHOULD A PILOT START DESCENT?



An approach path for an aircraft landing is shown in the figure and satisfies the following conditions:

(i) The cruising altitude is h when descent starts at a horizontal distance ℓ from touchdown at the origin.

(ii) The pilot must maintain a constant horizontal speed v throughout descent.

(iii) The absolute value of the vertical acceleration should not exceed a constant k (which is much less than the acceleration due to gravity).

1. Find a cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ that satisfies condition (i) by imposing suitable conditions on $P(x)$ and $P'(x)$ at the start of descent and at touchdown.

2. Use conditions (ii) and (iii) to show that

$$\frac{6hv^2}{\ell^2} \leqslant k$$

3. Suppose that an airline decides not to allow vertical acceleration of a plane to exceed $k = 1385 \text{ km/h}^2$. If the cruising altitude of a plane is 11,000 m and the speed is 480 km/h, how far away from the airport should the pilot start descent?

4. Graph the approach path if the conditions stated in Problem 3 are satisfied.

3.5 Implicit Differentiation

Implicitly Defined Functions

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable—for example,

$$y = \sqrt{x^3 + 1} \qquad \text{or} \qquad y = x \sin x$$

or, in general, $y = f(x)$. Some functions, however, are defined implicitly by a relation between x and y such as

$$\boxed{1} \qquad x^2 + y^2 = 25$$

or

$$\boxed{2} \qquad x^3 + y^3 = 6xy$$

In some cases it is possible to solve such an equation for y as an explicit function (or several functions) of x. For instance, if we solve Equation 1 for y, we get $y = \pm\sqrt{25 - x^2}$, so two of the functions determined by the implicit Equation l are $f(x) = \sqrt{25 - x^2}$ and $g(x) = -\sqrt{25 - x^2}$. The graphs of f and g are the upper and lower semicircles of the circle $x^2 + y^2 = 25$. (See Figure 1.)

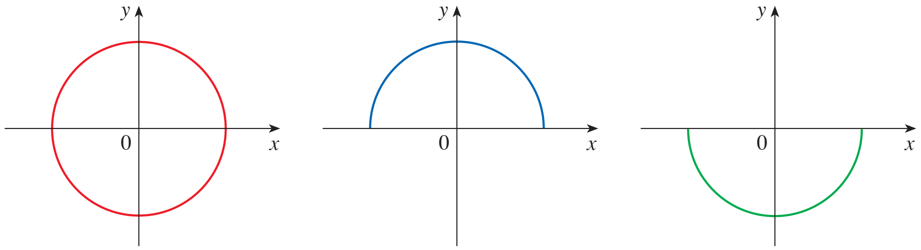


FIGURE 1

(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

It's not easy to solve Equation 2 for y explicitly as a function of x by hand. (Even if we use technology, the resulting expressions are very complicated.) Nonetheless, (2) is the equation of a curve called the **folium of Descartes** shown in Figure 2 and it implicitly defines y as several functions of x. The graphs of three such functions are shown in Figure 3. When we say that f is a function defined implicitly by Equation 2, we mean that the equation

$$x^3 + [f(x)]^3 = 6xf(x)$$

is true for all values of x in the domain of f.

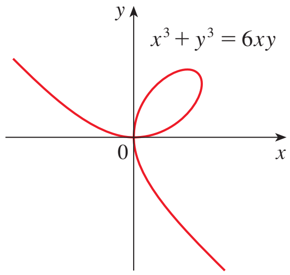


FIGURE 2 The folium of Descartes

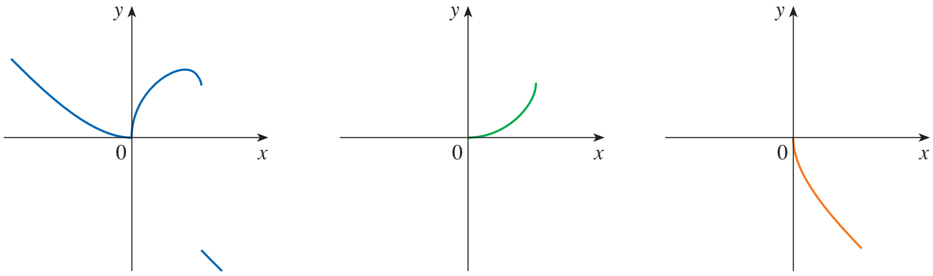


FIGURE 3 Graphs of three functions defined implicitly by the folium of Descartes

Implicit Differentiation

Fortunately, we don't need to solve an equation for y in terms of x in order to find the derivative of y. Instead we can use the method of **implicit differentiation**. This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for dy/dx. In the examples and exercises of this section it is always assumed that the given equation determines y implicitly as a differentiable function of x so that the method of implicit differentiation can be applied.

EXAMPLE 1

If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$. Then find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$.

SOLUTION 1

Differentiate both sides of the equation $x^2 + y^2 = 25$:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Remembering that y is a function of x and using the Chain Rule, we have

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$$

Thus

$$2x + 2y\frac{dy}{dx} = 0$$

Now we solve this equation for dy/dx:

$$\frac{dy}{dx} = -\frac{x}{y}$$

At the point $(3, 4)$ we have $x = 3$ and $y = 4$, so

$$\frac{dy}{dx} = -\frac{3}{4}$$

An equation of the tangent to the circle at $(3, 4)$ is therefore

$$y - 4 = -\tfrac{3}{4}(x - 3) \qquad \text{or} \qquad 3x + 4y = 25$$

SOLUTION 2

Solving the equation $x^2 + y^2 = 25$ for y, we get $y = \pm\sqrt{25 - x^2}$. The point $(3, 4)$ lies on the upper semicircle $y = \sqrt{25 - x^2}$ and so we consider the function $f(x) = \sqrt{25 - x^2}$. Differentiating f using the Chain Rule, we have

$$\begin{aligned} f'(x) &= \tfrac{1}{2}(25 - x^2)^{-1/2}\frac{d}{dx}(25 - x^2) \\ &= \tfrac{1}{2}(25 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{25 - x^2}} \end{aligned}$$

At the point $(3, 4)$ we have

$$f'(3) = -\frac{3}{\sqrt{25 - 3^2}} = -\frac{3}{4}$$

and, as in Solution 1, an equation of the tangent is $3x + 4y = 25$. ■

Example 1 illustrates that even when it is possible to solve an equation explicitly for y in terms of x, it may be easier to use implicit differentiation.

NOTE 1 The expression $dy/dx = -x/y$ in Solution 1 gives the derivative in terms of both x and y. It is correct no matter which function y is determined by the given equation. For instance, for $y = f(x) = \sqrt{25 - x^2}$ we have

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{\sqrt{25 - x^2}}$$

whereas for $y = g(x) = -\sqrt{25 - x^2}$ we have

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{-\sqrt{25 - x^2}} = \frac{x}{\sqrt{25 - x^2}}$$

EXAMPLE 2

(a) Find y' if $x^3 + y^3 = 6xy$.
(b) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point (3, 3).
(c) At what point in the first quadrant is the tangent line horizontal?

SOLUTION

(a) Differentiating both sides of $x^3 + y^3 = 6xy$ with respect to x, regarding y as a function of x, and using the Chain Rule on the term y^3 and the Product Rule on the term $6xy$, we get

We can use either of the notations dy/dx or y' for the derivative of y with respect to x.

$$3x^2 + 3y^2y' = 6xy' + 6y$$

or

$$x^2 + y^2y' = 2xy' + 2y$$

We now solve for y':

$$y^2y' - 2xy' = 2y - x^2$$

$$(y^2 - 2x)y' = 2y - x^2$$

$$y' = \frac{2y - x^2}{y^2 - 2x}$$

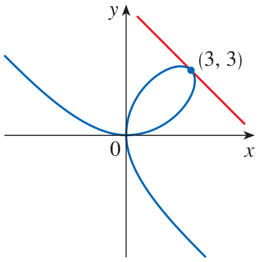


FIGURE 4

(b) When $x = y = 3$,

$$y' = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = -1$$

and a glance at Figure 4 confirms that this is a reasonable value for the slope at (3, 3). So an equation of the tangent to the folium at (3, 3) is

$$y - 3 = -1(x - 3) \qquad \text{or} \qquad x + y = 6$$

(c) The tangent line is horizontal if $y' = 0$. Using the expression for y' from part (a), we see that $y' = 0$ when $2y - x^2 = 0$ (provided that $y^2 - 2x \neq 0$). Substituting $y = \frac{1}{2}x^2$ in the equation of the curve, we get

$$x^3 + \left(\tfrac{1}{2}x^2\right)^3 = 6x\left(\tfrac{1}{2}x^2\right)$$

which simplifies to $x^6 = 16x^3$. Since $x \neq 0$ in the first quadrant, we have $x^3 = 16$. If $x = 16^{1/3} = 2^{4/3}$, then $y = \frac{1}{2}(2^{8/3}) = 2^{5/3}$. Thus the tangent is horizontal at $(2^{4/3}, 2^{5/3})$, which is approximately (2.5198, 3.1748). Looking at Figure 5, we see that our answer is reasonable. ■

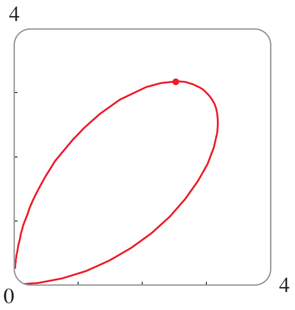


FIGURE 5

NOTE 2 There is a formula for the three solutions of a cubic equation that is like the quadratic formula but much more complicated. If we use this formula (or a computer) to solve the equation $x^3 + y^3 = 6xy$ for y in terms of x, we get three functions determined by the equations:

$$y = f(x) = \sqrt[3]{-\tfrac{1}{2}x^3 + \sqrt{\tfrac{1}{4}x^6 - 8x^3}} + \sqrt[3]{-\tfrac{1}{2}x^3 - \sqrt{\tfrac{1}{4}x^6 - 8x^3}}$$

and

$$y = \tfrac{1}{2}\left[-f(x) \pm \sqrt{-3}\left(\sqrt[3]{-\tfrac{1}{2}x^3 + \sqrt{\tfrac{1}{4}x^6 - 8x^3}} - \sqrt[3]{-\tfrac{1}{2}x^3 - \sqrt{\tfrac{1}{4}x^6 - 8x^3}}\right)\right]$$

(These are the three functions whose graphs are shown in Figure 3.) You can see that the method of implicit differentiation saves an enormous amount of work in cases such as this. Moreover, implicit differentiation works just as easily for equations such as

$$y^5 + 3x^2y^2 + 5x^4 = 12$$

for which it is *impossible* to find an expression for y in terms of x.

Abel and Galois

The Norwegian mathematician Niels Abel proved in 1824 that no general formula in terms of radicals can be given for the solutions of the equation $p(x) = 0$, where p is a polynomial of degree 5 with integer coefficients. Later the French mathematician Evariste Galois proved that it is impossible to find a general formula in terms of radicals for the solutions of an equation $p(x) = 0$, where p is a polynomial of degree $n \geqslant 5$.

EXAMPLE 3 Find y' if $\sin(x + y) = y^2 \cos x$.

SOLUTION Differentiating implicitly with respect to x and remembering that y is a function of x, we get

$$\cos(x + y) \cdot (1 + y') = y^2(-\sin x) + (\cos x)(2yy')$$

(Note that we have used the Chain Rule on the left side and the Product Rule and Chain Rule on the right side.) If we collect the terms that involve y', we get

$$\cos(x + y) + y^2 \sin x = (2y \cos x)y' - \cos(x + y) \cdot y'$$

So

$$y' = \frac{y^2 \sin x + \cos(x + y)}{2y \cos x - \cos(x + y)}$$

Figure 6, drawn by a computer, shows part of the curve $\sin(x + y) = y^2 \cos x$. As a check on our calculation, notice that $y' = -1$ when $x = y = 0$ and it appears from the graph that the slope is approximately -1 at the origin. ■

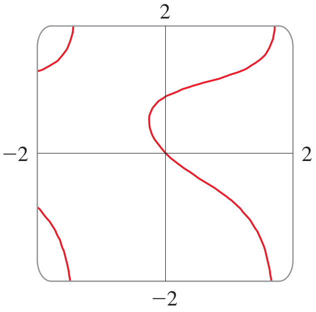


FIGURE 6

Figures 7, 8, and 9 show three more curves produced by a computer. In Exercises 45–46 you will have an opportunity to create and examine unusual curves of this nature.

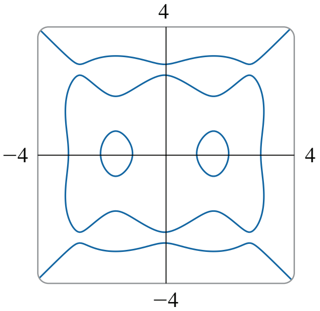


FIGURE 7
$(x^2 - 1)(x^2 - 4)(x^2 - 9) = y^2(y^2 - 4)(y^2 - 9)$

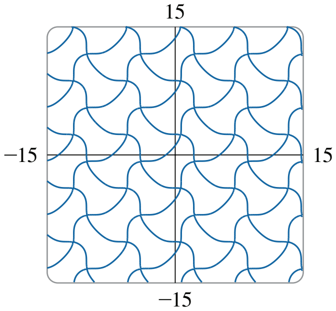


FIGURE 8
$\cos(x - \sin y) = \sin(y - \sin x)$

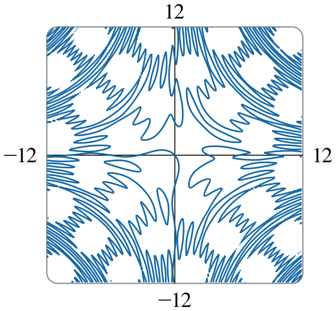


FIGURE 9
$\sin(xy) = \sin x + \sin y$

Second Derivatives of Implicit Functions

The following example shows how to find the second derivative of a function that is defined implicitly.

EXAMPLE 4 Find y'' if $x^4 + y^4 = 16$.

SOLUTION Differentiating the equation implicitly with respect to x, we get

$$4x^3 + 4y^3y' = 0$$

Solving for y' gives

$$y' = -\frac{x^3}{y^3} \tag{3}$$

To find y'' we differentiate this expression for y' using the Quotient Rule and remembering that y is a function of x:

$$y'' = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -\frac{y^3\,(d/dx)(x^3) - x^3\,(d/dx)(y^3)}{(y^3)^2}$$

$$= -\frac{y^3 \cdot 3x^2 - x^3(3y^2y')}{y^6}$$

If we now substitute Equation 3 into this expression, we get

$$y'' = -\frac{3x^2y^3 - 3x^3y^2\left(-\dfrac{x^3}{y^3}\right)}{y^6}$$

$$= -\frac{3(x^2y^4 + x^6)}{y^7} = -\frac{3x^2(y^4 + x^4)}{y^7}$$

But the values of x and y must satisfy the original equation $x^4 + y^4 = 16$. So the answer simplifies to

$$y'' = -\frac{3x^2(16)}{y^7} = -48\,\frac{x^2}{y^7}$$ ■

Figure 10 shows the graph of the curve $x^4 + y^4 = 16$ of Example 4. Notice that it's a stretched and flattened version of the circle $x^2 + y^2 = 4$. For this reason it's sometimes called a *fat circle*. It starts out very steep on the left but quickly becomes very flat. This can be seen from the expression

$$y' = -\frac{x^3}{y^3} = -\left(\frac{x}{y}\right)^3$$

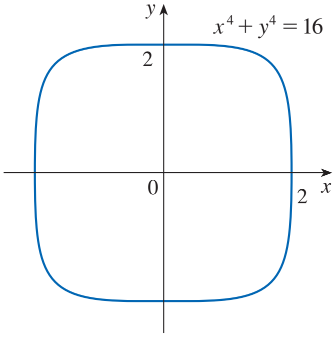


FIGURE 10

3.5 Exercises

1–4

(a) Find y' by implicit differentiation.

(b) Solve the equation explicitly for y and differentiate to get y' in terms of x.

(c) Check that your solutions to parts (a) and (b) are consistent by substituting the expression for y into your solution for part (a).

1. $5x^2 - y^3 = 7$

2. $6x^4 + y^5 = 2x$

3. $\sqrt{x} + \sqrt{y} = 1$

4. $\dfrac{2}{x} - \dfrac{1}{y} = 4$

5–22 Find dy/dx by implicit differentiation.

5. $x^2 - 4xy + y^2 = 4$

6. $2x^2 + xy - y^2 = 2$

7. $x^4 + x^2y^2 + y^3 = 5$

8. $x^3 - xy^2 + y^3 = 1$

9. $\dfrac{x^2}{x + y} = y^2 + 1$

10. $xe^y = x - y$

11. $\sin x + \cos y = 2x - 3y$

12. $e^x \sin y = x + y$

13. $\sin(x + y) = \cos x + \cos y$

14. $\tan(x - y) = 2xy^3 + 1$

15. $y \cos x = x^2 + y^2$

16. $\sin(xy) = \cos(x + y)$

17. $2xe^y + ye^x = 3$

18. $\sin x \cos y = x^2 - 5y$

19. $\sqrt{x + y} = x^4 + y^4$

20. $xy = \sqrt{x^2 + y^2}$

21. $e^{x/y} = x - y$

22. $\cos(x^2 + y^2) = xe^y$

23. If $f(x) + x^2[f(x)]^3 = 10$ and $f(1) = 2$, find $f'(1)$.

24. If $g(x) + x \sin g(x) = x^2$, find $g'(0)$.

25–26 Regard y as the independent variable and x as the dependent variable and use implicit differentiation to find dx/dy.

25. $x^4y^2 - x^3y + 2xy^3 = 0$ **26.** $y \sec x = x \tan y$

27–36 Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

27. $ye^{\sin x} = x \cos y, \quad (0, 0)$

28. $\tan(x + y) + \sec(x - y) = 2, \quad (\pi/8, \pi/8)$

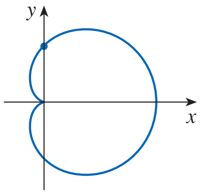
29. $x^{2/3} + y^{2/3} = 4, \quad \left(-3\sqrt{3}, 1\right)$ (astroid)

30. $y^2(6 - x) = x^3, \quad \left(2, \sqrt{2}\right)$ (cissoid of Diocles)

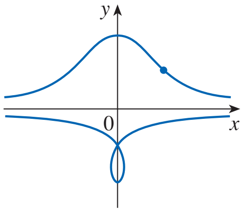
31. $x^2 - xy - y^2 = 1, \quad (2, 1)$ (hyperbola)

32. $x^2 + 2xy + 4y^2 = 12, \quad (2, 1)$ (ellipse)

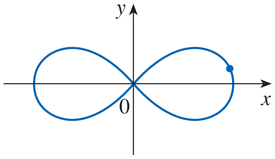
33. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2, \quad \left(0, \frac{1}{2}\right)$ (cardioid)



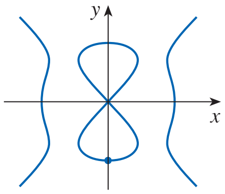
34. $x^2y^2 = (y + 1)^2(4 - y^2), \quad \left(2\sqrt{3}, 1\right)$
(conchoid of Nicomedes)



35. $2(x^2 + y^2)^2 = 25(x^2 - y^2), \quad (3, 1)$ (lemniscate)



36. $y^2(y^2 - 4) = x^2(x^2 - 5), \quad (0, -2)$ (devil's curve)



37. (a) The curve with equation $y^2 = 5x^4 - x^2$ is called a **kampyle of Eudoxus**. Find an equation of the tangent line to this curve at the point $(1, 2)$.

(b) Illustrate part (a) by graphing the curve and the tangent line on a common screen. (Graph the implicitly defined curve if possible, or you can graph the upper and lower halves separately.)

38. (a) The curve with equation $y^2 = x^3 + 3x^2$ is called the **Tschirnhausen cubic**. Find an equation of the tangent line to this curve at the point $(1, -2)$.

(b) At what points does this curve have horizontal tangents?

(c) Illustrate parts (a) and (b) by graphing the curve and the tangent lines on a common screen.

39–42 Find y'' by implicit differentiation. Simplify where possible.

39. $x^2 + 4y^2 = 4$ **40.** $x^2 + xy + y^2 = 3$

41. $\sin y + \cos x = 1$ **42.** $x^3 - y^3 = 7$

---

43. If $xy + e^y = e$, find the value of y'' at the point where $x = 0$.

44. If $x^2 + xy + y^3 = 1$, find the value of y''' at the point where $x = 1$.

45. Fanciful shapes can be created by using software that can graph implicitly defined curves.

(a) Graph the curve with equation

$$y(y^2 - 1)(y - 2) = x(x - 1)(x - 2)$$

At how many points does this curve have horizontal tangents? Estimate the x-coordinates of these points.

(b) Find equations of the tangent lines at the points $(0, 1)$ and $(0, 2)$.

(c) Find the exact x-coordinates of the points in part (a).

(d) Create even more fanciful curves by modifying the equation in part (a).

46. (a) The curve with equation

$$2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$$

has been likened to a bouncing wagon. Graph this curve and discover why.

(b) At how many points does this curve have horizontal tangent lines? Find the x-coordinates of these points.

47. Find the points on the lemniscate in Exercise 35 where the tangent is horizontal.

48. Show by implicit differentiation that the tangent line to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) has equation

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1$$

49. Find an equation of the tangent line to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0).

50. Show that the sum of the x- and y-intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to c.

51. Show, using implicit differentiation, that any tangent line at a point P to a circle with center O is perpendicular to the radius OP.

52. The Power Rule can be proved using implicit differentiation for the case where n is a rational number, $n = p/q$, and $y = f(x) = x^n$ is assumed beforehand to be a differentiable function. If $y = x^{p/q}$, then $y^q = x^p$. Use implicit differentiation to show that

$$y' = \frac{p}{q}x^{(p/q)-1}$$

53–56 Orthogonal Trajectories Two curves are *orthogonal* if their tangent lines are perpendicular at each point of intersection. Show that the given families of curves are *orthogonal trajectories* of each other; that is, every curve in one family is orthogonal to every curve in the other family. Sketch both families of curves on the same axes.

53. $x^2 + y^2 = r^2, \quad ax + by = 0$

54. $x^2 + y^2 = ax, \quad x^2 + y^2 = by$

55. $y = cx^2, \quad x^2 + 2y^2 = k$

56. $y = ax^3, \quad x^2 + 3y^2 = b$

57. Show that the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the hyperbola $x^2/A^2 - y^2/B^2 = 1$ are orthogonal trajectories if $A^2 < a^2$ and $a^2 - b^2 = A^2 + B^2$ (so the ellipse and hyperbola have the same foci).

58. Find the value of the number a such that the families of curves $y = (x + c)^{-1}$ and $y = a(x + k)^{1/3}$ are orthogonal trajectories.

59. The *van der Waals equation* for n moles of a gas is

$$\left(P + \frac{n^2a}{V^2}\right)(V - nb) = nRT$$

where P is the pressure, V is the volume, and T is the temperature of the gas. The constant R is the universal gas constant and a and b are positive constants that are characteristic of a particular gas.

(a) If T remains constant, use implicit differentiation to find dV/dP.

(b) Find the rate of change of volume with respect to pressure of 1 mole of carbon dioxide at a volume of $V = 10$ L and a pressure of $P = 2.5$ atm. Use $a = 3.592$ L^2-atm/mole^2 and $b = 0.04267$ L/mole.

60. (a) Use implicit differentiation to find y' if

$$x^2 + xy + y^2 + 1 = 0$$

(b) Plot the curve in part (a). What do you see? Prove that what you see is correct.

(c) In view of part (b), what can you say about the expression for y' that you found in part (a)?

61. The equation $x^2 - xy + y^2 = 3$ represents a “rotated ellipse,” that is, an ellipse whose axes are not parallel to the coordinate axes. Find the points at which this ellipse crosses the x-axis and show that the tangent lines at these points are parallel.

62. (a) Where does the normal line to the ellipse $x^2 - xy + y^2 = 3$ at the point $(-1, 1)$ intersect the ellipse a second time?

(b) Illustrate part (a) by graphing the ellipse and the normal line.

63. Find all points on the curve $x^2y^2 + xy = 2$ where the slope of the tangent line is -1.

64. Find equations of both the tangent lines to the ellipse $x^2 + 4y^2 = 36$ that pass through the point $(12, 3)$.

65. Use implicit differentiation to find dy/dx for the equation

$$\frac{x}{y} = y^2 + 1 \qquad y \neq 0$$

and for the equivalent equation

$$x = y^3 + y \qquad y \neq 0$$

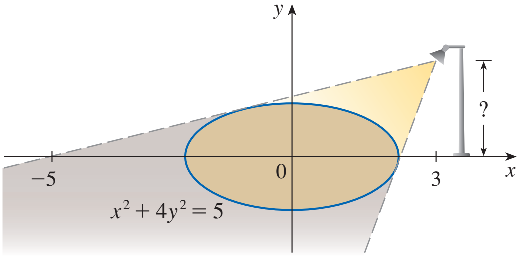
Show that although the expressions you get for dy/dx look different, they agree for all points that satisfy the given equation.

66. The *Bessel function* of order 0, $y = J(x)$, satisfies the differential equation $xy'' + y' + xy = 0$ for all values of x and its value at 0 is $J(0) = 1$.

(a) Find $J'(0)$.

(b) Use implicit differentiation to find $J''(0)$.

67. The figure shows a lamp located three units to the right of the y-axis and a shadow created by the elliptical region $x^2 + 4y^2 \leqslant 5$. If the point $(-5, 0)$ is on the edge of the shadow, how far above the x-axis is the lamp located?



DISCOVERY PROJECT | FAMILIES OF IMPLICIT CURVES

In this project you will explore the changing shapes of implicitly defined curves as you vary the constants in a family, and determine which features are common to all members of the family.

1. Consider the family of curves

$$y^2 - 2x^2(x + 8) = c[(y + 1)^2(y + 9) - x^2]$$

(a) By graphing the curves with $c = 0$ and $c = 2$, determine how many points of intersection there are. (You might have to zoom in to find all of them.)

(b) Now add the curves with $c = 5$ and $c = 10$ to your graphs in part (a). What do you notice? What about other values of c?

2. (a) Graph several members of the family of curves

$$x^2 + y^2 + cx^2y^2 = 1$$

Describe how the graph changes as you change the value of c.

(b) What happens to the curve when $c = -1$? Describe what appears on the screen. Can you prove it algebraically?

(c) Find y' by implicit differentiation. For the case $c = -1$, is your expression for y' consistent with what you discovered in part (b)?

3.6 Derivatives of Logarithmic and Inverse Trigonometric Functions

In this section we use implicit differentiation to find derivatives of logarithmic functions and of inverse trigonometric functions.

Derivatives of Logarithmic Functions

In Appendix F we prove that if f is a one-to-one differentiable function, then its inverse function f^{-1} is also differentiable, except where its tangents are vertical. This is plausible because, geometrically, we can think of a differentiable function as one whose graph has no corner or cusp. We obtain the graph of f^{-1} by reflecting the graph of f about the line $y = x$, so the graph of f^{-1} has no corner or cusp either. (Note that if f has a horizontal tangent at a point, then f^{-1} has a vertical tangent at the corresponding reflected point and so f^{-1} is not differentiable there.)

Because the logarithmic function $y = \log_b x$ is the inverse of the exponential function $y = b^x$, which we know is differentiable from Section 3.1, it follows that the logarithmic function is also differentiable. We now state and prove the formula for the derivative of a logarithmic function.

$$\boxed{1} \qquad \boxed{\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}}$$

Formula 3.4.5 says that

$$\frac{d}{dx}(b^x) = b^x \ln b$$

PROOF Let $y = \log_b x$. Then

$$b^y = x$$

Differentiating this equation implicitly with respect to x, and using Formula 3.4.5, we get

$$(b^y \ln b)\frac{dy}{dx} = 1$$

and so

$$\frac{dy}{dx} = \frac{1}{b^y \ln b} = \frac{1}{x \ln b} \qquad \blacksquare$$

If we put $b = e$ in Formula 1, then the factor $\ln b$ on the right side becomes $\ln e = 1$ and we get the formula for the derivative of the natural logarithmic function $\log_e x = \ln x$:

$$\boxed{2} \qquad \boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

By comparing Formulas 1 and 2, we see one of the main reasons that natural logarithms (logarithms with base e) are used in calculus: the differentiation formula is simplest when $b = e$ because $\ln e = 1$.

EXAMPLE 1 Differentiate $y = \ln(x^3 + 1)$.

SOLUTION To use the Chain Rule, we let $u = x^3 + 1$. Then $y = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{1}{x^3 + 1}(3x^2) = \frac{3x^2}{x^3 + 1} \qquad \blacksquare$$

In general, if we combine Formula 2 with the Chain Rule as in Example 1, we get

$$\boxed{3} \qquad \boxed{\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}} \qquad \text{or} \qquad \boxed{\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}}$$

EXAMPLE 2 Find $\dfrac{d}{dx}\ln(\sin x)$.

SOLUTION Using (3), we have

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x = \cot x \qquad \blacksquare$$

EXAMPLE 3 Differentiate $f(x) = \sqrt{\ln x}$.

SOLUTION This time the logarithm is the inner function, so the Chain Rule gives

$$f'(x) = \tfrac{1}{2}(\ln x)^{-1/2}\frac{d}{dx}(\ln x) = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}} \qquad \blacksquare$$

EXAMPLE 4 Differentiate $f(x) = \log_{10}(2 + \sin x)$.

SOLUTION Using Formula 1 with $b = 10$, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\log_{10}(2 + \sin x) \\ &= \frac{1}{(2 + \sin x)\ln 10}\,\frac{d}{dx}(2 + \sin x) \\ &= \frac{\cos x}{(2 + \sin x)\ln 10} \end{aligned}$$

■

Figure 1 shows the graph of the function f of Example 5 together with the graph of its derivative. It gives a visual check on our calculation. Notice that $f'(x)$ is large negative when f is decreasing rapidly.

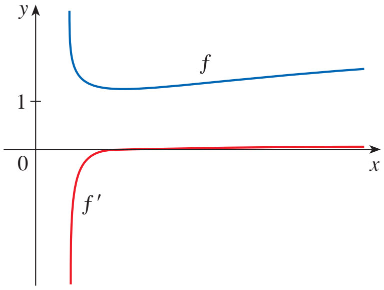


FIGURE 1

EXAMPLE 5 Find $\dfrac{d}{dx}\ln\dfrac{x+1}{\sqrt{x-2}}$.

SOLUTION 1

$$\begin{aligned} \frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}} &= \frac{1}{\dfrac{x+1}{\sqrt{x-2}}}\,\frac{d}{dx}\,\frac{x+1}{\sqrt{x-2}} \\ &= \frac{\sqrt{x-2}}{x+1}\;\frac{\sqrt{x-2}\cdot 1 - (x+1)\left(\frac{1}{2}\right)(x-2)^{-1/2}}{x-2} \\ &= \frac{x - 2 - \frac{1}{2}(x+1)}{(x+1)(x-2)} \\ &= \frac{x-5}{2(x+1)(x-2)} \end{aligned}$$

SOLUTION 2 If we first expand the given function using the laws of logarithms, then the differentiation becomes easier:

$$\begin{aligned} \frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}} &= \frac{d}{dx}\left[\ln(x+1) - \tfrac{1}{2}\ln(x-2)\right] \\ &= \frac{1}{x+1} - \frac{1}{2}\left(\frac{1}{x-2}\right) \end{aligned}$$

(This answer can be left as written, but if we used a common denominator we would see that it gives the same answer as in Solution 1.) ■

Figure 2 shows the graph of the function $f(x) = \ln|x|$ in Example 6 and its derivative $f'(x) = 1/x$. Notice that when x is small, the graph of $y = \ln|x|$ is steep and so $f'(x)$ is large (positive or negative).

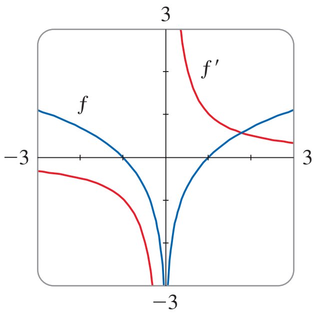


FIGURE 2

EXAMPLE 6 Find $f'(x)$ if $f(x) = \ln|x|$.

SOLUTION Since

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

it follows that

$$f'(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\[2ex] \dfrac{1}{-x}(-1) = \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

Thus $f'(x) = 1/x$ for all $x \neq 0$. ■

The result of Example 6 is worth remembering:

$$\boxed{4} \qquad \boxed{\frac{d}{dx}\ln|x| = \frac{1}{x}}$$

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called **logarithmic differentiation**.

EXAMPLE 7 Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

SOLUTION We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

If we hadn't used logarithmic differentiation in Example 7, we would have had to use both the Quotient Rule and the Product Rule. The resulting calculation would have been horrendous.

Because we have an explicit expression for y, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right) \qquad ■$$

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to expand the expression.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y' and replace y by $f(x)$.

If $f(x) < 0$ for some values of x, then $\ln f(x)$ is not defined, but we can still use logarithmic differentiation by first writing $|y| = |f(x)|$ and then using Equation 4. We illustrate this procedure by proving the general version of the Power Rule, as promised in Section 3.1. Recall that the general version of the Power Rule states that if n is any real number and $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

PROOF OF THE POWER RULE (GENERAL VERSION) Let $y = x^n$ and use logarithmic differentiation:

$$\ln|y| = \ln|x|^n = n\ln|x| \qquad x \neq 0$$

If $x = 0$, we can show that $f'(0) = 0$ for $n > 1$ directly from the definition of a derivative.

Therefore

$$\frac{y'}{y} = \frac{n}{x}$$

Hence

$$y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1} \qquad ■$$

You should distinguish carefully between the Power Rule $[(x^n)' = nx^{n-1}]$, where the base is variable and the exponent is constant, and the rule for differentiating exponential functions $[(b^x)' = b^x \ln b]$, where the base is constant and the exponent is variable.

In general there are four cases for exponents and bases:

Constant base, constant exponent

1. $\dfrac{d}{dx}(b^n) = 0 \qquad$ (b and n are constants)

Variable base, constant exponent

2. $\dfrac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x)$

Constant base, variable exponent

3. $\dfrac{d}{dx}[b^{g(x)}] = b^{g(x)}(\ln b)g'(x)$

Variable base, variable exponent

4. To find $(d/dx)[f(x)]^{g(x)}$, logarithmic differentiation can be used, as in the next example.

EXAMPLE 8 Differentiate $y = x^{\sqrt{x}}$.

SOLUTION 1 Since both the base and the exponent are variable, we use logarithmic differentiation:

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x}\,\ln x$$

$$\frac{y'}{y} = \sqrt{x}\cdot\frac{1}{x} + (\ln x)\frac{1}{2\sqrt{x}}$$

$$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right)$$

Figure 3 illustrates Example 8 by showing the graphs of $f(x) = x^{\sqrt{x}}$ and its derivative.

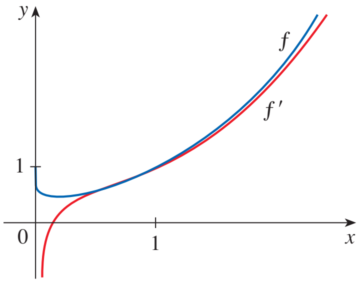


FIGURE 3

SOLUTION 2 Another method is to use Equation 1.5.10 to write $x^{\sqrt{x}} = e^{\sqrt{x}\,\ln x}$:

$$\frac{d}{dx}\left(x^{\sqrt{x}}\right) = \frac{d}{dx}\left(e^{\sqrt{x}\,\ln x}\right) = e^{\sqrt{x}\,\ln x}\frac{d}{dx}\left(\sqrt{x}\,\ln x\right)$$

$$= x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right) \qquad \text{(as in Solution 1)}$$

■

■ The Number *e* as a Limit

We have shown that if $f(x) = \ln x$, then $f'(x) = 1/x$. Thus $f'(1) = 1$. We now use this fact to express the number e as a limit.

From the definition of a derivative as a limit, we have

$$f'(1) = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h} = \lim_{x\to 0}\frac{f(1+x) - f(1)}{x}$$

$$= \lim_{x\to 0}\frac{\ln(1+x) - \ln 1}{x} = \lim_{x\to 0}\frac{1}{x}\ln(1+x)$$

$$= \lim_{x\to 0}\ln(1+x)^{1/x}$$

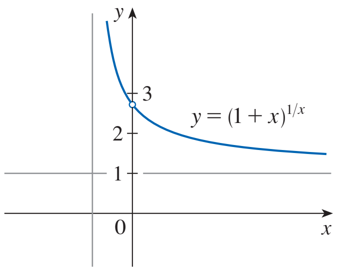


FIGURE 4

x	$(1 + x)^{1/x}$
0.1	2.59374246
0.01	2.70481383
0.001	2.71692393
0.0001	2.71814593
0.00001	2.71826824
0.000001	2.71828047
0.0000001	2.71828169
0.00000001	2.71828181

Because $f'(1) = 1$, we have

$$\lim_{x \to 0} \ln(1 + x)^{1/x} = 1$$

Then, by Theorem 2.5.8 and the continuity of the exponential function, we have

$$e = e^1 = e^{\lim_{x \to 0} \ln(1+x)^{1/x}} = \lim_{x \to 0} e^{\ln(1+x)^{1/x}} = \lim_{x \to 0} (1 + x)^{1/x}$$

5

$$\boxed{e = \lim_{x \to 0} (1 + x)^{1/x}}$$

Formula 5 is illustrated by the graph of the function $y = (1 + x)^{1/x}$ in Figure 4 and a table of values for small values of x. This illustrates the fact that, correct to seven decimal places,

$$e \approx 2.7182818$$

If we put $n = 1/x$ in Formula 5, then $n \to \infty$ as $x \to 0^+$ and so an alternative expression for e is

6

$$\boxed{e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n}$$

Derivatives of Inverse Trigonometric Functions

The inverse trigonometric functions were reviewed in Section 1.5. We discussed their continuity in Section 2.5 and their asymptotes in Section 2.6. Here we use implicit differentiation to find their derivatives. At the beginning of this section we observed that if f is a one-to-one differentiable function, then its inverse function f^{-1} is also differentiable (except where its tangents are vertical). Because the trigonometric functions—with the restricted domains that we used to define their inverses—are one-to-one and differentiable, it follows that the inverse trigonometric functions are also differentiable.

Recall the definition of the arcsine function:

$$y = \sin^{-1}x \qquad \text{means} \qquad \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

Differentiating $\sin y = x$ implicitly with respect to x, we obtain

$$\cos y \frac{dy}{dx} = 1 \qquad \text{or} \qquad \frac{dy}{dx} = \frac{1}{\cos y}$$

Now $\cos y \geqslant 0$ because $-\pi/2 \leqslant y \leqslant \pi/2$, so

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2} \qquad (\cos^2 y + \sin^2 y = 1)$$

Therefore

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

$$\boxed{\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}}}$$

Figure 5 shows the graph of $f(x) = \tan^{-1}x$ and its derivative $f'(x) = 1/(1 + x^2)$. Notice that f is increasing and $f'(x)$ is always positive. The fact that $\tan^{-1}x \to \pm\pi/2$ as $x \to \pm\infty$ is reflected in the fact that $f'(x) \to 0$ as $x \to \pm\infty$.

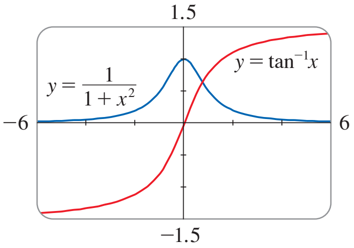


FIGURE 5

The formula for the derivative of the arctangent function is derived in a similar way. If $y = \tan^{-1}x$, then $\tan y = x$. Differentiating $\tan y = x$ implicitly with respect to x, we have

$$\sec^2 y \, \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

$$\boxed{\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2}}$$

The inverse trigonometric functions $\sin^{-1}x$ and $\tan^{-1}x$ occur most frequently. The derivatives of the remaining four are given in the following table. The proofs of the formulas are left as exercises.

The formulas for the derivatives of $\csc^{-1}x$ and $\sec^{-1}x$ depend on the definitions that are used for these functions. See Exercise 82.

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1 + x^2}$$

EXAMPLE 9 Differentiate (a) $y = \dfrac{1}{\sin^{-1}x}$ and (b) $f(x) = x \arctan\sqrt{x}$.

SOLUTION

(a)
$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)^{-1} = -(\sin^{-1}x)^{-2}\frac{d}{dx}(\sin^{-1}x)$$

$$= -\frac{1}{(\sin^{-1}x)^2\sqrt{1 - x^2}}$$

Recall that $\arctan x$ is an alternative notation for $\tan^{-1}x$.

(b)
$$f'(x) = x\,\frac{1}{1 + (\sqrt{x})^2}\left(\tfrac{1}{2}x^{-1/2}\right) + \arctan\sqrt{x}$$

$$= \frac{\sqrt{x}}{2(1 + x)} + \arctan\sqrt{x}$$ ■

EXAMPLE 10 Differentiate $g(x) = \sec^{-1}(x^2)$.

SOLUTION

$$g'(x) = \frac{1}{x^2\sqrt{(x^2)^2 - 1}}(2x) = \frac{2}{x\sqrt{x^4 - 1}}$$ ■

3.6 Exercises

1. Explain why the natural logarithmic function $y = \ln x$ is used much more frequently in calculus than the other logarithmic functions $y = \log_b x$.

2–26 Differentiate the function.

2. $g(t) = \ln(3 + t^2)$

3. $f(x) = \ln(x^2 + 3x + 5)$

4. $f(x) = x \ln x - x$

5. $f(x) = \sin(\ln x)$

6. $f(x) = \ln(\sin^2 x)$

7. $f(x) = \ln \dfrac{1}{x}$

8. $y = \dfrac{1}{\ln x}$

9. $g(x) = \ln(xe^{-2x})$

10. $g(t) = \sqrt{1 + \ln t}$

11. $F(t) = (\ln t)^2 \sin t$

12. $p(t) = \ln \sqrt{t^2 + 1}$

13. $y = \log_8(x^2 + 3x)$

14. $y = \log_{10} \sec x$

15. $F(s) = \ln \ln s$

16. $P(v) = \dfrac{\ln v}{1 - v}$

17. $T(z) = 2^z \log_2 z$

18. $g(t) = \ln \dfrac{t(t^2 + 1)^4}{\sqrt[3]{2t - 1}}$

19. $y = \ln |3 - 2x^5|$

20. $y = \ln(\csc x - \cot x)$

21. $y = \ln(e^{-x} + xe^{-x})$

22. $g(x) = e^{x^2 \ln x}$

23. $h(x) = e^{x^2 + \ln x}$

24. $y = \ln \sqrt{\dfrac{1 + 2x}{1 - 2x}}$

25. $y = \ln \dfrac{x^a}{b^x}$

26. $y = \log_2(x \log_5 x)$

---

27. Show that $\dfrac{d}{dx} \ln\left(x + \sqrt{x^2 + 1}\right) = \dfrac{1}{\sqrt{x^2 + 1}}$.

28. Show that $\dfrac{d}{dx} \ln \sqrt{\dfrac{1 - \cos x}{1 + \cos x}} = \csc x$.

29–32 Find y' and y''.

29. $y = \sqrt{x} \ln x$

30. $y = \dfrac{\ln x}{1 + \ln x}$

31. $y = \ln |\sec x|$

32. $y = \ln(1 + \ln x)$

33–36 Differentiate f and find the domain of f.

33. $f(x) = \dfrac{x}{1 - \ln(x - 1)}$

34. $f(x) = \sqrt{2 + \ln x}$

35. $f(x) = \ln(x^2 - 2x)$

36. $f(x) = \ln \ln \ln x$

---

37. If $f(x) = \ln(x + \ln x)$, find $f'(1)$.

38. If $f(x) = \cos(\ln x^2)$, find $f'(1)$.

39–40 Find an equation of the tangent line to the curve at the given point.

39. $y = \ln(x^2 - 3x + 1)$, $(3, 0)$

40. $y = x^2 \ln x$, $(1, 0)$

---

41. If $f(x) = \sin x + \ln x$, find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

42. Find equations of the tangent lines to the curve $y = (\ln x)/x$ at the points $(1, 0)$ and $(e, 1/e)$. Illustrate by graphing the curve and its tangent lines.

43. Let $f(x) = cx + \ln(\cos x)$. For what value of c is $f'(\pi/4) = 6$?

44. Let $f(x) = \log_b(3x^2 - 2)$. For what value of b is $f'(1) = 3$?

45–56 Use logarithmic differentiation to find the derivative of the function.

45. $y = (x^2 + 2)^2(x^4 + 4)^4$

46. $y = \dfrac{e^{-x} \cos^2 x}{x^2 + x + 1}$

47. $y = \sqrt{\dfrac{x - 1}{x^4 + 1}}$

48. $y = \sqrt{x}\, e^{x^2 - x}(x + 1)^{2/3}$

49. $y = x^x$

50. $y = x^{1/x}$

51. $y = x^{\sin x}$

52. $y = \left(\sqrt{x}\right)^x$

53. $y = (\cos x)^x$

54. $y = (\sin x)^{\ln x}$

55. $y = x^{\ln x}$

56. $y = (\ln x)^{\cos x}$

---

57. Find y' if $y = \ln(x^2 + y^2)$.

58. Find y' if $x^y = y^x$.

59. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x - 1)$.

60. Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$.

61. Use the definition of a derivative to prove that

$$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$$

62. Show that $\displaystyle\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$.

63–78 Find the derivative of the function. Simplify where possible.

63. $f(x) = \sin^{-1}(5x)$

64. $g(x) = \sec^{-1}(e^x)$

65. $y = \tan^{-1}\sqrt{x - 1}$

66. $y = \tan^{-1}(x^2)$

67. $y = (\tan^{-1} x)^2$

68. $g(x) = \arccos \sqrt{x}$

69. $h(x) = (\arcsin x) \ln x$

70. $g(t) = \ln(\arctan(t^4))$

71. $f(z) = e^{\arcsin(z^2)}$

72. $y = \tan^{-1}\left(x - \sqrt{1 + x^2}\right)$

73. $h(t) = \cot^{-1}(t) + \cot^{-1}(1/t)$

74. $R(t) = \arcsin(1/t)$

75. $y = x\sin^{-1}x + \sqrt{1 - x^2}$

76. $y = \cos^{-1}(\sin^{-1}t)$

77. $y = \tan^{-1}\left(\dfrac{x}{a}\right) + \ln\sqrt{\dfrac{x - a}{x + a}}$

78. $y = \arctan\sqrt{\dfrac{1 - x}{1 + x}}$

79–80 Find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

79. $f(x) = \sqrt{1 - x^2}\arcsin x$ **80.** $f(x) = \arctan(x^2 - x)$

---

81. Prove the formula for $(d/dx)(\cos^{-1}x)$ by the same method as for $(d/dx)(\sin^{-1}x)$.

82. (a) One way of defining $\sec^{-1}x$ is to say that $y = \sec^{-1}x \iff \sec y = x$ and $0 \leqslant y < \pi/2$ or $\pi \leqslant y < 3\pi/2$. Show that, with this definition,

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2 - 1}}$$

(b) Another way of defining $\sec^{-1}x$ that is sometimes used is to say that $y = \sec^{-1}x \iff \sec y = x$ and $0 \leqslant y \leqslant \pi$, $y \neq \pi/2$. Show that, with this definition,

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2 - 1}}$$

83. Derivatives of Inverse Functions Suppose that f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Use implicit differentiation to show that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided that the denominator is not 0.

84–86 Use the formula in Exercise 83.

84. If $f(4) = 5$ and $f'(4) = \frac{2}{3}$, find $(f^{-1})'(5)$.

85. If $f(x) = x + e^x$, find $(f^{-1})'(1)$.

86. If $f(x) = x^3 + 3\sin x + 2\cos x$, find $(f^{-1})'(2)$.

---

87. Suppose that f and g are differentiable functions and let $h(x) = f(x)^{g(x)}$. Use logarithmic differentiation to derive the formula

$$h' = g \cdot f^{g-1} \cdot f' + (\ln f) \cdot f^g \cdot g'$$

88. Use the formula in Exercise 87 to find the derivative.
(a) $h(x) = x^3$ (b) $h(x) = 3^x$ (c) $h(x) = (\sin x)^x$

3.7 Rates of Change in the Natural and Social Sciences

We know that if $y = f(x)$, then the derivative dy/dx can be interpreted as the rate of change of y with respect to x. In this section we examine some of the applications of this idea to physics, chemistry, biology, economics, and other sciences.

Let's recall from Section 2.7 the basic idea behind rates of change. If x changes from x_1 to x_2, then the change in x is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

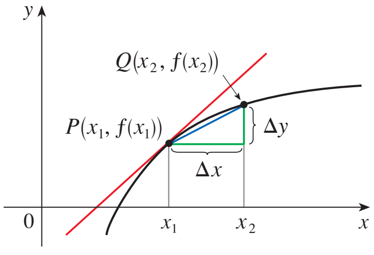
$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is **the average rate of change of y with respect to x** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line PQ in Figure 1. Its limit as $\Delta x \to 0$ is the derivative $f'(x_1)$, which can therefore be interpreted as the **instantaneous rate of change of y with respect to x** or the slope of the tangent line at $P(x_1, f(x_1))$. Using Leibniz notation, we write the process in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$



m_{PQ} = average rate of change
$m = f'(x_1)$ = instantaneous rate of change

FIGURE 1

Whenever the function $y = f(x)$ has a specific interpretation in one of the sciences, its derivative will have a specific interpretation as a rate of change. (As we discussed in Section 2.7, the units for dy/dx are the units for y divided by the units for x.) We now look at some of these interpretations in the natural and social sciences.

Physics

If $s = f(t)$ is the position function of a particle that is moving in a straight line, then $\Delta s/\Delta t$ represents the average velocity over a time period Δt, and $v = ds/dt$ represents the instantaneous **velocity** (the rate of change of displacement with respect to time). The instantaneous rate of change of velocity with respect to time is **acceleration**: $a(t) = v'(t) = s''(t)$. This was discussed in Sections 2.7 and 2.8, but now that we know the differentiation formulas, we are able to more easily solve problems involving the motion of objects.

EXAMPLE 1 The position of a particle is given by the equation

$$s = f(t) = t^3 - 6t^2 + 9t$$

where t is measured in seconds and s in meters.
(a) Find the velocity at time t.
(b) What is the velocity after 2 s? After 4 s?
(c) When is the particle at rest?
(d) When is the particle moving forward (that is, in the positive direction)?
(e) Draw a diagram to represent the motion of the particle.
(f) Find the total distance traveled by the particle during the first five seconds.
(g) Find the acceleration at time t and after 4 s.
(h) Graph the position, velocity, and acceleration functions for $0 \leqslant t \leqslant 5$.
(i) When is the particle speeding up? When is it slowing down?

SOLUTION

(a) The velocity function is the derivative of the position function:

$$s = f(t) = t^3 - 6t^2 + 9t$$

$$v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9$$

(b) The velocity after 2 s means the instantaneous velocity when $t = 2$, that is,

$$v(2) = \left.\frac{ds}{dt}\right|_{t=2} = 3(2)^2 - 12(2) + 9 = -3 \text{ m/s}$$

The velocity after 4 s is

$$v(4) = 3(4)^2 - 12(4) + 9 = 9 \text{ m/s}$$

(c) The particle is at rest when $v(t) = 0$, that is,

$$3t^2 - 12t + 9 = 3(t^2 - 4t + 3) = 3(t - 1)(t - 3) = 0$$

and this is true when $t = 1$ or $t = 3$. Thus the particle is at rest after 1 s and after 3 s.

(d) The particle moves in the positive direction when $v(t) > 0$, that is,

$$3t^2 - 12t + 9 = 3(t - 1)(t - 3) > 0$$

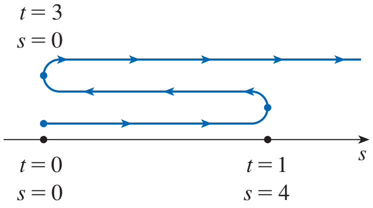


FIGURE 2

This inequality is true when both factors are positive ($t > 3$) or when both factors are negative ($t < 1$). Thus the particle moves in the positive direction in the time intervals $t < 1$ and $t > 3$. It moves backward (in the negative direction) when $1 < t < 3$.

(e) Using the information from part (d) we make a schematic sketch in Figure 2 of the motion of the particle back and forth along a line (the s-axis).

(f) Because of what we learned in parts (d) and (e), we need to calculate the distances traveled during the time intervals [0, 1], [1, 3], and [3, 5] separately.

The distance traveled in the first second is

$$|f(1) - f(0)| = |4 - 0| = 4 \text{ m}$$

From $t = 1$ to $t = 3$ the distance traveled is

$$|f(3) - f(1)| = |0 - 4| = 4 \text{ m}$$

From $t = 3$ to $t = 5$ the distance traveled is

$$|f(5) - f(3)| = |20 - 0| = 20 \text{ m}$$

The total distance is $4 + 4 + 20 = 28$ m.

(g) The acceleration is the derivative of the velocity function:

$$a(t) = \frac{d^2s}{dt^2} = \frac{dv}{dt} = 6t - 12$$

$$a(4) = 6(4) - 12 = 12 \text{ m/s}^2$$

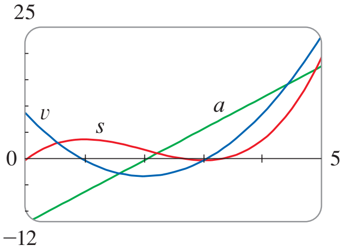


FIGURE 3

(h) Figure 3 shows the graphs of s, v, and a.

(i) The particle speeds up when the velocity is positive and increasing (v and a are both positive) and also when the velocity is negative and decreasing (v and a are both negative). In other words, the particle speeds up when the velocity and acceleration have the same sign. (The particle is pushed in the same direction it is moving.) From Figure 3 we see that this happens when $1 < t < 2$ and when $t > 3$. The particle slows down when v and a have opposite signs, that is, when $0 \leqslant t < 1$ and when $2 < t < 3$. Figure 4 summarizes the motion of the particle.

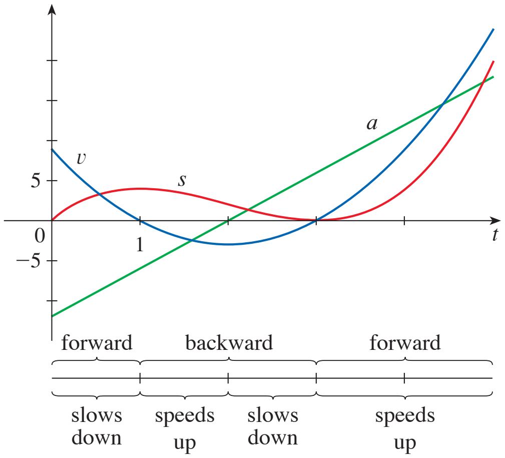
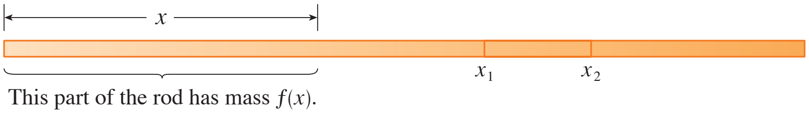


FIGURE 4

■

EXAMPLE 2 If a rod or piece of wire is homogeneous, then its *linear density* is uniform and is defined as the mass per unit length ($\rho = m/l$) and measured in kilograms per meter. Suppose, however, that the rod is not homogeneous but that its mass measured from its left end to a point x is $m = f(x)$, as shown in Figure 5.

FIGURE 5



The mass of the part of the rod that lies between $x = x_1$ and $x = x_2$ is given by $\Delta m = f(x_2) - f(x_1)$, so the average density of that part of the rod is

$$\text{average density} = \frac{\Delta m}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

If we now let $\Delta x \to 0$ (that is, $x_2 \to x_1$), we are computing the average density over smaller and smaller intervals. The **linear density** ρ at x_1 is the limit of these average densities as $\Delta x \to 0$; that is, the linear density is the rate of change of mass with respect to length. Symbolically,

$$\rho = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} = \frac{dm}{dx}$$

Thus the linear density of the rod is the derivative of mass with respect to length.

For instance, if $m = f(x) = \sqrt{x}$, where x is measured in meters and m in kilograms, then the average density of the part of the rod given by $1 \leqslant x \leqslant 1.2$ is

$$\frac{\Delta m}{\Delta x} = \frac{f(1.2) - f(1)}{1.2 - 1} = \frac{\sqrt{1.2} - 1}{0.2} \approx 0.48 \text{ kg/m}$$

while the density right at $x = 1$ is

$$\rho = \left.\frac{dm}{dx}\right|_{x=1} = \left.\frac{1}{2\sqrt{x}}\right|_{x=1} = 0.50 \text{ kg/m}$$ ■

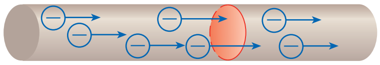


FIGURE 6

EXAMPLE 3 A current exists whenever electric charges move. Figure 6 shows part of a wire and electrons moving through a plane surface, shaded red. If ΔQ is the net charge that passes through this surface during a time period Δt, then the average current during this time interval is defined as

$$\text{average current} = \frac{\Delta Q}{\Delta t} = \frac{Q_2 - Q_1}{t_2 - t_1}$$

If we take the limit of this average current over smaller and smaller time intervals, we get what is called the **current** I at a given time t_1:

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

Thus the current is the rate at which charge flows through a surface. It is measured in units of charge per unit time (often coulombs per second, called amperes). ■

Velocity, density, and current are not the only rates of change that are important in physics. Others include power (the rate at which work is done), the rate of heat flow, temperature gradient (the rate of change of temperature with respect to position), and the rate of decay of a radioactive substance in nuclear physics.

Chemistry

EXAMPLE 4 A chemical reaction results in the formation of one or more substances (called *products*) from one or more starting materials (called *reactants*). For instance, the "equation"

$$2H_2 + O_2 \rightarrow 2H_2O$$

indicates that two molecules of hydrogen and one molecule of oxygen form two molecules of water. Let's consider the reaction

$$\text{A} + \text{B} \rightarrow \text{C}$$

where A and B are the reactants and C is the product. The **concentration** of a reactant A is the number of moles (1 mole $= 6.022 \times 10^{23}$ molecules) per liter and is denoted by [A]. The concentration varies during a reaction, so [A], [B], and [C] are all functions of time (t). The average rate of reaction of the product C over a time interval $t_1 \leqslant t \leqslant t_2$ is

$$\frac{\Delta[\text{C}]}{\Delta t} = \frac{[\text{C}](t_2) - [\text{C}](t_1)}{t_2 - t_1}$$

But chemists are interested in the **instantaneous rate of reaction** because it gives information about the mechanism of the chemical reaction. The instantaneous rate of reaction is obtained by taking the limit of the average rate of reaction as the time interval Δt approaches 0:

$$\text{rate of reaction} = \lim_{\Delta t \to 0} \frac{\Delta[\text{C}]}{\Delta t} = \frac{d[\text{C}]}{dt}$$

Since the concentration of the product increases as the reaction proceeds, the derivative $d[\text{C}]/dt$ will be positive, and so the rate of reaction of C is positive. The concentrations of the reactants, however, decrease during the reaction, so, to make the rates of reaction of A and B positive numbers, we put minus signs in front of the derivatives $d[\text{A}]/dt$ and $d[\text{B}]/dt$. Since [A] and [B] each decrease at the same rate that [C] increases, we have

$$\text{rate of reaction} = \frac{d[\text{C}]}{dt} = -\frac{d[\text{A}]}{dt} = -\frac{d[\text{B}]}{dt}$$

More generally, it turns out that for a reaction of the form

$$a\text{A} + b\text{B} \rightarrow c\text{C} + d\text{D}$$

we have

$$-\frac{1}{a}\frac{d[\text{A}]}{dt} = -\frac{1}{b}\frac{d[\text{B}]}{dt} = \frac{1}{c}\frac{d[\text{C}]}{dt} = \frac{1}{d}\frac{d[\text{D}]}{dt}$$

The rate of reaction can be determined from data and graphical methods. In some cases there are explicit formulas for the concentrations as functions of time that enable us to compute the rate of reaction (see Exercise 26). ■

EXAMPLE 5 One of the quantities of interest in thermodynamics is compressibility. If a given substance is kept at a constant temperature, then its volume V depends on its pressure P. We can consider the rate of change of volume with respect to pressure—namely, the derivative dV/dP. As P increases, V decreases, so $dV/dP < 0$. The **compressibility** is defined by introducing a minus sign and dividing this derivative by the volume V:

$$\text{isothermal compressibility} = \beta = -\frac{1}{V}\frac{dV}{dP}$$

Thus β measures how fast, per unit volume, the volume of a substance decreases as the pressure on it increases at constant temperature.

For instance, the volume V (in cubic meters) of a sample of air at 25°C was found to be related to the pressure P (in kilopascals) by the equation

$$V = \frac{5.3}{P}$$

The rate of change of V with respect to P when $P = 50$ kPa is

$$\begin{aligned} \left.\frac{dV}{dP}\right|_{P=50} &= \left.-\frac{5.3}{P^2}\right|_{P=50} \\ &= -\frac{5.3}{2500} = -0.00212 \text{ m}^3/\text{kPa} \end{aligned}$$

The compressibility at that pressure is

$$\beta = \left.-\frac{1}{V}\frac{dV}{dP}\right|_{P=50} = \frac{0.00212}{\dfrac{5.3}{50}} = 0.02 \text{ (m}^3/\text{kPa)/m}^3$$ ■

Biology

EXAMPLE 6 Let $n = f(t)$ be the number of individuals in an animal or plant population at time t. The change in the population size between the times $t = t_1$ and $t = t_2$ is $\Delta n = f(t_2) - f(t_1)$, and so the average rate of growth during the time period $t_1 \leqslant t \leqslant t_2$ is

$$\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

The **instantaneous rate of growth** is obtained from this average rate of growth by letting the time period Δt approach 0:

$$\text{growth rate} = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

Strictly speaking, this is not quite accurate because the actual graph of a population function $n = f(t)$ would be a step function that is discontinuous whenever a birth or death occurs and therefore not differentiable. However, for a large animal or plant population, we can replace the graph by a smooth approximating curve as in Figure 7.

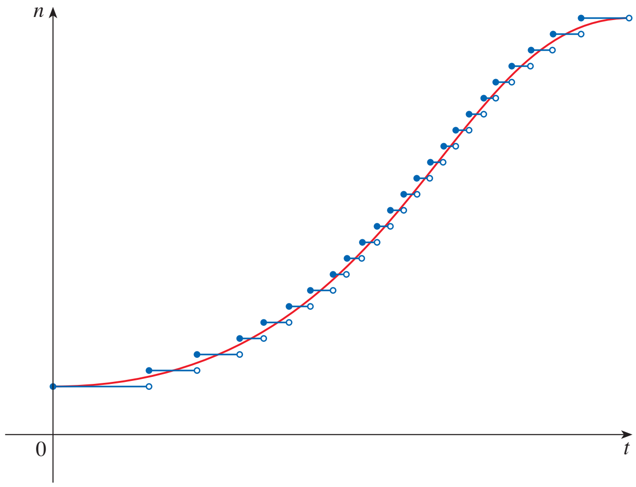


FIGURE 7
A smooth curve approximating a growth function

To be more specific, consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour. If the initial population is n_0 and the time t is measured in hours, then

$$f(1) = 2f(0) = 2n_0$$

$$f(2) = 2f(1) = 2^2 n_0$$

$$f(3) = 2f(2) = 2^3 n_0$$

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E. coli bacteria measure about 2 micrometers (μm) long and 0.75 μm wide. The image was produced with a scanning electron microscope.

and, in general,

$$f(t) = 2^t n_0$$

The population function is $n = n_0 2^t$.

In Section 3.4 we showed that

$$\frac{d}{dx}(b^x) = b^x \ln b$$

So the rate of growth of the bacteria population at time t is

$$\frac{dn}{dt} = \frac{d}{dt}(n_0 2^t) = n_0 2^t \ln 2$$

For example, suppose that we start with an initial population of $n_0 = 100$ bacteria. Then the rate of growth after 4 hours is

$$\left.\frac{dn}{dt}\right|_{t=4} = 100 \cdot 2^4 \ln 2 = 1600 \ln 2 \approx 1109$$

This means that, after 4 hours, the bacteria population is growing at a rate of about 1109 bacteria per hour. ■

EXAMPLE 7 When we consider the flow of blood through a blood vessel, such as a vein or artery, we can model the shape of the blood vessel by a cylindrical tube with radius R and length l as illustrated in Figure 8.

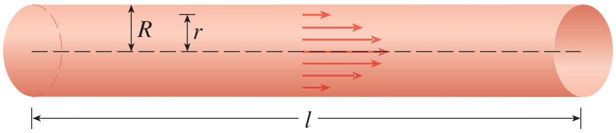


FIGURE 8
Blood flow in an artery

Because of friction at the walls of the tube, the velocity v of the blood is greatest along the central axis of the tube and decreases as the distance r from the axis increases until v becomes 0 at the wall. The relationship between v and r is given by the **law of laminar flow**, which was experimentally derived by the French physicist Jean Léonard Marie Poiseuille in 1838. This law states that

$$\boxed{1} \qquad v = \frac{P}{4\eta l}(R^2 - r^2)$$

where η is the viscosity of the blood and P is the pressure difference between the ends of the tube. If P and l are constant, then v is a function of r with domain $[0, R]$.

For more detailed information, see W. Nichols, M. O'Rourke, and C. Vlachopoulos (eds.), *McDonald's Blood Flow in Arteries: Theoretical, Experimental, and Clinical Principles*, 6th ed. (Boca Raton, FL, 2011).

The average rate of change of the velocity as we move from $r = r_1$ outward to $r = r_2$ is given by

$$\frac{\Delta v}{\Delta r} = \frac{v(r_2) - v(r_1)}{r_2 - r_1}$$

and if we let $\Delta r \to 0$, we obtain the **velocity gradient**, that is, the instantaneous rate of change of velocity with respect to r:

$$\text{velocity gradient} = \lim_{\Delta r \to 0} \frac{\Delta v}{\Delta r} = \frac{dv}{dr}$$

Using Equation 1, we obtain

$$\frac{dv}{dr} = \frac{P}{4\eta l}(0 - 2r) = -\frac{Pr}{2\eta l}$$

For one of the smaller human arteries we can take $\eta = 0.027$, $R = 0.008$ cm, $l = 2$ cm, and $P = 4000$ dynes/cm^2, which gives

$$v = \frac{4000}{4(0.027)2}(0.000064 - r^2)$$
$$\approx 1.85 \times 10^4(6.4 \times 10^{-5} - r^2)$$

At $r = 0.002$ cm the blood is flowing at a speed of

$$v(0.002) \approx 1.85 \times 10^4(64 \times 10^{-6} - 4 \times 10^{-6}) = 1.11 \text{ cm/s}$$

and the velocity gradient at that point is

$$\left.\frac{dv}{dr}\right|_{r=0.002} = -\frac{4000(0.002)}{2(0.027)2} \approx -74 \text{ (cm/s)/cm}$$

To get a feeling for what this statement means, let's change our units from centimeters to micrometers (1 cm = 10,000 μm). Then the radius of the artery is 80 μm. The velocity at the central axis is 11,850 μm/s, which decreases to 11,110 μm/s at a distance of $r = 20$ μm. The fact that $dv/dr = -74$ (μm/s)/μm means that, when $r = 20$ μm, the velocity is decreasing at a rate of about 74 μm/s for each micrometer that we proceed away from the center. ■

■ Economics

EXAMPLE 8 Suppose $C(x)$ is the total cost that a company incurs in producing x units of a certain commodity. The function C is called a **cost function**. If the number of items produced is increased from x_1 to x_2, then the additional cost is $\Delta C = C(x_2) - C(x_1)$, and the average rate of change of the cost is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}$$

The limit of this quantity as $\Delta x \to 0$, that is, the instantaneous rate of change of cost with respect to the number of items produced, is called the **marginal cost** by economists:

$$\text{marginal cost} = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

[Since x often takes on only integer values, it may not make literal sense to let Δx approach 0, but we can always replace $C(x)$ by a smooth approximating function as in Example 6.]

Taking $\Delta x = 1$ and n large (so that Δx is small compared to n), we have

$$C'(n) \approx C(n + 1) - C(n)$$

Thus the marginal cost of producing n units is approximately equal to the cost of producing one more unit [the $(n + 1)$st unit].

It is often appropriate to represent a total cost function by a polynomial

$$C(x) = a + bx + cx^2 + dx^3$$

where a represents the overhead cost (rent, heat, maintenance) and the other terms represent the cost of raw materials, labor, and so on. (The cost of raw materials may be proportional to x, but labor costs might depend partly on higher powers of x because of overtime costs and inefficiencies involved in large-scale operations.)

For instance, suppose a company has estimated that the cost (in dollars) of producing x items is

$$C(x) = 10{,}000 + 5x + 0.01x^2$$

Then the marginal cost function is

$$C'(x) = 5 + 0.02x$$

The marginal cost at the production level of 500 items is

$$C'(500) = 5 + 0.02(500) = \$15/\text{item}$$

This gives the rate at which costs are increasing with respect to the production level when $x = 500$ and predicts the cost of the 501st item.

The actual cost of producing the 501st item is

$$\begin{aligned} C(501) - C(500) &= [10{,}000 + 5(501) + 0.01(501)^2] \\ &\qquad - [10{,}000 + 5(500) + 0.01(500)^2] \\ &= \$15.01 \end{aligned}$$

Notice that $C'(500) \approx C(501) - C(500)$. ■

Economists also study marginal demand, marginal revenue, and marginal profit, the derivatives of the demand, revenue, and profit functions. These will be considered in Chapter 4 after we have developed techniques for finding the maximum and minimum values of functions.

Other Sciences

Rates of change occur in all the sciences. A geologist is interested in knowing the rate at which an intruded body of molten rock cools by conduction of heat into surrounding rocks. An engineer wants to know the rate at which water flows out of a reservoir. An urban geographer is interested in the rate of change of the population density in a city as the distance from the city center increases. A meteorologist is concerned with the rate of change of atmospheric pressure with respect to height (see Exercise 3.8.19).

In psychology, those interested in learning theory study the learning curve, which graphs the performance $P(t)$ of someone learning a skill as a function of the training time t. Of particular interest is the rate at which performance improves as time passes, that is, dP/dt. Psychologists have also studied the phenomenon of memory and have developed models for the rate of memory retention (see Exercise 42). They also study the difficulty involved in performing certain tasks and the rate at which difficulty increases when a given parameter is changed (see Exercise 43).

In sociology, differential calculus is used in analyzing the spread of rumors (or innovations or fads or fashions). If $p(t)$ denotes the proportion of a population that knows a rumor by time t, then the derivative dp/dt represents the rate of spread of the rumor (see Exercise 3.4.90).

A Single Idea, Many Interpretations

Velocity, density, current, power, and temperature gradient in physics; rate of reaction and compressibility in chemistry; rate of growth and blood velocity gradient in biology; marginal cost and marginal profit in economics; rate of heat flow in geology; rate of improvement of performance in psychology; rate of spread of a rumor in sociology—these are all special cases of a single mathematical concept, the derivative.

All of these different applications of the derivative illustrate the fact that part of the power of mathematics lies in its abstractness. A single abstract mathematical concept (such as the derivative) can have different interpretations in each of the sciences. When we develop the properties of the mathematical concept once and for all, we can then turn around and apply these results to all of the sciences. This is much more efficient than developing properties of special concepts in each separate science. The French mathematician Joseph Fourier (1768–1830) put it succinctly: “Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them.”

3.7 Exercises

1–4 A particle moves according to a law of motion $s = f(t)$, $t \geqslant 0$, where t is measured in seconds and s in meters.

(a) Find the velocity at time t.
(b) What is the velocity after 1 second?
(c) When is the particle at rest?
(d) When is the particle moving in the positive direction?
(e) Find the total distance traveled during the first 6 seconds.
(f) Draw a diagram like Figure 2 to illustrate the motion of the particle.
(g) Find the acceleration at time t and after 1 second.
(h) Graph the position, velocity, and acceleration functions for $0 \leqslant t \leqslant 6$.
(i) When is the particle speeding up? When is it slowing down?

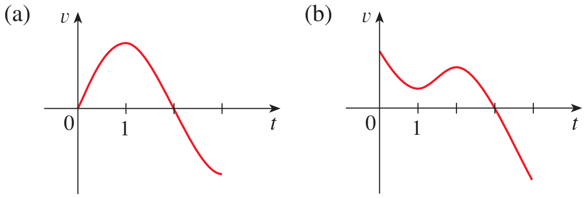
1. $f(t) = t^3 - 8t^2 + 24t$

2. $f(t) = \dfrac{9t}{t^2 + 9}$

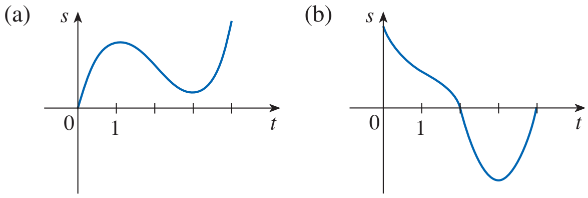
3. $f(t) = \sin(\pi t/2)$

4. $f(t) = t^2 e^{-t}$

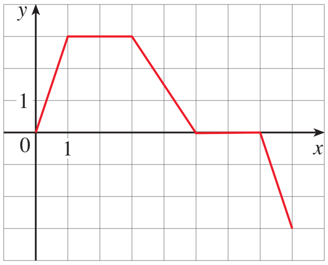
5. Graphs of the *velocity* functions of two particles are shown, where t is measured in seconds. When is each particle speeding up? When is it slowing down? Explain.



6. Graphs of the *position* functions of two particles are shown, where t is measured in seconds. When is the velocity of each particle positive? When is it negative? When is each particle speeding up? When is it slowing down? Explain.



7. Suppose that the graph of the velocity function of a particle is as shown in the figure, where t is measured in seconds. When is the particle traveling forward (in the positive direction)? When is it traveling backward? What is happening when $5 < t < 7$?



8. For the particle described in Exercise 7, sketch a graph of the acceleration function. When is the particle speeding up? When is it slowing down? When is it traveling at a constant speed?

9. The height (in meters) of a projectile shot vertically upward from a point 2 m above ground level with an initial velocity of 24.5 m/s is $h = 2 + 24.5t - 4.9t^2$ after t seconds.
(a) Find the velocity after 2 s and after 4 s.
(b) When does the projectile reach its maximum height?
(c) What is the maximum height?
(d) When does it hit the ground?
(e) With what velocity does it hit the ground?

10. If a ball is thrown vertically upward with a velocity of 24.5 m/s, then its height after t seconds is $s = 24.5t - 4.9t^2$.
(a) What is the maximum height reached by the ball?
(b) What is the velocity of the ball when it is 29.4 above the ground on its way up? On its way down?

11. If a rock is thrown vertically upward from the surface of Mars with velocity 15 m/s, its height after t seconds is $h = 15t - 1.86t^2$.
(a) What is the velocity of the rock after 2 s?
(b) What is the velocity of the rock when its height is 25 m on its way up? On its way down?

12. A particle moves with position function

$$s = t^4 - 4t^3 - 20t^2 + 20t \qquad t \geqslant 0$$

(a) At what time does the particle have a velocity of 20 m/s?
(b) At what time is the acceleration 0? What is the significance of this value of t?

13. (a) A company makes computer chips from square wafers of silicon. A process engineer wants to keep the side length of a wafer very close to 15 mm and needs to know how the area $A(x)$ of a wafer changes when the side length x changes. Find $A'(15)$ and explain its meaning in this situation.
(b) Show that the rate of change of the area of a square with respect to its side length is half its perimeter. Try to explain geometrically why this is true by drawing a square whose side length x is increased by an amount Δx. How can you approximate the resulting change in area ΔA if Δx is small?

14. (a) Sodium chlorate crystals are easy to grow in the shape of cubes by allowing a solution of water and sodium chlorate to evaporate slowly. If V is the volume of such a cube with side length x, calculate dV/dx when $x = 3$ mm and explain its meaning.

(b) Show that the rate of change of the volume of a cube with respect to its edge length is equal to half the surface area of the cube. Explain geometrically why this result is true by arguing by analogy with Exercise 13(b).

15. (a) Find the average rate of change of the area of a circle with respect to its radius r as r changes from

(i) 2 to 3 (ii) 2 to 2.5 (iii) 2 to 2.1

(b) Find the instantaneous rate of change when $r = 2$.

(c) Show that the rate of change of the area of a circle with respect to its radius (at any r) is equal to the circumference of the circle. Try to explain geometrically why this is true by drawing a circle whose radius is increased by an amount Δr. How can you approximate the resulting change in area ΔA if Δr is small?

16. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s. Find the rate at which the area within the circle is increasing after (a) 1 s, (b) 3 s, and (c) 5 s. What can you conclude?

17. A spherical balloon is being inflated. Find the rate of increase of the surface area ($S = 4\pi r^2$) with respect to the radius r when r is (a) 20 cm, (b) 40 cm, and (c) 60 cm. What conclusion can you make?

18. (a) The volume of a growing spherical cell is $V = \frac{4}{3}\pi r^3$, where the radius r is measured in micrometers ($1\ \mu\text{m} = 10^{-6}$ m). Find the average rate of change of V with respect to r when r changes from

(i) 5 to 8 μm (ii) 5 to 6 μm (iii) 5 to 5.1 μm

(b) Find the instantaneous rate of change of V with respect to r when $r = 5\ \mu\text{m}$.

(c) Show that the rate of change of the volume of a sphere with respect to its radius is equal to its surface area. Explain geometrically why this result is true. Argue by analogy with Exercise 15(c).

19. The mass of the part of a metal rod that lies between its left end and a point x meters to the right is $3x^2$ kg. Find the linear density (see Example 2) when x is (a) 1 m, (b) 2 m, and (c) 3 m. Where is the density the highest? The lowest?

20. If a cylindrical water tank holds 5000 liters, and the water drains from the bottom of the tank in 40 minutes, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V = 5000\left(1 - \tfrac{1}{40}t\right)^2 \qquad 0 \leqslant t \leqslant 40$$

Find the rate at which water is draining from the tank after (a) 5 min, (b) 10 min, (c) 20 min, and (d) 40 min. At what time is the water flowing out the fastest? The slowest? Summarize your findings.

21. The quantity of charge Q in coulombs (C) that has passed through a point in a wire up to time t (measured in seconds) is given by $Q(t) = t^3 - 2t^2 + 6t + 2$. Find the current when (a) $t = 0.5$ s and (b) $t = 1$ s. (See Example 3. The unit of current is an ampere [1 A = 1 C/s].) At what time is the current lowest?

22. Newton's Law of Gravitation says that the magnitude F of the force exerted by a body of mass m on a body of mass M is

$$F = \frac{GmM}{r^2}$$

where G is the gravitational constant and r is the distance between the bodies.

(a) Find dF/dr and explain its meaning. What does the minus sign indicate?

(b) Suppose it is known that the earth attracts an object with a force that decreases at the rate of 2 N/km when $r = 20{,}000$ km. How fast does this force change when $r = 10{,}000$ km?

23. The force F acting on a body with mass m and velocity v is the rate of change of momentum: $F = (d/dt)(mv)$. If m is constant, this becomes $F = ma$, where $a = dv/dt$ is the acceleration. But in the theory of relativity, the mass of a particle varies with v as follows: $m = m_0/\sqrt{1 - v^2/c^2}$ where m_0 is the mass of the particle at rest and c is the speed of light. Show that

$$F = \frac{m_0 a}{(1 - v^2/c^2)^{3/2}}$$

24. Some of the highest tides in the world occur in the Bay of Fundy on the Atlantic Coast of Canada. At Hopewell Cape the water depth at low tide is about 2.0 m and at high tide it is about 12.0 m. The natural period of oscillation is a little more than 12 hours and on a day in June, high tide occurred at 6:45 AM. This helps explain the following model for the water depth D (in meters) as a function of the time t (in hours after midnight) on that day:

$$D(t) = 7 + 5\cos[0.503(t - 6.75)]$$

How fast was the tide rising (or falling) at the following times?

(a) 3:00 AM (b) 6:00 AM

(c) 9:00 AM (d) Noon

25. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the product of the pressure and the volume remains constant: $PV = C$.

(a) Find the rate of change of volume with respect to pressure.

(b) A sample of gas is in a container at low pressure and is steadily compressed at constant temperature for 10 minutes. Is the volume decreasing more rapidly at the beginning or the end of the 10 minutes? Explain.

(c) Prove that the isothermal compressibility (see Example 5) is given by $\beta = 1/P$.

26. If, in Example 4, one molecule of the product C is formed from one molecule of the reactant A and one molecule of the reactant B, and the initial concentrations of A and B have a common value $[A] = [B] = a$ moles/L, then

$$[C] = a^2kt/(akt + 1)$$

where k is a constant.

(a) Find the rate of reaction at time t.

(b) Show that if $x = [C]$, then

$$\frac{dx}{dt} = k(a - x)^2$$

(c) What happens to the concentration as $t \to \infty$?

(d) What happens to the rate of reaction as $t \to \infty$?

(e) What do the results of parts (c) and (d) mean in practical terms?

27. In Example 6 we considered a bacteria population that doubles every hour. Suppose that another population of bacteria triples every hour and starts with 400 bacteria. Find an expression for the number n of bacteria after t hours and use it to estimate the rate of growth of the bacteria population after 2.5 hours.

28. The number of yeast cells in a laboratory culture increases rapidly initially but levels off eventually. The population is modeled by the function

$$n = f(t) = \frac{a}{1 + be^{-0.7t}}$$

where t is measured in hours. At time $t = 0$ the population is 20 cells and is increasing at a rate of 12 cells/hour. Find the values of a and b. According to this model, what happens to the yeast population in the long run?

T **29.** The table gives the world population $P(t)$, in millions, where t is measured in years and $t = 0$ corresponds to the year 1900.

t	Population (millions)	t	Population (millions)
0	1650	60	3040
10	1750	70	3710
20	1860	80	4450
30	2070	90	5280
40	2300	100	6080
50	2560	110	6870

(a) Estimate the rate of population growth in 1920 and in 1980 by averaging the slopes of two secant lines.

(b) Use a graphing calculator or computer to find a cubic function (a third-degree polynomial) that models the data.

(c) Use your model in part (b) to find a model for the rate of population growth.

(d) Use part (c) to estimate the rates of growth in 1920 and 1980. Compare with your estimates in part (a).

(e) In Section 1.1 we modeled $P(t)$ with the exponential function

$$f(t) = (1.43653 \times 10^9) \cdot (1.01395)^t$$

Use this model to find a model for the rate of population growth.

(f) Use your model in part (e) to estimate the rate of growth in 1920 and 1980. Compare with your estimates in parts (a) and (d).

(g) Estimate the rate of growth in 1985.

T **30.** The table shows how the average age of first marriage of Japanese women has varied since 1950.

t	$A(t)$	t	$A(t)$
1950	23.0	1985	25.5
1955	23.8	1990	25.9
1960	24.4	1995	26.3
1965	24.5	2000	27.0
1970	24.2	2005	28.0
1975	24.7	2010	28.8
1980	25.2	2015	29.4

(a) Use a graphing calculator or computer to model these data with a fourth-degree polynomial.

(b) Use part (a) to find a model for $A'(t)$.

(c) Estimate the rate of change of marriage age for women in 1990.

(d) Graph the data points and the models for A and A'.

31. Refer to the law of laminar flow given in Example 7. Consider a blood vessel with radius 0.01 cm, length 3 cm, pressure difference 3000 dynes/cm^2, and viscosity $\eta = 0.027$.

(a) Find the velocity of the blood along the centerline $r = 0$, at radius $r = 0.005$ cm, and at the wall $r = R = 0.01$ cm.

(b) Find the velocity gradient at $r = 0$, $r = 0.005$, and $r = 0.01$.

(c) Where is the velocity the greatest? Where is the velocity changing most?

32. The frequency of vibrations of a vibrating violin string is given by

$$f = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$$

where L is the length of the string, T is its tension, and ρ is its linear density. [See Chapter 11 in D. E. Hall, *Musical Acoustics*, 3rd ed. (Pacific Grove, CA, 2002).]

(a) Find the rate of change of the frequency with respect to
  (i) the length (when T and ρ are constant),
  (ii) the tension (when L and ρ are constant), and
  (iii) the linear density (when L and T are constant).

(b) The pitch of a note (how high or low the note sounds) is determined by the frequency f. (The higher the frequency, the higher the pitch.) Use the signs of the derivatives in part (a) to determine what happens to the pitch of a note

(i) when the effective length of a string is decreased by placing a finger on the string so a shorter portion of the string vibrates,

(ii) when the tension is increased by turning a tuning peg,

(iii) when the linear density is increased by switching to another string.

33. Suppose that the cost (in dollars) for a company to produce x pairs of a new line of jeans is

$$C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3$$

(a) Find the marginal cost function.

(b) Find $C'(100)$ and explain its meaning. What does it predict?

(c) Compare $C'(100)$ with the cost of manufacturing the 101st pair of jeans.

34. The cost function for a certain commodity is

$$C(q) = 84 + 0.16q - 0.0006q^2 + 0.000003q^3$$

(a) Find and interpret $C'(100)$.

(b) Compare $C'(100)$ with the cost of producing the 101st item.

35. If $p(x)$ is the total value of the production when there are x workers in a plant, then the *average productivity* of the workforce at the plant is

$$A(x) = \frac{p(x)}{x}$$

(a) Find $A'(x)$. Why does the company want to hire more workers if $A'(x) > 0$?

(b) Show that $A'(x) > 0$ if $p'(x)$ is greater than the average productivity.

36. If R denotes the reaction of the body to some stimulus of strength x, the *sensitivity* S is defined to be the rate of change of the reaction with respect to x. A particular example is that when the brightness x of a light source is increased, the eye reacts by decreasing the area R of the pupil. The experimental formula

$$R = \frac{40 + 24x^{0.4}}{1 + 4x^{0.4}}$$

has been used to model the dependence of R on x when R is measured in square millimeters and x is measured in appropriate units of brightness.

(a) Find the sensitivity.

(b) Illustrate part (a) by graphing both R and S as functions of x. Comment on the values of R and S at low levels of brightness. Is this what you would expect?

37. Patients undergo dialysis treatment to remove urea from their blood when their kidneys are not functioning properly. Blood is diverted from the patient through a machine that filters out urea. Under certain conditions, the duration of dialysis required, given that the initial urea concentration is $c > 1$, is given by the equation

$$t = \ln\left(\frac{3c + \sqrt{9c^2 - 8c}}{2}\right)$$

Calculate the derivative of t with respect to c and interpret it.

38. Invasive species often display a wave of advance as they colonize new areas. Mathematical models based on random dispersal and reproduction have demonstrated that the speed with which such waves move is given by the function $f(r) = 2\sqrt{Dr}$, where r is the reproductive rate of individuals and D is a parameter quantifying dispersal. Calculate the derivative of the wave speed with respect to the reproductive rate r and explain its meaning.

39. The gas law for an ideal gas at absolute temperature T (in kelvins), pressure P (in atmospheres), and volume V (in liters) is $PV = nRT$, where n is the number of moles of the gas and $R = 0.0821$ is the gas constant. Suppose that, at a certain instant, $P = 8.0$ atm and is increasing at a rate of 0.10 atm/min and $V = 10$ L and is decreasing at a rate of 0.15 L/min. Find the rate of change of T with respect to time at that instant if $n = 10$ mol.

40. In a fish farm, a population of fish is introduced into a pond and harvested regularly. A model for the rate of change of the fish population is given by the equation

$$\frac{dP}{dt} = r_0\left(1 - \frac{P(t)}{P_c}\right)P(t) - \beta P(t)$$

where r_0 is the birth rate of the fish, P_c is the maximum population that the pond can sustain (called the *carrying capacity*), and β is the percentage of the population that is harvested.

(a) What value of dP/dt corresponds to a stable population?

(b) If the pond can sustain 10,000 fish, the birth rate is 5%, and the harvesting rate is 4%, find the stable population level.

(c) What happens if β is raised to 5%?

41. In the study of ecosystems, *predator-prey models* are often used to study the interaction between species. Consider populations of tundra wolves, given by $W(t)$, and caribou, given by $C(t)$, in northern Canada. The interaction has been modeled by the equations

$$\frac{dC}{dt} = aC - bCW \qquad \frac{dW}{dt} = -cW + dCW$$

(a) What values of dC/dt and dW/dt correspond to stable populations?

(b) How would the statement "The caribou go extinct" be represented mathematically?

(c) Suppose that $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$. Find all population pairs (C, W) that lead to stable populations. According to this model, is it possible for the two species to live in balance or will one or both species become extinct?

42. Hermann Ebbinghaus (1850–1909) pioneered the study of memory. A 2011 article in the *Journal of Mathematical Psychology* presents the mathematical model

$$R(t) = a + b(1 + ct)^{-\beta}$$

for the *Ebbinghaus forgetting curve*, where $R(t)$ is the fraction of memory retained t days after learning a task; a, b, and c are experimentally determined constants between 0 and 1; β is a positive constant; and $R(0) = 1$. The constants depend on the type of task being learned.

(a) What is the rate of change of retention t days after a task is learned?

(b) Do you forget how to perform a task faster soon after learning it or a long time after you have learned it?

(c) What fraction of memory is permanent?

43. The difficulty of "acquiring a target" (such as using a mouse to click on an icon on a computer screen) depends on the ratio between the distance D to the target and width W of the target. According to *Fitts' law*, the index I of difficulty is modeled by

$$I = \log_2\left(\frac{2D}{W}\right)$$

This law is used for designing products that involve human–computer interactions.

(a) If W is held constant, what is the rate of change of I with respect to D? Does this rate increase or decrease with increasing values of D?

(b) If D is held constant, what is the rate of change of I with respect to W? What does the negative sign in your answer indicate? Does this rate increase or decrease with increasing values of W?

(c) Do your answers to parts (a) and (b) agree with your intuition?

3.8 Exponential Growth and Decay

In many natural phenomena, quantities grow or decay at a rate proportional to their size. For instance, if $y = f(t)$ is the number of individuals in a population of animals or bacteria at time t, then it seems reasonable to expect that the rate of growth $f'(t)$ is proportional to the population $f(t)$; that is, $f'(t) = kf(t)$ for some constant k. Indeed, under ideal conditions (unlimited environment, adequate nutrition, immunity to disease) the mathematical model given by the equation $f'(t) = kf(t)$ predicts what actually happens fairly accurately. Another example occurs in nuclear physics: the mass of a radioactive substance decays at a rate proportional to the mass. In chemistry, the rate of a unimolecular first-order reaction is proportional to the concentration of the substance. In finance, the value of a savings account with continuously compounded interest increases at a rate proportional to that value.

In general, if $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at any time, then

$$\boxed{1} \qquad \boxed{\frac{dy}{dt} = ky}$$

where k is a constant. Equation 1 is sometimes called the **law of natural growth** (if $k > 0$) or the **law of natural decay** (if $k < 0$). It is called a *differential equation* because it involves an unknown function y and its derivative dy/dt.

It's not hard to think of a solution of Equation 1. This equation asks us to find a function whose derivative is a constant multiple of itself. We have met such functions in this chapter. Any exponential function of the form $y(t) = Ce^{kt}$, where C is a constant, satisfies

$$\frac{dy}{dt} = C(ke^{kt}) = k(Ce^{kt}) = ky$$

We will see in Section 9.4 that *any* function that satisfies $dy/dt = ky$ must be of the form $y = Ce^{kt}$. To see the significance of the constant C, we observe that

$$y(0) = Ce^{k \cdot 0} = C$$

Therefore C is the **initial value** of the function.

2 Theorem The only solutions of the differential equation $dy/dt = ky$ are the exponential functions

$$y(t) = y(0)e^{kt}$$

Population Growth

What is the significance of the proportionality constant k? In the context of population growth, where $P(t)$ is the size of a population at time t, we can write

$$\boxed{3} \qquad \frac{dP}{dt} = kP \qquad \text{or} \qquad \frac{dP/dt}{P} = k$$

The quantity

$$\frac{dP/dt}{P}$$

is the growth rate divided by the population size; it is called the **relative growth rate**. According to (3), instead of saying "the growth rate is proportional to population size" we could say "the relative growth rate is constant." Then Theorem 2 says that a population with constant relative growth rate must grow exponentially. Notice that the relative growth rate k appears as the coefficient of t in the exponential function Ce^{kt}. For instance, if

$$\frac{dP}{dt} = 0.02P$$

and t is measured in years, then the relative growth rate is $k = 0.02$ and the population grows at a relative rate of 2% per year. If the population at time 0 is P_0, then the expression for the population is

$$P(t) = P_0e^{0.02t}$$

EXAMPLE 1 Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the growth rate is proportional to the population size.) What is the relative growth rate? Use the model to estimate the world population in 1993 and to predict the population in the year 2025.

SOLUTION We measure the time t in years and let $t = 0$ in the year 1950. We measure the population $P(t)$ in millions of people. Then $P(0) = 2560$ and $P(10) = 3040$. Since we are assuming that $dP/dt = kP$, Theorem 2 gives

$$P(t) = P(0)e^{kt} = 2560e^{kt}$$

$$P(10) = 2560e^{10k} = 3040$$

$$k = \frac{1}{10}\ln\frac{3040}{2560} \approx 0.017185$$

The relative growth rate is about 1.7% per year and the model is

$$P(t) = 2560e^{0.017185t}$$

We estimate that the world population in 1993 was

$$P(43) = 2560e^{0.017185(43)} \approx 5360 \text{ million}$$

The model predicts that the population in 2025 will be

$$P(75) = 2560e^{0.017185(75)} \approx 9289 \text{ million}$$

The graph in Figure 1 shows that the model is fairly accurate to the end of the 20th century (the dots represent the actual population), so the estimate for 1993 is quite reliable. But the prediction for 2025 may not be so accurate.

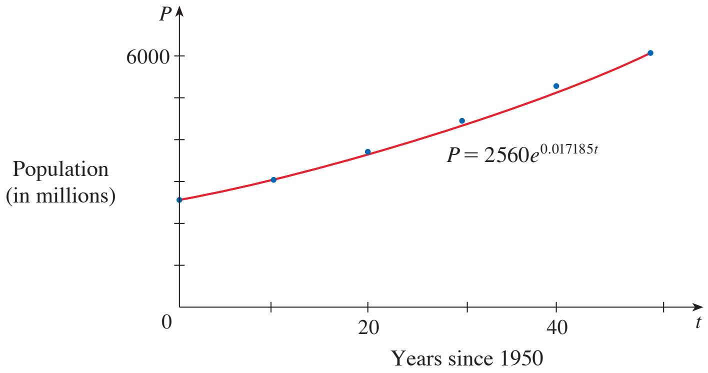


FIGURE 1
A model for world population growth in the second half of the 20th century

■

Radioactive Decay

Radioactive substances decay by spontaneously emitting radiation. If $m(t)$ is the mass remaining from an initial mass m_0 of the substance after time t, then the relative decay rate

$$-\frac{dm/dt}{m}$$

has been found experimentally to be constant. (Since dm/dt is negative, the relative decay rate is positive.) It follows that

$$\frac{dm}{dt} = km$$

where k is a negative constant. In other words, radioactive substances decay at a rate proportional to the remaining mass. This means that we can use Theorem 2 to show that the mass decays exponentially:

$$m(t) = m_0 e^{kt}$$

Physicists express the rate of decay in terms of **half-life**, the time required for half of any given quantity to decay.

EXAMPLE 2 The half-life of radium-226 is 1590 years.
(a) A sample of radium-226 has mass 100 mg. Find a formula for the mass of the sample that remains after t years.
(b) Find the mass remaining after 1000 years correct to the nearest milligram.
(c) When will the mass be reduced to 30 mg?

SOLUTION
(a) Let $m(t)$ be the mass of radium-226 (in milligrams) that remains after t years. Then $dm/dt = km$ and $m(0) = 100$, so Theorem 2 gives

$$m(t) = m(0)e^{kt} = 100e^{kt}$$

In order to determine the value of k, we use the fact that $m(1590) = \frac{1}{2}(100)$. Thus

$$100e^{1590k} = 50 \qquad \text{so} \qquad e^{1590k} = \tfrac{1}{2}$$

and

$$1590k = \ln \tfrac{1}{2} = -\ln 2$$

$$k = -\frac{\ln 2}{1590}$$

Therefore

$$m(t) = 100e^{-(\ln 2)t/1590}$$

We could use the fact that $e^{\ln 2} = 2$ to write the expression for $m(t)$ in the alternative form

$$m(t) = 100 \times 2^{-t/1590}$$

(b) The mass remaining after 1000 years is

$$m(1000) = 100e^{-(\ln 2)1000/1590} \approx 65 \text{ mg}$$

(c) We want to find the value of t such that $m(t) = 30$, that is,

$$100e^{-(\ln 2)t/1590} = 30 \qquad \text{or} \qquad e^{-(\ln 2)t/1590} = 0.3$$

We solve this equation for t by taking the natural logarithm of both sides:

$$-\frac{\ln 2}{1590}t = \ln 0.3$$

Thus

$$t = -1590\frac{\ln 0.3}{\ln 2} \approx 2762 \text{ years} \qquad \blacksquare$$

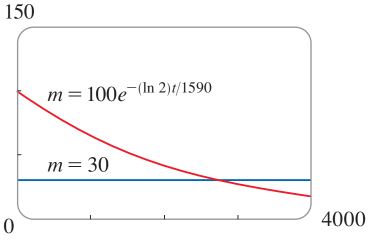


FIGURE 2

As a check on our work in Example 2, we use a calculator or computer to draw the graph of $m(t)$ in Figure 2 together with the horizontal line $m = 30$. These curves intersect when $t \approx 2800$, and this agrees with the answer to part (c).

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. (This law also applies to warming.) If we let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings, then we can formulate Newton's Law of Cooling as a differential equation:

$$\frac{dT}{dt} = k(T - T_s)$$

where k is a constant. This equation is not quite the same as Equation 1, so we make the change of variable $y(t) = T(t) - T_s$. Because T_s is constant, we have $y'(t) = T'(t)$ and so the equation becomes

$$\frac{dy}{dt} = ky$$

We can then use Theorem 2 to find an expression for y, from which we can find T.

EXAMPLE 3 A bottle of iced tea at room temperature (24°C) is placed in a refrigerator where the temperature is 7°C. After half an hour the tea has cooled to 16°C.
(a) What is the temperature of the tea after another half hour?
(b) How long does it take for the tea to cool to 10°C?

SOLUTION

(a) Let $T(t)$ be the temperature of the tea after t minutes. The surrounding temperature is $T_s = 7°\text{C}$, so Newton's Law of Cooling states that

$$\frac{dT}{dt} = k(T - 7)$$

If we let $y = T - 7$, then $y(0) = T(0) - 7 = 24 - 7 = 17$, so y satisfies

$$\frac{dy}{dt} = ky \qquad y(0) = 17$$

and by (2) we have

$$y(t) = y(0)e^{kt} = 17e^{kt}$$

We are given that $T(30) = 16$, so $y(30) = 16 - 7 = 9$ and

$$17e^{30k} = 9 \qquad e^{30k} = \tfrac{9}{17}$$

Taking logarithms, we have

$$k = \frac{\ln\left(\frac{9}{17}\right)}{30} \approx -0.02120$$

Thus

$$y(t) = 17e^{-0.02120t}$$
$$T(t) = 7 + 17e^{-0.02120t}$$
$$T(60) = 7 + 17e^{-0.02120(60)} \approx 11.8$$

So after another half hour the tea has cooled to about 11.8°C.

(b) We have $T(t) = 10$ when

$$7 + 17e^{-0.02120t} = 10$$
$$e^{-0.02120t} = \tfrac{3}{17}$$
$$t = \frac{\ln\left(\frac{3}{17}\right)}{-0.02120} \approx 81.8$$

The tea cools to 10°C after about 1 hour 22 minutes. ■

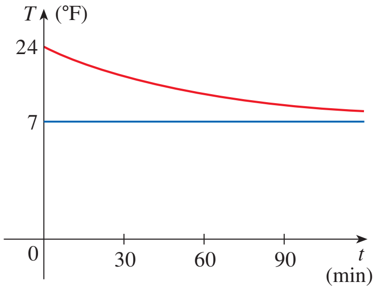


FIGURE 3

Notice that in Example 3 we have

$$\lim_{t \to \infty} T(t) = \lim_{t \to \infty} (7 + 17e^{-0.02120t}) = 7 + 17 \cdot 0 = 7$$

This means that, as expected, the temperature of the tea approaches the ambient temperature inside the refrigerator. The graph of the temperature function is shown in Figure 3.

Continuously Compounded Interest

EXAMPLE 4 If \$5000 is invested at 2% interest, compounded annually, then after 1 year the investment is worth $\$5000(1.02) = \$5100.00$, after 2 years it's worth $[\$5000(1.02)](1.02) = \$5202.00$, and after t years it's worth $\$5000(1.02)^t$. In general, if an amount A_0 is invested at an interest rate r ($r = 0.02$ in this example), then after t years it's worth $A_0(1 + r)^t$. Usually, interest is compounded more frequently, say, n times per year. Then in each compounding period the rate is r/n, and there are nt compounding periods in t years, so the value of the investment is

$$A = A_0\left(1 + \frac{r}{n}\right)^{nt}$$

For instance, after 3 years at 2% interest a $5000 investment will be worth

$$\$5000(1.02)^3 = \$5306.04 \text{ with annual compounding}$$

$$\$5000(1.01)^6 = \$5307.60 \text{ with semiannual compounding}$$

$$\$5000(1.005)^{12} = \$5308.39 \text{ with quarterly compounding}$$

$$\$5000\left(1 + \frac{0.02}{12}\right)^{36} = \$5308.92 \text{ with monthly compounding}$$

$$\$5000\left(1 + \frac{0.02}{365}\right)^{365\cdot 3} = \$5309.17 \text{ with daily compounding}$$

You can see that the interest paid increases as the number of compounding periods (n) increases. If we let $n \to \infty$, then we will be compounding the interest **continuously** and the value of the investment will be

$$\begin{aligned} A(t) &= \lim_{n\to\infty} A_0\left(1 + \frac{r}{n}\right)^{nt} \\ &= \lim_{n\to\infty} A_0\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} \\ &= A_0\left[\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} \\ &= A_0\left[\lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^{m}\right]^{rt} \qquad \text{(where } m = n/r\text{)} \end{aligned}$$

Equation 3.6.6:

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

But the limit in this expression is equal to the number e (see Equation 3.6.6). So with continuous compounding of interest at interest rate r, the amount after t years is

$$A(t) = A_0 e^{rt}$$

If we differentiate this equation, we get

$$\frac{dA}{dt} = rA_0 e^{rt} = rA(t)$$

which says that, with continuous compounding of interest, the rate of increase of an investment is proportional to its size.

Returning to the example of $5000 invested for 3 years at 2% interest, we see that with continuously compounding of interest the value of the investment will be

$$A(3) = \$5000\text{e}^{(0.02)3} = \$5309.18$$

Notice how close this is to the amount we calculated for daily compounding, $5309.17. But the amount is easier to compute if we use continuous compounding. ■

3.8 Exercises

1. A population of the yeast cell *Saccharomyces cerevisiae* (a yeast used for fermentation) develops with a constant relative growth rate of 0.4159 per hour. The initial population consists of 3.8 million cells. Find the population size after 2 hours.

2. A common inhabitant of human intestines is the bacterium *Escherichia coli*, named after the German pediatrician Theodor Escherich, who identified it in 1885. A cell of this bacterium in a nutrient-broth medium divides into two cells every 20 minutes. The initial population of a culture is 50 cells.

(a) Find the relative growth rate.
(b) Find an expression for the number of cells after t hours.
(c) Find the number of cells after 6 hours.
(d) Find the rate of growth after 6 hours.
(e) When will the population reach a million cells?

3. A culture of the bacterium *Salmonella enteritidis* initially contains 50 cells. When introduced into a nutrient broth, the culture grows at a rate proportional to its size. After 1.5 hours the population has increased to 975.

(a) Find an expression for the number of bacteria after t hours.
(b) Find the number of bacteria after 3 hours.
(c) Find the rate of growth after 3 hours.
(d) After how many hours will the population reach 250,000?

4. A bacteria culture grows with constant relative growth rate. The bacteria count was 400 after 2 hours and 25,600 after 6 hours.

(a) What is the relative growth rate? Express your answer as a percentage.
(b) What was the initial size of the culture?
(c) Find an expression for the number of bacteria after t hours.
(d) Find the number of bacteria after 4.5 hours.
(e) Find the rate of growth after 4.5 hours.
(f) When will the population reach 50,000?

5. The table gives estimates of the world population, in millions, from 1750 to 2000.

Year	Population (millions)	Year	Population (millions)
1750	790	1900	1650
1800	980	1950	2560
1850	1260	2000	6080

(a) Use the exponential model and the population figures for 1750 and 1800 to predict the world population in 1900 and 1950. Compare with the actual figures.
(b) Use the exponential model and the population figures for 1850 and 1900 to predict the world population in 1950. Compare with the actual population.
(c) Use the exponential model and the population figures for 1900 and 1950 to predict the world population in 2000. Compare with the actual population and try to explain the discrepancy.

6. The table gives census data for the population of Indonesia, in millions, during the second half of the 20th century.

Year	Population (millions)
1950	83
1960	100
1970	122
1980	150
1990	182
2000	214

(a) Assuming the population grows at a rate proportional to its size, use the census data for 1950 and 1960 to predict the population in 1980. Compare with the actual figure.
(b) Use the census data for 1960 and 1980 to predict the population in 2000. Compare with the actual population.
(c) Use the census data for 1980 and 2000 to predict the population in 2010 and compare with the actual population of 243 million.
(d) Use the model in part (c) to predict the population in 2025. Do you think the prediction will be too high or too low? Why?

7. Experiments show that if the chemical reaction

$$N_2O_5 \longrightarrow 2NO_2 + \tfrac{1}{2}O_2$$

takes place at 45°C, the rate of reaction of dinitrogen pentoxide is proportional to its concentration as follows:

$$-\frac{d[N_2O_5]}{dt} = 0.0005[N_2O_5]$$

(a) Find an expression for the concentration $[N_2O_5]$ after t seconds if the initial concentration is C.
(b) How long will the reaction take to reduce the concentration of N_2O_5 to 90% of its original value?

8. Strontium-90 has a half-life of 28 days.
  (a) A sample has initial mass 50 mg. Find a formula for the mass remaining after t days.
  (b) Find the mass remaining after 40 days.
  (c) How long does it take the sample to decay to a mass of 2 mg?
  (d) Sketch the graph of the mass function.

9. The half-life of cesium-137 is 30 years. Suppose we have a 100-mg sample.
  (a) Find the mass that remains after t years.
  (b) How much of the sample remains after 100 years?
  (c) After how long will only 1 mg remain?

10. A sample of einsteinium-252 decayed to 64.3% of its original mass after 300 days.
  (a) What is the half-life of einsteinium-252?
  (b) How long would it take the sample to decay to one-third of its original mass?

11–13 Radiocarbon Dating Scientists can determine the age of ancient objects by the method of *radiocarbon dating*. The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, ^{14}C, with a half-life of about 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates ^{14}C through food chains. When a plant or animal dies, it stops replacing its carbon and the amount of ^{14}C present begins to decrease through radioactive decay. Therefore the level of radioactivity must also decay exponentially.

11. A discovery revealed a parchment fragment that had about 74% as much ^{14}C radioactivity as does plant material on the earth today. Estimate the age of the parchment.

12. Dinosaur fossils are too old to be reliably dated using carbon-14. Suppose we had a 68-million-year-old dinosaur fossil. What fraction of the living dinosaur's ^{14}C would be remaining today? Suppose the minimum detectable mass is 0.1%. What is the maximum age of a fossil that could be dated using ^{14}C?

13. Dinosaur fossils are often dated by using an element other than carbon, such as potassium-40, that has a longer half-life (in this case, approximately 1.25 billion years). Suppose the minimum detectable mass is 0.1% and a dinosaur is dated with ^{40}K to be 68 million years old. Is such a dating possible? In other words, what is the maximum age of a fossil that could be dated using ^{40}K?

14. A curve passes through the point (0, 5) and has the property that the slope of the curve at every point P is twice the y-coordinate of P. What is the equation of the curve?

15. A roast turkey is removed from an oven when its temperature has reached 85°C and is placed on a table in a room where the ambient temperature is 22°C.
  (a) If the temperature of the turkey is 65°C after half an hour, what is the temperature after 45 minutes?
  (b) When will the turkey have cooled to 40°C?

16. In a murder investigation, the temperature of the corpse was 32.5°C at 1:30 PM and 30.3°C an hour later. Normal body temperature is 37.0°C and the ambient temperature was 20.0°C. When did the murder take place?

17. When a cold drink is taken from a refrigerator, its temperature is 5°C. After 25 minutes in a 20°C room its temperature has increased to 10°C.
  (a) What is the temperature of the drink after 50 minutes?
  (b) When will its temperature reach 15°C?

18. A freshly brewed cup of coffee has temperature 95°C in a 20°C room. When its temperature is 70°C, it is cooling at a rate of 1°C per minute. When does this occur?

19. The rate of change of atmospheric pressure P with respect to altitude h is proportional to P, provided that the temperature is constant. At 15°C the pressure is 101.3 kPa at sea level and 87.14 kPa at $h = 1000$ m.
  (a) What is the pressure at an altitude of 3000 m?
  (b) What is the pressure at the top of Mount McKinley, at an altitude of 6187 m?

20. (a) If \$2500 is borrowed at 4.5% interest, find the amounts due at the end of 3 years if the interest is compounded (i) annually, (ii) quarterly, (iii) monthly, (iv) weekly, (v) daily, (vi) hourly, and (vii) continuously.
  (b) Suppose \$2500 is borrowed and the interest is compounded continuously. If $A(t)$ is the amount due after t years, where $0 \leqslant t \leqslant 3$, graph $A(t)$ for each of the interest rates 5%, 6%, and 7% on a common screen.

21. (a) If \$4000 is invested at 1.75% interest, find the value of the investment at the end of 5 years if the interest is compounded (i) annually, (ii) semiannually, (iii) monthly, (iv) weekly, (v) daily, and (vi) continuously.
  (b) If $A(t)$ is the amount of the investment at time t for the case of continuous compounding, write a differential equation and an initial condition satisfied by $A(t)$.

22. (a) How long will it take an investment to double in value if the interest rate is 3%, compounded continuously?
  (b) What is the equivalent annual interest rate?

APPLIED PROJECT | CONTROLLING RED BLOOD CELL LOSS DURING SURGERY

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A typical volume of blood in the human body is about 5 L. A certain percentage of that volume (called the *hematocrit*) consists of red blood cells (RBCs); typically the hematocrit is about 45% in males. Suppose that a surgery takes four hours and a male patient bleeds 2.5 L of blood. During surgery the patient's blood volume is maintained at 5 L by injection of saline solution, which mixes quickly with the blood but dilutes it so that the hematocrit decreases as time passes.

1. Assuming that the rate of RBC loss is proportional to the volume of RBCs, determine the patient's volume of RBCs by the end of the operation.
2. A procedure called *acute normovolemic hemodilution* (ANH) has been developed to minimize RBC loss during surgery. In this procedure blood is extracted from the patient before the operation and replaced with saline solution. This dilutes the patient's blood, resulting in fewer RBCs being lost by bleeding during surgery. The extracted blood is then returned to the patient after surgery. Only a certain amount of blood can be extracted, however, because the RBC concentration can never be allowed to drop below 25% during surgery. What is the maximum amount of blood that can be extracted in the ANH procedure for the surgery described in this project?
3. What is the RBC loss without the ANH procedure? What is the loss if the procedure is carried out with the volume calculated in Problem 2?

3.9 Related Rates

If we are pumping air into a balloon, both the volume and the radius of the balloon are increasing and their rates of increase are related to each other. But it is much easier to measure directly the rate of increase of the volume than the rate of increase of the radius.

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

EXAMPLE 1 Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

PS According to the Principles of Problem Solving discussed following Chapter 1, the first step is to understand the problem. This includes reading the problem carefully, identifying the given and the unknown, and introducing suitable notation.

SOLUTION We start by identifying two things:

the *given information:*

the rate of increase of the volume of air is 100 cm^3/s

and the *unknown:*

the rate of increase of the radius when the diameter is 50 cm

In order to express these quantities mathematically, we introduce some suggestive *notation:*

Let V be the volume of the balloon and let r be its radius.

The key thing to remember is that rates of change are derivatives. In this problem, the volume and the radius are both functions of the time t. The rate of increase of the volume with respect to time is the derivative dV/dt, and the rate of increase of the radius is dr/dt. We can therefore restate the given and the unknown as follows:

$$\textit{Given:} \qquad \frac{dV}{dt} = 100 \text{ cm}^3\text{/s}$$

$$\textit{Unknown:} \qquad \frac{dr}{dt} \quad \text{when } r = 25 \text{ cm}$$

In order to connect dV/dt and dr/dt, we first relate V and r by a formula—in this case, the formula for the volume of a sphere:

$$V = \tfrac{4}{3}\pi r^3$$

PS The second stage of problem solving is to think of a plan for connecting the given and the unknown.

In order to use the given information, we differentiate each side of this equation with respect to t. To differentiate the right side, we need to use the Chain Rule:

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Now we solve for the unknown quantity:

Notice that, although dV/dt is constant, dr/dt is *not* constant.

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

If we put $r = 25$ and $dV/dt = 100$ in this equation, we obtain

$$\frac{dr}{dt} = \frac{1}{4\pi(25)^2}100 = \frac{1}{25\pi}$$

The radius of the balloon is increasing at the rate of $1/(25\pi) \approx 0.0127$ cm/s when the diameter is 50 cm. ■

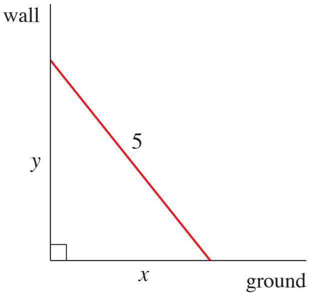


FIGURE 1

EXAMPLE 2 A ladder 5 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 m/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3 m from the wall?

SOLUTION We first draw a diagram and label it as in Figure 1. Let x meters be the distance from the bottom of the ladder to the wall and y meters the distance from the top of the ladder to the ground. Note that x and y are both functions of t (time, measured in seconds).

We are given that $dx/dt = 1$ m/s and we are asked to find dy/dt when $x = 3$ m (see Figure 2). In this problem, the relationship between x and y is given by the Pythagorean Theorem:

$$x^2 + y^2 = 25$$

Differentiating each side with respect to t using the Chain Rule, we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Solving this equation for the desired rate, we obtain

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

When $x = 3$, the Pythagorean Theorem gives $y = 4$ and so, substituting these values and $dx/dt = 1$, we have

$$\frac{dy}{dt} = -\frac{3}{4}(1) = -0.75 \text{ m/s}$$

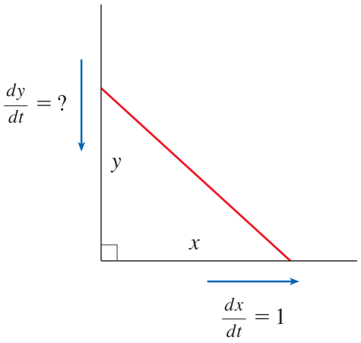


FIGURE 2

The fact that dy/dt is negative means that the distance from the top of the ladder to the ground is *decreasing* at a rate of 0.75 m/s. In other words, the top of the ladder is sliding down the wall at a rate of 0.75 m/s. ■

PS Look back: What have we learned from Examples 1 and 2 that will help us solve future problems?

WARNING A common error is to substitute the given numerical information (for quantities that vary with time) too early. This should be done only *after* the differentiation. (Step 7 follows Step 6.) For instance, in Example 1 we dealt with general values of r until we finally substituted $r = 25$ at the last step. (If we had put $r = 25$ earlier, we would have gotten $dV/dt = 0$, which is clearly wrong.)

Problem Solving Strategy

It is useful to recall some of the problem-solving principles and adapt them to related rates in light of our experience in Examples 1 and 2:

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution (see Example 3 below).
6. Use the Chain Rule to differentiate both sides of the equation with respect to t.
7. Substitute the given information into the resulting equation and solve for the unknown rate.

See also Principles of Problem Solving following Chapter 1.

The following examples further illustrate this strategy.

EXAMPLE 3 A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

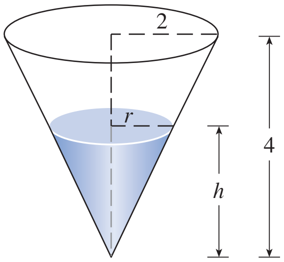


FIGURE 3

SOLUTION We first sketch the cone and label it as in Figure 3. Let V, r, and h be the volume of the water, the radius of the surface, and the height of the water at time t, where t is measured in minutes.

We are given that $dV/dt = 2$ m^3/min and we are asked to find dh/dt when h is 3 m. The quantities V and h are related by the equation

$$V = \tfrac{1}{3}\pi r^2 h$$

but it is very useful to express V as a function of h alone. In order to eliminate r, we use the similar triangles in Figure 3 to write

$$\frac{r}{h} = \frac{2}{4} \qquad r = \frac{h}{2}$$

and the expression for V becomes

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

Now we can differentiate each side with respect to t:

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

so

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

Substituting $h = 3$ m and $dV/dt = 2$ m^3/min, we have

$$\frac{dh}{dt} = \frac{4}{\pi(3)^2} \cdot 2 = \frac{8}{9\pi}$$

The water level is rising at a rate of $8/(9\pi) \approx 0.28$ m/min. ■

EXAMPLE 4 Car A is traveling west at 80 km/h and car B is traveling north at 100 km/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 km and car B is 0.4 km from the intersection?

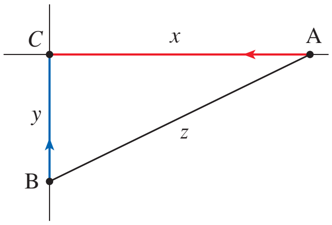


FIGURE 4

SOLUTION We draw Figure 4, where C is the intersection of the roads. At a given time t, let x be the distance from car A to C, let y be the distance from car B to C, and let z be the distance between the cars, where x, y, and z are measured in miles.

We are given that $dx/dt = -80$ km/h and $dy/dt = -700$ km/h. (The derivatives are negative because x and y are decreasing.) We are asked to find dz/dt. The equation that relates x, y, and z is given by the Pythagorean Theorem:

$$z^2 = x^2 + y^2$$

Differentiating each side with respect to t, we have

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

When $x = 0.3$ km and $y = 0.4$ km, the Pythagorean Theorem gives $z = 0.5$ km, so

$$\frac{dz}{dt} = \frac{1}{0.5}[0.3(-80) + 0.4(-100)] = -128 \text{ km/h}$$

The cars are approaching each other at a rate of 128 km/h. ■

EXAMPLE 5 A man walks along a straight path at a speed of 1 m/s. A spotlight is located on the ground 6 m from the path and is kept focused on the man. At what rate is the spotlight rotating when the man is 4.5 m from the point on the path closest to the light?

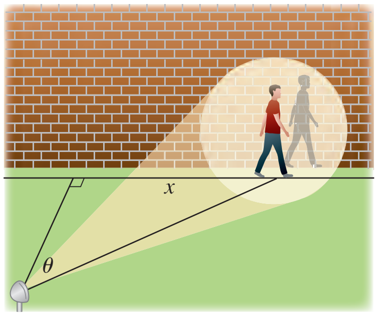


FIGURE 5

SOLUTION We draw Figure 5 and let x be the distance from the man to the point on the path closest to the spotlight. We let θ be the angle between the beam of the light and the perpendicular to the path.

We are given that $dx/dt = 1$ m/s and are asked to find $d\theta/dt$ when $x = 4.5$. The equation that relates x and θ can be written from Figure 5:

$$\frac{x}{6} = \tan\theta \qquad x = 6\tan\theta$$

Differentiating each side with respect to t, we get

$$\frac{dx}{dt} = 6\sec^2\theta\,\frac{d\theta}{dt}$$

so

$$\frac{d\theta}{dt} = \frac{1}{6}\cos^2\theta\,\frac{dx}{dt}$$

$$= \frac{1}{6}\cos^2\theta\,(1) = \frac{1}{6}\cos^2\theta$$

When $x = 4.5$, the length of the beam is 7.5, so $\cos\theta = \frac{20}{25} = \frac{4}{5}$ and

$$\frac{d\theta}{dt} = \frac{1}{6}\left(\frac{4}{5}\right)^2 = \frac{16}{150} = \sim 0.107$$

The spotlight is rotating at a rate of 0.107 rad/s. ■

$$0.107\,\frac{\text{rad}}{\text{s}} \times \frac{1\text{ rotation}}{2\pi\text{ rad}} \times \frac{60\text{ s}}{1\text{ min}}$$

$\approx$ 1.02 rotations per min

3.9 Exercises

1. (a) If V is the volume of a cube with edge length x and the cube expands as time passes, find dV/dt in terms of dx/dt.
(b) If the length of the edge of a cube is increasing at a rate of 4 cm/s, how fast is the volume of the cube increasing when the edge length is 15 cm?

2. (a) If A is the area of a circle with radius r and the circle expands as time passes, find dA/dt in terms of dr/dt.
(b) Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 2 m/s, how fast is the area of the spill increasing when the radius is 30 m?

3. Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 16 cm^2?

4. The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80 mm?

5. The radius of a spherical ball is increasing at a rate of 2 cm/min. At what rate is the surface area of the ball increasing when the radius is 8 cm?

6. The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

7. A cylindrical tank with radius 5 m is being filled with water at a rate of 3 m^3/min. How fast is the height of the water increasing?

8. The area of a triangle with sides of lengths a and b and contained angle θ is $A = \frac{1}{2}ab\sin\theta$. (See Formula 6 in Appendix D.)
(a) If $a = 2$ cm, $b = 3$ cm, and θ increases at a rate of 0.2 rad/min, how fast is the area increasing when $\theta = \pi/3$?
(b) If $a = 2$ cm, b increases at a rate of 1.5 cm/min, and θ increases at a rate of 0.2 rad/min, how fast is the area increasing when $b = 3$ cm and $\theta = \pi/3$?
(c) If a increases at a rate of 2.5 cm/min, b increases at a rate of 1.5 cm/min, and θ increases at a rate of 0.2 rad/min, how fast is the area increasing when $a = 2$ cm, $b = 3$ cm, and $\theta = \pi/3$?

9. Suppose $4x^2 + 9y^2 = 25$, where x and y are functions of t.
(a) If $dy/dt = \frac{1}{3}$, find dx/dt when $x = 2$ and $y = 1$.
(b) If $dx/dt = 3$, find dy/dt when $x = -2$ and $y = 1$.

10. If $x^2 + y^2 + z^2 = 9$, $dx/dt = 5$, and $dy/dt = 4$, find dz/dt when $(x, y, z) = (2, 2, 1)$.

11. The weight w of an astronaut (in newtons) is related to her height h above the surface of the earth (in kilometers) by

$$w = w_0\left(\frac{6370}{6370 + h}\right)^2$$

where w_0 is the weight of the astronaut on the surface of the earth. If the astronaut weighs 580 newtons on earth and is in a rocket, being propelled upward at a speed of 19 km/s, find the rate at which her weight is changing (in N/s) when she is 60 kilometers above the earth's surface.

12. A particle is moving along a hyperbola $xy = 8$. As it reaches the point (4, 2), the y-coordinate is decreasing at a rate of 3 cm/s. How fast is the x-coordinate of the point changing at that instant?

13–16
(a) What quantities are given in the problem?
(b) What is the unknown?
(c) Draw a picture of the situation for any time t.
(d) Write an equation that relates the quantities.
(e) Finish solving the problem.

13. A plane flying horizontally at an altitude of 2 km and a speed of 800 km/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when the plane is 3 km away from the station.

14. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

15. A street light is mounted at the top of a 6-meter-tall pole. A man 2 m tall walks away from the pole with a speed of 1.5 m/s along a straight path. How fast is the tip of his shadow moving when he is 10 m from the pole?

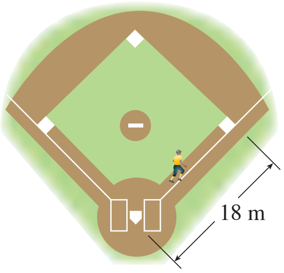
16. At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 PM?

17. Two cars start moving from the same point. One travels south at 30 km/h and the other travels west at 72 km/h. At what rate is the distance between the cars increasing two hours later?

18. A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?

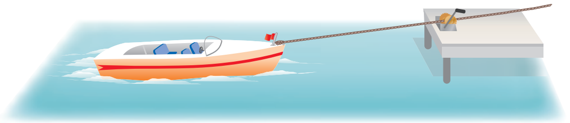
19. A man starts walking north at 1.2 m/s from a point *P*. Five minutes later a woman starts walking south at 1.6 m/s from a point 200 m due east of *P*. At what rate are the people moving apart 15 min after the woman starts walking?

20. A baseball diamond is a square with side 18 m. A batter hits the ball and runs toward first base with a speed of 7.5 m/s.
(a) At what rate is his distance from second base decreasing when he is halfway to first base?
(b) At what rate is his distance from third base increasing at the same moment?



21. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?

22. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?



23–24 Use the fact that the distance (in meters) a dropped stone falls after t seconds is $d = 4.9t^2$.

23. A woman stands near the edge of a cliff and drops a stone over the edge. Exactly one second later she drops another stone. One second after that, how fast is the distance between the two stones changing?

24. Two men stand 10 m apart on level ground near the edge of a cliff. One man drops a stone and one second later the other man drops a stone. One second after that, how fast is the distance between the two stones changing?

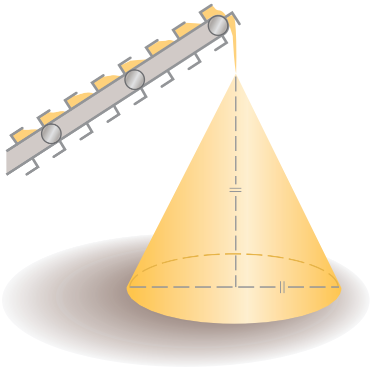
25. Water is leaking out of an inverted conical tank at a rate of 10,000 cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.

26. A particle moves along the curve $y = 2\sin(\pi x/2)$. As the particle passes through the point $\left(\frac{1}{3}, 1\right)$, its x-coordinate increases at a rate of $\sqrt{10}$ cm/s. How fast is the distance from the particle to the origin changing at this instant?

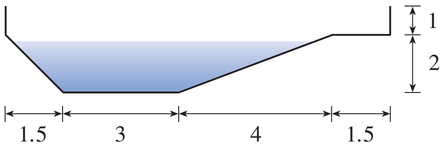
27. A water trough is 10 m long and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top, and has height 50 cm. If the trough is being filled with water at the rate of 0.2 m^3/min, how fast is the water level rising when the water is 30 cm deep?

28. A trough is 6 m long and its ends have the shape of isosceles triangles that are 1 m across at the top and have a height of 50 cm. If the trough is being filled with water at a rate of 1.2 m^3/min, how fast is the water level rising when the water is 30 centimeters deep?

29. Gravel is being dumped from a conveyor belt at a rate of 3 m^3/min, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 3 m high?



30. A swimming pool is 5 m wide, 10 m long, 1 m deep at the shallow end, and 3 m deep at its deepest point. A cross-section is shown in the figure. If the pool is being filled at a rate of 0.1 m^3/min, how fast is the water level rising when the depth at the deepest point is 1 m?



31. The sides of an equilateral triangle are increasing at a rate of 10 cm/min. At what rate is the area of the triangle increasing when the sides are 30 cm long?

32. A kite 50 m above the ground moves horizontally at a speed of 2 m/s. At what rate is the angle between the string and the horizontal decreasing when 100 m of string has been let out?

33. A car is traveling north on a straight road at 20 m/s and a drone is flying east at 6 m/s at an elevation of 25 m. At one instant the drone passes directly over the car. How fast is the distance between the drone and the car changing 5 seconds later?

34. If the minute hand of a clock has length r (in centimeters), find the rate at which it sweeps out area as a function of r.

35. How fast is the angle between the ladder and the ground changing in Example 2 when the bottom of the ladder is 3 m from the wall?

36. According to the model we used to solve Example 2, what happens as the top of the ladder approaches the ground? Is the model appropriate for small values of y?

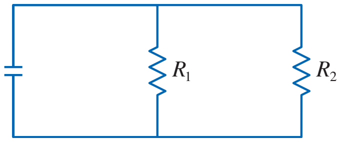
37. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy the equation $PV = C$, where C is a constant. Suppose that at a certain instant the volume is 600 cm^3, the pressure is 150 kPa, and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume decreasing at this instant?

38. A faucet is filling a hemispherical basin of diameter 60 cm with water at a rate of 2 L/min. Find the rate at which the water is rising in the basin when it is half full. [Use the following facts: 1 L is 1000 cm^3. The volume of the portion of a sphere with radius r from the bottom to a height h is $V = \pi\left(rh^2 - \frac{1}{3}h^3\right)$, as we will show in Chapter 6.]

39. If two resistors with resistances R_1 and R_2 are connected in parallel, as shown in the figure, then the total resistance R, measured in ohms (Ω), is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

If R_1 and R_2 are increasing at rates of 0.3 Ω/s and 0.2 Ω/s, respectively, how fast is R changing when $R_1 = 80\ \Omega$ and $R_2 = 100\ \Omega$?



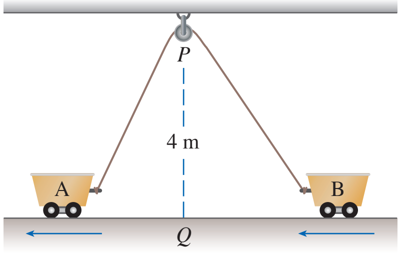
40. When air expands adiabatically (without gaining or losing heat), its pressure P and volume V are related by the equation $PV^{1.4} = C$, where C is a constant. Suppose that at a certain instant the volume is 400 cm^3 and the pressure is 80 kPa and is decreasing at a rate of 10 kPa/min. At what rate is the volume increasing at this instant?

41. Two straight roads diverge from an intersection at an angle of 60°. Two cars leave the intersection at the same time, the first traveling down one road at 60 km/h and the second traveling down the other road at 100 km/h. How fast is the distance between the cars changing after half an hour? [*Hint:* Use the Law of Cosines (Formula 21 in Appendix D).]

42. Brain weight B as a function of body weight W in fish has been modeled by the power function $B = 0.007W^{2/3}$, where B and W are measured in grams. A model for body weight as a function of body length L (measured in centimeters) is $W = 0.12L^{2.53}$. If, over 10 million years, the average length of a certain species of fish evolved from 15 cm to 20 cm at a constant rate, how fast was this species' brain growing when its average length was 18 cm?

43. Two sides of a triangle have lengths 12 m and 15 m. The angle between them is increasing at a rate of 2°/min. How fast is the length of the third side increasing when the angle between the sides of fixed length is 60°? [*Hint:* Use the Law of Cosines (Formula 21 in Appendix D).]

44. Two carts, A and B, are connected by a rope 12 m long that passes over a pulley P. (See the figure.) The point Q is on the floor 4 m directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of 0.5 m/s. How fast is cart B moving toward Q at the instant when cart A is 3 m from Q?



45. A television camera is positioned 1200 m from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also, the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Let's assume the rocket rises vertically and its speed is 200 m/s when it has risen 900 m.

(a) How fast is the distance from the television camera to the rocket changing at that moment?

(b) If the television camera is always kept aimed at the rocket, how fast is the camera's angle of elevation changing at that same moment?

46. A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

47. A plane flies horizontally at an altitude of 5 km and passes directly over a tracking telescope on the ground. When the angle of elevation is $\pi/3$, this angle is decreasing at a rate of $\pi/6$ rad/min. How fast is the plane traveling at that time?

48. A Ferris wheel with a radius of 10 m is rotating at a rate of one revolution every 2 minutes. How fast is a rider rising when his seat is 16 m above ground level?

49. A plane flying with a constant speed of 300 km/h passes over a ground radar station at an altitude of 1 km and climbs at an angle of 30°. At what rate is the distance from the plane to the radar station increasing a minute later?

50. Two people start from the same point. One walks east at 4 km/h and the other walks northeast at 2 km/h. How fast is the distance between the people changing after 15 minutes?

51. A runner sprints around a circular track of radius 100 m at a constant speed of 7 m/s. The runner's friend is standing at a distance 200 m from the center of the track. How fast is the distance between the friends changing when the distance between them is 200 m?

52. The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?

53. Suppose that the volume V of a rolling snowball increases so that dV/dt is proportional to the surface area of the snowball at time t. Show that the radius r increases at a constant rate, that is, dr/dt is constant.

3.10 Linear Approximations and Differentials

We have seen that a curve lies very close to its tangent line near the point of tangency. In fact, by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like its tangent line. (See Figure 2.7.2.) This observation is the basis for a method of finding approximate values of functions.

Linearization and Approximation

It might be easy to calculate a value $f(a)$ of a function, but difficult (or even impossible) to compute nearby values of f. So we settle for the easily computed values of the linear function L whose graph is the tangent line of f at $(a, f(a))$. (See Figure 1.)

In other words, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. An equation of this tangent line is

$$y = f(a) + f'(a)(x - a)$$

The linear function whose graph is this tangent line, that is,

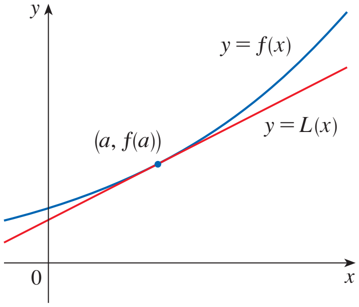


FIGURE 1

$$\boxed{1} \qquad \boxed{L(x) = f(a) + f'(a)(x - a)}$$

is called the **linearization** of f at a. The approximation $f(x) \approx L(x)$ or

$$\boxed{2} \qquad \boxed{f(x) \approx f(a) + f'(a)(x - a)}$$

is called the **linear approximation** or **tangent line approximation** of f at a.

EXAMPLE 1 Find the linearization of the function $f(x) = \sqrt{x + 3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

SOLUTION The derivative of $f(x) = (x + 3)^{1/2}$ is

$$f'(x) = \tfrac{1}{2}(x + 3)^{-1/2} = \frac{1}{2\sqrt{x + 3}}$$

and so we have $f(1) = 2$ and $f'(1) = \frac{1}{4}$. Putting these values into Equation 1, we see that the linearization is

$$L(x) = f(1) + f'(1)(x - 1) = 2 + \tfrac{1}{4}(x - 1) = \frac{7}{4} + \frac{x}{4}$$

The corresponding linear approximation (2) is

$$\sqrt{x + 3} \approx \frac{7}{4} + \frac{x}{4} \qquad \text{(when } x \text{ is near 1)}$$

In particular, we have

$$\sqrt{3.98} \approx \tfrac{7}{4} + \tfrac{0.98}{4} = 1.995 \qquad \text{and} \qquad \sqrt{4.05} \approx \tfrac{7}{4} + \tfrac{1.05}{4} = 2.0125$$

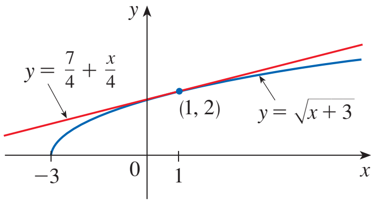


FIGURE 2

The linear approximation is illustrated in Figure 2. We see that, indeed, the tangent line approximation is a good approximation to the given function when x is near 1. We also see that our approximations are overestimates because the tangent line lies above the curve.

Of course, a calculator could give us approximations for $\sqrt{3.98}$ and $\sqrt{4.05}$, but the linear approximation gives an approximation *over an entire interval.* ■

In the following table we compare the estimates from the linear approximation in Example 1 with the true values. Notice from this table, and also from Figure 2, that the tangent line approximation gives good estimates when x is close to 1 but the accuracy of the approximation deteriorates when x is farther away from 1.

	x	From $L(x)$	Actual value
$\sqrt{3.9}$	0.9	1.975	1.97484176. . .
$\sqrt{3.98}$	0.98	1.995	1.99499373. . .
$\sqrt{4}$	1	2	2.00000000. . .
$\sqrt{4.05}$	1.05	2.0125	2.01246117. . .
$\sqrt{4.1}$	1.1	2.025	2.02484567. . .
$\sqrt{5}$	2	2.25	2.23606797. . .
$\sqrt{6}$	3	2.5	2.44948974. . .

How good is the approximation that we obtained in Example 1? The next example shows that by using a graphing calculator or computer we can determine an interval throughout which a linear approximation provides a specified accuracy.

EXAMPLE 2 For what values of x is the linear approximation

$$\sqrt{x + 3} \approx \frac{7}{4} + \frac{x}{4}$$

accurate to within 0.5? What about accuracy to within 0.1?

SOLUTION Accuracy to within 0.5 means that the functions should differ by less than 0.5:

$$\left| \sqrt{x + 3} - \left(\frac{7}{4} + \frac{x}{4} \right) \right| < 0.5$$

Equivalently, we could write

$$\sqrt{x + 3} - 0.5 < \frac{7}{4} + \frac{x}{4} < \sqrt{x + 3} + 0.5$$

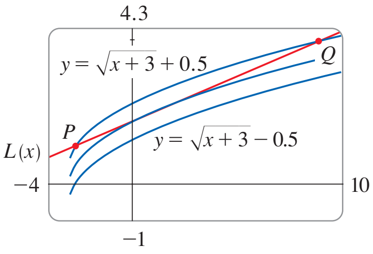


FIGURE 3

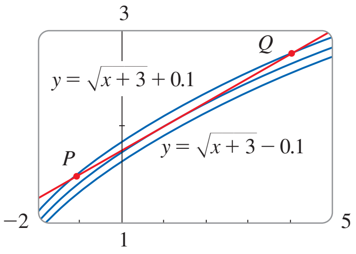


FIGURE 4

This says that the linear approximation should lie between the curves obtained by shifting the curve $y = \sqrt{x+3}$ upward and downward by an amount 0.5. Figure 3 shows the tangent line $y = (7 + x)/4$ intersecting the upper curve $y = \sqrt{x+3} + 0.5$ at P and Q. We estimate that the x-coordinate of P is about -2.66 and the x-coordinate of Q is about 8.66. Thus we see from the graph that the approximation

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

is accurate to within 0.5 when $-2.6 < x < 8.6$. (We have rounded the smaller value up and the larger value down.)

Similarly, from Figure 4 we see that the approximation is accurate to within 0.1 when $-1.1 < x < 3.9$. ■

Applications to Physics

Linear approximations are often used in physics. In analyzing the consequences of an equation, a physicist sometimes needs to simplify a function by replacing it with its linear approximation. For instance, in deriving a formula for the period of a pendulum, physics textbooks obtain an expression involving $\sin\theta$ and then replace $\sin\theta$ by θ with the remark that $\sin\theta$ is very close to θ if θ is not too large. You can verify that the linearization of the function $f(x) = \sin x$ at $a = 0$ is $L(x) = x$ and so the linear approximation at 0 is

$$\sin x \approx x$$

(see Exercise 50). So, in effect, the derivation of the formula for the period of a pendulum uses the tangent line approximation for the sine function.

Another example occurs in the theory of optics, where light rays that arrive at shallow angles relative to the optical axis are called *paraxial rays*. In paraxial (or Gaussian) optics, both $\sin\theta$ and $\cos\theta$ are replaced by their linearizations. In other words, the linear approximations

$$\sin\theta \approx \theta \qquad \text{and} \qquad \cos\theta \approx 1$$

are used because θ is close to 0. The results of calculations made with these approximations became the basic theoretical tool used to design lenses. (See *Optics*, 5th ed., by Eugene Hecht [Boston, 2017], p. 164.)

In Section 11.11 we will present several other applications of the idea of linear approximations to physics and engineering.

Differentials

The ideas behind linear approximations are sometimes formulated in the terminology and notation of *differentials*. If $y = f(x)$, where f is a differentiable function, then the **differential** dx is an independent variable; that is, dx can be given the value of any real number. The **differential** dy is then defined in terms of dx by the equation

$$\boxed{3} \qquad dy = f'(x)\, dx$$

So dy is a dependent variable; it depends on the values of x and dx. If dx is given a specific value and x is taken to be some specific number in the domain of f, then the numerical value of dy is determined.

If $dx \neq 0$, we can divide both sides of Equation 3 by dx to obtain

$$\frac{dy}{dx} = f'(x)$$

We have seen similar equations before, but now the left side can genuinely be interpreted as a ratio of differentials.

The geometric meaning of differentials is shown in Figure 5. Let $P(x, f(x))$ and $Q(x + \Delta x, f(x + \Delta x))$ be points on the graph of f and let $dx = \Delta x$. The corresponding change in y is

$$\Delta y = f(x + \Delta x) - f(x)$$

The slope of the tangent line PR is the derivative $f'(x)$. Thus the directed distance from S to R is $f'(x)\,dx = dy$. Therefore dy represents the amount that the tangent line rises or falls (the change in the linearization), whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount dx.

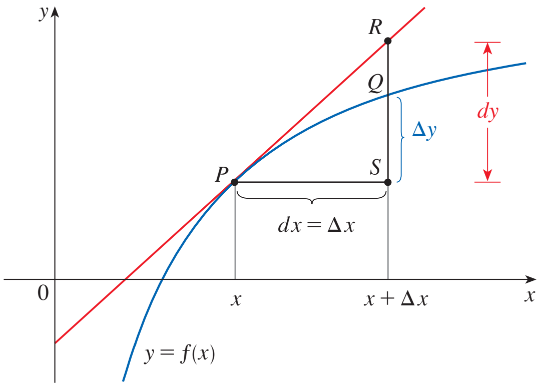


FIGURE 5

EXAMPLE 3 Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes (a) from 2 to 2.05 and (b) from 2 to 2.01.

SOLUTION

(a) We have

$$f(2) = 2^3 + 2^2 - 2(2) + 1 = 9$$
$$f(2.05) = (2.05)^3 + (2.05)^2 - 2(2.05) + 1 = 9.717625$$
$$\Delta y = f(2.05) - f(2) = 0.717625$$

In general, $$dy = f'(x)\,dx = (3x^2 + 2x - 2)\,dx$$

When $x = 2$ and $dx = \Delta x = 0.05$, this becomes

$$dy = [3(2)^2 + 2(2) - 2]0.05 = 0.7$$

(b) $$f(2.01) = (2.01)^3 + (2.01)^2 - 2(2.01) + 1 = 9.140701$$
$$\Delta y = f(2.01) - f(2) = 0.140701$$

When $dx = \Delta x = 0.01$,

$$dy = [3(2)^2 + 2(2) - 2]0.01 = 0.14$$ ■

Figure 6 shows the function in Example 3 and a comparison of dy and Δy when $a = 2$. The viewing rectangle is $[1.8, 2.5]$ by $[6, 18]$.

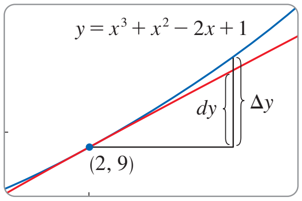


FIGURE 6

Notice that the approximation $\Delta y \approx dy$ becomes better as Δx becomes smaller in Example 3. Notice also that dy was easier to compute than Δy.

In the notation of differentials, the linear approximation $f(x) \approx f(a) + f'(a)(x - a)$ can be written as

$$f(a + dx) \approx f(a) + dy$$

by taking $dx = x - a$, so $x = a + dx$. For instance, for the function $f(x) = \sqrt{x + 3}$ in Example 1, we have

$$dy = f'(x)\,dx = \frac{dx}{2\sqrt{x + 3}}$$

If $a = 1$ and $dx = \Delta x = 0.05$, then

$$dy = \frac{0.05}{2\sqrt{1 + 3}} = 0.0125$$

and $$\sqrt{4.05} = f(1.05) = f(1 + 0.05) \approx f(1) + dy = 2.0125$$

just as we found in Example 1.

Our final example illustrates the use of differentials in estimating the errors that occur because of approximate measurements.

EXAMPLE 4 The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

SOLUTION If the radius of the sphere is r, then its volume is $V = \frac{4}{3}\pi r^3$. If the error in the measured value of r is denoted by $dr = \Delta r$, then the corresponding error in the calculated value of V is ΔV, which can be approximated by the differential

$$dV = 4\pi r^2\,dr$$

When $r = 21$ and $dr = 0.05$, this becomes

$$dV = 4\pi(21)^2\,0.05 \approx 277$$

The maximum error in the calculated volume is about 277 cm^3. ■

NOTE Although the possible error in Example 4 may appear to be rather large, a better picture of the error is given by the **relative error**, which is computed by dividing the error by the total volume:

$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{4\pi r^2\,dr}{\frac{4}{3}\pi r^3} = 3\,\frac{dr}{r}$$

Thus the relative error in the volume is about three times the relative error in the radius. In Example 4 the relative error in the radius is approximately $dr/r = 0.05/21 \approx 0.0024$ and it produces a relative error of about 0.007 in the volume. The errors could also be expressed as percentage errors of 0.24% in the radius and 0.7% in the volume.

3.10 Exercises

1–4 Find the linearization $L(x)$ of the function at a.

1. $f(x) = x^3 - x^2 + 3, \quad a = -2$

2. $f(x) = e^{3x}, \quad a = 0$

3. $f(x) = \sqrt[3]{x}, \quad a = 8$

4. $f(x) = \cos 2x, \quad a = \pi/6$

5. Find the linear approximation of the function $f(x) = \sqrt{1 - x}$ at $a = 0$ and use it to approximate the numbers $\sqrt{0.9}$ and $\sqrt{0.99}$. Illustrate by graphing f and the tangent line.

6. Find the linear approximation of the function $g(x) = \sqrt[3]{1 + x}$ at $a = 0$ and use it to approximate the numbers $\sqrt[3]{0.95}$ and $\sqrt[3]{1.1}$. Illustrate by graphing g and the tangent line.

7–10 Verify the given linear approximation at $a = 0$. Then determine the values of x for which the linear approximation is accurate to within 0.1.

7. $\tan^{-1}x \approx x$

8. $(1 + x)^{-3} \approx 1 - 3x$

9. $\sqrt[4]{1 + 2x} \approx 1 + \frac{1}{2}x$

10. $\dfrac{2}{1 + e^x} \approx 1 - \frac{1}{2}x$

11–18 Find the differential of the function.

11. $y = e^{5x}$

12. $y = \sqrt{1 - t^4}$

13. $y = \dfrac{1 + 2u}{1 + 3u}$

14. $y = \theta^2 \sin 2\theta$

15. $y = \dfrac{1}{x^2 - 3x}$

16. $y = \sqrt{1 + \cos\theta}$

17. $y = \ln(\sin\theta)$

18. $y = \dfrac{e^x}{1 - e^x}$

19–22 (a) Find the differential dy and (b) evaluate dy for the given values of x and dx.

19. $y = e^{x/10}$, $x = 0$, $dx = 0.1$

20. $y = \cos \pi x$, $x = \frac{1}{3}$, $dx = -0.02$

21. $y = \sqrt{3 + x^2}$, $x = 1$, $dx = -0.1$

22. $y = \dfrac{x + 1}{x - 1}$, $x = 2$, $dx = 0.05$

23–26 Compute Δy and dy for the given values of x and $dx = \Delta x$. Then sketch a diagram like Figure 5 showing the line segments with lengths dx, dy, and Δy.

23. $y = x^2 - 4x$, $x = 3$, $\Delta x = 0.5$

24. $y = x - x^3$, $x = 0$, $\Delta x = -0.3$

25. $y = \sqrt{x - 2}$, $x = 3$, $\Delta x = 0.8$

26. $y = e^x$, $x = 0$, $\Delta x = 0.5$

27–30 Compare the values of Δy and dy if x changes from 1 to 1.05. What if x changes from 1 to 1.01? Does the approximation $\Delta y \approx dy$ become better as Δx gets smaller?

27. $f(x) = x^4 - x + 1$

28. $f(x) = e^{2x-2}$

29. $f(x) = \sqrt{5 - x}$

30. $f(x) = \dfrac{1}{x^2 + 1}$

31–36 Use a linear approximation (or differentials) to estimate the given number.

31. $(1.999)^4$

32. $1/4.002$

33. $\sqrt[3]{1001}$

34. $\sqrt{100.5}$

35. $e^{0.1}$

36. $\cos 29°$

37–39 Explain, in terms of linear approximations or differentials, why the approximation is reasonable.

37. $\ln 1.04 \approx 0.04$

38. $\sqrt{4.02} \approx 2.005$

39. $\dfrac{1}{9.98} \approx 0.1002$

---

40. Let $f(x) = (x - 1)^2 \qquad g(x) = e^{-2x}$

and $h(x) = 1 + \ln(1 - 2x)$

(a) Find the linearizations of f, g, and h at $a = 0$. What do you notice? How do you explain what happened?

(b) Graph f, g, and h and their linear approximations. For which function is the linear approximation best? For which is it worst? Explain.

41. The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum possible error, relative error, and percentage error in computing (a) the volume of the cube and (b) the surface area of the cube.

42. The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm.

(a) Use differentials to estimate the maximum error in the calculated area of the disk.

(b) What is the relative error and the percentage error?

43. The circumference of a sphere was measured to be 84 cm with a possible error of 0.5 cm.

(a) Use differentials to estimate the maximum error in the calculated surface area. What is the relative error?

(b) Use differentials to estimate the maximum error in the calculated volume. What is the relative error?

44. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.

45. (a) Use differentials to find a formula for the approximate volume of a thin cylindrical shell with height h, inner radius r, and thickness Δr.

(b) What is the error involved in using the formula from part (a)?

46. One side of a right triangle is known to be 20 cm long and the opposite angle is measured as 30°, with a possible error of ±1°.

(a) Use differentials to estimate the error in computing the length of the hypotenuse.

(b) What is the percentage error?

47. If a current I passes through a resistor with resistance R, Ohm's Law states that the voltage drop is $V = RI$. If V is constant and R is measured with a certain error, use differentials to show that the relative error in calculating I is approximately the same (in magnitude) as the relative error in R.

48. When blood flows along a blood vessel, the flux F (the volume of blood per unit time that flows past a given point) is proportional to the fourth power of the radius R of the blood vessel:

$$F = kR^4$$

(This is known as Poiseuille's Law; we will show why it is true in Section 8.4.) A partially clogged artery can be expanded by an operation called angioplasty, in which a balloon-tipped catheter is inflated inside the artery in order to widen it and restore normal blood flow.

Show that the relative change in F is about four times the relative change in R. How will a 5% increase in the radius affect the flow of blood?

49. Establish the following rules for working with differentials (where c denotes a constant and u and v are functions of x).

(a) $dc = 0$ (b) $d(cu) = c\,du$

(c) $d(u + v) = du + dv$ (d) $d(uv) = u\,dv + v\,du$

(e) $d\left(\dfrac{u}{v}\right) = \dfrac{v\,du - u\,dv}{v^2}$ (f) $d(x^n) = nx^{n-1}\,dx$

50. In physics textbooks, the period T of a pendulum of length L is often given as $T \approx 2\pi\sqrt{L/g}$, provided that the pendulum swings through a relatively small arc. In the course of deriving this formula, the equation $a_T = -g\sin\theta$ for the tangential acceleration of the bob of the pendulum is obtained, and then $\sin\theta$ is replaced by θ with the remark that for small angles, θ (in radians) is very close to $\sin\theta$.

(a) Verify the linear approximation at 0 for the sine function:

$$\sin\theta \approx \theta$$

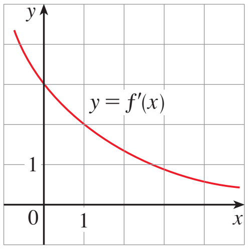
(b) If $\theta = \pi/18$ (equivalent to 10°) and we approximate $\sin\theta$ by θ, what is the percentage error?

(c) Use a graph to determine the values of θ for which $\sin\theta$ and θ differ by less than 2%. What are the values in degrees?

51. Suppose that the only information we have about a function f is that $f(1) = 5$ and the graph of its *derivative* is as shown.

(a) Use a linear approximation to estimate $f(0.9)$ and $f(1.1)$.

(b) Are your estimates in part (a) too large or too small? Explain.



52. Suppose that we don't have a formula for $g(x)$ but we know that $g(2) = -4$ and $g'(x) = \sqrt{x^2 + 5}$ for all x.

(a) Use a linear approximation to estimate $g(1.95)$ and $g(2.05)$.

(b) Are your estimates in part (a) too large or too small? Explain.

DISCOVERY PROJECT POLYNOMIAL APPROXIMATIONS

The tangent line approximation $L(x)$ is the best first-degree (linear) approximation to $f(x)$ near $x = a$ because $f(x)$ and $L(x)$ have the same rate of change (derivative) at a. For a better approximation than a linear one, let's try a second-degree (quadratic) approximation $P(x)$. In other words, we approximate a curve by a parabola instead of by a straight line. To make sure that the approximation is a good one, we stipulate the following:

(i) $P(a) = f(a)$ (P and f should have the same value at a.)

(ii) $P'(a) = f'(a)$ (P and f should have the same rate of change at a.)

(iii) $P''(a) = f''(a)$ (The slopes of P and f should change at the same rate at a.)

1. Find the quadratic approximation $P(x) = A + Bx + Cx^2$ to the function $f(x) = \cos x$ that satisfies conditions (i), (ii), and (iii) with $a = 0$. Graph P, f, and the linear approximation $L(x) = 1$ on a common screen. Comment on how well the functions P and L approximate f.

2. Determine the values of x for which the quadratic approximation $f(x) \approx P(x)$ in Problem 1 is accurate to within 0.1. [*Hint:* Graph $y = P(x)$, $y = \cos x - 0.1$, and $y = \cos x + 0.1$ on a common screen.]

3. To approximate a function f by a quadratic function P near a number a, it is best to write P in the form

$$P(x) = A + B(x - a) + C(x - a)^2$$

Show that the quadratic function that satisfies conditions (i), (ii), and (iii) is

$$P(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2}f''(a)(x - a)^2$$

4. Find the quadratic approximation to $f(x) = \sqrt{x + 3}$ near $a = 1$. Graph f, the quadratic approximation, and the linear approximation from Example 3.10.2 on a common screen. What do you conclude?

5. Instead of being satisfied with a linear or quadratic approximation to $f(x)$ near $x = a$, let's try to find better approximations with higher-degree polynomials. We look for an nth-degree polynomial

$$T_n(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \cdots + c_n(x - a)^n$$

such that T_n and its first n derivatives have the same values at $x = a$ as f and its first n derivatives. By differentiating repeatedly and setting $x = a$, show that these conditions are satisfied if $c_0 = f(a)$, $c_1 = f'(a)$, $c_2 = \frac{1}{2}f''(a)$, and in general

$$c_k = \frac{f^{(k)}(a)}{k!}$$

where $k! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot k$. The resulting polynomial

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

is called the ***n*th-degree Taylor polynomial of *f* centered at *a***. (We will study Taylor polynomials in more detail in Chapter 11.)

6. Find the 8th-degree Taylor polynomial centered at $a = 0$ for the function $f(x) = \cos x$. Graph f together with the Taylor polynomials T_2, T_4, T_6, T_8 in the viewing rectangle $[-5, 5]$ by $[-1.4, 1.4]$ and comment on how well they approximate f.

3.11 Hyperbolic Functions

Hyperbolic Functions and Their Derivatives

Certain combinations of the exponential functions e^x and e^{-x} arise so frequently in mathematics and its applications that they deserve to be given special names. In many ways they are analogous to the trigonometric functions, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle. For this reason they are collectively called **hyperbolic functions** and individually called **hyperbolic sine**, **hyperbolic cosine**, and so on.

Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

The graphs of hyperbolic sine and hyperbolic cosine can be sketched using graphical addition as in Figures 1 and 2.

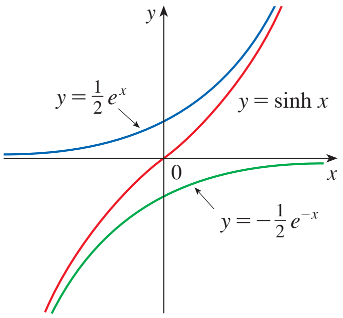


FIGURE 1
$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

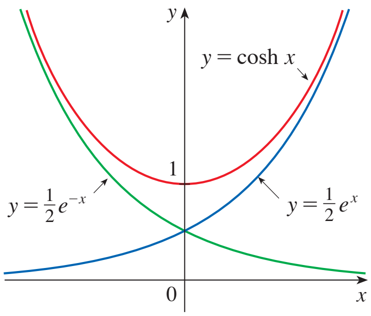


FIGURE 2
$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

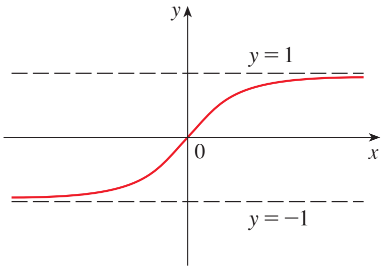


FIGURE 3
$y = \tanh x$

Note that sinh has domain $\mathbb{R}$ and range $\mathbb{R}$, whereas cosh has domain $\mathbb{R}$ and range $[1, \infty)$. The graph of tanh is shown in Figure 3. It has horizontal asymptotes $y = \pm 1$. (See Exercise 27.)

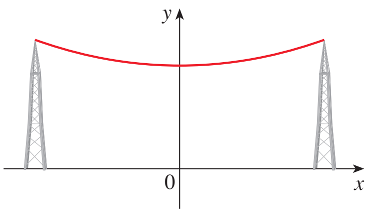


FIGURE 4
A catenary $y = c + a\cosh(x/a)$

Some of the mathematical uses of hyperbolic functions will be seen in Chapter 7. Applications to science and engineering occur whenever an entity such as light, velocity, electricity, or radioactivity is gradually absorbed or extinguished because the decay can be represented by hyperbolic functions. The most famous application is the use of hyperbolic cosine to describe the shape of a hanging wire. It can be proved that if a heavy flexible cable (such as an overhead power line) is suspended between two points at the same height, then it takes the shape of a curve with equation

$$y = c + a\cosh(x/a)$$

called a *catenary* (see Figure 4). (The Latin word *catena* means "chain.")

Another application of hyperbolic functions occurs in the description of ocean waves: the velocity of a water wave with length L moving across a body of water with depth d is modeled by the function

$$v = \sqrt{\frac{gL}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)}$$

where g is the acceleration due to gravity (see Figure 5 and Exercise 57).

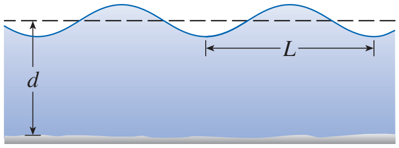


FIGURE 5
Idealized water wave

The hyperbolic functions satisfy a number of identities that are similar to well-known trigonometric identities. We list some of them here and leave most of the proofs to the exercises.

Hyperbolic Identities

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

Stockbyte/Stockbyte/Getty Images

The Gateway Arch in St. Louis was designed using a hyperbolic cosine function (see Exercise 56).

EXAMPLE 1 Prove (a) $\cosh^2 x - \sinh^2 x = 1$ and (b) $1 - \tanh^2 x = \operatorname{sech}^2 x$.

SOLUTION

(a)
$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$
$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}$$
$$= \frac{4}{4} = 1$$

(b) We start with the identity proved in part (a):

$$\cosh^2 x - \sinh^2 x = 1$$

If we divide both sides by $\cosh^2 x$, we get

$$1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

or
$$1 - \tanh^2 x = \operatorname{sech}^2 x$$
■

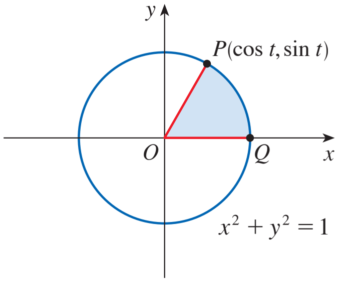


FIGURE 6

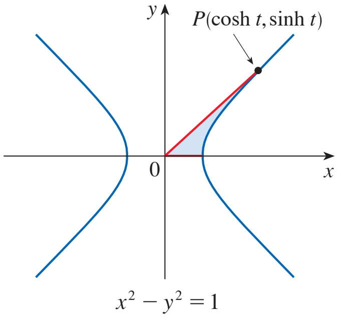


FIGURE 7

The identity proved in Example 1(a) gives a clue to the reason for the name "hyperbolic" functions:

If t is any real number, then the point $P(\cos t, \sin t)$ lies on the unit circle $x^2 + y^2 = 1$ because $\cos^2 t + \sin^2 t = 1$. In fact, t can be interpreted as the radian measure of $\angle POQ$ in Figure 6. For this reason the trigonometric functions are sometimes called *circular* functions.

Likewise, if t is any real number, then the point $P(\cosh t, \sinh t)$ lies on the right branch of the hyperbola $x^2 - y^2 = 1$ because $\cosh^2 t - \sinh^2 t = 1$ and $\cosh t \geqslant 1$. This time, t does not represent the measure of an angle. However, it turns out that t represents twice the area of the shaded hyperbolic sector in Figure 7, just as in the trigonometric case t represents twice the area of the shaded circular sector in Figure 6.

The derivatives of the hyperbolic functions are easily computed. For example,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

We list the differentiation formulas for the hyperbolic functions as Table 1. The remaining proofs are left as exercises. Note the analogy with the differentiation formulas for trigonometric functions, but note that the signs are different in some cases.

1 **Derivatives of Hyperbolic Functions**

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Any of these differentiation rules can be combined with the Chain Rule, as in the next example.

EXAMPLE 2 If $y = \cosh \sqrt{x}$, find dy/dx.

SOLUTION Using (1) and the Chain Rule, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cosh \sqrt{x}\right) = \sinh \sqrt{x} \cdot \frac{d}{dx}\sqrt{x} = \frac{\sinh \sqrt{x}}{2\sqrt{x}} \qquad ■$$

Inverse Hyperbolic Functions and Their Derivatives

You can see from Figures 1 and 3 that sinh and tanh are one-to-one functions and so they have inverse functions denoted by $\sinh^{-1}$ and $\tanh^{-1}$. Figure 2 shows that cosh is not one-to-one, but if we restrict the domain to the interval $[0, \infty)$, then the function $y = \cosh x$ is one-to-one and attains all the values in its range $[1, \infty)$. The inverse hyperbolic cosine function is defined as the inverse of this restricted function.

2

$$\begin{aligned} y = \sinh^{-1}x &\iff \sinh y = x \\ y = \cosh^{-1}x &\iff \cosh y = x \quad \text{and} \quad y \geqslant 0 \\ y = \tanh^{-1}x &\iff \tanh y = x \end{aligned}$$

The remaining inverse hyperbolic functions are defined similarly (see Exercise 32).

We sketch the graphs of $\sinh^{-1}$, $\cosh^{-1}$, and $\tanh^{-1}$ in Figures 8, 9, and 10 by referring to Figures 1, 2, and 3.

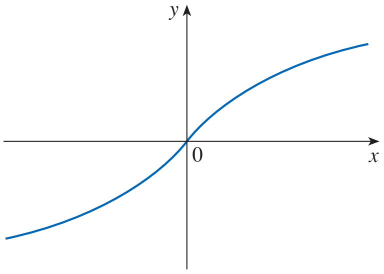


FIGURE 8 $y = \sinh^{-1}x$
domain $= \mathbb{R}$ range $= \mathbb{R}$

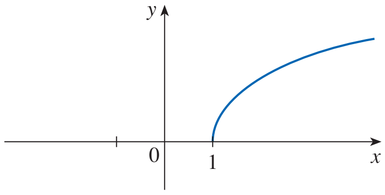


FIGURE 9 $y = \cosh^{-1}x$
domain $= [1, \infty)$ range $= [0, \infty)$

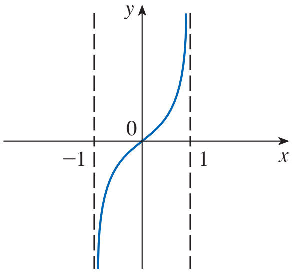


FIGURE 10 $y = \tanh^{-1}x$
domain $= (-1, 1)$ range $= \mathbb{R}$

Since the hyperbolic functions are defined in terms of exponential functions, it's not surprising to learn that the inverse hyperbolic functions can be expressed in terms of logarithms. In particular, we have:

Formula 3 is proved in Example 3. The proofs of Formulas 4 and 5 are requested in Exercises 30 and 31.

3 $$\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad x \in \mathbb{R}$$

4 $$\cosh^{-1}x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad x \geqslant 1$$

5 $$\tanh^{-1}x = \tfrac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) \qquad -1 < x < 1$$

EXAMPLE 3 Show that $\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right)$.

SOLUTION Let $y = \sinh^{-1}x$. Then

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

so
$$e^y - 2x - e^{-y} = 0$$

or, multiplying by e^y,
$$e^{2y} - 2xe^y - 1 = 0$$

This is really a quadratic equation in e^y:

$$(e^y)^2 - 2x(e^y) - 1 = 0$$

Solving by the quadratic formula, we get

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$$

Note that $e^y > 0$, but $x - \sqrt{x^2 + 1} < 0$ $\left(\text{because } x < \sqrt{x^2 + 1}\right)$. Thus the minus sign is inadmissible and we have

$$e^y = x + \sqrt{x^2 + 1}$$

Therefore
$$y = \ln(e^y) = \ln\left(x + \sqrt{x^2 + 1}\right)$$

This shows that
$$\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

(See Exercise 29 for another method.) ■

Notice that the formulas for the derivatives of $\tanh^{-1}x$ and $\coth^{-1}x$ appear to be identical. But the domains of these functions have no numbers in common: $\tanh^{-1}x$ is defined for $|x| < 1$, whereas $\coth^{-1}x$ is defined for $|x| > 1$.

6 Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1 + x^2}} \qquad \frac{d}{dx}(\operatorname{csch}^{-1}x) = -\frac{1}{|x|\sqrt{x^2 + 1}}$$

$$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2 - 1}} \qquad \frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1 - x^2} \qquad \frac{d}{dx}(\coth^{-1}x) = \frac{1}{1 - x^2}$$

The inverse hyperbolic functions are all differentiable because the hyperbolic functions are differentiable (see Appendix F). The formulas in Table 6 can be proved either by the method for inverse functions or by differentiating Formulas 3, 4, and 5.

EXAMPLE 4 Prove that $\dfrac{d}{dx}(\sinh^{-1}x) = \dfrac{1}{\sqrt{1 + x^2}}$.

SOLUTION 1 Let $y = \sinh^{-1}x$. Then $\sinh y = x$. If we differentiate this equation implicitly with respect to x, we get

$$\cosh y \frac{dy}{dx} = 1$$

Since $\cosh^2 y - \sinh^2 y = 1$ and $\cosh y \geqslant 0$, we have $\cosh y = \sqrt{1 + \sinh^2 y}$, so

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}$$

SOLUTION 2 From Equation 3 (proved in Example 3), we have

$$\begin{aligned}
\frac{d}{dx}(\sinh^{-1}x) &= \frac{d}{dx}\ln\left(x + \sqrt{x^2 + 1}\right) \\
&= \frac{1}{x + \sqrt{x^2 + 1}}\frac{d}{dx}\left(x + \sqrt{x^2 + 1}\right) \\
&= \frac{1}{x + \sqrt{x^2 + 1}}\left(1 + \frac{x}{\sqrt{x^2 + 1}}\right) \\
&= \frac{\sqrt{x^2 + 1} + x}{\left(x + \sqrt{x^2 + 1}\right)\sqrt{x^2 + 1}} \\
&= \frac{1}{\sqrt{x^2 + 1}}
\end{aligned}$$

■

EXAMPLE 5 Find $\dfrac{d}{dx}[\tanh^{-1}(\sin x)]$.

SOLUTION Using Table 6 and the Chain Rule, we have

$$\begin{aligned}
\frac{d}{dx}[\tanh^{-1}(\sin x)] &= \frac{1}{1 - (\sin x)^2}\frac{d}{dx}(\sin x) \\
&= \frac{1}{1 - \sin^2 x}\cos x = \frac{\cos x}{\cos^2 x} = \sec x
\end{aligned}$$

■

3.11 Exercises

1–6 Find the numerical value of each expression.

1. (a) $\sinh 0$ (b) $\cosh 0$

2. (a) $\tanh 0$ (b) $\tanh 1$

3. (a) $\cosh(\ln 5)$ (b) $\cosh 5$

4. (a) $\sinh 4$ (b) $\sinh(\ln 4)$

5. (a) $\operatorname{sech} 0$ (b) $\cosh^{-1} 1$

6. (a) $\sinh 1$ (b) $\sinh^{-1} 1$

---

7. Write $8 \sinh x + 5 \cosh x$ in terms of e^x and e^{-x}.

8. Write $2e^{2x} + 3e^{-2x}$ in terms of $\sinh 2x$ and $\cosh 2x$.

9. Write $\sinh(\ln x)$ as a rational function of x.

10. Write $\cosh(4 \ln x)$ as a rational function of x.

11–23 Prove the identity.

11. $\sinh(-x) = -\sinh x$
(This shows that sinh is an odd function.)

12. $\cosh(-x) = \cosh x$
(This shows that cosh is an even function.)

13. $\cosh x + \sinh x = e^x$

14. $\cosh x - \sinh x = e^{-x}$

15. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

16. $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

17. $\coth^2 x - 1 = \operatorname{csch}^2 x$

18. $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

19. $\sinh 2x = 2 \sinh x \cosh x$

20. $\cosh 2x = \cosh^2 x + \sinh^2 x$

21. $\tanh(\ln x) = \dfrac{x^2 - 1}{x^2 + 1}$

22. $\dfrac{1 + \tanh x}{1 - \tanh x} = e^{2x}$

23. $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$
(n any real number)

---

24. If $\tanh x = \frac{12}{13}$, find the values of the other hyperbolic functions at x.

25. If $\cosh x = \frac{5}{3}$ and $x > 0$, find the values of the other hyperbolic functions at x.

26. (a) Use the graphs of sinh, cosh, and tanh in Figures 1–3 to draw the graphs of csch, sech, and coth.
(b) Check the graphs that you sketched in part (a) by using a graphing calculator or computer to produce them.

27. Use the definitions of the hyperbolic functions to find each of the following limits.
(a) $\lim_{x \to \infty} \tanh x$ (b) $\lim_{x \to -\infty} \tanh x$
(c) $\lim_{x \to \infty} \sinh x$ (d) $\lim_{x \to -\infty} \sinh x$
(e) $\lim_{x \to \infty} \operatorname{sech} x$ (f) $\lim_{x \to \infty} \coth x$
(g) $\lim_{x \to 0^+} \coth x$ (h) $\lim_{x \to 0^-} \coth x$
(i) $\lim_{x \to -\infty} \operatorname{csch} x$ (j) $\lim_{x \to \infty} \dfrac{\sinh x}{e^x}$

28. Prove the formulas given in Table 1 for the derivatives of the functions (a) cosh, (b) tanh, (c) csch, (d) sech, and (e) coth.

29. Give an alternative solution to Example 3 by letting $y = \sinh^{-1}x$ and then using Exercise 13 and Example 1(a) with x replaced by y.

30. Prove Equation 4.

31. Prove Equation 5 using (a) the method of Example 3 and (b) Exercise 22 with x replaced by y.

32. For each of the following functions (i) give a definition like those in (2), (ii) sketch the graph, and (iii) find a formula similar to Equation 3.
(a) csch^{-1} (b) sech^{-1} (c) $\coth^{-1}$

33. Prove the formula given in Table 6 for the derivative of each of the following functions.
(a) $\cosh^{-1}$ (b) $\tanh^{-1}$ (c) $\coth^{-1}$

34. Prove the formula given in Table 6 for the derivative of each of the following functions.
(a) sech^{-1} (b) csch^{-1}

35–53 Find the derivative. Simplify where possible.

35. $f(x) = \cosh 3x$

36. $f(x) = e^x \cosh x$

37. $h(x) = \sinh(x^2)$

38. $g(x) = \sinh^2 x$

39. $G(t) = \sinh(\ln t)$

40. $F(t) = \ln(\sinh t)$

41. $f(x) = \tanh \sqrt{x}$

42. $H(v) = e^{\tanh 2v}$

43. $y = \operatorname{sech} x \tanh x$

44. $y = \operatorname{sech}(\tanh x)$

45. $g(t) = t \coth \sqrt{t^2 + 1}$

46. $f(t) = \dfrac{1 + \sinh t}{1 - \sinh t}$

47. $f(x) = \sinh^{-1}(-2x)$

48. $g(x) = \tanh^{-1}(x^3)$

49. $y = \cosh^{-1}(\sec \theta), \quad 0 \leqslant \theta < \pi/2$

50. $y = \operatorname{sech}^{-1}(\sin \theta), \quad 0 < \theta < \pi/2$

51. $G(u) = \cosh^{-1}\sqrt{1 + u^2}, \quad u > 0$

52. $y = x \tanh^{-1}x + \ln\sqrt{1 - x^2}$

53. $y = x \sinh^{-1}(x/3) - \sqrt{9 + x^2}$

---

54. Show that $\dfrac{d}{dx} \sqrt[4]{\dfrac{1 + \tanh x}{1 - \tanh x}} = \frac{1}{2}e^{x/2}$.

55. Show that $\dfrac{d}{dx} \arctan(\tanh x) = \operatorname{sech} 2x$.

56. The Gateway Arch The Gateway Arch in St. Louis was designed by Eero Saarinen and was constructed using the equation

$$y = 211.49 - 20.96 \cosh 0.03291765x$$

for the central curve of the arch, where x and y are measured in meters and $|x| \leqslant 91.20$.
(a) Graph the central curve.
(b) What is the height of the arch at its center?
(c) At what points is the height 100 m?
(d) What is the slope of the arch at the points in part (c)?

57. If a water wave with length L moves with velocity v in a body of water with depth d, then

$$v = \sqrt{\frac{gL}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)}$$

where g is the acceleration due to gravity. (See Figure 5.) Explain why the approximation

$$v \approx \sqrt{\frac{gL}{2\pi}}$$

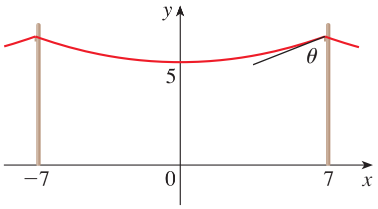
is appropriate in deep water.

58. A flexible cable always hangs in the shape of a catenary $y = c + a\cosh(x/a)$, where c and a are constants and $a > 0$ (see Figure 4 and Exercise 60). Graph several members of the family of functions $y = a\cosh(x/a)$. How does the graph change as a varies?

59. A telephone line hangs between two poles 14 m apart in the shape of the catenary $y = 20\cosh(x/20) - 15$, where x and y are measured in meters.

(a) Find the slope of this curve where it meets the right-hand pole.

(b) Find the angle θ between the line and the pole.



60. Using principles from physics it can be shown that when a cable is hung between two poles, it takes the shape of a curve $y = f(x)$ that satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{\rho g}{T}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

where ρ is the linear density of the cable, g is the acceleration due to gravity, T is the tension in the cable at its lowest point, and the coordinate system is chosen appropriately. Verify that the function

$$y = f(x) = \frac{T}{\rho g}\cosh\left(\frac{\rho g x}{T}\right)$$

is a solution of this differential equation.

61. A cable with linear density $\rho = 2$ kg/m is strung from the tops of two poles that are 200 m apart.

(a) Use Exercise 60 to find the tension T so that the cable is 60 m above the ground at its lowest point. How tall are the poles?

(b) If the tension is doubled, what is the new low point of the cable? How tall are the poles now?

62. A model for the velocity of a falling object after time t is

$$v(t) = \sqrt{\frac{mg}{k}}\tanh\left(t\sqrt{\frac{gk}{m}}\right)$$

where m is the mass of the object, $g = 9.8$ m/s^2 is the acceleration due to gravity, k is a constant, t is measured in seconds, and v in m/s.

(a) Calculate the terminal velocity of the object, that is, $\lim_{t\to\infty} v(t)$.

(b) If a person skydives from a plane, the value of the constant k depends on his or her position. For a "belly-to-earth" position, $k = 0.515$ kg/s, but for a "feet-first" position, $k = 0.067$ kg/s. If a 60-kg person descends in belly-to-earth position, what is the terminal velocity? What about feet-first?

Source: L. Long et al., "How Terminal Is Terminal Velocity?" *American Mathematical Monthly* 113 (2006): 752–55.

63. (a) Show that any function of the form

$$y = A\sinh mx + B\cosh mx$$

satisfies the differential equation $y'' = m^2y$.

(b) Find $y = y(x)$ such that $y'' = 9y$, $y(0) = -4$, and $y'(0) = 6$.

64. If $x = \ln(\sec\theta + \tan\theta)$, show that $\sec\theta = \cosh x$.

65. At what point of the curve $y = \cosh x$ does the tangent have slope 1?

66. Investigate the family of functions

$$f_n(x) = \tanh(n\sin x)$$

where n is a positive integer. Describe what happens to the graph of f_n when n becomes large.

67. Show that if $a \neq 0$ and $b \neq 0$, then there exist numbers α and β such that $ae^x + be^{-x}$ equals either

$$\alpha\sinh(x + \beta) \qquad \text{or} \qquad \alpha\cosh(x + \beta)$$

In other words, almost every function of the form $f(x) = ae^x + be^{-x}$ is a shifted and stretched hyperbolic sine or hyperbolic cosine function.

3 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. State each differentiation rule both in symbols and in words.
  (a) The Power Rule
  (b) The Constant Multiple Rule
  (c) The Sum Rule
  (d) The Difference Rule
  (e) The Product Rule
  (f) The Quotient Rule
  (g) The Chain Rule

2. State the derivative of each function.
  (a) $y = x^n$
  (b) $y = e^x$
  (c) $y = b^x$
  (d) $y = \ln x$
  (e) $y = \log_b x$
  (f) $y = \sin x$
  (g) $y = \cos x$
  (h) $y = \tan x$
  (i) $y = \csc x$
  (j) $y = \sec x$
  (k) $y = \cot x$
  (l) $y = \sin^{-1}x$
  (m) $y = \cos^{-1}x$
  (n) $y = \tan^{-1}x$
  (o) $y = \sinh x$
  (p) $y = \cosh x$
  (q) $y = \tanh x$
  (r) $y = \sinh^{-1}x$
  (s) $y = \cosh^{-1}x$
  (t) $y = \tanh^{-1}x$

3. (a) How is the number e defined?
  (b) Express e as a limit.
  (c) Why is the natural exponential function $y = e^x$ used more often in calculus than the other exponential functions $y = b^x$?
  (d) Why is the natural logarithmic function $y = \ln x$ used more often in calculus than the other logarithmic functions $y = \log_b x$?

4. (a) Explain how implicit differentiation works.
  (b) Explain how logarithmic differentiation works.

5. Give several examples of how the derivative can be interpreted as a rate of change in physics, chemistry, biology, economics, or other sciences.

6. (a) Write a differential equation that expresses the law of natural growth.
  (b) Under what circumstances is this an appropriate model for population growth?
  (c) What are the solutions of this equation?

7. (a) Write an expression for the linearization of f at a.
  (b) If $y = f(x)$, write an expression for the differential dy.
  (c) If $dx = \Delta x$, draw a picture showing the geometric meanings of Δy and dy.

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If f and g are differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

2. If f and g are differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$$

3. If f and g are differentiable, then

$$\frac{d}{dx}\big[f(g(x))\big] = f'(g(x))g'(x)$$

4. If f is differentiable, then $\dfrac{d}{dx}\sqrt{f(x)} = \dfrac{f'(x)}{2\sqrt{f(x)}}$.

5. If f is differentiable, then $\dfrac{d}{dx}f(\sqrt{x}) = \dfrac{f'(x)}{2\sqrt{x}}$.

6. If $y = e^2$, then $y' = 2e$.

7. $\dfrac{d}{dx}(10^x) = x10^{x-1}$

8. $\dfrac{d}{dx}(\ln 10) = \dfrac{1}{10}$

9. $\dfrac{d}{dx}(\tan^2 x) = \dfrac{d}{dx}(\sec^2 x)$

10. $\dfrac{d}{dx}\left|x^2 + x\right| = \left|2x + 1\right|$

11. The derivative of a polynomial is a polynomial.

12. If $f(x) = (x^6 - x^4)^5$, then $f^{(31)}(x) = 0$.

13. The derivative of a rational function is a rational function.

14. An equation of the tangent line to the parabola $y = x^2$ at $(-2, 4)$ is $y - 4 = 2x(x + 2)$.

15. If $g(x) = x^5$, then $\displaystyle\lim_{x \to 2} \frac{g(x) - g(2)}{x - 2} = 80$.

EXERCISES

1–54 Calculate y'.

1. $y = (x^2 + x^3)^4$

2. $y = \dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt[5]{x^3}}$

3. $y = \dfrac{x^2 - x + 2}{\sqrt{x}}$

4. $y = \dfrac{\tan x}{1 + \cos x}$

5. $y = x^2 \sin \pi x$

6. $y = x \cos^{-1} x$

7. $y = \dfrac{t^4 - 1}{t^4 + 1}$

8. $xe^y = y \sin x$

9. $y = \ln(x \ln x)$

10. $y = e^{mx} \cos nx$

11. $y = \sqrt{x} \cos \sqrt{x}$

12. $y = (\arcsin 2x)^2$

13. $y = \dfrac{e^{1/x}}{x^2}$

14. $y = \ln \sec x$

15. $y + x \cos y = x^2 y$

16. $y = \left(\dfrac{u - 1}{u^2 + u + 1}\right)^4$

17. $y = \sqrt{\arctan x}$

18. $y = \cot(\csc x)$

19. $y = \tan\left(\dfrac{t}{1 + t^2}\right)$

20. $y = e^{x \sec x}$

21. $y = 3^{x \ln x}$

22. $y = \sec(1 + x^2)$

23. $y = (1 - x^{-1})^{-1}$

24. $y = 1/\sqrt[3]{x + \sqrt{x}}$

25. $\sin(xy) = x^2 - y$

26. $y = \sqrt{\sin \sqrt{x}}$

27. $y = \log_5(1 + 2x)$

28. $y = (\cos x)^x$

29. $y = \ln \sin x - \frac{1}{2} \sin^2 x$

30. $y = \dfrac{(x^2 + 1)^4}{(2x + 1)^3 (3x - 1)^5}$

31. $y = x \tan^{-1}(4x)$

32. $y = e^{\cos x} + \cos(e^x)$

33. $y = \ln|\sec 5x + \tan 5x|$

34. $y = 10^{\tan \pi\theta}$

35. $y = \cot(3x^2 + 5)$

36. $y = \sqrt{t \ln(t^4)}$

37. $y = \sin\left(\tan \sqrt{1 + x^3}\right)$

38. $y = x \sec^{-1} x$

39. $y = 5 \arctan(1/x)$

40. $y = \sin^{-1}(\cos \theta), \quad 0 < \theta < \pi$

41. $y = x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2)$

42. $y = \ln(\arcsin x^2)$

43. $y = \tan^2(\sin \theta)$

44. $y + \ln y = xy^2$

45. $y = \dfrac{\sqrt{x + 1}\,(2 - x)^5}{(x + 3)^7}$

46. $y = \dfrac{(x + \lambda)^4}{x^4 + \lambda^4}$

47. $y = x \sinh(x^2)$

48. $y = \dfrac{\sin mx}{x}$

49. $y = \ln(\cosh 3x)$

50. $y = \ln\left|\dfrac{x^2 - 4}{2x + 5}\right|$

51. $y = \cosh^{-1}(\sinh x)$

52. $y = x \tanh^{-1} \sqrt{x}$

53. $y = \cos\left(e^{\sqrt{\tan 3x}}\right)$

54. $y = \sin^2\left(\cos \sqrt{\sin \pi x}\right)$

---

55. If $f(t) = \sqrt{4t + 1}$, find $f''(2)$.

56. If $g(\theta) = \theta \sin \theta$, find $g''(\pi/6)$.

57. Find y'' if $x^6 + y^6 = 1$.

58. Find $f^{(n)}(x)$ if $f(x) = 1/(2 - x)$.

59. Use mathematical induction to show that if $f(x) = xe^x$, then $f^{(n)}(x) = (x + n)e^x$. (*Note:* See Principles of Problem Solving following Chapter 1.)

60. Evaluate $\displaystyle\lim_{t \to 0} \frac{t^3}{\tan^3(2t)}$.

61–63 Find an equation of the tangent line to the curve at the given point.

61. $y = 4 \sin^2 x, \quad (\pi/6, 1)$

62. $y = \dfrac{x^2 - 1}{x^2 + 1}, \quad (0, -1)$

63. $y = \sqrt{1 + 4 \sin x}, \quad (0, 1)$

64–65 Find equations of the tangent line and normal line to the curve at the given point.

64. $x^2 + 4xy + y^2 = 13, \quad (2, 1)$

65. $y = (2 + x)e^{-x}, \quad (0, 2)$

---

66. If $f(x) = xe^{\sin x}$, find $f'(x)$. Graph f and f' on the same screen and comment.

67. (a) If $f(x) = x\sqrt{5 - x}$, find $f'(x)$.
(b) Find equations of the tangent lines to the curve $y = x\sqrt{5 - x}$ at the points $(1, 2)$ and $(4, 4)$.
(c) Illustrate part (b) by graphing the curve and tangent lines on the same screen.
(d) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

68. (a) If $f(x) = 4x - \tan x$, $-\pi/2 < x < \pi/2$, find f' and f''.
(b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f, f', and f''.

69. At what points on the curve $y = \sin x + \cos x$, $0 \leqslant x \leqslant 2\pi$, is the tangent line horizontal?

70. Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1.

71. If $f(x) = (x - a)(x - b)(x - c)$, show that

$$\frac{f'(x)}{f(x)} = \frac{1}{x - a} + \frac{1}{x - b} + \frac{1}{x - c}$$

72. (a) By differentiating the double-angle formula

$$\cos 2x = \cos^2 x - \sin^2 x$$

obtain the double-angle formula for the sine function.
(b) By differentiating the addition formula

$$\sin(x + a) = \sin x \cos a + \cos x \sin a$$

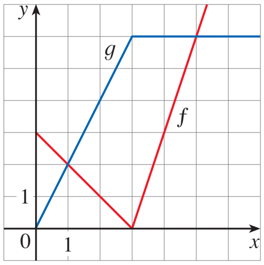
obtain the addition formula for the cosine function.

73. Suppose that

$$\begin{array}{llll} f(1) = 2 & f'(1) = 3 & f(2) = 1 & f'(2) = 2 \\ g(1) = 3 & g'(1) = 1 & g(2) = 1 & g'(2) = 4 \end{array}$$

(a) If $S(x) = f(x) + g(x)$, find $S'(1)$.
(b) If $P(x) = f(x)g(x)$, find $P'(2)$.
(c) If $Q(x) = f(x)/g(x)$, find $Q'(1)$.
(d) If $C(x) = f(g(x))$, find $C'(2)$.

74. If f and g are the functions whose graphs are shown, let $P(x) = f(x)g(x)$, $Q(x) = f(x)/g(x)$, and $C(x) = f(g(x))$. Find (a) $P'(2)$, (b) $Q'(2)$, and (c) $C'(2)$.



75–82 Find f' in terms of g'.

75. $f(x) = x^2 g(x)$

76. $f(x) = g(x^2)$

77. $f(x) = [g(x)]^2$

78. $f(x) = g(g(x))$

79. $f(x) = g(e^x)$

80. $f(x) = e^{g(x)}$

81. $f(x) = \ln|g(x)|$

82. $f(x) = g(\ln x)$

83–85 Find h' in terms of f' and g'.

83. $h(x) = \dfrac{f(x)g(x)}{f(x) + g(x)}$

84. $h(x) = \sqrt{\dfrac{f(x)}{g(x)}}$

85. $h(x) = f(g(\sin 4x))$

---

86. (a) Graph the function $f(x) = x - 2 \sin x$ in the viewing rectangle $[0, 8]$ by $[-2, 8]$.
(b) On which interval is the average rate of change larger: $[1, 2]$ or $[2, 3]$?
(c) At which value of x is the instantaneous rate of change larger: $x = 2$ or $x = 5$?
(d) Check your visual estimates in part (c) by computing $f'(x)$ and comparing the numerical values of $f'(2)$ and $f'(5)$.

87. At what point on the curve

$$y = [\ln(x + 4)]^2$$

is the tangent horizontal?

88. (a) Find an equation of the tangent line to the curve $y = e^x$ that is parallel to the line $x - 4y = 1$.
(b) Find an equation of the tangent to the curve $y = e^x$ that passes through the origin.

89. Find a parabola $y = ax^2 + bx + c$ that passes through the point $(1, 4)$ and whose tangent lines at $x = -1$ and $x = 5$ have slopes 6 and -2, respectively.

90. The function $C(t) = K(e^{-at} - e^{-bt})$, where a, b, and K are positive constants and $b > a$, is used to model the concentration at time t of a drug injected into the bloodstream.
(a) Show that $\lim_{t \to \infty} C(t) = 0$.
(b) Find $C'(t)$, the rate of change of drug concentration in the blood.
(c) When is this rate equal to 0?

91. An equation of motion of the form $s = Ae^{-ct}\cos(\omega t + \delta)$ represents damped oscillation of an object. Find the velocity and acceleration of the object.

92. A particle moves along a horizontal line so that its coordinate at time t is $x = \sqrt{b^2 + c^2 t^2}$, $t \geqslant 0$, where b and c are positive constants.
(a) Find the velocity and acceleration functions.
(b) Show that the particle always moves in the positive direction.

93. A particle moves on a vertical line so that its coordinate at time t is $y = t^3 - 12t + 3$, $t \geq 0$.
(a) Find the velocity and acceleration functions.
(b) When is the particle moving upward and when is it moving downward?
(c) Find the distance that the particle travels in the time interval $0 \leq t \leq 3$.
(d) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 3$.
(e) When is the particle speeding up? When is it slowing down?

94. The volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height.
(a) Find the rate of change of the volume with respect to the height if the radius is constant.
(b) Find the rate of change of the volume with respect to the radius if the height is constant.

95. The mass of part of a wire is $x(1 + \sqrt{x})$ kilograms, where x is measured in meters from one end of the wire. Find the linear density of the wire when $x = 4$ m.

96. The cost, in dollars, of producing x units of a certain commodity is

$$C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3$$

(a) Find the marginal cost function.
(b) Find $C'(100)$ and explain its meaning.
(c) Compare $C'(100)$ with the cost of producing the 101st item.

97. A bacteria culture contains 200 cells initially and grows at a rate proportional to its size. After half an hour the population has increased to 360 cells.
(a) Find the number of cells after t hours.
(b) Find the number of cells after 4 hours.
(c) Find the rate of growth after 4 hours.
(d) When will the population reach 10,000?

98. Cobalt-60 has a half-life of 5.24 years.
(a) Find the mass that remains from a 100-mg sample after 20 years.
(b) How long would it take for the mass to decay to 1 mg?

99. Let $C(t)$ be the concentration of a drug in the bloodstream. As the body eliminates the drug, $C(t)$ decreases at a rate that is proportional to the amount of the drug that is present at the time. Thus $C'(t) = -kC(t)$, where k is a positive number called the *elimination constant* of the drug.
(a) If C_0 is the concentration at time $t = 0$, find the concentration at time t.
(b) If the body eliminates half the drug in 30 hours, how long does it take to eliminate 90% of the drug?

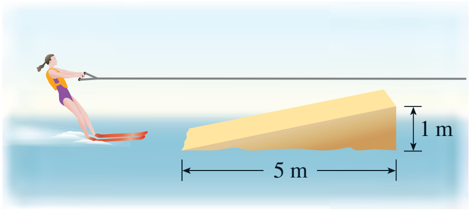
100. A cup of hot chocolate has temperature 80°C in a room kept at 20°C. After half an hour the hot chocolate cools to 60°C.
(a) What is the temperature of the chocolate after another half hour?
(b) When will the chocolate have cooled to 40°C?

101. The volume of a cube is increasing at a rate of 10 cm^3/min. How fast is the surface area increasing when the length of an edge is 30 cm?

102. A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm^3/s, how fast is the water level rising when the water is 5 cm deep?

103. A balloon is rising at a constant speed of 2 m/s. A boy is cycling along a straight road at a speed of 5 m/s. When he passes under the balloon, it is 15 m above him. How fast is the distance between the boy and the balloon increasing 3 s later?

104. A waterskier skis over the ramp shown in the figure at a speed of 10 m/s. How fast is she rising as she leaves the ramp?



105. The angle of elevation of the sun is decreasing at a rate of 0.25 rad/h. How fast is the shadow cast by a 400-meter-tall building increasing when the angle of elevation of the sun is $\pi/6$?

106. (a) Find the linear approximation to $f(x) = \sqrt{25 - x^2}$ near 3.
(b) Illustrate part (a) by graphing f and the linear approximation.
(c) For what values of x is the linear approximation accurate to within 0.1?

107. (a) Find the linearization of $f(x) = \sqrt[3]{1 + 3x}$ at $a = 0$. State the corresponding linear approximation and use it to give an approximate value for $\sqrt[3]{1.03}$.
(b) Determine the values of x for which the linear approximation given in part (a) is accurate to within 0.1.

108. Evaluate dy if $y = x^3 - 2x^2 + 1$, $x = 2$, and $dx = 0.2$.

109. A window has the shape of a square surmounted by a semicircle. The base of the window is measured as having width 60 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum error possible in computing the area of the window.

110–112 Express the limit as a derivative and evaluate.

110. $\lim_{x \to 1} \frac{x^{17} - 1}{x - 1}$

111. $\lim_{h \to 0} \frac{\sqrt[4]{16 + h} - 2}{h}$

112. $\lim_{\theta \to \pi/3} \frac{\cos \theta - 0.5}{\theta - \pi/3}$

113. Evaluate $\lim_{x \to 0} \frac{\sqrt{1 + \tan x} - \sqrt{1 + \sin x}}{x^3}$.

114. Suppose f is a differentiable function such that $f(g(x)) = x$ and $f'(x) = 1 + [f(x)]^2$. Show that $g'(x) = 1/(1 + x^2)$.

115. Find $f'(x)$ if it is known that

$$\frac{d}{dx}[f(2x)] = x^2$$

116. Show that the length of the portion of any tangent line to the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ cut off by the coordinate axes is constant.

Problems Plus

Try to solve the following examples yourself before reading the solutions.

EXAMPLE 1 How many lines are tangent to both of the parabolas $y = -1 - x^2$ and $y = 1 + x^2$? Find the coordinates of the points at which these tangents touch the parabolas.

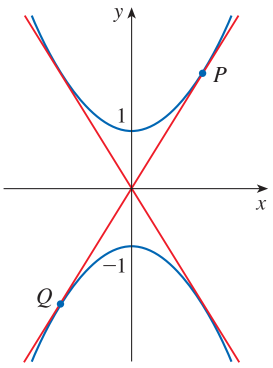


FIGURE 1

SOLUTION To gain insight into this problem, it is essential to draw a diagram. So we sketch the parabolas $y = 1 + x^2$ (which is the standard parabola $y = x^2$ shifted 1 unit upward) and $y = -1 - x^2$ (which is obtained by reflecting the first parabola about the x-axis). If we try to draw a line tangent to both parabolas, we soon discover that there are only two possibilities, as illustrated in Figure 1.

Let P be a point at which one of these tangents touches the upper parabola and let a be its x-coordinate. (The choice of notation for the unknown is important. Of course we could have used b or c or x_0 or x_1 instead of a. However, it's not advisable to use x in place of a because that x could be confused with the variable x in the equation of the parabola.) Then, since P lies on the parabola $y = 1 + x^2$, its y-coordinate must be $1 + a^2$. Because of the symmetry shown in Figure 1, the coordinates of the point Q where the tangent touches the lower parabola must be $(-a, -(1 + a^2))$.

To use the given information that the line is a tangent, we equate the slope of the line PQ to the slope of the tangent line at P. We have

$$m_{PQ} = \frac{1 + a^2 - (-1 - a^2)}{a - (-a)} = \frac{1 + a^2}{a}$$

If $f(x) = 1 + x^2$, then the slope of the tangent line at P is $f'(a) = 2a$. Thus the condition that we need to use is that

$$\frac{1 + a^2}{a} = 2a$$

Solving this equation, we get $1 + a^2 = 2a^2$, so $a^2 = 1$ and $a = \pm 1$. Therefore the points are $(1, 2)$ and $(-1, -2)$. By symmetry, the two remaining points are $(-1, 2)$ and $(1, -2)$. ■

EXAMPLE 2 For what values of c does the equation $\ln x = cx^2$ have exactly one solution?

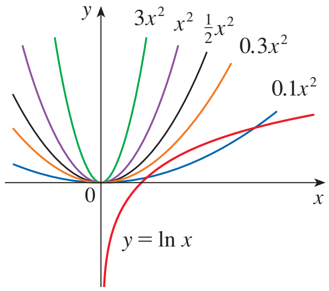


FIGURE 2

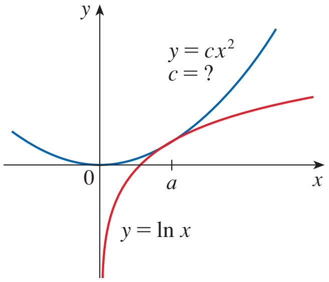


FIGURE 3

SOLUTION One of the most important principles of problem solving is to draw a diagram, even if the problem as stated doesn't explicitly mention a geometric situation. Our present problem can be reformulated geometrically as follows: for what values of c does the curve $y = \ln x$ intersect the curve $y = cx^2$ in exactly one point?

Let's start by graphing $y = \ln x$ and $y = cx^2$ for various values of c. We know that, for $c \neq 0$, $y = cx^2$ is a parabola that opens upward if $c > 0$ and downward if $c < 0$. Figure 2 shows the parabolas $y = cx^2$ for several positive values of c. Most of them don't intersect $y = \ln x$ at all and one intersects twice. We have the feeling that there must be a value of c (somewhere between 0.1 and 0.3) for which the curves intersect exactly once, as in Figure 3.

To find that particular value of c, we let a be the x-coordinate of the single point of intersection. In other words, $\ln a = ca^2$, so a is the unique solution of the given equation. We see from Figure 3 that the curves just touch, so they have a common tangent line when $x = a$. That means the curves $y = \ln x$ and $y = cx^2$ have the same slope when $x = a$. Therefore

$$\frac{1}{a} = 2ca$$

Solving the equations $\ln a = ca^2$ and $1/a = 2ca$, we get

$$\ln a = ca^2 = c \cdot \frac{1}{2c} = \frac{1}{2}$$

Thus $a = e^{1/2}$ and

$$c = \frac{\ln a}{a^2} = \frac{\ln e^{1/2}}{e} = \frac{1}{2e}$$

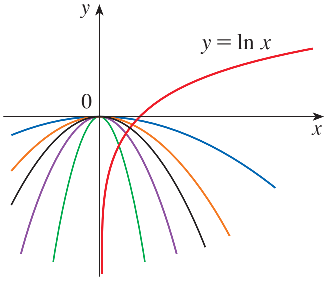


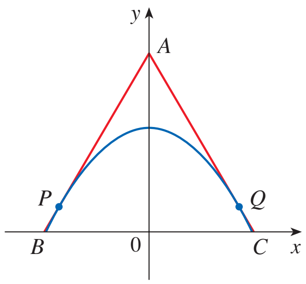
FIGURE 4

For negative values of c we have the situation illustrated in Figure 4: all parabolas $y = cx^2$ with negative values of c intersect $y = \ln x$ exactly once. And let's not forget about $c = 0$: the curve $y = 0x^2 = 0$ is just the x-axis, which intersects $y = \ln x$ exactly once.

To summarize, the required values of c are $c = 1/(2e)$ and $c \leqslant 0$. ■

Problems

1. Find points P and Q on the parabola $y = 1 - x^2$ so that the triangle ABC formed by the x-axis and the tangent lines at P and Q is an equilateral triangle (see the figure).



2. Find the point where the curves $y = x^3 - 3x + 4$ and $y = 3(x^2 - x)$ are tangent to each other, that is, have a common tangent line. Illustrate by sketching both curves and the common tangent.

3. Show that the tangent lines to the parabola $y = ax^2 + bx + c$ at any two points with x-coordinates p and q must intersect at a point whose x-coordinate is halfway between p and q.

4. Show that $\dfrac{d}{dx}\left(\dfrac{\sin^2 x}{1 + \cot x} + \dfrac{\cos^2 x}{1 + \tan x}\right) = -\cos 2x$.

5. If $f(x) = \lim\limits_{t \to x} \dfrac{\sec t - \sec x}{t - x}$, find the value of $f'(\pi/4)$.

6. Find the values of the constants a and b such that

$$\lim_{x \to 0} \frac{\sqrt[3]{ax + b} - 2}{x} = \frac{5}{12}$$

7. Show that $\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x)$.

8. A car is traveling at night along a highway shaped like a parabola with its vertex at the origin (see the figure). The car starts at a point 100 m west and 100 m north of the origin and travels in an easterly direction. There is a statue located 100 m east and 50 m north of the origin. At what point on the highway will the car's headlights illuminate the statue?

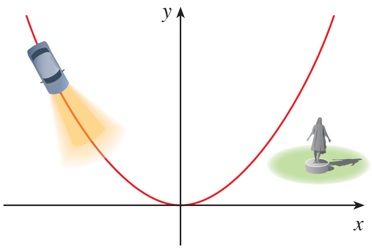


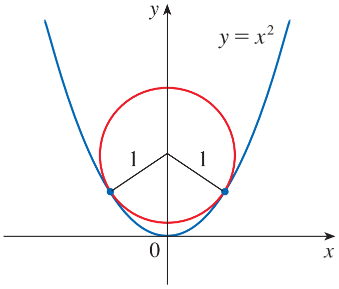
FIGURE FOR PROBLEM 8

9. Prove that $\dfrac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = 4^{n-1} \cos(4x + n\pi/2)$.

10. If f is differentiable at a, where $a > 0$, evaluate the following limit in terms of $f'(a)$:

$$\lim_{x \to a} \frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}}$$

11. The figure shows a circle with radius 1 inscribed in the parabola $y = x^2$. Find the center of the circle.



12. Find all values of c such that the parabolas $y = 4x^2$ and $x = c + 2y^2$ intersect each other at right angles.

13. How many lines are tangent to both of the circles $x^2 + y^2 = 4$ and $x^2 + (y - 3)^2 = 1$? At what points do these tangent lines touch the circles?

14. If $f(x) = \dfrac{x^{46} + x^{45} + 2}{1 + x}$, calculate $f^{(46)}(3)$. Express your answer using factorial notation:

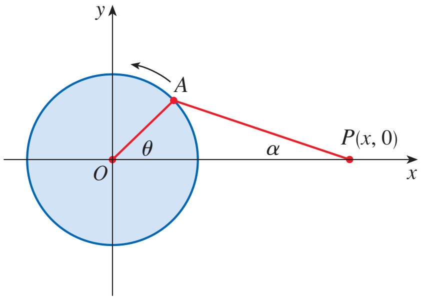
$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n - 1) \cdot n$$

15. The figure shows a rotating wheel with radius 40 cm and a connecting rod AP with length 1.2 m. The pin P slides back and forth along the x-axis as the wheel rotates counterclockwise at a rate of 360 revolutions per minute.

(a) Find the angular velocity of the connecting rod, $d\alpha/dt$, in radians per second, when $\theta = \pi/3$.

(b) Express the distance $x = |OP|$ in terms of θ.

(c) Find an expression for the velocity of the pin P in terms of θ.



16. Tangent lines T_1 and T_2 are drawn at two points P_1 and P_2 on the parabola $y = x^2$ and they intersect at a point P. Another tangent line T is drawn at a point between P_1 and P_2; it intersects T_1 at Q_1 and T_2 at Q_2. Show that

$$\frac{|PQ_1|}{|PP_1|} + \frac{|PQ_2|}{|PP_2|} = 1$$

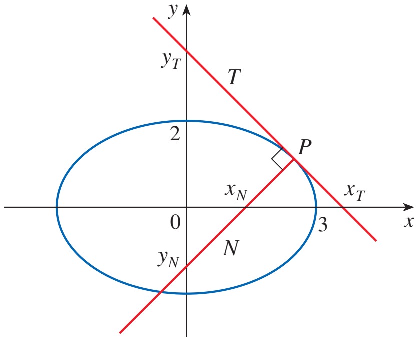
17. Show that

$$\frac{d^n}{dx^n}(e^{ax} \sin bx) = r^n e^{ax} \sin(bx + n\theta)$$

where a and b are positive numbers, $r^2 = a^2 + b^2$, and $\theta = \tan^{-1}(b/a)$.

18. Evaluate $\lim_{x \to \pi} \frac{e^{\sin x} - 1}{x - \pi}$.

19. Let T and N be the tangent and normal lines to the ellipse $x^2/9 + y^2/4 = 1$ at any point P on the ellipse in the first quadrant. Let x_T and y_T be the x- and y-intercepts of T and x_N and y_N be the intercepts of N. As P moves along the ellipse in the first quadrant (but not on the axes), what values can x_T, y_T, x_N, and y_N take on? First try to guess the answers just by looking at the figure. Then use calculus to solve the problem and see how good your intuition is.



20. Evaluate $\lim_{x \to 0} \frac{\sin(3 + x)^2 - \sin 9}{x}$.

21. (a) Use the identity for $\tan(x - y)$ (see Equation 15b in Appendix D) to show that if two lines L_1 and L_2 intersect at an angle α, then

$$\tan \alpha = \frac{m_2 - m_1}{1 + m_1 m_2}$$

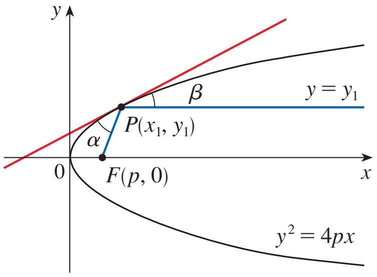
where m_1 and m_2 are the slopes of L_1 and L_2, respectively.

(b) The **angle between the curves** C_1 and C_2 at a point of intersection P is defined to be the angle between the tangent lines to C_1 and C_2 at P (if these tangent lines exist). Use part (a) to find, correct to the nearest degree, the angle between each pair of curves at each point of intersection.

(i) $y = x^2$ and $y = (x - 2)^2$

(ii) $x^2 - y^2 = 3$ and $x^2 - 4x + y^2 + 3 = 0$

22. Let $P(x_1, y_1)$ be a point on the parabola $y^2 = 4px$ with focus $F(p, 0)$. Let α be the angle between the parabola and the line segment FP, and let β be the angle between the horizontal line $y = y_1$ and the parabola as in the figure. Prove that $\alpha = \beta$. (Thus, by a principle of geometrical optics, light from a source placed at F will be reflected along a line parallel to the x-axis. This explains why *paraboloids,* the surfaces obtained by rotating parabolas about their axes, are used as the shape of some automobile headlights and mirrors for telescopes.)



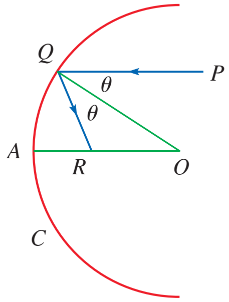


FIGURE FOR PROBLEM 23

23. Suppose that we replace the parabolic mirror of Problem 22 by a spherical mirror. Although the mirror has no focus, we can show the existence of an *approximate* focus. In the figure, C is a semicircle with center O. A ray of light coming in toward the mirror parallel to the axis along the line PQ will be reflected to the point R on the axis so that $\angle PQO = \angle OQR$ (the angle of incidence is equal to the angle of reflection). What happens to the point R as P is taken closer and closer to the axis?

24. If f and g are differentiable functions with $f(0) = g(0) = 0$ and $g'(0) \neq 0$, show that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}$$

25. Evaluate $\displaystyle\lim_{x \to 0} \frac{\sin(a + 2x) - 2\sin(a + x) + \sin a}{x^2}$.

T **26.** (a) The cubic function $f(x) = x(x - 2)(x - 6)$ has three distinct zeros: 0, 2, and 6. Graph f and its tangent lines at the *average* of each pair of zeros. What do you notice?

(b) Suppose the cubic function $f(x) = (x - a)(x - b)(x - c)$ has three distinct zeros: a, b, and c. Prove, with the help of a computer algebra system, that a tangent line drawn at the average of the zeros a and b intersects the graph of f at the third zero.

27. For what value of k does the equation $e^{2x} = k\sqrt{x}$ have exactly one solution?

28. For which positive numbers a is it true that $a^x \geqslant 1 + x$ for all x?

29. If

$$y = \frac{x}{\sqrt{a^2 - 1}} - \frac{2}{\sqrt{a^2 - 1}} \arctan \frac{\sin x}{a + \sqrt{a^2 - 1} + \cos x}$$

show that $y' = \dfrac{1}{a + \cos x}$.

30. Given an ellipse $x^2/a^2 + y^2/b^2 = 1$, where $a \neq b$, find the equation of the set of all points from which there are two tangents to the curve whose slopes are (a) reciprocals and (b) negative reciprocals.

31. Find the two points on the curve $y = x^4 - 2x^2 - x$ that have a common tangent line.

32. Suppose that three points on the parabola $y = x^2$ have the property that their normal lines intersect at a common point. Show that the sum of their x-coordinates is 0.

33. A *lattice point* in the plane is a point with integer coordinates. Suppose that circles with radius r are drawn using all lattice points as centers. Find the smallest value of r such that any line with slope $\frac{2}{5}$ intersects some of these circles.

34. A cone of radius r centimeters and height h centimeters is lowered point first at a rate of 1 cm/s into a tall cylinder of radius R centimeters that is partially filled with water. How fast is the water level rising at the instant the cone is completely submerged?

35. A container in the shape of an inverted cone has height 16 cm and radius 5 cm at the top. It is partially filled with a liquid that oozes through the sides at a rate proportional to the area of the container that is in contact with the liquid. (The surface area of a cone is πrl, where r is the radius and l is the slant height.) If we pour the liquid into the container at a rate of 2 cm^3/min, then the height of the liquid decreases at a rate of 0.3 cm/min when the height is 10 cm. If our goal is to keep the liquid at a constant height of 10 cm, at what rate should we pour the liquid into the container?

The great mathematician Leonard Euler observed "… nothing at all takes place in the universe in which some rule of maximum or minimum does not appear." In Exercise 4.7.53 you will use calculus to show that bees construct the cells in their hive in a shape that minimizes surface area.

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4

Applications of Differentiation

WE HAVE ALREADY INVESTIGATED SOME of the applications of derivatives, but now that we know the differentiation rules we are in a better position to pursue the applications of differentiation in greater depth. Here we learn what derivatives tell us about the shape of a graph of a function and, in particular, how they help us locate maximum and minimum values of functions. Many practical problems require us to minimize a cost or maximize an area or somehow find the best possible outcome of a situation. In particular, we will be able to investigate the optimal shape of a can and to explain the location of rainbows in the sky.

4.1 Maximum and Minimum Values

Some of the most important applications of differential calculus are *optimization problems*, in which we are required to find the optimal (best) way of doing something. Here are examples of such problems that we will solve in this chapter:

- What is the shape of a can that minimizes manufacturing costs?
- What is the maximum acceleration of a spacecraft? (This is an important question for the astronauts who have to withstand the effects of acceleration.)
- What is the radius of a contracted windpipe that expels air most rapidly during a cough?
- At what angle should blood vessels branch so as to minimize the energy expended by the heart in pumping blood?

These problems can be reduced to finding the maximum or minimum values of a function. Let's first explain exactly what we mean by maximum and minimum values.

Absolute and Local Extreme Values

We see that the highest point on the graph of the function f shown in Figure 1 is the point (3, 5). In other words, the largest value of f is $f(3) = 5$. Likewise, the smallest value is $f(6) = 2$. We say that $f(3) = 5$ is the *absolute maximum* of f and $f(6) = 2$ is the *absolute minimum*. In general, we use the following definition.

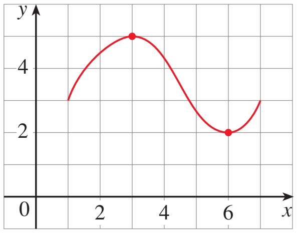


FIGURE 1

1 Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

An absolute maximum or minimum is sometimes called a **global** maximum or minimum. The maximum and minimum values of f are called **extreme values** of f.

Figure 2 shows the graph of a function f with absolute maximum at d and absolute minimum at a. Note that $(d, f(d))$ is the highest point on the graph and $(a, f(a))$ is the lowest point. In Figure 2, if we consider only values of x near b [for instance, if we restrict our attention to the interval (a, c)], then $f(b)$ is the largest of those values of $f(x)$ and is called a *local maximum value* of f. Likewise, $f(c)$ is called a *local minimum value* of f because $f(c) \leqslant f(x)$ for x near c [in the interval (b, d), for instance]. The function f also has a local minimum at e. In general, we have the following definition.

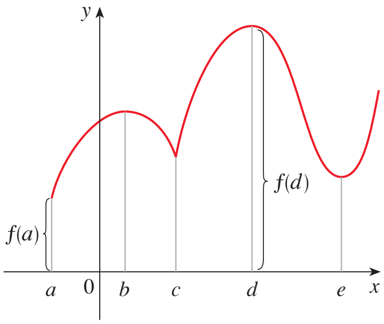


FIGURE 2
Abs min $f(a)$, abs max $f(d)$,
loc min $f(c), f(e)$, loc max $f(b), f(d)$

2 Definition The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geqslant f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leqslant f(x)$ when x is near c.

In Definition 2 (and elsewhere), if we say that something is true **near** c, we mean that it is true on some open interval containing c. (Thus a local maximum or minimum can't occur at an endpoint.) For instance, in Figure 3 we see that $f(4) = 5$ is a local minimum because it's the smallest value of f on the interval I. It's not the absolute minimum because $f(x)$ takes on smaller values when x is near 12 (in the interval K, for instance). In fact $f(12) = 3$ is both a local minimum and the absolute minimum. Similarly, $f(8) = 7$ is a local maximum, but not the absolute maximum because f takes on larger values near 1.

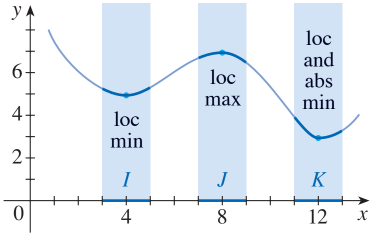


FIGURE 3

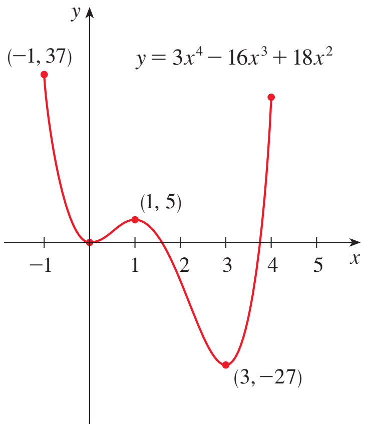


FIGURE 4

EXAMPLE 1 The graph of the function

$$f(x) = 3x^4 - 16x^3 + 18x^2 \qquad -1 \leqslant x \leqslant 4$$

is shown in Figure 4. You can see that $f(1) = 5$ is a local maximum, whereas the absolute maximum is $f(-1) = 37$. (This absolute maximum is not a local maximum because it occurs at an endpoint.) Also, $f(0) = 0$ is a local minimum and $f(3) = -27$ is both a local and an absolute minimum. Note that f has neither a local nor an absolute maximum at $x = 4$. ■

EXAMPLE 2 The function $f(x) = \cos x$ takes on its (local and absolute) maximum value of 1 infinitely many times, because $\cos 2n\pi = 1$ for any integer n and $-1 \leqslant \cos x \leqslant 1$ for all x. (See Figure 5.) Likewise, $\cos(2n + 1)\pi = -1$ is its minimum value, where n is any integer.

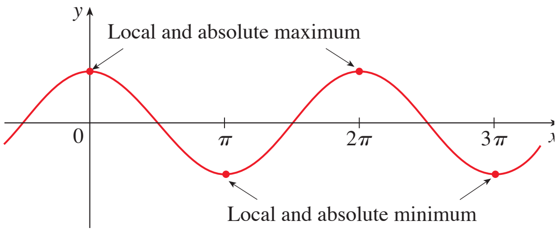


FIGURE 5
$y = \cos x$

■

EXAMPLE 3 If $f(x) = x^2$, then $f(x) \geqslant f(0)$ because $x^2 \geqslant 0$ for all x. Therefore $f(0) = 0$ is the absolute (and local) minimum value of f. This corresponds to the fact that the origin is the lowest point on the parabola $y = x^2$. (See Figure 6.) However, there is no highest point on the parabola and so this function has no maximum value. ■

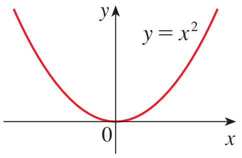


FIGURE 6
Mimimum value 0, no maximum

EXAMPLE 4 From the graph of the function $f(x) = x^3$, shown in Figure 7, we see that this function has neither an absolute maximum value nor an absolute minimum value. In fact, it has no local extreme values either. ■

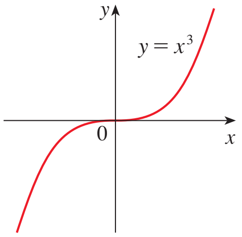


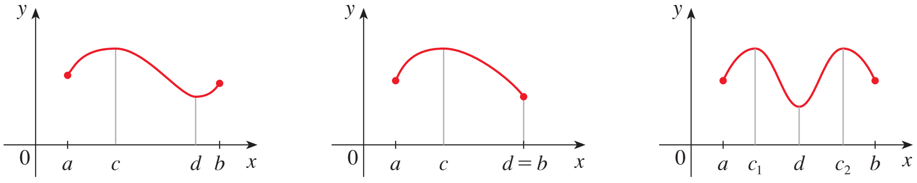
FIGURE 7
No mimimum, no maximum

We have seen that some functions have extreme values, whereas others do not. The following theorem gives conditions under which a function is guaranteed to possess extreme values.

$\boxed{3}$ **The Extreme Value Theorem** If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

The Extreme Value Theorem is illustrated in Figure 8. Note that an extreme value can be taken on more than once. Although the Extreme Value Theorem is intuitively very plausible, it is difficult to prove and so we omit the proof.

FIGURE 8
Functions continuous on a closed interval always attain extreme values.



Figures 9 and 10 show that a function need not possess extreme values if either hypothesis (continuity or closed interval) is omitted from the Extreme Value Theorem.

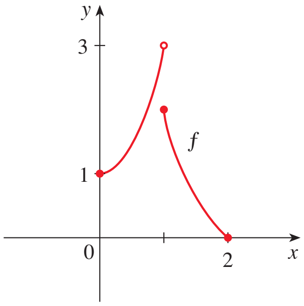


FIGURE 9
This function has minimum value $f(2) = 0$, but no maximum value.

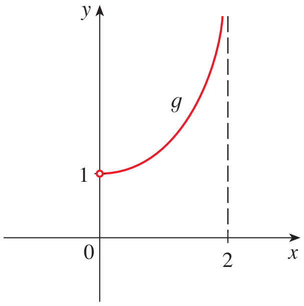


FIGURE 10
This continuous function g has no maximum or minimum.

The function f whose graph is shown in Figure 9 is defined on the closed interval $[0, 2]$ but has no maximum value. (Notice that the range of f is $[0, 3)$. The function takes on values arbitrarily close to 3, but never actually attains the value 3.) This does not contradict the Extreme Value Theorem because f is not continuous. [Nonetheless, a discontinuous function *could* have maximum and minimum values. See Exercise 13(b).]

The function g shown in Figure 10 is continuous on the open interval $(0, 2)$ but has neither a maximum nor a minimum value. [The range of g is $(1, \infty)$. The function takes on arbitrarily large values.] This does not contradict the Extreme Value Theorem because the interval $(0, 2)$ is not closed.

Critical Numbers and the Closed Interval Method

The Extreme Value Theorem says that a continuous function on a closed interval has a maximum value and a minimum value, but it does not tell us how to find these extreme values. Notice in Figure 8 that the absolute maximum and minimum values that are *between* a and b occur at local maximum or minimum values, so we start by looking for local extreme values.

Figure 11 shows the graph of a function f with a local maximum at c and a local minimum at d. It appears that at the maximum and minimum points the tangent lines are horizontal and therefore each has slope 0. We know that the derivative is the slope of the tangent line, so it appears that $f'(c) = 0$ and $f'(d) = 0$. The following theorem says that this is always true for differentiable functions.

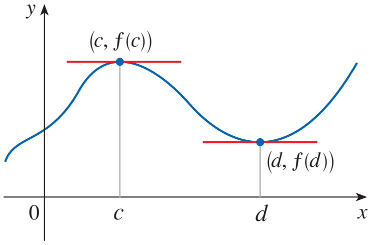


FIGURE 11

4 **Fermat's Theorem** If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

PROOF Suppose, for the sake of definiteness, that f has a local maximum at c. Then, according to Definition 2, $f(c) \geqslant f(x)$ if x is sufficiently close to c. This implies that if h is sufficiently close to 0, with h being positive or negative, then

$$f(c) \geqslant f(c + h)$$

and therefore

$$f(c + h) - f(c) \leqslant 0 \tag{5}$$

Fermat

Fermat's Theorem is named after Pierre Fermat (1601–1665), a French lawyer who took up mathematics as a hobby. Despite his amateur status, Fermat was one of the two inventors of analytic geometry (Descartes was the other). His methods for finding tangents to curves and maximum and minimum values (before the invention of limits and derivatives) made him a forerunner of Newton in the creation of differential calculus.

We can divide both sides of an inequality by a positive number. Thus, if $h > 0$ and h is sufficiently small, we have

$$\frac{f(c+h)-f(c)}{h} \leqslant 0$$

Taking the right-hand limit of both sides of this inequality (using Theorem 2.3.2), we get

$$\lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h} \leqslant \lim_{h \to 0^+} 0 = 0$$

But since $f'(c)$ exists, we have

$$f'(c) = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h}$$

and so we have shown that $f'(c) \leqslant 0$.

If $h < 0$, then the direction of the inequality (5) is reversed when we divide by h:

$$\frac{f(c+h)-f(c)}{h} \geqslant 0$$

So, taking the left-hand limit, we have

$$f'(c) = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h} \geqslant 0$$

We have shown that $f'(c) \geqslant 0$ and also that $f'(c) \leqslant 0$. Since both of these inequalities must be true, the only possibility is that $f'(c) = 0$.

We have proved Fermat's Theorem for the case of a local maximum. The case of a local minimum can be proved in a similar manner, or see Exercise 81 for an alternate method. ■

The following examples caution us against reading too much into Fermat's Theorem: we can't expect to locate extreme values simply by setting $f'(x) = 0$ and solving for x.

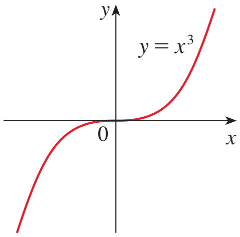


FIGURE 12
If $f(x) = x^3$, then $f'(0) = 0$, but f has no maximum or minimum.

EXAMPLE 5 If $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(0) = 0$. But f has no maximum or minimum at 0, as you can see from its graph in Figure 12. (Or observe that $x^3 > 0$ for $x > 0$ but $x^3 < 0$ for $x < 0$.) The fact that $f'(0) = 0$ simply means that the curve $y = x^3$ has a horizontal tangent at $(0, 0)$. Instead of having a maximum or minimum at $(0, 0)$, the curve crosses its horizontal tangent there. ■

EXAMPLE 6 The function $f(x) = |x|$ has its (local and absolute) minimum value at 0, but that value can't be found by setting $f'(x) = 0$ because, as was shown in Example 2.8.5, $f'(0)$ does not exist. (See Figure 13.) ■

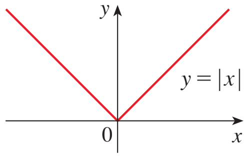


FIGURE 13
If $f(x) = |x|$, then $f(0) = 0$ is a minimum value, but $f'(0)$ does not exist.

WARNING Examples 5 and 6 show that we must be careful when using Fermat's Theorem. Example 5 demonstrates that even when $f'(c) = 0$ there need not be a maximum or minimum at c. (In other words, the converse of Fermat's Theorem is false in general.) Furthermore, there may be an extreme value even when $f'(c)$ does not exist (as in Example 6).

Fermat's Theorem does suggest that we should at least *start* looking for extreme values of f at the numbers c where $f'(c) = 0$ or where $f'(c)$ does not exist. Such numbers are given a special name.

6 Definition A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

EXAMPLE 7 Find the critical numbers of (a) $f(x) = x^3 - 3x^2 + 1$ and (b) $f(x) = x^{3/5}(4 - x)$.

Figure 14 shows a graph of the function f in Example 7(b). It supports our answer because there is a horizontal tangent when $x = 1.5$ [where $f'(x) = 0$] and a vertical tangent when $x = 0$ [where $f'(x)$ is undefined].

SOLUTION

(a) The derivative of f is $f'(x) = 3x^2 - 6x = 3x(x - 2)$. Since $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x) = 0$, that is, when $x = 0$ or $x = 2$.

(b) First note that the domain of f is $\mathbb{R}$. The Product Rule gives

$$f'(x) = x^{3/5}(-1) + (4 - x)\left(\tfrac{3}{5}x^{-2/5}\right) = -x^{3/5} + \frac{3(4 - x)}{5x^{2/5}}$$

$$= \frac{-5x + 3(4 - x)}{5x^{2/5}} = \frac{12 - 8x}{5x^{2/5}}$$

[The same result could be obtained by first writing $f(x) = 4x^{3/5} - x^{8/5}$.] Therefore $f'(x) = 0$ if $12 - 8x = 0$, that is, $x = \frac{3}{2}$, and $f'(x)$ does not exist when $x = 0$. Thus the critical numbers are $\frac{3}{2}$ and 0. ■

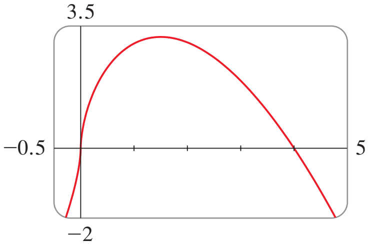


FIGURE 14

In terms of critical numbers, Fermat's Theorem can be rephrased as follows (compare Definition 6 with Theorem 4):

7 If f has a local maximum or minimum at c, then c is a critical number of f.

To find an absolute maximum or minimum of a continuous function on a closed interval, we note that either it is local [in which case it occurs at a critical number by (7)] or it occurs at an endpoint of the interval, as we see from the examples in Figure 8. Thus the following three-step procedure always works.

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

EXAMPLE 8 Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad -\tfrac{1}{2} \leqslant x \leqslant 4$$

SOLUTION Since f is continuous on $\left[-\frac{1}{2}, 4\right]$, we can use the Closed Interval Method.

In Example 7(a) we found the critical numbers $x = 0$ and $x = 2$. Notice that each of these critical numbers lies in the interval $\left(-\frac{1}{2}, 4\right)$. The values of f at these critical numbers are

$$f(0) = 1 \qquad f(2) = -3$$

The values of f at the endpoints of the interval are

$$f\left(-\tfrac{1}{2}\right) = \tfrac{1}{8} \qquad f(4) = 17$$

Comparing these four numbers, we see that the absolute maximum value is $f(4) = 17$ and the absolute minimum value is $f(2) = -3$.

In this example the absolute maximum occurs at an endpoint, whereas the absolute minimum occurs at a critical number. The graph of f is sketched in Figure 15. ■

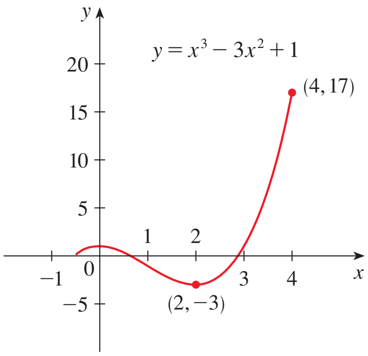


FIGURE 15

With graphing software or a graphing calculator it is possible to estimate maximum and minimum values very easily. But, as the next example shows, calculus is needed to find the *exact* values.

EXAMPLE 9

(a) Use a calculator or computer to estimate the absolute minimum and maximum values of the function $f(x) = x - 2\sin x$, $0 \leqslant x \leqslant 2\pi$.
(b) Use calculus to find the exact minimum and maximum values.

SOLUTION

(a) Figure 16 shows a graph of f in the viewing rectangle $[0, 2\pi]$ by $[-1, 8]$. The absolute maximum value is about 6.97 and it occurs when $x \approx 5.24$. Similarly, the absolute minimum value is about -0.68 and it occurs when $x \approx 1.05$. It is possible to get more accurate numerical estimates, but for exact values we must use calculus.

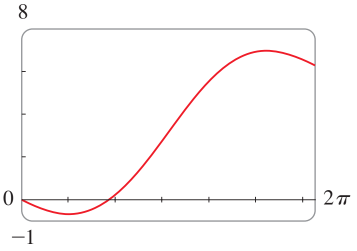


FIGURE 16

(b) The function $f(x) = x - 2\sin x$ is continuous on $[0, 2\pi]$. Since $f'(x) = 1 - 2\cos x$, we have $f'(x) = 0$ when $\cos x = \frac{1}{2}$ and this occurs when $x = \pi/3$ or $5\pi/3$. The values of f at these critical numbers are

$$f(\pi/3) = \frac{\pi}{3} - 2\sin\frac{\pi}{3} = \frac{\pi}{3} - \sqrt{3} \approx -0.684853$$

and $$f(5\pi/3) = \frac{5\pi}{3} - 2\sin\frac{5\pi}{3} = \frac{5\pi}{3} + \sqrt{3} \approx 6.968039$$

The values of f at the endpoints are

$$f(0) = 0 \qquad f(2\pi) = 2\pi \approx 6.28$$

Comparing these four numbers and using the Closed Interval Method, we see that the absolute minimum value is $f(\pi/3) = \pi/3 - \sqrt{3}$ and the absolute maximum value is $f(5\pi/3) = 5\pi/3 + \sqrt{3}$. The values from part (a) serve as a check on our work. ■

NASA

EXAMPLE 10 The Hubble Space Telescope was deployed on April 24, 1990, by the space shuttle *Discovery*. A model for the velocity of the shuttle during this mission, from liftoff at $t = 0$ until the solid rocket boosters were jettisoned at $t = 126$ seconds, is given by

$$v(t) = 0.000397t^3 - 0.02752t^2 + 7.196t - 0.9397$$

(in meters per second). Using this model, estimate the absolute maximum and minimum values of the *acceleration* of the shuttle between liftoff and the jettisoning of the boosters.

SOLUTION We are asked for the extreme values not of the given velocity function, but rather of the acceleration function. So we first need to differentiate to find the acceleration:

$$\begin{aligned} a(t) = v'(t) &= \frac{d}{dt}(0.000397t^3 - 0.02752t^2 + 7.196t - 0.9397) \\ &= 0.001191t^2 - 0.05504t + 7.196 \end{aligned}$$

We now apply the Closed Interval Method to the continuous function a on the interval $0 \leqslant t \leqslant 126$. Its derivative is

$$a'(t) = 0.0023808t - 0.05504$$

The only critical number occurs when $a'(t) = 0$:

$$t_1 = \frac{0.05504}{0.0023808} \approx 23.12$$

Evaluating $a(t)$ at the critical number and at the endpoints, we have

$$a(0) = 7.196 \qquad a(t_1) = a(23.12) = 6.56 \qquad a(126) \approx 19.16$$

So the maximum acceleration is about 19.16 m/s^2 and the minimum acceleration is about 6.56 m/s^2. ■

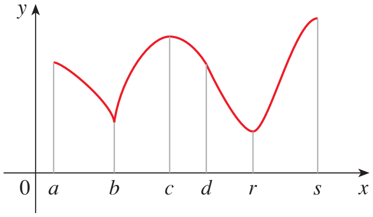
4.1 Exercises

1. Explain the difference between an absolute minimum and a local minimum.

2. Suppose f is a continuous function defined on a closed interval $[a, b]$.
- (a) What theorem guarantees the existence of an absolute maximum value and an absolute minimum value for f?
- (b) What steps would you take to find those maximum and minimum values?

3–4 For each of the numbers a, b, c, d, r, and s, state whether the function whose graph is shown has an absolute maximum or minimum, a local maximum or minimum, or neither a maximum nor a minimum.

3.



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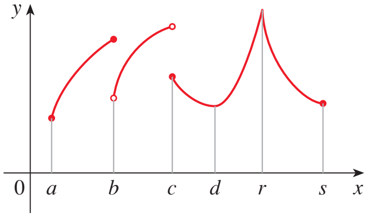
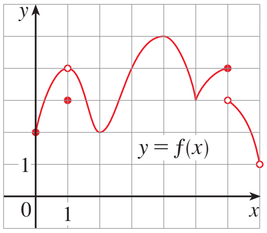
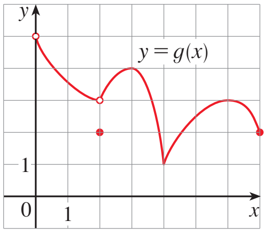


5–6 Use the graph to state the absolute and local maximum and minimum values of the function.

5.



6.



7–10 Sketch the graph of a function f that is continuous on [1, 5] and has the given properties.

7. Absolute maximum at 5, absolute minimum at 2, local maximum at 3, local minima at 2 and 4

8. Absolute maximum at 4, absolute minimum at 5, local maximum at 2, local minimum at 3

9. Absolute minimum at 3, absolute maximum at 4, local maximum at 2

10. Absolute maximum at 2, absolute minimum at 5, 4 is a critical number but there is no local maximum or minimum there.

11. (a) Sketch the graph of a function that has a local maximum at 2 and is differentiable at 2.
- (b) Sketch the graph of a function that has a local maximum at 2 and is continuous but not differentiable at 2.
- (c) Sketch the graph of a function that has a local maximum at 2 and is not continuous at 2.

12. (a) Sketch the graph of a function on $[-1, 2]$ that has an absolute maximum but no local maximum.
(b) Sketch the graph of a function on $[-1, 2]$ that has a local maximum but no absolute maximum.

13. (a) Sketch the graph of a function on $[-1, 2]$ that has an absolute maximum but no absolute minimum.
(b) Sketch the graph of a function on $[-1, 2]$ that is discontinuous but has both an absolute maximum and an absolute minimum.

14. (a) Sketch the graph of a function that has two local maxima, one local minimum, and no absolute minimum.
(b) Sketch the graph of a function that has three local minima, two local maxima, and seven critical numbers.

15–28 Sketch the graph of f by hand and use your sketch to find the absolute and local maximum and minimum values of f. (Use the graphs and transformations of Sections 1.2 and 1.3.)

15. $f(x) = 3 - 2x, \quad x \geqslant -1$

16. $f(x) = x^2, \quad -1 \leqslant x < 2$

17. $f(x) = 1/x, \quad x \geqslant 1$

18. $f(x) = 1/x, \quad 1 < x < 3$

19. $f(x) = \sin x, \quad 0 \leqslant x < \pi/2$

20. $f(x) = \sin x, \quad 0 < x \leqslant \pi/2$

21. $f(x) = \sin x, \quad -\pi/2 \leqslant x \leqslant \pi/2$

22. $f(t) = \cos t, \quad -3\pi/2 \leqslant t \leqslant 3\pi/2$

23. $f(x) = \ln x, \quad 0 < x \leqslant 2$

24. $f(x) = |x|$

25. $f(x) = 1 - \sqrt{x}$

26. $f(x) = e^x$

27. $f(x) = \begin{cases} x^2 & \text{if } -1 \leqslant x \leqslant 0 \\ 2 - 3x & \text{if } 0 < x \leqslant 1 \end{cases}$

28. $f(x) = \begin{cases} 2x + 1 & \text{if } 0 \leqslant x < 1 \\ 4 - 2x & \text{if } 1 \leqslant x \leqslant 3 \end{cases}$

29–48 Find the critical numbers of the function.

29. $f(x) = 3x^2 + x - 2$

30. $g(v) = v^3 - 12v + 4$

31. $f(x) = 3x^4 + 8x^3 - 48x^2$

32. $f(x) = 2x^3 + x^2 + 8x$

33. $g(t) = t^5 + 5t^3 + 50t$

34. $A(x) = |3 - 2x|$

35. $g(y) = \dfrac{y - 1}{y^2 - y + 1}$

36. $h(p) = \dfrac{p - 1}{p^2 + 4}$

37. $p(x) = \dfrac{x^2 + 2}{2x - 1}$

38. $q(t) = \dfrac{t^2 + 9}{t^2 - 9}$

39. $h(t) = t^{3/4} - 2t^{1/4}$

40. $g(x) = \sqrt[3]{4 - x^2}$

41. $F(x) = x^{4/5}(x - 4)^2$

42. $h(x) = x^{-1/3}(x - 2)$

43. $f(x) = x^{1/3}(4 - x)^{2/3}$

44. $f(\theta) = \theta + \sqrt{2} \cos \theta$

45. $f(\theta) = 2 \cos \theta + \sin^2\theta$

46. $p(t) = te^{4t}$

47. $g(x) = x^2 \ln x$

48. $B(u) = 4 \tan^{-1}u - u$

49–50 A formula for the *derivative* of a function f is given. How many critical numbers does f have?

49. $f'(x) = 5e^{-0.1|x|} \sin x - 1$

50. $f'(x) = \dfrac{100 \cos^2 x}{10 + x^2} - 1$

51–66 Find the absolute maximum and absolute minimum values of f on the given interval.

51. $f(x) = 12 + 4x - x^2, \quad [0, 5]$

52. $f(x) = 5 + 54x - 2x^3, \quad [0, 4]$

53. $f(x) = 2x^3 - 3x^2 - 12x + 1, \quad [-2, 3]$

54. $f(x) = x^3 - 6x^2 + 5, \quad [-3, 5]$

55. $f(x) = 3x^4 - 4x^3 - 12x^2 + 1, \quad [-2, 3]$

56. $f(t) = (t^2 - 4)^3, \quad [-2, 3]$

57. $f(x) = x + \dfrac{1}{x}, \quad [0.2, 4]$

58. $f(x) = \dfrac{x}{x^2 - x + 1}, \quad [0, 3]$

59. $f(t) = t - \sqrt[3]{t}, \quad [-1, 4]$

60. $f(x) = \dfrac{e^x}{1 + x^2}, \quad [0, 3]$

61. $f(t) = 2\cos t + \sin 2t, \quad [0, \pi/2]$

62. $f(\theta) = 1 + \cos^2\theta, \quad [\pi/4, \pi]$

63. $f(x) = x^{-2} \ln x, \quad \left[\frac{1}{2}, 4\right]$

64. $f(x) = xe^{x/2}, \quad [-3, 1]$

65. $f(x) = \ln(x^2 + x + 1), \quad [-1, 1]$

66. $f(x) = x - 2 \tan^{-1}x, \quad [0, 4]$

67. If a and b are positive numbers, find the maximum value of $f(x) = x^a(1 - x)^b$, $0 \leqslant x \leqslant 1$.

68. Use a graph to estimate the critical numbers of $f(x) = |1 + 5x - x^3|$ correct to one decimal place.

69–72
(a) Use a graph to estimate the absolute maximum and minimum values of the function to two decimal places.
(b) Use calculus to find the exact maximum and minimum values.

69. $f(x) = x^5 - x^3 + 2, \quad -1 \leqslant x \leqslant 1$

70. $f(x) = e^x + e^{-2x}, \quad 0 \leqslant x \leqslant 1$

71. $f(x) = x\sqrt{x - x^2}$

72. $f(x) = x - 2 \cos x, \quad -2 \leqslant x \leqslant 0$

73. After an alcoholic beverage is consumed, the concentration of alcohol in the bloodstream (blood alcohol concentration, or BAC) surges as the alcohol is absorbed, followed by a gradual decline as the alcohol is metabolized. The function

$$C(t) = 0.135te^{-2.802t}$$

models the average BAC, measured in g/dL, of a group of eight male subjects t hours after rapid consumption of 15 mL of ethanol (corresponding to one alcoholic drink). What is the maximum average BAC during the first 3 hours? When does it occur?

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

74. After an antibiotic tablet is taken, the concentration of the antibiotic in the bloodstream is modeled by the function

$$C(t) = 8(e^{-0.4t} - e^{-0.6t})$$

where the time t is measured in hours and C is measured in μg/mL. What is the maximum concentration of the antibiotic during the first 12 hours?

75. Between 0°C and 30°C, the volume V (in cubic centimeters) of 1 kg of water at a temperature T is given approximately by the formula

$$V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3$$

Find the temperature at which water has its maximum density.

76. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin \theta + \cos \theta}$$

where μ is a positive constant called the *coefficient of friction* and where $0 \leqslant \theta \leqslant \pi/2$. Show that F is minimized when $\tan \theta = \mu$.

77. The water level, measured in meters above mean sea level, of Lake Lanier in Georgia, USA, during 2012 can be modeled by the function

$$L(t) = 0.00439t^3 - 0.1273t^2 + 0.8239t + 323.1$$

where t is measured in months since January 1, 2012. Estimate when the water level was highest during 2012.

T **78.** In 1992 the space shuttle *Endeavour* was launched on mission STS-49 in order to install a new perigee kick motor in an Intelsat communications satellite. The table gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters.

(a) Use a graphing calculator or computer to find the cubic polynomial that best models the velocity of the shuttle for the time interval $t \in [0, 125]$. Then graph this polynomial.

(b) Find a model for the acceleration of the shuttle and use it to estimate the maximum and minimum values of the acceleration during the first 125 seconds.

Event	Time (s)	Velocity (m/s)
Launch	0	0
Begin roll maneuver	10	56.4
End roll maneuver	15	97.2
Throttle to 89%	20	136.2
Throttle to 67%	32	226.2
Throttle to 104%	59	403.9
Maximum dynamic pressure	62	440.4
Solid rocket booster separation	125	1265.2

79. When a foreign object lodged in the trachea forces a person to cough, the diaphragm thrusts upward, causing an increase in pressure in the lungs. This is accompanied by a contraction of the trachea, making a narrower channel for the expelled air to flow through. For a given amount of air to escape in a fixed time, it must move faster through the narrower channel than the wider one. The greater the velocity of the airstream, the greater the force on the foreign object. X-rays show that the radius of the circular tracheal tube contracts to about two-thirds of its normal radius during a cough. According to a mathematical model of coughing, the velocity v of the airstream is related to the radius r of the trachea by the equation

$$v(r) = k(r_0 - r)r^2 \qquad \tfrac{1}{2}r_0 \leqslant r \leqslant r_0$$

where k is a constant and r_0 is the normal radius of the trachea. The restriction on r is due to the fact that the tracheal wall stiffens under pressure and a contraction greater than $\frac{1}{2}r_0$ is prevented (otherwise the person would suffocate).

(a) Determine the value of r in the interval $\left[\frac{1}{2}r_0, r_0\right]$ at which v has an absolute maximum. How does this compare with experimental evidence?

(b) What is the absolute maximum value of v on the interval?

(c) Sketch the graph of v on the interval $[0, r_0]$.

80. Prove that the function

$$f(x) = x^{101} + x^{51} + x + 1$$

has neither a local maximum nor a local minimum.

81. (a) If f has a local minimum value at c, show that the function $g(x) = -f(x)$ has a local maximum value at c.

(b) Use part (a) to prove Fermat's Theorem for the case in which f has a local minimum at c.

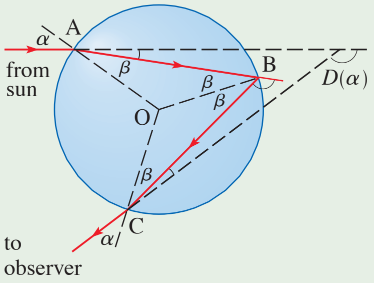
82. A cubic function is a polynomial of degree 3; that is, it has the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$.

(a) Show that a cubic function can have two, one, or no critical number(s). Give examples and sketches to illustrate the three possibilities.

(b) How many local extreme values can a cubic function have?

APPLIED PROJECT | THE CALCULUS OF RAINBOWS

Rainbows are created when raindrops scatter sunlight. They have fascinated mankind since ancient times and have inspired attempts at scientific explanation since the time of Aristotle. In this project we use the ideas of Descartes and Newton to explain the shape, location, and colors of rainbows.

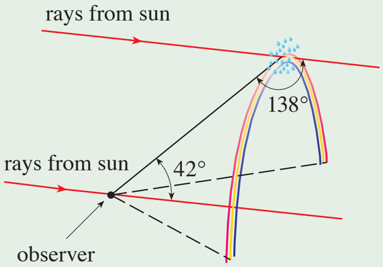


Formation of the primary rainbow

1. The figure shows a ray of sunlight entering a spherical raindrop at A. Some of the light is reflected, but the line AB shows the path of the part that enters the drop. Notice that the light is refracted toward the normal line AO and in fact Snell's Law says that $\sin \alpha = k \sin \beta$, where α is the angle of incidence, β is the angle of refraction, and $k \approx \frac{4}{3}$ is the index of refraction for water. At B some of the light passes through the drop and is refracted into the air, but the line BC shows the part that is reflected. (The angle of incidence equals the angle of reflection.) When the ray reaches C, part of it is reflected, but for the time being we are more interested in the part that leaves the raindrop at C. (Notice that it is refracted away from the normal line.) The *angle of deviation* $D(\alpha)$ is the amount of clockwise rotation that the ray has undergone during this three-stage process. Thus

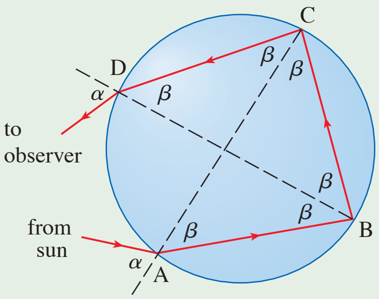
$$D(\alpha) = (\alpha - \beta) + (\pi - 2\beta) + (\alpha - \beta) = \pi + 2\alpha - 4\beta$$

Show that the minimum value of the deviation is $D(\alpha) \approx 138°$ and occurs when $\alpha \approx 59.4°$.



The significance of the minimum deviation is that when $\alpha \approx 59.4°$ we have $D'(\alpha) \approx 0$, so $\Delta D/\Delta\alpha \approx 0$. This means that many rays with $\alpha \approx 59.4°$ become deviated by approximately the same amount. It is the *concentration* of rays coming from near the direction of minimum deviation that creates the brightness of the primary rainbow. The figure at the left shows that the angle of elevation from the observer up to the highest point on the rainbow is $180° - 138° = 42°$. (This angle is called the *rainbow angle.*)

2. Problem 1 explains the location of the primary rainbow, but how do we explain the colors? Sunlight comprises a range of wavelengths, from the red range through orange, yellow, green, blue, indigo, and violet. As Newton discovered in his prism experiments of 1666, the index of refraction is different for each color. (The effect is called *dispersion.*) For red light the refractive index is $k \approx 1.3318$, whereas for violet light it is $k \approx 1.3435$. By repeating the calculation of Problem 1 for these values of k, show that the rainbow angle is about 42.3° for the red bow and 40.6° for the violet bow. So the rainbow really consists of seven individual bows corresponding to the seven colors.



Formation of the secondary rainbow

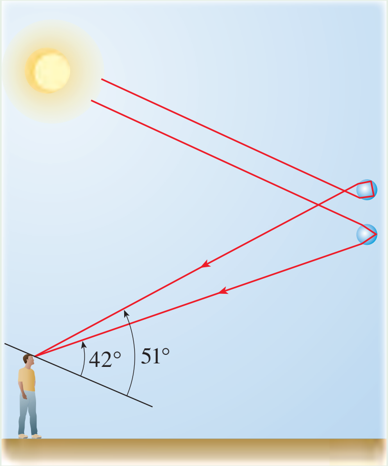
3. Perhaps you have seen a fainter secondary rainbow above the primary bow. That results from the part of a ray that enters a raindrop and is refracted at A, reflected twice (at B and C), and refracted as it leaves the drop at D (see the figure at the left). This time the deviation angle $D(\alpha)$ is the total amount of counterclockwise rotation that the ray undergoes in this four-stage process. Show that

$$D(\alpha) = 2\alpha - 6\beta + 2\pi$$

and $D(a)$ has a minimum value when

$$\cos \alpha = \sqrt{\frac{k^2 - 1}{8}}$$

(*continued*)



Taking $k = \frac{4}{3}$, show that the minimum deviation is about 129° and so the rainbow angle for the secondary rainbow is about 51°, as shown in the figure at the left.

4. Show that the colors in the secondary rainbow appear in the opposite order from those in the primary rainbow.

Leonid Andronov / Shutterstock.com

4.2 The Mean Value Theorem

We will see that many of the results of this chapter depend on one central fact, which is called the Mean Value Theorem.

Rolle's Theorem

To arrive at the Mean Value Theorem, we first need the following result.

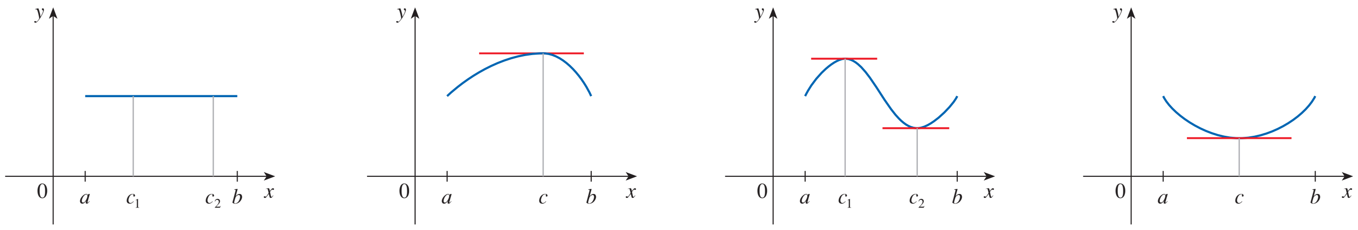
Rolle
Rolle's Theorem was first published i 1691 by the French mathematician Michel Rolle (1652–1719) in a book entitled *Méthode pour resoudre les egalitez*. He was a vocal critic of the methods of his day and attacked calculus as being a "collection of ingenious fallacies." Later, however, he became convinced of the essential correctness of the methods of calculus.

Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

Before giving the proof let's take a look at the graphs of some typical functions that satisfy the three hypotheses. Figure 1 shows the graphs of four such functions. In each case it appears that there is at least one point $(c, f(c))$ on the graph where the tangent is horizontal and therefore $f'(c) = 0$. Thus Rolle's Theorem is plausible.



(a) (b) (c) (d)

FIGURE 1

PS Take cases

PROOF There are three cases:

CASE I $f(x) = k$**, a constant**
Then $f'(x) = 0$, so the number c can be taken to be *any* number in (a, b).

CASE II $f(x) > f(a)$ **for some** x **in** (a, b) [as in Figure 1(b) or (c)]
By the Extreme Value Theorem (which we can apply by hypothesis 1), f has a maximum value somewhere in $[a, b]$. Since $f(a) = f(b)$, it must attain this maximum value at a number c in the open interval (a, b). Then f has a *local* maximum at c and, by hypothesis 2, f is differentiable at c. Therefore $f'(c) = 0$ by Fermat's Theorem.

CASE III $f(x) < f(a)$ **for some** x **in** (a, b) [as in Figure 1(c) or (d)]
By the Extreme Value Theorem, f has a minimum value in $[a, b]$ and, since $f(a) = f(b)$, it attains this minimum value at a number c in (a, b). Again $f'(c) = 0$ by Fermat's Theorem. ■

EXAMPLE 1 Let's apply Rolle's Theorem to the position function $s = f(t)$ of a moving object. If the object is in the same place at two different instants $t = a$ and $t = b$, then $f(a) = f(b)$. Rolle's Theorem says that there is some instant of time $t = c$ between a and b when $f'(c) = 0$; that is, the velocity is 0. (In particular, you can see that this is true when a ball is thrown directly upward.) ■

Figure 2 shows a graph of the function $f(x) = x^3 + x - 1$ discussed in Example 2. Rolle's Theorem shows that, no matter how much we enlarge the viewing rectangle, we can never find a second x-intercept.

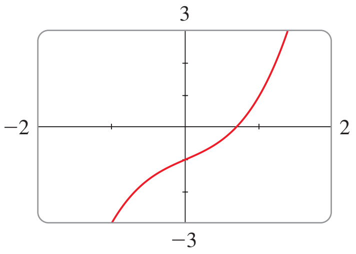


FIGURE 2

EXAMPLE 2 Prove that the equation $x^3 + x - 1 = 0$ has exactly one real solution.

SOLUTION First we use the Intermediate Value Theorem (2.5.10) to show that a solution exists. Let $f(x) = x^3 + x - 1$. Then $f(0) = -1 < 0$ and $f(1) = 1 > 0$. Since f is a polynomial, it is continuous, so the Intermediate Value Theorem states that there is a number c between 0 and 1 such that $f(c) = 0$. Thus the given equation has a solution.

To show that the equation has no other real solution, we use Rolle's Theorem and argue by contradiction. Suppose that it had two solutions a and b. Then $f(a) = 0 = f(b)$ and, since f is a polynomial, it is differentiable on (a, b) and continuous on $[a, b]$. Thus, by Rolle's Theorem, there is a number c between a and b such that $f'(c) = 0$. But

$$f'(x) = 3x^2 + 1 \geqslant 1 \qquad \text{for all } x$$

(since $x^2 \geqslant 0$) so $f'(x)$ can never be 0. This gives a contradiction. Therefore the equation can't have two real solutions. ■

The Mean Value Theorem

Our main use of Rolle's Theorem is in proving the following important theorem, which was first stated by another French mathematician, Joseph-Louis Lagrange.

The Mean Value Theorem is an example of what is called an existence theorem. Like the Intermediate Value Theorem, the Extreme Value Theorem, and Rolle's Theorem, it guarantees that there *exists* a number with a certain property, but it doesn't tell us how to find the number.

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$\boxed{1} \qquad f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$\boxed{2} \qquad f(b) - f(a) = f'(c)(b - a)$$

Before proving this theorem, we can see that it is reasonable by interpreting it geometrically. Figures 3 and 4 show the points $A(a, f(a))$ and $B(b, f(b))$ on the graphs of two differentiable functions. The slope of the secant line AB is

$$\boxed{3} \qquad m_{AB} = \frac{f(b) - f(a)}{b - a}$$

which is the same expression as on the right side of Equation 1. Since $f'(c)$ is the slope of the tangent line at the point $(c, f(c))$, the Mean Value Theorem, in the form given by Equation 1, says that there is at least one point $P(c, f(c))$ on the graph where the slope of the tangent line is the same as the slope of the secant line AB. In other words, there is a point P where the tangent line is parallel to the secant line AB. (Imagine a line far away that stays parallel to AB while moving toward AB until it touches the graph for the first time.)

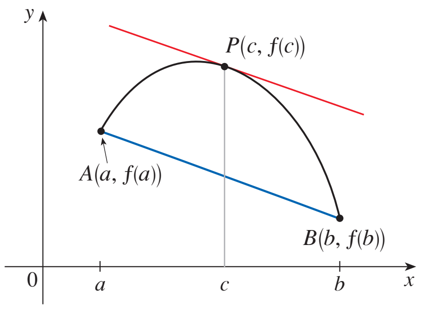


FIGURE 3

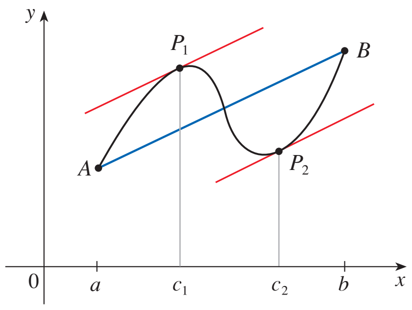


FIGURE 4

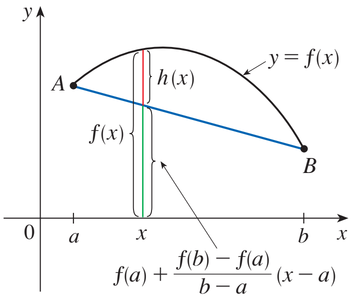


FIGURE 5

PROOF We apply Rolle's Theorem to a new function h defined as the difference between f and the function whose graph is the secant line AB. Using Equation 3 and the point-slope equation of a line, we see that the equation of the line AB can be written as

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

or as

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

So, as shown in Figure 5,

$$\boxed{4} \qquad h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

First we must verify that h satisfies the three hypotheses of Rolle's Theorem.

1. The function h is continuous on $[a, b]$ because it is the sum of f and a first-degree polynomial, both of which are continuous.
2. The function h is differentiable on (a, b) because both f and the first-degree polynomial are differentiable. In fact, we can compute h' directly from Equation 4:

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

(Note that $f(a)$ and $[f(b) - f(a)]/(b - a)$ are constants.)

Lagrange and the Mean Value Theorem

The Mean Value Theorem was first formulated by Joseph-Louis Lagrange (1736–1813), born in Italy of a French father and an Italian mother. He was a child prodigy and became a professor in Turin at the age of 19. Lagrange made great contributions to number theory, theory of functions, theory of equations, and analytical and celestial mechanics. In particular, he applied calculus to the analysis of the stability of the solar system. At the invitation of Frederick the Great, he succeeded Euler at the Berlin Academy and, when Frederick died, Lagrange accepted King Louis XVI's invitation to Paris, where he was given apartments in the Louvre and became a professor at the Ecole Polytechnique. Despite all the trappings of luxury and fame, he was a kind and quiet man, living only for science.

3.
$$h(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0$$

$$\begin{aligned} h(b) &= f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) \\ &= f(b) - f(a) - [f(b) - f(a)] = 0 \end{aligned}$$

Therefore $h(a) = h(b)$.

Since h satisfies all the hypotheses of Rolle's Theorem, that theorem says there is a number c in (a, b) such that $h'(c) = 0$. Therefore

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

and so
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$ ■

EXAMPLE 3 To illustrate the Mean Value Theorem with a specific function, let's consider $f(x) = x^3 - x$, $a = 0$, $b = 2$. Since f is a polynomial, it is continuous and differentiable for all x, so it is certainly continuous on $[0, 2]$ and differentiable on $(0, 2)$. Therefore, by the Mean Value Theorem, there is a number c in $(0, 2)$ such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

Now $f(2) = 6$, $f(0) = 0$, and $f'(x) = 3x^2 - 1$, so the above equation becomes

$$6 = (3c^2 - 1)2 = 6c^2 - 2$$

which gives $c^2 = \frac{4}{3}$, that is, $c = \pm 2/\sqrt{3}$. But c must lie in $(0, 2)$, so $c = 2/\sqrt{3}$. Figure 6 illustrates this calculation: the tangent line at this value of c is parallel to the secant line OB. ■

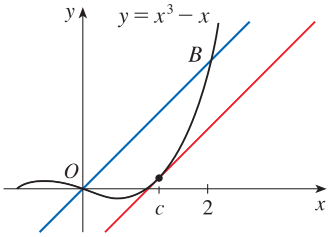


FIGURE 6

EXAMPLE 4 If an object moves in a straight line with position function $s = f(t)$, then the average velocity between $t = a$ and $t = b$ is

$$\frac{f(b) - f(a)}{b - a}$$

and the velocity at $t = c$ is $f'(c)$. Thus the Mean Value Theorem (in the form of Equation 1) tells us that at some time $t = c$ between a and b the instantaneous velocity $f'(c)$ is equal to that average velocity. For instance, if a car traveled 180 km in 2 hours, then the speedometer must have read 90 km/h at least once.

In general, the Mean Value Theorem can be interpreted as saying that there is a number at which the instantaneous rate of change is equal to the average rate of change over an interval. ■

The main significance of the Mean Value Theorem is that it enables us to obtain information about a function from information about its derivative. The next example provides an instance of this principle.

EXAMPLE 5 Suppose that $f(0) = -3$ and $f'(x) \leqslant 5$ for all values of x. How large can $f(2)$ possibly be?

SOLUTION We are given that f is differentiable (and therefore continuous) everywhere. In particular, we can apply the Mean Value Theorem on the interval $[0, 2]$. There exists a number c such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

so

$$f(2) = f(0) + 2f'(c) = -3 + 2f'(c)$$

We are given that $f'(x) \leqslant 5$ for all x, so in particular we know that $f'(c) \leqslant 5$. Multiplying both sides of this inequality by 2, we have $2f'(c) \leqslant 10$, so

$$f(2) = -3 + 2f'(c) \leqslant -3 + 10 = 7$$

The largest possible value for $f(2)$ is 7. ■

The Mean Value Theorem can be used to establish some of the basic facts of differential calculus. One of these basic facts is the following theorem. Others will be discussed in the following sections.

5 Theorem If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

PROOF Let x_1 and x_2 be any two numbers in (a, b) with $x_1 < x_2$. Since f is differentiable on (a, b), it must be differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$. By applying the Mean Value Theorem to f on the interval $[x_1, x_2]$, we get a number c such that $x_1 < c < x_2$ and

$$\boxed{6} \qquad f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Since $f'(x) = 0$ for all x, we have $f'(c) = 0$, and so Equation 6 becomes

$$f(x_2) - f(x_1) = 0 \qquad \text{or} \qquad f(x_2) = f(x_1)$$

Therefore f has the same value at *any* two numbers x_1 and x_2 in (a, b). This means that f is constant on (a, b). ■

Corollary 7 says that if two functions have the same derivatives on an interval, then their graphs must be vertical translations of each other there. In other words, the graphs have the same shape, but they could be shifted up or down.

7 Corollary If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

PROOF Let $F(x) = f(x) - g(x)$. Then

$$F'(x) = f'(x) - g'(x) = 0$$

for all x in (a, b). Thus, by Theorem 5, F is constant; that is, $f - g$ is constant. ■

NOTE Care must be taken in applying Theorem 5. Let

$$f(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The domain of f is $D = \{x \mid x \neq 0\}$ and $f'(x) = 0$ for all x in D. But f is obviously not a constant function. This does not contradict Theorem 5 because D is not an interval.

EXAMPLE 6 Prove the identity $\tan^{-1}x + \cot^{-1}x = \pi/2$.

SOLUTION Although calculus isn't needed to prove this identity, the proof using calculus is quite simple. If $f(x) = \tan^{-1}x + \cot^{-1}x$, then

$$f'(x) = \frac{1}{1 + x^2} - \frac{1}{1 + x^2} = 0$$

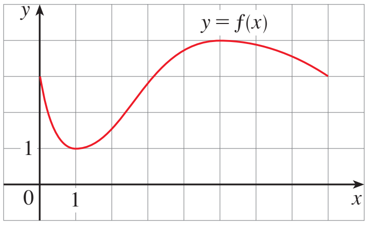
for all values of x. Therefore $f(x) = C$, a constant. To determine the value of C, we put $x = 1$ [because we can evaluate $f(1)$ exactly]. Then

$$C = f(1) = \tan^{-1}1 + \cot^{-1}1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

Thus $\tan^{-1}x + \cot^{-1}x = \pi/2$. ■

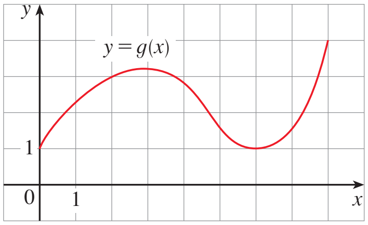
4.2 Exercises

1. The graph of a function f is shown. Verify that f satisfies the hypotheses of Rolle's Theorem on the interval $[0, 8]$. Then estimate the value(s) of c that satisfy the conclusion of Rolle's Theorem on that interval.



2. Draw the graph of a function defined on $[0, 8]$ such that $f(0) = f(8) = 3$ and the function does not satisfy the conclusion of Rolle's Theorem on $[0, 8]$.

3. The graph of a function g is shown.

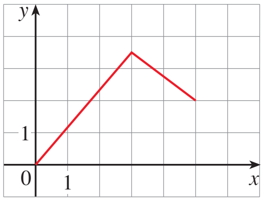


(a) Verify that g satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 8]$.
(b) Estimate the value(s) of c that satisfy the conclusion of the Mean Value Theorem on the interval $[0, 8]$.
(c) Estimate the value(s) of c that satisfy the conclusion of the Mean Value Theorem on the interval $[2, 6]$.

4. Draw the graph of a function that is continuous on $[0, 8]$ where $f(0) = 1$ and $f(8) = 4$ and that does not satisfy the conclusion of the Mean Value Theorem on $[0, 8]$.

5–8 The graph of a function f is shown. Does f satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 5]$? If so, find a value c that satisfies the conclusion of the Mean Value Theorem on that interval.

5.



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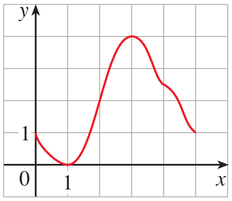

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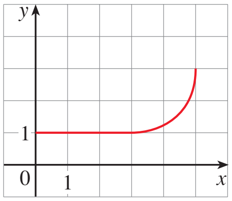

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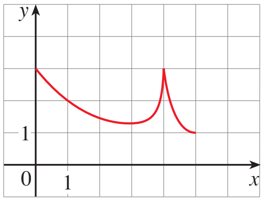


9–12 Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

9. $f(x) = 2x^2 - 4x + 5$, $[-1, 3]$

10. $f(x) = x^3 - 2x^2 - 4x + 2$, $[-2, 2]$

11. $f(x) = \sin(x/2)$, $[\pi/2, 3\pi/2]$

12. $f(x) = x + 1/x$, $\left[\frac{1}{2}, 2\right]$

13. Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1, 1)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

14. Let $f(x) = \tan x$. Show that $f(0) = f(\pi)$ but there is no number c in $(0, \pi)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

15–18 Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

15. $f(x) = 2x^2 - 3x + 1, \quad [0, 2]$

16. $f(x) = x^3 - 3x + 2, \quad [-2, 2]$

17. $f(x) = \ln x, \quad [1, 4]$

18. $f(x) = 1/x, \quad [1, 3]$

19–20 Find the number c that satisfies the conclusion of the Mean Value Theorem on the given interval. Graph the function, the secant line through the endpoints, and the tangent line at $(c, f(c))$. Are the secant line and the tangent line parallel?

19. $f(x) = \sqrt{x}, \quad [0, 4]$

20. $f(x) = e^{-x}, \quad [0, 2]$

---

21. Let $f(x) = (x - 3)^{-2}$. Show that there is no value of c in $(1, 4)$ such that $f(4) - f(1) = f'(c)(4 - 1)$. Why does this not contradict the Mean Value Theorem?

22. Let $f(x) = 2 - |2x - 1|$. Show that there is no value of c such that $f(3) - f(0) = f'(c)(3 - 0)$. Why does this not contradict the Mean Value Theorem?

23–24 Show that the equation has exactly one real solution.

23. $2x + \cos x = 0$

24. $x^3 + e^x = 0$

---

25. Show that the equation $x^3 - 15x + c = 0$ has at most one solution in the interval $[-2, 2]$.

26. Show that the equation $x^4 + 4x + c = 0$ has at most two real solutions.

27. (a) Show that a polynomial of degree 3 has at most three real zeros.
(b) Show that a polynomial of degree n has at most n real zeros.

28. (a) Suppose that f is differentiable on $\mathbb{R}$ and has two zeros. Show that f' has at least one zero.
(b) Suppose f is twice differentiable on $\mathbb{R}$ and has three zeros. Show that f'' has at least one real zero.
(c) Can you generalize parts (a) and (b)?

29. If $f(1) = 10$ and $f'(x) \geqslant 2$ for $1 \leqslant x \leqslant 4$, how small can $f(4)$ possibly be?

30. Suppose that $3 \leqslant f'(x) \leqslant 5$ for all values of x. Show that $18 \leqslant f(8) - f(2) \leqslant 30$.

31. Does there exist a function f such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leqslant 2$ for all x?

32. Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b). Suppose also that $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$. Prove that $f(b) < g(b)$. [*Hint:* Apply the Mean Value Theorem to the function $h = f - g$.]

33. Show that $\sin x < x$ if $0 < x < 2\pi$.

34. Suppose f is an odd function and is differentiable everywhere. Prove that for every positive number b, there exists a number c in $(-b, b)$ such that $f'(c) = f(b)/b$.

35. Use the Mean Value Theorem to prove the inequality

$$|\sin a - \sin b| \leqslant |a - b| \qquad \text{for all } a \text{ and } b$$

36. If $f'(x) = c$ (c a constant) for all x, use Corollary 7 to show that $f(x) = cx + d$ for some constant d.

37. Let $f(x) = 1/x$ and

$$g(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\[2mm] 1 + \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

Show that $f'(x) = g'(x)$ for all x in their domains. Can we conclude from Corollary 7 that $f - g$ is constant?

38–39 Use the method of Example 6 to prove the identity.

38. $\arctan x + \arctan\left(\dfrac{1}{x}\right) = \dfrac{\pi}{2}, \quad x > 0$

39. $2\sin^{-1}x = \cos^{-1}(1 - 2x^2), \quad x \geqslant 0$

---

40. At 2:00 PM a car's speedometer reads 50 km/h. At 2:10 PM it reads 65 km/h. Show that at some time between 2:00 and 2:10 the acceleration is exactly 90 km/h^2.

41. Two runners start a race at the same time and finish in a tie. Prove that at some time during the race they have the same speed. [*Hint:* Consider $f(t) = g(t) - h(t)$, where g and h are the position functions of the two runners.]

42. **Fixed Points** A number a is called a *fixed point* of a function f if $f(a) = a$. Prove that if $f'(x) \neq 1$ for all real numbers x, then f has at most one fixed point.

4.3 What Derivatives Tell Us about the Shape of a Graph

Many of the applications of calculus depend on our ability to deduce facts about a function f from information concerning its derivatives. Because $f'(x)$ represents the slope of the curve $y = f(x)$ at the point $(x, f(x))$, it tells us the direction in which the curve proceeds at each point. So it is reasonable to expect that information about $f'(x)$ will provide us with information about $f(x)$.

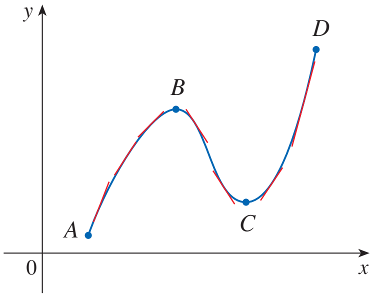


FIGURE 1

What Does f' Say about f?

To see how the derivative of f can tell us where a function is increasing or decreasing, look at Figure 1. (Increasing functions and decreasing functions were defined in Section 1.1.) Between A and B and between C and D, the tangent lines have positive slope and so $f'(x) > 0$. Between B and C, the tangent lines have negative slope and so $f'(x) < 0$. Thus it appears that f increases when $f'(x)$ is positive and decreases when $f'(x)$ is negative. To prove that this is always the case, we use the Mean Value Theorem.

Notation
Let's abbreviate the name of this test to the I/D Test.

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

PROOF

(a) Let x_1 and x_2 be any two numbers in the interval with $x_1 < x_2$. According to the definition of an increasing function (Section 1.1), we have to show that $f(x_1) < f(x_2)$.

Because we are given that $f'(x) > 0$, we know that f is differentiable on $[x_1, x_2]$. So, by the Mean Value Theorem, there is a number c between x_1 and x_2 such that

$$\boxed{1} \qquad f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Now $f'(c) > 0$ by assumption and $x_2 - x_1 > 0$ because $x_1 < x_2$. Thus the right side of Equation 1 is positive, and so

$$f(x_2) - f(x_1) > 0 \qquad \text{or} \qquad f(x_1) < f(x_2)$$

This shows that f is increasing.

Part (b) is proved similarly. ■

EXAMPLE 1 Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

SOLUTION We start by differentiating f:

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x - 2)(x + 1)$$

FIGURE 2

To use the I/D Test we have to know where $f'(x) > 0$ and where $f'(x) < 0$. To solve these inequalities we first find where $f'(x) = 0$, namely at $x = 0$, 2, and -1. These are the critical numbers of f, and they divide the domain into four intervals (see the number line in Figure 2). Within each interval, $f'(x)$ must be always positive or always negative. (See Examples 3 and 4 in Appendix A.) We can determine which is the case for each interval from the signs of the three factors of $f'(x)$, namely, $12x$, $x - 2$, and $x + 1$, as shown in the following chart. A plus sign indicates that the given expression is positive, and a minus sign indicates that it is negative. The last column of the chart gives the conclusion based on the I/D Test. For instance,

$f'(x) < 0$ for $0 < x < 2$, so f is decreasing on $(0, 2)$. (It would also be true to say that f is decreasing on the closed interval $[0, 2]$.)

Interval	$12x$	$x - 2$	$x + 1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$

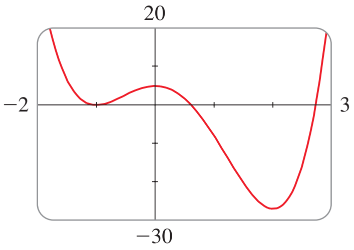


FIGURE 3

The graph of f shown in Figure 3 confirms the information in the chart. ■

The First Derivative Test

Recall from Section 4.1 that if f has a local maximum or minimum at c, then c must be a critical number of f (by Fermat's Theorem), but not every critical number gives rise to a maximum or a minimum. We therefore need a test that will tell us whether or not f has a local maximum or minimum at a critical number.

You can see from Figure 3 that for the function f in Example 1, $f(0) = 5$ is a local maximum value of f because f increases on $(-1, 0)$ and decreases on $(0, 2)$. Or, in terms of derivatives, $f'(x) > 0$ for $-1 < x < 0$ and $f'(x) < 0$ for $0 < x < 2$. In other words, the sign of $f'(x)$ changes from positive to negative at 0. This observation is the basis of the following test.

The First Derivative Test Suppose that c is a critical number of a continuous function f.

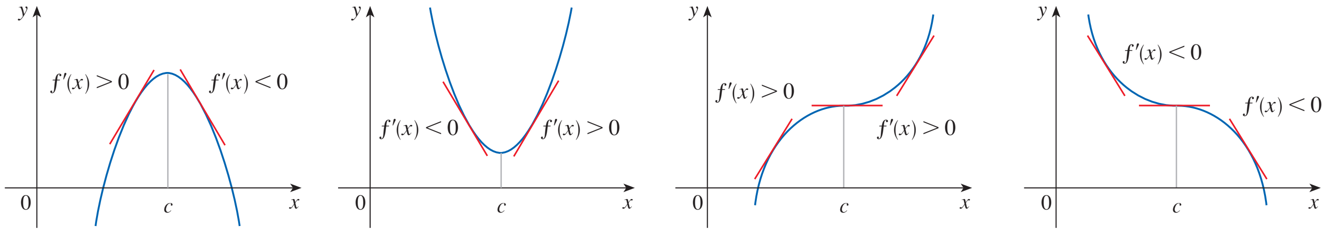
(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

The First Derivative Test is a consequence of the I/D Test. In part (a), for instance, because the sign of $f'(x)$ changes from positive to negative at c, f is increasing to the left of c and decreasing to the right of c. It follows that f has a local maximum at c.

It is easy to remember the First Derivative Test by visualizing diagrams such as those in Figure 4.



(a) Local maximum at c (b) Local minimum at c (c) No maximum or minimum at c (d) No maximum or minimum at c

FIGURE 4

EXAMPLE 2 Find the local minimum and maximum values of the function f in Example 1.

SOLUTION From the chart in the solution to Example 1 we see that $f'(x)$ changes from negative to positive at -1, so $f(-1) = 0$ is a local minimum value by the First Derivative Test. Similarly, f' changes from negative to positive at 2, so $f(2) = -27$ is also a local minimum value. As noted previously, $f(0) = 5$ is a local maximum value because $f'(x)$ changes from positive to negative at 0. ■

EXAMPLE 3 Find the local maximum and minimum values of the function

$$g(x) = x + 2\sin x \qquad 0 \leqslant x \leqslant 2\pi$$

SOLUTION As in Example 1, we start by finding the critical numbers. The derivative is:

$$g'(x) = 1 + 2\cos x$$

so $g'(x) = 0$ when $\cos x = -\frac{1}{2}$. The solutions of this equation are $2\pi/3$ and $4\pi/3$. Because g is differentiable everywhere, the only critical numbers are $2\pi/3$ and $4\pi/3$. We split the domain into intervals according to the critical numbers. Within each interval, $g'(x)$ is either always positive or always negative and so we analyze g in the following chart.

The + signs in the chart come from the fact that $g'(x) > 0$ when $\cos x > -\frac{1}{2}$. From the graph of $y = \cos x$, this is true in the indicated intervals. Alternatively, we can choose a test value within each interval and check the sign of $g'(x)$ using that value.

Interval	$g'(x) = 1 + 2\cos x$	g
$0 < x < 2\pi/3$	+	increasing on $(0, 2\pi/3)$
$2\pi/3 < x < 4\pi/3$	–	decreasing on $(2\pi/3, 4\pi/3)$
$4\pi/3 < x < 2\pi$	+	increasing on $(4\pi/3, 2\pi)$

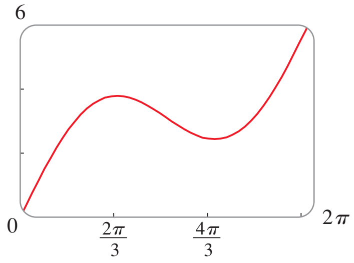


FIGURE 5
$g(x) = x + 2\sin x$

Because $g'(x)$ changes from positive to negative at $2\pi/3$, the First Derivative Test tells us that there is a local maximum at $2\pi/3$ and the local maximum value is

$$g(2\pi/3) = \frac{2\pi}{3} + 2\sin\frac{2\pi}{3} = \frac{2\pi}{3} + 2\left(\frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3} + \sqrt{3} \approx 3.83$$

Likewise, $g'(x)$ changes from negative to positive at $4\pi/3$ and so

$$g(4\pi/3) = \frac{4\pi}{3} + 2\sin\frac{4\pi}{3} = \frac{4\pi}{3} + 2\left(-\frac{\sqrt{3}}{2}\right) = \frac{4\pi}{3} - \sqrt{3} \approx 2.46$$

is a local minimum value. The graph of g in Figure 5 supports our conclusion. ■

■ What Does f'' Say about f?

Figure 6 shows the graphs of two increasing functions on (a, b). Both graphs join point A to point B but they look different because they bend in different directions. How can we distinguish between these two types of behavior?

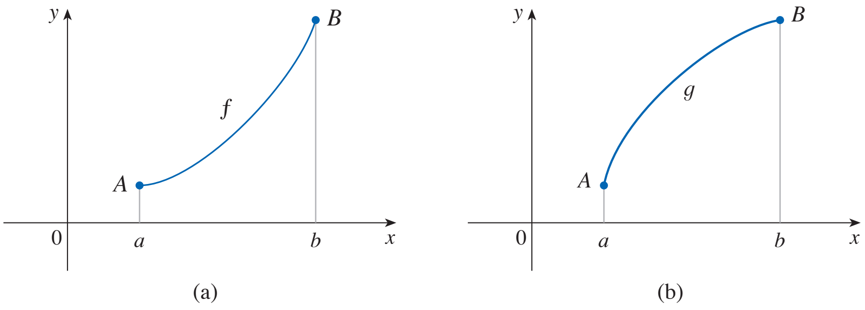


FIGURE 6

In Figure 7 tangents to these curves have been drawn at several points. In part (a) the curve lies above the tangents and f is called *concave upward* on (a, b). In part (b) the curve lies below the tangents and g is called *concave downward* on (a, b).

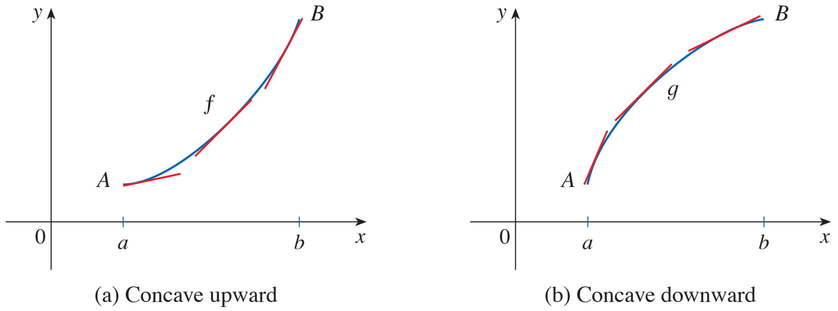


FIGURE 7

(a) Concave upward (b) Concave downward

Notation
We use the abbreviation CU for concave upward and CD for concave downward.

Definition If the graph of f lies above all of its tangents on an interval I, then f is called **concave upward** on I. If the graph of f lies below all of its tangents on I, then f is called **concave downward** on I.

Figure 8 shows the graph of a function that is concave upward (CU) on the intervals (b, c), (d, e), and (e, p) and concave downward (CD) on the intervals (a, b), (c, d), and (p, q).

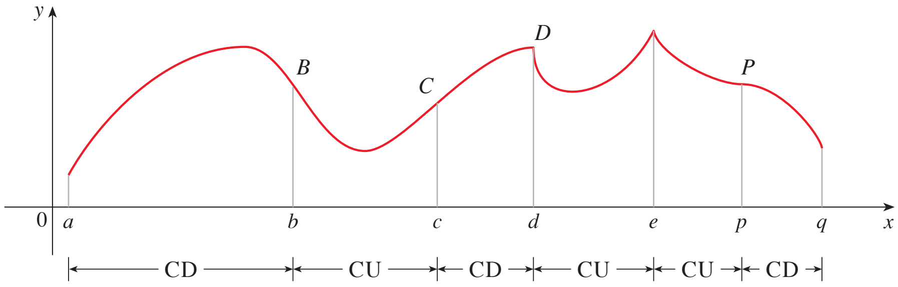


FIGURE 8

Let's see how the second derivative helps determine the intervals of concavity. Looking at Figure 7(a), you can see that, going from left to right, the slope of the tangent increases. This means that the derivative f' is an increasing function and therefore its derivative f'' is positive. Likewise, in Figure 7(b) the slope of the tangent decreases from left to right, so f' decreases and therefore f'' is negative. This reasoning can be reversed and suggests that the following theorem is true. A proof is given in Appendix F with the help of the Mean Value Theorem.

Concavity Test

(a) If $f''(x) > 0$ on an interval I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ on an interval I, then the graph of f is concave downward on I.

EXAMPLE 4 Figure 9 shows a population graph for honeybees raised in an apiary. How does the rate of population growth change over time? When is this rate highest? Over what intervals is P concave upward or concave downward?

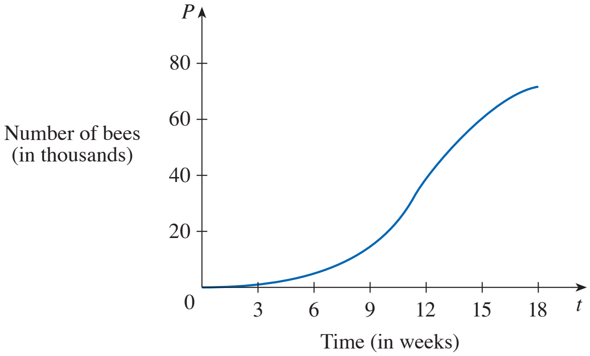


FIGURE 9

SOLUTION By looking at the slope of the curve as t increases, we see that the rate of growth of the population is initially very small, then gets larger until it reaches a maximum at about $t = 12$ weeks, and decreases as the population begins to level off. As the population approaches its maximum value of about 75,000 (called the *carrying capacity*), the rate of growth, $P'(t)$, approaches 0. The curve appears to be concave upward on $(0, 12)$ and concave downward on $(12, 18)$. ■

In Example 4, the population curve changed from concave upward to concave downward at approximately the point $(12, 38{,}000)$. This point is called an *inflection point* of the curve. The significance of this point is that the rate of population increase has its maximum value there. In general, an inflection point is a point where a curve changes its direction of concavity.

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downwardor from concave downward to concave upward at P.

For instance, in Figure 8, B, C, D, and P are the points of inflection. Notice that if a curve has a tangent at a point of inflection, then the curve crosses its tangent there.

In view of the Concavity Test, there is a point of inflection at any point where the function is continuous and the second derivative changes sign.

EXAMPLE 5 Sketch a possible graph of a function f that satisfies the following conditions:

(i) $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$

(ii) $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $f''(x) < 0$ on $(-2, 2)$

(iii) $\lim_{x \to -\infty} f(x) = -2$, $\lim_{x \to \infty} f(x) = 0$

SOLUTION Condition (i) tells us that f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. Condition (ii) says that f is concave upward on $(-\infty, -2)$ and $(2, \infty)$, and concave downward on $(-2, 2)$. From condition (iii) we know that the graph of f has two horizontal asymptotes: $y = -2$ (to the left) and $y = 0$ (to the right).

We first draw the horizontal asymptote $y = -2$ as a dashed line (see Figure 10). We then draw the graph of f approaching this asymptote at the far left, increasing to its maximum point at $x = 1$, and decreasing toward the x-axis at the far right. We also make sure that the graph has inflection points when $x = -2$ and 2. Notice that we made the curve bend upward for $x < -2$ and $x > 2$, and bend downward when x is between -2 and 2. ■

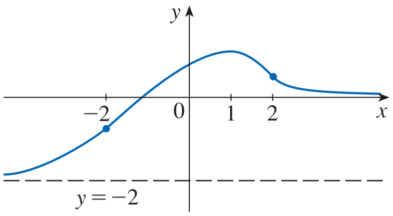


FIGURE 10

The Second Derivative Test

Another application of the second derivative is the following test for identifying local maximum and minimum values. It is a consequence of the Concavity Test, and it serves as an alternative to the First Derivative Test.

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

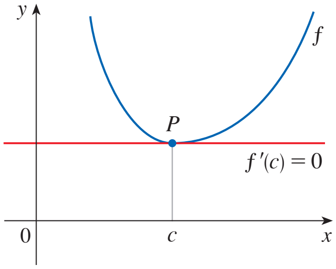


FIGURE 11
$f''(c) > 0$, f is concave upward

For instance, part (a) is true because $f''(x) > 0$ near c and so f is concave upward near c. This means that the graph of f lies *above* its horizontal tangent at c and so f has a local minimum at c. (See Figure 11.)

NOTE The Second Derivative Test is inconclusive when $f''(c) = 0$. In other words, at such a point there might be a maximum, there might be a minimum, or there might be neither. This test also fails when $f''(c)$ does not exist. In such cases the First Derivative Test must be used. In fact, even when both tests apply, the First Derivative Test is often the easier one to use.

EXAMPLE 6 Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima.

SOLUTION If $f(x) = x^4 - 4x^3$, then

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$
$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

To find the critical numbers we set $f'(x) = 0$ and obtain $x = 0$ and $x = 3$. (Note that f' is a polynomial and hence defined everywhere.) To use the Second Derivative Test we evaluate f'' at these critical numbers:

$$f''(0) = 0 \qquad f''(3) = 36 > 0$$

Since $f'(3) = 0$ and $f''(3) > 0$, the Second Derivative Test tells us that $f(3) = -27$ is a local minimum. Because $f''(0) = 0$, the Second Derivative Test gives no information about the critical number 0. But since $f'(x) < 0$ for $x < 0$ and also for $0 < x < 3$, the First Derivative Test tells us that f does not have a local maximum or minimum at 0.

Since $f''(x) = 0$ when $x = 0$ or 2, we divide the real line into intervals with these numbers as endpoints and complete the following chart.

Interval	$f''(x) = 12x(x - 2)$	Concavity
$(-\infty, 0)$	$+$	upward
$(0, 2)$	$-$	downward
$(2, \infty)$	$+$	upward

The point $(0, 0)$ is an inflection point because the curve changes from concave upward to concave downward there. Also $(2, -16)$ is an inflection point because the curve changes from concave downward to concave upward there.

The graph of $y = x^4 - 4x^3$ in Figure 12 supports our conclusions. ■

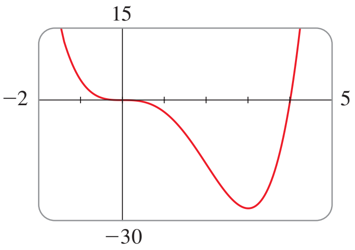


FIGURE 12
$y = x^4 - 4x^3$

■ Curve Sketching

We now use the information we obtain from the first and second derivatives to sketch the graph of a function.

EXAMPLE 7 Sketch the graph of the function $f(x) = x^{2/3}(6 - x)^{1/3}$.

SOLUTION First note that the domain of f is $\mathbb{R}$. Calculation of the first two derivatives gives

Use the differentiation rules to check these calculations.

$$f'(x) = \frac{4 - x}{x^{1/3}(6 - x)^{2/3}} \qquad f''(x) = \frac{-8}{x^{4/3}(6 - x)^{5/3}}$$

Since $f'(x) = 0$ when $x = 4$ and $f'(x)$ does not exist when $x = 0$ or $x = 6$, the critical numbers are 0, 4, and 6.

Interval	$4 - x$	$x^{1/3}$	$(6 - x)^{2/3}$	$f'(x)$	f
$x < 0$	$+$	$-$	$+$	$-$	decreasing on $(-\infty, 0)$
$0 < x < 4$	$+$	$+$	$+$	$+$	increasing on $(0, 4)$
$4 < x < 6$	$-$	$+$	$+$	$-$	decreasing on $(4, 6)$
$x > 6$	$-$	$+$	$+$	$-$	decreasing on $(6, \infty)$

To find the local extreme values we use the First Derivative Test. Since f' changes from negative to positive at 0, $f(0) = 0$ is a local minimum. Since f' changes from positive to negative at 4, $f(4) = 2^{5/3}$ is a local maximum. The sign of f' does not change at 6, so there is no minimum or maximum there. (The Second Derivative Test could be used at 4 but not at 0 or 6 because f'' does not exist at either of these numbers.)

Looking at the expression for $f''(x)$ and noting that $x^{4/3} \geqslant 0$ for all x, we have $f''(x) < 0$ for $x < 0$ and for $0 < x < 6$ and $f''(x) > 0$ for $x > 6$. So f is concave downward on $(-\infty, 0)$ and $(0, 6)$ and concave upward on $(6, \infty)$, and the only inflection point is $(6, 0)$. Using all of the information we gathered about f from its first and second derivatives, we sketch the graph in Figure 13. Note that the curve has vertical tangents at $(0, 0)$ and $(6, 0)$ because $|f'(x)| \to \infty$ as $x \to 0$ and as $x \to 6$.

Try reproducing the graph in Figure 13 with a graphing calculator or computer. Some machines produce the complete graph, some produce only the portion to the right of the y-axis, and some produce only the portion between $x = 0$ and $x = 6$. For an explanation, see Example 7 in Graphing Calculators and Computers at www.StewartCalculus.com.

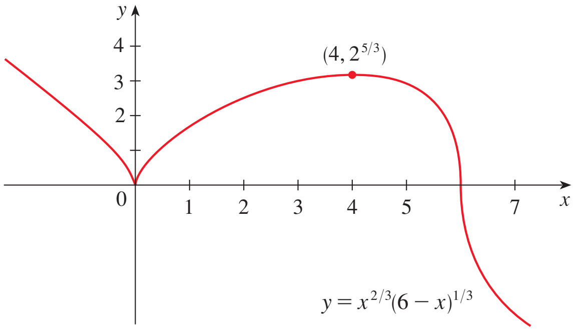


FIGURE 13

■

EXAMPLE 8 Use the first and second derivatives of $f(x) = e^{1/x}$, together with asymptotes, to sketch its graph.

SOLUTION Notice that the domain of f is $\{x \mid x \neq 0\}$, so we check for vertical asymptotes by computing the left and right limits as $x \to 0$. As $x \to 0^+$, we know that $t = 1/x \to \infty$, so

$$\lim_{x \to 0^+} e^{1/x} = \lim_{t \to \infty} e^t = \infty$$

and this shows that $x = 0$ is a vertical asymptote. As $x \to 0^-$, we have $t = 1/x \to -\infty$, so

$$\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0$$

As $x \to \pm\infty$, we have $1/x \to 0$ and so

$$\lim_{x \to \pm\infty} e^{1/x} = e^0 = 1$$

This shows that $y = 1$ is a horizontal asymptote (to both the left and right).

Now let's compute the derivative. The Chain Rule gives

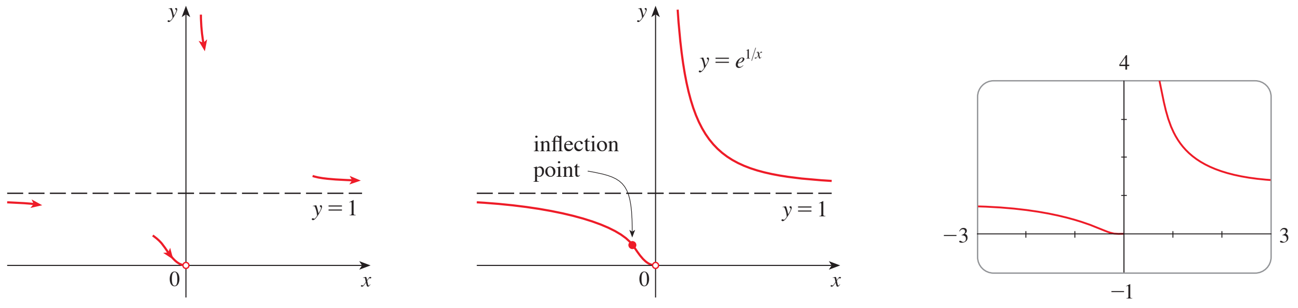
$$f'(x) = -\frac{e^{1/x}}{x^2}$$

Since $e^{1/x} > 0$ and $x^2 > 0$ for all $x \neq 0$, we have $f'(x) < 0$ for all $x \neq 0$. Thus f is decreasing on $(-\infty, 0)$ and on $(0, \infty)$. There is no critical number, so the function has no local maximum or minimum. The second derivative is

$$f''(x) = -\frac{x^2 e^{1/x}(-1/x^2) - e^{1/x}(2x)}{x^4} = \frac{e^{1/x}(2x + 1)}{x^4}$$

Since $e^{1/x} > 0$ and $x^4 > 0$, we have $f''(x) > 0$ when $x > -\frac{1}{2}$ $(x \neq 0)$ and $f''(x) < 0$ when $x < -\frac{1}{2}$. So the curve is concave downward on $\left(-\infty, -\frac{1}{2}\right)$ and concave upward on $\left(-\frac{1}{2}, 0\right)$ and on $(0, \infty)$. There is one inflection point: $\left(-\frac{1}{2}, e^{-2}\right)$.

To sketch the graph of f we first draw the horizontal asymptote $y = 1$ (as a dashed line), together with the parts of the curve near the asymptotes in a preliminary sketch [Figure 14(a)]. These parts reflect the information concerning limits and the fact that f is decreasing on both $(-\infty, 0)$ and $(0, \infty)$. Notice that we have indicated that $f(x) \to 0$ as $x \to 0^-$ even though $f(0)$ does not exist. In Figure 14(b) we finish the sketch by incorporating the information concerning concavity and the inflection point. In Figure 14(c) we check our work with a computer.



(a) Preliminary sketch (b) Finished sketch (c) Computer confirmation

FIGURE 14

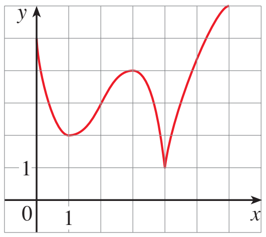
■

4.3 Exercises

1–2 Use the given graph of f to find the following.

(a) The open intervals on which f is increasing.
(b) The open intervals on which f is decreasing.
(c) The open intervals on which f is concave upward.
(d) The open intervals on which f is concave downward.
(e) The coordinates of the points of inflection.

1.



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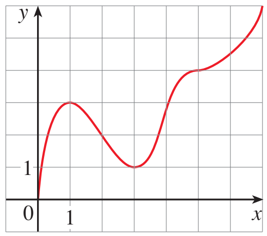


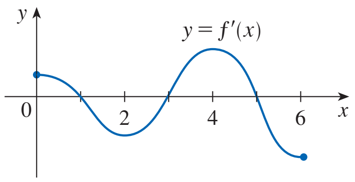
3. Suppose you are given a formula for a function f.
(a) How do you determine where f is increasing or decreasing?
(b) How do you determine where the graph of f is concave upward or concave downward?
(c) How do you locate inflection points?

4. (a) State the First Derivative Test.
(b) State the Second Derivative Test. Under what circumstances is it inconclusive? What do you do if it fails?

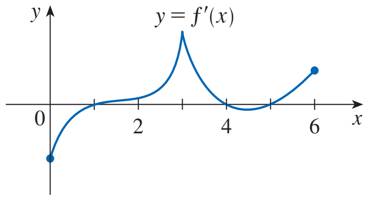
5–6 The graph of the *derivative* f' of a function f is shown.

(a) On what intervals is f increasing? Decreasing?
(b) At what values of x does f have a local maximum? Local minimum?

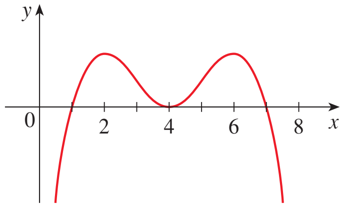
5.



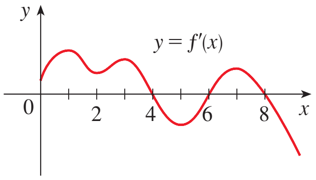
6.



7. In each part state the x-coordinates of the inflection points of f. Give reasons for your answers.
(a) The curve is the graph of f.
(b) The curve is the graph of f'.
(c) The curve is the graph of f''.



8. The graph of the first derivative f' of a function f is shown.
(a) On what intervals is f increasing? Explain.
(b) At what values of x does f have a local maximum or minimum? Explain.
(c) On what intervals is f concave upward or concave downward? Explain.
(d) What are the x-coordinates of the inflection points of f? Why?



9–16 Find the intervals on which f is increasing or decreasing, and find the local maximum and minimum values of f.

9. $f(x) = 2x^3 - 15x^2 + 24x - 5$

10. $f(x) = x^3 - 6x^2 - 135x$

11. $f(x) = 6x^4 - 16x^3 + 1$

12. $f(x) = x^{2/3}(x - 3)$

13. $f(x) = \dfrac{x^2 - 24}{x - 5}$

14. $f(x) = x + \dfrac{4}{x^2}$

15. $f(x) = \sin x + \cos x, \quad 0 \leqslant x \leqslant 2\pi$

16. $f(x) = x^4 e^{-x}$

17–22 Find the intervals on which f is concave upward or concave downward, and find the inflection points of f.

17. $f(x) = x^3 - 3x^2 - 9x + 4$

18. $f(x) = 2x^3 - 9x^2 + 12x - 3$

19. $f(x) = \sin^2 x - \cos 2x, \quad 0 \leqslant x \leqslant \pi$

20. $f(x) = \ln(2 + \sin x), \quad 0 \leqslant x \leqslant 2\pi$

21. $f(x) = \ln(x^2 + 5)$

22. $f(x) = \dfrac{e^x}{e^x + 2}$

23–28

(a) Find the intervals on which f is increasing or decreasing.
(b) Find the local maximum and minimum values of f.
(c) Find the intervals of concavity and the inflection points.

23. $f(x) = x^4 - 2x^2 + 3$

24. $f(x) = \dfrac{x}{x^2 + 1}$

25. $f(x) = x^2 - x - \ln x$

26. $f(x) = x^2 \ln x$

27. $f(x) = xe^{2x}$

28. $f(x) = \cos^2 x - 2\sin x, \quad 0 \leqslant x \leqslant 2\pi$

29–30 Find the local maximum and minimum values of f using both the First and Second Derivative Tests. Which method do you prefer?

29. $f(x) = 1 + 3x^2 - 2x^3$

30. $f(x) = \dfrac{x^2}{x - 1}$

31. Suppose the derivative of a function f is

$$f'(x) = (x - 4)^2(x + 3)^7(x - 5)^8$$

On what interval(s) is f increasing?

32. (a) Find the critical numbers of $f(x) = x^4(x - 1)^3$.
(b) What does the Second Derivative Test tell you about the behavior of f at these critical numbers?
(c) What does the First Derivative Test tell you?

33. Suppose f'' is continuous on $(-\infty, \infty)$.
(a) If $f'(2) = 0$ and $f''(2) = -5$, what can you say about f?
(b) If $f'(6) = 0$ and $f''(6) = 0$, what can you say about f?

34–41 Sketch the graph of a function that satisfies all of the given conditions.

34. (a) $f'(x) < 0$ and $f''(x) < 0$ for all x
(b) $f'(x) > 0$ and $f''(x) > 0$ for all x

35. (a) $f'(x) > 0$ and $f''(x) < 0$ for all x
(b) $f'(x) < 0$ and $f''(x) > 0$ for all x

36. Vertical asymptote $x = 0$, $\quad f'(x) > 0$ if $x < -2$,
$f'(x) < 0$ if $x > -2$ $(x \neq 0)$,
$f''(x) < 0$ if $x < 0$, $\quad f''(x) > 0$ if $x > 0$

37. $f'(0) = f'(2) = f'(4) = 0$,
$f'(x) > 0$ if $x < 0$ or $2 < x < 4$,
$f'(x) < 0$ if $0 < x < 2$ or $x > 4$,
$f''(x) > 0$ if $1 < x < 3$, $\quad f''(x) < 0$ if $x < 1$ or $x > 3$

38. $f'(x) > 0$ for all $x \neq 1$, $\quad$ vertical asymptote $x = 1$,
$f''(x) > 0$ if $x < 1$ or $x > 3$, $\quad f''(x) < 0$ if $1 < x < 3$

39. $f'(5) = 0$, $\quad f'(x) < 0$ when $x < 5$,
$f'(x) > 0$ when $x > 5$, $\quad f''(2) = 0$, $\quad f''(8) = 0$,
$f''(x) < 0$ when $x < 2$ or $x > 8$,
$f''(x) > 0$ for $2 < x < 8$, $\quad \lim_{x \to \infty} f(x) = 3$, $\quad \lim_{x \to -\infty} f(x) = 3$

40. $f'(0) = f'(4) = 0$, $\quad f'(x) = 1$ if $x < -1$,
$f'(x) > 0$ if $0 < x < 2$,
$f'(x) < 0$ if $-1 < x < 0$ or $2 < x < 4$ or $x > 4$,
$\lim_{x \to 2^-} f'(x) = \infty$, $\quad \lim_{x \to 2^+} f'(x) = -\infty$,
$f''(x) > 0$ if $-1 < x < 2$ or $2 < x < 4$,
$f''(x) < 0$ if $x > 4$

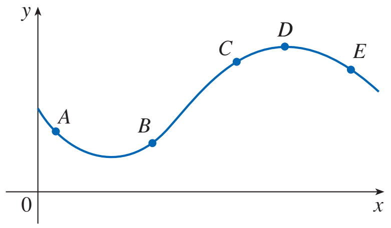
41. $f'(x) > 0$ if $x \neq 2$, $\quad f''(x) > 0$ if $x < 2$,
$f''(x) < 0$ if $x > 2$, $\quad f$ has inflection point $(2, 5)$,
$\lim_{x \to \infty} f(x) = 8$, $\quad \lim_{x \to -\infty} f(x) = 0$

42. The graph of a function $y = f(x)$ is shown. At which point(s) are the following true?

(a) $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ are both positive.

(b) $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ are both negative.

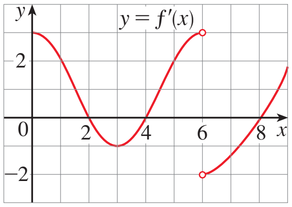
(c) $\dfrac{dy}{dx}$ is negative but $\dfrac{d^2y}{dx^2}$ is positive.



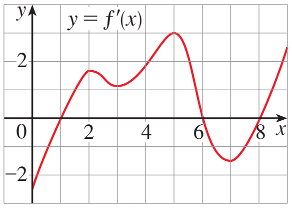
43–44 The graph of the derivative f' of a continuous function f is shown.

(a) On what intervals is f increasing? Decreasing?
(b) At what values of x does f have a local maximum? Local minimum?
(c) On what intervals is f concave upward? Concave downward?
(d) State the x-coordinate(s) of the point(s) of inflection.
(e) Assuming that $f(0) = 0$, sketch a graph of f.

43.



44.



45–58

(a) Find the intervals of increase or decrease.
(b) Find the local maximum and minimum values.
(c) Find the intervals of concavity and the inflection points.
(d) Use the information from parts (a)–(c) to sketch the graph. You may want to check your work with a graphing calculator or computer.

45. $f(x) = x^3 - 3x^2 + 4$

46. $f(x) = 36x + 3x^2 - 2x^3$

47. $f(x) = \frac{1}{2}x^4 - 4x^2 + 3$

48. $g(x) = 200 + 8x^3 + x^4$

49. $g(t) = 3t^4 - 8t^3 + 12$

50. $h(x) = 5x^3 - 3x^5$

51. $f(z) = z^7 - 112z^2$

52. $f(x) = (x^2 - 4)^3$

53. $F(x) = x\sqrt{6 - x}$

54. $G(x) = 5x^{2/3} - 2x^{5/3}$

55. $C(x) = x^{1/3}(x + 4)$

56. $f(x) = \ln(x^2 + 9)$

57. $f(\theta) = 2\cos\theta + \cos^2\theta, \quad 0 \leqslant \theta \leqslant 2\pi$

58. $S(x) = x - \sin x, \quad 0 \leqslant x \leqslant 4\pi$

59–66

(a) Find the vertical and horizontal asymptotes.
(b) Find the intervals of increase or decrease.
(c) Find the local maximum and minimum values.
(d) Find the intervals of concavity and the inflection points.
(e) Use the information from parts (a)–(d) to sketch the graph of f.

59. $f(x) = 1 + \dfrac{1}{x} - \dfrac{1}{x^2}$

60. $f(x) = \dfrac{x^2 - 4}{x^2 + 4}$

61. $f(x) = e^{-2/x}$

62. $f(x) = \dfrac{e^x}{1 - e^x}$

63. $f(x) = e^{-x^2}$

64. $f(x) = x - \frac{1}{6}x^2 - \frac{2}{3}\ln x$

65. $f(x) = \ln(1 - \ln x)$

66. $f(x) = e^{\arctan x}$

67–68 Use the methods of this section to sketch several members of the given family of curves. What do the members have in common? How do they differ from each other?

67. $f(x) = x^4 - cx, \quad c > 0$

68. $f(x) = x^3 - 3c^2x + 2c^3, \quad c > 0$

69–70

(a) Use a graph of f to estimate the maximum and minimum values. Then find the exact values.
(b) Estimate the value of x at which f increases most rapidly. Then find the exact value.

69. $f(x) = \dfrac{x + 1}{\sqrt{x^2 + 1}}$

70. $f(x) = x^2e^{-x}$

71–72

(a) Use a graph of f to give a rough estimate of the intervals of concavity and the coordinates of the points of inflection.
(b) Use a graph of f'' to give better estimates.

71. $f(x) = \sin 2x + \sin 4x, \quad 0 \leqslant x \leqslant \pi$

72. $f(x) = (x - 1)^2(x + 1)^3$

73–74 Estimate the intervals of concavity to one decimal place by using a computer algebra system to compute and graph f''.

73. $f(x) = \dfrac{x^4 + x^3 + 1}{\sqrt{x^2 + x + 1}}$

74. $f(x) = \dfrac{x^2\tan^{-1}x}{1 + x^3}$

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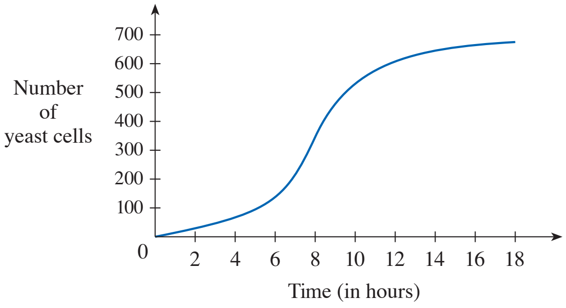
75. A graph of a population of yeast cells in a new laboratory culture as a function of time is shown.



(a) Describe how the rate of population increase varies.
(b) When is this rate highest?
(c) On what intervals is the population function concave upward or downward?
(d) Estimate the coordinates of the inflection point.

76. In an episode of *The Simpsons* television show, Homer reads from a newspaper and announces "Here's good news! According to this eye-catching article, SAT scores are declining at a slower rate." Interpret Homer's statement in terms of a function and its first and second derivatives.

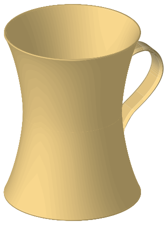
77. The president announces that the national deficit is increasing, but at a decreasing rate. Interpret this statement in terms of a function and its first and second derivatives.

78. Let $f(t)$ be the temperature at time t where you live and suppose that at time $t = 3$ you feel uncomfortably hot. How do you feel about the given data in each case?

(a) $f'(3) = 2, \quad f''(3) = 4$
(b) $f'(3) = 2, \quad f''(3) = -4$
(c) $f'(3) = -2, \quad f''(3) = 4$
(d) $f'(3) = -2, \quad f''(3) = -4$

79. Let $K(t)$ be a measure of the knowledge you gain by studying for a test for t hours. Which do you think is larger, $K(8) - K(7)$ or $K(3) - K(2)$? Is the graph of K concave upward or concave downward? Why?

80. Coffee is being poured into the mug shown in the figure at a constant rate (measured in volume per unit time). Sketch a rough graph of the depth of the coffee in the mug as a function of time. Account for the shape of the graph in terms of concavity. What is the significance of the inflection point?



81. A *drug response curve* describes the level of medication in the bloodstream after a drug is administered. A surge function $S(t) = At^pe^{-kt}$ is often used to model the response curve, reflecting an initial surge in the drug level and then a more gradual decline. If, for a particular drug, $A = 0.01$, $p = 4$, $k = 0.07$, and t is measured in minutes, estimate the times corresponding to the inflection points and explain their significance. Then graph the drug response curve.

82. Normal Density Functions The family of bell-shaped curves

$$y = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}$$

occurs in probability and statistics, where it is called the *normal density function*. The constant μ is called the *mean* and the positive constant σ is called the *standard deviation*. For simplicity, let's scale the function so as to remove the factor $1/(\sigma\sqrt{2\pi})$ and let's analyze the special case where $\mu = 0$. So we study the function

$$f(x) = e^{-x^2/(2\sigma^2)}$$

(a) Find the asymptote, maximum value, and inflection points of f.
(b) What role does σ play in the shape of the curve?
(c) Illustrate by graphing four members of this family on the same screen.

83. Find a cubic function $f(x) = ax^3 + bx^2 + cx + d$ that has a local maximum value of 3 at $x = -2$ and a local minimum value of 0 at $x = 1$.

84. For what values of the numbers a and b does the function

$$f(x) = axe^{bx^2}$$

have the maximum value $f(2) = 1$?

85. Show that the curve $y = (1 + x)/(1 + x^2)$ has three points of inflection and they all lie on one straight line.

86. Show that the curves $y = e^{-x}$ and $y = -e^{-x}$ touch the curve $y = e^{-x} \sin x$ at its inflection points.

87. Show that the inflection points of the curve $y = x \sin x$ lie on the curve $y^2(x^2 + 4) = 4x^2$.

88–90 Assume that all of the functions are twice differentiable and the second derivatives are never 0.

88. (a) If f and g are concave upward on an interval I, show that $f + g$ is concave upward on I.
(b) If f is positive and concave upward on I, show that the function $g(x) = [f(x)]^2$ is concave upward on I.

89. (a) If f and g are positive, increasing, concave upward functions on an interval I, show that the product function fg is concave upward on I.
(b) Show that part (a) remains true if f and g are both decreasing.
(c) Suppose f is increasing and g is decreasing. Show, by giving three examples, that fg may be concave upward, concave downward, or linear. Why doesn't the argument in parts (a) and (b) work in this case?

90. Suppose f and g are both concave upward on $(-\infty, \infty)$. Under what condition on f will the composite function $h(x) = f(g(x))$ be concave upward?

---

91. Show that a cubic function (a third-degree polynomial) always has exactly one point of inflection. If its graph has three x-intercepts x_1, x_2, and x_3, show that the x-coordinate of the inflection point is $(x_1 + x_2 + x_3)/3$.

92. For what values of c does the polynomial $P(x) = x^4 + cx^3 + x^2$ have two inflection points? One inflection point? None? Illustrate by graphing P for several values of c. How does the graph change as c decreases?

93. Prove that if $(c, f(c))$ is a point of inflection of the graph of f and f'' exists in an open interval that contains c, then $f''(c) = 0$. [*Hint:* Apply the First Derivative Test and Fermat's Theorem to the function $g = f'$.]

94. Show that if $f(x) = x^4$, then $f''(0) = 0$, but $(0, 0)$ is not an inflection point of the graph of f.

95. Show that the function $g(x) = x|x|$ has an inflection point at $(0, 0)$ but $g''(0)$ does not exist.

96. Suppose that f''' is continuous and $f'(c) = f''(c) = 0$, but $f'''(c) > 0$. Does f have a local maximum or minimum at c? Does f have a point of inflection at c?

97. Suppose f is differentiable on an interval I and $f'(x) > 0$ for all numbers x in I except for a single number c. Prove that f is increasing on the entire interval I.

98. For what values of c is the function

$$f(x) = cx + \frac{1}{x^2 + 3}$$

increasing on $(-\infty, \infty)$?

99. The three cases in the First Derivative Test cover the situations commonly encountered but do not exhaust all possibilities. Consider the functions f, g, and h whose values at 0 are all 0 and, for $x \neq 0$,

$$f(x) = x^4 \sin \frac{1}{x} \qquad g(x) = x^4\left(2 + \sin \frac{1}{x}\right)$$

$$h(x) = x^4\left(-2 + \sin \frac{1}{x}\right)$$

(a) Show that 0 is a critical number of all three functions but their derivatives change sign infinitely often on both sides of 0.

(b) Show that f has neither a local maximum nor a local minimum at 0, g has a local minimum, and h has a local maximum.

4.4 Indeterminate Forms and l'Hospital's Rule

Suppose we are trying to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x - 1}$$

Although F is not defined when $x = 1$, we need to know how F behaves *near* 1. In particular, we would like to know the value of the limit

$$\boxed{1} \qquad \lim_{x \to 1} \frac{\ln x}{x - 1}$$

In computing this limit we can't apply Law 5 of limits (the limit of a quotient is the quotient of the limits, see Section 2.3) because the limit of the denominator is 0. In fact, although the limit in (1) exists, its value is not obvious because both numerator and denominator approach 0 and $\frac{0}{0}$ is not defined.

Indeterminate Forms (Types $\frac{0}{0}, \frac{\infty}{\infty}$)

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then this limit may or may not exist and is called an **indeterminate form of type $\frac{0}{0}$**. We met some limits of this type in Chapter 2. For rational functions, we can cancel common factors:

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x(x - 1)}{(x + 1)(x - 1)} = \lim_{x \to 1} \frac{x}{x + 1} = \frac{1}{2}$$

In Section 3.3 we used a geometric argument to show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

But these methods do not work for limits such as (1).

Another situation in which a limit is not obvious occurs when we look for a horizontal asymptote of F and need to evaluate the limit

$$\boxed{2} \qquad \lim_{x \to \infty} \frac{\ln x}{x - 1}$$

It isn't obvious how to evaluate this limit because both numerator and denominator become large as $x \to \infty$. There is a struggle between numerator and denominator. If the numerator wins, the limit will be ∞ (the numerator was increasing significantly faster than the denominator); if the denominator wins, the answer will be 0. Or there may be some compromise, in which case the answer will be some finite positive number.

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$), then the limit may or may not exist and is called an **indeterminate form of type $\frac{\infty}{\infty}$**. We saw in Section 2.6 that this type of limit can be evaluated for certain functions, including rational functions, by dividing numerator and denominator by the highest power of x that occurs in the denominator. For instance,

$$\lim_{x \to \infty} \frac{x^2 - 1}{2x^2 + 1} = \lim_{x \to \infty} \frac{1 - \dfrac{1}{x^2}}{2 + \dfrac{1}{x^2}} = \frac{1 - 0}{2 + 0} = \frac{1}{2}$$

But this method does not work for limits such as (2).

L'Hospital's Rule

We now introduce a systematic method, known as *l'Hospital's Rule*, for the evaluation of indeterminate forms of type $\frac{0}{0}$ or type $\frac{\infty}{\infty}$.

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

NOTE 1 L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. It is especially important to verify the conditions regarding the limits of f and g before using l'Hospital's Rule.

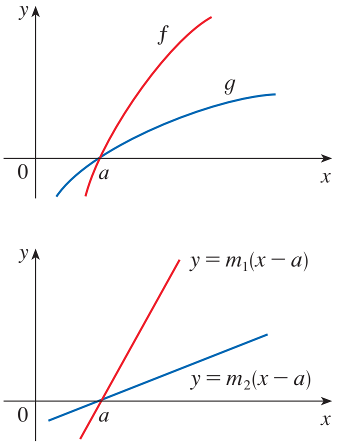


FIGURE 1

Figure 1 suggests visually why l'Hospital's Rule might be true. The first graph shows two differentiable functions f and g, each of which approaches 0 as $x \to a$. If we were to zoom in toward the point $(a, 0)$, the graphs would start to look almost linear. But if the functions actually *were* linear, as in the second graph, then their ratio would be

$$\frac{m_1(x - a)}{m_2(x - a)} = \frac{m_1}{m_2}$$

which is the ratio of their derivatives. This suggests that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

L'Hospital

L'Hospital's Rule is named after a French nobleman, the Marquis de l'Hospital (1661–1704), but was discovered by a Swiss mathematician, John Bernoulli (1667–1748). You might sometimes see l'Hospital spelled as l'Hôpital, but he spelled his own name l'Hospital, as was common in the 17th century. See Exercise 85 for the example that the Marquis used to illustrate his rule. See the project following this section for further historical details.

NOTE 2 L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, "$x \to a$" can be replaced by any of the symbols $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$.

NOTE 3 For the special case in which $f(a) = g(a) = 0$, f' and g' are continuous, and $g'(a) \neq 0$, it is easy to see why l'Hospital's Rule is true. In fact, using the alternative form of the definition of a derivative (2.7.5), we have

$$\begin{aligned}
\lim_{x\to a}\frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} &= \frac{\displaystyle\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\displaystyle\lim_{x\to a}\frac{g(x)-g(a)}{x-a}} \\
&= \lim_{x\to a}\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}} \\
&= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a}\frac{f(x)}{g(x)} \qquad [\text{because } f(a) = g(a) = 0]
\end{aligned}$$

It is more difficult to prove the general version of l'Hospital's Rule. See Appendix F.

EXAMPLE 1 Find $\displaystyle\lim_{x\to 1}\frac{\ln x}{x-1}$.

SOLUTION Since

$$\lim_{x\to 1}\ln x = \ln 1 = 0 \qquad \text{and} \qquad \lim_{x\to 1}(x-1) = 0$$

the limit is an indeterminate form of type $\frac{0}{0}$, so we can apply l'Hospital's Rule:

Notice that when using l'Hospital's Rule we differentiate the numerator and denominator *separately*. We do *not* use the Quotient Rule.

$$\begin{aligned}
\lim_{x\to 1}\frac{\ln x}{x-1} &= \lim_{x\to 1}\frac{\dfrac{d}{dx}(\ln x)}{\dfrac{d}{dx}(x-1)} = \lim_{x\to 1}\frac{1/x}{1} \\
&= \lim_{x\to 1}\frac{1}{x} = 1
\end{aligned}$$

■

EXAMPLE 2 Calculate $\displaystyle\lim_{x\to\infty}\frac{e^x}{x^2}$.

SOLUTION We have $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} x^2 = \infty$, so the limit is an indeterminate form of type $\frac{\infty}{\infty}$, and l'Hospital's Rule gives

$$\lim_{x\to\infty}\frac{e^x}{x^2} = \lim_{x\to\infty}\frac{\dfrac{d}{dx}(e^x)}{\dfrac{d}{dx}(x^2)} = \lim_{x\to\infty}\frac{e^x}{2x}$$

Since $e^x \to \infty$ and $2x \to \infty$ as $x \to \infty$, the limit on the right side is also indeterminate. A second application of l'Hospital's Rule gives

$$\lim_{x\to\infty}\frac{e^x}{x^2} = \lim_{x\to\infty}\frac{e^x}{2x} = \lim_{x\to\infty}\frac{e^x}{2} = \infty$$

■

The graph of the function of Example 2 is shown in Figure 2. We have noticed previously that exponential functions grow far more rapidly than power functions, so the result of Example 2 is not unexpected. See also Exercise 75.

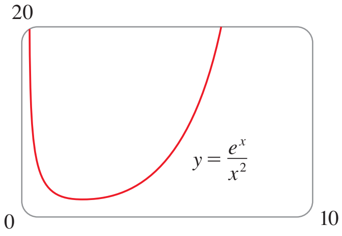


FIGURE 2

The graph of the function of Example 3 is shown in Figure 3. We have discussed previously the slow growth of logarithms, so it isn't surprising that this ratio approaches 0 as $x \to \infty$. See also Exercise 76.

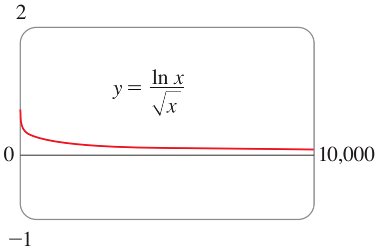


FIGURE 3

EXAMPLE 3 Calculate $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$.

SOLUTION Since $\ln x \to \infty$ and $\sqrt{x} \to \infty$ as $x \to \infty$, l'Hospital's Rule applies:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{1/x}{1/(2\sqrt{x}\,)}$$

Notice that the limit on the right side is now indeterminate of type $\frac{0}{0}$. But instead of applying l'Hospital's Rule a second time as we did in Example 2, we simplify the expression and see that a second application is unnecessary:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/(2\sqrt{x}\,)} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$ ■

In both Examples 2 and 3 we evaluated limits of type $\frac{\infty}{\infty}$, but we got two different results. In Example 2, the infinite limit tells us that the numerator e^x increases significantly faster than the denominator x^2, resulting in larger and larger ratios. In fact, $y = e^x$ grows more quickly than all the power functions $y = x^n$ (see Exercise 75). In Example 3 we have the opposite situation; the limit of 0 means that the denominator outpaces the numerator, and the ratio eventually approaches 0.

EXAMPLE 4 Find $\lim_{x\to 0} \frac{\tan x - x}{x^3}$. (See Exercise 2.2.48.)

SOLUTION Noting that both $\tan x - x \to 0$ and $x^3 \to 0$ as $x \to 0$, we use l'Hospital's Rule:

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}$$

Since the limit on the right side is still indeterminate of type $\frac{0}{0}$, we apply l'Hospital's Rule again:

$$\lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x}$$

Because $\lim_{x\to 0} \sec^2 x = 1$, we simplify the calculation by writing

$$\lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} = \frac{1}{3} \lim_{x\to 0} \sec^2 x \cdot \lim_{x\to 0} \frac{\tan x}{x} = \frac{1}{3} \lim_{x\to 0} \frac{\tan x}{x}$$

We can evaluate this last limit either by using l'Hospital's Rule a third time or by writing $\tan x$ as $(\sin x)/(\cos x)$ and making use of our knowledge of trigonometric limits. Putting together all the steps, we get

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x}$$

$$= \frac{1}{3} \lim_{x\to 0} \frac{\tan x}{x} = \frac{1}{3} \lim_{x\to 0} \frac{\sec^2 x}{1} = \frac{1}{3}$$ ■

The graph in Figure 4 gives visual confirmation of the result of Example 4. If we were to zoom in too far, however, we would get an inaccurate graph because $\tan x$ is close to x when x is small. See Exercise 2.2.48(d).

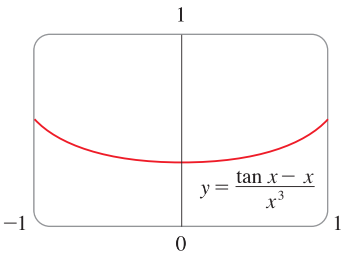


FIGURE 4

EXAMPLE 5 Find $\displaystyle\lim_{x\to\pi^-}\frac{\sin x}{1-\cos x}$.

SOLUTION If we blindly attempted to use l'Hospital's Rule, we might think that an equivalent limit is

$$\lim_{x\to\pi^-}\frac{\cos x}{\sin x}=-\infty$$

This is wrong! Although the numerator $\sin x \to 0$ as $x\to\pi^-$, notice that the denominator $(1-\cos x)$ does not approach 0, so l'Hospital's Rule can't be applied here.

The required limit can, in fact, be found by direct substitution because the function is continuous at π and the denominator is nonzero there:

$$\lim_{x\to\pi^-}\frac{\sin x}{1-\cos x}=\frac{\sin\pi}{1-\cos\pi}=\frac{0}{1-(-1)}=0$$ ■

Example 5 shows what can go wrong if you use l'Hospital's Rule without thinking. Some limits *can* be found using l'Hospital's Rule but are more easily found by other methods. (See Examples 2.3.3, 2.3.5, and 2.6.3, and the discussion at the beginning of this section.) When evaluating any limit, you should consider other methods before using l'Hospital's Rule.

Indeterminate Products (Type $0\cdot\infty$)

If $\lim_{x\to a}f(x)=0$ and $\lim_{x\to a}g(x)=\infty$ (or $-\infty$), then it isn't clear what the value of $\lim_{x\to a}[f(x)\,g(x)]$, if any, will be. There is a struggle between f and g. If f wins, the answer will be 0; if g wins, the answer will be ∞ (or $-\infty$). Or there may be a compromise where the answer is a finite nonzero number. For instance,

$$\lim_{x\to0^+}x^2=0,\qquad \lim_{x\to0^+}\frac{1}{x}=\infty\qquad\text{and}\qquad \lim_{x\to0^+}x^2\cdot\frac{1}{x}=\lim_{x\to0^+}x=0$$

$$\lim_{x\to0^+}x=0,\qquad \lim_{x\to0^+}\frac{1}{x^2}=\infty\qquad\text{and}\qquad \lim_{x\to0^+}x\cdot\frac{1}{x^2}=\lim_{x\to0^+}\frac{1}{x}=\infty$$

$$\lim_{x\to0^+}x=0,\qquad \lim_{x\to0^+}\frac{1}{x}=\infty\qquad\text{and}\qquad \lim_{x\to0^+}x\cdot\frac{1}{x}=\lim_{x\to0^+}1=1$$

This kind of limit is called an **indeterminate form of type $0\cdot\infty$**. We can deal with it by writing the product fg as a quotient:

$$fg=\frac{f}{1/g}\qquad\text{or}\qquad fg=\frac{g}{1/f}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so that we can use l'Hospital's Rule.

EXAMPLE 6 Evaluate $\displaystyle\lim_{x\to0^+}x\ln x$.

SOLUTION The given limit is indeterminate because, as $x\to0^+$, the first factor (x) approaches 0 while the second factor $(\ln x)$ approaches $-\infty$. Writing $x=1/(1/x)$, we have $1/x\to\infty$ as $x\to0^+$, so l'Hospital's Rule gives

$$\lim_{x\to0^+}x\ln x=\lim_{x\to0^+}\frac{\ln x}{1/x}=\lim_{x\to0^+}\frac{1/x}{-1/x^2}=\lim_{x\to0^+}(-x)=0$$ ■

Figure 5 shows the graph of the function in Example 6. Notice that the function is undefined at $x=0$; the graph approaches the origin but never quite reaches it.

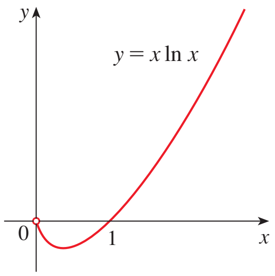


FIGURE 5

NOTE In solving Example 6 another possible option would have been to write

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{x}{1/\ln x}$$

This gives an indeterminate form of the type $\frac{0}{0}$, but if we apply l'Hospital's Rule we get a more complicated expression than the one we started with. In general, when we rewrite an indeterminate product, we try to choose the option that leads to the simpler limit.

Indeterminate Differences (Type $\infty - \infty$)

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a} [f(x) - g(x)]$$

is called an **indeterminate form of type $\infty - \infty$**. Again there is a contest between f and g. Will the answer be ∞ (f wins) or will it be $-\infty$ (g wins) or will they compromise on a finite number? To find out, we try to convert the difference into a quotient (for instance, by using a common denominator, or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

EXAMPLE 7 Compute $\displaystyle\lim_{x\to 1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$.

SOLUTION First notice that $1/(\ln x) \to \infty$ and $1/(x-1) \to \infty$ as $x \to 1^+$, so the limit is indeterminate of type $\infty - \infty$. Here we can start with a common denominator:

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to 1^+} \frac{x-1-\ln x}{(x-1)\ln x}$$

Both numerator and denominator have a limit of 0, so l'Hospital's Rule applies, giving

$$\lim_{x\to 1^+} \frac{x-1-\ln x}{(x-1)\ln x} = \lim_{x\to 1^+} \frac{1 - \dfrac{1}{x}}{(x-1)\cdot\dfrac{1}{x} + \ln x} = \lim_{x\to 1^+} \frac{x-1}{x-1+x\ln x}$$

Again we have an indeterminate limit of type $\frac{0}{0}$, so we apply l'Hospital's Rule a second time:

$$\lim_{x\to 1^+} \frac{x-1}{x-1+x\ln x} = \lim_{x\to 1^+} \frac{1}{1 + x\cdot\dfrac{1}{x} + \ln x}$$

$$= \lim_{x\to 1^+} \frac{1}{2+\ln x} = \frac{1}{2}$$

■

EXAMPLE 8 Calculate $\displaystyle\lim_{x\to\infty} (e^x - x)$.

SOLUTION This is an indeterminate difference because both e^x and x approach infinity. We would expect the limit to be infinity because $e^x \to \infty$ much faster than x. But we can verify this by factoring out x:

$$e^x - x = x\left(\frac{e^x}{x} - 1\right)$$

The term $e^x/x \to \infty$ as $x \to \infty$ by l'Hospital's Rule and so we now have a product in which both factors grow large:

$$\lim_{x\to\infty}(e^x - x) = \lim_{x\to\infty}\left[x\left(\frac{e^x}{x} - 1\right)\right] = \infty \qquad ■$$

■ Indeterminate Powers (Types 0^0, ∞^0, 1^∞)

Several indeterminate forms arise from the limit

$$\lim_{x\to a}[f(x)]^{g(x)}$$

1. $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ type 0^0
2. $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ type ∞^0
3. $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$ type 1^∞

Although forms of the type 0^0, ∞^0, and 1^∞ are indeterminate, the form 0^∞ is not indeterminate. (See Exercise 88.)

Each of these three cases can be treated either by taking the natural logarithm:

$$\text{let} \quad y = [f(x)]^{g(x)}, \quad \text{then} \quad \ln y = g(x)\ln f(x)$$

or by using Formula 1.5.10 to write the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x)\ln f(x)}$$

(Recall that both of these methods were used in differentiating such functions.) In either method we are led to the indeterminate product $g(x)\ln f(x)$, which is of type $0 \cdot \infty$.

EXAMPLE 9 Calculate $\lim_{x\to 0^+}(1 + \sin 4x)^{\cot x}$.

SOLUTION First notice that as $x \to 0^+$, we have $1 + \sin 4x \to 1$ and $\cot x \to \infty$, so the given limit is indeterminate (type 1^∞). Let

$$y = (1 + \sin 4x)^{\cot x}$$

Then $$\ln y = \ln[(1 + \sin 4x)^{\cot x}] = \cot x \ln(1 + \sin 4x) = \frac{\ln(1 + \sin 4x)}{\tan x}$$

so l'Hospital's Rule gives

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln(1 + \sin 4x)}{\tan x} = \lim_{x\to 0^+}\frac{\dfrac{4\cos 4x}{1 + \sin 4x}}{\sec^2 x} = 4$$

So far we have computed the limit of $\ln y$, but what we want is the limit of y. To find this we use the fact that $y = e^{\ln y}$:

$$\lim_{x\to 0^+}(1 + \sin 4x)^{\cot x} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^4 \qquad ■$$

The graph of the function $y = x^x$, $x > 0$, is shown in Figure 6. Notice that although 0^0 is not defined, the values of the function approach 1 as $x \to 0^+$. This confirms the result of Example 10.

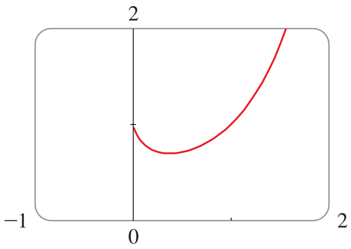


FIGURE 6

EXAMPLE 10 Find $\lim_{x\to 0^+} x^x$.

SOLUTION Notice that this limit is indeterminate since $0^x = 0$ for any $x > 0$ but $x^0 = 1$ for any $x \neq 0$. (Recall that 0^0 is undefined.) We could proceed as in Example 9 or by writing the function as an exponential:

$$x^x = (e^{\ln x})^x = e^{x\ln x}$$

In Example 6 we used l'Hospital's Rule to show that

$$\lim_{x\to 0^+} x \ln x = 0$$

Therefore $$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^0 = 1$$ ■

4.4 Exercises

1–4 Given that

$$\lim_{x\to a} f(x) = 0 \qquad \lim_{x\to a} g(x) = 0 \qquad \lim_{x\to a} h(x) = 1$$
$$\lim_{x\to a} p(x) = \infty \qquad \lim_{x\to a} q(x) = \infty$$

which of the following limits are indeterminate forms? For any limit that is not an indeterminate form, evaluate it where possible.

1. (a) $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)}$ (b) $\displaystyle\lim_{x\to a} \frac{f(x)}{p(x)}$

(c) $\displaystyle\lim_{x\to a} \frac{h(x)}{p(x)}$ (d) $\displaystyle\lim_{x\to a} \frac{p(x)}{f(x)}$

(e) $\displaystyle\lim_{x\to a} \frac{p(x)}{q(x)}$

2. (a) $\lim_{x\to a} [f(x)p(x)]$ (b) $\lim_{x\to a} [h(x)p(x)]$

(c) $\lim_{x\to a} [p(x)q(x)]$

3. (a) $\lim_{x\to a} [f(x) - p(x)]$ (b) $\lim_{x\to a} [p(x) - q(x)]$

(c) $\lim_{x\to a} [p(x) + q(x)]$

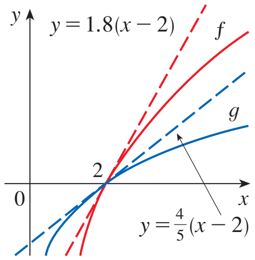
4. (a) $\lim_{x\to a} [f(x)]^{g(x)}$ (b) $\lim_{x\to a} [f(x)]^{p(x)}$

(c) $\lim_{x\to a} [h(x)]^{p(x)}$ (d) $\lim_{x\to a} [p(x)]^{f(x)}$

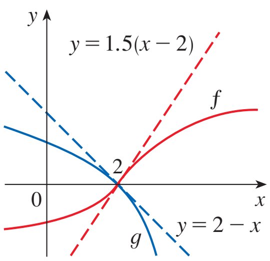
(e) $\lim_{x\to a} [p(x)]^{q(x)}$ (f) $\lim_{x\to a} \sqrt[q(x)]{p(x)}$

5–6 Use the graphs of f and g and their tangent lines at (2, 0) to find $\displaystyle\lim_{x\to 2} \frac{f(x)}{g(x)}$.

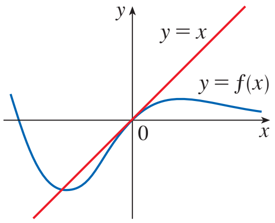
5.



6.



7. The graph of a function f and its tangent line at 0 are shown. What is the value of $\displaystyle\lim_{x\to 0} \frac{f(x)}{e^x - 1}$?



8–70 Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

8. $\displaystyle\lim_{x\to 3} \frac{x - 3}{x^2 - 9}$

9. $\displaystyle\lim_{x\to 4} \frac{x^2 - 2x - 8}{x - 4}$ **10.** $\displaystyle\lim_{x\to -2} \frac{x^3 + 8}{x + 2}$

11. $\displaystyle\lim_{x\to 1} \frac{x^7 - 1}{x^3 - 1}$ **12.** $\displaystyle\lim_{x\to 4} \frac{\sqrt{x} - 2}{x - 4}$

13. $\displaystyle\lim_{x\to \pi/4} \frac{\sin x - \cos x}{\tan x - 1}$ **14.** $\displaystyle\lim_{x\to 0} \frac{\tan 3x}{\sin 2x}$

15. $\displaystyle\lim_{t\to 0} \frac{e^{2t} - 1}{\sin t}$ **16.** $\displaystyle\lim_{x\to 0} \frac{x^2}{1 - \cos x}$

17. $\displaystyle\lim_{x\to 1} \frac{\sin(x - 1)}{x^3 + x - 2}$ **18.** $\displaystyle\lim_{\theta\to \pi} \frac{1 + \cos\theta}{1 - \cos\theta}$

19. $\displaystyle\lim_{x\to \infty} \frac{\sqrt{x}}{1 + e^x}$ **20.** $\displaystyle\lim_{x\to \infty} \frac{x + x^2}{1 - 2x^2}$

21. $\displaystyle\lim_{x\to 0^+} \frac{\ln x}{x}$ **22.** $\displaystyle\lim_{x\to \infty} \frac{\ln\sqrt{x}}{x^2}$

23. $\displaystyle\lim_{x\to 3} \frac{\ln(x/3)}{3 - x}$ **24.** $\displaystyle\lim_{t\to 0} \frac{8^t - 5^t}{t}$

25. $\displaystyle\lim_{x\to 0} \frac{\sqrt{1 + 2x} - \sqrt{1 - 4x}}{x}$ **26.** $\displaystyle\lim_{u\to \infty} \frac{e^{u/10}}{u^3}$

27. $\displaystyle\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{e^x - x - 1}$ **28.** $\displaystyle\lim_{x\to 0} \frac{\sinh x - x}{x^3}$

29. $\displaystyle\lim_{x\to0}\frac{\tanh x}{\tan x}$

30. $\displaystyle\lim_{x\to0}\frac{x-\sin x}{x-\tan x}$

31. $\displaystyle\lim_{x\to0}\frac{\sin^{-1}x}{x}$

32. $\displaystyle\lim_{x\to\infty}\frac{(\ln x)^2}{x}$

33. $\displaystyle\lim_{x\to0}\frac{x3^x}{3^x-1}$

34. $\displaystyle\lim_{x\to0}\frac{e^x+e^{-x}-2\cos x}{x\sin x}$

35. $\displaystyle\lim_{x\to0}\frac{\ln(1+x)}{\cos x+e^x-1}$

36. $\displaystyle\lim_{x\to1}\frac{x\sin(x-1)}{2x^2-x-1}$

37. $\displaystyle\lim_{x\to0^+}\frac{\arctan 2x}{\ln x}$

38. $\displaystyle\lim_{x\to0}\frac{x^2\sin x}{\sin x-x}$

39. $\displaystyle\lim_{x\to1}\frac{x^a-1}{x^b-1},\quad b\neq0$

40. $\displaystyle\lim_{x\to\infty}\frac{e^{-x}}{(\pi/2)-\tan^{-1}x}$

41. $\displaystyle\lim_{x\to0}\frac{\cos x-1+\frac{1}{2}x^2}{x^4}$

42. $\displaystyle\lim_{x\to0}\frac{x-\sin x}{x\sin(x^2)}$

43. $\displaystyle\lim_{x\to\infty}x\sin(\pi/x)$

44. $\displaystyle\lim_{x\to\infty}\sqrt{x}\,e^{-x/2}$

45. $\displaystyle\lim_{x\to0}\sin5x\csc3x$

46. $\displaystyle\lim_{x\to-\infty}x\ln\left(1-\frac{1}{x}\right)$

47. $\displaystyle\lim_{x\to\infty}x^3e^{-x^2}$

48. $\displaystyle\lim_{x\to\infty}x^{3/2}\sin(1/x)$

49. $\displaystyle\lim_{x\to1^+}\ln x\tan(\pi x/2)$

50. $\displaystyle\lim_{x\to(\pi/2)^-}\cos x\sec5x$

51. $\displaystyle\lim_{x\to1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$

52. $\displaystyle\lim_{x\to0}(\csc x-\cot x)$

53. $\displaystyle\lim_{x\to0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

54. $\displaystyle\lim_{x\to0^+}\left(\frac{1}{x}-\frac{1}{\tan^{-1}x}\right)$

55. $\displaystyle\lim_{x\to0^+}\frac{1}{x}-\frac{1}{\tan x}$

56. $\displaystyle\lim_{x\to\infty}(x-\ln x)$

57. $\displaystyle\lim_{x\to0^+}x^{\sqrt{x}}$

58. $\displaystyle\lim_{x\to0^+}(\tan2x)^x$

59. $\displaystyle\lim_{x\to0}(1-2x)^{1/x}$

60. $\displaystyle\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$

61. $\displaystyle\lim_{x\to1^+}x^{1/(1-x)}$

62. $\displaystyle\lim_{x\to\infty}(e^x+10x)^{1/x}$

63. $\displaystyle\lim_{x\to\infty}x^{1/x}$

64. $\displaystyle\lim_{x\to\infty}x^{e^{-x}}$

65. $\displaystyle\lim_{x\to0^+}(4x+1)^{\cot x}$

66. $\displaystyle\lim_{x\to0^+}(1-\cos x)^{\sin x}$

67. $\displaystyle\lim_{x\to0^+}(1+\sin3x)^{1/x}$

68. $\displaystyle\lim_{x\to0}(\cos x)^{1/x^2}$

69. $\displaystyle\lim_{x\to0^+}\frac{x^x-1}{\ln x+x-1}$

70. $\displaystyle\lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1}$

71–72 Use a graph to estimate the value of the limit. Then use l'Hospital's Rule to find the exact value.

71. $\displaystyle\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

72. $\displaystyle\lim_{x\to0}\frac{5^x-4^x}{3^x-2^x}$

73–74 Illustrate l'Hospital's Rule by graphing both $f(x)/g(x)$ and $f'(x)/g'(x)$ near $x=0$ to see that these ratios have the same limit as $x\to0$. Also, calculate the exact value of the limit.

73. $f(x)=e^x-1,\quad g(x)=x^3+4x$

74. $f(x)=2x\sin x,\quad g(x)=\sec x-1$

---

75. Prove that

$$\lim_{x\to\infty}\frac{e^x}{x^n}=\infty$$

for any positive integer n. This shows that the exponential function approaches infinity faster than any power of x.

76. Prove that

$$\lim_{x\to\infty}\frac{\ln x}{x^p}=0$$

for any number $p>0$. This shows that the logarithmic function approaches infinity more slowly than any power of x.

77–78 What happens if you try to use l'Hospital's Rule to find the limit? Evaluate the limit using another method.

77. $\displaystyle\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}$

78. $\displaystyle\lim_{x\to(\pi/2)^-}\frac{\sec x}{\tan x}$

---

79. Investigate the family of curves $f(x)=e^x-cx$. In particular, find the limits as $x\to\pm\infty$ and determine the values of c for which f has an absolute minimum. What happens to the minimum points as c increases?

80. If an object with mass m is dropped from rest, one model for its speed v after t seconds, taking air resistance into account, is

$$v=\frac{mg}{c}\left(1-e^{-ct/m}\right)$$

where g is the acceleration due to gravity and c is a positive constant. (In Chapter 9 we will be able to deduce this equation from the assumption that the air resistance is proportional to the speed of the object; c is the proportionality constant.)

(a) Calculate $\lim_{t\to\infty}v$. What is the meaning of this limit?

(b) For fixed t, use l'Hospital's Rule to calculate $\lim_{c\to0^+}v$. What can you conclude about the velocity of a falling object in a vacuum?

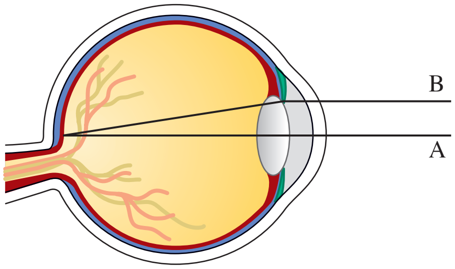
81. If an initial amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is

$$A=A_0\left(1+\frac{r}{n}\right)^{nt}$$

If we let $n\to\infty$, we refer to the *continuous compounding* of interest. Use l'Hospital's Rule to show that if interest is compounded continuously, then the amount after t years is

$$A=A_0e^{rt}$$

82. Light enters the eye through the pupil and strikes the retina, where photoreceptor cells sense light and color. W. Stanley Stiles and B. H. Crawford studied the phenomenon in which measured brightness decreases as light enters farther from the center of the pupil (see the figure).



A light beam A that enters through the center of the pupil measures brighter than a beam B entering near the edge of the pupil.

They detailed their findings of this phenomenon, known as the *Stiles–Crawford effect of the first kind*, in an important paper published in 1933. In particular, they observed that the amount of luminance sensed was *not* proportional to the area of the pupil as they expected. The percentage P of the total luminance entering a pupil of radius r mm that is sensed at the retina can be described by

$$P = \frac{1 - 10^{-\rho r^2}}{\rho r^2 \ln 10}$$

where ρ is an experimentally determined constant, typically about 0.05.

(a) What is the percentage of luminance sensed by a pupil of radius 3 mm? Use $\rho = 0.05$.

(b) Compute the percentage of luminance sensed by a pupil of radius 2 mm. Does it make sense that it is larger than the answer to part (a)?

(c) Compute $\lim\limits_{r \to 0^+} P$. Is the result what you would expect? Is this result physically possible?

Source: Adapted from W. Stiles and B. Crawford, "The Luminous Efficiency of Rays Entering the Eye Pupil at Different Points." *Proceedings of the Royal Society of London, Series B: Biological Sciences* 112 (1933): 428–50.

83. Logistic Equations Some populations initally grow exponentially but eventually level off. Equations of the form

$$P(t) = \frac{M}{1 + Ae^{-kt}}$$

where M, A, and k are positive constants, are called *logistic equations* and are often used to model such populations. (We will investigate these in detail in Chapter 9.) Here M is called the *carrying capacity* and represents the maximum population size that can be supported, and

$$A = \frac{M - P_0}{P_0}$$

where P_0 is the initial population.

(a) Compute $\lim_{t \to \infty} P(t)$. Explain why your answer is to be expected.

(b) Compute $\lim_{M \to \infty} P(t)$. (Note that A is defined in terms of M.) What kind of function is your result?

84. A metal cable has radius r and is covered by insulation so that the distance from the center of the cable to the exterior of the insulation is R. The velocity v of an electrical impulse in the cable is

$$v = -c\left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right)$$

where c is a positive constant. Find the following limits and interpret your answers.

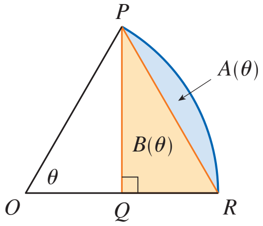
(a) $\lim\limits_{R \to r^+} v$ (b) $\lim\limits_{r \to 0^+} v$

85. The first appearance in print of l'Hospital's Rule was in the book *Analyse des infiniment petits* published by the Marquis de l'Hospital in 1696. This was the first calculus textbook ever published. The example that the Marquis used in that book to illustrate his rule was to find the limit of the function

$$y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}$$

as x approaches a, where $a > 0$. (At that time it was common to write aa instead of a^2.) Solve this problem.

86. The figure shows a sector of a circle with central angle θ. Let $A(\theta)$ be the area of the segment between the chord PR and the arc PR. Let $B(\theta)$ be the area of the triangle PQR. Find $\lim_{\theta \to 0^+} A(\theta)/B(\theta)$.



87. Evaluate

$$\lim_{x \to \infty}\left[x - x^2 \ln\left(\frac{1 + x}{x}\right)\right]$$

88. Suppose f is a positive function. If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$, show that

$$\lim_{x \to a}\,[f(x)]^{g(x)} = 0$$

This shows that 0^∞ is not an indeterminate form.

89. Find functions f and g where $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = \infty$ and

(a) $\displaystyle\lim_{x\to 0} \frac{f(x)}{g(x)} = 7$ (b) $\displaystyle\lim_{x\to 0} [f(x) - g(x)] = 7$

90. For what values of a and b is the following equation true?

$$\lim_{x\to 0}\left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2}\right) = 0$$

91. Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Use the definition of derivative to compute $f'(0)$.

(b) Show that f has derivatives of all orders that are defined on $\mathbb{R}$. [*Hint:* First show by induction that there is a polynomial $p_n(x)$ and a nonnegative integer k_n such that $f^{(n)}(x) = p_n(x)f(x)/x^{k_n}$ for $x \neq 0$.]

92. Let

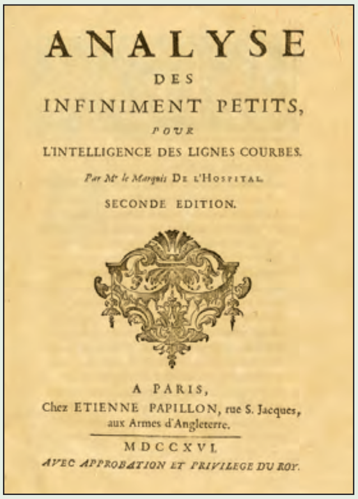
$$f(x) = \begin{cases} |x|^x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

(a) Show that f is continuous at 0.

(b) Investigate graphically whether f is differentiable at 0 by zooming in several times toward the point (0, 1) on the graph of f.

(c) Show that f is not differentiable at 0. How can you reconcile this fact with the appearance of the graphs in part (b)?

WRITING PROJECT | THE ORIGINS OF L'HOSPITAL'S RULE



ANALYSE
DES
INFINIMENT PETITS,
POUR
L'INTELLIGENCE DES LIGNES COURBES.
Par M^r le Marquis De l'Hospital.
SECONDE EDITION.

A PARIS,
Chez ETIENNE PAPILLON, rue S. Jacques,
aux Armes d'Angleterre.
M D C C X V I.
AVEC APPROBATION ET PRIVILEGE DU ROY.

Thomas Fisher Rare Book Library

ital's Rule was first published in 1696 in the Marquis de l'Hospital's calculus text- *nalyse des infiniment petits*, but the rule was discovered in 1694 by the Swiss math- ian John (Johann) Bernoulli. The explanation is that these two mathematicians had into a curious business arrangement whereby the Marquis de l'Hospital bought hts to Bernoulli's mathematical discoveries. The details, including a translation of ital's letter to Bernoulli proposing the arrangement, can be found in the book by 1].

te an essay on the historical and mathematical origins of l'Hospital's Rule. Start by ng brief biographical details of both men (the dictionary edited by Gillispie [2] is a ource) and outline the business deal between them. Then give l'Hospital's statement of , which is found in Struik's source book [4] and more briefly in the book of Katz [3]. that l'Hospital and Bernoulli formulated the rule geometrically and gave the answer in f differentials. Compare their statement with the version of l'Hospital's Rule given in 4.4 and show that the two statements are essentially the same.

The Internet is another source of information for this project. Click on *History of Mathematics* for a list of reliable websites.

ward W. Eves, *Mathematical Circles: Volume 1* (Washington, D.C.: Mathematical sociation of America, 2003). First published 1969 as *In Mathematical Circles (Vol- e 2: Quadrants III and IV)* by Prindle Weber and Schmidt.

C. Gillispie, ed., *Dictionary of Scientific Biography,* 8 vols. (New York: Scribner, 1981). See the article on Johann Bernoulli by E. A. Fellmann and J. O. Fleckstein in Volume II and the article on the Marquis de l'Hospital by Abraham Robinson in Volume III.

3. Victor J. Katz, *A History of Mathematics: An Introduction.* 3rd ed. (New York: Pearson, 2018).

4. Dirk Jan Stuik, ed. *A Source Book in Mathematics, 1200–1800* (1969; repr., Princeton, NJ: Princeton University Press, 2016).

4.5 Summary of Curve Sketching

So far we have been concerned with some particular aspects of curve sketching: domain, range, and symmetry in Chapter 1; limits, continuity, and asymptotes in Chapter 2; derivatives and tangents in Chapters 2 and 3; and extreme values, intervals of increase and decrease, concavity, points of inflection, and l'Hospital's Rule in this chapter. It is now time to put all of this information together to sketch graphs that reveal the important features of functions.

You might ask: Why don't we just use a graphing calculator or computer to graph a curve? Why do we need to use calculus?

It's true that technology is capable of producing very accurate graphs. But even the best graphing devices have to be used intelligently. It is easy to arrive at a misleading graph, or to miss important details of a curve, when relying solely on technology. (See the topic Graphing Calculators and Computers at *www.StewartCalculus.com*, especially Examples 1, 3, 4, and 5. See also Section 4.6.) The use of calculus enables us to discover the most interesting aspects of graphs and in many cases to calculate maximum and minimum points and inflection points *exactly* instead of approximately.

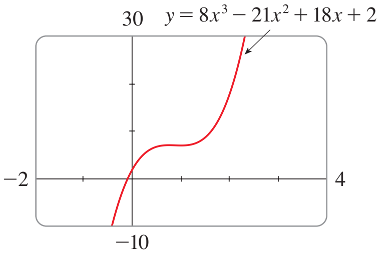


FIGURE 1

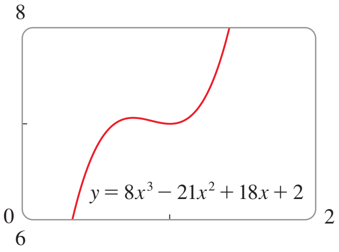


FIGURE 2

For instance, Figure 1 shows the graph of $f(x) = 8x^3 - 21x^2 + 18x + 2$. At first glance it seems reasonable: It has the same shape as cubic curves like $y = x^3$, and it appears to have no maximum or minimum point. But if you compute the derivative, you will see that there is a maximum when $x = 0.75$ and a minimum when $x = 1$. Indeed, if we zoom in to this portion of the graph, we see that behavior exhibited in Figure 2. Without calculus, we could easily have overlooked it.

In the next section we will graph functions by using the interaction between calculus and technology. In this section we draw graphs (by hand) by first considering the following information. A graph produced by a calculator or computer can serve as a check on your work.

Guidelines for Sketching a Curve

The following checklist is intended as a guide to sketching a curve $y = f(x)$ by hand. Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess symmetry.) But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

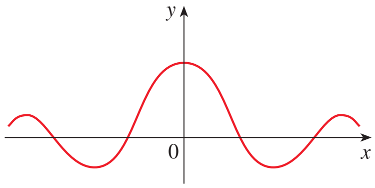
A. Domain It's often useful to start by determining the domain D of f, that is, the set of values of x for which $f(x)$ is defined.

B. Intercepts The y-intercept is $f(0)$ and this tells us where the curve intersects the y-axis. To find the x-intercepts, we set $y = 0$ and solve for x. (You can omit this step if the equation is difficult to solve.)

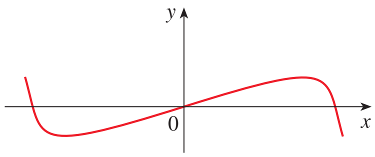
C. Symmetry

(i) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an *even function* and the curve is symmetric about the y-axis. (See Section 1.1.) This means that our work is cut in half. If we know what the curve looks like for $x \geqslant 0$, then we need only reflect about the y-axis to obtain the complete curve [see Figure 3(a)]. Here are some examples: $y = x^2$, $y = x^4$, $y = |x|$, and $y = \cos x$.

(ii) If $f(-x) = -f(x)$ for all x in D, then f is an *odd function* and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x \geqslant 0$. [Rotate 180° about the origin; see Figure 3(b).] Some simple examples of odd functions are $y = x$, $y = x^3$, $y = 1/x$, and $y = \sin x$.



(a) Even function: reflectional symmetry



(b) Odd function: rotational symmetry

FIGURE 3

(iii) If $f(x + p) = f(x)$ for all x in D, where p is a positive constant, then f is a **periodic function** and the smallest such number p is called the **period**. For instance, $y = \sin x$ has period 2π and $y = \tan x$ has period π. If we know what the graph looks like in an interval of length p, then we can use translation to visualize the entire graph (see Figure 4).

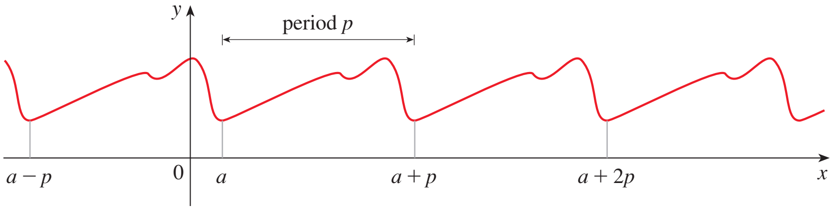


FIGURE 4
Periodic function: translational symmetry

D. Asymptotes

(i) *Horizontal Asymptotes.* Recall from Section 2.6 that if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$. If it turns out that $\lim_{x\to\infty} f(x) = \infty$ (or $-\infty$), then we do not have an asymptote to the right, but this fact is still useful information for sketching the curve.

(ii) *Vertical Asymptotes.* Recall from Section 2.2 that the line $x = a$ is a vertical asymptote if at least one of the following statements is true:

$$\boxed{1} \qquad \lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty$$

$$\lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$

(For rational functions you can locate the vertical asymptotes by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.) Furthermore, in sketching the curve it is useful to know exactly which of the statements in (1) is true. If $f(a)$ is not defined but a is an endpoint of the domain of f, then you should compute $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$, whether or not this limit is infinite.

(iii) *Slant Asymptotes.* These are discussed at the end of this section.

E. Intervals of Increase or Decrease Use the I/D Test. Compute $f'(x)$ and find the intervals on which $f'(x)$ is positive (f is increasing) and the intervals on which $f'(x)$ is negative (f is decreasing).

F. Local Maximum or Minimum Values Find the critical numbers of f [the numbers c where $f'(c) = 0$ or $f'(c)$ does not exist]. Then use the First Derivative Test. If f' changes from positive to negative at a critical number c, then $f(c)$ is a local maximum. If f' changes from negative to positive at c, then $f(c)$ is a local minimum. Although it is usually preferable to use the First Derivative Test, you can use the Second Derivative Test if $f'(c) = 0$ and $f''(c) \neq 0$. Then $f''(c) > 0$ implies that $f(c)$ is a local minimum, whereas $f''(c) < 0$ implies that $f(c)$ is a local maximum.

G. Concavity and Points of Inflection Compute $f''(x)$ and use the Concavity Test. The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$. Inflection points occur where the direction of concavity changes.

H. Sketch the Curve Using the information in items A–G, draw the graph. Sketch the asymptotes as dashed lines. Plot the intercepts, maximum and minimum points, and inflection points. Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes.

If additional accuracy is desired near any point, you can compute the value of the derivative there. The tangent indicates the direction in which the curve proceeds.

EXAMPLE 1 Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

A. Domain The domain is

$$\{x \mid x^2 - 1 \neq 0\} = \{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

B. Intercepts The x- and y-intercepts are both 0.

C. Symmetry Since $f(-x) = f(x)$, the function f is even. The curve is symmetric about the y-axis.

D. Asymptotes $$\lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{2}{1 - 1/x^2} = 2$$

Therefore the line $y = 2$ is a horizontal asymptote (at both the left and right).

Since the denominator is 0 when $x = \pm 1$, we compute the following limits:

$$\lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = \infty \qquad \lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = -\infty$$

$$\lim_{x \to -1^+} \frac{2x^2}{x^2 - 1} = -\infty \qquad \lim_{x \to -1^-} \frac{2x^2}{x^2 - 1} = \infty$$

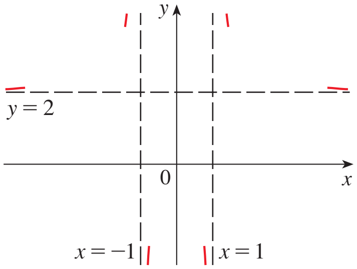


FIGURE 5
Preliminary sketch

We have shown the curve approaching its horizontal asymptote from above in Figure 5. This is confirmed by the intervals of increase and decrease.

Therefore the lines $x = 1$ and $x = -1$ are vertical asymptotes. This information about limits and asymptotes enables us to draw the preliminary sketch in Figure 5, showing the parts of the curve near the asymptotes.

E. Intervals of Increase or Decrease

$$f'(x) = \frac{(x^2 - 1)(4x) - 2x^2 \cdot 2x}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}$$

Since $f'(x) > 0$ when $x < 0$ $(x \neq -1)$ and $f'(x) < 0$ when $x > 0$ $(x \neq 1)$, f is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, \infty)$.

F. Local Maximum or Minimum Values The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = 0$ is a local maximum by the First Derivative Test.

G. Concavity and Points of Inflection

$$f''(x) = \frac{(x^2 - 1)^2(-4) + 4x \cdot 2(x^2 - 1)2x}{(x^2 - 1)^4} = \frac{12x^2 + 4}{(x^2 - 1)^3}$$

Since $12x^2 + 4 > 0$ for all x, we have

$$f''(x) > 0 \iff x^2 - 1 > 0 \iff |x| > 1$$

and $f''(x) < 0 \iff |x| < 1$. Thus the curve is concave upward on the intervals $(-\infty, -1)$ and $(1, \infty)$ and concave downward on $(-1, 1)$. It has no point of inflection because 1 and -1 are not in the domain of f.

H. Sketch the Curve Using the information in E–G, we finish the sketch in Figure 6. ■

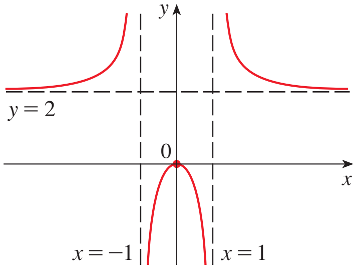


FIGURE 6
Finished sketch of $y = \dfrac{2x^2}{x^2 - 1}$

EXAMPLE 2 Sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x + 1}}$.

A. Domain The domain is $\{x \mid x + 1 > 0\} = \{x \mid x > -1\} = (-1, \infty)$.

B. Intercepts The x- and y-intercepts are both 0.

C. Symmetry None

D. Asymptotes Since

$$\lim_{x\to\infty} \frac{x^2}{\sqrt{x+1}} = \infty$$

there is no horizontal asymptote. Since $\sqrt{x+1} \to 0$ as $x \to -1^+$ and $f(x)$ is always positive, we have

$$\lim_{x\to -1^+} \frac{x^2}{\sqrt{x+1}} = \infty$$

and so the line $x = -1$ is a vertical asymptote.

E. Intervals of Increase or Decrease

$$f'(x) = \frac{\sqrt{x+1}\,(2x) - x^2 \cdot 1/\big(2\sqrt{x+1}\big)}{x+1} = \frac{3x^2 + 4x}{2(x+1)^{3/2}} = \frac{x(3x+4)}{2(x+1)^{3/2}}$$

We see that $f'(x) = 0$ when $x = 0$ (notice that $-\frac{4}{3}$ is not in the domain of f), so the only critical number is 0. Since $f'(x) < 0$ when $-1 < x < 0$ and $f'(x) > 0$ when $x > 0$, f is decreasing on $(-1, 0)$ and increasing on $(0, \infty)$.

F. Local Maximum or Minimum Values Since $f'(0) = 0$ and f' changes from negative to positive at 0, $f(0) = 0$ is a local (and absolute) minimum by the First Derivative Test.

G. Concavity and Points of Inflection

$$f''(x) = \frac{2(x+1)^{3/2}(6x+4) - (3x^2+4x)3(x+1)^{1/2}}{4(x+1)^3} = \frac{3x^2 + 8x + 8}{4(x+1)^{5/2}}$$

Note that the denominator is always positive. The numerator is the quadratic $3x^2 + 8x + 8$, which is always positive because its discriminant is $b^2 - 4ac = -32$, which is negative, and the coefficient of x^2 is positive. Thus $f''(x) > 0$ for all x in the domain of f, which means that f is concave upward on $(-1, \infty)$ and there is no point of inflection.

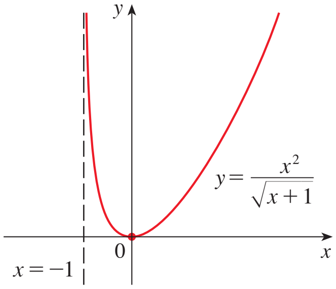


FIGURE 7

H. Sketch the Curve The curve is sketched in Figure 7. ■

EXAMPLE 3 Sketch the graph of $f(x) = xe^x$.

A. Domain The domain is $\mathbb{R}$.

B. Intercepts The x- and y-intercepts are both 0.

C. Symmetry None

D. Asymptotes Because both x and e^x become large as $x \to \infty$, we have $\lim_{x\to\infty} xe^x = \infty$. As $x \to -\infty$, however, $e^x \to 0$ and so we have an indeterminate product that requires the use of l'Hospital's Rule:

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \lim_{x\to-\infty} (-e^x) = 0$$

Thus the x-axis is a horizontal asymptote.

E. Intervals of Increase or Decrease

$$f'(x) = xe^x + e^x = (x + 1)e^x$$

Since e^x is always positive, we see that $f'(x) > 0$ when $x + 1 > 0$, and $f'(x) < 0$ when $x + 1 < 0$. So f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$.

F. Local Maximum or Minimum Values Because $f'(-1) = 0$ and f' changes from negative to positive at $x = -1$, $f(-1) = -e^{-1} \approx -0.37$ is a local (and absolute) minimum.

G. Concavity and Points of Inflection

$$f''(x) = (x + 1)e^x + e^x = (x + 2)e^x$$

Since $f''(x) > 0$ if $x > -2$ and $f''(x) < 0$ if $x < -2$, f is concave upward on $(-2, \infty)$ and concave downward on $(-\infty, -2)$. The inflection point is $(-2, -2e^{-2}) \approx (-2, -0.27)$.

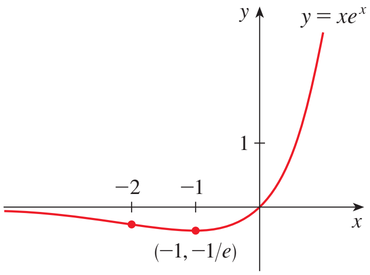


FIGURE 8

H. Sketch the Curve We use this information to sketch the curve in Figure 8. ■

EXAMPLE 4 Sketch the graph of $f(x) = \dfrac{\cos x}{2 + \sin x}$.

A. Domain The domain is $\mathbb{R}$.

B. Intercepts The y-intercept is $f(0) = \frac{1}{2}$. The x-intercepts occur when $\cos x = 0$, that is, $x = (\pi/2) + n\pi$, where n is an integer.

C. Symmetry f is neither even nor odd, but $f(x + 2\pi) = f(x)$ for all x and so f is periodic and has period 2π. Thus, in what follows, we need to consider only $0 \leqslant x \leqslant 2\pi$ and then extend the curve by translation in part H.

D. Asymptotes None

E. Intervals of Increase or Decrease

$$f'(x) = \frac{(2 + \sin x)(-\sin x) - \cos x\,(\cos x)}{(2 + \sin x)^2} = -\frac{2\sin x + 1}{(2 + \sin x)^2}$$

The denominator is always positive, so $f'(x) > 0$ when $2\sin x + 1 < 0 \iff \sin x < -\frac{1}{2} \iff 7\pi/6 < x < 11\pi/6$. So f is increasing on $(7\pi/6, 11\pi/6)$ and decreasing on $(0, 7\pi/6)$ and $(11\pi/6, 2\pi)$.

F. Local Maximum or Minimum Values From part E and the First Derivative Test, we see that the local minimum value is $f(7\pi/6) = -1/\sqrt{3}$ and the local maximum value is $f(11\pi/6) = 1/\sqrt{3}$.

G. Concavity and Points of Inflection If we use the Quotient Rule again and simplify, we get

$$f''(x) = -\frac{2\cos x\,(1 - \sin x)}{(2 + \sin x)^3}$$

Because $(2 + \sin x)^3 > 0$ and $1 - \sin x \geqslant 0$ for all x, we know that $f''(x) > 0$ when $\cos x < 0$, that is, $\pi/2 < x < 3\pi/2$. So f is concave upward on $(\pi/2, 3\pi/2)$ and concave downward on $(0, \pi/2)$ and $(3\pi/2, 2\pi)$. The inflection points are $(\pi/2, 0)$ and $(3\pi/2, 0)$.

H. Sketch the Curve The graph of the function restricted to $0 \leqslant x \leqslant 2\pi$ is shown in Figure 9. Then we extend it, using periodicity, to arrive at the graph in Figure 10.

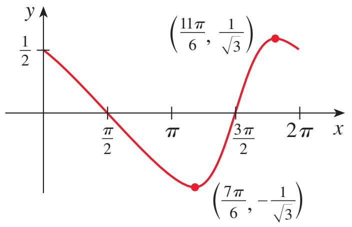


FIGURE 9

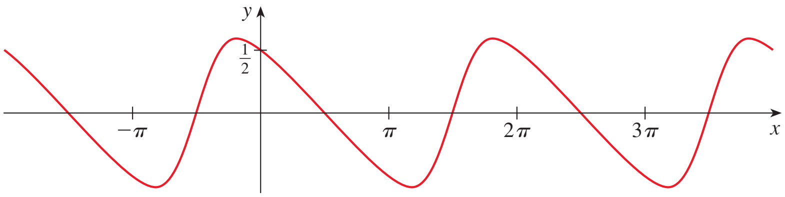


FIGURE 10 ■

EXAMPLE 5 Sketch the graph of $y = \ln(4 - x^2)$.

A. Domain The domain is

$$\{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\} = (-2, 2)$$

B. Intercepts The y-intercept is $f(0) = \ln 4$. To find the x-intercept we set

$$y = \ln(4 - x^2) = 0$$

We know that $\ln 1 = 0$, so we have $4 - x^2 = 1 \Rightarrow x^2 = 3$ and therefore the x-intercepts are $\pm\sqrt{3}$.

C. Symmetry Since $f(-x) = f(x)$, f is even and the curve is symmetric about the y-axis.

D. Asymptotes We look for vertical asymptotes at the endpoints of the domain. Since $4 - x^2 \to 0^+$ as $x \to 2^-$ and also as $x \to -2^+$, we have

$$\lim_{x \to 2^-} \ln(4 - x^2) = -\infty \qquad \lim_{x \to -2^+} \ln(4 - x^2) = -\infty$$

Thus the lines $x = 2$ and $x = -2$ are vertical asymptotes.

E. Intervals of Increase or Decrease

$$f'(x) = \frac{-2x}{4 - x^2}$$

Since $f'(x) > 0$ when $-2 < x < 0$ and $f'(x) < 0$ when $0 < x < 2$, f is increasing on $(-2, 0)$ and decreasing on $(0, 2)$.

F. Local Maximum or Minimum Values The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = \ln 4$ is a local maximum by the First Derivative Test.

G. Concavity and Points of Inflection

$$f''(x) = \frac{(4 - x^2)(-2) + 2x(-2x)}{(4 - x^2)^2} = \frac{-8 - 2x^2}{(4 - x^2)^2}$$

Since $f''(x) < 0$ for all x, the curve is concave downward on $(-2, 2)$ and has no inflection point.

H. Sketch the Curve Using this information, we sketch the curve in Figure 11. ■

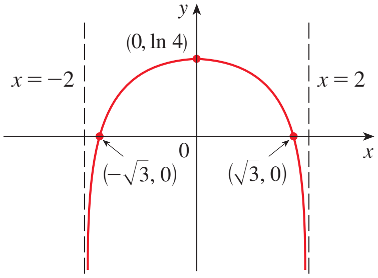


FIGURE 11
$y = \ln(4 - x^2)$

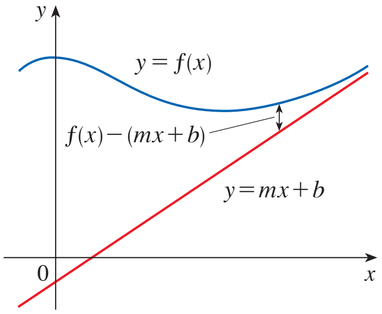


FIGURE 12

Slant Asymptotes

Some curves have asymptotes that are *oblique*, that is, neither horizontal nor vertical. If

$$\lim_{x \to \infty} [f(x) - (mx + b)] = 0$$

where $m \neq 0$, then the line $y = mx + b$ is called a **slant asymptote** because the vertical distance between the curve $y = f(x)$ and the line $y = mx + b$ approaches 0, as in Figure 12. (A similar situation exists if we let $x \to -\infty$.) In the case of rational functions, slant asymptotes occur when the degree of the numerator is one more than the degree of the denominator. In such a case the equation of the slant asymptote can be found by long division as in the following example.

EXAMPLE 6 Sketch the graph of $f(x) = \dfrac{x^3}{x^2 + 1}$.

A. Domain The domain is $\mathbb{R}$.

B. Intercepts The x- and y-intercepts are both 0.

C. Symmetry Since $f(-x) = -f(x)$, f is odd and its graph is symmetric about the origin.

D. Asymptotes Since $x^2 + 1$ is never 0, there is no vertical asymptote. Since $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$, there is no horizontal asymptote. But long division gives

$$f(x) = \frac{x^3}{x^2 + 1} = x - \frac{x}{x^2 + 1}$$

This equation suggests that $y = x$ is a candidate for a slant asymptote. In fact,

$$f(x) - x = -\frac{x}{x^2 + 1} = -\frac{\dfrac{1}{x}}{1 + \dfrac{1}{x^2}} \to 0 \quad \text{as} \quad x \to \pm\infty$$

So the line $y = x$ is indeed a slant asymptote.

E. Intervals of Increase or Decrease

$$f'(x) = \frac{(x^2 + 1)(3x^2) - x^3 \cdot 2x}{(x^2 + 1)^2} = \frac{x^2(x^2 + 3)}{(x^2 + 1)^2}$$

Since $f'(x) > 0$ for all x (except 0), f is increasing on $(-\infty, \infty)$.

F. Local Maximum or Minimum Values Although $f'(0) = 0$, f' does not change sign at 0, so there is no local maximum or minimum.

G. Concavity and Points of Inflection

$$f''(x) = \frac{(x^2 + 1)^2(4x^3 + 6x) - (x^4 + 3x^2) \cdot 2(x^2 + 1)2x}{(x^2 + 1)^4} = \frac{2x(3 - x^2)}{(x^2 + 1)^3}$$

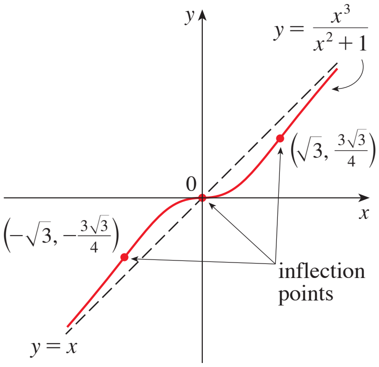


FIGURE 13

Since $f''(x) = 0$ when $x = 0$ or $x = \pm\sqrt{3}$, we set up the following chart:

Interval	x	$3 - x^2$	$(x^2 + 1)^3$	$f''(x)$	f
$x < -\sqrt{3}$	$-$	$-$	$+$	$+$	CU on $(-\infty, -\sqrt{3})$
$-\sqrt{3} < x < 0$	$-$	$+$	$+$	$-$	CD on $(-\sqrt{3}, 0)$
$0 < x < \sqrt{3}$	$+$	$+$	$+$	$+$	CU on $(0, \sqrt{3})$
$x > \sqrt{3}$	$+$	$-$	$+$	$-$	CD on $(\sqrt{3}, \infty)$

The points of inflection are $\left(-\sqrt{3}, -\frac{3}{4}\sqrt{3}\right)$, $(0, 0)$, and $\left(\sqrt{3}, \frac{3}{4}\sqrt{3}\right)$.

H. Sketch the Curve The graph of f is sketched in Figure 13. ■

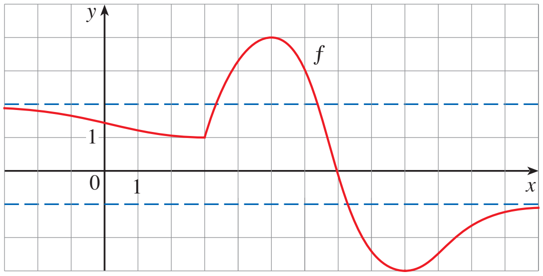
4.5 Exercises

1–54 Use the guidelines of this section to sketch the curve.

1. $y = x^3 + 3x^2$
2. $y = 2x^3 - 12x^2 + 18x$
3. $y = x^4 - 4x$
4. $y = x^4 - 8x^2 + 8$
5. $y = x(x - 4)^3$
6. $y = x^5 - 5x$
7. $y = \frac{1}{5}x^5 - \frac{8}{3}x^3 + 16x$
8. $y = (4 - x^2)^5$
9. $y = \dfrac{2x + 3}{x + 2}$
10. $y = \dfrac{x^2 + 5x}{25 - x^2}$
11. $y = \dfrac{x - x^2}{2 - 3x + x^2}$
12. $y = 1 + \dfrac{1}{x} + \dfrac{1}{x^2}$
13. $y = \dfrac{x}{x^2 - 4}$
14. $y = \dfrac{1}{x^2 - 4}$
15. $y = \dfrac{x^2}{x^2 + 3}$
16. $y = \dfrac{(x - 1)^2}{x^2 + 1}$
17. $y = \dfrac{x - 1}{x^2}$
18. $y = \dfrac{x}{x^3 - 1}$
19. $y = \dfrac{x^3}{x^3 + 1}$
20. $y = \dfrac{x^3}{x - 2}$
21. $y = (x - 3)\sqrt{x}$
22. $y = (x - 4)\sqrt[3]{x}$
23. $y = \sqrt{x^2 + x - 2}$
24. $y = \sqrt{x^2 + x} - x$
25. $y = \dfrac{x}{\sqrt{x^2 + 1}}$
26. $y = x\sqrt{2 - x^2}$
27. $y = \dfrac{\sqrt{1 - x^2}}{x}$
28. $y = \dfrac{x}{\sqrt{x^2 - 1}}$
29. $y = x - 3x^{1/3}$
30. $y = x^{5/3} - 5x^{2/3}$
31. $y = \sqrt[3]{x^2 - 1}$
32. $y = \sqrt[3]{x^3 + 1}$
33. $y = \sin^3 x$
34. $y = x + \cos x$
35. $y = x \tan x, \quad -\pi/2 < x < \pi/2$
36. $y = 2x - \tan x, \quad -\pi/2 < x < \pi/2$
37. $y = \sin x + \sqrt{3} \cos x, \quad -2\pi \leqslant x \leqslant 2\pi$
38. $y = \csc x - 2\sin x, \quad 0 < x < \pi$
39. $y = \dfrac{\sin x}{1 + \cos x}$
40. $y = \dfrac{\sin x}{2 + \cos x}$
41. $y = \arctan(e^x)$
42. $y = (1 - x)e^x$
43. $y = 1/(1 + e^{-x})$
44. $y = e^{-x} \sin x, \quad 0 \leqslant x \leqslant 2\pi$
45. $y = \dfrac{1}{x} + \ln x$
46. $y = x(\ln x)^2$
47. $y = (1 + e^x)^{-2}$
48. $y = e^x/x^2$
49. $y = \ln(\sin x)$
50. $y = \ln(1 + x^3)$
51. $y = xe^{-1/x}$
52. $y = \dfrac{\ln x}{x^2}$
53. $y = e^{\arctan x}$
54. $y = \tan^{-1}\left(\dfrac{x - 1}{x + 1}\right)$

55–58 The graph of a function f is shown. (The dashed lines indicate horizontal asymptotes.) Find each of the following for the given function g.
(a) The domains of g and g'
(b) The critical numbers of g
(c) The approximate value of $g'(6)$

(d) All vertical and horizontal asymptotes of g



55. $g(x) = \sqrt{f(x)}$

56. $g(x) = \sqrt[3]{f(x)}$

57. $g(x) = |f(x)|$

58. $g(x) = 1/f(x)$

---

59. In the theory of relativity, the mass of a particle is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle, m is the mass when the particle moves with speed v relative to the observer, and c is the speed of light. Sketch the graph of m as a function of v.

60. In the theory of relativity, the energy of a particle is

$$E = \sqrt{m_0^2c^4 + h^2c^2/\lambda^2}$$

where m_0 is the rest mass of the particle, λ is its wave length, and h is Planck's constant. Sketch the graph of E as a function of λ. What does the graph say about the energy?

61. A model for the spread of a rumor is given by the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that knows the rumor at time t and a and k are positive constants.

(a) When will half the population have heard the rumor?
(b) When is the rate of spread of the rumor greatest?
(c) Sketch the graph of p.

62. A model for the concentration at time t of a drug injected into the bloodstream is

$$C(t) = K(e^{-at} - e^{-bt})$$

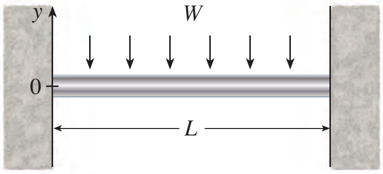
where a, b, and K are positive constants and $b > a$. Sketch the graph of the concentration function. What does the graph tell us about how the concentration varies as time passes?

63. The figure shows a beam of length L embedded in concrete walls. If a constant load W is distributed evenly along its length, the beam takes the shape of the deflection curve

$$y = -\frac{W}{24EI}x^4 + \frac{WL}{12EI}x^3 - \frac{WL^2}{24EI}x^2$$

where E and I are positive constants. (E is Young's modulus of elasticity and I is the moment of inertia of a cross-

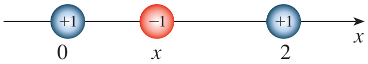
etch the graph of the deflection



64. Coulomb's Law states that the force of attraction between two charged particles is directly proportional to the product of the charges and inversely proportional to the square of the distance between them. The figure shows particles with charge 1 located at positions 0 and 2 on a coordinate line and a particle with charge -1 at a position x between them. It follows from Coulomb's Law that the net force acting on the middle particle is

$$F(x) = -\frac{k}{x^2} + \frac{k}{(x-2)^2} \qquad 0 < x < 2$$

where k is a positive constant. Sketch the graph of the net force function. What does the graph say about the force?



65–68 Find an equation of the slant asymptote. Do not sketch the curve.

65. $y = \dfrac{x^2 + 1}{x + 1}$

66. $y = \dfrac{4x^3 - 10x^2 - 11x + 1}{x^2 - 3x}$

67. $y = \dfrac{2x^3 - 5x^2 + 3x}{x^2 - x - 2}$

68. $y = \dfrac{-6x^4 + 2x^3 + 3}{2x^3 - x}$

69–74 Use the guidelines of this section to sketch the curve. In guideline D, find an equation of the slant asymptote.

69. $y = \dfrac{x^2}{x - 1}$

70. $y = \dfrac{1 + 5x - 2x^2}{x - 2}$

71. $y = \dfrac{x^3 + 4}{x^2}$

72. $y = \dfrac{x^3}{(x + 1)^2}$

73. $y = 1 + \frac{1}{2}x + e^{-x}$

74. $y = 1 - x + e^{1 + x/3}$

---

75. Show that the curve $y = x - \tan^{-1}x$ has two slant asymptotes: $y = x + \pi/2$ and $y = x - \pi/2$. Use this fact to help sketch the curve.

76. Show that the curve $y = \sqrt{x^2 + 4x}$ has two slant asymptotes: $y = x + 2$ and $y = -x - 2$. Use this fact to help sketch the curve.

77. Show that the lines $y = (b/a)x$ and $y = -(b/a)x$ are slant asymptotes of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$.

78. Let $f(x) = (x^3 + 1)/x$. Show that

$$\lim_{x \to \pm\infty} [f(x) - x^2] = 0$$

This shows that the graph of f approaches the graph of $y = x^2$, and we say that the curve $y = f(x)$ is *asymptotic* to the parabola $y = x^2$. Use this fact to help sketch the graph of f.

79. Discuss the asymptotic behavior of $f(x) = (x^4 + 1)/x$ in the same manner as in Exercise 78. Then use your results to help sketch the graph of f.

80. Use the asymptotic behavior of $f(x) = \sin x + e^{-x}$ to sketch its graph without going through the curve-sketching procedure of this section.

4.6 Graphing with Calculus *and* Technology

You may want to read Graphing Calculators and Computers at www.StewartCalculus.com if you haven't already. In particular, it explains how to avoid some of the pitfalls of graphing devices by choosing appropriate viewing rectangles.

The method we used to sketch curves in the preceding section was a culmination of much of our study of differential calculus. The graph was the final object that we produced. In this section our point of view is completely different. Here we *start* with a graph produced by a graphing calculator or computer and then we refine it. We use calculus to make sure that we reveal all the important aspects of the curve. And with the use of graphing devices we can tackle curves that would be far too complicated to consider without technology. The theme is the *interaction* between calculus and technology.

EXAMPLE 1 Graph the polynomial $f(x) = 2x^6 + 3x^5 + 3x^3 - 2x^2$. Use the graphs of f' and f'' to estimate all maximum and minimum points and intervals of concavity.

SOLUTION If we specify a domain but not a range, graphing software will often deduce a suitable range from the values computed. Figure 1 shows a plot that may result if we specify that $-5 \leqslant x \leqslant 5$. Although this viewing rectangle is useful for showing that the asymptotic behavior (or end behavior) is the same as for $y = 2x^6$, it is obviously hiding some finer detail. So we change to the viewing rectangle $[-3, 2]$ by $[-50, 100]$ in Figure 2.

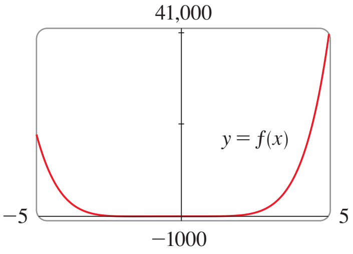


FIGURE 1

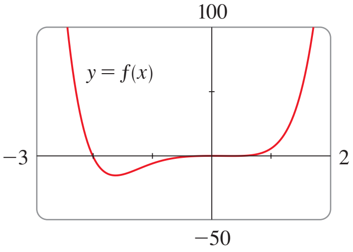


FIGURE 2

Most graphing calculators and graphing software allow us to "trace" along a curve and see approximate coordinates of points. (Some also have features to identify the approximate locations of local maximum and minimum points.) Here it appears that there is an absolute minimum value of about -15.33 when $x \approx -1.62$ and f is decreasing on $(-\infty, -1.62)$ and increasing on $(-1.62, \infty)$. Also, there appears to be a horizontal tangent at the origin and inflection points when $x = 0$ and when x is somewhere between -2 and -1.

Now let's try to confirm these impressions using calculus. We differentiate and get

$$f'(x) = 12x^5 + 15x^4 + 9x^2 - 4x$$

$$f''(x) = 60x^4 + 60x^3 + 18x - 4$$

When we graph f' in Figure 3 we see that $f'(x)$ changes from negative to positive when $x \approx -1.62$; this confirms (by the First Derivative Test) the minimum value that we found earlier. But, perhaps to our surprise, we also notice that $f'(x)$ changes from positive to negative when $x = 0$ and from negative to positive when $x \approx 0.35$. This means that f has a local maximum at 0 and a local minimum when $x \approx 0.35$, but these were hidden in Figure 2. Indeed, if we now zoom in toward the origin in Figure 4, we

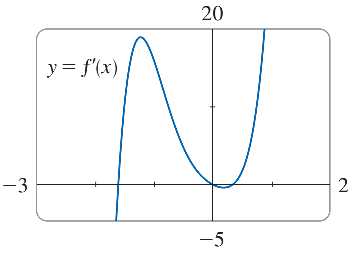


FIGURE 3

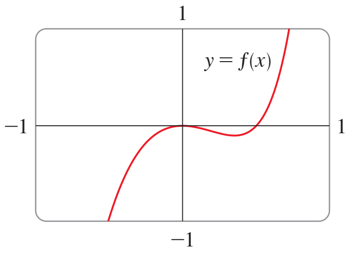


FIGURE 4

see what we missed before: a local maximum value of 0 when $x = 0$ and a local minimum value of about -0.1 when $x \approx 0.35$.

What about concavity and inflection points? From Figures 2 and 4 there appear to be inflection points when x is a little to the left of -1 and when x is a little to the right of 0. But it's difficult to determine inflection points from the graph of f, so we graph the second derivative f'' in Figure 5. We see that f'' changes from positive to negative when $x \approx -1.23$ and from negative to positive when $x \approx 0.19$. So, correct to two decimal places, f is concave upward on $(-\infty, -1.23)$ and $(0.19, \infty)$ and concave downward on $(-1.23, 0.19)$. The inflection points are $(-1.23, -10.18)$ and $(0.19, -0.05)$.

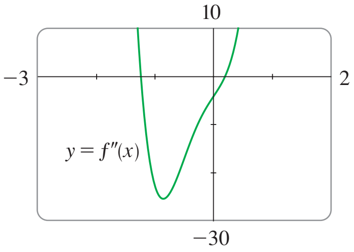


FIGURE 5

We have discovered that no single graph reveals all the important features of this polynomial. But Figures 2 and 4, when taken together, do provide an accurate picture. ■

EXAMPLE 2 Draw the graph of the function

$$f(x) = \frac{x^2 + 7x + 3}{x^2}$$

in a viewing rectangle that shows all the important features of the function. Estimate the local maximum and minimum values and the intervals of concavity. Then use calculus to find these quantities exactly.

SOLUTION Figure 6—produced by graphing software with automatic scaling—is a disaster. Some graphing calculators use $[-10, 10]$ by $[-10, 10]$ as the default viewing rectangle, so let's try it. We get the graph shown in Figure 7; it's a major improvement.

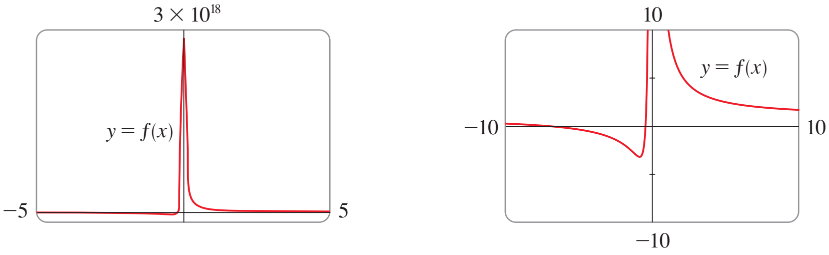


FIGURE 6 **FIGURE 7**

The y-axis appears to be a vertical asymptote and indeed it is because

$$\lim_{x \to 0} \frac{x^2 + 7x + 3}{x^2} = \infty$$

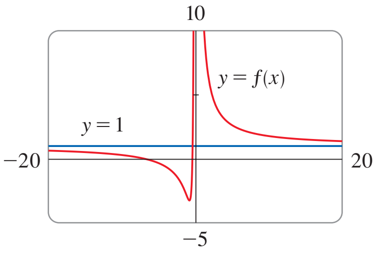


FIGURE 8

Figure 7 also allows us to estimate the x-intercepts: about -0.5 and -6.5. The exact values are obtained by using the quadratic formula to solve the equation $x^2 + 7x + 3 = 0$; we get $x = \left(-7 \pm \sqrt{37}\right)/2$.

To get a better look at horizontal asymptotes, we change to the viewing rectangle $[-20, 20]$ by $[-5, 10]$ in Figure 8. It appears that $y = 1$ is the horizontal asymptote and this is easily confirmed:

$$\lim_{x \to \pm\infty} \frac{x^2 + 7x + 3}{x^2} = \lim_{x \to \pm\infty} \left(1 + \frac{7}{x} + \frac{3}{x^2}\right) = 1$$

To estimate the minimum value we zoom in to the viewing rectangle $[-3, 0]$ by $[-4, 2]$ in Figure 9. We find that the absolute minimum value is about -3.1 when $x \approx -0.9$, and we see that the function decreases on $(-\infty, -0.9)$ and $(0, \infty)$ and increases on $(-0.9, 0)$. The exact values are obtained by differentiating:

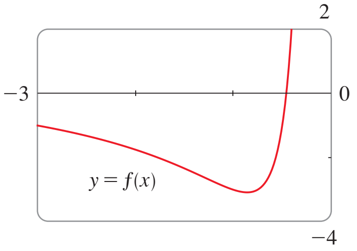


FIGURE 9

$$f'(x) = -\frac{7}{x^2} - \frac{6}{x^3} = -\frac{7x + 6}{x^3}$$

This shows that $f'(x) > 0$ when $-\frac{6}{7} < x < 0$ and $f'(x) < 0$ when $x < -\frac{6}{7}$ and when $x > 0$. The exact minimum value is $f\left(-\frac{6}{7}\right) = -\frac{37}{12} \approx -3.08$.

Figure 9 also shows that an inflection point occurs somewhere between $x = -1$ and $x = -2$. We could estimate it much more accurately using the graph of the second derivative, but in this case it's just as easy to find exact values. Since

$$f''(x) = \frac{14}{x^3} + \frac{18}{x^4} = \frac{2(7x + 9)}{x^4}$$

we see that $f''(x) > 0$ when $x > -\frac{9}{7}$ $(x \neq 0)$ and $f''(x) < 0$ when $x < -\frac{9}{7}$. So f is concave upward on $\left(-\frac{9}{7}, 0\right)$ and $(0, \infty)$ and concave downward on $\left(-\infty, -\frac{9}{7}\right)$. The inflection point is $\left(-\frac{9}{7}, -\frac{71}{27}\right)$.

The analysis using the first two derivatives shows that Figure 8 displays all the major aspects of the curve. ■

EXAMPLE 3 Graph the function $f(x) = \dfrac{x^2(x + 1)^3}{(x - 2)^2(x - 4)^4}$.

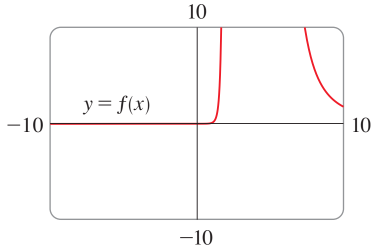


FIGURE 10

SOLUTION Drawing on our experience with a rational function in Example 2, let's start by graphing f in the viewing rectangle $[-10, 10]$ by $[-10, 10]$. From Figure 10 we have the feeling that we are going to have to zoom in to see some finer detail and also zoom out to see the larger picture. But, as a guide to intelligent zooming, let's first take a close look at the expression for $f(x)$. Because of the factors $(x - 2)^2$ and $(x - 4)^4$ in the denominator, we expect $x = 2$ and $x = 4$ to be the vertical asymptotes. Indeed

$$\lim_{x \to 2} \frac{x^2(x + 1)^3}{(x - 2)^2(x - 4)^4} = \infty \qquad \text{and} \qquad \lim_{x \to 4} \frac{x^2(x + 1)^3}{(x - 2)^2(x - 4)^4} = \infty$$

To find the horizontal asymptotes, we divide numerator and denominator by x^6:

$$\frac{x^2(x + 1)^3}{(x - 2)^2(x - 4)^4} = \frac{\dfrac{x^2}{x^3} \cdot \dfrac{(x + 1)^3}{x^3}}{\dfrac{(x - 2)^2}{x^2} \cdot \dfrac{(x - 4)^4}{x^4}} = \frac{\dfrac{1}{x}\left(1 + \dfrac{1}{x}\right)^3}{\left(1 - \dfrac{2}{x}\right)^2\left(1 - \dfrac{4}{x}\right)^4}$$

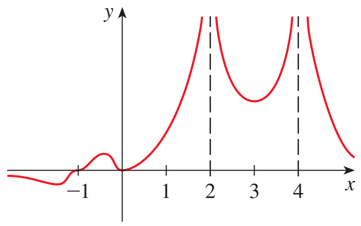


FIGURE 11

This shows that $f(x) \to 0$ as $x \to \pm\infty$, so the x-axis is a horizontal asymptote.

It is also very useful to consider the behavior of the graph near the x-intercepts using an analysis like that in Example 2.6.12. Since x^2 is positive, $f(x)$ does not change sign at 0 and so its graph doesn't cross the x-axis at 0. But, because of the factor $(x + 1)^3$, the graph does cross the x-axis at -1 and has a horizontal tangent there. Putting all this information together, but without using derivatives, we see that the curve has to look something like the one in Figure 11.

Now that we know what to look for, we zoom in (several times) to produce the graphs in Figures 12 and 13 and zoom out (several times) to get Figure 14.

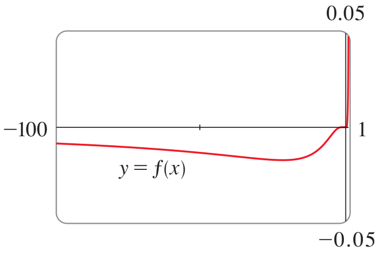


FIGURE 12

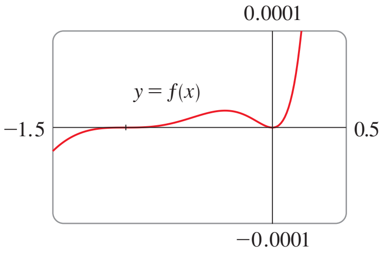


FIGURE 13

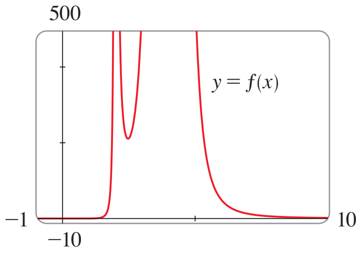


FIGURE 14

We can read from these graphs that the absolute minimum is about -0.02 and occurs when $x \approx -20$. There is also a local maximum ≈ 0.00002 when $x \approx -0.3$ and a local minimum ≈ 211 when $x \approx 2.5$. These graphs also show three inflection points near -35, -5, and -1 and two between -1 and 0. To estimate the inflection points closely we would need to graph f'', but to compute f'' by hand is an unreasonable chore. If you have a computer algebra system, then it's easy to do (see Exercise 15).

We have seen that, for this particular function, *three* graphs (Figures 12, 13, and 14) are necessary to convey all the useful information. The only way to display all these features of the function on a single graph is to draw it by hand. Despite the exaggerations and distortions, Figure 11 does manage to summarize the essential nature of the function. ■

EXAMPLE 4 Graph the function $f(x) = \sin(x + \sin 2x)$. For $0 \leqslant x \leqslant \pi$, estimate all maximum and minimum values, intervals of increase and decrease, and inflection points.

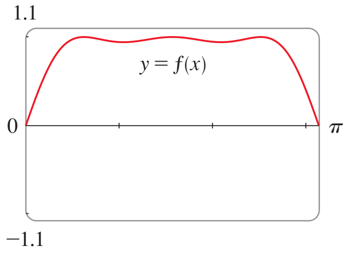


FIGURE 15

SOLUTION We first note that f is periodic with period 2π. Also, f is odd and $|f(x)| \leqslant 1$ for all x. So the choice of a viewing rectangle is not a problem for this function: we start with $[0, \pi]$ by $[-1.1, 1.1]$. (See Figure 15.) It appears that there are three local maximum values and two local minimum values in that window. To confirm these values and locate them more accurately, we calculate that

$$f'(x) = \cos(x + \sin 2x) \cdot (1 + 2\cos 2x)$$

and graph both f and f' in Figure 16.

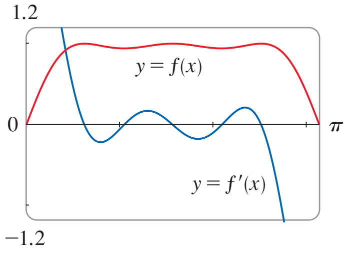


FIGURE 16

After estimating the values of the x-intercepts of f', we use the First Derivative Test to find the following approximate values:

Intervals of increase:	$(0, 0.6),\ (1.0, 1.6),\ (2.1, 2.5)$
Intervals of decrease:	$(0.6, 1.0),\ (1.6, 2.1),\ (2.5, \pi)$
Local maximum values:	$f(0.6) \approx 1,\ f(1.6) \approx 1,\ f(2.5) \approx 1$
Local minimum values:	$f(1.0) \approx 0.94,\ f(2.1) \approx 0.94$

The second derivative is

$$f''(x) = -(1 + 2\cos 2x)^2 \sin(x + \sin 2x) - 4 \sin 2x \cos(x + \sin 2x)$$

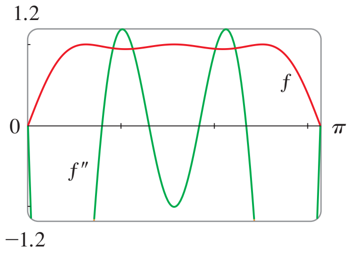


FIGURE 17

The family of functions

$$f(x) = \sin(x + \sin cx)$$

where c is a constant, occurs in applications to frequency modulation (FM) synthesis. A sine wave is modulated by a wave with a different frequency ($\sin cx$). The case where $c = 2$ is studied in Example 4. Exercise 27 explores another special case.

Graphing both f and f'' in Figure 17, we obtain the following approximate values:

Concave upward on:	$(0.8, 1.3)$, $(1.8, 2.3)$
Concave downward on:	$(0, 0.8)$, $(1.3, 1.8)$, $(2.3, \pi)$
Inflection points:	$(0, 0)$, $(0.8, 0.97)$, $(1.3, 0.97)$, $(1.8, 0.97)$, $(2.3, 0.97)$

Having checked that Figure 15 does indeed represent f accurately for $0 \leqslant x \leqslant \pi$, we can state that the extended graph in Figure 18 represents f accurately for $-2\pi \leqslant x \leqslant 2\pi$.

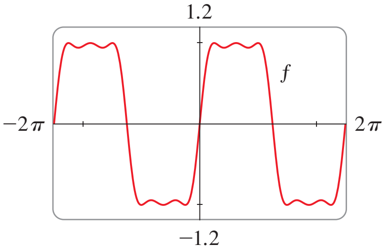


FIGURE 18

■

Our final example is concerned with *families* of functions. This means that the functions in the family are related to each other by a formula that contains one or more arbitrary constants. Each value of the constant gives rise to a member of the family and the idea is to see how the graph of the function changes as the constant changes.

EXAMPLE 5 How does the graph of $f(x) = 1/(x^2 + 2x + c)$ vary as c varies?

SOLUTION The graphs in Figures 19 and 20 (the special cases $c = 2$ and $c = -2$) show two very different-looking curves.

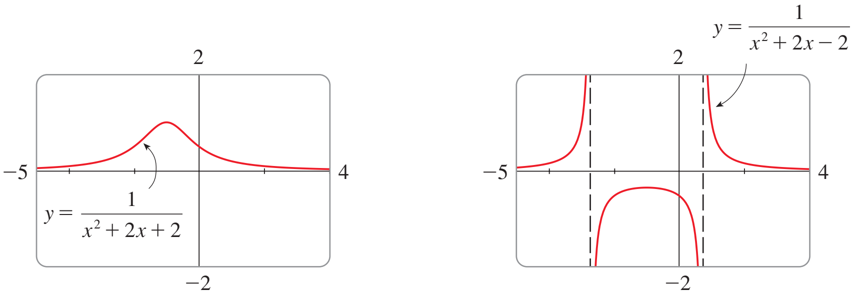


FIGURE 19
$c = 2$

FIGURE 20
$c = -2$

Before drawing any more graphs, let's see what members of this family have in common. Since

$$\lim_{x \to \pm\infty} \frac{1}{x^2 + 2x + c} = 0$$

for any value of c, they all have the x-axis as a horizontal asymptote. A vertical asymptote will occur when $x^2 + 2x + c = 0$. Solving this quadratic equation, we get $x = -1 \pm \sqrt{1 - c}$. When $c > 1$, there is no vertical asymptote (as in Figure 19). When $c = 1$, the graph has a single vertical asymptote $x = -1$ because

$$\lim_{x \to -1} \frac{1}{x^2 + 2x + 1} = \lim_{x \to -1} \frac{1}{(x + 1)^2} = \infty$$

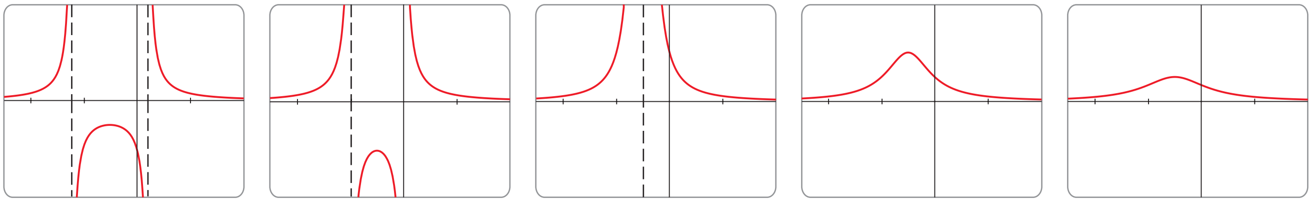
When $c < 1$, there are two vertical asymptotes: $x = -1 \pm \sqrt{1 - c}$ (as in Figure 20).

Now we compute the derivative:

$$f'(x) = -\frac{2x + 2}{(x^2 + 2x + c)^2}$$

This shows that $f'(x) = 0$ when $x = -1$ (if $c \neq 1$), $f'(x) > 0$ when $x < -1$, and $f'(x) < 0$ when $x > -1$. For $c \geqslant 1$, this means that f increases on $(-\infty, -1)$ and decreases on $(-1, \infty)$. For $c > 1$, there is an absolute maximum value $f(-1) = 1/(c - 1)$. For $c < 1$, $f(-1) = 1/(c - 1)$ is a local maximum value and the intervals of increase and decrease are interrupted at the vertical asymptotes.

Figure 21 is a "slide show" displaying five members of the family, all graphed in the viewing rectangle $[-5, 4]$ by $[-2, 2]$. As predicted, a transition takes place from two vertical asymptotes to one for $c = 1$, and then to none for $c > 1$. As c increases from 1, we see that the maximum point becomes lower; this is explained by the fact that $1/(c - 1) \to 0$ as $c \to \infty$. As c decreases from 1, the vertical asymptotes become more widely separated because the distance between them is $2\sqrt{1 - c}$, which becomes large as $c \to -\infty$. Again, the maximum point approaches the x-axis because $1/(c - 1) \to 0$ as $c \to -\infty$.



$c = -1$ $c = 0$ $c = 1$ $c = 2$ $c = 3$

FIGURE 21

The family of functions $f(x) = 1/(x^2 + 2x + c)$

There is clearly no inflection point when $c \leqslant 1$. For $c > 1$ we calculate that

$$f''(x) = \frac{2(3x^2 + 6x + 4 - c)}{(x^2 + 2x + c)^3}$$

and deduce that inflection points occur when $x = -1 \pm \sqrt{3(c - 1)}/3$. So the inflection points become more spread out as c increases and this seems plausible from the last two graphs of Figure 21. ■

4.6 Exercises

1–8 Produce graphs of f that reveal all the important aspects of the curve. In particular, you should use graphs of f' and f'' to estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points.

1. $f(x) = x^5 - 5x^4 - x^3 + 28x^2 - 2x$

2. $f(x) = -2x^6 + 5x^5 + 140x^3 - 110x^2$

3. $f(x) = x^6 - 5x^5 + 25x^3 - 6x^2 - 48x$

4. $f(x) = \dfrac{x^4 - x^3 - 8}{x^2 - x - 6}$

5. $f(x) = \dfrac{x}{x^3 + x^2 + 1}$

6. $f(x) = 6 \sin x - x^2, \quad -5 \leqslant x \leqslant 3$

7. $f(x) = 6 \sin x + \cot x, \quad -\pi \leqslant x \leqslant \pi$

8. $f(x) = e^x - 0.186x^4$

9–10 Produce graphs of f that reveal all the important aspects of the curve. Estimate the intervals of increase and decrease and intervals of concavity, and use calculus to find these intervals exactly.

9. $f(x) = 1 + \dfrac{1}{x} + \dfrac{8}{x^2} + \dfrac{1}{x^3}$

10. $f(x) = \dfrac{1}{x^8} - \dfrac{2 \times 10^8}{x^4}$

11–12
(a) Graph the function.
(b) Use l'Hospital's Rule to explain the behavior as $x \to 0$.
(c) Estimate the minimum value and intervals of concavity. Then use calculus to find the exact values.

11. $f(x) = x^2 \ln x$ **12.** $f(x) = xe^{1/x}$

13–14 Sketch the graph by hand using asymptotes and inter-cepts, but not derivatives. Then use your sketch as a guide to producing graphs using a calculator or computer that display the major features of the curve. Use these graphs to estimate the maximum and minimum values.

13. $f(x) = \dfrac{(x+4)(x-3)^2}{x^4(x-1)}$ **14.** $f(x) = \dfrac{(2x+3)^2(x-2)^5}{x^3(x-5)^2}$

T **15.** For the function f of Example 3, use a computer algebra system to calculate f' and then graph it to confirm that all the maximum and minimum values are as given in the example. Calculate f'' and use it to estimate the intervals of concavity and inflection points.

T **16.** For the function f of Exercise 14, use a computer algebra system to find f' and f'' and use their graphs to estimate the intervals of increase and decrease and concavity of f.

T **17–22** Use a computer algebra system to graph f and to find f' and f''. Use graphs of these derivatives to estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points of f.

17. $f(x) = \dfrac{x^3 + 5x^2 + 1}{x^4 + x^3 - x^2 + 2}$

18. $f(x) = \dfrac{x^{2/3}}{1 + x + x^4}$

19. $f(x) = \sqrt{x + 5\sin x}, \quad x \leqslant 20$

20. $f(x) = x - \tan^{-1}(x^2)$

21. $f(x) = \dfrac{1 - e^{1/x}}{1 + e^{1/x}}$

22. $f(x) = \dfrac{3}{3 + 2\sin x}$

23–24 Graph the function using as many viewing rectangles as you need to depict the true nature of the function.

23. $f(x) = \dfrac{1 - \cos(x^4)}{x^8}$ **24.** $f(x) = e^x + \ln|x - 4|$

25–26
(a) Graph the function.
(b) Explain the shape of the graph by computing the limit as $x \to 0^+$ or as $x \to \infty$.
(c) Estimate the maximum and minimum values and then use calculus to find the exact values.
T (d) Use a computer algebra system to compute f''. Then use a graph of f'' to estimate the x-coordinates of the inflection points.

25. $f(x) = x^{1/x}$ **26.** $f(x) = (\sin x)^{\sin x}$

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27. In Example 4 we considered a member of the family of functions $f(x) = \sin(x + \sin cx)$ that occur in FM synthesis. Here we investigate the function with $c = 3$. Start by graphing f in the viewing rectangle $[0, \pi]$ by $[-1.2, 1.2]$. How many local maximum points do you see? The graph has more than are visible to the naked eye. To discover the hidden maximum and minimum points you will need to examine the graph of f' very carefully. In fact, it helps to look at the graph of f'' at the same time. Find all the maximum and minimum values and inflection points. Then graph f in the viewing rectangle $[-2\pi, 2\pi]$ by $[-1.2, 1.2]$ and comment on symmetry.

28–35 Describe how the graph of f varies as c varies. Graph several members of the family to illustrate the trends that you discover. In particular, you should investigate how maximum and minimum points and inflection points move when c changes. You should also identify any transitional values of c at which the basic shape of the curve changes.

28. $f(x) = x^3 + cx$

29. $f(x) = x^2 + 6x + c/x$ (trident of Newton)

30. $f(x) = x\sqrt{c^2 - x^2}$ **31.** $f(x) = e^x + ce^{-x}$

32. $f(x) = \ln(x^2 + c)$ **33.** $f(x) = \dfrac{cx}{1 + c^2x^2}$

34. $f(x) = \dfrac{\sin x}{c + \cos x}$ **35.** $f(x) = cx + \sin x$

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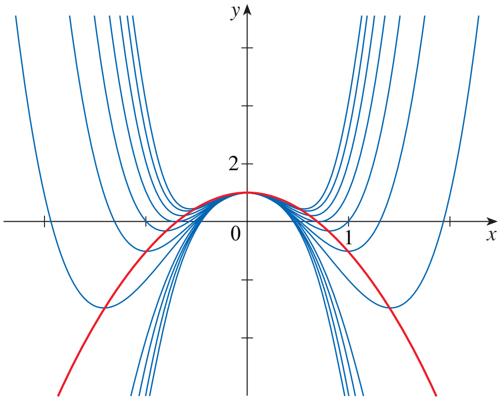
36. The family of functions $f(t) = C(e^{-at} - e^{-bt})$, where a, b, and C are positive numbers and $b > a$, has been used to model the concentration of a drug injected into the bloodstream at time $t = 0$. Graph several members of this family. What do they have in common? For fixed values of C and a, discover graphically what happens as b increases. Then use calculus to prove what you have discovered.

37. Investigate the family of curves given by $f(x) = xe^{-cx}$, where c is a real number. Start by computing the limits as $x \to \pm\infty$. Identify any transitional values of c where the basic shape changes. What happens to the maximum or minimum points and inflection points as c changes? Illustrate by graphing several members of the family.

38. The figure shows graphs (in blue) of several members of the family of polynomials $f(x) = cx^4 - 4x^2 + 1$.
(a) For which values of c does the curve have minimum points?

(b) Show that the minimum and maximum points of every curve in the family lie on the parabola $y = -2x^2 + 1$ (shown in red).



39. Investigate the family of curves given by the equation $f(x) = x^4 + cx^2 + x$. Start by determining the transitional value of c at which the number of inflection points changes. Then graph several members of the family to see what shapes are possible. There is another transitional value of c at which the number of critical numbers changes. Try to discover it graphically. Then prove what you have discovered.

40. (a) Investigate the family of polynomials given by the equation

$$f(x) = 2x^3 + cx^2 + 2x$$

For what values of c does the curve have maximum and minimum points?

(b) Show that the minimum and maximum points of every curve in the family lie on the curve $y = x - x^3$. Illustrate by graphing this curve and several members of the family.

4.7 Optimization Problems

The methods we have learned in this chapter for finding extreme values have practical applications in many areas of life: A businessperson wants to minimize costs and maximize profits. A traveler wants to minimize transportation time. Fermat's Principle in optics states that light follows the path that takes the least time. In this section we solve such problems as maximizing areas, volumes, and profits and minimizing distances, times, and costs.

PS

In solving such practical problems the greatest challenge is often to convert the word problem into a mathematical optimization problem by setting up the function that is to be maximized or minimized. Let's recall the problem-solving principles discussed in the Principles of Problem Solving following Chapter 1 and adapt them to this situation:

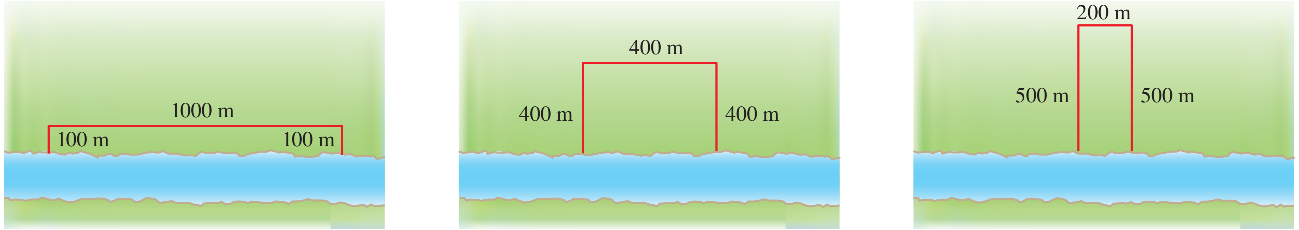
Steps In Solving Optimization Problems

1. **Understand the Problem** The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?
2. **Draw a Diagram** In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.
3. **Introduce Notation** Assign a symbol to the quantity that is to be maximized or minimized (let's call it Q for now). Also select symbols $(a, b, c, \ldots, x, y)$ for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols—for example, A for area, h for height, t for time.
4. Express Q in terms of some of the other symbols from Step 3.
5. If Q has been expressed as a function of more than one variable in Step 4, use the given information to find relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q. Thus Q will be expressed as a function of *one* variable x, say, $Q = f(x)$. Write the domain of this function in the given context.
6. Use the methods of Sections 4.1 and 4.3 to find the *absolute* maximum or minimum value of f. In particular, if the domain of f is a closed interval, then the Closed Interval Method in Section 4.1 can be used.

EXAMPLE 1 A farmer has 1200 m of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

PS Understand the problem
PS Analogy: Try special cases
PS Draw diagrams

SOLUTION In order to get a feeling for what is happening in this problem, let's experiment with some specific cases. Figure 1 (not to scale) shows three possible ways of laying out the 1200 m of fencing.



Area $= 100 \cdot 1000 = 100{,}000\ \text{m}^2$ Area $= 400 \cdot 400 = 160{,}000\ \text{m}^2$ Area $= 500 \cdot 200 = 100{,}000\ \text{m}^2$

FIGURE 1

We see that when we try shallow, wide fields or deep, narrow fields, we get relatively small areas. It seems plausible that there is some intermediate configuration that produces the largest area.

PS Introduce notation

Figure 2 illustrates the general case. We wish to maximize the area A of the rectangle. Let x and y be the depth and width of the rectangle (in meters). Then we express A in terms of x and y:

$$A = xy$$

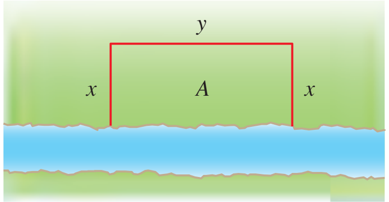


FIGURE 2

We want to express A as a function of just one variable, so we eliminate y by expressing it in terms of x. To do this we use the given information that the total length of the fencing is1200 m. Thus

$$2x + y = 1200$$

From this equation we have $y = 1200 - 2x$, which gives

$$A = xy = x(1200 - 2x) = 1200x - 2x^2$$

Note that the largest x can be is 600 (this uses all the fence for the depth and none for the width) and x can't be negative, so the function that we wish to maximize is

$$A(x) = 1200x - 2x^2 \qquad 0 \leqslant x \leqslant 600$$

The derivative is $A'(x) = 1200 - 4x$, so to find the critical numbers we solve the equation

$$1200 - 4x = 0$$

which gives $x = 300$. The maximum value of A must occur either at this critical number or at an endpoint of the interval. Since $A(0) = 0$, $A(300) = 180{,}000$, and $A(600) = 0$, the Closed Interval Method gives the maximum value as $A(300) = 180{,}000$.

[Alternatively, we could have observed that $A''(x) = -4 < 0$ for all x, so A is always concave downward and the local maximum at $x = 300$ must be an absolute maximum.]

The corresponding y-value is $y = 1200 - 2(300) = 600$, so the rectangular field should be 300 m deep and 600 m wide. ■

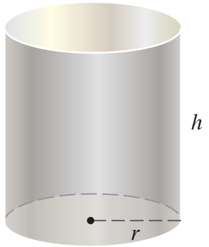


FIGURE 3

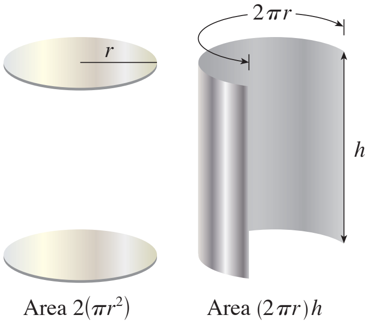


FIGURE 4

EXAMPLE 2 A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

SOLUTION Draw a diagram as in Figure 3, where r is the radius and h the height (both in centimeters). In order to minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, and sides). From Figure 4 we see that the sides are made from a rectangular sheet with dimensions $2\pi r$ and h. So the surface area is

$$A = 2\pi r^2 + 2\pi rh$$

We would like to express A in terms of one variable, r. To eliminate h we use the fact that the volume is given as 1 L, which is equivalent to 1000 cm^3. Thus

$$\pi r^2 h = 1000$$

which gives $h = 1000/(\pi r^2)$. Substitution of this into the expression for A gives

$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

We know that r must be positive, and there are no limitations on how large r can be. Therefore the function that we want to minimize is

$$A(r) = 2\pi r^2 + \frac{2000}{r} \qquad r > 0$$

To find the critical numbers, we differentiate:

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}$$

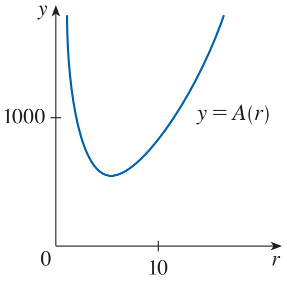


FIGURE 5

In the Applied Project following this section we investigate the most economical shape for a can by taking into account other manufacturing costs.

Then $A'(r) = 0$ when $\pi r^3 = 500$, so the only critical number is $r = \sqrt[3]{500/\pi}$.

Since the domain of A is $(0, \infty)$, we can't use the argument of Example 1 concerning endpoints. But we can observe that $A'(r) < 0$ for $r < \sqrt[3]{500/\pi}$ and $A'(r) > 0$ for $r > \sqrt[3]{500/\pi}$, so A is decreasing for *all* r to the left of the critical number and increasing for *all* r to the right. Thus $r = \sqrt[3]{500/\pi}$ must give rise to an *absolute* minimum.

[Alternatively, we could argue that $A(r) \to \infty$ as $r \to 0^+$ and $A(r) \to \infty$ as $r \to \infty$, so there must be a minimum value of $A(r)$, which must occur at the critical number. See Figure 5.]

The value of h corresponding to $r = \sqrt[3]{500/\pi}$ is

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi(500/\pi)^{2/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$$

Thus, to minimize the cost of the can, the radius should be $\sqrt[3]{500/\pi}$ cm and the height should be equal to twice the radius, namely, the diameter. ■

NOTE 1 The argument used in Example 2 to justify the absolute minimum is a variant of the First Derivative Test (which applies only to *local* maximum or minimum values) and is stated here for future reference.

First Derivative Test for Absolute Extreme Values Suppose that c is a critical number of a continuous function f defined on an interval.

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

NOTE 2 An alternative method for solving optimization problems is to use implicit differentiation. Let's look at Example 2 again to illustrate the method. We work with the same equations

$$A = 2\pi r^2 + 2\pi rh \qquad \pi r^2 h = 1000$$

but instead of eliminating h, we differentiate both equations implicitly with respect to r (treating both A and h as functions of r):

$$A' = 4\pi r + 2\pi rh' + 2\pi h \qquad \pi r^2 h' + 2\pi rh = 0$$

The minimum occurs at a critical number, so we set $A' = 0$, simplify, and arrive at the equations

$$2r + rh' + h = 0 \qquad rh' + 2h = 0$$

and subtraction gives $2r - h = 0$, or $h = 2r$.

EXAMPLE 3 Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

SOLUTION The distance between the point $(1, 4)$ and the point (x, y) is

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

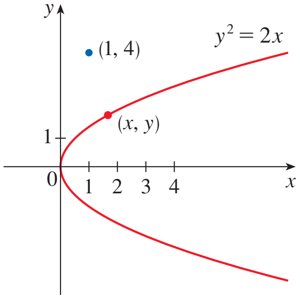


FIGURE 6

(See Figure 6.) But if (x, y) lies on the parabola, then $x = \frac{1}{2}y^2$, so the expression for d becomes

$$d = \sqrt{\left(\tfrac{1}{2}y^2 - 1\right)^2 + (y-4)^2}$$

(Alternatively, we could have substituted $y = \sqrt{2x}$ to get d in terms of x alone.)

Instead of minimizing d, we minimize its square:

$$d^2 = f(y) = \left(\tfrac{1}{2}y^2 - 1\right)^2 + (y-4)^2$$

(You should convince yourself that the minimum of d occurs at the same point as the minimum of d^2, but d^2 is easier to work with.) Note that there is no restriction on y, so the domain is all real numbers. Differentiating, we obtain

$$f'(y) = 2\left(\tfrac{1}{2}y^2 - 1\right)y + 2(y-4) = y^3 - 8$$

so $f'(y) = 0$ when $y = 2$. Observe that $f'(y) < 0$ when $y < 2$ and $f'(y) > 0$ when $y > 2$, so by the First Derivative Test for Absolute Extreme Values, the absolute minimum occurs when $y = 2$. (Or we could simply say that because of the geometric nature of the problem, it's obvious that there is a closest point but not a farthest point.) The corresponding value of x is $x = \frac{1}{2}y^2 = 2$. Thus the point on $y^2 = 2x$ closest to $(1, 4)$ is $(2, 2)$. [The distance between the points is $d = \sqrt{f(2)} = \sqrt{5}$.] ■

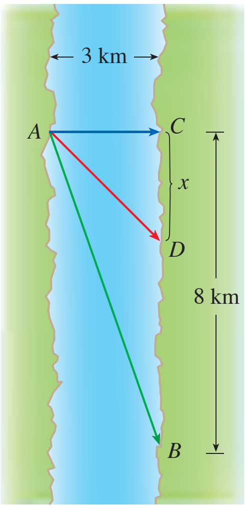


FIGURE 7

EXAMPLE 4 A woman launches her boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible (see Figure 7). She could row her boat directly across the river to point C and then run to B, or she could row directly to B, or she could row to some point D between C and B and then run to B. If she can row 6 km/h and run 8 km/h, where should she land to reach B as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the woman rows.)

SOLUTION If we let x be the distance from C to D, then the running distance is $|DB| = 8 - x$ and the Pythagorean Theorem gives the rowing distance as $|AD| = \sqrt{x^2 + 9}$. We use the equation

$$\text{time} = \frac{\text{distance}}{\text{rate}}$$

Then the rowing time is $\sqrt{x^2 + 9}/6$ and the running time is $(8 - x)/8$, so the total time T as a function of x is

$$T(x) = \frac{\sqrt{x^2 + 9}}{6} + \frac{8 - x}{8}$$

The domain of this function T is $[0, 8]$. Notice that if $x = 0$, she rows to C and if $x = 8$, she rows directly to B. The derivative of T is

$$T'(x) = \frac{x}{6\sqrt{x^2 + 9}} - \frac{1}{8}$$

Thus, using the fact that $x \geqslant 0$, we have

$$T'(x) = 0 \iff \frac{x}{6\sqrt{x^2 + 9}} = \frac{1}{8} \iff 4x = 3\sqrt{x^2 + 9}$$

$$\iff 16x^2 = 9(x^2 + 9) \iff 7x^2 = 81 \iff x = \frac{9}{\sqrt{7}}$$

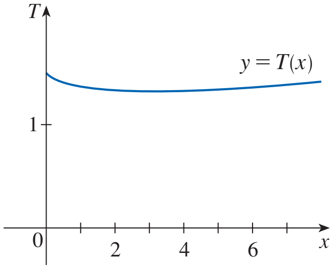


FIGURE 8

The only critical number is $x = 9/\sqrt{7}$. To see whether the minimum occurs at this critical number or at an endpoint of the domain $[0, 8]$, we follow the Closed Interval Method by evaluating T at all three points:

$$T(0) = 1.5 \qquad T\left(\frac{9}{\sqrt{7}}\right) = 1 + \frac{\sqrt{7}}{8} \approx 1.33 \qquad T(8) = \frac{\sqrt{73}}{6} \approx 1.42$$

Since the smallest of these values of T occurs when $x = 9/\sqrt{7}$, the absolute minimum value of T must occur there. Figure 8 illustrates this calculation by showing the graph of T.

Thus the woman should land the boat at a point $9/\sqrt{7}$ km ($\approx$3.4 km) downstream from her starting point. ■

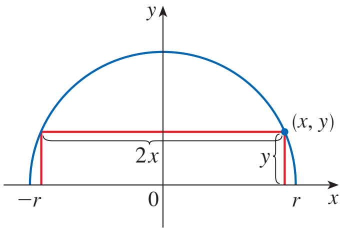


FIGURE 9

EXAMPLE 5 Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

SOLUTION 1 Let's take the semicircle to be the upper half of the circle $x^2 + y^2 = r^2$ with center the origin. Then the word *inscribed* means that the rectangle has two vertices on the semicircle and two vertices on the x-axis as shown in Figure 9.

Let (x, y) be the vertex that lies in the first quadrant. Then the rectangle has sides of lengths $2x$ and y, so its area is

$$A = 2xy$$

To eliminate y we use the fact that (x, y) lies on the circle $x^2 + y^2 = r^2$ and so $y = \sqrt{r^2 - x^2}$. Thus

$$A = 2x\sqrt{r^2 - x^2}$$

The domain of this function is $0 \leqslant x \leqslant r$. Its derivative is

$$A' = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}}$$

which is 0 when $2x^2 = r^2$, that is, $x = r/\sqrt{2}$ (since $x \geqslant 0$). This value of x gives a maximum value of A since $A(0) = 0$ and $A(r) = 0$. Therefore the area of the largest inscribed rectangle is

$$A\left(\frac{r}{\sqrt{2}}\right) = 2\frac{r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}} = r^2$$

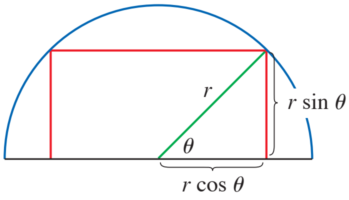


FIGURE 10

SOLUTION 2 A simpler solution is possible if we think of using an angle as a variable. Let θ be the angle shown in Figure 10. Then the area of the rectangle is

$$A(\theta) = (2r\cos\theta)(r\sin\theta) = r^2(2\sin\theta\cos\theta) = r^2\sin 2\theta$$

We know that $\sin 2\theta$ has a maximum value of 1 and it occurs when $2\theta = \pi/2$. So $A(\theta)$ has a maximum value of r^2 and it occurs when $\theta = \pi/4$.

Notice that this trigonometric solution doesn't involve differentiation. In fact, we didn't need to use calculus at all. ■

■ Applications to Business and Economics

In Section 3.7 we introduced the idea of marginal cost. Recall that if $C(x)$, the **cost function**, is the cost of producing x units of a certain product, then the **marginal cost** is the rate of change of C with respect to x. In other words, the marginal cost function is the derivative, $C'(x)$, of the cost function.

Now let's consider marketing. Let $p(x)$ be the price per unit that the company can charge if it sells x units. Then p is called the **demand function** (or **price function**) and we would expect it to be a decreasing function of x. (More units sold corresponds to a lower price.) If x units are sold and the price per unit is $p(x)$, then the total revenue is

$$R(x) = \text{quantity} \times \text{price} = xp(x)$$

and R is called the **revenue function**. The derivative R' of the revenue function is called the **marginal revenue function** and it is the rate of change of revenue with respect to the number of units sold.

If x units are sold, then the total profit is

$$P(x) = R(x) - C(x)$$

and P is called the **profit function**. The **marginal profit function** is P', the derivative of the profit function. In Exercises 65–69 you are asked to use the marginal cost, revenue, and profit functions to minimize costs and maximize revenues and profits.

EXAMPLE 6 A store has been selling 200 TV monitors a week at \$350 each. A market survey indicates that for each \$10 rebate offered to buyers, the number of monitors sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize revenue?

SOLUTION If x is the number of monitors sold per week, then the weekly increase in sales is $x - 200$. For each increase of 20 units sold, the price is decreased by \$10. So for each additional unit sold, the decrease in price will be $\frac{1}{20} \times 10$ and the demand function is

$$p(x) = 350 - \tfrac{10}{20}(x - 200) = 450 - \tfrac{1}{2}x$$

The revenue function is

$$R(x) = xp(x) = 450x - \tfrac{1}{2}x^2$$

Since $R'(x) = 450 - x$, we see that $R'(x) = 0$ when $x = 450$. This value of x gives an absolute maximum by the First Derivative Test (or simply by observing that the graph of R is a parabola that opens downward). The corresponding price is

$$p(450) = 450 - \tfrac{1}{2}(450) = 225$$

and the rebate is $350 - 225 = 125$. Therefore, to maximize revenue, the store should offer a rebate of $125. ■

4.7 Exercises

1. Consider the following problem: find two numbers whose sum is 23 and whose product is a maximum.
   (a) Make a table of values, like the one below, so that the sum of the numbers in the first two columns is always 23. On the basis of the evidence in your table, estimate the answer to the problem.
   (b) Use calculus to solve the problem and compare with your answer to part (a).

First number	Second number	Product
1	22	22
2	21	42
3	20	60
⋮	⋮	⋮

2. Find two numbers whose difference is 100 and whose product is a minimum.

3. Find two positive numbers whose product is 100 and whose sum is a minimum.

4. The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

5. What is the maximum vertical distance between the line $y = x + 2$ and the parabola $y = x^2$ for $-1 \leqslant x \leqslant 2$?

6. What is the minimum vertical distance between the parabolas $y = x^2 + 1$ and $y = x - x^2$?

7. Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

8. Find the dimensions of a rectangle with area 1000 m^2 whose perimeter is as small as possible.

9. A model used for the yield Y of an agricultural crop as a function of the nitrogen level N in the soil (measured in appropriate units) is

$$Y = \frac{kN}{1 + N^2}$$

where k is a positive constant. What nitrogen level gives the best yield?

10. The rate (in mg carbon/m^3/h) at which photosynthesis takes place for a species of phytoplankton is modeled by the function

$$P = \frac{100I}{I^2 + I + 4}$$

where I is the light intensity (measured in thousands of foot-candles). For what light intensity is P a maximum?

11. Consider the following problem: a farmer with 300 m of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?
   (a) Draw several diagrams illustrating the situation, some with shallow, wide pens and some with deep, narrow pens. Find the total areas of these configurations. Does it appear that there is a maximum area? If so, estimate it.
   (b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.
   (c) Write an expression for the total area.
   (d) Use the given information to write an equation that relates the variables.
   (e) Use part (d) to write the total area as a function of one variable.
   (f) Finish solving the problem and compare the answer with your estimate in part (a).

12. Consider the following problem: a box with an open top is to be constructed from a square piece of cardboard, 3 m wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.

(a) Draw several diagrams to illustrate the situation, some short boxes with large bases and some tall boxes with small bases. Find the volumes of several such boxes. Does it appear that there is a maximum volume? If so, estimate it.

(b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.

(c) Write an expression for the volume.

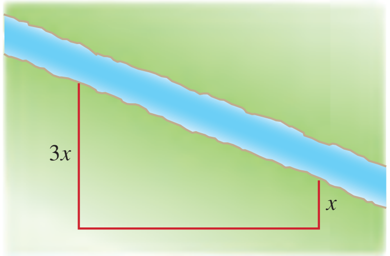
(d) Use the given information to write an equation that relates the variables.

(e) Use part (d) to write the volume as a function of one variable.

(f) Finish solving the problem and compare the answer with your estimate in part (a).

13. A farmer wants to fence in an area of 15,000 m^2 in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this so as to minimize the cost of the fence?

14. A farmer has 400 m of fencing for enclosing a trapezoidal field along a river as shown. One of the parallel sides is three times longer than the other. No fencing is needed along the river. Find the largest area the farmer can enclose.



15. A farmer wants to fence in a rectangular plot of land adjacent to the north wall of his barn. No fencing is needed along the barn, and the fencing along the west side of the plot is shared with a neighbor who will split the cost of that portion of the fence. If the fencing costs \$30 per linear meter to install and the farmer is not willing to spend more than \$1800, find the dimensions for the plot that would enclose the most area.

16. If the farmer in Exercise 15 wants to enclose 750 square meters of land, what dimensions will minimize the cost of the fence?

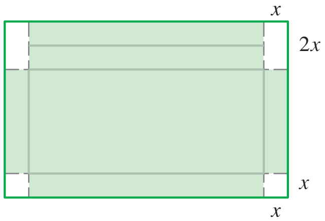
17. (a) Show that of all the rectangles with a given area, the one with smallest perimeter is a square.

(b) Show that of all the rectangles with a given perimeter, the one with greatest area is a square.

18. A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used.

19. If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

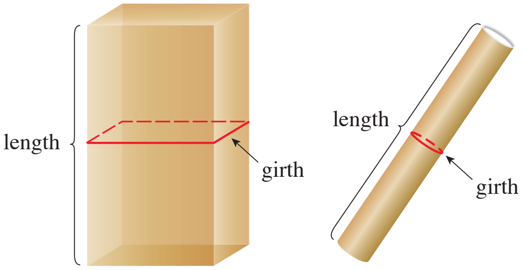
20. A box with an open top is to be constructed from a 2 m by 1 m rectangular piece of cardboard by cutting out squares or rectangles from each of the four corners, as shown in the figure, and bending up the sides. One of the longer sides of the box is to have a double layer of cardboard, which is obtained by folding the side twice. Find the largest volume that such a box can have.



21. A rectangular storage container without a lid is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the least expensive such container.

22. Rework Exercise 21 assuming the container has a lid that is made from the same material as the sides.

23. A package to be mailed using the US postal service may not measure more than 274 cm in length plus girth. (Length is the longest dimension and girth is the largest distance around the package, perpendicular to the length.) Find the dimensions of the rectangular box with square base of greatest volume that may be mailed.



24. Refer to Exercise 23. Find the dimensions of the cylindrical mailing tube of greatest volume that may be mailed using the US postal service.

25. Find the point on the line $y = 2x + 3$ that is closest to the origin.

26. Find the point on the curve $y = \sqrt{x}$ that is closest to the point (3, 0).

27. Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point (1, 0).

28. Find, correct to two decimal places, the coordinates of the point on the curve $y = \sin x$ that is closest to the point $(4, 2)$.

29. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r.

30. Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$.

31. Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side L if one side of the rectangle lies on the base of the triangle.

32. Find the area of the largest trapezoid that can be inscribed in a circle of radius 1 and whose base is a diameter of the circle.

33. Find the dimensions of the isosceles triangle of largest area that can be inscribed in a circle of radius r.

34. If the two equal sides of an isosceles triangle have length a, find the length of the third side that maximizes the area of the triangle.

35. If one side of a triangle has length a and another has length $2a$, show that the largest possible area of the triangle is a^2.

36. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 4 - x^2$. What is the largest possible area of the rectangle?

37. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

38. A right circular cylinder is inscribed in a cone with height h and base radius r. Find the largest possible volume of such a cylinder.

39. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible surface area of such a cylinder.

40. A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus the diameter of the semicircle is equal to the width of the rectangle. See Exercise 1.1.72.) If the perimeter of the window is 10 m, find the
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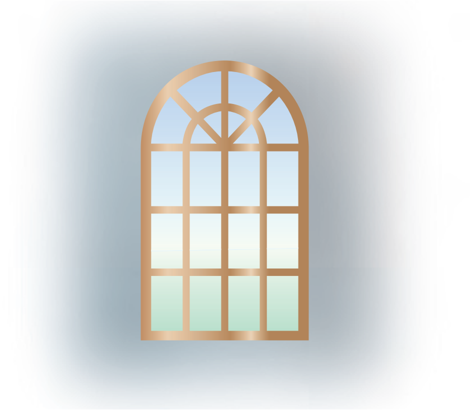
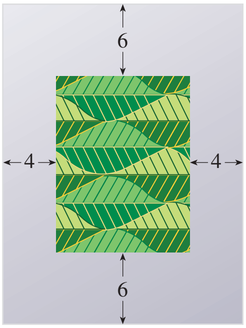


41. The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the area of printed material on the poster is fixed at 384 cm², find the dimensions of the poster with the smallest area.



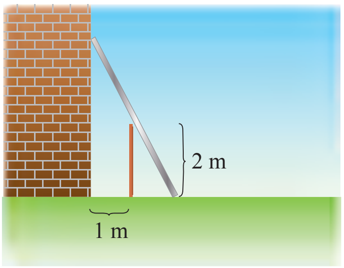
42. A poster is to have an area of 900 cm² with 2.5 cm margins at the bottom and sides and a 5 cm margin at the top. What dimensions will give the largest printed area?

43. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) A minimum?

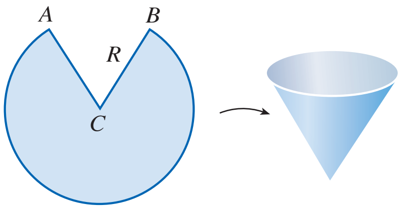
44. Answer Exercise 43 if one piece is bent into a square and the other into a circle.

45. If you are offered one slice from a round pizza (in other words, a sector of a circle) and the slice must have a perimeter of 60 cm, what diameter pizza will reward you with the largest slice?

46. A fence 2 m tall runs parallel to a tall building at a distance of 1 m from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?



47. A cone-shaped drinking cup is made from a circular piece of paper of radius R by cutting out a sector and joining the edges CA and CB. Find the maximum capacity of such a cup.



48. A cone-shaped paper drinking cup is to be made to hold 27 cm^3 of water. Find the height and radius of the cup that will use the smallest amount of paper.

49. A cone with height h is inscribed in a larger cone with height H so that its vertex is at the center of the base of the larger cone. Show that the inner cone has maximum volume when $h = \frac{1}{3}H$.

50. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with a plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin\theta + \cos\theta}$$

where μ is a constant called the coefficient of friction. For what value of θ is F smallest?

51. If a resistor of R ohms is connected across a battery of E volts with internal resistance r ohms, then the power (in watts) in the external resistor is

$$P = \frac{E^2 R}{(R + r)^2}$$

If E and r are fixed but R varies, what is the maximum value of the power?

52. For a fish swimming at a speed v relative to the water, the energy expenditure per unit time is proportional to v^3. It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against a current u $(u < v)$, then the time required to swim a distance L is $L/(v - u)$ and the total energy E required to swim the distance is given by

$$E(v) = av^3 \cdot \frac{L}{v - u}$$

where a is the proportionality constant.

(a) Determine the value of v that minimizes E.

(b) Sketch the graph of E.

Note: This result has been verified experimentally; migrating fish swim against a current at a speed 50% greater than the current speed.

53. In a beehive, each cell is a regular hexagonal prism, open at one end; the other end is capped by three congruent rhombi forming a trihedral angle at the apex, as in the figure. Let θ be the angle at which each rhombus meets the altitude, s the side length of the hexagon, and h the length of the longer base of the trapezoids on the sides of the cell. It can be shown that if s and h are held fixed, then the volume of the cell is constant (independent of θ), and for a given value of θ the surface area S of the cell is

$$S = 6sh - \tfrac{3}{2}s^2 \cot\theta + \tfrac{3}{2}\sqrt{3}s^2 \csc\theta$$

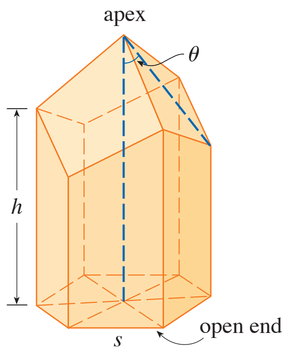
It is believed that bees form their cells in such a way as to minimize surface area, thus using the least amount of wax in cell construction.

(a) Calculate $dS/d\theta$.

(b) What angle θ should the bees prefer?

(c) Determine the minimum surface area of the cell in terms of s and h.

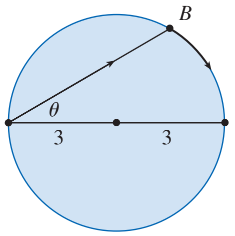
Note: Actual measurements of the angle θ in beehives have been made, and the measures of these angles seldom differ from the calculated value by more than 2°.



54. A boat leaves a dock at 2:00 PM and travels due south at a speed of 20 km/h. Another boat has been heading due east at 15 km/h and reaches the same dock at 3:00 PM. At what time were the two boats closest together?

55. Solve the problem in Example 4 if the river is 5 km wide and point B is only 5 km downstream from A.

56. A woman at a point A on the shore of a circular lake with radius 3 km wants to arrive at the point C diametrically opposite A on the other side of the lake in the shortest possible time (see the figure). She can walk at the rate of 6 km/h and row a boat at 3 km/h. How should she proceed?



57. An oil refinery is located on the north bank of a straight river that is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 6 km east of the refinery. The cost of laying pipe is \$400,000/km over land to a point P on the north bank and \$800,000/km under the river to the tanks. To minimize the cost of the pipeline, where should P be located?

T **58.** Suppose the refinery in Exercise 57 is located 1 km north of the river. Where should P be located?

59. The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. If two light sources, one three times as strong as the other, are placed 4 m apart, where should an object be placed on the line between the sources so as to receive the least illumination?

60. Find an equation of the line through the point (3, 5) that cuts off the least area from the first quadrant.

61. Let a and b be positive numbers. Find the length of the shortest line segment that is cut off by the first quadrant and passes through the point (a, b).

62. At which points on the curve $y = 1 + 40x^3 - 3x^5$ does the tangent line have the largest slope?

63. What is the shortest possible length of the line segment that is cut off by the first quadrant and is tangent to the curve $y = 3/x$ at some point?

64. What is the smallest possible area of the triangle that is cut off by the first quadrant and whose hypotenuse is tangent to the parabola $y = 4 - x^2$ at some point?

65. (a) If $C(x)$ is the cost of producing x units of a commodity, then the **average cost** per unit is $c(x) = C(x)/x$. Show that if the average cost is a minimum, then the marginal cost equals the average cost.
(b) If $C(x) = 16{,}000 + 200x + 4x^{3/2}$, in dollars, find (i) the cost, average cost, and marginal cost at a production level of 1000 units; (ii) the production level that will minimize the average cost; and (iii) the minimum average cost.

66. (a) Show that if the profit $P(x)$ is a maximum, then the marginal revenue equals the marginal cost.
(b) If $C(x) = 16{,}000 + 500x - 1.6x^2 + 0.004x^3$ is the cost function and $p(x) = 1700 - 7x$ is the demand function, find the production level that will maximize profit.

67. A baseball team plays in a stadium that seats 55,000 spectators. With ticket prices at \$10, the average attendance had been 27,000. When ticket prices were lowered to \$8, the average attendance rose to 33,000.
(a) Find the demand function, assuming that it is linear.
(b) How should ticket prices be set to maximize revenue?

68. During the summer months Terry makes and sells necklaces on the beach. Last summer she sold the necklaces for \$10 each and her sales averaged 20 per day. When she increased the price by \$1, she found that the average decreased by two sales per day.
(a) Find the demand function, assuming that it is linear.
(b) If the material for each necklace costs \$6, what selling price should Terry set to maximize her profit?

69. A retailer has been selling 1200 tablet computers a week at \$350 each. The marketing department estimates that an additional 80 tablets will sell each week for every \$10 that the price is lowered.
(a) Find the demand function.
(b) What should the price be set at in order to maximize revenue?
(c) If the retailer's weekly cost function is

$$C(x) = 35{,}000 + 120x$$

what price should it choose in order to maximize its profit?

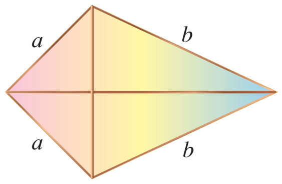
70. A company operates 16 oil wells in a designated area. Each pump, on average, extracts 240 barrels of oil daily. The company can add more wells but every added well reduces the average daily ouput of each of the wells by 8 barrels. How many wells should the company add in order to maximize daily production?

71. Show that of all the isosceles triangles with a given perimeter, the one with the greatest area is equilateral.

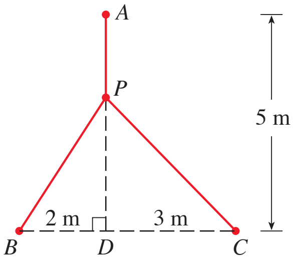
72. Consider the situation in Exercise 57 if the cost of laying pipe under the river is considerably higher than the cost of laying pipe over land (\$400,000/km). You may suspect that in some instances, the minimum distance possible under the river should be used, and P should be located 6 km from the refinery, directly across from the storage tanks. Show that this is *never* the case, no matter what the "under river" cost is.

73. Consider the tangent line to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at a point (p, q) in the first quadrant.
(a) Show that the tangent line has x-intercept a^2/p and y-intercept b^2/q.
(b) Show that the portion of the tangent line cut off by the coordinate axes has minimum length $a + b$.
(c) Show that the triangle formed by the tangent line and the coordinate axes has minimum area ab.

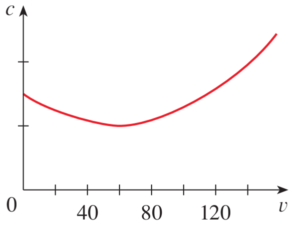
74. The frame for a kite is to be made from six pieces of wood. The four exterior pieces have been cut with the lengths indicated in the figure. To maximize the area of the kite, how long should the diagonal pieces be?



75. A point P needs to be located somewhere on the line AD so that the total length L of cables linking P to the points A, B, and C is minimized (see the figure). Express L as a function of $x = |AP|$ and use the graphs of L and dL/dx to estimate the minimum value of L.



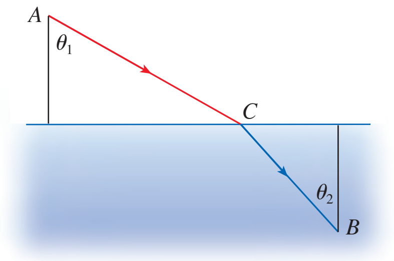
76. The graph shows the fuel consumption c of a car (measured in liters per hour) as a function of the speed v of the car. At very low speeds the engine runs inefficiently, so initially c decreases as the speed increases. But at high speeds the fuel consumption increases. You can see that $c(v)$ is minimized for this car when $v \approx 48$ km/h. However, for fuel efficiency, what must be minimized is not the consumption in gallons per hour but rather the fuel consumption in liters *per kilometer.* Let's call this consumption G. Using the graph, estimate the speed at which G has its minimum value.



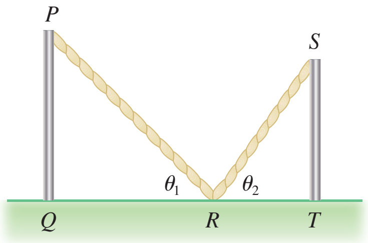
77. Let v_1 be the velocity of light in air and v_2 the velocity of light in water. According to Fermat's Principle, a ray of light will travel from a point A in the air to a point B in the water by a path ACB that minimizes the time taken. Show that

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}$$

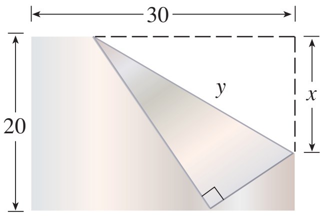
where θ_1 (the angle of incidence) and θ_2 (the angle of refraction) are as shown. This equation is known as Snell's Law.



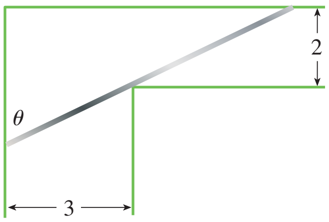
78. Two vertical poles PQ and ST are secured by a rope PRS going from the top of the first pole to a point R on the ground between the poles and then to the top of the second pole as in the figure. Show that the shortest length of such a rope occurs when $\theta_1 = \theta_2$.



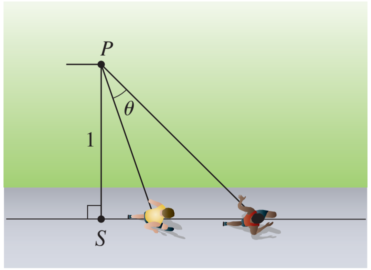
79. The upper right-hand corner of a piece of paper, 30 centimeters by 20 centimeters , as in the figure, is folded over to the bottom edge. How would you fold the paper so as to minimize the length of the fold? In other words, how would you choose x to minimize y?



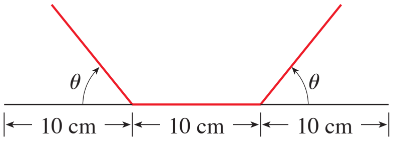
80. A steel pipe is being carried down a hallway that is 3 m wide. At the end of the hall there is a right-angled turn into a narrower hallway, 2 m wide. What is the length of the longest pipe that can be carried horizontally around the corner?



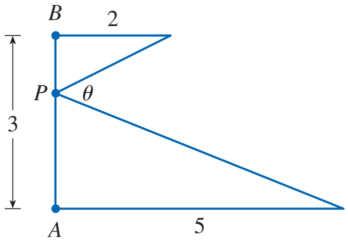
81. An observer stands at a point P, one unit away from a track. Two runners start at the point S in the figure and run along the track. One runner runs three times as fast as the other. Find the maximum value of the observer's angle of sight θ between the runners.



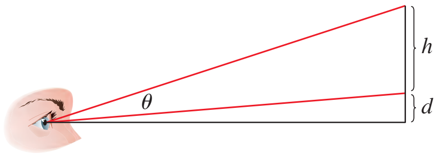
82. A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle θ. How should θ be chosen so that the gutter will carry the maximum amount of water?



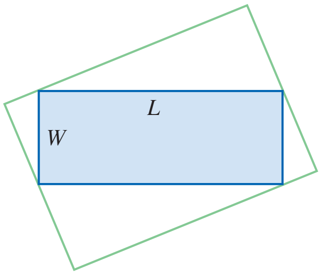
83. Where should the point P be chosen on the line segment AB so as to maximize the angle θ?



84. A painting in an art gallery has height h and is hung so that its lower edge is a distance d above the eye of an observer (as in the figure). How far from the wall should the observer stand to get the best view? (In other words, where should the observer stand so as to maximize the angle θ subtended at his eye by the painting?)



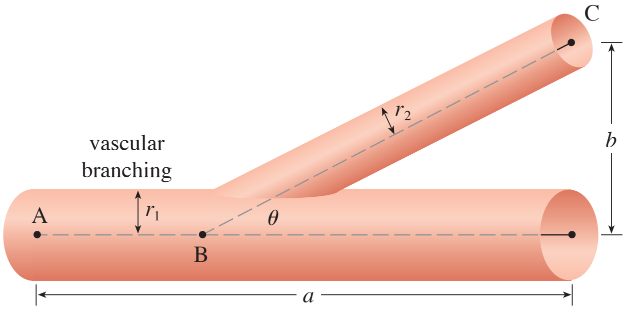
85. Find the maximum area of a rectangle that can be circumscribed about a given rectangle with length L and width W. [*Hint:* Express the area as a function of an angle θ.]



86. The blood vascular system consists of blood vessels (arteries, arterioles, capillaries, and veins) that convey blood from the heart to the organs and back to the heart. This system should work so as to minimize the energy expended by the heart in pumping the blood. In particular, this energy is reduced when the resistance of the blood is lowered. One of Poiseuille's Laws gives the resistance R of the blood as

$$R = C\frac{L}{r^4}$$

where L is the length of the blood vessel, r is the radius, and C is a positive constant determined by the viscosity of the blood. (Poiseuille established this law experimentally, but it also follows from Equation 8.4.2.) The figure shows a main blood vessel with radius r_1 branching at an angle θ into a smaller vessel with radius r_2.



(a) Use Poiseuille's Law to show that the total resistance of the blood along the path ABC is

$$R = C\left(\frac{a - b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4}\right)$$

where a and b are the distances shown in the figure.

(b) Prove that this resistance is minimized when

$$\cos\theta = \frac{r_2^4}{r_1^4}$$

(c) Find the optimal branching angle (correct to the nearest degree) when the radius of the smaller blood vessel is two-thirds the radius of the larger vessel.

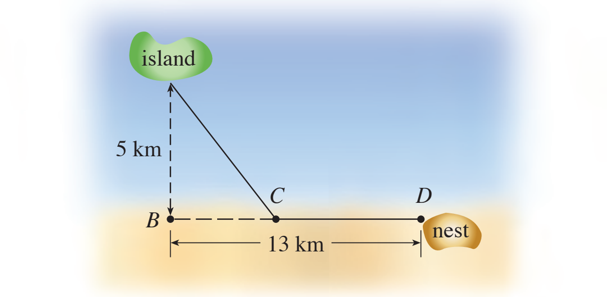
87. Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours. It is believed that more energy is required to fly over water than over land because air generally rises over land and falls over water during the day. A bird with these tendencies is released from an island that is 5 km from the nearest point B on a straight shoreline, flies to a point C on the shoreline, and then flies along the shoreline to its nesting area D. Assume that the bird instinctively chooses a path that will minimize its energy expenditure. Points B and D are 13 km apart.

(a) In general, if it takes 1.4 times as much energy to fly over water as it does over land, to what point C should the bird fly in order to minimize the total energy expended in returning to its nesting area?

(b) Let W and L denote the energy (in joules) per kilometer flown over water and land, respectively. What would a large value of the ratio W/L mean in terms of the bird's flight? What would a small value mean? Determine the ratio W/L corresponding to the minimum expenditure of energy.

(c) What should the value of W/L be in order for the bird to fly directly to its nesting area D? What should the value of W/L be for the bird to fly to B and then along the shore to D?

(d) If the ornithologists observe that birds of a certain species reach the shore at a point 4 km from B, how many times r



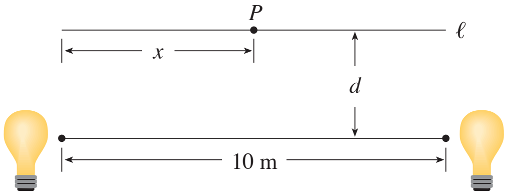
88. Two light sources of identical strength are placed 10 m apart. An object is to be placed at a point P on a line ℓ, parallel to the line joining the light sources and at a distance d meters from it (see the figure). We want to locate P on ℓ so that the intensity of illumination is minimized. We need to use the fact that the intensity of illumination for a single source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source.

(a) Find an expression for the intensity $I(x)$ at the point P.

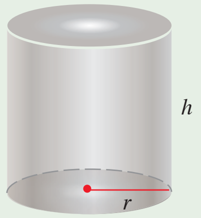
(b) If $d = 5$ m, use graphs of $I(x)$ and $I'(x)$ to show that the intensity is minimized when $x = 5$ m, that is, when P is at the midpoint of ℓ.

(c) If $d = 10$ m, show that the intensity (perhaps surprisingly) is *not* minimized at the midpoint.

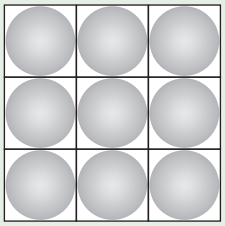
(d) Somewhere between $d = 5$ m and $d = 10$ m there is a transitional value of d at which the point of minimal illumination abruptly changes. Estimate this value of d by graphical methods. Then find the exact value of d.



APPLIED PROJECT | THE SHAPE OF A CAN

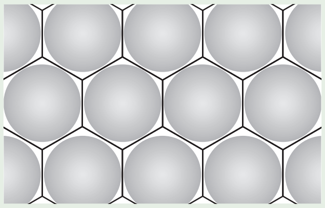


In this project we investigate the most economical shape for a can. We first interpret this to mean that the volume V of a cylindrical can is given and we need to find the height h and radius r that minimize the cost of the metal to construct the can (see the figure). If we disregard any waste metal in the manufacturing process, then the problem is to minimize the surface area of the cylinder. We solved this problem in Example 4.7.2 and we found that $h = 2r$; that is, the height should be the same as the diameter. But if you go to your cupboard or your supermarket with a ruler, you will discover that the height is usually greater than the diameter and the ratio h/r varies from 2 up to about 3.8. Let's see if we can explain this phenomenon.


Discs cut from squares

1. The material for the cans is cut from sheets of metal. The cylindrical sides are formed by bending rectangles; these rectangles are cut from the sheet with little or no waste. But if the top and bottom discs are cut from squares of side $2r$ (as in the figure), this leaves considerable waste metal, which may be recycled but has little or no value to the can makers. If this is the case, show that the amount of metal used is minimized when

$$\frac{h}{r} = \frac{8}{\pi} \approx 2.55$$


Discs cut from hexagons

2. A more efficient packing of the discs is obtained by dividing the metal sheet into hexagons and cutting the circular lids and bases from the hexagons (see the figure). Show that if this strategy is adopted, then

$$\frac{h}{r} = \frac{4\sqrt{3}}{\pi} \approx 2.21$$

3. The values of h/r that we found in Problems 1 and 2 are a little closer to the ones that actually occur on supermarket shelves, but they still don't account for everything. If we look more closely at some real cans, we see that the lid and the base are formed from discs with radius larger than r that are bent over the ends of the can. If we allow for this we would increase h/r. More significantly, in addition to the cost of the metal we need to incorporate the manufacturing of the can into the cost. Let's assume that most of the

(*continued*)

expense is incurred in joining the sides to the rims of the cans. If we cut the discs from hexagons as in Problem 2, then the total cost is proportional to

$$4\sqrt{3}\,r^2 + 2\pi rh + k(4\pi r + h)$$

where k is the reciprocal of the length that can be joined for the cost of one unit area of metal. Show that this expression is minimized when

$$\frac{\sqrt[3]{V}}{k} = \sqrt[3]{\frac{\pi h}{r}} \cdot \frac{2\pi - h/r}{\pi h/r - 4\sqrt{3}}$$

4. Plot $\sqrt[3]{V}/k$ as a function of $x = h/r$ and use your graph to argue that when a can is large or joining is cheap, we should make h/r approximately 2.21 (as in Problem 2). But when the can is small or joining is costly, h/r should be substantially larger.

5. Our analysis shows that large cans should be almost square but small cans should be tall and thin. Take a look at the relative shapes of the cans in a supermarket. Is our conclusion usually true in practice? Are there exceptions? Can you suggest reasons why small cans are not always tall and thin?

APPLIED PROJECT | PLANES AND BIRDS: MINIMIZING ENERGY

Small birds like finches alternate between flapping their wings and keeping them folded while gliding (see Figure 1). In this project we analyze this phenomenon and try to determine how frequently a bird should flap its wings. Some of the principles are the same as for fixed-wing aircraft and so we begin by considering how required power and energy depend on the speed of airplanes.[1]

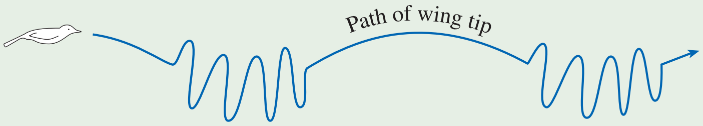


FIGURE 1

1. The power needed to propel an airplane forward at velocity v is

$$P = Av^3 + \frac{BL^2}{v}$$

where A and B are positive constants specific to the particular aircraft and L is the lift, the upward force supporting the weight of the plane. Find the speed that minimizes the required power.

2. The speed found in Problem 1 minimizes power but a faster speed might use less fuel. The energy needed to propel the airplane a unit distance is $E = P/v$. At what speed is energy minimized?

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[1] Adapted from R. McNeill Alexander, *Optima for Animals* (Princeton, NJ: Princeton University Press, 1996.)

3. How much faster is the speed for minimum energy than the speed for minimum power?

4. In applying the equation of Problem 1 to bird flight we split the term Av^3 into two parts: $A_b v^3$ for the bird's body and $A_w v^3$ for its wings. Let x be the fraction of flying time spent in flapping mode. If m is the bird's mass and all the lift occurs during flapping, then the lift is mg/x and so the power needed during flapping is

$$P_{\text{flap}} = (A_b + A_w)v^3 + \frac{B(mg/x)^2}{v}$$

The power while wings are folded is $P_{\text{fold}} = A_b v^3$. Show that the average power over an entire flight cycle is

$$\overline{P} = xP_{\text{flap}} + (1 - x)P_{\text{fold}} = A_b v^3 + xA_w v^3 + \frac{Bm^2g^2}{xv}$$

5. For what value of x is the average power a minimum? What can you conclude if the bird flies slowly? What can you conclude if the bird flies faster and faster?

6. The average energy over a cycle is $\overline{E} = \overline{P}/v$. What value of x minimizes $\overline{E}$?

4.8 Newton's Method

Suppose that a car dealer offers to sell you a car for \$18,000 or for payments of \$375 per month for five years. You would like to know what monthly interest rate the dealer is, in effect, charging you. To find the answer, you have to solve the equation

$$\boxed{1} \qquad 48x(1 + x)^{60} - (1 + x)^{60} + 1 = 0$$

(The details are explained in Exercise 41.) How would you solve such an equation?

For a quadratic equation $ax^2 + bx + c = 0$ there is a well-known formula for the solutions. For third- and fourth-degree equations there are also formulas for the solutions, but they are extremely complicated. If f is a polynomial of degree 5 or higher, there is no such formula. Likewise, there is no formula that will enable us to find the exact solutions of a transcendental equation such as $\cos x = x$.

We can find an *approximate* solution to Equation 1 by plotting the left side of the equation and finding the x-intercepts. Using a graphing calculator (or computer), and after experimenting with viewing rectangles, we produce the graph in Figure 1.

We see that in addition to the solution $x = 0$, which doesn't interest us, there is a solution between 0.007 and 0.008. Zooming in shows that the x-intercept is approximately 0.0076. If we need more accuracy than graphing provides, we can use a calculator or computer algebra system to solve the equation numerically. If we do so, we find that the solution, correct to nine decimal places, is 0.007628603.

How do these devices solve equations? They use a variety of methods, but most of them make some use of **Newton's method**, also called the **Newton-Raphson method**. We will explain how this method works, partly to show what happens inside a calculator or computer, and partly as an application of the idea of linear approximation.

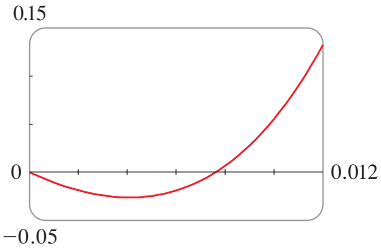


FIGURE 1

Try to solve Equation 1 numerically using a calculator or computer. Some machines are not able to solve it. Others are successful but require you to specify a starting point for the search.

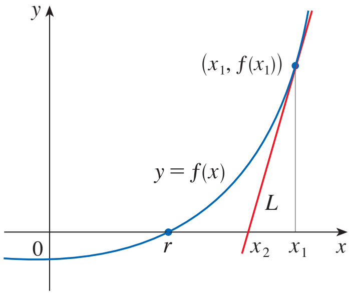


FIGURE 2

The geometry behind Newton's method is shown in Figure 2. We wish to solve an equation of the form $f(x) = 0$, so the solutions of the equation correspond to the x-intercepts of the graph of f. The solution that we are trying to find is labeled r in the figure. We start with a first approximation x_1, which is obtained by guessing, or from a rough sketch of the graph of f, or from a computer-generated graph of f. Consider the tangent line L to the curve $y = f(x)$ at the point $(x_1, f(x_1))$ and look at the x-intercept of L, labeled x_2. The idea behind Newton's method is that the tangent line is close to the curve and so its x-intercept, x_2, is close to the x-intercept of the curve (namely, the solution r that we are seeking). Because the tangent is a line, we can easily find its x-intercept.

To find a formula for x_2 in terms of x_1 we use the fact that the slope of L is $f'(x_1)$, so its equation is

$$y - f(x_1) = f'(x_1)(x - x_1)$$

Since the x-intercept of L is x_2, we know that the point $(x_2, 0)$ is on the line, and so

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

If $f'(x_1) \neq 0$, we can solve this equation for x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

We use x_2 as a second approximation to r.

Next we repeat this procedure with x_1 replaced by the second approximation x_2, using the tangent line at $(x_2, f(x_2))$. This gives a third approximation:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

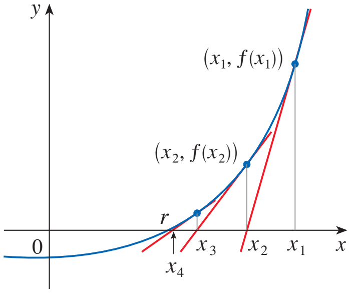


FIGURE 3

If we keep repeating this process, we obtain a sequence of approximations $x_1, x_2, x_3, x_4, \ldots$ as shown in Figure 3. In general, if the nth approximation is x_n and $f'(x_n) \neq 0$, then the next approximation is given by

2
$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Sequences are discussed in more detail in Section 11.1.

If the numbers x_n become closer and closer to r as n becomes large, then we say that the sequence *converges* to r and we write

$$\lim_{n \to \infty} x_n = r$$

Although the sequence of successive approximations converges to the desired solution for functions of the type illustrated in Figure 3, in certain circumstances the sequence may not converge. For example, consider the situation shown in Figure 4. You can see that x_2 is a worse approximation than x_1. This is likely to be the case when $f'(x_1)$ is close to 0. It might even happen that an approximation (such as x_3 in Figure 4) falls outside the domain of f. Then Newton's method fails and a better initial approximation x_1 should be chosen. See Exercises 31–34 for specific examples in which Newton's method works very slowly or does not work at all.

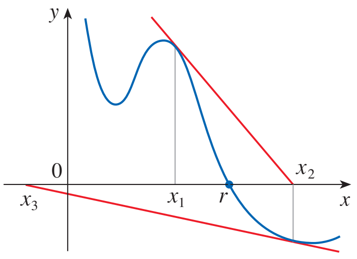


FIGURE 4

EXAMPLE 1 Starting with $x_1 = 2$, find the third approximation x_3 to the solution of the equation $x^3 - 2x - 5 = 0$.

SOLUTION We apply Newton's method with

$$f(x) = x^3 - 2x - 5 \qquad \text{and} \qquad f'(x) = 3x^2 - 2$$

Newton himself used this equation to illustrate his method and he chose $x_1 = 2$ after some experimentation because $f(1) = -6$, $f(2) = -1$, and $f(3) = 16$. Equation 2 becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

With $n = 1$ we have

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2}$$

$$= 2 - \frac{2^3 - 2(2) - 5}{3(2)^2 - 2} = 2.1$$

Then with $n = 2$ we obtain

$$x_3 = x_2 - \frac{x_2^3 - 2x_2 - 5}{3x_2^2 - 2} = 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2} \approx 2.0946$$

It turns out that this third approximation $x_3 \approx 2.0946$ is accurate to four decimal places. ■

Figure 5 shows the geometry behind the first step in Newton's method in Example 1. Since $f'(2) = 10$, the tangent line to $y = x^3 - 2x - 5$ at $(2, -1)$ has equation $y = 10x - 21$ so its x-intercept is $x_2 = 2.1$.

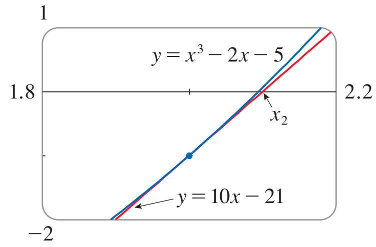


FIGURE 5

Suppose that we want to achieve a given accuracy, say to eight decimal places, using Newton's method. How do we know when to stop? The rule of thumb that is generally used is: stop when successive approximations x_n and x_{n+1} agree to eight decimal places. (A precise statement concerning accuracy in Newton's method will be given in Exercise 11.11.39.)

Notice that the procedure in going from n to $n + 1$ is the same for all values of n. (It is called an *iterative* process.) This means that Newton's method is particularly convenient for use with a programmable calculator or a computer.

EXAMPLE 2 Use Newton's method to find $\sqrt[6]{2}$ correct to eight decimal places.

SOLUTION First we observe that finding $\sqrt[6]{2}$ is equivalent to finding the positive solution of the equation

$$x^6 - 2 = 0$$

so we take $f(x) = x^6 - 2$. Then $f'(x) = 6x^5$ and Formula 2 (Newton's method) becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^6 - 2}{6x_n^5}$$

If we choose $x_1 = 1$ as the initial approximation, then we obtain

$$x_2 \approx 1.16666667$$

$$x_3 \approx 1.12644368$$

$$x_4 \approx 1.12249707$$

$$x_5 \approx 1.12246205$$

$$x_6 \approx 1.12246205$$

Since x_5 and x_6 agree to eight decimal places, we conclude that

$$\sqrt[6]{2} \approx 1.12246205$$

to eight decimal places. ■

EXAMPLE 3 Find, correct to six decimal places, the solution of the equation $\cos x = x$.

SOLUTION We first rewrite the equation in standard form: $\cos x - x = 0$. Therefore we let $f(x) = \cos x - x$. Then $f'(x) = -\sin x - 1$, so Formula 2 becomes

$$x_{n+1} = x_n - \frac{\cos x_n - x_n}{-\sin x_n - 1} = x_n + \frac{\cos x_n - x_n}{\sin x_n + 1}$$

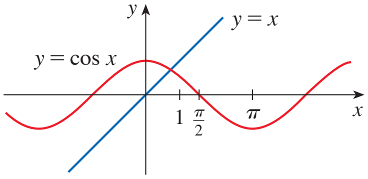


FIGURE 6

In order to guess a suitable value for x_1 we sketch the graphs of $y = \cos x$ and $y = x$ in Figure 6. It appears that they intersect at a point whose x-coordinate is somewhat less than 1, so let's take $x_1 = 1$ as a convenient first approximation. Then, remembering to put our calculator in radian mode, we get

$$x_2 \approx 0.75036387$$

$$x_3 \approx 0.73911289$$

$$x_4 \approx 0.73908513$$

$$x_5 \approx 0.73908513$$

Since x_4 and x_5 agree to six decimal places (eight, in fact), we conclude that the solution of the equation, correct to six decimal places, is 0.739085. ■

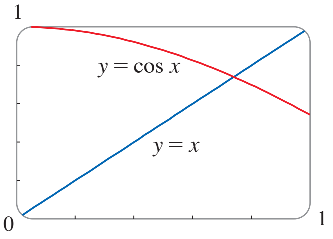


FIGURE 7

Instead of using the rough sketch in Figure 6 to get a starting approximation for Newton's method in Example 3, we could have used the more accurate graph that a calculator or computer provides. Figure 7 suggests that we use $x_1 = 0.75$ as the initial approximation. Then Newton's method gives

$$x_2 \approx 0.73911114$$

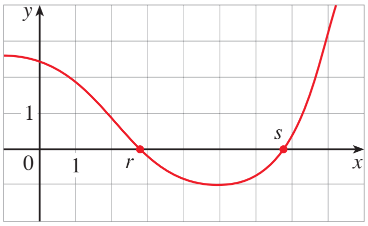
$$x_3 \approx 0.73908513$$

$$x_4 \approx 0.73908513$$

and so we obtain the same answer as before, but with one fewer step.

4.8 Exercises

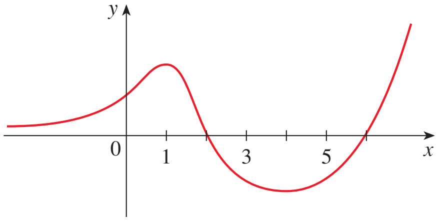
1. The figure shows the graph of a function f. Suppose that Newton's method is used to approximate the solution s of the equation $f(x) = 0$ with initial approximation $x_1 = 6$.
(a) Draw the tangent lines that are used to find x_2 and x_3, and estimate the numerical values of x_2 and x_3.
(b) Would $x_1 = 8$ be a better first approximation? Explain.



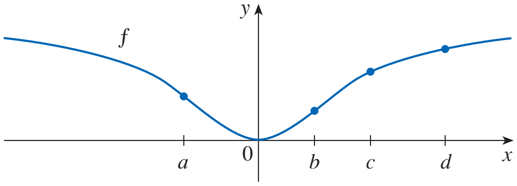
2. Follow the instructions for Exercise 1(a) but use $x_1 = 1$ as the starting approximation for finding the solution r.

3. Suppose the tangent line to the curve $y = f(x)$ at the point $(2, 5)$ has the equation $y = 9 - 2x$. If Newton's method is used to locate a solution of the equation $f(x) = 0$ and the initial approximation is $x_1 = 2$, find the second approximation x_2.

4. For each initial approximation, determine graphically what happens if Newton's method is used for the function whose graph is shown.
(a) $x_1 = 0$ (b) $x_1 = 1$ (c) $x_1 = 3$
(d) $x_1 = 4$ (e) $x_1 = 5$



5. For which of the initial approximations $x_1 = a$, b, c, and d do you think Newton's method will work and lead to the solution of the equation $f(x) = 0$?



6–8 Use Newton's method with the specified initial approximation x_1 to find x_3, the third approximation to the solution of the given equation. (Give your answer to four decimal places.)

6. $2x^3 - 3x^2 + 2 = 0, \quad x_1 = -1$

7. $\dfrac{2}{x} - x^2 + 1 = 0, \quad x_1 = 2$

8. $x^5 = x^2 + 1, \quad x_1 = 1$

9. Use Newton's method with initial approximation $x_1 = -1$ to find x_2, the second approximation to the solution of the equation $x^3 + x + 3 = 0$. Explain how the method works by first graphing the function and its tangent line at $(-1, 1)$.

10. Use Newton's method with initial approximation $x_1 = 1$ to find x_2, the second approximation to the solution of the equation $x^4 - x - 1 = 0$. Explain how the method works by first graphing the function and its tangent line at $(1, -1)$.

11–12 Use Newton's method to approximate the given number correct to eight decimal places.

11. $\sqrt[4]{75}$

12. $\sqrt[8]{500}$

13–14 (a) Explain how we know that the given equation must have a solution in the given interval. (b) Use Newton's method to approximate the solution correct to six decimal places.

13. $3x^4 - 8x^3 + 2 = 0, \quad [2, 3]$

14. $-2x^5 + 9x^4 - 7x^3 - 11x = 0, \quad [3, 4]$

15–16 Use Newton's method to approximate the indicated solution of the equation correct to six decimal places.

15. The negative solution of $\cos x = x^2 - 4$

16. The positive solution of $e^{2x} = x + 3$

17–22 Use Newton's method to find all solutions of the equation correct to six decimal places.

17. $\sin x = x - 1$

18. $\cos 2x = x^3$

19. $2^x = 2 - x^2$

20. $\ln x = \dfrac{1}{x - 3}$

21. $\arctan x = x^2 - 3$

22. $x^3 = 5x - 3$

23–28 Use Newton's method to find all the solutions of the equation correct to eight decimal places. Start by looking at a graph to find initial approximations.

23. $-2x^7 - 5x^4 + 9x^3 + 5 = 0$

24. $x^5 - 3x^4 + x^3 - x^2 - x + 6 = 0$

25. $\dfrac{x}{x^2 + 1} = \sqrt{1 - x}$

26. $\cos(x^2 - x) = x^4$

27. $\sqrt{4 - x^3} = e^{x^2}$

28. $\ln(x^2 + 2) = \dfrac{3x}{\sqrt{x^2 + 1}}$

29. (a) Apply Newton's method to the equation $x^2 - a = 0$ to derive the following square-root algorithm (used by the ancient Babylonians to compute $\sqrt{a}$):

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

(b) Use part (a) to compute $\sqrt{1000}$ correct to six decimal places.

30. (a) Apply Newton's method to the equation $1/x - a = 0$ to derive the following reciprocal algorithm:

$$x_{n+1} = 2x_n - ax_n^2$$

(This algorithm enables a computer to find reciprocals without actually dividing.)

(b) Use part (a) to compute $1/1.6984$ correct to six decimal places.

31. Explain why Newton's method doesn't work for finding the solution of the equation $x^3 - 3x + 6 = 0$ if the initial approximation is chosen to be $x_1 = 1$.

32. (a) Use Newton's method with $x_1 = 1$ to find the solution of the equation $x^3 - x = 1$ correct to six decimal places.

(b) Solve the equation in part (a) using $x_1 = 0.6$ as the initial approximation.

(c) Solve the equation in part (a) using $x_1 = 0.57$. (You definitely need a programmable calculator for this part.)

(d) Graph $f(x) = x^3 - x - 1$ and its tangent lines at $x_1 = 1$, 0.6, and 0.57 to explain why Newton's method is so sensitive to the value of the initial approximation.

33. Explain why Newton's method fails when applied to the equation $\sqrt[3]{x} = 0$ with any initial approximation $x_1 \neq 0$. Illustrate your explanation with a sketch.

34. If

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geqslant 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$$

then the solution of the equation $f(x) = 0$ is $x = 0$. Explain why Newton's method fails to find the solution no matter which initial approximation $x_1 \neq 0$ is used. Illustrate your explanation with a sketch.

35. (a) Use Newton's method to find the critical numbers of the function $f(x) = x^6 - x^4 + 3x^3 - 2x$ correct to six decimal places.
(b) Find the absolute minimum value of f correct to four decimal places.

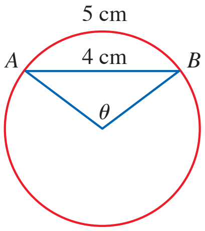
36. Use Newton's method to find the absolute maximum value of the function $f(x) = x \cos x$, $0 \leqslant x \leqslant \pi$, correct to six decimal places.

37. Use Newton's method to find the coordinates of the inflection point of the curve $y = x^2 \sin x$, $0 \leqslant x \leqslant \pi$, correct to six decimal places.

38. Of the infinitely many lines that are tangent to the curve $y = -\sin x$ and pass through the origin, there is one that has the largest slope. Use Newton's method to find the slope of that line correct to six decimal places.

39. Use Newton's method to find the coordinates, correct to six decimal places, of the point on the parabola $y = (x - 1)^2$ that is closest to the origin.

40. In the figure, the length of the chord AB is 4 cm and the length of the arc AB is 5 cm. Find the central angle θ, in radians, correct to four decimal places. Then give the answer to the nearest degree.



41. A car dealer sells a new car for \$18,000. He also offers to sell the same car for payments of \$375 per month for five years. What monthly interest rate is this dealer charging?

To solve this problem you will need to use the formula for the present value A of an annuity consisting of n equal payments of size R with interest rate i per time period:

$$A = \frac{R}{i}\left[1 - (1 + i)^{-n}\right]$$

Replacing i by x, show that

$$48x(1 + x)^{60} - (1 + x)^{60} + 1 = 0$$

Use Newton's method to solve this equation.

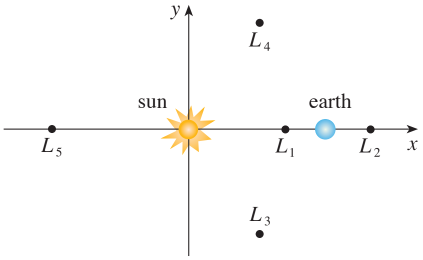
42. The figure shows the sun located at the origin and the earth at the point $(1, 0)$. (The unit here is the distance between the centers of the earth and the sun, called an *astronomical unit:* 1 AU $\approx 1.496 \times 10^8$ km.) There are five locations L_1, L_2, L_3, L_4, and L_5 in this plane of rotation of the earth about the sun where a satellite remains motionless with respect to the earth because the forces acting on the satellite (including the gravitational attractions of the earth and the sun) balance each other. These locations are called *libration points*. (A solar research satellite has been placed at one of these libration points.) If m_1 is the mass of the sun, m_2 is the mass of the earth, and $r = m_2/(m_1 + m_2)$, it turns out that the x-coordinate of L_1 is the unique solution of the fifth-degree equation

$$p(x) = x^5 - (2 + r)x^4 + (1 + 2r)x^3 - (1 - r)x^2 + 2(1 - r)x + r - 1 = 0$$

and the x-coordinate of L_2 is the solution of the equation

$$p(x) - 2rx^2 = 0$$

Using the value $r \approx 3.04042 \times 10^{-6}$, find the locations of the libration points (a) L_1 and (b) L_2.



4.9 Antiderivatives

A physicist who knows the velocity of a particle might wish to know its position at a given time. An engineer who can measure the variable rate at which water is leaking from a tank wants to know the amount leaked over a certain time period. A biologist who knows the rate at which a bacteria population is increasing might want to deduce what the size of the population will be at some future time. In each case, the problem is to find a function whose derivative is a known function.

The Antiderivative of a Function

If we have a function F whose derivative is the function f, then F is called an *antiderivative* of f.

Definition A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

For instance, let $f(x) = x^2$. It isn't difficult to discover an antiderivative of f if we keep the Power Rule in mind. In fact, if $F(x) = \frac{1}{3}x^3$, then $F'(x) = x^2 = f(x)$. But the function $G(x) = \frac{1}{3}x^3 + 100$ also satisfies $G'(x) = x^2$. Therefore both F and G are antiderivatives of f. Indeed, any function of the form $H(x) = \frac{1}{3}x^3 + C$, where C is a constant, is an antiderivative of f. The question arises: are there any others?

To answer this question, recall that in Section 4.2 we used the Mean Value Theorem to prove that if two functions have identical derivatives on an interval, then they must differ by a constant (Corollary 4.2.7). Thus if F and G are any two antiderivatives of f, then

$$F'(x) = f(x) = G'(x)$$

so $G(x) - F(x) = C$, where C is a constant. We can write this as $G(x) = F(x) + C$, so we have the following result.

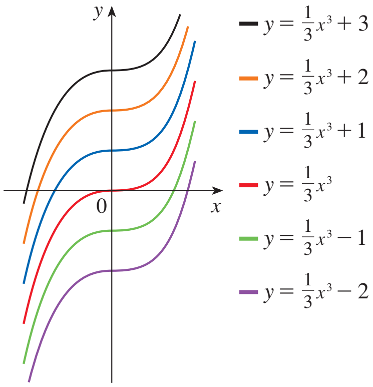


FIGURE 1
Members of the family of antiderivatives of $f(x) = x^2$

1 Theorem If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Going back to the function $f(x) = x^2$, we see that the general antiderivative of f is $\frac{1}{3}x^3 + C$. By assigning specific values to the constant C, we obtain a family of functions whose graphs are vertical translates of one another (see Figure 1). This makes sense because each curve must have the same slope at any given value of x.

EXAMPLE 1 Find the most general antiderivative of each of the following functions.

(a) $f(x) = \sin x$ (b) $f(x) = 1/x$ (c) $f(x) = x^n$, $n \neq -1$

SOLUTION

(a) If $F(x) = -\cos x$, then $F'(x) = \sin x$, so an antiderivative of $\sin x$ is $-\cos x$. By Theorem 1, the most general antiderivative is $G(x) = -\cos x + C$.

(b) Recall from Section 3.6 that

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

So on the interval $(0, \infty)$ the general antiderivative of $1/x$ is $\ln x + C$. We also learned that

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

for all $x \neq 0$. Theorem 1 then tells us that the general antiderivative of $f(x) = 1/x$ is $\ln|x| + C$ on any interval that doesn't contain 0. In particular, this is true on each of the intervals $(-\infty, 0)$ and $(0, \infty)$. So the general antiderivative of f is

$$F(x) = \begin{cases} \ln x + C_1 & \text{if } x > 0 \\ \ln(-x) + C_2 & \text{if } x < 0 \end{cases}$$

(c) We use the Power Rule to discover an antiderivative of x^n. In fact, if $n \neq -1$, then

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{n+1} = x^n$$

Therefore the general antiderivative of $f(x) = x^n$ is

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

This is valid for $n \geqslant 0$ since then $f(x) = x^n$ is defined on an interval. If n is negative (but $n \neq -1$), it is valid on any interval that doesn't contain 0. ■

■ Antidifferentiation Formulas

As in Example 1, every differentiation formula, when read from right to left, gives rise to an antidifferentiation formula. In Table 2 we list some particular antiderivatives. Each formula in the table is true because the derivative of the function in the right column appears in the left column. In particular, the first formula says that the antiderivative of a constant times a function is the constant times the antiderivative of the function. The second formula says that the antiderivative of a sum is the sum of the antiderivatives. (We use the notation $F' = f$, $G' = g$.)

To obtain the most general antiderivative from the particular ones in Table 2, we have to add a constant (or constants), as in Example 1.

2 Table of Antidifferentiation Formulas

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sin x$	$-\cos x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sec^2 x$	$\tan x$
$x^n \ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec x \tan x$	$\sec x$
$\dfrac{1}{x}$	$\ln\|x\|$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1}x$
e^x	e^x	$\dfrac{1}{1+x^2}$	$\tan^{-1}x$
b^x	$\dfrac{b^x}{\ln b}$	$\cosh x$	$\sinh x$
$\cos x$	$\sin x$	$\sinh x$	$\cosh x$

EXAMPLE 2 Find all functions g such that

$$g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}$$

SOLUTION We first rewrite the given function as follows:

$$g'(x) = 4\sin x + \frac{2x^5}{x} - \frac{\sqrt{x}}{x} = 4\sin x + 2x^4 - \frac{1}{\sqrt{x}}$$

Thus we want to find an antiderivative of

$$g'(x) = 4\sin x + 2x^4 - x^{-1/2}$$

Using the formulas in Table 2 together with Theorem 1, we obtain

$$\begin{aligned} g(x) &= 4(-\cos x) + 2\frac{x^5}{5} - \frac{x^{1/2}}{\frac{1}{2}} + C \\ &= -4\cos x + \tfrac{2}{5}x^5 - 2\sqrt{x} + C \end{aligned}$$

■

We often use a capital letter F to represent an antiderivative of a function f. If we begin with derivative notation, f', an antiderivative is f, of course.

In applications of calculus it is very common to have a situation as in Example 2, where it is required to find a function, given knowledge about its derivatives. An equation that involves the derivatives of a function is called a **differential equation**. These will be studied in some detail in Chapter 9, but for the present we can solve some elementary differential equations. The general solution of a differential equation involves an arbitrary constant (or constants) as in Example 2. However, there may be some extra conditions given that will determine the constants and therefore uniquely specify the solution.

EXAMPLE 3 Find f if $f'(x) = e^x + 20(1 + x^2)^{-1}$ and $f(0) = -2$.

SOLUTION The general antiderivative of

$$f'(x) = e^x + \frac{20}{1 + x^2}$$

is

$$f(x) = e^x + 20\tan^{-1}x + C$$

To determine C we use the fact that $f(0) = -2$:

$$f(0) = e^0 + 20\tan^{-1}0 + C = -2$$

Thus we have $C = -2 - 1 = -3$, so the particular solution is

$$f(x) = e^x + 20\tan^{-1}x - 3$$

■

Figure 2 shows the graphs of the function f' in Example 3 and its antiderivative f. Notice that $f'(x) > 0$, so f is always increasing. Also notice that when f' has a maximum or minimum, f appears to have an inflection point. So the graph serves as a check on our calculation.

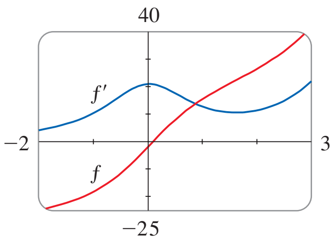


FIGURE 2

EXAMPLE 4 Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, and $f(1) = 1$.

SOLUTION The general antiderivative of $f''(x) = 12x^2 + 6x - 4$ is

$$f'(x) = 12\frac{x^3}{3} + 6\frac{x^2}{2} - 4x + C = 4x^3 + 3x^2 - 4x + C$$

Using the antidifferentiation rules once more, we find that

$$f(x) = 4\frac{x^4}{4} + 3\frac{x^3}{3} - 4\frac{x^2}{2} + Cx + D = x^4 + x^3 - 2x^2 + Cx + D$$

To determine C and D we use the given conditions that $f(0) = 4$ and $f(1) = 1$. Since $f(0) = 0 + D = 4$, we have $D = 4$. Since

$$f(1) = 1 + 1 - 2 + C + 4 = 1$$

we have $C = -3$. Therefore the required function is

$$f(x) = x^4 + x^3 - 2x^2 - 3x + 4$$

■

Graphing Antiderivatives

If we are given the graph of a function f, it seems reasonable that we should be able to sketch the graph of an antiderivative F. Suppose, for instance, that we are given that $F(0) = 1$. Then we have a place to start, the point $(0, 1)$, and the direction in which we move our pencil is given at each stage by the derivative $F'(x) = f(x)$. In the next example we use the principles of this chapter to show how to graph F even when we don't have a formula for f. This would be the case, for instance, when $f(x)$ is determined by experimental data.

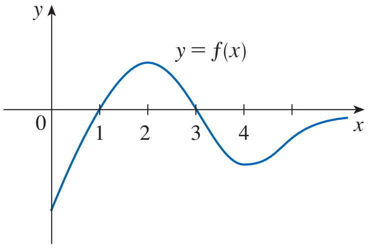


FIGURE 3

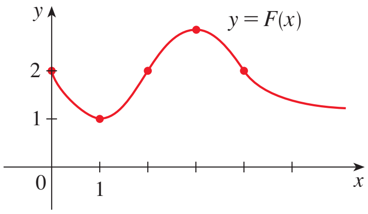


FIGURE 4

EXAMPLE 5 The graph of a function f is given in Figure 3. Make a rough sketch of an antiderivative F, given that $F(0) = 2$.

SOLUTION We are guided by the fact that the slope of $y = F(x)$ is $f(x)$. We start at the point $(0, 2)$ and draw F as an initially decreasing function since $f(x)$ is negative when $0 < x < 1$. Notice that $f(1) = f(3) = 0$, so F has horizontal tangents when $x = 1$ and $x = 3$. For $1 < x < 3$, $f(x)$ is positive and so F is increasing. We see that F has a local minimum when $x = 1$ and a local maximum when $x = 3$. For $x > 3$, $f(x)$ is negative and so F is decreasing on $(3, \infty)$. Since $f(x) \to 0$ as $x \to \infty$, the graph of F becomes flatter as $x \to \infty$. Also notice that $F''(x) = f'(x)$ changes from positive to negative at $x = 2$ and from negative to positive at $x = 4$, so F has inflection points when $x = 2$ and $x = 4$. We use this information to sketch the graph of the antiderivative in Figure 4. ■

Linear Motion

Antidifferentiation is particularly useful in analyzing the motion of an object moving in a straight line. Recall that if the object has position function $s = f(t)$, then the velocity function is $v(t) = s'(t)$. This means that the position function is an antiderivative of the velocity function. Likewise, the acceleration function is $a(t) = v'(t)$, so the velocity function is an antiderivative of the acceleration. If the acceleration and the initial values $s(0)$ and $v(0)$ are known, then the position function can be found by antidifferentiating twice.

EXAMPLE 6 A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

SOLUTION Since $v'(t) = a(t) = 6t + 4$, antidifferentiation gives

$$v(t) = 6\frac{t^2}{2} + 4t + C = 3t^2 + 4t + C$$

Note that $v(0) = C$. But we are given that $v(0) = -6$, so $C = -6$ and

$$v(t) = 3t^2 + 4t - 6$$

Since $v(t) = s'(t)$, s is the antiderivative of v:

$$s(t) = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + D = t^3 + 2t^2 - 6t + D$$

This gives $s(0) = D$. We are given that $s(0) = 9$, so $D = 9$ and the required position function is

$$s(t) = t^3 + 2t^2 - 6t + 9$$ ■

An object near the surface of the earth is subject to a gravitational force that produces a downward acceleration denoted by g. For motion close to the ground we may assume that g is constant, its value being about 9.8 m/s^2 (or 32 ft/s^2). It is remarkable that from the single fact that the acceleration due to gravity is constant, we can use calculus to deduce the position and velocity of any object moving under the force of gravity, as illustrated in the next example.

EXAMPLE 7 A ball is thrown upward with a speed of 15 m/s from the edge of a cliff, 130 m above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground?

SOLUTION The motion is vertical and we choose the positive direction to be upward. At time t the distance above the ground is $s(t)$ and the velocity $v(t)$ is decreasing. Therefore the acceleration must be negative and we have

$$a(t) = \frac{dv}{dt} = -9.8$$

Taking antiderivatives, we have

$$v(t) = -9.8t + C$$

To determine C we use the given information that $v(0) = 15$. This gives $15 = 0 + C$, so

$$v(t) = -9.8t + 15$$

The maximum height is reached when $v(t) = 0$, that is, after 1.5 seconds. Since $s'(t) = v(t)$, we antidifferentiate again and obtain

$$s(t) = -4.9t^2 + 15t + D$$

Using the fact that $s(0) = 130$, we have $130 = 0 + D$ and so

$$s(t) = -4.9t^2 + 15t + 130$$

The expression for $s(t)$ is valid until the ball hits the ground. This happens when $s(t) = 0$, that is, when

$$-4.9t^2 + 15t + 130 = 0$$

or, equivalently,

$$4.9t^2 - 15t - 130 = 0$$

Using the quadratic formula to solve this equation, we get

$$t = \frac{15 \pm \sqrt{2773}}{9.8}$$

We reject the solution with the minus sign because it gives a negative value for t. Therefore the ball hits the ground after $15 + \sqrt{2773}/9.8 \approx 6.9$ seconds. ■

Figure 5 shows the position function of the ball in Example 7. The graph corroborates the conclusions we reached: the ball reaches its maximum height after 1.5 seconds and hits the ground after about 6.9 seconds.

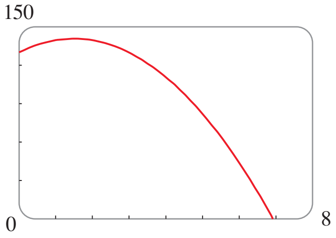


FIGURE 5

4.9 Exercises

1–4 Find an antiderivative of the function.

1. (a) $f(x) = 6$ (b) $g(t) = 3t^2$

2. (a) $f(x) = 2x$ (b) $g(x) = -1/x^2$

3. (a) $h(q) = \cos q$ (b) $f(x) = e^x$

4. (a) $g(t) = 1/t$ (b) $r(\theta) = \sec^2\theta$

5–26 Find the most general antiderivative of the function. (Check your answer by differentiation.)

5. $f(x) = 4x + 7$

6. $f(x) = x^2 - 3x + 2$

7. $f(x) = 2x^3 - \frac{2}{3}x^2 + 5x$

8. $f(x) = 6x^5 - 8x^4 - 9x^2$

9. $f(x) = x(12x + 8)$

10. $f(x) = (x - 5)^2$

11. $g(x) = 4x^{-2/3} - 2x^{5/3}$

12. $h(z) = 3z^{0.8} + z^{-2.5}$

13. $f(x) = 3\sqrt{x} - 2\sqrt[3]{x}$

14. $g(x) = \sqrt{x}\,(2 - x + 6x^2)$

15. $f(t) = \dfrac{2t - 4 + 3\sqrt{t}}{\sqrt{t}}$

16. $f(x) = \sqrt[4]{5} + \sqrt[4]{x}$

17. $f(x) = \dfrac{2}{5x} - \dfrac{3}{x^2}$

18. $f(x) = \dfrac{5x^2 - 6x + 4}{x^2}, \quad x > 0$

19. $g(t) = 7e^t - e^3$

20. $f(x) = \dfrac{10}{x^6} - 2e^x + 3$

21. $f(\theta) = 2\sin\theta - 3\sec\theta\tan\theta$

22. $h(x) = \sec^2 x + \pi\cos x$

23. $f(r) = \dfrac{4}{1 + r^2} - \sqrt[5]{r^4}$

24. $g(v) = 2\cos v - \dfrac{3}{\sqrt{1 - v^2}}$

25. $f(x) = 2^x + 4\sinh x$

26. $f(x) = \dfrac{2x^2 + 5}{x^2 + 1}$

27–28 Find the antiderivative F of f that satisfies the given condition. Check your answer by comparing the graphs of f and F.

27. $f(x) = 2e^x - 6x, \quad F(0) = 1$

28. $f(x) = 4 - 3(1 + x^2)^{-1}, \quad F(1) = 0$

29–54 Find f.

29. $f''(x) = 24x$

30. $f''(t) = t^2 - 4$

31. $f''(x) = 4x^3 + 24x - 1$

32. $f''(x) = 6x - x^4 + 3x^5$

33. $f''(x) = 2x + 3e^x$

34. $f''(x) = 1/x^2, \quad x > 0$

35. $f'''(t) = 12 + \sin t$

36. $f'''(t) = \sqrt{t} - 2\cos t$

37. $f'(x) = 8x^3 + \dfrac{1}{x}, \quad x > 0, \quad f(1) = -3$

38. $f'(x) = \sqrt{x} - 2, \quad f(9) = 4$

39. $f'(t) = 4/(1 + t^2), \quad f(1) = 0$

40. $f'(t) = t + 1/t^3, \quad t > 0, \quad f(1) = 6$

41. $f'(x) = 5x^{2/3}, \quad f(8) = 21$

42. $f'(x) = (x + 1)/\sqrt{x}, \quad f(1) = 5$

43. $f'(t) = \sec t\,(\sec t + \tan t), \quad -\pi/2 < t < \pi/2,$ $f(\pi/4) = -1$

44. $f'(t) = 3^t - 3/t, \quad f(1) = 2, \quad f(-1) = 1$

45. $f''(x) = -2 + 12x - 12x^2, \quad f(0) = 4, \quad f'(0) = 12$

46. $f''(x) = 8x^3 + 5, \quad f(1) = 0, \quad f'(1) = 8$

47. $f''(\theta) = \sin\theta + \cos\theta, \quad f(0) = 3, \quad f'(0) = 4$

48. $f''(t) = t^2 + 1/t^2, \quad t > 0, \quad f(2) = 3, \quad f'(1) = 2$

49. $f''(x) = 4 + 6x + 24x^2, \quad f(0) = 3, \quad f(1) = 10$

50. $f''(x) = x^3 + \sinh x, \quad f(0) = 1, \quad f(2) = 2.6$

51. $f''(x) = e^x - 2\sin x, \quad f(0) = 3, \quad f(\pi/2) = 0$

52. $f''(t) = \sqrt[3]{t} - \cos t, \quad f(0) = 2, \quad f(1) = 2$

53. $f''(x) = x^{-2}, \quad x > 0, \quad f(1) = 0, \quad f(2) = 0$

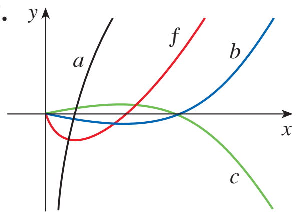
54. $f'''(x) = \cos x, \quad f(0) = 1, \quad f'(0) = 2, \quad f''(0) = 3$

55. Given that the graph of f passes through the point $(2, 5)$ and that the slope of its tangent line at $(x, f(x))$ is $3 - 4x$, find $f(1)$.

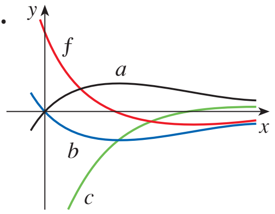
56. Find a function f such that $f'(x) = x^3$ and the line $x + y = 0$ is tangent to the graph of f.

57–58 The graph of a function f is shown. Which graph is an antiderivative of f and why?

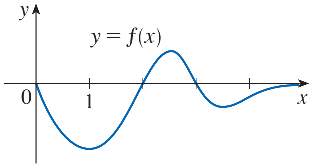
57.



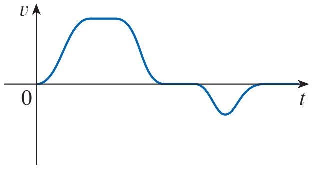
58.



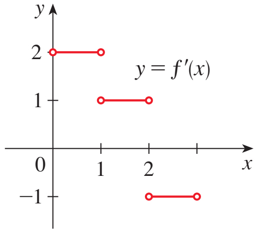
59. The graph of a function is shown in the figure. Make a rough sketch of an antiderivative F, given that $F(0) = 1$.



60. The graph of the velocity function of a particle is shown in the figure. Sketch the graph of a position function.



61. The graph of f' is shown in the figure. Sketch the graph of f if f is continuous on $[0, 3]$ and $f(0) = -1$.



62. (a) Graph $f(x) = 2x - 3\sqrt{x}$.
(b) Starting with the graph in part (a), sketch a rough graph of the antiderivative F that satisfies $F(0) = 1$.
(c) Use the rules of this section to find an expression for $F(x)$.
(d) Graph F using the expression in part (c). Compare with your sketch in part (b).

63–64 Draw a graph of f and use it to make a rough sketch of the antiderivative that passes through the origin.

63. $f(x) = \dfrac{\sin x}{1 + x^2}, \quad -2\pi \leqslant x \leqslant 2\pi$

64. $f(x) = \sqrt{x^4 - 2x^2 + 2} - 2, \quad -3 \leqslant x \leqslant 3$

65–70 A particle is moving with the given data. Find the position of the particle.

65. $v(t) = 2\cos t + 4\sin t, \quad s(0) = 3$

66. $v(t) = t^2 - 3\sqrt{t}, \quad s(4) = 8$

67. $a(t) = 2t + 1, \quad s(0) = 3, \quad v(0) = -2$

68. $a(t) = 3\cos t - 2\sin t, \quad s(0) = 0, \quad v(0) = 4$

69. $a(t) = \sin t - \cos t, \quad s(0) = 0, \quad s(\pi) = 6$

70. $a(t) = t^2 - 4t + 6, \quad s(0) = 0, \quad s(1) = 20$

---

71. A stone is dropped from the upper observation deck (the Space Deck) of the CN Tower, 450 m above the ground.
(a) Find the distance of the stone above ground level at time t.
(b) How long does it take the stone to reach the ground?
(c) With what velocity does it strike the ground?
(d) If the stone is thrown downward with a speed of 5 m/s, how long does it take to reach the ground?

72. Show that for motion in a straight line with constant acceleration a, initial velocity v_0, and initial displacement s_0, the displacement after time t is $s = \frac{1}{2}at^2 + v_0 t + s_0$.

73. An object is projected upward with initial velocity v_0 meters per second from a point s_0 meters above the ground. Show that

$$[v(t)]^2 = v_0^2 - 19.6[s(t) - s_0]$$

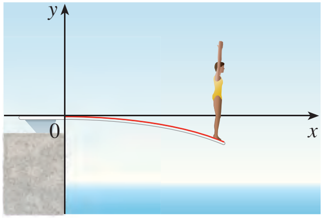
74. Two balls are thrown upward from the edge of the cliff in Example 7. The first is thrown with a speed of 15 m/s and the other is thrown a second later with a speed of 8 m/s. Do the balls ever pass each other?

75. A stone was dropped off a cliff and hit the ground with a speed of 40 m/s. What is the height of the cliff?

76. If a diver of mass m stands at the end of a diving board with length L and linear density ρ, then the board takes on the shape of a curve $y = f(x)$, where

$$EIy'' = mg(L - x) + \tfrac{1}{2}\rho g(L - x)^2$$

E and I are positive constants that depend on the material of the board and g (< 0) is the acceleration due to gravity.
(a) Find an expression for the shape of the curve.
(b) Use $f(L)$ to estimate the distance below the horizontal at the end of the board.



77. A company estimates that the marginal cost (in dollars per item) of producing x items is $1.92 - 0.002x$. If the cost of producing one item is \$562, find the cost of producing 100 items.

78. The linear density of a rod of length 1 m is given by $\rho(x) = 1/\sqrt{x}$, in grams per centimeter, where x is measured in centimeters from one end of the rod. Find the mass of the rod.

79. Since raindrops grow as they fall, their surface area increases and therefore the resistance to their falling increases. A raindrop has an initial downward velocity of 10 m/s and its downward acceleration is

$$a = \begin{cases} 9 - 0.9t & \text{if } 0 \leqslant t \leqslant 10 \\ 0 & \text{if } t > 10 \end{cases}$$

If the raindrop forms 500 m above the ground, how long does it take to fall?

80. A car is traveling at 80 km/h when the brakes are fully applied, producing a constant deceleration of 7 m/s^2. What is the distance traveled before the car comes to a stop?

81. What constant acceleration is required to increase the speed of a car from 50 km/h to 80 km/h in 5 seconds?

82. A car braked with a constant deceleration of 5 m/s^2, producing skid marks measuring 60 m before coming to a stop. How fast was the car traveling when the brakes were first applied?

83. A car is traveling at 100 km/h when the driver sees an accident 80 m ahead and slams on the brakes. What constant deceleration is required to stop the car in time to avoid a multicar pileup?

84. A model rocket is fired vertically upward from rest. Its acceleration for the first three seconds is $a(t) = 18t$, at which time the fuel is exhausted and it becomes a freely "falling" body. Fourteen seconds later, the rocket's parachute opens, and the (downward) velocity slows linearly to -5.5 m/s in 5 seconds. The rocket then "floats" to the ground at that rate.
(a) Determine the position function s and the velocity function v (for all times t). Sketch the graphs of s and v.
(b) At what time does the rocket reach its maximum height, and what is that height?
(c) At what time does the rocket land?

85. A particular bullet train accelerates and decelerates at the rate of 1.2 m/s^2. Its maximum cruising speed is 145 km/h.
(a) What is the maximum distance the train can travel if it accelerates from rest until it reaches its cruising speed and then runs at that speed for 20 minutes?
(b) Suppose that the train starts from rest and must come to a complete stop in 20 minutes. What is the maximum distance it can travel under these conditions?
(c) Find the minimum time that the train takes to travel between two consecutive stations that are 72 km apart.
(d) The trip from one station to the next takes 37.5 minutes. How far apart are the stations?

4 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. Explain the difference between an absolute maximum and a local maximum. Illustrate with a sketch.

2. (a) What does the Extreme Value Theorem say?
(b) Explain how the Closed Interval Method works.

3. (a) State Fermat's Theorem.
(b) Define a critical number of f.

4. (a) State Rolle's Theorem.
(b) State the Mean Value Theorem and give a geometric interpretation.

5. (a) State the Increasing/Decreasing Test.
(b) What does it mean to say that f is concave upward on an interval I?
(c) State the Concavity Test.
(d) What are inflection points? How do you find them?

6. (a) State the First Derivative Test.
(b) State the Second Derivative Test.
(c) What are the relative advantages and disadvantages of these tests?

7. (a) What does l'Hospital's Rule say?
(b) How can you use l'Hospital's Rule if you have a product $f(x)\,g(x)$ where $f(x) \to 0$ and $g(x) \to \infty$ as $x \to a$?
(c) How can you use l'Hospital's Rule if you have a difference $f(x) - g(x)$ where $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$?
(d) How can you use l'Hospital's Rule if you have a power $[f(x)]^{g(x)}$ where $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$?

8. State whether each of the following limit forms is indeterminate. Where possible, state the limit.

(a) $\dfrac{0}{0}$ (b) $\dfrac{\infty}{\infty}$ (c) $\dfrac{0}{\infty}$ (d) $\dfrac{\infty}{0}$

(e) $\infty + \infty$ (f) $\infty - \infty$ (g) $\infty \cdot \infty$ (h) $\infty \cdot 0$

(i) 0^0 (j) 0^∞ (k) ∞^0 (l) 1^∞

9. If you have a graphing calculator or computer, why do you need calculus to graph a function?

10. (a) Given an initial approximation x_1 to a solution of the equation $f(x) = 0$, explain geometrically, with a diagram, how the second approximation x_2 in Newton's method is obtained.
(b) Write an expression for x_2 in terms of x_1, $f(x_1)$, and $f'(x_1)$.
(c) Write an expression for x_{n+1} in terms of x_n, $f(x_n)$, and $f'(x_n)$.
(d) Under what circumstances is Newton's method likely to fail or to work very slowly?

11. (a) What is an antiderivative of a function f?
(b) Suppose F_1 and F_2 are both antiderivatives of f on an interval I. How are F_1 and F_2 related?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If $f'(c) = 0$, then f has a local maximum or minimum at c.

2. If f has an absolute minimum value at c, then $f'(c) = 0$.

3. If f is continuous on (a, b), then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in (a, b).

4. If f is differentiable and $f(-1) = f(1)$, then there is a number c such that $|c| < 1$ and $f'(c) = 0$.

5. If $f'(x) < 0$ for $1 < x < 6$, then f is decreasing on $(1, 6)$.

6. If $f''(2) = 0$, then $(2, f(2))$ is an inflection point of the curve $y = f(x)$.

7. If $f'(x) = g'(x)$ for $0 < x < 1$, then $f(x) = g(x)$ for $0 < x < 1$.

8. There exists a function f such that $f(1) = -2$, $f(3) = 0$, and $f'(x) > 1$ for all x.

9. There exists a function f such that $f(x) > 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.

10. There exists a function f such that $f(x) < 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.

11. If f and g are increasing on an interval I, then $f + g$ is increasing on I.

12. If f and g are increasing on an interval I, then $f - g$ is increasing on I.

13. If f and g are increasing on an interval I, then fg is increasing on I.

14. If f and g are positive increasing functions on an interval I, then fg is increasing on I.

15. If f is increasing and $f(x) > 0$ on I, then $g(x) = 1/f(x)$ is decreasing on I.

16. If f is even, then f' is even.

17. If f is periodic, then f' is periodic.

18. The most general antiderivative of $f(x) = x^{-2}$ is

$$F(x) = -\frac{1}{x} + C$$

19. If $f'(x)$ exists and is nonzero for all x, then $f(1) \neq f(0)$.

20. If $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to\infty} g(x) = \infty$, then

$$\lim_{x\to\infty} [f(x)]^{g(x)} = 1$$

21. $\lim_{x\to 0} \dfrac{x}{e^x} = 1$

EXERCISES

1–6 Find the local and absolute extreme values of the function on the given interval.

1. $f(x) = x^3 - 9x^2 + 24x - 2$, $[0, 5]$

2. $f(x) = x\sqrt{1 - x}$, $[-1, 1]$

3. $f(x) = \dfrac{3x - 4}{x^2 + 1}$, $[-2, 2]$

4. $f(x) = \sqrt{x^2 + x + 1}$, $[-2, 1]$

5. $f(x) = x + 2\cos x$, $[-\pi, \pi]$

6. $f(x) = x^2 e^{-x}$, $[-1, 3]$

7–14 Evaluate the limit.

7. $\lim_{x\to 0} \dfrac{e^x - 1}{\tan x}$

8. $\lim_{x\to 0} \dfrac{\tan 4x}{x + \sin 2x}$

9. $\lim_{x\to 0} \dfrac{e^{2x} - e^{-2x}}{\ln(x + 1)}$

10. $\lim_{x\to\infty} \dfrac{e^{2x} - e^{-2x}}{\ln(x + 1)}$

11. $\lim_{x\to -\infty} (x^2 - x^3)e^{2x}$

12. $\lim_{x\to \pi^-} (x - \pi)\csc x$

13. $\lim_{x\to 1^+} \left(\dfrac{x}{x - 1} - \dfrac{1}{\ln x}\right)$

14. $\lim_{x\to (\pi/2)^-} (\tan x)^{\cos x}$

15–17 Sketch the graph of a function that satisfies the given conditions.

15. $f(0) = 0$, $f'(-2) = f'(1) = f'(9) = 0$,

$\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to 6} f(x) = -\infty$,

$f'(x) < 0$ on $(-\infty, -2)$, $(1, 6)$, and $(9, \infty)$,

$f'(x) > 0$ on $(-2, 1)$ and $(6, 9)$,

$f''(x) > 0$ on $(-\infty, 0)$ and $(12, \infty)$,

$f''(x) < 0$ on $(0, 6)$ and $(6, 12)$

16. $f(0) = 0$, f is continuous and even,

$f'(x) = 2x$ if $0 < x < 1$, $f'(x) = -1$ if $1 < x < 3$,

$f'(x) = 1$ if $x > 3$

17. f is odd, $f'(x) < 0$ for $0 < x < 2$,

$f'(x) > 0$ for $x > 2$, $f''(x) > 0$ for $0 < x < 3$,

$f''(x) < 0$ for $x > 3$, $\lim_{x\to\infty} f(x) = -2$

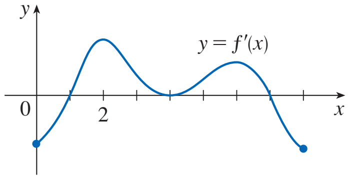
18. The figure shows the graph of the *derivative* f' of a function f.

(a) On what intervals is f increasing or decreasing?

(b) For what values of x does f have a local maximum or minimum?

(c) Sketch the graph of f''.

(d) Sketch a possible graph of f.



19–34 Use the guidelines of Section 4.5 to sketch the curve.

19. $y = 2 - 2x - x^3$

20. $y = -2x^3 - 3x^2 + 12x + 5$

21. $y = 3x^4 - 4x^3 + 2$

22. $y = \dfrac{x}{1 - x^2}$

23. $y = \dfrac{1}{x(x - 3)^2}$

24. $y = \dfrac{1}{x^2} - \dfrac{1}{(x - 2)^2}$

25. $y = \dfrac{(x - 1)^3}{x^2}$

26. $y = \sqrt{1 - x} + \sqrt{1 + x}$

27. $y = x\sqrt{2 + x}$

28. $y = x^{2/3}(x - 3)^2$

29. $y = e^x \sin x, \quad -\pi \leqslant x \leqslant \pi$

30. $y = 4x - \tan x, \quad -\pi/2 < x < \pi/2$

31. $y = \sin^{-1}(1/x)$

32. $y = e^{2x - x^2}$

33. $y = (x - 2)e^{-x}$

34. $y = x + \ln(x^2 + 1)$

35–38 Produce graphs of f that reveal all the important aspects of the curve. Use graphs of f' and f'' to estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points. In Exercise 35 use calculus to find these quantities exactly.

35. $f(x) = \dfrac{x^2 - 1}{x^3}$

36. $f(x) = \dfrac{x^3 + 1}{x^6 + 1}$

37. $f(x) = 3x^6 - 5x^5 + x^4 - 5x^3 - 2x^2 + 2$

38. $f(x) = x^2 + 6.5 \sin x, \quad -5 \leqslant x \leqslant 5$

---

39. Graph $f(x) = e^{-1/x^2}$ in a viewing rectangle that shows all the main aspects of this function. Estimate the inflection points. Then use calculus to find them exactly.

40. (a) Graph the function $f(x) = 1/(1 + e^{1/x})$.
  (b) Explain the shape of the graph by computing the limits of $f(x)$ as x approaches ∞, $-\infty$, 0^+, and 0^-.
  (c) Use the graph of f to estimate the coordinates of the inflection points.
  (d) Use a computer algebra system to compute and graph f''.
  (e) Use the graph in part (d) to estimate the inflection points more accurately.

41–42 Use the graphs of f, f', and f'' to estimate the x-coordinates of the maximum and minimum points and inflection points of f.

41. $f(x) = \dfrac{\cos^2 x}{\sqrt{x^2 + x + 1}}, \quad -\pi \leqslant x \leqslant \pi$

42. $f(x) = e^{-0.1x} \ln(x^2 - 1)$

---

43. Investigate the family of functions $f(x) = \ln(\sin x + c)$. What features do the members of this family have in common? How do they differ? For which values of c is f continuous on $(-\infty, \infty)$? For which values of c does f have no graph at all? What happens as $c \to \infty$?

44. Investigate the family of functions $f(x) = cxe^{-cx^2}$. What happens to the maximum and minimum points and the inflection points as c changes? Illustrate your conclusions by graphing several members of the family.

45. Show that the equation $3x + 2\cos x + 5 = 0$ has exactly one real solution.

46. Suppose that f is continuous on $[0, 4]$, $f(0) = 1$, and $2 \leqslant f'(x) \leqslant 5$ for all x in $(0, 4)$. Show that $9 \leqslant f(4) \leqslant 21$.

47. By applying the Mean Value Theorem to the function $f(x) = x^{1/5}$ on the interval $[32, 33]$, show that

$$2 < \sqrt[5]{33} < 2.0125$$

48. For what values of the constants a and b is $(1, 3)$ a point of inflection of the curve $y = ax^3 + bx^2$?

49. Let $g(x) = f(x^2)$, where f is twice differentiable for all x, $f'(x) > 0$ for all $x \neq 0$, and f is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$.
  (a) At what numbers does g have an extreme value?
  (b) Discuss the concavity of g.

50. Find two positive integers such that the sum of the first number and four times the second number is 1000 and the product of the numbers is as large as possible.

51. Show that the shortest distance from the point (x_1, y_1) to the straight line $Ax + By + C = 0$ is

$$\frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

52. Find the point on the hyperbola $xy = 8$ that is closest to the point $(3, 0)$.

53. Find the smallest possible area of an isosceles triangle that is circumscribed about a circle of radius r.

54. Find the volume of the largest circular cone that can be inscribed in a sphere of radius r.

55. In ΔABC, D lies on AB, $CD \perp AB$, $|AD| = |BD| = 4$ cm, and $|CD| = 5$ cm. Where should a point P be chosen on CD so that the sum $|PA| + |PB| + |PC|$ is a minimum?

56. Solve Exercise 55 when $|CD| = 2$ cm.

57. The velocity of a wave of length L in deep water is

$$v = K\sqrt{\frac{L}{C} + \frac{C}{L}}$$

where K and C are known positive constants. What is the length of the wave that gives the minimum velocity?

58. A metal storage tank with volume V is to be constructed in the shape of a right circular cylinder surmounted by a hemisphere. What dimensions will require the least amount of metal?

59. A hockey team plays in an arena that seats 15,000 spectators. With the ticket price set at \$12, average attendance at a game has been 11,000. A market survey indicates that for each dollar the ticket price is lowered, average attendance will increase by 1000. How should the owners of the team set the ticket price to maximize their revenue from ticket sales?

60. A manufacturer determines that the cost of making x units of a commodity is

$$C(x) = 1800 + 25x - 0.2x^2 + 0.001x^3$$

and the demand function is $p(x) = 48.2 - 0.03x$.

(a) Graph the cost and revenue functions and use the graphs to estimate the production level for maximum profit.

(b) Use calculus to find the production level for maximum profit.

(c) Estimate the production level that minimizes the average cost.

61. Use Newton's method to find the solution of the equation

$$x^5 - x^4 + 3x^2 - 3x - 2 = 0$$

in the interval $[1, 2]$ correct to six decimal places.

62. Use Newton's method to find all solutions of the equation $\sin x = x^2 - 3x + 1$ correct to six decimal places.

63. Use Newton's method to find the absolute maximum value of the function $f(t) = \cos t + t - t^2$ correct to eight decimal places.

64. Use the guidelines in Section 4.5 to sketch the curve $y = x \sin x$, $0 \leqslant x \leqslant 2\pi$. Use Newton's method when necessary.

65–68 Find the most general antiderivative of the function.

65. $f(x) = 4\sqrt{x} - 6x^2 + 3$

66. $g(x) = \dfrac{1}{x} + \dfrac{1}{x^2 + 1}$

67. $f(t) = 2 \sin t - 3e^t$

68. $f(x) = x^{-3} + \cosh x$

69–72 Find f.

69. $f'(t) = 2t - 3 \sin t, \quad f(0) = 5$

70. $f'(u) = \dfrac{u^2 + \sqrt{u}}{u}, \quad f(1) = 3$

71. $f''(x) = 1 - 6x + 48x^2, \quad f(0) = 1, \quad f'(0) = 2$

72. $f''(x) = 5x^3 + 6x^2 + 2, \quad f(0) = 3, \quad f(1) = -2$

73–74 A particle is moving along a straight line with the given data. Find the position of the particle.

73. $v(t) = 2t - 1/(1 + t^2), \quad s(0) = 1$

74. $a(t) = \sin t + 3 \cos t, \quad s(0) = 0, \quad v(0) = 2$

75. (a) If $f(x) = 0.1e^x + \sin x$, $-4 \leqslant x \leqslant 4$, use a graph of f to sketch a rough graph of the antiderivative F of f that satisfies $F(0) = 0$.

(b) Find an expression for $F(x)$.

(c) Graph F using the expression in part (b). Compare with your sketch in part (a).

76. Investigate the family of curves given by

$$f(x) = x^4 + x^3 + cx^2$$

In particular you should determine the transitional value of c at which the number of critical numbers changes and the transitional value at which the number of inflection points changes. Illustrate the various possible shapes with graphs.

77. A canister is dropped from a helicopter hovering 500 m above the ground. Its parachute does not open, but the canister has been designed to withstand an impact velocity of 100 m/s. Will it burst?

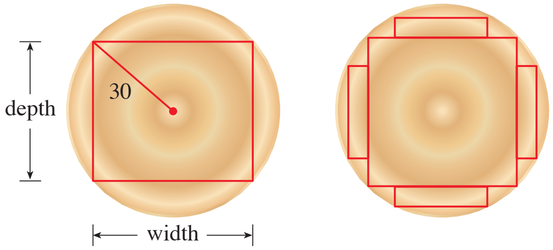
78. In an automobile race along a straight road, car A passed car B twice. Prove that at some time during the race their accelerations were equal. State the assumptions that you make.

79. A rectangular beam will be cut from a cylindrical log of radius 30 centimeters.

(a) Show that the beam of maximal cross-sectional area is a square.

(b) Four rectangular planks will be cut from the four sections of the log that remain after cutting the square beam. Determine the dimensions of the planks that will have maximal cross-sectional area.

(c) Suppose that the strength of a rectangular beam is proportional to the product of its width and the square of its depth. Find the dimensions of the strongest beam that can be cut from the cylindrical log.



80. If a projectile is fired with an initial velocity v at an angle of inclination θ from the horizontal, then its trajectory, neglecting air resistance, is the parabola

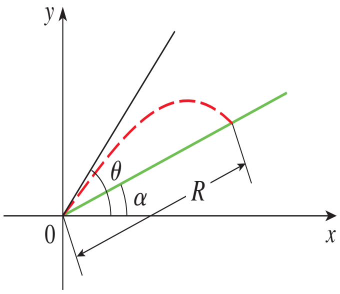
$$y = (\tan\theta)x - \frac{g}{2v^2\cos^2\theta}x^2 \qquad 0 < \theta < \frac{\pi}{2}$$

(a) Suppose the projectile is fired from the base of a plane that is inclined at an angle α, $\alpha > 0$, from the horizontal, as shown in the figure. Show that the range of the projectile, measured up the slope, is given by

$$R(\theta) = \frac{2v^2\cos\theta\,\sin(\theta - \alpha)}{g\cos^2\alpha}$$

(b) Determine θ so that R is a maximum.

(c) Suppose the plane is at an angle α *below* the horizontal. Determine the range R in this case, and determine the angle at which the projectile should be fired to maximize R.



81. If an electrostatic field E acts on a liquid or a gaseous polar dielectric, the net dipole moment P per unit volume is

$$P(E) = \frac{e^E + e^{-E}}{e^E - e^{-E}} - \frac{1}{E}$$

Show that $\lim_{E\to 0^+} P(E) = 0$.

82. If a metal ball with mass m is projected in water and the force of resistance is proportional to the square of the velocity, then the distance the ball travels in time t is

$$s(t) = \frac{m}{c}\ln\cosh\sqrt{\frac{gc}{mt}}$$

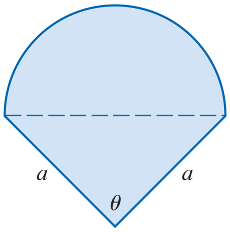
where c is a positive constant. Find $\lim_{c\to 0^+} s(t)$.

83. Show that, for $x > 0$,

$$\frac{x}{1 + x^2} < \tan^{-1}x < x$$

84. Sketch the graph of a function f such that $f'(x) < 0$ for all x, $f''(x) > 0$ for $|x| > 1$, $f''(x) < 0$ for $|x| < 1$, and $\lim_{x\to\pm\infty}[f(x) + x] = 0$.

85. The figure shows an isosceles triangle with equal sides of length a surmounted by a semicircle. What should the measure of angle θ be in order to maximize the total area?



86. Water is flowing at a constant rate into a spherical tank. Let $V(t)$ be the volume of water in the tank and $H(t)$ be the height of the water in the tank at time t.

(a) What are the meanings of $V'(t)$ and $H'(t)$? Are these derivatives positive, negative, or zero?

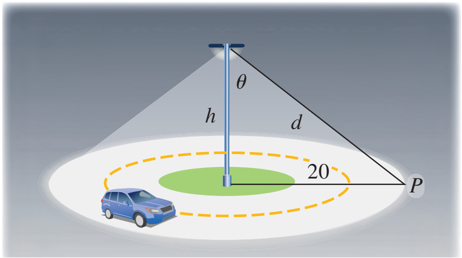
(b) Is $V''(t)$ positive, negative, or zero? Explain.

(c) Let t_1, t_2, and t_3 be the times when the tank is one-quarter full, half full, and three-quarters full, respectively. Are each of the values $H''(t_1)$, $H''(t_2)$, and $H''(t_3)$ positive, negative, or zero? Why?

87. A light is to be mounted atop a pole of height h feet to illuminate a busy traffic circle, which has a radius of 20 m. The intensity of illumination I at any point P on the circle is directly proportional to the cosine of the angle θ (see the figure) and inversely proportional to the square of the distance d from the source.

(a) How tall should the light pole be to maximize I?

(b) Suppose that the light pole is h meters tall and that a woman is walking away from the base of the pole at the rate of 1 m/s. At what rate is the intensity of the light at the point on her back 1 m above the ground decreasing when she reaches the outer edge of the traffic circle?



Problems Plus

1. If a rectangle has its base on the x-axis and two vertices on the curve $y = e^{-x^2}$, show that the rectangle has the largest possible area when the two vertices are at the points of inflection of the curve.

2. Show that $|\sin x - \cos x| \leqslant \sqrt{2}$ for all x.

3. Does the function $f(x) = e^{10|x-2|-x^2}$ have an absolute maximum? If so, find it. What about an absolute minimum?

4. Show that $x^2y^2(4 - x^2)(4 - y^2) \leqslant 16$ for all numbers x and y such that $|x| \leqslant 2$ and $|y| \leqslant 2$.

5. Show that the inflection points of the curve $y = (\sin x)/x$ lie on the curve $y^2(x^4 + 4) = 4$.

6. Find the point on the parabola $y = 1 - x^2$ at which the tangent line cuts from the first quadrant the triangle with the smallest area.

7. If a, b, c, and d are constants such that $\displaystyle\lim_{x\to 0} \frac{ax^2 + \sin bx + \sin cx + \sin dx}{3x^2 + 5x^4 + 7x^6} = 8$, find the value of the sum $a + b + c + d$.

8. Evaluate $\displaystyle\lim_{x\to\infty} \frac{(x + 2)^{1/x} - x^{1/x}}{(x + 3)^{1/x} - x^{1/x}}$.

9. Find the highest and lowest points on the curve $x^2 + xy + y^2 = 12$.

10. Show that if f is a differentiable function that satisfies

$$\frac{f(x + n) - f(x)}{n} = f'(x)$$

for all real numbers x and all positive integers n, then f is a linear function.

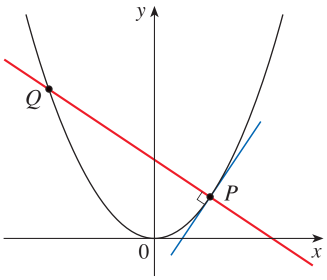


FIGURE FOR PROBLEM 11

11. If $P(a, a^2)$ is any first-quadrant point on the parabola $y = x^2$, let Q be the point where the normal line at P intersects the parabola again (see the figure).
(a) Show that the y-coordinate of Q is smallest when $a = 1/\sqrt{2}$.
(b) Show that the line segment PQ has the shortest possible length when $a = 1/\sqrt{2}$.

12. For what values of c does the curve $y = cx^3 + e^x$ have inflection points?

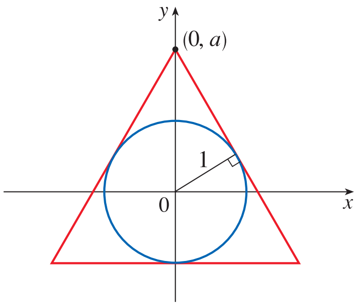


FIGURE FOR PROBLEM 13

13. An isosceles triangle is circumscribed about the unit circle so that the equal sides meet at the point $(0, a)$ on the y-axis (see the figure). Find the value of a that minimizes the lengths of the equal sides. (You may be surprised that the result does not give an equilateral triangle.).

14. Sketch the region in the plane consisting of all points (x, y) such that

$$2xy \leqslant |x - y| \leqslant x^2 + y^2$$

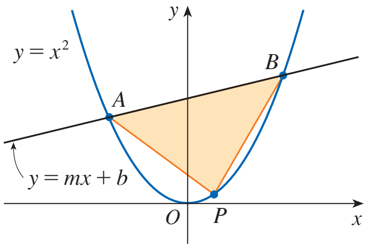


FIGURE FOR PROBLEM 15

15. The line $y = mx + b$ intersects the parabola $y = x^2$ in points A and B. (See the figure.) Find the point P on the arc AOB of the parabola that maximizes the area of the triangle PAB.

16. $ABCD$ is a square piece of paper with sides of length 1 m. A quarter-circle is drawn from B to D with center A. The piece of paper is folded along EF, with E on AB and F on AD, so that A falls on the quarter-circle. Determine the maximum and minimum areas that the triangle AEF can have.

17. For which positive numbers a does the curve $y = a^x$ intersect the line $y = x$?

18. For what value of a is the following equation true?

$$\lim_{x\to\infty} \left(\frac{x + a}{x - a}\right)^x = e$$

19. Let $f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$, where $a_1, a_2, \ldots, a_n$ are real numbers and n is a positive integer. If it is given that $|f(x)| \leqslant |\sin x|$ for all x, show that

$$|a_1 + 2a_2 + \cdots + na_n| \leqslant 1$$

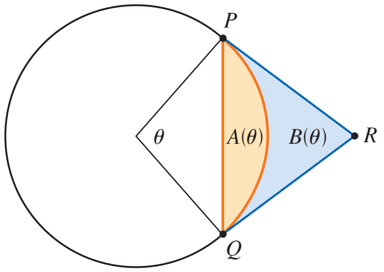


FIGURE FOR PROBLEM 20

20. An arc PQ of a circle subtends a central angle θ as in the figure. Let $A(\theta)$ be the area between the chord PQ and the arc PQ. Let $B(\theta)$ be the area between the tangent lines PR, QR, and the arc. Find

$$\lim_{\theta \to 0^+} \frac{A(\theta)}{B(\theta)}$$

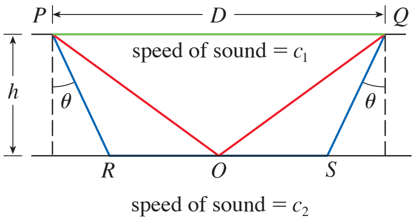


FIGURE FOR PROBLEM 21

21. The speeds of sound c_1 in an upper layer and c_2 in a lower layer of rock and the thickness h of the upper layer can be determined by seismic exploration if the speed of sound in the lower layer is greater than the speed in the upper layer. A dynamite charge is detonated at a point P and the transmitted signals are recorded at a point Q, which is a distance D from P. The first signal to arrive at Q travels along the surface and takes T_1 seconds. The next signal travels from P to a point R, from R to S in the lower layer, and then to Q, taking T_2 seconds. The third signal is reflected off the lower layer at the midpoint O of RS and takes T_3 seconds to reach Q. (See the figure.)

(a) Express T_1, T_2, and T_3 in terms of D, h, c_1, c_2, and θ.
(b) Show that T_2 is a minimum when $\sin \theta = c_1/c_2$.
(c) Suppose that $D = 1$ km, $T_1 = 0.26$ s, $T_2 = 0.32$ s, and $T_3 = 0.34$ s. Find c_1, c_2, and h.

Note: Geophysicists use this technique when studying the structure of the earth's crust—searching for oil or examining fault lines, for example.

22. For what values of c is there a straight line that intersects the curve

$$y = x^4 + cx^3 + 12x^2 - 5x + 2$$

in four distinct points?

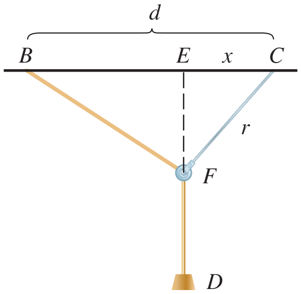


FIGURE FOR PROBLEM 23

23. One of the problems posed by the Marquis de l'Hospital in his calculus textbook *Analyse des infiniment petits* concerns a pulley that is attached to the ceiling of a room at a point C by a rope of length r. At another point B on the ceiling, at a distance d from C (where $d > r$), a rope of length ℓ is attached and passed through the pulley at F and connected to a weight W. The weight is released and comes to rest at its equilibrium position D. (See the figure.) As l'Hospital argued, this happens when the distance $|ED|$ is maximized. Show that when the system reaches equilibrium, the value of x is

$$\frac{r}{4d}\left(r + \sqrt{r^2 + 8d^2}\right)$$

Notice that this expression is independent of both W and ℓ.

24. Given a sphere with radius r, find the height of a pyramid of minimum volume whose base is a square and whose base and triangular faces are all tangent to the sphere. What if the base of the pyramid is a regular n-gon? (A regular n-gon is a polygon with n equal sides and angles.) (Use the fact that the volume of a pyramid is $\frac{1}{3}Ah$, where A is the area of the base.)

25. Assume that a snowball melts so that its volume decreases at a rate proportional to its surface area. If it takes three hours for the snowball to decrease to half its original volume, how much longer will it take for the snowball to melt completely?

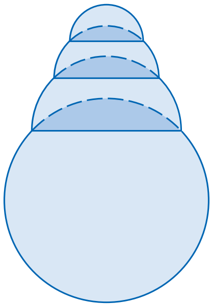


FIGURE FOR PROBLEM 26

26. A hemispherical bubble is placed on a spherical bubble of radius 1. A smaller hemispherical bubble is then placed on the first one. This process is continued until n chambers, including the sphere, are formed. (The figure shows the case $n = 4$.) Use mathematical induction to prove that the maximum height of any bubble tower with n chambers is $1 + \sqrt{n}$.

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In Exercise 5.4.83 we see how to use electric power consumption data and an integral to compute the amount of electric energy used in a typical day in the New England states.

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5 Integrals

IN CHAPTER 2 WE USED the tangent and velocity problems to introduce the derivative. In this chapter we use the area and distance problems to introduce the other central idea in calculus—the integral. The all-important relationship between the derivative and the integral is expressed in the Fundamental Theorem of Calculus, which says that differentiation and integration are in a sense inverse processes. We learn in this chapter, and in Chapters 6 and 8, how integration can be used to solve problems involving volumes, length of curves, population predictions, cardiac output, forces on a dam, work, consumer surplus, and baseball, among many others.

5.1 The Area and Distance Problems

Now is a good time to read (or reread) A Preview of Calculus, which discusses the unifying ideas of calculus and helps put in perspective where we have been and where we are going.

In this section we discover that in trying to find the area under a curve or the distance traveled by a car, we end up with the same special type of limit.

The Area Problem

We begin by attempting to solve the *area problem:* find the area of the region S that lies under the curve $y = f(x)$ from a to b. This means that S, illustrated in Figure 1, is bounded by the graph of a continuous function f [where $f(x) \geqslant 0$], the vertical lines $x = a$ and $x = b$, and the x-axis.

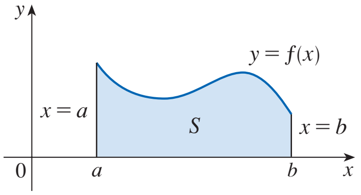


FIGURE 1
$S = \{(x, y) \mid a \leqslant x \leqslant b, 0 \leqslant y \leqslant f(x)\}$

In trying to solve the area problem we have to ask ourselves: what is the meaning of the word *area*? This question is easy to answer for regions with straight sides. For a rectangle, the area is defined as the product of the length and the width. The area of a triangle is half the base times the height. The area of a polygon is found by dividing it into triangles (as in Figure 2) and adding the areas of the triangles.

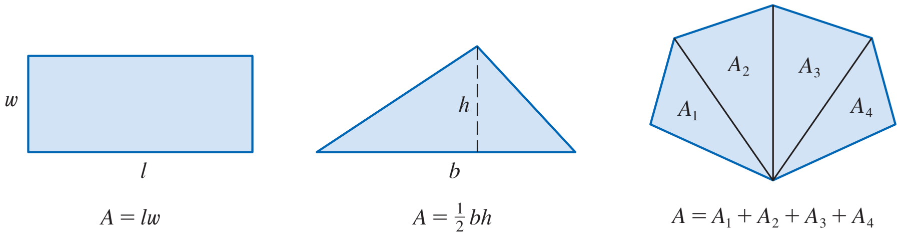


FIGURE 2

It isn't so easy, however, to find the area of a region with curved sides. We all have an intuitive idea of what the area of a region is. But part of the area problem is to make this intuitive idea precise by giving an exact definition of area.

Recall that in defining a tangent we first approximated the slope of the tangent line by slopes of secant lines and then we took the limit of these approximations. We pursue a similar idea for areas. We first approximate the region S by rectangles and then we take the limit of the sum of the areas of the approximating rectangles as we increase the number of rectangles. The following example illustrates the procedure.

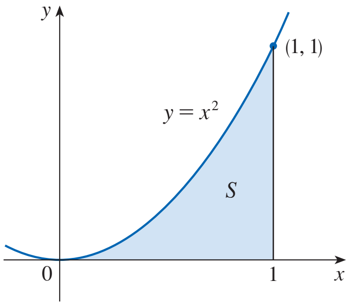
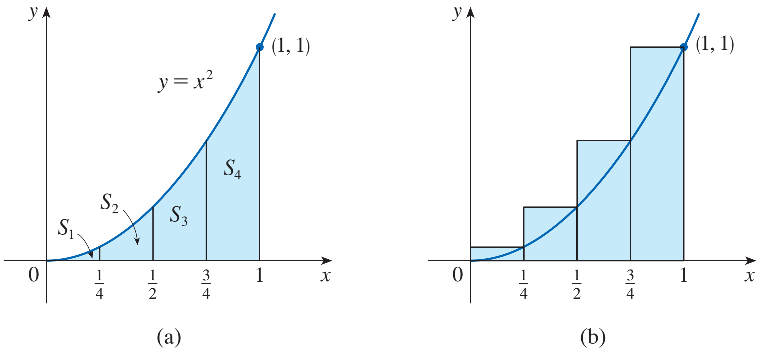


FIGURE 3

EXAMPLE 1 Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1 (the parabolic region S illustrated in Figure 3).

SOLUTION We first notice that the area of S must be somewhere between 0 and 1 because S is contained in a square with side length 1, but we can certainly do better than that. Suppose we divide S into four strips S_1, S_2, S_3, and S_4 by drawing the vertical lines $x = \frac{1}{4}$, $x = \frac{1}{2}$, and $x = \frac{3}{4}$ as in Figure 4(a).



(a) (b)

FIGURE 4

We can approximate each strip by a rectangle that has the same base as the strip and whose height is the same as the right edge of the strip [see Figure 4(b)]. In other words, the heights of these rectangles are the values of the function $f(x) = x^2$ at the *right* endpoints of the subintervals $\left[0, \frac{1}{4}\right]$, $\left[\frac{1}{4}, \frac{1}{2}\right]$, $\left[\frac{1}{2}, \frac{3}{4}\right]$, and $\left[\frac{3}{4}, 1\right]$.

Each rectangle has width $\frac{1}{4}$ and the heights are $\left(\frac{1}{4}\right)^2$, $\left(\frac{1}{2}\right)^2$, $\left(\frac{3}{4}\right)^2$, and 1^2. If we let R_4 be the sum of the areas of these approximating rectangles, we get

$$R_4 = \tfrac{1}{4} \cdot \left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{3}{4}\right)^2 + \tfrac{1}{4} \cdot 1^2 = \tfrac{15}{32} = 0.46875$$

From Figure 4(b) we see that the area A of S is less than R_4, so

$$A < 0.46875$$

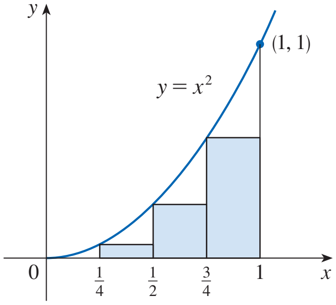


FIGURE 5

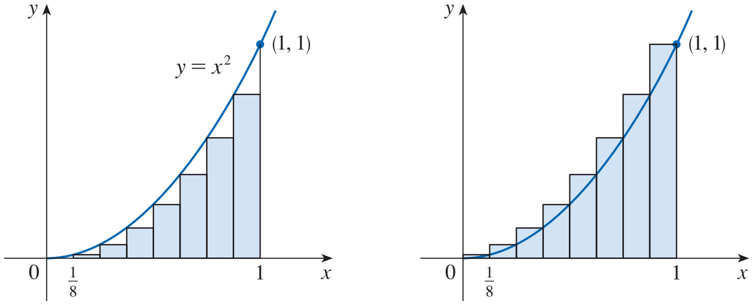
Instead of using the rectangles in Figure 4(b) we could use the smaller rectangles in Figure 5 whose heights are the values of f at the *left* endpoints of the subintervals. (The leftmost rectangle has collapsed because its height is 0.) The sum of the areas of these approximating rectangles is

$$L_4 = \tfrac{1}{4} \cdot 0^2 + \tfrac{1}{4} \cdot \left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{3}{4}\right)^2 = \tfrac{7}{32} = 0.21875$$

We see that the area of S is larger than L_4, so we have lower and upper estimates for A:

$$0.21875 < A < 0.46875$$

We can repeat this procedure with a larger number of strips. Figure 6 shows what happens when we divide the region S into eight strips of equal width.



(a) Using left endpoints

(b) Using right endpoints

FIGURE 6
Approximating S with eight rectangles

By computing the sum of the areas of the smaller rectangles (L_8) and the sum of the areas of the larger rectangles (R_8), we obtain better lower and upper estimates for A:

$$0.2734375 < A < 0.3984375$$

So one possible answer to the question is to say that the true area of S lies somewhere between 0.2734375 and 0.3984375.

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

We could obtain better estimates by increasing the number of strips. The table at the left shows the results of similar calculations (with a computer) using n rectangles whose heights are found with left endpoints (L_n) or right endpoints (R_n). In particular, we see by using 50 strips that the area lies between 0.3234 and 0.3434. With 1000 strips we narrow it down even more: A lies between 0.3328335 and 0.3338335. A good estimate is obtained by averaging these numbers: $A \approx 0.3333335$. ■

From the values listed in the table in Example 1, it looks as if R_n is approaching $\frac{1}{3}$ as n increases. We confirm this in the next example.

EXAMPLE 2 For the region S in Example 1, show that the approximating sums R_n approach $\frac{1}{3}$, that is,

$$\lim_{n\to\infty} R_n = \tfrac{1}{3}$$

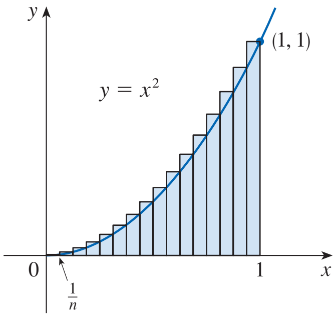


FIGURE 7

SOLUTION R_n is the sum of the areas of the n rectangles in Figure 7. Each rectangle has width $1/n$ and the heights are the values of the function $f(x) = x^2$ at the points $1/n, 2/n, 3/n, \ldots, n/n$; that is, the heights are $(1/n)^2, (2/n)^2, (3/n)^2, \ldots, (n/n)^2$. Thus

$$\begin{aligned}R_n &= \frac{1}{n} f\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{2}{n}\right) + \frac{1}{n} f\left(\frac{3}{n}\right) + \cdots + \frac{1}{n} f\left(\frac{n}{n}\right)\\ &= \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2\\ &= \frac{1}{n} \cdot \frac{1}{n^2}(1^2 + 2^2 + 3^2 + \cdots + n^2)\\ &= \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2)\end{aligned}$$

Here we need the formula for the sum of the squares of the first n positive integers:

$$\boxed{1} \qquad 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Perhaps you have seen this formula before. It is proved in Example 5 in Appendix E.

Putting Formula 1 into our expression for R_n, we get

$$R_n = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

Here we are computing the limit of the sequence $\{R_n\}$. Sequences and their limits will be studied in detail in Section 11.1. The idea is very similar to a limit at infinity (Section 2.6) except that in writing $\lim_{n\to\infty}$ we restrict n to be a positive integer. In particular, we know that

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

When we write $\lim_{n\to\infty} R_n = \frac{1}{3}$ we mean that we can make R_n as close to $\frac{1}{3}$ as we like by taking n sufficiently large.

Thus we have

$$\begin{aligned}\lim_{n\to\infty} R_n &= \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2}\\ &= \lim_{n\to\infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)\\ &= \lim_{n\to\infty} \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\\ &= \frac{1}{6} \cdot 1 \cdot 2 = \frac{1}{3}\end{aligned}$$ ■

It can be shown that the approximating sums L_n in Example 2 also approach $\frac{1}{3}$, that is,

$$\lim_{n\to\infty} L_n = \tfrac{1}{3}$$

From Figures 8 and 9 it appears that as n increases, both L_n and R_n become better and better approximations to the area of S. Therefore we *define* the area A to be the limit of the sums of the areas of the approximating rectangles, that is,

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} L_n = \tfrac{1}{3}$$

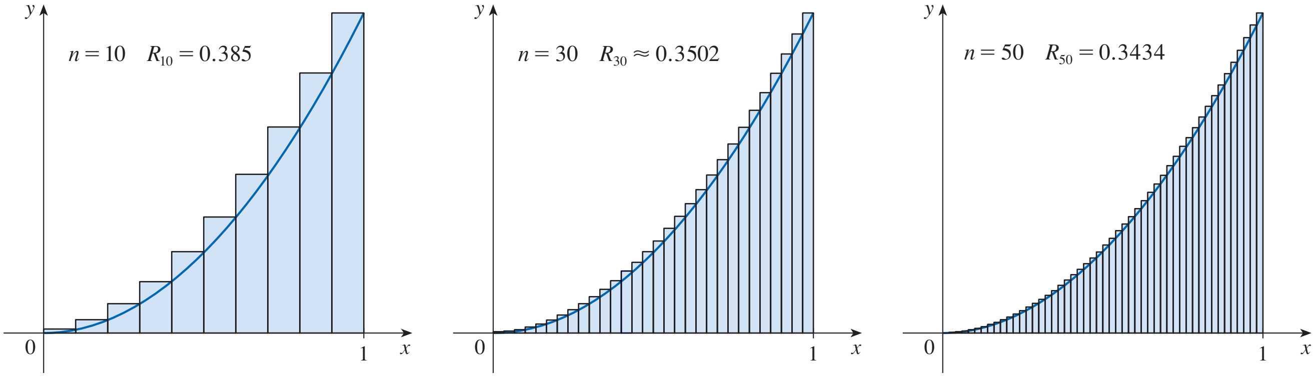


FIGURE 8 Right endpoints produce upper estimates because $f(x) = x^2$ is increasing.

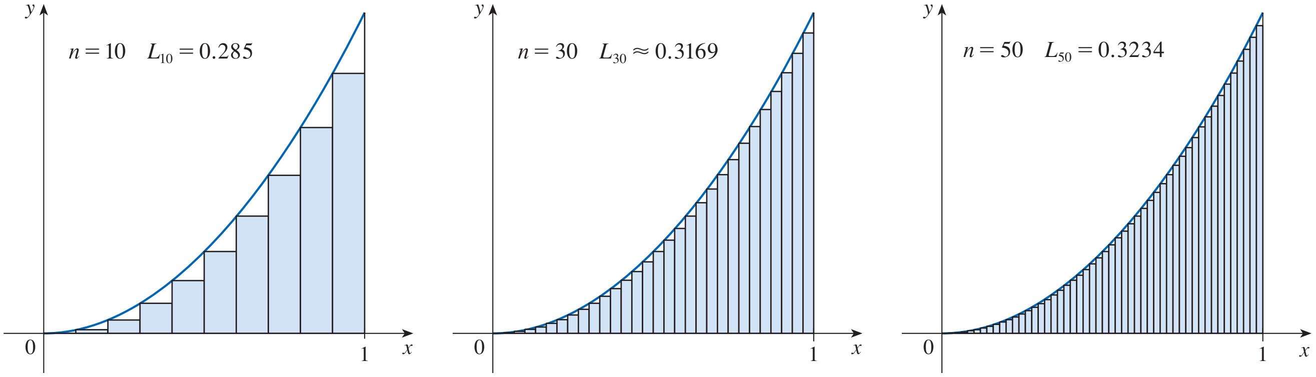


FIGURE 9 Left endpoints produce lower estimates because $f(x) = x^2$ is increasing.

Let's apply the idea of Examples 1 and 2 to the more general region S of Figure 1. We start by subdividing S into n strips $S_1, S_2, \ldots, S_n$ of equal width as in Figure 10.

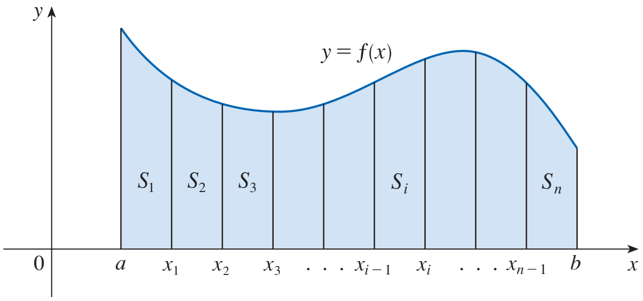


FIGURE 10

The width of the interval $[a, b]$ is $b - a$, so the width of each of the n strips is

$$\Delta x = \frac{b - a}{n}$$

These strips divide the interval $[a, b]$ into n subintervals

$$[x_0, x_1], \quad [x_1, x_2], \quad [x_2, x_3], \quad \ldots, \quad [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$. The right endpoints of the subintervals are

$$\begin{aligned} x_1 &= a + \Delta x, \\ x_2 &= a + 2\,\Delta x, \\ x_3 &= a + 3\,\Delta x, \\ &\vdots \end{aligned}$$

and, in general, $x_i = a + i\,\Delta x$. Now let's approximate the ith strip S_i by a rectangle with width Δx and height $f(x_i)$, which is the value of f at the right endpoint (see Figure 11). Then the area of the ith rectangle is $f(x_i)\,\Delta x$. What we think of intuitively as the area of S is approximated by the sum of the areas of these rectangles, which is

$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

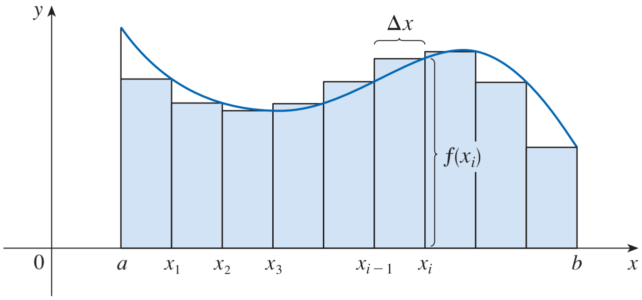
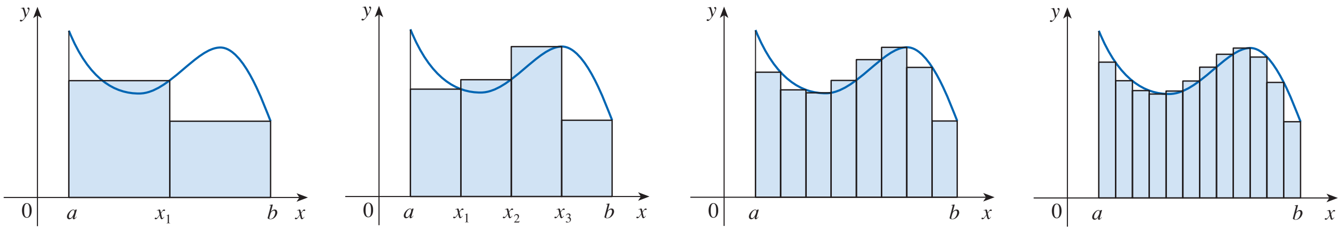


FIGURE 11

Figure 12 shows this approximation for $n = 2$, 4, 8, and 12. Notice that this approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$. Therefore we define the area A of the region S in the following way.



(a) $n = 2$ (b) $n = 4$ (c) $n = 8$ (d) $n = 12$

FIGURE 12

2 **Definition** The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

It can be proved that the limit in Definition 2 always exists, since we are assuming that f is continuous. It can also be shown that we get the same value if we use left endpoints:

3
$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x]$$

In fact, instead of using left endpoints or right endpoints, we could take the height of the ith rectangle to be the value of f at *any* number x_i^* in the ith subinterval $[x_{i-1}, x_i]$. We call the numbers $x_1^*, x_2^*, \ldots, x_n^*$ the **sample points**. Figure 13 shows approximating rectangles when the sample points are not chosen to be endpoints. So a more general expression for the area of S is

4
$$A = \lim_{n\to\infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$

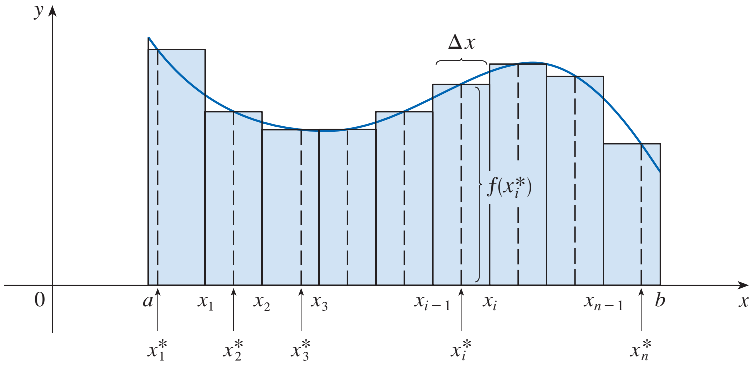
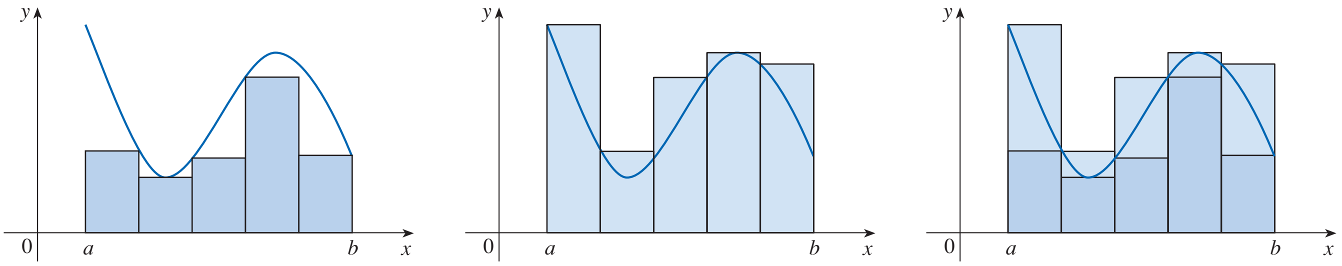


FIGURE 13

NOTE To approximate the area under the graph of f we can form **lower sums** (or **upper sums**) by choosing the sample points x_i^* so that $f(x_i^*)$ is the minimum (or maximum) value of f on the ith subinterval (see Figure 14). [Since f is continuous, we know that the minimum and maximum values of f exist on each subinterval by the Extreme Value Theorem.] It can be shown that an equivalent definition of area is the following: *A is the unique number that is smaller than all the upper sums and bigger than all the lower sums.*



(a) Lower sums (b) Upper sums (c) Upper and lower sums

FIGURE 14

We saw in Examples 1 and 2, for instance, that the area $\left(A = \frac{1}{3}\right)$ is trapped between all the left approximating sums L_n and all the right approximating sums R_n. The function in those examples, $f(x) = x^2$, happens to be increasing on $[0, 1]$ and so the lower sums arise from left endpoints and the upper sums from right endpoints. (See Figures 8 and 9.)

This tells us to end with $i = n$.

This tells us to add.

$$\sum_{i=m}^{n} f(x_i)\,\Delta x$$

This tells us to start with $i = m$.

If you need practice with sigma notation, look at the examples and try some of the exercises in Appendix E.

We often use **sigma notation** to write sums with many terms more compactly. For instance,

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

So the expressions for area in Equations 2, 3, and 4 can be written as follows:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

We can also rewrite Formula 1 in the following way:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

EXAMPLE 3 Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$.

(a) Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.

(b) Estimate the area by taking the sample points to be midpoints and using four subintervals and then ten subintervals.

SOLUTION

(a) Since $a = 0$ and $b = 2$, the width of a subinterval is

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

So $x_1 = 2/n$, $x_2 = 4/n$, $x_3 = 6/n$, $x_i = 2i/n$, and $x_n = 2n/n$. The sum of the areas of the approximating rectangles is

$$\begin{aligned} R_n &= f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x \\ &= e^{-x_1}\,\Delta x + e^{-x_2}\,\Delta x + \cdots + e^{-x_n}\,\Delta x \\ &= e^{-2/n}\left(\frac{2}{n}\right) + e^{-4/n}\left(\frac{2}{n}\right) + \cdots + e^{-2n/n}\left(\frac{2}{n}\right) \end{aligned}$$

According to Definition 2, the area is

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{2}{n}\left(e^{-2/n} + e^{-4/n} + e^{-6/n} + \cdots + e^{-2n/n}\right)$$

Using sigma notation we could write

$$A = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} e^{-2i/n}$$

It is difficult to evaluate this limit directly by hand, but with the aid of a computer algebra system it isn't hard (see Exercise 32). In Section 5.3 we will be able to find A more easily using a different method.

(b) With $n = 4$ the subintervals of equal width $\Delta x = 0.5$ are $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, and $[1.5, 2]$. The midpoints of these subintervals are $x_1^* = 0.25$, $x_2^* = 0.75$, $x_3^* = 1.25$, and $x_4^* = 1.75$, and the sum M_4 of the areas of the four approximating rectangles (see Figure 15) is

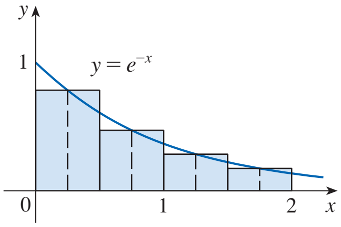


FIGURE 15

$$\begin{aligned}M_4 &= \sum_{i=1}^{4} f(x_i^*)\,\Delta x\\ &= f(0.25)\,\Delta x + f(0.75)\,\Delta x + f(1.25)\,\Delta x + f(1.75)\,\Delta x\\ &= e^{-0.25}(0.5) + e^{-0.75}(0.5) + e^{-1.25}(0.5) + e^{-1.75}(0.5)\\ &= 0.5\left(e^{-0.25} + e^{-0.75} + e^{-1.25} + e^{-1.75}\right) \approx 0.8557\end{aligned}$$

So an estimate for the area is

$$A \approx 0.8557$$

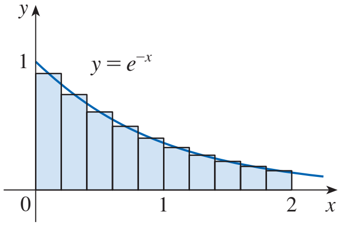


FIGURE 16

With $n = 10$ the subintervals are $[0, 0.2]$, $[0.2, 0.4]$, . . . , $[1.8, 2]$ and the midpoints are $x_1^* = 0.1$, $x_2^* = 0.3$, $x_3^* = 0.5$, . . . , $x_{10}^* = 1.9$. Thus

$$\begin{aligned}A \approx M_{10} &= f(0.1)\,\Delta x + f(0.3)\,\Delta x + f(0.5)\,\Delta x + \cdots + f(1.9)\,\Delta x\\ &= 0.2\left(e^{-0.1} + e^{-0.3} + e^{-0.5} + \cdots + e^{-1.9}\right) \approx 0.8632\end{aligned}$$

From Figure 16 it appears that this estimate is better than the estimate with $n = 4$. ■

The Distance Problem

In Section 2.1 we considered the *velocity problem:* find the velocity of a moving object at a given instant if the distance of the object (from a starting point) is known at all times. Now let's consider the *distance problem:* find the distance traveled by an object during a certain time period if the velocity of the object is known at all times. (In a sense this is the inverse problem of the velocity problem.) If the velocity remains constant, then the distance problem is easy to solve by means of the formula

$$\text{distance} = \text{velocity} \times \text{time}$$

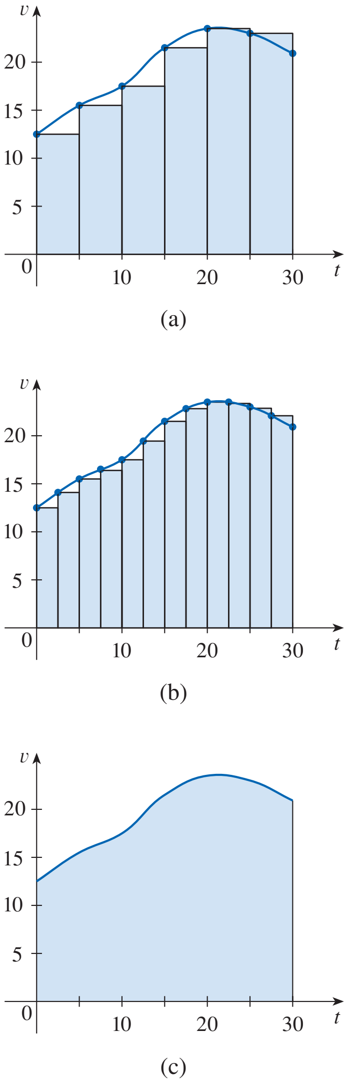
But if the velocity varies, it's not so easy to find the distance traveled. We investigate the problem in the following example.

EXAMPLE 4 Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

Time (s)	0	5	10	15	20	25	30
Velocity (km/h)	27	34	39	47	51	50	45

In order to have the time and the velocity in consistent units, let's convert the velocity readings to meters per second (1 km/h = 1000/3600 m/s):

Time (s)	0	5	10	15	20	25	30
Velocity (m/s)	8	9	10	12	13	12	11



(a)

(b)

(c)

FIGURE 17

During the first five seconds the velocity doesn't change very much, so we can estimate the distance traveled during that time by assuming that the velocity is constant. If we take the velocity during that time interval to be the initial velocity (8 m/s), then we obtain the approximate distance traveled during the first five seconds:

$$8 \text{ m/s} \times 5 \text{ s} = 40 \text{ m}$$

Similarly, during the second time interval the velocity is approximately constant and we take it to be the velocity when $t = 5$ s. So our estimate for the distance traveled from $t = 5$ s to $t = 10$ s is

$$9 \text{ m/s} \times 5 \text{ s} = 45 \text{ m}$$

If we add similar estimates for the other time intervals, we obtain an estimate for the total distance traveled:

$$(8 \times 5) + (9 \times 5) + (10 \times 5) + (12 \times 5) + (13 \times 5) + (12 \times 5) = 320 \text{ m}$$

We could just as well have used the velocity at the *end* of each time period instead of the velocity at the beginning as our assumed constant velocity. Then our estimate becomes

$$(9 \times 5) + (10 \times 5) + (12 \times 5) + (13 \times 5) + (12 \times 5) + (11 \times 5) = 335 \text{ m}$$

Now let's sketch an approximate graph of the velocity function of the car along with rectangles whose heights are the initial velocities for each time interval [see Figure 17(a)]. The area of the first rectangle is $8 \times 5 = 40$, which is also our estimate for the distance traveled in the first five seconds. In fact, the area of each rectangle can be interpreted as a distance because the height represents velocity and the width represents time. The sum of the areas of the rectangles in Figure 17(a) is $L_6 = 320$, which is our initial estimate for the total distance traveled.

If we want a more accurate estimate, we could take velocity readings more often, as illustrated in Figure 17(b). You can see that the more velocity readings we take, the closer the sum of the areas of the rectangles gets to the exact area under the velocity curve [see Figure 17(c)]. This suggests that the total distance traveled is equal to the area under the velocity graph. ■

In general, suppose an object moves with velocity $v = f(t)$, where $a \leqslant t \leqslant b$ and $f(t) \geqslant 0$ (so the object always moves in the positive direction). We take velocity readings at times $t_0 \, (= a), t_1, t_2, \ldots, t_n \, (= b)$ so that the velocity is approximately constant on each subinterval. If these times are equally spaced, then the time between consecutive readings is $\Delta t = (b - a)/n$. During the first time interval the velocity is approximately $f(t_0)$ and so the distance traveled is approximately $f(t_0)\,\Delta t$. Similarly, the distance traveled during the second time interval is about $f(t_1)\,\Delta t$ and the total distance traveled during the time interval $[a, b]$ is approximately

$$f(t_0)\,\Delta t + f(t_1)\,\Delta t + \cdots + f(t_{n-1})\,\Delta t = \sum_{i=1}^{n} f(t_{i-1})\,\Delta t$$

If we use the velocity at right endpoints instead of left endpoints, our estimate for the total distance becomes

$$f(t_1)\,\Delta t + f(t_2)\,\Delta t + \cdots + f(t_n)\,\Delta t = \sum_{i=1}^{n} f(t_i)\,\Delta t$$

The more frequently we measure the velocity, the more accurate our estimates become, so it seems plausible that the *exact* distance d traveled is the *limit* of such expressions:

$$\boxed{5} \qquad d = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_{i-1})\,\Delta t = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i)\,\Delta t$$

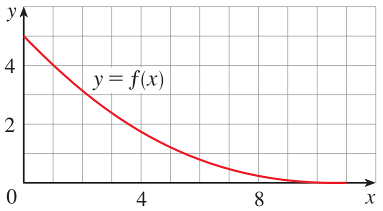
We will see in Section 5.4 that this is indeed true.

Because Equation 5 has the same form as our expressions for area in Equations 2 and 3, it follows that the distance traveled is equal to the area under the graph of the velocity function. In Chapters 6 and 8 we will see that other quantities of interest in the natural and social sciences—such as the work done by a variable force or the cardiac output of the heart—can also be interpreted as the area under a curve. So when we compute areas in this chapter, bear in mind that they can be interpreted in a variety of practical ways.

5.1 Exercises

1. (a) By reading values from the given graph of f, use five rectangles to find a lower estimate and an upper estimate for the area under the given graph of f from $x = 0$ to $x = 10$. In each case sketch the rectangles that you use.

(b) Find new estimates using ten rectangles in each case.



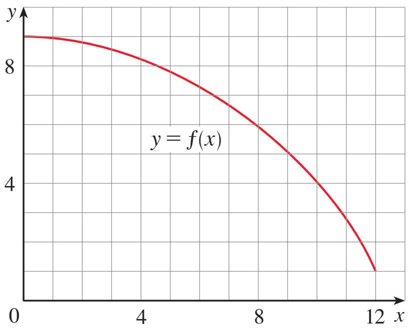
2. (a) Use six rectangles to find estimates of each type for the area under the given graph of f from $x = 0$ to $x = 12$.

(i) L_6 (sample points are left endpoints)
(ii) R_6 (sample points are right endpoints)
(iii) M_6 (sample points are midpoints)

(b) Is L_6 an underestimate or overestimate of the true area?

(c) Is R_6 an underestimate or overestimate of the true area?

(d) Which of the numbers L_6, R_6, or M_6 gives the best estimate? Explain.



3. (a) Estimate the area under the graph of $f(x) = 1/x$ from $x = 1$ to $x = 2$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?

(b) Repeat part (a) using left endpoints.

4. (a) Estimate the area under the graph of $f(x) = \sin x$ from $x = 0$ to $x = \pi/2$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?

(b) Repeat part (a) using left endpoints.

5. (a) Estimate the area under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$ using three rectangles and right endpoints. Then improve your estimate by using six rectangles. Sketch the curve and the approximating rectangles.

(b) Repeat part (a) using left endpoints.

(c) Repeat part (a) using midpoints.

(d) From your sketches in parts (a)–(c), which estimate appears to be the most accurate?

6. (a) Graph the function

$$f(x) = e^{x-x^2} \qquad 0 \leqslant x \leqslant 2$$

(b) Estimate the area under the graph of f using four approximating rectangles and taking the sample points to be (i) right endpoints and (ii) midpoints. In each case sketch the curve and the rectangles.

(c) Improve your estimates in part (b) by using eight rectangles.

7. Evaluate the upper and lower sums for $f(x) = 6 - x^2$, $-2 \leqslant x \leqslant 2$, with $n = 2$, 4, and 8. Illustrate with diagrams like Figure 14.

8. Evaluate the upper and lower sums for

$$f(x) = 1 + \cos(x/2) \qquad -\pi \leqslant x \leqslant \pi$$

with $n = 3$, 4, and 6. Illustrate with diagrams like Figure 14.

9. The speed of a runner increased steadily during the first three seconds of a race. Her speed at half-second intervals is given in the table. Find lower and upper estimates for the distance that she traveled during these three seconds.

t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0
v (m/s)	0	1.9	3.3	4.5	5.5	5.9	6.2

10. The table shows speedometer readings at 10-second intervals during a 1-minute period for a car racing at the Daytona International Speedway in Florida.

(a) Estimate the distance the race car traveled during this time period using the velocities at the beginning of the time intervals.

(b) Give another estimate using the velocities at the end of the time periods.

(c) Are your estimates in parts (a) and (b) upper and lower estimates? Explain.

Time(s)	Velocity (mi/h)
0	182.9
10	168.0
20	106.6
30	99.8
40	124.5
50	176.1
60	175.6

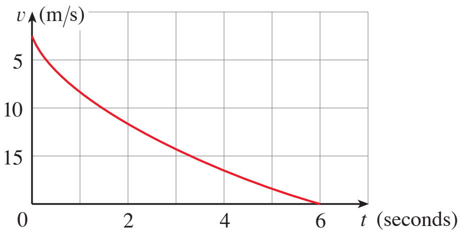
11. Oil leaked from a tank at a rate of $r(t)$ liters per hour. The rate decreased as time passed and values of the rate at two-hour time intervals are shown in the table. Find lower and upper estimates for the total amount of oil that leaked out.

t (h)	0	2	4	6	8	10
$r(t)$ (L/h)	8.7	7.6	6.8	6.2	5.7	5.3

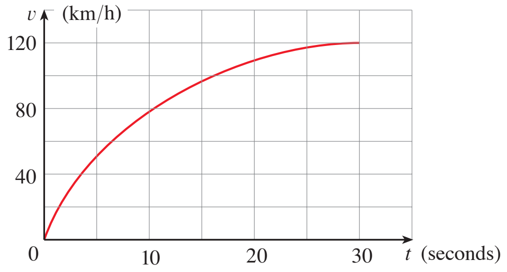
12. When we estimate distances from velocity data, it is sometimes necessary to use times $t_0, t_1, t_2, t_3, \ldots$ that are not equally spaced. We can still estimate distances using the time periods $\Delta t_i = t_i - t_{i-1}$. For example, in 1992 the space shuttle *Endeavour* was launched on mission STS-49 in order to install a new perigee kick motor in an Intelsat communications satellite. The table, provided by NASA, gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters. Use these data to estimate the height above the earth's surface of the *Endeavour*, 62 seconds after liftoff.

Event	Time (s)	Velocity (m/s)
Launch	0	0
Begin roll maneuver	10	56
End roll maneuver	15	97
Throttle to 89%	20	136
Throttle to 67%	32	226
Throttle to 104%	59	404
Maximum dynamic pressure	62	440
Solid rocket booster separation	125	1265

13. The velocity graph of a braking car is shown. Use it to estimate the distance traveled by the car while the brakes are applied.



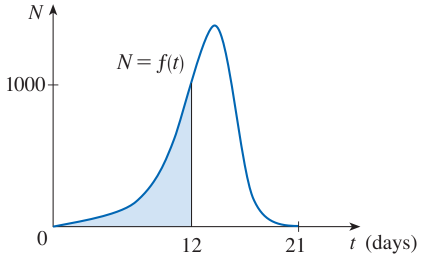
14. The velocity graph of a car accelerating from rest to a speed of 120 km/h over a period of 30 seconds is shown. Estimate the distance traveled during this period.



15. In a person infected with measles, the virus level N (measured in number of infected cells per mL of blood plasma) reaches a peak density at about $t = 12$ days (when a rash appears) and then decreases fairly rapidly as a result of immune response. The area under the graph of $N(t)$ from $t = 0$ to $t = 12$ (as shown in the figure) is equal to the total amount of infection needed to develop symptoms (measured in density of infected cells × time). The function N has been modeled by the function

$$f(t) = -t(t - 21)(t + 1)$$

Use this model with six subintervals and their midpoints to estimate the total amount of infection needed to develop symptoms of measles.



Source: J. M. Heffernan et al., "An In-Host Model of Acute Infection: Measles as a Case Study," *Theoretical Population Biology* 73 (2006): 134–47.

16–19 Use Definition 2 to find an expression for the area under the graph of f as a limit. Do not evaluate the limit.

16. $f(x) = x^2e^x, \quad 0 \leqslant x \leqslant 4$

17. $f(x) = 2 + \sin^2 x, \quad 0 \leqslant x \leqslant \pi$

18. $f(x) = x + \ln x, \quad 3 \leqslant x \leqslant 8$

19. $f(x) = x\sqrt{x^3 + 8}, \quad 1 \leqslant x \leqslant 5$

20–23 Determine a region whose area is equal to the given limit. Do not evaluate the limit.

20. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^3$

21. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\,\frac{1}{1 + (2i/n)}$

22. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1 + \frac{3i}{n}}$

23. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n}$

---

24. (a) Use Definition 2 to express the area under the curve $y = x^3$ from 0 to 1 as a limit.
(b) The following formula for the sum of the cubes of the first n integers is proved in Appendix E. Use it to evaluate the limit in part (a).

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

25. Let A be the area under the graph of an increasing continuous function f from a to b, and let L_n and R_n be the approximations to A with n subintervals using left and right endpoints, respectively.
(a) How are A, L_n, and R_n related?
(b) Show that

$$R_n - L_n = \frac{b-a}{n}[f(b) - f(a)]$$

Then draw a diagram to illustrate this equation by showing that the n rectangles representing $R_n - L_n$ can be reassembled to form a single rectangle whose area is the right-hand side of the equation.
(c) Deduce that

$$R_n - A < \frac{b-a}{n}[f(b) - f(a)]$$

26. If A is the area under the curve $y = e^x$ from 1 to 3, use Exercise 25 to find a value of n such that $R_n - A < 0.0001$.

T **27–28** With a programmable calculator (or a computer), it is possible to evaluate the expressions for the sums of areas of approximating rectangles, even for large values of n, using looping. (On a TI use the Is> command or a For-EndFor loop, on a Casio use Isz, on an HP or in BASIC use a FOR-NEXT loop.) Compute the sum of the areas of approximating rectangles using equal subintervals and right endpoints for $n = 10$, 30, 50, and 100. Then guess the value of the exact area.

27. The region under $y = x^4$ from 0 to 1

28. The region under $y = \cos x$ from 0 to $\pi/2$

T **29–30** Some computer algebra systems have commands that will draw approximating rectangles and evaluate the sums of their areas, at least if x_i^* is a left or right endpoint. (For instance, in Maple use `leftbox`, `rightbox`, `leftsum`, and `rightsum`.)

29. Let $f(x) = 1/(x^2 + 1)$, $0 \leqslant x \leqslant 1$.
(a) Find the left and right sums for $n = 10$, 30, and 50.
(b) Illustrate by graphing the rectangles in part (a).
(c) Show that the exact area under f lies between 0.780 and 0.791.

30. Let $f(x) = \ln x$, $1 \leqslant x \leqslant 4$.
(a) Find the left and right sums for $n = 10$, 30, and 50.
(b) Illustrate by graphing the rectangles in part (a).
(c) Show that the exact area under f lies between 2.50 and 2.59.

T **31.** (a) Express the area under the curve $y = x^5$ from 0 to 2 as a limit.
(b) Use a computer algebra system to find the sum in your expression from part (a).
(c) Evaluate the limit in part (a).

T **32.** Find the exact area of the region under the graph of $y = e^{-x}$ from 0 to 2 by using a computer algebra system to evaluate the sum and then the limit in Example 3(a). Compare your answer with the estimate obtained in Example 3(b).

T **33.** Find the exact area under the cosine curve $y = \cos x$ from $x = 0$ to $x = b$, where $0 \leqslant b \leqslant \pi/2$. (Use a computer algebra system both to evaluate the sum and compute the limit.) In particular, what is the area if $b = \pi/2$?

34. (a) Let A_n be the area of a polygon with n equal sides inscribed in a circle with radius r. By dividing the polygon into n congruent triangles with central angle $2\pi/n$, show that

$$A_n = \tfrac{1}{2}nr^2 \sin\frac{2\pi}{n}$$

(b) Show that $\lim_{n\to\infty} A_n = \pi r^2$. [*Hint:* Use Equation 3.3.5.]

5.2 The Definite Integral

We saw in Section 5.1 that a limit of the form

$$\boxed{1} \qquad \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} \left[f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x \right]$$

arises when we compute an area. We also saw that it arises when we try to find the distance traveled by an object. It turns out that this same type of limit occurs in a wide variety of situations even when f is not necessarily a positive function. In Chapters 6 and 8 we will see that limits of this type also arise in finding lengths of curves, volumes of solids, centers of mass, force due to water pressure, and work, as well as other quantities.

The Definite Integral

We give limits of the form (1) a special name and notation.

2 **Definition of a Definite Integral** If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\ (= a), x_1, x_2, \ldots, x_n\ (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

The precise meaning of the limit that defines the integral is as follows:

For every number $\varepsilon > 0$ there is an integer N such that

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_i^*)\,\Delta x \right| < \varepsilon$$

for every integer $n > N$ and for every choice of x_i^* in $[x_{i-1}, x_i]$.

NOTE 1 The symbol $\int$ was introduced by Leibniz and is called an **integral sign**. It is an elongated S and was chosen because an integral is a limit of sums. In the notation $\int_a^b f(x)\,dx$, $f(x)$ is called the **integrand** and a and b are called the **limits of integration**; a is the **lower limit** and b is the **upper limit**. For now, the symbol dx has no meaning by itself; $\int_a^b f(x)\,dx$ is all one symbol. The dx simply indicates that the independent variable is x. The procedure of calculating an integral is called **integration**.

NOTE 2 The definite integral $\int_a^b f(x)\,dx$ is a number; it does not depend on x. In fact, we could use any letter in place of x without changing the value of the integral:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$$

NOTE 3 The sum

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

that occurs in Definition 2 is called a **Riemann sum** after the German mathematician Bernhard Riemann (1826–1866). So Definition 2 says that the definite integral of an integrable function can be approximated to within any desired degree of accuracy by a Riemann sum.

Riemann

Bernhard Riemann received his Ph.D. under the direction of the legendary Gauss at the University of Göttingen and remained there to teach. Gauss, who was not in the habit of praising other mathematicians, spoke of Riemann's "creative, active, truly mathematical mind and gloriously fertile originality." The definition (2) of an integral that we use is due to Riemann. He also made major contributions to the theory of functions of a complex variable, mathematical physics, number theory, and the foundations of geometry. Riemann's broad concept of space and geometry turned out to be the right setting, 50 years later, for Einstein's general relativity theory. Riemann's health was poor throughout his life, and he died of tuberculosis at the age of 39.

We know that if f happens to be positive, then the Riemann sum can be interpreted as a sum of areas of approximating rectangles (see Figure 1). By comparing Definition 2 with the definition of area in Section 5.1, we see that the definite integral $\int_a^b f(x)\,dx$ can be interpreted as the area under the curve $y = f(x)$ from a to b. (See Figure 2.)

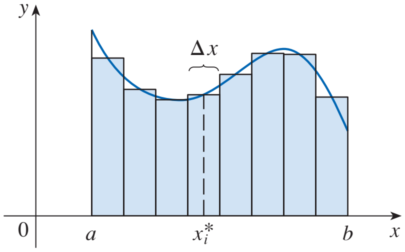


FIGURE 1
If $f(x) \geqslant 0$, the Riemann sum $\sum f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.

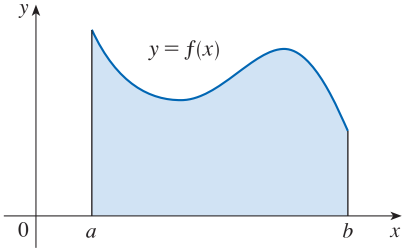


FIGURE 2
If $f(x) \geqslant 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

If f takes on both positive and negative values, as in Figure 3, then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the *negatives* of the areas of the rectangles that lie below the x-axis (the areas of the blue rectangles *minus* the areas of the gold rectangles). When we take the limit of such Riemann sums, we get the situation illustrated in Figure 4. A definite integral can be interpreted as a **net area**, that is, a difference of areas:

$$\int_a^b f(x)\,dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f.

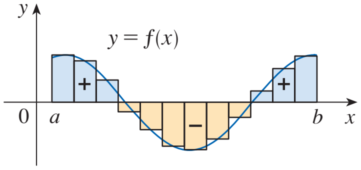


FIGURE 3
$\sum f(x_i^*)\,\Delta x$ is an approximation to the net area.

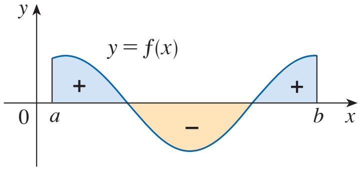


FIGURE 4
$\int_a^b f(x)\,dx$ is the net area.

EXAMPLE 1 Evaluate the Riemann sum for $f(x) = x^3 - 6x$, $0 \leqslant x \leqslant 3$, with $n = 6$ subintervals and taking the sample endpoints to be right endpoints.

SOLUTION

With $n = 6$ subintervals, the interval width is $\Delta x = (3 - 0)/6 = \frac{1}{2}$ and the right endpoints are

$$x_1 = 0.5 \qquad x_2 = 1.0 \qquad x_3 = 1.5 \qquad x_4 = 2.0 \qquad x_5 = 2.5 \qquad x_6 = 3.0$$

So the Riemann sum is

$$\begin{aligned} R_6 &= \sum_{i=1}^{6} f(x_i)\,\Delta x \\ &= f(0.5)\,\Delta x + f(1.0)\,\Delta x + f(1.5)\,\Delta x + f(2.0)\,\Delta x + f(2.5)\,\Delta x + f(3.0)\,\Delta x \\ &= \tfrac{1}{2}(-2.875 - 5 - 5.625 - 4 + 0.625 + 9) \\ &= -3.9375 \end{aligned}$$

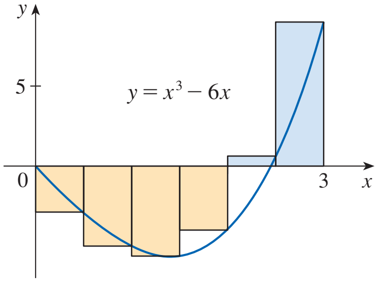


FIGURE 5

Notice that f is not a positive function and so the Riemann sum does not represent a sum of areas of rectangles. But it does represent the sum of the areas of the blue rectangles (above the x-axis) minus the sum of the areas of the gold rectangles (below the x-axis) in Figure 5. ■

NOTE 4 Although we have defined $\int_a^b f(x)\,dx$ by dividing $[a, b]$ into subintervals of equal width, there are situations in which it is advantageous to work with subintervals of unequal width. For instance, in Exercise 5.1.12, NASA provided velocity data at times that were not equally spaced, but we were still able to estimate the distance traveled. And there are methods for numerical integration that take advantage of unequal subintervals. If the subinterval widths are $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, we have to ensure that all these widths approach 0 in the limiting process. This happens if the largest width, $\max \Delta x_i$, approaches 0. So in this case the definition of a definite integral becomes

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

We have defined the definite integral for an integrable function, but not all functions are integrable (see Exercises 81–82). The following theorem shows that the most commonly occurring functions are in fact integrable. The theorem is proved in more advanced courses.

3 Theorem If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

If f is integrable on $[a, b]$, then the limit in Definition 2 exists and gives the same value no matter how we choose the sample points x_i^*. To simplify the calculation of the integral we often take the sample points to be right endpoints. Then $x_i^* = x_i$ and the definition of an integral simplifies as follows.

4 Theorem If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $$\Delta x = \frac{b-a}{n} \qquad \text{and} \qquad x_i = a + i\,\Delta x$$

EXAMPLE 2 Express

$$\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\,\Delta x$$

as an integral on the interval $[0, \pi]$.

SOLUTION Comparing the given limit with the limit in Theorem 4, we see that they will be identical if we choose $f(x) = x^3 + x \sin x$. We are given that $a = 0$ and $b = \pi$. Therefore, by Theorem 4 we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\,\Delta x = \int_0^{\pi} (x^3 + x \sin x)\,dx$$ ■

Later, when we apply the definite integral to physical situations, it will be important to recognize limits of sums as integrals, as we did in Example 2. When Leibniz chose the notation for an integral, he chose the ingredients as reminders of the limiting process. In general, when we write

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \int_a^b f(x)\,dx$$

we replace $\lim \Sigma$ by $\int$, x_i^* by x, and Δx by dx.

Evaluating Definite Integrals

In order to use a limit to evaluate a definite integral, we need to know how to work with sums. The following four equations give formulas for sums of powers of positive integers. Equation 6 may be familiar to you from a course in algebra. Equations 7 and 8 were discussed in Section 5.1 and are proved in Appendix E.

Sums of Powers

5 $$\sum_{i=1}^{n} 1 = n$$

6 $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

7 $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

8 $$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

The remaining formulas are simple rules for working with sigma notation:

Properties of Sums

9 $$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

10 $$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

11 $$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

Formulas 9–11 are proved by writing out each side in expanded form. The left side of Equation 9 is

$$ca_1 + ca_2 + \cdots + ca_n$$

The right side is

$$c(a_1 + a_2 + \cdots + a_n)$$

These are equal by the distributive property. The other formulas are discussed in Appendix E.

In the next example we calculate a definite integral of the function f from Example 1.

EXAMPLE 3 Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

SOLUTION We use Theorem 4. We have $f(x) = x^3 - 6x$, $a = 0$, $b = 3$, and

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

Then the endpoints of the subintervals are $x_0 = 0$, $x_1 = 0 + 1(3/n) = 3/n$, $x_2 = 0 + 2(3/n) = 6/n$, $x_3 = 0 + 3(3/n) = 9/n$, and in general,

$$x_i = 0 + i\left(\frac{3}{n}\right) = \frac{3i}{n}$$

Thus

$$\begin{aligned}
\int_0^3 (x^3 - 6x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n} \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right] && \text{(Equation 9 with } c = 3/n\text{)} \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{27}{n^3} i^3 - \frac{18}{n} i\right] \\
&= \lim_{n\to\infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right] && \text{(Equations 11 and 9)} \\
&= \lim_{n\to\infty} \left\{\frac{81}{n^4} \left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\,\frac{n(n+1)}{2}\right\} && \text{(Equations 8 and 6)} \\
&= \lim_{n\to\infty} \left[\frac{81}{4}\left(1 + \frac{1}{n}\right)^2 - 27\left(1 + \frac{1}{n}\right)\right] \\
&= \frac{81}{4} - 27 = -\frac{27}{4} = -6.75
\end{aligned}$$

In the sum, n is a constant (unlike i), so we can move $3/n$ in front of the $\sum$ sign.

This integral can't be interpreted as an area because f takes on both positive and negative values. But it can be interpreted as the difference of areas $A_1 - A_2$, where A_1 and A_2 are shown in Figure 6. ■

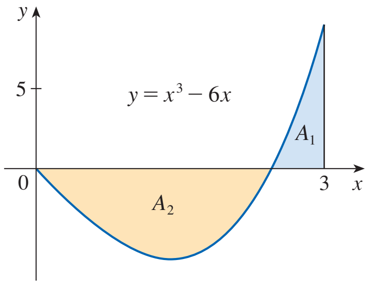


FIGURE 6
$\int_0^3 (x^3 - 6x)\,dx = A_1 - A_2 = -6.75$

Figure 7 illustrates the calculation in Example 3 by showing the positive and negative terms in the right Riemann sum R_n for $n = 40$. The values in the table show the Riemann sums approaching the exact value of the integral, -6.75, as $n \to \infty$.

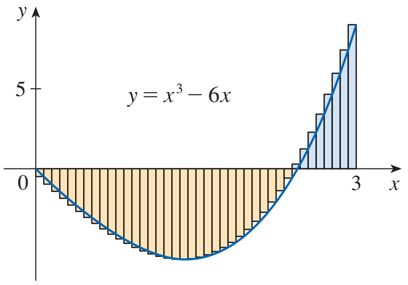


FIGURE 7
$R_{40} \approx -6.3998$

n	R_n
40	-6.3998
100	-6.6130
500	-6.7229
1000	-6.7365
5000	-6.7473

A much simpler method (made possible by the Fundamental Theorem of Calculus) for evaluating integrals like the one in Example 3 will be given in Section 5.3.

Because $f(x) = e^x$ is positive, the integral in Example 4 represents the area shown in Figure 8.

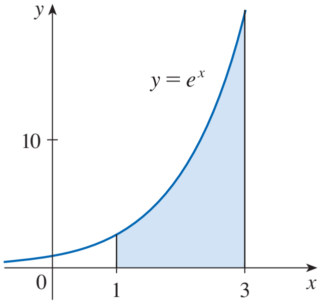


FIGURE 8

EXAMPLE 4

(a) Set up an expression for $\int_1^3 e^x\,dx$ as a limit of sums.
(b) Use a computer algebra system to evaluate the expression.

SOLUTION

(a) Here we have $f(x) = e^x$, $a = 1$, $b = 3$, and

$$\Delta x = \frac{b - a}{n} = \frac{2}{n}$$

So $x_0 = 1$, $x_1 = 1 + 2/n$, $x_2 = 1 + 4/n$, $x_3 = 1 + 6/n$, and

$$x_i = 1 + \frac{2i}{n}$$

From Theorem 4, we get

$$\begin{aligned}\int_1^3 e^x\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\frac{2}{n} \\ &= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} e^{1+2i/n}\end{aligned}$$

A computer algebra system is able to find an explicit expression for this sum because it is a geometric series. The limit could be found using l'Hospital's Rule.

(b) If we ask a computer algebra system to evaluate the sum and simplify, we obtain

$$\sum_{i=1}^{n} e^{1+2i/n} = \frac{e^{(3n+2)/n} - e^{(n+2)/n}}{e^{2/n} - 1}$$

Now we ask the computer algebra system to evaluate the limit:

$$\int_1^3 e^x\,dx = \lim_{n\to\infty} \frac{2}{n} \cdot \frac{e^{(3n+2)/n} - e^{(n+2)/n}}{e^{2/n} - 1} = e^3 - e \qquad \blacksquare$$

EXAMPLE 5 Evaluate the following integrals by interpreting each in terms of areas.

(a) $\int_0^1 \sqrt{1 - x^2}\, dx$ (b) $\int_0^3 (x - 1)\, dx$

SOLUTION

(a) Since $f(x) = \sqrt{1 - x^2} \geqslant 0$, we can interpret this integral as the area under the curve $y = \sqrt{1 - x^2}$ from 0 to 1. But, because $y^2 = 1 - x^2$, we get $x^2 + y^2 = 1$, which shows that the graph of f is the quarter-circle with radius 1 in Figure 9. Therefore

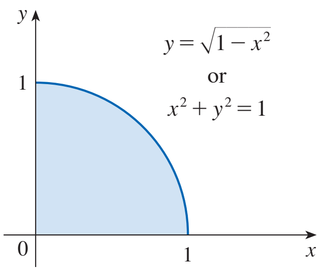


FIGURE 9

$$\int_0^1 \sqrt{1 - x^2}\, dx = \tfrac{1}{4}\pi(1)^2 = \frac{\pi}{4}$$

(In Section 7.3 we will be able to *prove* that the area of a circle of radius r is πr^2.)

(b) The graph of $y = x - 1$ is the line with slope 1 shown in Figure 10. We compute the integral as the difference of the areas of the two triangles:

$$\int_0^3 (x - 1)\, dx = A_1 - A_2 = \tfrac{1}{2}(2 \cdot 2) - \tfrac{1}{2}(1 \cdot 1) = 1.5 \qquad ■$$

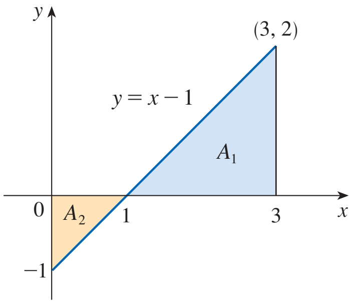


FIGURE 10

The Midpoint Rule

We often choose the sample point x_i^* to be the right endpoint of the ith subinterval because it is convenient for computing the limit. But if the purpose is to find an *approximation* to an integral, it is usually better to choose x_i^* to be the midpoint of the interval, which we denote by $\bar{x}_i$. Any Riemann sum is an approximation to an integral, but if we use midpoints we get the following approximation.

Midpoint Rule

$$\int_a^b f(x)\, dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\, \Delta x = \Delta x \left[f(\bar{x}_1) + \cdots + f(\bar{x}_n) \right]$$

where $\Delta x = \dfrac{b - a}{n}$

and $\bar{x}_i = \tfrac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$

EXAMPLE 6 Use the Midpoint Rule with $n = 5$ to approximate $\int_1^2 \frac{1}{x}\, dx$.

SOLUTION The endpoints of the five subintervals are 1, 1.2, 1.4, 1.6, 1.8, and 2.0, so the midpoints are 1.1, 1.3, 1.5, 1.7, and 1.9. (See Figure 11.) The width of the subintervals is $\Delta x = (2 - 1)/5 = \frac{1}{5}$, so the Midpoint Rule gives

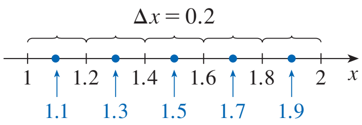


FIGURE 11
The endpoints and midpoints of the subintervals used in Example 6

$$\int_1^2 \frac{1}{x}\, dx \approx \Delta x \left[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9) \right]$$

$$= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9} \right)$$

$$\approx 0.691908$$

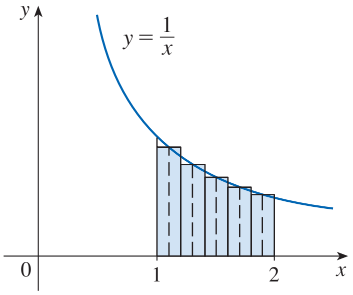


FIGURE 12

Since $f(x) = 1/x > 0$ for $1 \leqslant x \leqslant 2$, the integral represents an area, and the approximation given by the Midpoint Rule is the sum of the areas of the rectangles shown in Figure 12. ■

At the moment we don't know how accurate the approximation in Example 6 is, but in Section 7.7 we will learn a method for estimating the error involved in using the Midpoint Rule. At that time we will discuss other methods for approximating definite integrals.

If we apply the Midpoint Rule to the integral in Example 3, we get the picture in Figure 13. The approximation $M_{40} \approx -6.7563$ is much closer to the true value -6.75 than the right endpoint approximation, $R_{40} \approx -6.3998$, shown in Figure 7.

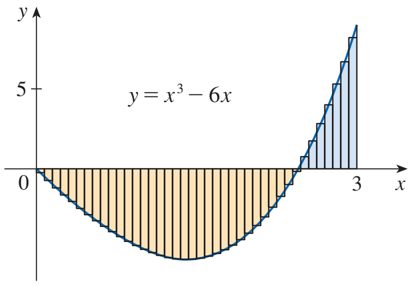


FIGURE 13
$M_{40} \approx -6.7563$

Properties of the Definite Integral

When we defined the definite integral $\int_a^b f(x)\,dx$, we implicitly assumed that $a < b$. But the definition as a limit of Riemann sums makes sense even if $a > b$. Notice that if we interchange a and b, then Δx changes from $(b - a)/n$ to $(a - b)/n$. Therefore

$$\boxed{\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx}$$

If $a = b$, then $\Delta x = 0$ and so

$$\boxed{\int_a^a f(x)\,dx = 0}$$

We now develop some basic properties of integrals that will help us to evaluate integrals in a simple manner. We assume that f and g are continuous functions.

Properties of the Integral

1. $\int_a^b c\,dx = c(b - a)$, where c is any constant

2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Property 1 says that the integral of a constant function $f(x) = c$ is the constant times the length of the interval. If $c > 0$ and $a < b$, this is to be expected because $c(b - a)$ is the area of the shaded rectangle in Figure 14.

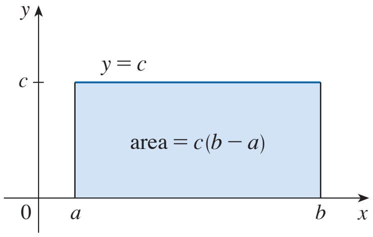


FIGURE 14
$\int_a^b c\,dx = c(b - a)$

Property 2 says that the integral of a sum is the sum of the integrals. For positive functions it says that the area under $f + g$ is the area under f plus the area under g. Figure 15 helps us understand why this is true: in view of how graphical addition works, the corresponding vertical line segments have equal height.

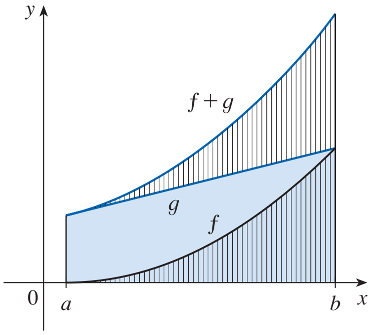


FIGURE 15
$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

In general, Property 2 follows from Theorem 4 and the fact that the limit of a sum is the sum of the limits:

$$\begin{aligned}\int_a^b [f(x) + g(x)]\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) + g(x_i)]\,\Delta x \\ &= \lim_{n\to\infty} \left[\sum_{i=1}^{n} f(x_i)\,\Delta x + \sum_{i=1}^{n} g(x_i)\,\Delta x \right] \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x + \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\,\Delta x \\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx\end{aligned}$$

Property 3 seems intuitively reasonable because we know that multiplying a function by a positive number c stretches or shrinks its graph vertically by a factor of c. So it stretches or shrinks each approximating rectangle by a factor of c and therefore it has the effect of multiplying the area by c.

Property 3 can be proved in a similar manner and says that the integral of a constant times a function is the constant times the integral of the function. In other words, a constant (but *only* a constant) can be taken in front of an integral sign. Property 4 is proved by writing $f - g = f + (-g)$ and using Properties 2 and 3 with $c = -1$.

EXAMPLE 7 Use the properties of integrals to evaluate $\int_0^1 (4 + 3x^2)\,dx$.

SOLUTION Using Properties 2 and 3 of integrals, we have

$$\int_0^1 (4 + 3x^2)\,dx = \int_0^1 4\,dx + \int_0^1 3x^2\,dx = \int_0^1 4\,dx + 3\int_0^1 x^2\,dx$$

We know from Property 1 that

$$\int_0^1 4\,dx = 4(1 - 0) = 4$$

and we found in Example 5.1.2 that $\int_0^1 x^2\,dx = \frac{1}{3}$. So

$$\begin{aligned}\int_0^1 (4 + 3x^2)\,dx &= \int_0^1 4\,dx + 3\int_0^1 x^2\,dx \\ &= 4 + 3 \cdot \tfrac{1}{3} = 5\end{aligned}$$ ■

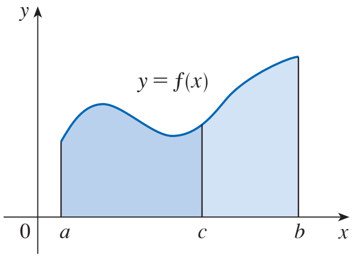


FIGURE 16

The next property tells us how to combine integrals of the same function over adjacent intervals.

5. $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

This is not easy to prove in general, but for the case where $f(x) \geqslant 0$ and $a < c < b$ Property 5 can be seen from the geometric interpretation in Figure 16: the area under $y = f(x)$ from a to c plus the area from c to b is equal to the total area from a to b.

EXAMPLE 8 If it is known that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^8 f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$.

SOLUTION By Property 5, we have

$$\int_0^8 f(x)\,dx + \int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx$$

so $$\int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx - \int_0^8 f(x)\,dx = 17 - 12 = 5$$ ■

Properties 1–5 are true whether $a < b$, $a = b$, or $a > b$. The following properties, in which we compare sizes of functions and sizes of integrals, are true only if $a \leqslant b$.

Comparison Properties of the Integral

6. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant 0$.

7. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.

8. If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b - a) \leqslant \int_a^b f(x)\,dx \leqslant M(b - a)$$

If $f(x) \geqslant 0$, then $\int_a^b f(x)\,dx$ represents the area under the graph of f, so the geometric interpretation of Property 6 is simply that areas are positive. (It also follows directly from the definition because all the quantities involved are positive.) Property 7 says that a bigger function has a bigger integral. It follows from Properties 6 and 4 because $f - g \geqslant 0$.

Property 8 is illustrated by Figure 17 for the case where $f(x) \geqslant 0$. If f is continuous, we could take m and M to be the absolute minimum and maximum values of f on the interval $[a, b]$. In this case Property 8 says that the area under the graph of f is greater than the area of the rectangle with height m and less than the area of the rectangle with height M.

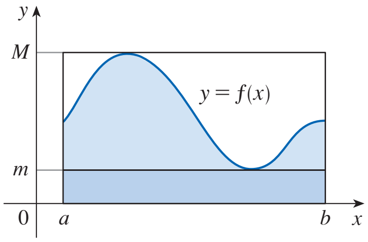


FIGURE 17

PROOF OF PROPERTY 8 Since $m \leqslant f(x) \leqslant M$, Property 7 gives

$$\int_a^b m\,dx \leqslant \int_a^b f(x)\,dx \leqslant \int_a^b M\,dx$$

Using Property 1 to evaluate the integrals on the left and right sides, we obtain

$$m(b - a) \leqslant \int_a^b f(x)\,dx \leqslant M(b - a)$$ ■

Property 8 is useful when all we want is a rough estimate of the size of an integral without going to the bother of using the Midpoint Rule.

EXAMPLE 9 Use Property 8 to estimate $\int_0^1 e^{-x^2}\,dx$.

SOLUTION Because $f(x) = e^{-x^2}$ is a decreasing function on $[0, 1]$, its absolute maximum value is $M = f(0) = 1$ and its absolute minimum value is $m = f(1) = e^{-1}$. Thus, by Property 8,

$$e^{-1}(1 - 0) \leqslant \int_0^1 e^{-x^2}\,dx \leqslant 1(1 - 0)$$

or

$$e^{-1} \leqslant \int_0^1 e^{-x^2}\,dx \leqslant 1$$

Since $e^{-1} \approx 0.3679$, we can write

$$0.367 \leqslant \int_0^1 e^{-x^2}\,dx \leqslant 1$$

■

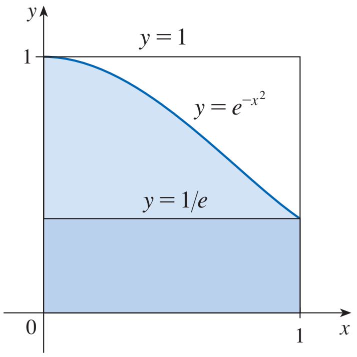


FIGURE 18

The result of Example 9 is illustrated in Figure 18. The integral is greater than the area of the lower rectangle and less than the area of the square.

5.2 Exercises

1. Evaluate the Riemann sum for $f(x) = x - 1$, $-6 \leqslant x \leqslant 4$, with five subintervals, taking the sample points to be right endpoints. Explain, with the aid of a diagram, what the Riemann sum represents.

2. If

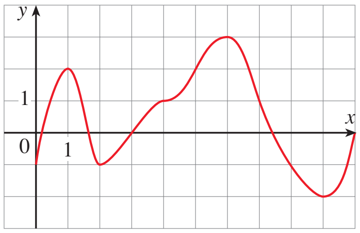
$$f(x) = \cos x \qquad 0 \leqslant x \leqslant 3\pi/4$$

evaluate the Riemann sum with $n = 6$, taking the sample points to be left endpoints. (Give your answer correct to six decimal places.) What does the Riemann sum represent? Illustrate with a diagram.

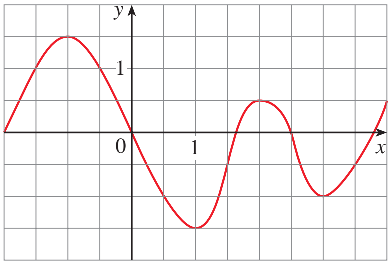
3. If $f(x) = x^2 - 4$, $0 \leqslant x \leqslant 3$, evaluate the Riemann sum with $n = 6$, taking the sample points to be midpoints. What does the Riemann sum represent? Illustrate with a diagram.

4. (a) Evaluate the Riemann sum for $f(x) = 1/x$, $1 \leqslant x \leqslant 2$, with four terms, taking the sample points to be right endpoints. (Give your answer correct to six decimal places.) Explain what the Riemann sum represents with the aid of a sketch.
(b) Repeat part (a) with midpoints as the sample points.

5. The graph of a function f is given. Estimate $\int_0^{10} f(x)\,dx$ using five subintervals with (a) right endpoints, (b) left endpoints, and (c) midpoints.



6. The graph of a function g is shown. Estimate $\int_{-2}^{4} g(x)\,dx$ with six subintervals using (a) right endpoints, (b) left endpoints, and (c) midpoints.



7. A table of values of an increasing function f is shown. Use the table to find lower and upper estimates for $\int_{10}^{30} f(x)\,dx$.

x	10	14	18	22	26	30
$f(x)$	-12	-6	-2	1	3	8

8. The table gives the values of a function obtained from an experiment. Use them to estimate $\int_3^9 f(x)\,dx$ using three equal subintervals with (a) right endpoints, (b) left endpoints, and (c) midpoints. If the function is known to be an increasing function, can you say whether your estimates are less than or greater than the exact value of the integral?

x	3	4	5	6	7	8	9
$f(x)$	-3.4	-2.1	-0.6	0.3	0.9	1.4	1.8

9–10 Use the Midpoint Rule with $n = 4$ to approximate the integral.

9. $\int_0^8 x^2\,dx$

10. $\int_0^2 (8x + 3)\,dx$

11–14 Use the Midpoint Rule with the given value of n to approximate the integral. Round the answer to four decimal places.

11. $\int_0^3 e^{\sqrt{x}}\,dx, \quad n = 6$

12. $\int_0^1 \sqrt{x^3 + 1}\,dx, \quad n = 5$

13. $\int_1^3 \frac{x}{x^2 + 8}\,dx, \quad n = 5$

14. $\int_0^\pi x \sin^2 x\,dx, \quad n = 4$

T **15.** Use a computer algebra system that evaluates midpoint approximations and graphs the corresponding rectangles (use `RiemannSum` or `middlesum` and `middlebox` commands in Maple) to check the answer to Exercise 13 and illustrate with a graph. Then repeat with $n = 10$ and $n = 20$.

T **16.** Use a computer algebra system to compute the left and right Riemann sums for the function $f(x) = x/(x + 1)$ on the interval $[0, 2]$ with $n = 100$. Explain why these estimates show that

$$0.8946 < \int_0^2 \frac{x}{x + 1}\,dx < 0.9081$$

T **17.** Use a calculator or computer to make a table of values of right Riemann sums R_n for the integral $\int_0^\pi \sin x\,dx$ with $n = 5$, 10, 50, and 100. What value do these numbers appear to be approaching?

T **18.** Use a calculator or computer to make a table of values of left and right Riemann sums L_n and R_n for the integral $\int_0^2 e^{-x^2}\,dx$ with $n = 5$, 10, 50, and 100. Between what two numbers must the value of the integral lie? Can you make a similar statement for the integral $\int_{-1}^2 e^{-x^2}\,dx$? Explain.

19–22 Express the limit as a definite integral on the given interval.

19. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{e^{x_i}}{1 + x_i}\,\Delta x, \quad [0, 1]$

20. $\lim_{n\to\infty} \sum_{i=1}^{n} x_i\sqrt{1 + x_i^3}\,\Delta x, \quad [2, 5]$

21. $\lim_{n\to\infty} \sum_{i=1}^{n} [5(x_i^*)^3 - 4x_i^*]\,\Delta x, \quad [2, 7]$

22. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i^*}{(x_i^*)^2 + 4}\,\Delta x, \quad [1, 3]$

23–24 Show that the definite integral is equal to $\lim_{n\to\infty} R_n$ and then evaluate the limit.

23. $\int_0^4 (x - x^2)\,dx, \quad R_n = \frac{4}{n}\sum_{i=1}^{n}\left[\frac{4i}{n} - \frac{16i^2}{n^2}\right]$

24. $\int_1^3 (x^3 + 5x^2)\,dx, \quad R_n = \frac{2}{n}\sum_{i=1}^{n}\left[6 + \frac{26i}{n} + \frac{32i^2}{n^2} + \frac{8i^3}{n^3}\right]$

25–26 Express the integral as a limit of Riemann sums using right endpoints. Do not evaluate the limit.

25. $\int_1^3 \sqrt{4 + x^2}\,dx$

26. $\int_2^5 \left(x^2 + \frac{1}{x}\right) dx$

27–34 Use the form of the definition of the integral given in Theorem 4 to evaluate the integral.

27. $\int_0^2 3x\,dx$

28. $\int_0^3 x^2\,dx$

29. $\int_0^3 (5x + 2)\,dx$

30. $\int_0^4 (6 - x^2)\,dx$

31. $\int_1^5 (3x^2 + 7x)\,dx$

32. $\int_{-1}^2 (4x^2 + x + 2)\,dx$

33. $\int_0^1 (x^3 - 3x^2)\,dx$

34. $\int_0^2 (2x - x^3)\,dx$

---

35. The graph of f is shown. Evaluate each integral by interpreting it in terms of areas.

(a) $\int_0^2 f(x)\,dx$

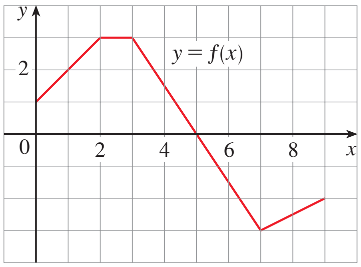
(b) $\int_0^5 f(x)\,dx$

(c) $\int_5^7 f(x)\,dx$

(d) $\int_3^7 f(x)\,dx$

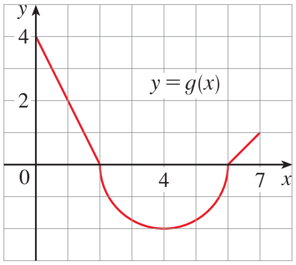
(e) $\int_3^7 |f(x)|\,dx$

(f) $\int_2^0 f(x)\,dx$



36. The graph of g consists of two straight lines and a semicircle. Evaluate each integral by interpreting it in terms of areas.

(a) $\int_0^2 g(x)\,dx$ (b) $\int_2^6 g(x)\,dx$ (c) $\int_0^7 g(x)\,dx$



37–38

(a) Use the form of the definition of the integral given in Theorem 4 to evaluate the given integral.

(b) Confirm your answer to part (a) graphically by interpreting the integral in terms of areas.

37. $\int_0^3 4x\,dx$ **38.** $\int_{-1}^4 \left(2 - \frac{1}{2}x\right) dx$

39–40

(a) Find an approximation to the integral using a Riemann sum with right endpoints and $n = 8$.

(b) Draw a diagram like Figure 3 to illustrate the approximation in part (a).

(c) Use Theorem 4 to evaluate the integral.

(d) Interpret the integral in part (c) as a difference of areas and illustrate with a diagram like Figure 4.

39. $\int_0^8 (3 - 2x)\,dx$ **40.** $\int_0^4 (x^2 - 3x)\,dx$

41–46 Evaluate the integral by interpreting it in terms of areas.

41. $\int_{-2}^5 (10 - 5x)\,dx$ **42.** $\int_{-1}^3 (2x - 1)\,dx$

43. $\int_{-4}^3 \left|\frac{1}{2}x\right| dx$ **44.** $\int_0^1 |2x - 1|\,dx$

45. $\int_{-3}^0 \left(1 + \sqrt{9 - x^2}\right) dx$ **46.** $\int_{-4}^4 \left(2x - \sqrt{16 - x^2}\right) dx$

---

47. Prove that $\displaystyle\int_a^b x\,dx = \frac{b^2 - a^2}{2}$.

48. Prove that $\displaystyle\int_a^b x^2\,dx = \frac{b^3 - a^3}{3}$.

T **49–50** Express the integral as a limit of sums. Then evaluate, using a computer algebra system to find both the sum and the limit.

49. $\int_0^\pi \sin 5x\,dx$ **50.** $\int_2^{10} x^6\,dx$

---

51. Evaluate $\int_1^1 \sqrt{1 + x^4}\,dx$.

52. Given that $\int_0^\pi \sin^4 x\,dx = \frac{3}{8}\pi$, what is $\int_\pi^0 \sin^4 \theta\,d\theta$?

53. In Example 5.1.2 we showed that $\int_0^1 x^2\,dx = \frac{1}{3}$. Use this fact and the properties of integrals to evaluate $\int_0^1 (5 - 6x^2)\,dx$.

54. Use the properties of integrals and the result of Example 4 to evaluate $\int_1^3 (2e^x - 1)\,dx$.

55. Use the result of Example 4 to evaluate $\int_1^3 e^{x+2}\,dx$.

56. Use the result of Exercise 47 and the fact that $\int_0^{\pi/2} \cos x\,dx = 1$ (from Exercise 5.1.33), together with the properties of integrals, to evaluate $\int_0^{\pi/2} (2\cos x - 5x)\,dx$.

57. Write as a single integral in the form $\int_a^b f(x)\,dx$:

$$\int_{-2}^2 f(x)\,dx + \int_2^5 f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$$

58. If $\int_2^8 f(x)\,dx = 7.3$ and $\int_2^4 f(x)\,dx = 5.9$, find $\int_4^8 f(x)\,dx$.

59. If $\int_0^9 f(x)\,dx = 37$ and $\int_0^9 g(x)\,dx = 16$, find

$$\int_0^9 [2f(x) + 3g(x)]\,dx$$

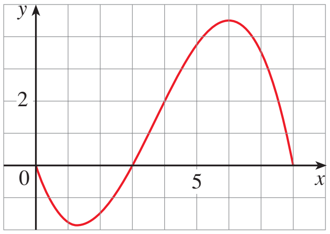
60. Find $\int_0^5 f(x)\,dx$ if

$$f(x) = \begin{cases} 3 & \text{for } x < 3 \\ x & \text{for } x \geqslant 3 \end{cases}$$

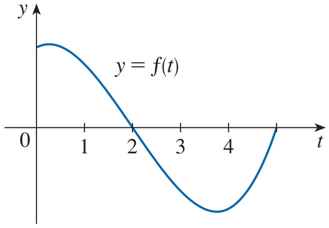
61. For the function f whose graph is shown, list the following quantities in increasing order, from smallest to largest, and explain your reasoning.

(A) $\int_0^8 f(x)\,dx$ (B) $\int_0^3 f(x)\,dx$ (C) $\int_3^8 f(x)\,dx$

(D) $\int_4^8 f(x)\,dx$ (E) $f'(1)$

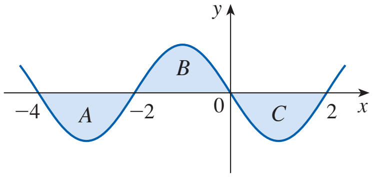


62. If $F(x) = \int_2^x f(t)\,dt$, where f is the function whose graph is given, which of the following values is largest?
(A) $F(0)$ (B) $F(1)$ (C) $F(2)$
(D) $F(3)$ (E) $F(4)$



63. Each of the regions A, B, and C bounded by the graph of f and the x-axis has area 3. Find the value of

$$\int_{-4}^{2} [f(x) + 2x + 5]\,dx$$



64. Suppose f has absolute minimum value m and absolute maximum value M. Between what two values must $\int_0^2 f(x)\,dx$ lie? Which property of integrals allows you to make your conclusion?

65–68 Use the properties of integrals to verify the inequality without evaluating the integrals.

65. $\displaystyle\int_0^4 (x^2 - 4x + 4)\,dx \geqslant 0$

66. $\displaystyle\int_0^1 \sqrt{1 + x^2}\,dx \leqslant \int_0^1 \sqrt{1 + x}\,dx$

67. $\displaystyle 2 \leqslant \int_{-1}^1 \sqrt{1 + x^2}\,dx \leqslant 2\sqrt{2}$

68. $\displaystyle \frac{\pi}{12} \leqslant \int_{\pi/6}^{\pi/3} \sin x\,dx \leqslant \frac{\sqrt{3}\,\pi}{12}$

69–74 Use Property 8 of integrals to estimate the value of the integral.

69. $\displaystyle\int_0^1 x^3\,dx$

70. $\displaystyle\int_0^3 \frac{1}{x + 4}\,dx$

71. $\displaystyle\int_{\pi/4}^{\pi/3} \tan x\,dx$

72. $\displaystyle\int_0^2 (x^3 - 3x + 3)\,dx$

73. $\displaystyle\int_0^2 xe^{-x}\,dx$

74. $\displaystyle\int_{\pi}^{2\pi} (x - 2\sin x)\,dx$

75–76 Use properties of integrals, together with Exercises 47 and 48, to prove the inequality.

75. $\displaystyle\int_1^3 \sqrt{x^4 + 1}\,dx \geqslant \frac{26}{3}$

76. $\displaystyle\int_0^{\pi/2} x \sin x\,dx \leqslant \frac{\pi^2}{8}$

---

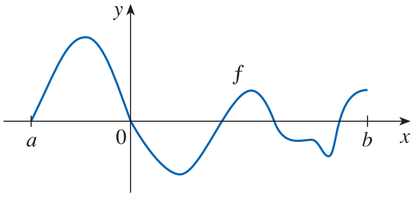
77. Which of the integrals $\int_1^2 \arctan x\,dx$, $\int_1^2 \arctan \sqrt{x}\,dx$, and $\int_1^2 \arctan(\sin x)\,dx$ has the largest value? Why?

78. Which of the integrals $\int_0^{0.5} \cos(x^2)\,dx$, $\int_0^{0.5} \cos \sqrt{x}\,dx$ is larger? Why?

79. Prove Property 3 of integrals.

80. (a) For the function f shown in the graph, verify graphically that the following inequality holds:

$$\left| \int_a^b f(x)\,dx \right| \leqslant \int_a^b |f(x)|\,dx$$



(b) Prove that the inequality from part (a) holds for any function f that is continuous on $[a, b]$.

(c) Show that

$$\left| \int_a^b f(x) \sin 2x\,dx \right| \leqslant \int_a^b |f(x)|\,dx$$

81. Let $f(x) = 0$ if x is any rational number and $f(x) = 1$ if x is any irrational number. Show that f is not integrable on $[0, 1]$.

82. Let $f(0) = 0$ and $f(x) = 1/x$ if $0 < x \leqslant 1$. Show that f is not integrable on $[0, 1]$. [*Hint:* Show that the first term in the Riemann sum, $f(x_1^*)\,\Delta x$, can be made arbitrarily large.]

83–84 Express the limit as a definite integral.

83. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5}$ [*Hint:* Consider $f(x) = x^4$.]

84. $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 + (i/n)^2}$

---

85. Find $\int_1^2 x^{-2}\,dx$. *Hint:* Choose x_i^* to be the geometric mean of x_{i-1} and x_i (that is, $x_i^* = \sqrt{x_{i-1}x_i}$) and use the identity

$$\frac{1}{m(m + 1)} = \frac{1}{m} - \frac{1}{m + 1}$$

DISCOVERY PROJECT | AREA FUNCTIONS

1. (a) Draw the line $y = 2t + 1$ and use geometry to find the area under this line, above the t-axis, and between the vertical lines $t = 1$ and $t = 3$.

(b) If $x > 1$, let $A(x)$ be the area of the region that lies under the line $y = 2t + 1$ between $t = 1$ and $t = x$. Sketch this region and use geometry to find an expression for $A(x)$.

(c) Differentiate the area function $A(x)$. What do you notice?

2. (a) If $x \geqslant -1$, let

$$A(x) = \int_{-1}^{x} (1 + t^2)\,dt$$

$A(x)$ represents the area of a region. Sketch that region.

(b) Use the result of Exercise 5.2.48 to find an expression for $A(x)$.

(c) Find $A'(x)$. What do you notice?

(d) If $x \geqslant -1$ and h is a small positive number, then $A(x + h) - A(x)$ represents the area of a region. Describe and sketch the region.

(e) Draw a rectangle that approximates the region in part (d). By comparing the areas of these two regions, show that

$$\frac{A(x + h) - A(x)}{h} \approx 1 + x^2$$

(f) Use part (e) to give an intuitive explanation for the result of part (c).

3. (a) Draw the graph of the function $f(x) = \cos(x^2)$ in the viewing rectangle $[0, 2]$ by $[-1.25, 1.25]$.

(b) If we define a new function g by

$$g(x) = \int_{0}^{x} \cos(t^2)\,dt$$

then $g(x)$ is the area under the graph of f from 0 to x [until $f(x)$ becomes negative, at which point $g(x)$ becomes a difference of areas]. Use part (a) to determine the value of x at which $g(x)$ starts to decrease. [Unlike the integral in Problem 2, it is impossible to evaluate the integral defining g to obtain an explicit expression for $g(x)$.]

(c) Use the integration command on a calculator or computer to estimate $g(0.2)$, $g(0.4)$, $g(0.6)$, . . . , $g(1.8)$, $g(2)$. Then use these values to sketch a graph of g.

(d) Use your graph of g from part (c) to sketch the graph of g' using the interpretation of $g'(x)$ as the slope of a tangent line. How does the graph of g' compare with the graph of f?

4. Suppose f is a continuous function on the interval $[a, b]$ and we define a new function g by the equation

$$g(x) = \int_{a}^{x} f(t)\,dt$$

Based on your results in Problems 1–3, conjecture an expression for $g'(x)$.

5.3 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus: differential calculus and integral calculus. Differential calculus arose from the tangent problem, whereas integral calculus arose from a seemingly unrelated problem, the area problem. Newton's mentor at Cambridge, Isaac Barrow (1630–1677), discovered that these two problems are actually closely related. In fact, he realized that differentiation and integration are inverse processes. The Fundamental Theorem of Calculus gives the precise inverse relationship between the derivative and the integral. It was Newton and Leibniz who exploited this relationship and used it to develop calculus into a systematic mathematical method. In particular, they saw that the Fundamental Theorem enabled them to compute areas and integrals very easily without having to compute them as limits of sums as we did in Sections 5.1 and 5.2.

The Fundamental Theorem of Calculus, Part 1

The first part of the Fundamental Theorem deals with functions defined by an equation of the form

$$\boxed{1} \qquad g(x) = \int_a^x f(t)\,dt$$

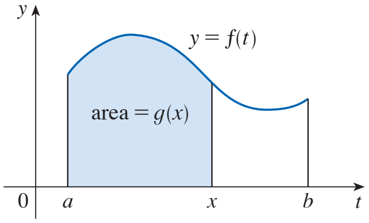


FIGURE 1

where f is a continuous function on $[a, b]$ and x varies between a and b. Observe that g depends only on x, which appears as the variable upper limit in the integral. If x is a fixed number, then the integral $\int_a^x f(t)\,dt$ is a definite number. If we then let x vary, the number $\int_a^x f(t)\,dt$ also varies and defines a function of x denoted by $g(x)$.

If f happens to be a positive function, then $g(x)$ can be interpreted as the area under the graph of f from a to x, where x can vary from a to b. (Think of g as the "area so far" function; see Figure 1.)

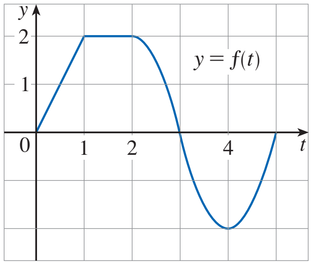


FIGURE 2

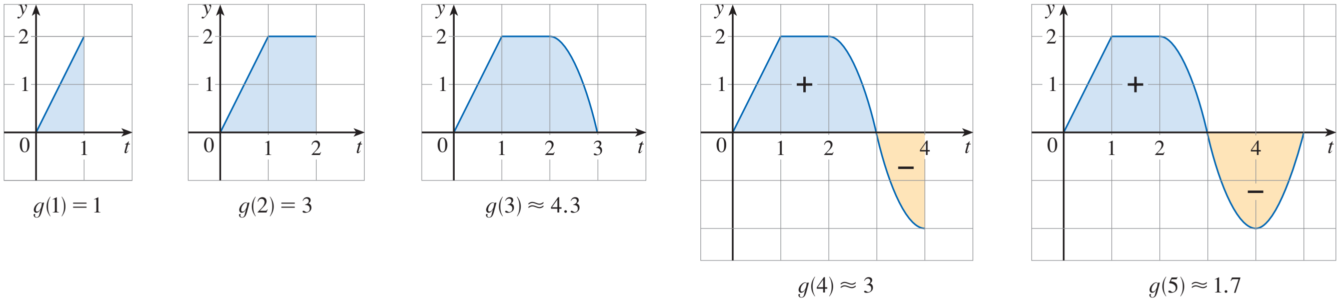
EXAMPLE 1 If f is the function whose graph is shown in Figure 2 and $g(x) = \int_0^x f(t)\,dt$, find the values of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, and $g(5)$. Then sketch a rough graph of g.

SOLUTION First we notice that $g(0) = \int_0^0 f(t)\,dt = 0$. From Figure 3 we see that $g(1)$ is the area of a triangle:

$$g(1) = \int_0^1 f(t)\,dt = \tfrac{1}{2}(1 \cdot 2) = 1$$

To find $g(2)$ we add to $g(1)$ the area of a rectangle:

$$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = 1 + (1 \cdot 2) = 3$$



$g(1) = 1$ $g(2) = 3$ $g(3) \approx 4.3$ $g(4) \approx 3$ $g(5) \approx 1.7$

FIGURE 3

We estimate that the area under f from 2 to 3 is about 1.3, so

$$g(3) = g(2) + \int_2^3 f(t)\,dt \approx 3 + 1.3 = 4.3$$

For $t > 3$, $f(t)$ is negative and so we start subtracting areas:

$$g(4) = g(3) + \int_3^4 f(t)\,dt \approx 4.3 + (-1.3) = 3.0$$

$$g(5) = g(4) + \int_4^5 f(t)\,dt \approx 3 + (-1.3) = 1.7$$

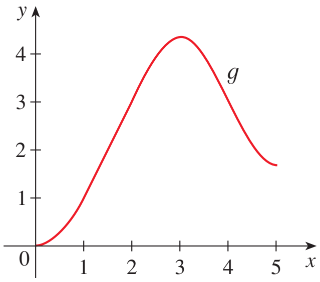


FIGURE 4

$g(x) = \int_0^x f(t)\,dt$

We use these values to sketch the graph of g in Figure 4. Notice that, because $f(t)$ is positive for $t < 3$, we keep adding area for $t < 3$ and so g is increasing up to $x = 3$, where it attains a maximum value. For $x > 3$, g decreases because $f(t)$ is negative. ■

If we take $f(t) = t$ and $a = 0$, then, using Exercise 5.2.47, we have

$$g(x) = \int_0^x t\,dt = \frac{x^2}{2}$$

Notice that $g'(x) = x$, that is, $g' = f$. In other words, if g is defined as the integral of f by Equation 1, then g turns out to be an antiderivative of f, at least in this case. And if we sketch the derivative of the function g shown in Figure 4 by estimating slopes of tangents, we get a graph like that of f in Figure 2. So we suspect that $g' = f$ in Example 1 too.

To see why this might be generally true we consider any continuous function f with $f(x) \geqslant 0$. Then $g(x) = \int_a^x f(t)\,dt$ can be interpreted as the area under the graph of f from a to x, as in Figure 1.

In order to compute $g'(x)$ from the definition of a derivative we first observe that, for $h > 0$, $g(x + h) - g(x)$ is obtained by subtracting areas, so it is the area under the graph of f from x to $x + h$ (the blue area in Figure 5). For small h you can see from the figure that this area is approximately equal to the area of the rectangle with height $f(x)$ and width h:

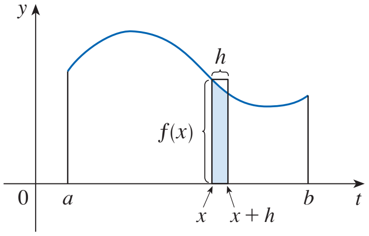


FIGURE 5

$$g(x + h) - g(x) \approx hf(x)$$

so
$$\frac{g(x + h) - g(x)}{h} \approx f(x)$$

Intuitively, we therefore expect that

$$g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = f(x)$$

The fact that this is true, even when f is not necessarily positive, is the first part of the Fundamental Theorem of Calculus.

We abbreviate the name of this theorem as FTC1. In words, it says that the derivative of a definite integral with respect to its upper limit is the integrand evaluated at the upper limit.

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

PROOF If x and $x + h$ are in (a, b), then

$$\begin{aligned} g(x+h) - g(x) &= \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \\ &= \left(\int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt\right) - \int_a^x f(t)\,dt \qquad \text{(by Property 5 of integrals)} \\ &= \int_x^{x+h} f(t)\,dt \end{aligned}$$

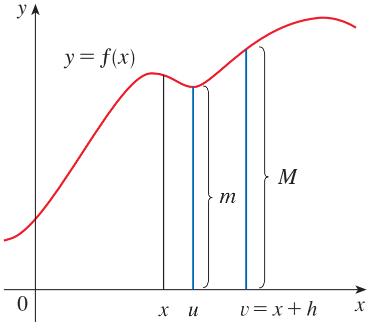


FIGURE 6

and so, for $h \neq 0$,

$$\boxed{2} \qquad \frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

For now let's assume that $h > 0$. Since f is continuous on $[x, x + h]$, the Extreme Value Theorem says that there are numbers u and v in $[x, x + h]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x, x + h]$. (See Figure 6.)

By Property 8 of integrals, we have

$$mh \leqslant \int_x^{x+h} f(t)\,dt \leqslant Mh$$

that is,

$$f(u)h \leqslant \int_x^{x+h} f(t)\,dt \leqslant f(v)h$$

Since $h > 0$, we can divide this inequality by h:

$$f(u) \leqslant \frac{1}{h}\int_x^{x+h} f(t)\,dt \leqslant f(v)$$

Now we use Equation 2 to replace the middle part of this inequality:

$$\boxed{3} \qquad f(u) \leqslant \frac{g(x+h) - g(x)}{h} \leqslant f(v)$$

Inequality 3 can be proved in a similar manner for the case where $h < 0$. (See Exercise 87.)

Now we let $h \to 0$. Then $u \to x$ and $v \to x$ because u and v lie between x and $x + h$. Therefore

$$\lim_{h \to 0} f(u) = \lim_{u \to x} f(u) = f(x) \qquad \text{and} \qquad \lim_{h \to 0} f(v) = \lim_{v \to x} f(v) = f(x)$$

because f is continuous at x. We conclude, from (3) and the Squeeze Theorem, that

$$\boxed{4} \qquad g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x)$$

If $x = a$ or b, then Equation 4 can be interpreted as a one-sided limit. Then Theorem 2.8.4 (modified for one-sided limits) shows that g is continuous on $[a, b]$. ■

Using Leibniz notation for derivatives, we can write FTC1 as

$$\boxed{5} \qquad \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

when f is continuous. Roughly speaking, Equation 5 says that if we first integrate f and then differentiate the result, we get back to the original function f.

EXAMPLE 2 Find the derivative of the function $g(x) = \int_0^x \sqrt{1 + t^2}\, dt$.

SOLUTION Since $f(t) = \sqrt{1 + t^2}$ is continuous, Part 1 of the Fundamental Theorem of Calculus gives

$$g'(x) = \sqrt{1 + x^2}$$ ■

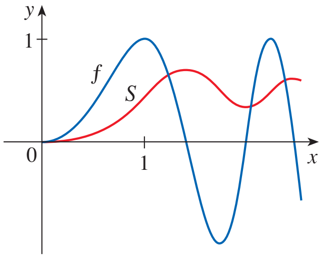


FIGURE 7
$f(x) = \sin(\pi x^2/2)$
$S(x) = \int_0^x \sin(\pi t^2/2)\, dt$

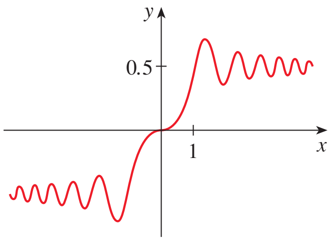


FIGURE 8
The Fresnel function
$S(x) = \int_0^x \sin(\pi t^2/2)\, dt$

EXAMPLE 3 Although a formula of the form $g(x) = \int_a^x f(t)\, dt$ may seem like a strange way of defining a function, books on physics, chemistry, and statistics are full of such functions. For instance, the **Fresnel function**

$$S(x) = \int_0^x \sin(\pi t^2/2)\, dt$$

is named after the French physicist Augustin Fresnel (1788–1827), who is famous for his works in optics. This function first appeared in Fresnel's theory of the diffraction of light waves, but more recently it has been applied to the design of highways.

Part 1 of the Fundamental Theorem tells us how to differentiate the Fresnel function:

$$S'(x) = \sin(\pi x^2/2)$$

This means that we can apply all the methods of differential calculus to analyze S (see Exercise 81).

Figure 7 shows the graphs of $f(x) = \sin(\pi x^2/2)$ and the Fresnel function $S(x) = \int_0^x f(t)\, dt$. A computer was used to graph S by computing the value of this integral for many values of x. It does indeed look as if $S(x)$ is the area under the graph of f from 0 to x [until $x \approx 1.4$ when $S(x)$ becomes a difference of areas]. Figure 8 shows a larger part of the graph of S.

If we now start with the graph of S in Figure 7 and think about what its derivative should look like, it seems reasonable that $S'(x) = f(x)$. [For instance, S is increasing when $f(x) > 0$ and decreasing when $f(x) < 0$.] So this gives a visual confirmation of Part 1 of the Fundamental Theorem of Calculus. ■

EXAMPLE 4 Find $\dfrac{d}{dx} \int_1^{x^4} \sec t\, dt$.

SOLUTION Here we have to be careful to use the Chain Rule in conjunction with FTC1. Let $u = x^4$. Then

$$\begin{aligned} \frac{d}{dx} \int_1^{x^4} \sec t\, dt &= \frac{d}{dx} \int_1^{u} \sec t\, dt \\ &= \frac{d}{du} \left[\int_1^{u} \sec t\, dt \right] \frac{du}{dx} && \text{(by the Chain Rule)} \\ &= \sec u \, \frac{du}{dx} && \text{(by FTC1)} \\ &= \sec(x^4) \cdot 4x^3 \end{aligned}$$ ■

■ The Fundamental Theorem of Calculus, Part 2

In Section 5.2 we computed integrals from the definition as a limit of Riemann sums and we saw that this procedure is sometimes long and difficult. The second part of the Fundamental Theorem of Calculus, which follows easily from the first part, provides us with a much simpler method for the evaluation of integrals.

We abbreviate this theorem as FTC2.

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function F such that $F' = f$.

PROOF Let $g(x) = \int_a^x f(t)\,dt$. We know from Part 1 that $g'(x) = f(x)$; that is, g is an antiderivative of f. If F is any other antiderivative of f on $[a, b]$, then we know from Corollary 4.2.7 that F and g differ by a constant:

$$\boxed{6} \qquad F(x) = g(x) + C$$

for $a < x < b$. But both F and g are continuous on $[a, b]$ and so, by taking limits of both sides of Equation 6 (as $x \to a^+$ and $x \to b^-$), we see that it also holds when $x = a$ and $x = b$. So $F(x) = g(x) + C$ for all x in $[a, b]$.

If we put $x = a$ in the formula for $g(x)$, we get

$$g(a) = \int_a^a f(t)\,dt = 0$$

So, using Equation 6 with $x = b$ and $x = a$, we have

$$\begin{aligned} F(b) - F(a) &= [g(b) + C] - [g(a) + C] \\ &= g(b) - g(a) = g(b) = \int_a^b f(t)\,dt \end{aligned}$$

■

Part 2 of the Fundamental Theorem states that if we know an antiderivative F of f, then we can evaluate $\int_a^b f(x)\,dx$ simply by subtracting the values of F at the endpoints of the interval $[a, b]$. It's very surprising that $\int_a^b f(x)\,dx$, which was defined by a complicated procedure involving all of the values of $f(x)$ for $a \leqslant x \leqslant b$, can be found by knowing the values of $F(x)$ at only two points, a and b.

Although the theorem may be surprising at first glance, it becomes plausible if we interpret it in physical terms. If $v(t)$ is the velocity of an object and $s(t)$ is its position at time t, then $v(t) = s'(t)$, so s is an antiderivative of v. In Section 5.1 we considered an object that always moves in the positive direction and made the observation that the area under the velocity curve is equal to the distance traveled. In symbols:

$$\int_a^b v(t)\,dt = s(b) - s(a)$$

That is exactly what FTC2 says in this context.

EXAMPLE 5 Evaluate the integral $\displaystyle\int_1^3 e^x\,dx$.

SOLUTION The function $f(x) = e^x$ is continuous everywhere and we know that an antiderivative is $F(x) = e^x$, so Part 2 of the Fundamental Theorem gives

Compare the calculation in Example 5 with the much harder one in Example 5.2.4.

$$\int_1^3 e^x\,dx = F(3) - F(1) = e^3 - e$$

Notice that FTC2 says we can use *any* antiderivative F of f. So we may as well use the simplest one, namely $F(x) = e^x$, instead of $e^x + 7$ or $e^x + C$. ■

Notation

We often use the notation

$$F(x)\Big]_a^b = F(b) - F(a)$$

So the equation of FTC2 can be written as

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b \qquad \text{where} \qquad F' = f$$

Other common notations are $F(x)\,\big|_a^b$ and $[F(x)]_a^b$.

EXAMPLE 6 Find the area under the parabola $y = x^2$ from 0 to 1.

SOLUTION An antiderivative of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$. The required area A is found using Part 2 of the Fundamental Theorem:

In applying the Fundamental Theorem we use a particular antiderivative F of f. It is not necessary to use the most general antiderivative.

$$A = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$ ■

If you compare the calculation in Example 6 with the one in Example 5.1.2, you will see that the Fundamental Theorem gives a *much* shorter method.

EXAMPLE 7 Evaluate $\displaystyle\int_3^6 \frac{dx}{x}$.

SOLUTION The given integral is another way of writing

$$\int_3^6 \frac{1}{x}\,dx$$

An antiderivative of $f(x) = 1/x$ is $F(x) = \ln|x|$ and, because $3 \leqslant x \leqslant 6$, we can write $F(x) = \ln x$. So

$$\int_3^6 \frac{1}{x}\,dx = \ln x\Big]_3^6 = \ln 6 - \ln 3 = \ln \frac{6}{3} = \ln 2$$ ■

EXAMPLE 8 Find the area under the cosine curve from 0 to b, where $0 \leqslant b \leqslant \pi/2$.

SOLUTION Since an antiderivative of $f(x) = \cos x$ is $F(x) = \sin x$, we have

$$A = \int_0^b \cos x\,dx = \sin x\Big]_0^b = \sin b - \sin 0 = \sin b$$

In particular, taking $b = \pi/2$, we have proved that the area under the cosine curve from 0 to $\pi/2$ is $\sin(\pi/2) = 1$. (See Figure 9.) ■

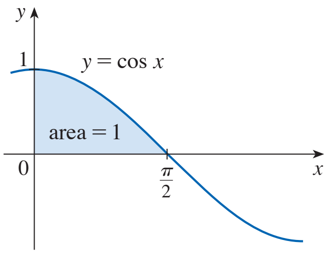


FIGURE 9

When the French mathematician Gilles de Roberval first found the area under the sine and cosine curves in 1635, this was a very challenging problem that required a great deal of ingenuity. If we didn't have the benefit of the Fundamental Theorem, we would have to compute a difficult limit of sums using obscure trigonometric identities (or use a computer algebra system as in Exercise 5.1.33). It was even more difficult for Roberval because the apparatus of limits had not been invented in 1635. But in the 1660s and 1670s, when the Fundamental Theorem was discovered by Barrow and exploited by Newton and Leibniz, such problems became very easy, as you can see from Example 8.

EXAMPLE 9 What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\Bigg]_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}$$

SOLUTION To start, we notice that this calculation must be wrong because the answer is negative but $f(x) = 1/x^2 \geqslant 0$ and Property 6 of integrals says that $\int_a^b f(x)\,dx \geqslant 0$ when $f \geqslant 0$. The Fundamental Theorem of Calculus applies to continuous functions. It can't be applied here because $f(x) = 1/x^2$ is not continuous on $[-1, 3]$. In fact, f has an infinite discontinuity at $x = 0$, and we will see in Section 7.8 that

$$\int_{-1}^{3} \frac{1}{x^2}\,dx \qquad \text{does not exist.}$$

■

Differentiation and Integration as Inverse Processes

We end this section by bringing together the two parts of the Fundamental Theorem.

The Fundamental Theorem of Calculus Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

We noted that Part 1 can be rewritten as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

This says that if we integrate a continuous function f and then differentiate the result, we arrive back at the original function f. We could use Part 2 to write

$$\int_a^x F'(t)\,dt = F(x) - F(a)$$

which says that if we differentiate a function F and then integrate the result, we arrive back at the original function F, except for the constant $F(a)$. So taken together, the two parts of the Fundamental Theorem of Calculus say that integration and differentiation are inverse processes.

The Fundamental Theorem of Calculus is unquestionably the most important theorem in calculus and, indeed, it ranks as one of the great accomplishments of the human mind. Before it was discovered, from the time of Eudoxus and Archimedes to the time of Galileo and Fermat, problems of finding areas, volumes, and lengths of curves were so difficult that only a genius could meet the challenge, and even then, only for very special cases. But now, armed with the systematic method that Newton and Leibniz fashioned out of the Fundamental Theorem, we will see in the chapters to come that these challenging problems are accessible to all of us.

5.3 Exercises

1. Explain exactly what is meant by the statement that "differentiation and integration are inverse processes."

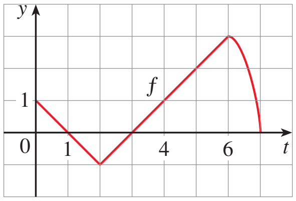
2. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.

(a) Evaluate $g(x)$ for $x = 0, 1, 2, 3, 4, 5$, and 6.

(b) Estimate $g(7)$.

(c) Where does g have a maximum value? Where does it have a minimum value?

(d) Sketch a rough graph of g.



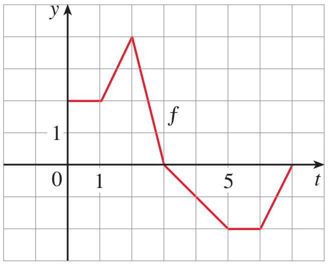
3. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.

(a) Evaluate $g(0)$, $g(1)$, $g(2)$, $g(3)$, and $g(6)$.

(b) On what interval is g increasing?

(c) Where does g have a maximum value?

(d) Sketch a rough graph of g.



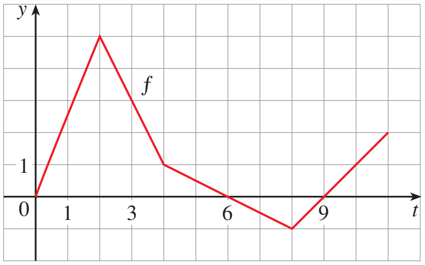
4. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.

(a) Use Part 1 of the Fundamental Theorem of Calculus to graph g'.

(b) Find $g(3)$, $g'(3)$, and $g''(3)$.

(c) Does g have a local maximum, a local minimum, or neither at $x = 6$?

(d) Does g have a local maximum, a local minimum, or neither at $x = 9$?

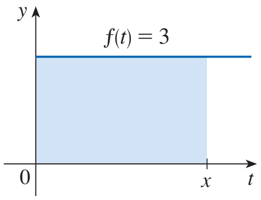


5–6 The graph of a function f is shown. Let g be the function that represents the area under the graph of f between 0 and x.

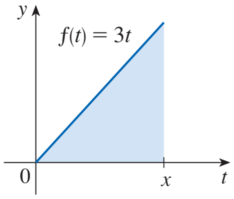
(a) Use geometry to find a formula for $g(x)$.

(b) Verify that g is an antiderivative of f and explain how this confirms Part 1 of the Fundamental Theorem of Calculus for the function f.

5.



6.



7–8 Sketch the area represented by $g(x)$. Then find $g'(x)$ in two ways: (a) by using Part 1 of the Fundamental Theorem and (b) by evaluating the integral using Part 2 and then differentiating.

7. $g(x) = \displaystyle\int_1^x t^2\,dt$

8. $g(x) = \displaystyle\int_0^x (2 + \sin t)\,dt$

9–20 Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the function.

9. $g(x) = \displaystyle\int_0^x \sqrt{t + t^3}\,dt$

10. $g(x) = \displaystyle\int_1^x \ln(1 + t^2)\,dt$

11. $g(w) = \displaystyle\int_0^w \sin(1 + t^3)\,dt$

12. $h(u) = \displaystyle\int_0^u \frac{\sqrt{t}}{t + 1}\,dt$

13. $F(x) = \displaystyle\int_x^0 \sqrt{1 + \sec t}\,dt$

$$\left[\textit{Hint: } \int_x^0 \sqrt{1 + \sec t}\,dt = -\int_0^x \sqrt{1 + \sec t}\,dt\right]$$

14. $A(w) = \displaystyle\int_w^{-1} e^{t + t^2}\,dt$

15. $h(x) = \displaystyle\int_1^{e^x} \ln t\,dt$

16. $h(x) = \displaystyle\int_1^{\sqrt{x}} \frac{z^2}{z^4 + 1}\,dz$

17. $y = \displaystyle\int_1^{3x+2} \frac{t}{1 + t^3}\,dt$

18. $y = \displaystyle\int_0^{\tan x} e^{-t^2}\,dt$

19. $y = \displaystyle\int_{\sqrt{x}}^{\pi/4} \theta \tan \theta\,d\theta$

20. $y = \displaystyle\int_{1/x}^4 \sqrt{1 + \frac{1}{t}}\,dt$

21–24 Use Part 2 of the Fundamental Theorem of Calculus to evaluate the integral and interpret the result as an area or a difference of areas. Illustrate with a sketch.

21. $\displaystyle\int_{-1}^2 x^3\,dx$

22. $\displaystyle\int_0^4 (x^2 - 4x)\,dx$

23. $\displaystyle\int_{\pi/2}^{2\pi} (2\sin x)\,dx$

24. $\displaystyle\int_{-1}^{2} (e^x + 2)\,dx$

25–54 Evaluate the integral.

25. $\displaystyle\int_1^3 (x^2 + 2x - 4)\,dx$

26. $\displaystyle\int_{-1}^{1} x^{100}\,dx$

27. $\displaystyle\int_0^2 \left(\tfrac{4}{5}t^3 - \tfrac{3}{4}t^2 + \tfrac{2}{5}t\right) dt$

28. $\displaystyle\int_0^1 \left(1 - 8v^3 + 16v^7\right) dv$

29. $\displaystyle\int_1^9 \sqrt{x}\,dx$

30. $\displaystyle\int_1^8 x^{-2/3}\,dx$

31. $\displaystyle\int_0^4 (t^2 + t^{3/2})\,dt$

32. $\displaystyle\int_1^3 \left(\frac{1}{z^2} + \frac{1}{z^3}\right) dz$

33. $\displaystyle\int_{\pi/2}^{0} \cos\theta\,d\theta$

34. $\displaystyle\int_{-5}^{5} e\,dx$

35. $\displaystyle\int_0^1 (u + 2)(u - 3)\,du$

36. $\displaystyle\int_0^4 (4 - t)\sqrt{t}\,dt$

37. $\displaystyle\int_1^4 \frac{2 + x^2}{\sqrt{x}}\,dx$

38. $\displaystyle\int_{-1}^{2} (3u - 2)(u + 1)\,du$

39. $\displaystyle\int_1^3 \left(2x + \frac{1}{x}\right) dx$

40. $\displaystyle\int_5^5 \sqrt{t^2 + \sin t}\,dt$

41. $\displaystyle\int_0^{\pi/3} \sec\theta\tan\theta\,d\theta$

42. $\displaystyle\int_1^3 \frac{y^3 - 2y^2 - y}{y^2}\,dy$

43. $\displaystyle\int_0^1 (1 + r)^3\,dr$

44. $\displaystyle\int_0^3 (2\sin x - e^x)\,dx$

45. $\displaystyle\int_1^2 \frac{v^3 + 3v^6}{v^4}\,dv$

46. $\displaystyle\int_1^{18} \sqrt{\frac{3}{z}}\,dz$

47. $\displaystyle\int_0^1 (x^e + e^x)\,dx$

48. $\displaystyle\int_0^1 \cosh t\,dt$

49. $\displaystyle\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1 + x^2}\,dx$

50. $\displaystyle\int_1^3 \frac{(3x + 1)^2}{x^3}\,dx$

51. $\displaystyle\int_0^4 2^s\,ds$

52. $\displaystyle\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1 - x^2}}\,dx$

53. $\displaystyle\int_0^\pi f(x)\,dx \quad \text{where } f(x) = \begin{cases} \sin x & \text{if } 0 \leqslant x < \pi/2 \\ \cos x & \text{if } \pi/2 \leqslant x \leqslant \pi \end{cases}$

54. $\displaystyle\int_{-2}^{2} f(x)\,dx \quad \text{where } f(x) = \begin{cases} 2 & \text{if } -2 \leqslant x \leqslant 0 \\ 4 - x^2 & \text{if } 0 < x \leqslant 2 \end{cases}$

55–58 Sketch the region enclosed by the given curves and calculate its area.

55. $y = \sqrt{x}, \quad y = 0, \quad x = 4$

56. $y = x^3, \quad y = 0, \quad x = 1$

57. $y = 4 - x^2, \quad y = 0$

58. $y = 2x - x^2, \quad y = 0$

59–62 Use a graph to give a rough estimate of the area of the region that lies beneath the given curve. Then find the exact area.

59. $y = \sqrt[3]{x}, \ 0 \leqslant x \leqslant 27$

60. $y = x^{-4}, \ 1 \leqslant x \leqslant 6$

61. $y = \sin x, \ 0 \leqslant x \leqslant \pi$

62. $y = \sec^2 x, \ 0 \leqslant x \leqslant \pi/3$

63–66 What is wrong with the equation?

63. $\displaystyle\int_{-2}^{1} x^{-4}\,dx = \frac{x^{-3}}{-3}\bigg]_{-2}^{1} = -\frac{3}{8}$

64. $\displaystyle\int_{-1}^{2} \frac{4}{x^3}\,dx = -\frac{2}{x^2}\bigg]_{-1}^{2} = \frac{3}{2}$

65. $\displaystyle\int_{\pi/3}^{\pi} \sec\theta\tan\theta\,d\theta = \sec\theta\Big]_{\pi/3}^{\pi} = -3$

66. $\displaystyle\int_0^\pi \sec^2 x\,dx = \tan x\Big]_0^\pi = 0$

67–71 Find the derivative of the function.

67. $\displaystyle g(x) = \int_{2x}^{3x} \frac{u^2 - 1}{u^2 + 1}\,du$

$$\left[\textit{Hint: } \int_{2x}^{3x} f(u)\,du = \int_{2x}^{0} f(u)\,du + \int_0^{3x} f(u)\,du\right]$$

68. $\displaystyle g(x) = \int_{1-2x}^{1+2x} t\sin t\,dt$

69. $\displaystyle F(x) = \int_x^{x^2} e^{t^2}\,dt$

70. $\displaystyle F(x) = \int_{\sqrt{x}}^{2x} \arctan t\,dt$

71. $\displaystyle y = \int_{\cos x}^{\sin x} \ln(1 + 2v)\,dv$

---

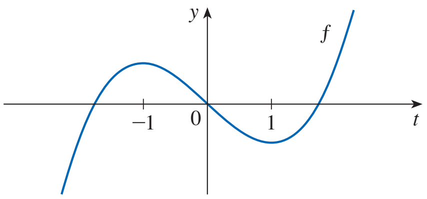
72. If $f(x) = \int_0^x (1 - t^2)e^{t^2}\,dt$, on what interval is f increasing?

73. On what interval is the curve

$$y = \int_0^x \frac{t^2}{t^2 + t + 2}\,dt$$

concave downward?

74. Let $F(x) = \int_1^x f(t)\,dt$, where f is the function whose graph is shown. Where is F concave downward?



75. Let $F(x) = \int_2^x e^{t^2}\,dt$. Find an equation of the tangent line to the curve $y = F(x)$ at the point with x-coordinate 2.

76. If $f(x) = \int_0^{\sin x} \sqrt{1 + t^2}\,dt$ and $g(y) = \int_3^y f(x)\,dx$, find $g''(\pi/6)$.

77–78 Use l'Hospital's Rule to evaluate the limit.

77. $\displaystyle \lim_{x \to 0} \frac{1}{x^2} \int_0^x \frac{2t}{\sqrt{t^3 + 1}}\,dt$

78. $\displaystyle \lim_{x \to \infty} \frac{1}{x^2} \int_0^x \ln(1 + e^t)\,dt$

---

79. If $f(1) = 12$, f' is continuous, and $\int_1^4 f'(x)\,dx = 17$, what is the value of $f(4)$?

80. The Error Function The *error function*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

is used in probability, statistics, and engineering.

(a) Show that $\int_a^b e^{-t^2}\,dt = \frac{1}{2}\sqrt{\pi}\,[\operatorname{erf}(b) - \operatorname{erf}(a)]$.

(b) Show that the function $y = e^{x^2}\operatorname{erf}(x)$ satisfies the differential equation $y' = 2xy + 2/\sqrt{\pi}$.

81. The Fresnel Function The Fresnel function S was defined in Example 3 and graphed in Figures 7 and 8.

(a) At what values of x does this function have local maximum values?

(b) On what intervals is the function concave upward?

T (c) Use a graph to solve the following equation correct to two decimal places:

$$\int_0^x \sin(\pi t^2/2)\,dt = 0.2$$

T **82. The Sine Integral Function** The *sine integral function*

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt$$

is important in electrical engineering. [The integrand $f(t) = (\sin t)/t$ is not defined when $t = 0$, but we know that its limit is 1 when $t \to 0$. So we define $f(0) = 1$ and this makes f a continuous function everywhere.]

(a) Draw the graph of Si.

(b) At what values of x does this function have local maximum values?

(c) Find the coordinates of the first inflection point to the right of the origin.

(d) Does this function have horizontal asymptotes?

(e) Solve the following equation correct to one decimal place:

$$\int_0^x \frac{\sin t}{t}\,dt = 1$$

83–84 Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.

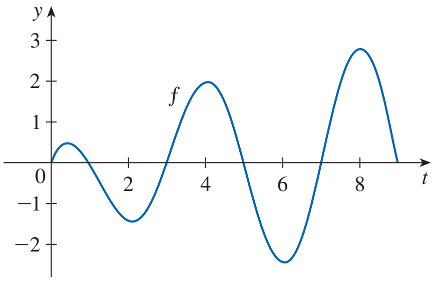
(a) At what values of x do the local maximum and minimum values of g occur?

(b) Where does g attain its absolute maximum value?

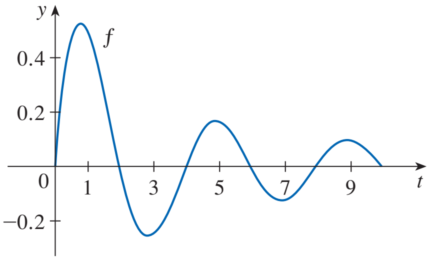
(c) On what intervals is g concave downward?

(d) Sketch the graph of g.

83.



84.



85–86 Evaluate the limit by first recognizing the sum as a Riemann sum for a function defined on $[0, 1]$.

85. $\displaystyle \lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{i^4}{n^5} + \frac{i}{n^2} \right)$

86. $\displaystyle \lim_{n \to \infty} \frac{1}{n} \left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \sqrt{\frac{3}{n}} + \cdots + \sqrt{\frac{n}{n}} \right)$

---

87. Justify (3) for the case $h < 0$.

88. If f is continuous and g and h are differentiable functions, show that

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))\,h'(x) - f(g(x))\,g'(x)$$

89. (a) Show that $1 \leqslant \sqrt{1 + x^3} \leqslant 1 + x^3$ for $x \geqslant 0$.

(b) Show that $1 \leqslant \int_0^1 \sqrt{1 + x^3}\,dx \leqslant 1.25$.

90. (a) Show that $\cos(x^2) \geqslant \cos x$ for $0 \leqslant x \leqslant 1$.

(b) Deduce that $\int_0^{\pi/6} \cos(x^2)\,dx \geqslant \frac{1}{2}$.

91. Show that

$$0 \leqslant \int_5^{10} \frac{x^2}{x^4 + x^2 + 1}\,dx \leqslant 0.1$$

by comparing the integrand to a simpler function.

92. Let

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leqslant x \leqslant 1 \\ 2 - x & \text{if } 1 < x \leqslant 2 \\ 0 & \text{if } x > 2 \end{cases}$$

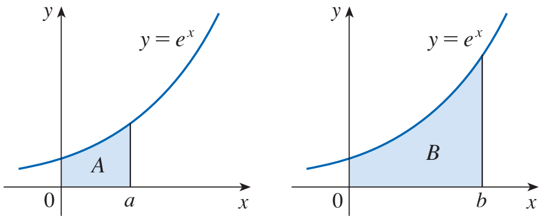
and $$g(x) = \int_0^x f(t)\,dt$$

(a) Find an expression for $g(x)$ similar to the one for $f(x)$.
(b) Sketch the graphs of f and g.
(c) Where is f differentiable? Where is g differentiable?

93. Find a function f and a number a such that

$$6 + \int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x} \qquad \text{for all } x > 0$$

94. The area labeled B is three times the area labeled A. Express b in terms of a.



95. A manufacturing company owns a major piece of equipment that depreciates at the (continuous) rate $f(t)$, where t is the time measured in months since its last overhaul. Because a fixed cost A is incurred each time the machine is overhauled, the company wants to determine the optimal time T (in months) between overhauls.

(a) Explain why $\int_0^t f(s)\,ds$ represents the loss in value of the machine over the period of time t since the last overhaul.

(b) Let $C = C(t)$ be given by

$$C(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right]$$

What does C represent and why would the company want to minimize C?

(c) Show that C has a minimum value at the numbers $t = T$ where $C(T) = f(T)$.

5.4 Indefinite Integrals and the Net Change Theorem

We saw in Section 5.3 that the second part of the Fundamental Theorem of Calculus provides a very powerful method for evaluating the definite integral of a function, assuming that we can find an antiderivative of the function. In this section we introduce a notation for antiderivatives, review the formulas for antiderivatives, and use them to evaluate definite integrals. We also reformulate FTC2 in a way that makes it easier to apply to science and engineering problems.

Indefinite Integrals

Both parts of the Fundamental Theorem establish connections between antiderivatives and definite integrals. Part 1 says that if f is continuous, then $\int_a^x f(t)\,dt$ is an antiderivative of f. Part 2 says that $\int_a^b f(x)\,dx$ can be found by evaluating $F(b) - F(a)$, where F is an antiderivative of f.

We need a convenient notation for antiderivatives that makes them easy to work with. Because of the relation between antiderivatives and integrals given by the Fundamental Theorem, the notation $\int f(x)\,dx$ is traditionally used for an antiderivative of f and is called an **indefinite integral**. Thus

$$\boxed{\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)}$$

For example, we can write

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

So we can regard an indefinite integral as representing an entire *family* of functions (one antiderivative for each value of the constant C).

You should distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)\,dx$ is a *number*, whereas an indefinite integral $\int f(x)\,dx$ is a *function* (or family of functions). The connection between them is given by Part 2 of the Fundamental Theorem: if f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big]_a^b$$

The effectiveness of the Fundamental Theorem depends on having a supply of antiderivatives of functions. We therefore restate the Table of Antidifferentiation Formulas from Section 4.9, together with a few others, in the notation of indefinite integrals. Any formula can be verified by differentiating the function on the right side and obtaining the integrand. For instance,

$$\int \sec^2 x\,dx = \tan x + C \qquad \text{because} \qquad \frac{d}{dx}(\tan x + C) = \sec^2 x$$

1 Table of Indefinite Integrals

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad\qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad\qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad\qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad\qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad\qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad\qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2 + 1}\,dx = \tan^{-1}x + C \qquad\qquad \int \frac{1}{\sqrt{1 - x^2}}\,dx = \sin^{-1}x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad\qquad \int \cosh x\,dx = \sinh x + C$$

Recall from Theorem 4.9.1 that the most general antiderivative *on a given interval* is obtained by adding a constant to a particular antiderivative. **We adopt the convention that when a formula for a general indefinite integral is given, it is valid only on an interval.** Thus we write

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$$

with the understanding that it is valid on the interval $(0, \infty)$ or on the interval $(-\infty, 0)$. This is true despite the fact that the general antiderivative of the function $f(x) = 1/x^2$, $x \neq 0$, is

$$F(x) = \begin{cases} -\dfrac{1}{x} + C_1 & \text{if } x < 0 \\[2ex] -\dfrac{1}{x} + C_2 & \text{if } x > 0 \end{cases}$$

EXAMPLE 1 Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)\, dx$$

SOLUTION Using our convention and Table 1, we have

$$\begin{aligned}\int (10x^4 - 2\sec^2 x)\, dx &= 10 \int x^4\, dx - 2 \int \sec^2 x\, dx \\ &= 10\,\frac{x^5}{5} - 2\tan x + C \\ &= 2x^5 - 2\tan x + C\end{aligned}$$

You should check this answer by differentiating it. ■

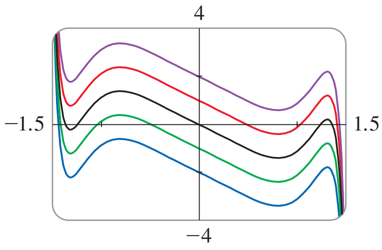


FIGURE 1

The indefinite integral in Example 1 is graphed in Figure 1 for several values of C. Here the value of C is the y-intercept.

EXAMPLE 2 Evaluate $\displaystyle\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$.

SOLUTION This indefinite integral isn't immediately apparent from Table 1, so we use trigonometric identities to rewrite the function before integrating:

$$\begin{aligned}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta &= \int \left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right) d\theta \\ &= \int \csc\theta\, \cot\theta\, d\theta = -\csc\theta + C\end{aligned}$$

■

EXAMPLE 3 Evaluate $\displaystyle\int_0^3 (x^3 - 6x)\, dx$.

SOLUTION Using FTC2 and Table 1, we have

$$\begin{aligned}\int_0^3 (x^3 - 6x)\, dx &= \frac{x^4}{4} - 6\,\frac{x^2}{2}\Bigg]_0^3 \\ &= \left(\tfrac{1}{4}\cdot 3^4 - 3\cdot 3^2\right) - \left(\tfrac{1}{4}\cdot 0^4 - 3\cdot 0^2\right) \\ &= \tfrac{81}{4} - 27 - 0 + 0 = -6.75\end{aligned}$$

Compare this calculation with Example 5.2.3. ■

EXAMPLE 4 Find $\displaystyle\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx$ and interpret the result in terms of areas.

SOLUTION The Fundamental Theorem gives

$$\begin{aligned}\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx &= 2\,\frac{x^4}{4} - 6\,\frac{x^2}{2} + 3\tan^{-1}x\Bigg]_0^2 \\ &= \tfrac{1}{2}x^4 - 3x^2 + 3\tan^{-1}x\Big]_0^2 \\ &= \tfrac{1}{2}(2^4) - 3(2^2) + 3\tan^{-1}2 - 0 \\ &= -4 + 3\tan^{-1}2\end{aligned}$$

Figure 2 shows the graph of the integrand in Example 4. We know from Section 5.2 that the value of the integral can be interpreted as a net area: the sum of the areas labeled with a plus sign minus the area labeled with a minus sign.

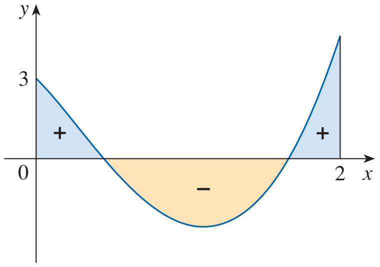


FIGURE 2

This is the exact value of the integral. If a decimal approximation is desired, we can use a calculator to approximate $\tan^{-1} 2$. Doing so, we get

$$\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2 + 1}\right) dx \approx -0.67855 \qquad ■$$

EXAMPLE 5 Evaluate $\displaystyle\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt$.

SOLUTION First we need to write the integrand in a simpler form by carrying out the division:

$$\begin{aligned}\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt &= \int_1^9 (2 + t^{1/2} - t^{-2})\,dt \\ &= 2t + \frac{t^{3/2}}{\frac{3}{2}} - \frac{t^{-1}}{-1}\Bigg]_1^9 = 2t + \tfrac{2}{3}t^{3/2} + \frac{1}{t}\Bigg]_1^9 \\ &= \left(2 \cdot 9 + \tfrac{2}{3} \cdot 9^{3/2} + \tfrac{1}{9}\right) - \left(2 \cdot 1 + \tfrac{2}{3} \cdot 1^{3/2} + \tfrac{1}{1}\right) \\ &= 18 + 18 + \tfrac{1}{9} - 2 - \tfrac{2}{3} - 1 = 32\tfrac{4}{9} \qquad ■\end{aligned}$$

The Net Change Theorem

Part 2 of the Fundamental Theorem says that if f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f. This means that $F' = f$, so the equation can be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

We know that $F'(x)$ represents the rate of change of $y = F(x)$ with respect to x and $F(b) - F(a)$ is the change in y when x changes from a to b. [Note that y could, for instance, increase, then decrease, then increase again. Although y might change in both directions, $F(b) - F(a)$ represents the *net* change in y.] So we can reformulate FTC2 in words as follows.

Net Change Theorem The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

The principle expressed in the Net Change Theorem applies to all of the rates of change in the natural and social sciences that we discussed in Section 3.7. These applications show that part of the power of mathematics is in its abstractness. A single abstract idea (in this case the integral) can have many different interpretations. Here are a few instances of how the Net Change Theorem can be applied.

- If $V(t)$ is the volume of water in a reservoir at time t, then its derivative $V'(t)$ is the rate at which water flows into the reservoir at time t. So

$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$$

is the change in the amount of water in the reservoir between time t_1 and time t_2.

- If $[\mathrm{C}](t)$ is the concentration of the product of a chemical reaction at time t, then the rate of reaction is the derivative $d[\mathrm{C}]/dt$. So

$$\int_{t_1}^{t_2} \frac{d[\mathrm{C}]}{dt}\,dt = [\mathrm{C}](t_2) - [\mathrm{C}](t_1)$$

  is the change in the concentration of C from time t_1 to time t_2.

- If the mass of a rod measured from the left end to a point x is $m(x)$, then the linear density is $\rho(x) = m'(x)$. So

$$\int_a^b \rho(x)\,dx = m(b) - m(a)$$

  is the mass of the segment of the rod that lies between $x = a$ and $x = b$.

- If the rate of growth of a population is dn/dt, then

$$\int_{t_1}^{t_2} \frac{dn}{dt}\,dt = n(t_2) - n(t_1)$$

  is the net change in population during the time period from t_1 to t_2. (The population increases when births happen and decreases when deaths occur. The net change takes into account both births and deaths.)

- If $C(x)$ is the cost of producing x units of a commodity, then the marginal cost is the derivative $C'(x)$. So

$$\int_{x_1}^{x_2} C'(x)\,dx = C(x_2) - C(x_1)$$

  is the increase in cost when production is increased from x_1 units to x_2 units.

- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

$$\boxed{2} \qquad \int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

  is the net change of position, or *displacement*, of the object during the time period from t_1 to t_2. In Section 5.1 we guessed that this was true for the case where the object moves in the positive direction, but now we have proved that it is always true.

- If we want to calculate the distance the object travels during a time interval, we have to consider the intervals when $v(t) \geqslant 0$ (the object moves to the right) and also the intervals when $v(t) \leqslant 0$ (the object moves to the left). In both cases the distance is computed by integrating $|v(t)|$, the speed. Therefore

$$\boxed{3} \qquad \int_{t_1}^{t_2} |v(t)|\,dt = \text{total distance traveled}$$

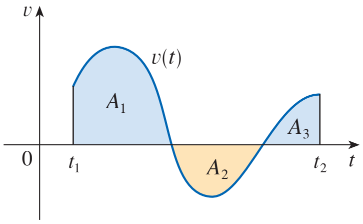


FIGURE 3

  Figure 3 shows how both displacement and distance traveled can be interpreted in terms of areas under a velocity curve.

$$\text{displacement} = \int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2 + A_3$$

$$\text{distance} = \int_{t_1}^{t_2} |v(t)|\,dt = A_1 + A_2 + A_3$$

- The acceleration of the object is $a(t) = v'(t)$, so

$$\int_{t_1}^{t_2} a(t)\,dt = v(t_2) - v(t_1)$$

  is the change in velocity from time t_1 to time t_2.

EXAMPLE 6 A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \leqslant t \leqslant 4$.

(b) Find the distance traveled during this time period.

SOLUTION

(a) By Equation 2, the displacement is

$$s(4) - s(1) = \int_1^4 v(t)\,dt = \int_1^4 (t^2 - t - 6)\,dt$$

$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 = -\frac{9}{2}$$

This means that the particle moved 4.5 m toward the left.

(b) Note that $v(t) = t^2 - t - 6 = (t - 3)(t + 2)$ and so $v(t) \leqslant 0$ on the interval $[1, 3]$ and $v(t) \geqslant 0$ on $[3, 4]$. Thus, from Equation 3, the distance traveled is

To integrate the absolute value of $v(t)$, we use Property 5 of integrals from Section 5.2 to split the integral into two parts, one where $v(t) \leqslant 0$ and one where $v(t) \geqslant 0$.

$$\int_1^4 |v(t)|\,dt = \int_1^3 [-v(t)]\,dt + \int_3^4 v(t)\,dt$$

$$= \int_1^3 (-t^2 + t + 6)\,dt + \int_3^4 (t^2 - t - 6)\,dt$$

$$= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4$$

$$= \frac{61}{6} \approx 10.17 \text{ m}$$

■

EXAMPLE 7 Figure 4 shows the power consumption in the city of San Francisco for a day in September (P is measured in megawatts; t is measured in hours starting at midnight). Estimate the energy used on that day.

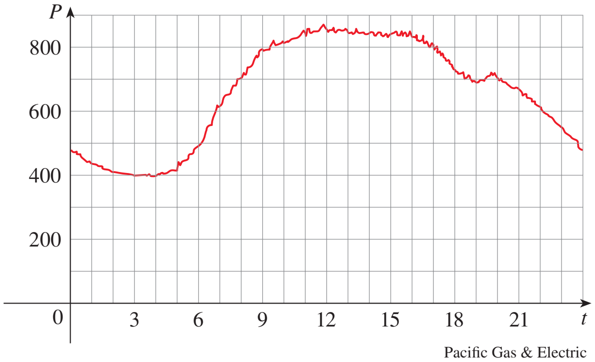


FIGURE 4

Pacific Gas & Electric

SOLUTION Power is the rate of change of energy: $P(t) = E'(t)$. So, by the Net Change Theorem,

$$\int_0^{24} P(t)\,dt = \int_0^{24} E'(t)\,dt = E(24) - E(0)$$

is the total amount of energy used on that day. We approximate the value of the integral using the Midpoint Rule with 12 subintervals and $\Delta t = 2$:

$$\begin{aligned}\int_0^{24} P(t)\,dt &\approx [P(1) + P(3) + P(5) + \cdots + P(21) + P(23)]\,\Delta t\\ &\approx (440 + 400 + 420 + 620 + 790 + 840 + 850\\ &\qquad + 840 + 810 + 690 + 670 + 550)(2)\\ &= 15{,}840\end{aligned}$$

The energy used was approximately 15,840 megawatt-hours. ■

A note on units

How did we know what units to use for energy in Example 7? The integral $\int_0^{24} P(t)\,dt$ is defined as the limit of sums of terms of the form $P(t_i^*)\,\Delta t$. Now $P(t_i^*)$ is measured in megawatts and Δt is measured in hours, so their product is measured in megawatt-hours. The same is true of the limit. In general, the unit of measurement for $\int_a^b f(x)\,dx$ is the product of the unit for $f(x)$ and the unit for x.

5.4 Exercises

1–4 Verify by differentiation that the formula is correct.

1. $\displaystyle\int \ln x\,dx = x\ln x - x + C$

2. $\displaystyle\int \tan^2 x\,dx = \tan x - x + C$

3. $\displaystyle\int \frac{1}{x^2\sqrt{1 + x^2}}\,dx = -\frac{\sqrt{1 + x^2}}{x} + C$

4. $\displaystyle\int x\sqrt{a + bx}\,dx = \frac{2}{15b^2}(3bx - 2a)(a + bx)^{3/2} + C$

5–24 Find the general indefinite integral.

5. $\displaystyle\int (3x^2 + 4x + 1)\,dx$

6. $\displaystyle\int (5 + 2\sqrt{x})\,dx$

7. $\displaystyle\int (x + \cos x)\,dx$

8. $\displaystyle\int \left(\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}\right)dx$

9. $\displaystyle\int (x^{1.3} + 7x^{2.5})\,dx$

10. $\displaystyle\int \sqrt[4]{x^5}\,dx$

11. $\displaystyle\int \left(5 + \tfrac{2}{3}x^2 + \tfrac{3}{4}x^3\right)dx$

12. $\displaystyle\int \left(u^6 - 2u^5 - u^3 + \tfrac{2}{7}\right)du$

13. $\displaystyle\int (u + 4)(2u + 1)\,du$

14. $\displaystyle\int \sqrt{t}\,(t^2 + 3t + 2)\,dt$

15. $\displaystyle\int \frac{1 + \sqrt{x} + x}{x}\,dx$

16. $\displaystyle\int \left(x^2 + 1 + \frac{1}{x^2 + 1}\right)dx$

17. $\displaystyle\int \left(e^x + \frac{1}{x}\right)dx$

18. $\displaystyle\int (2 + 3^x)\,dx$

19. $\displaystyle\int (\sin x + \sinh x)\,dx$

20. $\displaystyle\int \left(\frac{1 + r}{r}\right)^2 dr$

21. $\displaystyle\int (2 + \tan^2\theta)\,d\theta$

22. $\displaystyle\int \sec t\,(\sec t + \tan t)\,dt$

23. $\displaystyle\int 3\csc^2 t\,dt$

24. $\displaystyle\int \frac{\sin 2x}{\sin x}\,dx$

25–26 Find the general indefinite integral. Illustrate by graphing several members of the family on the same screen.

25. $\displaystyle\int \left(\cos x + \tfrac{1}{2}x\right)dx$

26. $\displaystyle\int (e^x - 2x^2)\,dx$

27–54 Evaluate the definite integral.

27. $\displaystyle\int_{-2}^{3} (x^2 - 3)\,dx$

28. $\displaystyle\int_1^2 (4x^3 - 3x^2 + 2x)\,dx$

29. $\displaystyle\int_1^4 (8t^3 - 6t^{-2})\,dt$

30. $\displaystyle\int_0^8 \left(\tfrac{1}{8} + \tfrac{1}{2}w + \tfrac{1}{3}w^{1/3}\right)dw$

31. $\displaystyle\int_0^2 (2x - 3)(4x^2 + 1)\,dx$

32. $\displaystyle\int_1^2 \left(\frac{1}{x^2} - \frac{4}{x^3}\right)dx$

33. $\displaystyle\int_1^3 \left(\frac{3x^2 + 4x + 1}{x}\right)dx$

34. $\displaystyle\int_{-1}^{1} t(1 - t)^2\,dt$

35. $\displaystyle\int_1^4 \left(\frac{4 + 6u}{\sqrt{u}}\right)du$

36. $\displaystyle\int_0^1 \frac{4}{1 + p^2}\,dp$

37. $\displaystyle\int_{\pi/6}^{\pi/3} (4\sec^2 y)\,dy$

38. $\displaystyle\int_0^{\pi/2} (\sqrt{t} - 3\cos t)\,dt$

39. $\int_0^1 x(\sqrt[3]{x} + \sqrt[4]{x})\,dx$ **40.** $\int_1^4 \frac{\sqrt{y} - y}{y^2}\,dy$

41. $\int_1^2 \left(\frac{x}{2} - \frac{2}{x}\right) dx$ **42.** $\int_0^1 (5x - 5^x)\,dx$

43. $\int_{-2}^2 (\sinh x + \cosh x)\,dx$ **44.** $\int_0^{\pi/4} (3e^x - 4\sec x \tan x)\,dx$

45. $\int_0^{\pi/4} \frac{1 + \cos^2\theta}{\cos^2\theta}\,d\theta$ **46.** $\int_0^{\pi/3} \frac{\sin\theta + \sin\theta\,\tan^2\theta}{\sec^2\theta}\,d\theta$

47. $\int_3^4 \sqrt{\frac{3}{x}}\,dx$ **48.** $\int_{-10}^{10} \frac{2e^x}{\sinh x + \cosh x}\,dx$

49. $\int_0^{\sqrt{3}/2} \frac{dr}{\sqrt{1 - r^2}}$ **50.** $\int_{\pi/6}^{\pi/2} \csc t \cot t\,dt$

51. $\int_0^{1/\sqrt{3}} \frac{t^2 - 1}{t^4 - 1}\,dt$ **52.** $\int_0^2 |2x - 1|\,dx$

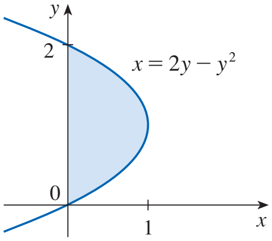
53. $\int_{-1}^2 (x - 2|x|)\,dx$ **54.** $\int_0^{3\pi/2} |\sin x|\,dx$

---

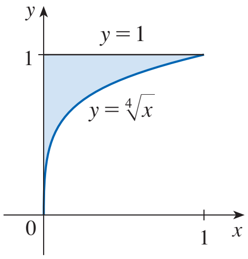
55. Use a graph to estimate the x-intercepts of the curve $y = 1 - 2x - 5x^4$. Then use this information to estimate the area of the region that lies under the curve and above the x-axis.

56. Repeat Exercise 55 for the curve $y = (x^2 + 1)^{-1} - x^4$.

57. The area of the region that lies to the right of the y-axis and to the left of the parabola $x = 2y - y^2$ (the shaded region in the figure) is given by the integral $\int_0^2 (2y - y^2)\,dy$. (Turn your head clockwise and think of the region as lying below the curve $x = 2y - y^2$ from $y = 0$ to $y = 2$.) Find the area of the region.



58. The boundaries of the shaded region are the y-axis, the line $y = 1$, and the curve $y = \sqrt[4]{x}$. Find the area of this region by writing x as a function of y and integrating with respect to y (as in Exercise 57).



59. If $w'(t)$ is the rate of growth of a child in kilograms per year, what does $\int_5^{10} w'(t)\,dt$ represent?

60. The current in a wire is defined as the derivative of the charge: $I(t) = Q'(t)$. (See Example 3.7.3.) What does $\int_a^b I(t)\,dt$ represent?

61. If oil leaks from a tank at a rate of $r(t)$ liters per minute at time t, what does $\int_0^{120} r(t)\,dt$ represent?

62. A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. What does $100 + \int_0^{15} n'(t)\,dt$ represent?

63. In Section 4.7 we defined the marginal revenue function $R'(x)$ as the derivative of the revenue function $R(x)$, where x is the number of units sold. What does $\int_{1000}^{5000} R'(x)\,dx$ represent?

64. If $f(x)$ is the slope of a trail at a distance of x kilometers from the start of the trail, what does $\int_3^5 f(x)\,dx$ represent?

65. If $h(t)$ is a person's heart rate in beats per minute t minutes into an exercise session, what does $\int_0^{30} h(t)\,dt$ represent?

66. If the units for x are meters and the units for $a(x)$ are kilograms per meter, what are the units for da/dx? What units does $\int_2^8 a(x)\,dx$ have?

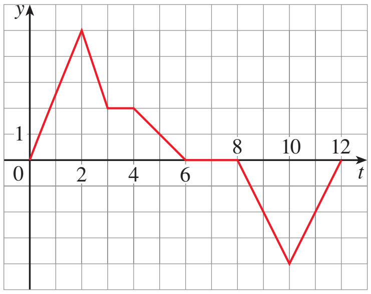
67. If x is measured in meters and $f(x)$ is measured in newtons, what are the units for $\int_0^{100} f(x)\,dx$?

68. The graph shows the velocity (in m/s) of an electric autonomous vehicle moving along a straight track. At $t = 0$ the vehicle is at the charging station.

(a) How far is the vehicle from the charging station when $t = 2, 4, 6, 8, 10$, and 12?

(b) At what times is the vehicle farthest from the charging station?

(c) What is the total distance traveled by the vehicle?



69–70 The velocity function (in m/s) is given for a particle moving along a line. Find (a) the displacement and (b) the distance traveled by the particle during the given time interval.

69. $v(t) = 3t - 5, \quad 0 \leqslant t \leqslant 3$

70. $v(t) = t^2 - 2t - 3, \quad 2 \leqslant t \leqslant 4$

71–72 The acceleration function (in m/s^2) and the initial velocity are given for a particle moving along a line. Find (a) the velocity at time t and (b) the distance traveled during the given time interval.

71. $a(t) = t + 4, \quad v(0) = 5, \quad 0 \leqslant t \leqslant 10$

72. $a(t) = 2t + 3, \quad v(0) = -4, \quad 0 \leqslant t \leqslant 3$

---

73. The linear density of a rod of length 4 m is given by $\rho(x) = 9 + 2\sqrt{x}$ measured in kilograms per meter, where x is measured in meters from one end of the rod. Find the total mass of the rod.

74. Water flows from the bottom of a storage tank at a rate of $r(t) = 200 - 4t$ liters per minute, where $0 \leqslant t \leqslant 50$. Find the amount of water that flows from the tank during the first 10 minutes.

75. The velocity of a car was read from its speedometer at 10-second intervals and recorded in the table. Use the Midpoint Rule to estimate the distance traveled by the car.

t (s)	v (km/h)	t (s)	v (km/h)
0	0	60	90
10	61	70	85
20	84	80	80
30	93	90	76
40	89	100	72
50	82		

76. Suppose that a volcano is erupting and readings of the rate $r(t)$ at which solid materials are spewed into the atmosphere are given in the table. The time t is measured in seconds and the units for $r(t)$ are tonnes (metric tons) per second.

t	0	1	2	3	4	5	6
$r(t)$	2	10	24	36	46	54	60

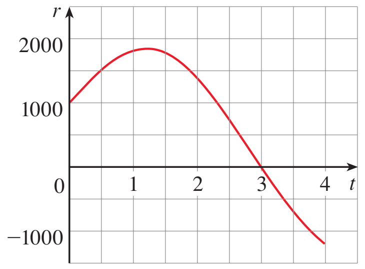
(a) Give upper and lower estimates for the total quantity $Q(6)$ of erupted materials after six seconds.
(b) Use the Midpoint Rule to estimate $Q(6)$.

77. The marginal cost of manufacturing x meter of a certain fabric is

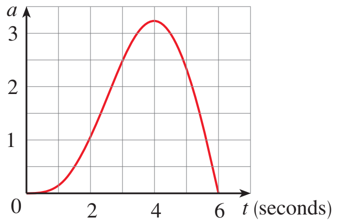
$$C'(x) = 3 - 0.01x + 0.000006x^2$$

(in dollars per meter). Find the increase in cost if the production level is raised from 2000 meters to 4000 meters.

78. Water flows into and out of a storage tank. A graph of the rate of change $r(t)$ of the volume of water in the tank, in liters per day, is shown. If the amount of water in the tank at time $t = 0$ is 25,000 L, use the Midpoint Rule to estimate the amount of water in the tank four days later.



79. The graph of a car's acceleration $a(t)$, measured in m/s^2, is shown. Use the Midpoint Rule to estimate the increase in the velocity of the car during the six-second time interval.

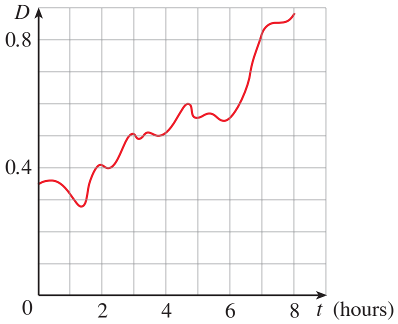


80. Lake Lanier in Georgia, USA, is a reservoir created by Buford Dam on the Chattahoochee River. The table shows the rate of inflow of water, in cubic meters per second, as measured every morning at 7:30 AM by the US Army Corps of Engineers. Use the Midpoint Rule to estimate the amount of water that flowed into Lake Lanier from July 18th, 2013, at 7:30 AM to July 26th at 7:30 AM.

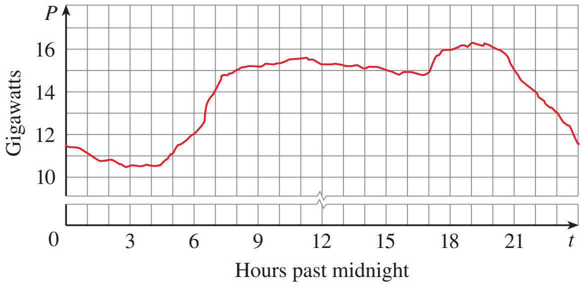
Day	Inflow rate (m^3/s)
July 18	149
July 19	181
July 20	72
July 21	120
July 22	85
July 23	108
July 24	70
July 25	74
July 26	85

81. A bacteria population is 4000 at time $t = 0$ and its rate of growth is $1000 \cdot 2^t$ bacteria per hour after t hours. What is the population after one hour?

82. Shown is the graph of traffic on an Internet service provider's T1 data line from midnight to 8:00 AM. D is the data throughput, measured in megabits per second. Use the Midpoint Rule to estimate the total amount of data transmitted during that time period.



83. Shown is a graph of the electric power consumption in the New England states (Connecticut, Maine, Massachusetts, New Hampshire, Rhode Island, and Vermont) for October 22, 2010 (P is measured in gigawatts and t is measured in hours starting at midnight). Use the fact that power is the rate of change of energy to estimate the electric energy used on that day.



Source: US Energy Information Administration

T **84.** In 1992 the space shuttle *Endeavour* was launched on mission STS-49 in order to install a new perigee kick motor in an Intelsat communications satellite. The table gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters.

(a) Use a graphing calculator or computer to model these data by a third-degree polynomial.

(b) Use the model in part (a) to estimate the height reached by the *Endeavour,* 125 seconds after liftoff.

Event	Time (s)	Velocity (m/s)
Launch	0	0
Begin roll maneuver	10	56
End roll maneuver	15	97
Throttle to 89%	20	136
Throttle to 67%	32	226
Throttle to 104%	59	404
Maximum dynamic pressure	62	440
Solid rocket booster separation	125	1265

WRITING PROJECT NEWTON, LEIBNIZ, AND THE INVENTION OF CALCULUS

We sometimes read that the inventors of calculus were Sir Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716). But we know that the basic ideas behind integration were investigated 2500 years ago by ancient Greeks such as Eudoxus and Archimedes, and methods for finding tangents were pioneered by Pierre Fermat (1601–1665), Isaac Barrow (1630–1677), and others. Barrow—who taught at Cambridge and was a major influence on Newton—was the first to understand the inverse relationship between differentiation and integration. What Newton and Leibniz did was to use this relationship, in the form of the Fundamental Theorem of Calculus, in order to develop calculus into a systematic mathematical discipline. It is in this sense that both Newton and Leibniz are credited with the invention of calculus.

Search the Internet to find out more about the contributions of these men, and consult one or more of the given references. Write an essay on one of the following three topics. You can include biographical details, but the main thrust of your report should be a description, in some detail, of their methods and notations. In particular, you should consult one of the source books, which give excerpts from the original publications of Newton and Leibniz, translated from Latin to English.

- The Role of Newton in the Development of Calculus
- The Role of Leibniz in the Development of Calculus
- The Controversy between the Followers of Newton and Leibniz over Priority in the Invention of Calculus

References

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3. C. H. Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979), Chapters 8 and 9.
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6. Victor Katz, *A History of Mathematics: An Introduction* (New York: HarperCollins, 1993), Chapter 12.
7. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), Chapter 17.

Source Books

1. John Fauvel and Jeremy Gray, eds., *The History of Mathematics: A Reader* (London: MacMillan Press, 1987), Chapters 12 and 13.
2. D. E. Smith, ed., *A Sourcebook in Mathematics* (New York: Dover, 1959), Chapter V.
3. D. J. Struik, ed., *A Sourcebook in Mathematics,* 1200–1800 (Princeton, NJ: Princeton University Press, 1969), Chapter V.

5.5 The Substitution Rule

Because of the Fundamental Theorem, it's important to be able to find antiderivatives. But our antidifferentiation formulas don't tell us how to evaluate integrals such as

$$\boxed{1} \qquad \int 2x\sqrt{1 + x^2}\,dx$$

PS To find this integral we use the problem-solving strategy of *introducing something extra.* Here the "something extra" is a new variable; we change from the variable x to a new variable u.

Substitution: Indefinite Integrals

Differentials were defined in Section 3.10. If $u = f(x)$, then

$$du = f'(x)\,dx$$

Suppose that we let u be the quantity under the root sign in (1), $u = 1 + x^2$. Then the differential of u is $du = 2x\,dx$. Notice that if the dx in the notation for an integral were to be interpreted as a differential, then the differential $2x\,dx$ would occur in (1) and so, formally, without justifying our calculation, we could write

$$\boxed{2} \qquad \int 2x\sqrt{1 + x^2}\,dx = \int \sqrt{1 + x^2}\,2x\,dx = \int \sqrt{u}\,du$$
$$= \tfrac{2}{3}u^{3/2} + C = \tfrac{2}{3}(1 + x^2)^{3/2} + C$$

But now we can check that we have the correct answer by using the Chain Rule to differentiate the final function of Equation 2:

$$\frac{d}{dx}\left[\tfrac{2}{3}(1+x^2)^{3/2}+C\right]=\tfrac{2}{3}\cdot\tfrac{3}{2}(1+x^2)^{1/2}\cdot 2x=2x\sqrt{1+x^2}$$

In general, this method works whenever we have an integral that we can write in the form $\int f(g(x))\,g'(x)\,dx$. Observe that if $F'=f$, then

$$\boxed{3}\qquad \int F'(g(x))\,g'(x)\,dx=F(g(x))+C$$

because, by the Chain Rule,

$$\frac{d}{dx}\left[F(g(x))\right]=F'(g(x))\,g'(x)$$

If we make the "change of variable" or "substitution" $u=g(x)$, then from Equation 3 we have

$$\int F'(g(x))\,g'(x)\,dx=F(g(x))+C=F(u)+C=\int F'(u)\,du$$

or, writing $F'=f$, we get

$$\int f(g(x))\,g'(x)\,dx=\int f(u)\,du$$

Thus we have proved the following rule.

4 The Substitution Rule If $u=g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\,g'(x)\,dx=\int f(u)\,du$$

Notice that the Substitution Rule for integration was proved using the Chain Rule for differentiation. Notice also that if $u=g(x)$, then $du=g'(x)\,dx$, so a way to remember the Substitution Rule is to think of dx and du in (4) as differentials.

Thus the Substitution Rule says: **it is permissible to operate with *dx* and *du* after integral signs as if they were differentials.**

EXAMPLE 1 Find $\int x^3\cos(x^4+2)\,dx$.

SOLUTION We make the substitution $u=x^4+2$ because its differential is $du=4x^3\,dx$, which, apart from the constant factor 4, occurs in the integral. Thus, using $x^3\,dx=\frac{1}{4}\,du$ and the Substitution Rule, we have

$$\begin{aligned}\int x^3\cos(x^4+2)\,dx&=\int \cos u\cdot\tfrac{1}{4}\,du=\tfrac{1}{4}\int\cos u\,du\\&=\tfrac{1}{4}\sin u+C\\&=\tfrac{1}{4}\sin(x^4+2)+C\end{aligned}$$

Check the answer by differentiating it.

Notice that at the final stage we had to return to the original variable x. ■

The idea behind the Substitution Rule is to replace a relatively complicated integral by a simpler integral. This is accomplished by changing from the original variable x to a new variable u that is a function of x. Thus in Example 1 we replaced the integral $\int x^3\cos(x^4 + 2)\,dx$ by the simpler integral $\frac{1}{4}\int \cos u\,du$.

The main challenge in using the Substitution Rule is to think of an appropriate substitution. You should try to choose u to be some function in the integrand whose differential also occurs (except for a constant factor). This was the case in Example 1. If that is not possible, try choosing u to be some complicated part of the integrand (perhaps the inner function in a composite function). Finding the right substitution is a bit of an art. It's not unusual to guess wrong; if your first guess doesn't work, try another substitution.

EXAMPLE 2 Evaluate $\int \sqrt{2x + 1}\,dx$.

SOLUTION 1 Let $u = 2x + 1$. Then $du = 2\,dx$, so $dx = \frac{1}{2}\,du$. Thus the Substitution Rule gives

$$\begin{aligned}\int \sqrt{2x+1}\,dx &= \int \sqrt{u}\cdot \tfrac{1}{2}\,du = \tfrac{1}{2}\int u^{1/2}\,du\\ &= \frac{1}{2}\cdot\frac{u^{3/2}}{3/2} + C = \tfrac{1}{3}u^{3/2} + C\\ &= \tfrac{1}{3}(2x+1)^{3/2} + C\end{aligned}$$

SOLUTION 2 Another possible substitution is $u = \sqrt{2x + 1}$. Then

$$du = \frac{dx}{\sqrt{2x+1}} \qquad \text{so} \qquad dx = \sqrt{2x+1}\,du = u\,du$$

(Or observe that $u^2 = 2x + 1$, so $2u\,du = 2\,dx$.) Therefore

$$\begin{aligned}\int \sqrt{2x+1}\,dx &= \int u\cdot u\,du = \int u^2\,du\\ &= \frac{u^3}{3} + C = \tfrac{1}{3}(2x+1)^{3/2} + C\end{aligned}$$ ■

EXAMPLE 3 Find $\int \frac{x}{\sqrt{1 - 4x^2}}\,dx$.

SOLUTION Let $u = 1 - 4x^2$. Then $du = -8x\,dx$, so $x\,dx = -\frac{1}{8}\,du$ and

$$\begin{aligned}\int \frac{x}{\sqrt{1-4x^2}}\,dx &= -\tfrac{1}{8}\int \frac{1}{\sqrt{u}}\,du = -\tfrac{1}{8}\int u^{-1/2}\,du\\ &= -\tfrac{1}{8}(2\sqrt{u}) + C = -\tfrac{1}{4}\sqrt{1-4x^2} + C\end{aligned}$$ ■

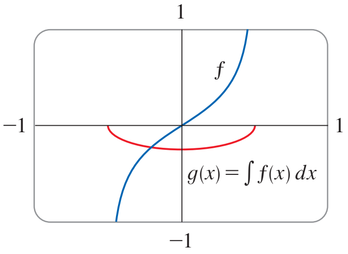


FIGURE 1

$f(x) = \dfrac{x}{\sqrt{1 - 4x^2}}$

$g(x) = \int f(x)\,dx = -\frac{1}{4}\sqrt{1 - 4x^2}$

The answer to Example 3 could be checked by differentiation, but instead let's check it with a graph. In Figure 1 we have used a computer to graph both the integrand $f(x) = x/\sqrt{1 - 4x^2}$ and its indefinite integral $g(x) = -\frac{1}{4}\sqrt{1 - 4x^2}$ (we take the case $C = 0$). Notice that $g(x)$ decreases when $f(x)$ is negative, increases when $f(x)$ is positive, and has its minimum value when $f(x) = 0$. So it seems reasonable, from the graphical evidence, that g is an antiderivative of f.

EXAMPLE 4 Evaluate $\int e^{5x}\,dx$.

SOLUTION If we let $u = 5x$, then $du = 5\,dx$, so $dx = \frac{1}{5}\,du$. Therefore

$$\int e^{5x}\,dx = \tfrac{1}{5}\int e^{u}\,du = \tfrac{1}{5}e^{u} + C = \tfrac{1}{5}e^{5x} + C \qquad \blacksquare$$

NOTE With some experience, you might be able to evaluate integrals like those in Examples 1–4 without going to the trouble of making an explicit substitution. By recognizing the pattern in Equation 3, where the integrand on the left side is the product of the derivative of an outer function and the derivative of the inner function, we could work Example 1 as follows:

$$\int x^3\cos(x^4+2)\,dx = \int \cos(x^4+2)\cdot x^3\,dx = \tfrac{1}{4}\int \cos(x^4+2)\cdot(4x^3)\,dx$$

$$= \tfrac{1}{4}\int \cos(x^4+2)\cdot\frac{d}{dx}(x^4+2)\,dx = \tfrac{1}{4}\sin(x^4+2) + C$$

Similarly, the solution to Example 4 could be written like this:

$$\int e^{5x}\,dx = \tfrac{1}{5}\int 5e^{5x}\,dx = \tfrac{1}{5}\int \frac{d}{dx}(e^{5x})\,dx = \tfrac{1}{5}e^{5x} + C$$

The following example, however, is more complicated and so an explicit substitution is advisable.

EXAMPLE 5 Find $\int \sqrt{1+x^2}\,x^5\,dx$.

SOLUTION An appropriate substitution becomes more apparent if we factor x^5 as $x^4\cdot x$. Let $u = 1 + x^2$. Then $du = 2x\,dx$, so $x\,dx = \frac{1}{2}\,du$. Also $x^2 = u - 1$, so $x^4 = (u-1)^2$:

$$\begin{aligned}\int \sqrt{1+x^2}\,x^5\,dx &= \int \sqrt{1+x^2}\,x^4\cdot x\,dx\\ &= \int \sqrt{u}\,(u-1)^2\cdot\tfrac{1}{2}\,du = \tfrac{1}{2}\int \sqrt{u}\,(u^2-2u+1)\,du\\ &= \tfrac{1}{2}\int (u^{5/2} - 2u^{3/2} + u^{1/2})\,du\\ &= \tfrac{1}{2}\left(\tfrac{2}{7}u^{7/2} - 2\cdot\tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right) + C\\ &= \tfrac{1}{7}(1+x^2)^{7/2} - \tfrac{2}{5}(1+x^2)^{5/2} + \tfrac{1}{3}(1+x^2)^{3/2} + C \qquad \blacksquare\end{aligned}$$

EXAMPLE 6 Evaluate $\int \tan x\,dx$.

SOLUTION First we write tangent in terms of sine and cosine:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

This suggests that we should substitute $u = \cos x$, since then $du = -\sin x\,dx$ and so $\sin x\,dx = -du$:

$$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du\\ &= -\ln|u| + C = -\ln|\cos x| + C \qquad \blacksquare\end{aligned}$$

Notice that $-\ln|\cos x| = \ln(|\cos x|^{-1}) = \ln(1/|\cos x|) = \ln|\sec x|$, so the result of Example 6 can also be written as

5
$$\boxed{\int \tan x\,dx = \ln|\sec x| + C}$$

Substitution: Definite Integrals

When evaluating a *definite* integral by substitution, two methods are possible. One method is to evaluate the indefinite integral first and then use the Fundamental Theorem. For instance, using the result of Example 2, we have

$$\begin{aligned}\int_0^4 \sqrt{2x+1}\,dx &= \int \sqrt{2x+1}\,dx\Big]_0^4\\ &= \tfrac{1}{3}(2x+1)^{3/2}\Big]_0^4 = \tfrac{1}{3}(9)^{3/2} - \tfrac{1}{3}(1)^{3/2}\\ &= \tfrac{1}{3}(27-1) = \tfrac{26}{3}\end{aligned}$$

Another method, which is usually preferable, is to change the limits of integration when the variable is changed.

This rule says that when using a substitution in a definite integral, we must put everything in terms of the new variable u, not only x and dx but also the limits of integration. The new limits of integration are the values of u that correspond to $x = a$ and $x = b$.

6 The Substitution Rule for Definite Integrals If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

PROOF Let F be an antiderivative of f. Then, by (3), $F(g(x))$ is an antiderivative of $f(g(x))\,g'(x)$, so by Part 2 of the Fundamental Theorem, we have

$$\int_a^b f(g(x))\,g'(x)\,dx = F(g(x))\Big]_a^b = F(g(b)) - F(g(a))$$

But, applying FTC2 a second time, we also have

$$\int_{g(a)}^{g(b)} f(u)\,du = F(u)\Big]_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$ ■

EXAMPLE 7 Evaluate $\int_0^4 \sqrt{2x+1}\,dx$ using (6).

SOLUTION Using the substitution from Solution 1 of Example 2, we have $u = 2x + 1$ and $dx = \frac{1}{2}\,du$. To find the new limits of integration we note that

when $x = 0$, $u = 2(0) + 1 = 1$ and when $x = 4$, $u = 2(4) + 1 = 9$

Therefore
$$\begin{aligned}\int_0^4 \sqrt{2x+1}\,dx &= \int_1^9 \tfrac{1}{2}\sqrt{u}\,du\\ &= \tfrac{1}{2}\cdot\tfrac{2}{3}u^{3/2}\Big]_1^9\\ &= \tfrac{1}{3}(9^{3/2} - 1^{3/2}) = \tfrac{26}{3}\end{aligned}$$ ■

Observe that when using (6) we do *not* return to the variable x after integrating. We simply evaluate the expression in u between the appropriate values of u.

Another way of writing the integral given in Example 8 is

$$\int_1^2 \frac{1}{(3-5x)^2}\,dx$$

EXAMPLE 8 Evaluate $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}$.

SOLUTION Let $u = 3 - 5x$. Then $du = -5\,dx$, so $dx = -\frac{1}{5}\,du$. When $x = 1$, $u = -2$ and when $x = 2$, $u = -7$. Thus

$$\begin{aligned}\int_1^2 \frac{dx}{(3-5x)^2} &= -\frac{1}{5}\int_{-2}^{-7} \frac{du}{u^2} = -\frac{1}{5}\left[-\frac{1}{u}\right]_{-2}^{-7} = \frac{1}{5u}\Bigg]_{-2}^{-7} \\ &= \frac{1}{5}\left(-\frac{1}{7} + \frac{1}{2}\right) = \frac{1}{14}\end{aligned}$$

■

Because the function $f(x) = (\ln x)/x$ in Example 9 is positive for $x > 1$, the integral represents the area of the shaded region in Figure 2.

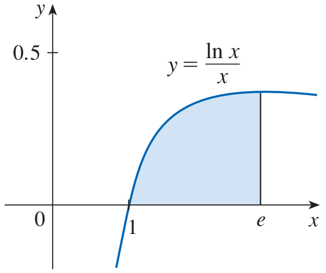


FIGURE 2

EXAMPLE 9 Evaluate $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$.

SOLUTION We let $u = \ln x$ because its differential $du = (1/x)\,dx$ occurs in the integral. When $x = 1$, $u = \ln 1 = 0$; when $x = e$, $u = \ln e = 1$. Thus

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \frac{u^2}{2}\Bigg]_0^1 = \frac{1}{2}$$

■

Symmetry

The next theorem uses the Substitution Rule for Definite Integrals (6) to simplify the calculation of integrals of functions that possess symmetry properties.

7 Integrals of Symmetric Functions Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$.

PROOF We split the integral in two:

$$\boxed{8} \qquad \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx$$

In the first integral on the far right side we make the substitution $u = -x$. Then $du = -dx$ and when $x = -a$, $u = a$. Therefore

$$-\int_0^{-a} f(x)\,dx = -\int_0^a f(-u)\,(-du) = \int_0^a f(-u)\,du$$

and so Equation 8 becomes

$$\boxed{9} \qquad \int_{-a}^{a} f(x)\,dx = \int_0^a f(-u)\,du + \int_0^a f(x)\,dx$$

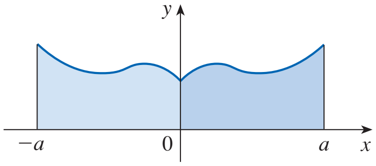
(a) If f is even, then $f(-u) = f(u)$ so Equation 9 gives

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(u)\,du + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

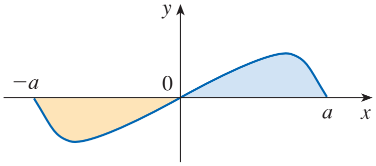
(b) If f is odd, then $f(-u) = -f(u)$ and so Equation 9 gives

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 0 \qquad \blacksquare$$

Theorem 7 is illustrated by Figure 3. For the case where f is positive and even, part (a) says that the area under $y = f(x)$ from $-a$ to a is twice the area from 0 to a because of symmetry. Recall that an integral $\int_a^b f(x)\,dx$ can be expressed as the area above the x-axis and below $y = f(x)$ minus the area below the axis and above the curve. So part (b) says the integral is 0 because the areas cancel.



(a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$



(b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

FIGURE 3

EXAMPLE 10 Because $f(x) = x^6 + 1$ satisfies $f(-x) = f(x)$, it is even and so

$$\int_{-2}^{2} (x^6 + 1)\,dx = 2\int_{0}^{2} (x^6 + 1)\,dx$$

$$= 2\Big[\tfrac{1}{7}x^7 + x\Big]_0^2 = 2\big(\tfrac{128}{7} + 2\big) = \tfrac{284}{7} \qquad \blacksquare$$

EXAMPLE 11 Because $f(x) = (\tan x)/(1 + x^2 + x^4)$ satisfies $f(-x) = -f(x)$, it is odd and so

$$\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\,dx = 0 \qquad \blacksquare$$

5.5 Exercises

1–8 Evaluate the integral by making the given substitution.

1. $\displaystyle\int \cos 2x\,dx, \quad u = 2x$

2. $\displaystyle\int xe^{-x^2}\,dx, \quad u = -x^2$

3. $\displaystyle\int x^2\sqrt{x^3 + 1}\,dx, \quad u = x^3 + 1$

4. $\displaystyle\int \sin^2\theta\cos\theta\,d\theta, \quad u = \sin\theta$

5. $\displaystyle\int \frac{x^3}{x^4 - 5}\,dx, \quad u = x^4 - 5$

6. $\displaystyle\int \frac{1}{x^2}\sqrt{1 + \frac{1}{x}}\,dx, \quad u = 1 + \frac{1}{x}$

7. $\displaystyle\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt, \quad u = \sqrt{t}$

8. $\displaystyle\int z\sqrt{z - 1}\,dz, \quad u = z - 1$

9–54 Evaluate the indefinite integral.

9. $\displaystyle\int x\sqrt{1 - x^2}\,dx$

10. $\displaystyle\int (5 - 3x)^{10}\,dx$

11. $\displaystyle\int t^3 e^{-t^4}\,dt$

12. $\displaystyle\int \sin t\sqrt{1 + \cos t}\,dt$

13. $\displaystyle\int \sin(\pi t/3)\,dt$

14. $\displaystyle\int \sec^2 2\theta\,d\theta$

15. $\displaystyle\int \frac{dx}{4x + 7}$

16. $\displaystyle\int y^2(4 - y^3)^{2/3}\,dy$

17. $\displaystyle\int \frac{\cos\theta}{1 + \sin\theta}\,d\theta$

18. $\displaystyle\int \frac{z^2}{z^3 + 1}\,dz$

19. $\displaystyle\int \cos^3\theta\,\sin\theta\,d\theta$

20. $\displaystyle\int e^{-5r}\,dr$

21. $\displaystyle\int \frac{e^u}{(1 - e^u)^2}\,du$

22. $\displaystyle\int \frac{\sin(1/x)}{x^2}\,dx$

23. $\displaystyle\int \frac{a + bx^2}{\sqrt{3ax + bx^3}}\,dx$

24. $\displaystyle\int \frac{t + 1}{3t^2 + 6t - 5}\,dt$

25. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

26. $\displaystyle\int \sin x\,\sin(\cos x)\,dx$

27. $\int \sec^2\theta \tan^3\theta \, d\theta$

28. $\int x\sqrt{x+2} \, dx$

29. $\int \left(x - \dfrac{1}{x^2}\right)\left(x^2 + \dfrac{2}{x}\right)^5 dx$

30. $\int \dfrac{dx}{ax+b} \quad (a \neq 0)$

31. $\int e^r(2+3e^r)^{3/2} \, dr$

32. $\int \dfrac{e^{\arcsin x}}{\sqrt{1-x^2}} \, dx$

33. $\int \dfrac{\sec^2\theta}{\tan\theta} \, d\theta$

34. $\int \dfrac{\sec^2 x}{\tan^2 x} \, dx$

35. $\int \dfrac{(\arctan x)^2}{x^2+1} \, dx$

36. $\int \dfrac{1}{(x^2+1)\arctan x} \, dx$

37. $\int 5^t \sin(5^t) \, dt$

38. $\int \dfrac{\sin\theta\cos\theta}{1+\sin^2\theta} \, d\theta$

39. $\int \cos(1+5t) \, dt$

40. $\int \dfrac{\cos(\pi/x)}{x^2} \, dx$

41. $\int \sqrt{\cot x} \, \csc^2 x \, dx$

42. $\int \dfrac{2^t}{2^t+3} \, dt$

43. $\int \sinh^2 x \cosh x \, dx$

44. $\int \dfrac{dt}{\cos^2 t\sqrt{1+\tan t}}$

45. $\int \dfrac{\sin 2x}{1+\cos^2 x} \, dx$

46. $\int \dfrac{\sin x}{1+\cos^2 x} \, dx$

47. $\int \cot x \, dx$

48. $\int \dfrac{\cos(\ln t)}{t} \, dt$

49. $\int \dfrac{dx}{\sqrt{1-x^2}\,\sin^{-1}x}$

50. $\int \dfrac{x}{1+x^4} \, dx$

51. $\int \dfrac{1+x}{1+x^2} \, dx$

52. $\int x^2\sqrt{2+x} \, dx$

53. $\int x(2x+5)^8 \, dx$

54. $\int x^3\sqrt{x^2+1} \, dx$

55–58 Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the function and its antiderivative (take $C = 0$).

55. $\int x(x^2-1)^3 \, dx$

56. $\int \tan^2\theta \sec^2\theta \, d\theta$

57. $\int e^{\cos x} \sin x \, dx$

58. $\int \sin x \cos^4 x \, dx$

59–80 Evaluate the definite integral.

59. $\int_0^1 \cos(\pi t/2) \, dt$

60. $\int_0^1 (3t-1)^{50} \, dt$

61. $\int_0^1 \sqrt[3]{1+7x} \, dx$

62. $\int_{\pi/3}^{2\pi/3} \csc^2\left(\tfrac{1}{2}t\right) dt$

63. $\int_0^{\pi/6} \dfrac{\sin t}{\cos^2 t} \, dt$

64. $\int_1^4 \dfrac{\sqrt{2+\sqrt{x}}}{\sqrt{x}} \, dx$

65. $\int_1^2 \dfrac{e^{1/x}}{x^2} \, dx$

66. $\int_0^1 \dfrac{e^x}{1+e^{2x}} \, dx$

67. $\int_{-\pi/4}^{\pi/4} (x^3 + x^4\tan x) \, dx$

68. $\int_0^{\pi/2} \cos x \sin(\sin x) \, dx$

69. $\int_0^{13} \dfrac{dx}{\sqrt[3]{(1+2x)^2}}$

70. $\int_0^a x\sqrt{a^2-x^2} \, dx$

71. $\int_0^a x\sqrt{x^2+a^2} \, dx \quad (a > 0)$

72. $\int_{-\pi/3}^{\pi/3} x^4 \sin x \, dx$

73. $\int_1^2 x\sqrt{x-1} \, dx$

74. $\int_0^4 \dfrac{x}{\sqrt{1+2x}} \, dx$

75. $\int_e^{e^4} \dfrac{dx}{x\sqrt{\ln x}}$

76. $\int_0^2 (x-1)e^{(x-1)^2} \, dx$

77. $\int_0^1 \dfrac{e^z+1}{e^z+z} \, dz$

78. $\int_1^4 \dfrac{1}{(x+1)\sqrt{x}} \, dx$

79. $\int_0^1 \dfrac{dx}{(1+\sqrt{x})^4}$

80. $\int_1^{16} \dfrac{x^{1/2}}{1+x^{3/4}} \, dx$

81–82 Use a graph to give a rough estimate of the area of the region that lies under the given curve. Then find the exact area.

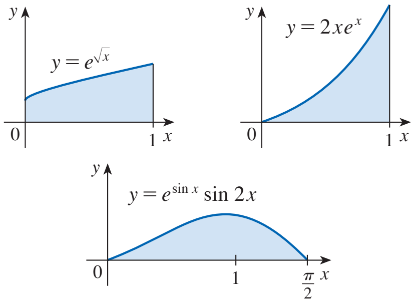
81. $y = \sqrt{2x+1}, \quad 0 \leqslant x \leqslant 1$

82. $y = 2\sin x - \sin 2x, \quad 0 \leqslant x \leqslant \pi$

83. Evaluate $\int_{-2}^{2} (x+3)\sqrt{4-x^2} \, dx$ by writing it as a sum of two integrals and interpreting one of those integrals in terms of an area.

84. Evaluate $\int_0^1 x\sqrt{1-x^4} \, dx$ by making a substitution and interpreting the resulting integral in terms of an area.

85. Which of the following areas are equal? Why?



86. A model for the basal metabolism rate, in kcal/h, of a young man is $R(t) = 85 - 0.18 \cos(\pi t/12)$, where t is the time in hours measured from 5:00 AM. What is the total basal metabolism of this man, $\int_0^{24} R(t)\, dt$, over a 24-hour time period?

87. An oil storage tank ruptures at time $t = 0$ and oil leaks from the tank at a rate of $r(t) = 100e^{-0.01t}$ liters per minute. How much oil leaks out during the first hour?

88. A bacteria population starts with 400 bacteria and grows at a rate of $r(t) = (450.268)e^{1.12567t}$ bacteria per hour. How many bacteria will there be after three hours?

89. Breathing is cyclic and a full respiratory cycle from the beginning of inhalation to the end of exhalation takes about 5 seconds. The maximum rate of air flow into the lungs is about 0.5 L/s. This explains, in part, why the function $f(t) = \frac{1}{2} \sin(2\pi t/5)$ has often been used to model the rate of air flow into the lungs. Use this model to find the volume of inhaled air in the lungs at time t.

90. The rate of growth of a fish population was modeled by the equation

$$G(t) = \frac{60{,}000e^{-0.6t}}{(1 + 5e^{-0.6t})^2}$$

where t is the number of years since 2000 and G is measured in kilograms per year. If the biomass was 25,000 kg in the year 2000, what is the predicted biomass for the year 2020?

91. Dialysis treatment removes urea and other waste products from a patient's blood by diverting some of the bloodflow externally through a machine called a dialyzer. The rate at which urea is removed from the blood (in mg/min) is often well described by the equation

$$u(t) = \frac{r}{V} C_0 e^{-rt/V}$$

where r is the rate of flow of blood through the dialyzer (in mL/min), V is the volume of the patient's blood (in mL), and C_0 is the amount of urea in the blood (in mg) at time $t = 0$. Evaluate the integral $\int_0^{30} u(t)\, dt$ and interpret it.

92. Alabama Instruments Company has set up a production line to manufacture a new calculator. The rate of production of these calculators after t weeks is

$$\frac{dx}{dt} = 5000\left(1 - \frac{100}{(t + 10)^2}\right) \text{ calculators/week}$$

(Notice that production approaches 5000 per week as time goes on, but the initial production is lower because of the workers' unfamiliarity with the new techniques.) Find the number of calculators produced from the beginning of the third week to the end of the fourth week.

93. If f is continuous and $\int_0^4 f(x)\, dx = 10$, find $\int_0^2 f(2x)\, dx$.

94. If f is continuous and $\int_0^9 f(x)\, dx = 4$, find $\int_0^3 xf(x^2)\, dx$.

95. If f is continuous on $\mathbb{R}$, prove that

$$\int_a^b f(-x)\, dx = \int_{-b}^{-a} f(x)\, dx$$

For the case where $f(x) \geqslant 0$ and $0 < a < b$, draw a diagram to interpret this equation geometrically as an equality of areas.

96. If f is continuous on $\mathbb{R}$, prove that

$$\int_a^b f(x + c)\, dx = \int_{a+c}^{b+c} f(x)\, dx$$

For the case where $f(x) \geqslant 0$, draw a diagram to interpret this equation geometrically as an equality of areas.

97. If a and b are positive numbers, show that

$$\int_0^1 x^a(1 - x)^b\, dx = \int_0^1 x^b(1 - x)^a\, dx$$

98. If f is continuous on $[0, \pi]$, use the substitution $u = \pi - x$ to show that

$$\int_0^\pi xf(\sin x)\, dx = \frac{\pi}{2} \int_0^\pi f(\sin x)\, dx$$

99. Use Exercise 98 to evaluate the integral

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x}\, dx$$

100. (a) If f is continuous, prove that

$$\int_0^{\pi/2} f(\cos x)\, dx = \int_0^{\pi/2} f(\sin x)\, dx$$

(b) Use part (a) to evaluate

$$\int_0^{\pi/2} \cos^2 x\, dx \quad \text{and} \quad \int_0^{\pi/2} \sin^2 x\, dx$$

5 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. (a) Write an expression for a Riemann sum of a function f. Explain the meaning of the notation that you use.
   (b) If $f(x) \geqslant 0$, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.
   (c) If $f(x)$ takes on both positive and negative values, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.

2. (a) Write the definition of the definite integral of a continuous function from a to b.
   (b) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x) \geqslant 0$?
   (c) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x)$ takes on both positive and negative values? Illustrate with a diagram.

3. State the Midpoint Rule.

4. State both parts of the Fundamental Theorem of Calculus.

5. (a) State the Net Change Theorem.
   (b) If $r(t)$ is the rate at which water flows into a reservoir, what does $\int_{t_1}^{t_2} r(t)\,dt$ represent?

6. Suppose a particle moves back and forth along a straight line with velocity $v(t)$, measured in meters per second, and acceleration $a(t)$.
   (a) What is the meaning of $\int_{60}^{120} v(t)\,dt$?
   (b) What is the meaning of $\int_{60}^{120} |v(t)|\,dt$?
   (c) What is the meaning of $\int_{60}^{120} a(t)\,dt$?

7. (a) Explain the meaning of the indefinite integral $\int f(x)\,dx$.
   (b) What is the connection between the definite integral $\int_a^b f(x)\,dx$ and the indefinite integral $\int f(x)\,dx$?

8. Explain exactly what is meant by the statement that "differentiation and integration are inverse processes."

9. State the Substitution Rule. In practice, how do you use it?

TRUE–FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If f and g are continuous on $[a, b]$, then

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

2. If f and g are continuous on $[a, b]$, then

$$\int_a^b [f(x)g(x)]\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$

3. If f is continuous on $[a, b]$, then

$$\int_a^b 5f(x)\,dx = 5\int_a^b f(x)\,dx$$

4. If f is continuous on $[a, b]$, then

$$\int_a^b xf(x)\,dx = x\int_a^b f(x)\,dx$$

5. If f is continuous on $[a, b]$ and $f(x) \geqslant 0$, then

$$\int_a^b \sqrt{f(x)}\,dx = \sqrt{\int_a^b f(x)\,dx}$$

6. $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(z)\,dz$

7. If f' is continuous on $[1, 3]$, then $\displaystyle\int_1^3 f'(v)\,dv = f(3) - f(1)$.

8. If $v(t)$ is the velocity at time t of a particle moving along a line, then $\int_a^b v(t)\,dt$ is the distance traveled during the time period $a \leqslant t \leqslant b$.

9. $\displaystyle\int_a^b f'(x)\,[f(x)]^4\,dx = \tfrac{1}{5}[f(x)]^5 + C$

10. If f and g are differentiable and $f(x) \geqslant g(x)$ for $a < x < b$, then $f'(x) \geqslant g'(x)$ for $a < x < b$.

11. If f and g are continuous and $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then

$$\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$$

12. $\displaystyle\int_{-5}^{5} (ax^2 + bx + c)\,dx = 2\int_0^5 (ax^2 + c)\,dx$

13. All continuous functions have derivatives.

14. All continuous functions have antiderivatives.

15. $\displaystyle\int_0^3 e^{x^2}\,dx = \int_0^5 e^{x^2}\,dx + \int_5^3 e^{x^2}\,dx$

16. If $\int_0^1 f(x)\,dx = 0$, then $f(x) = 0$ for $0 \leqslant x \leqslant 1$.

17. If f is continuous on $[a, b]$, then

$$\frac{d}{dx}\left(\int_a^b f(x)\,dx\right) = f(x)$$

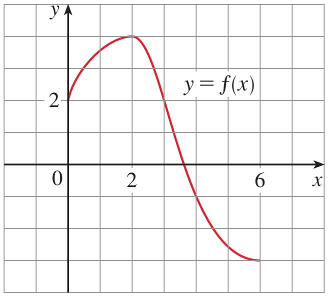
18. $\int_0^2 (x - x^3)\,dx$ represents the area under the curve $y = x - x^3$ from 0 to 2.

19. $\displaystyle\int_{-2}^{1} \frac{1}{x^4}\,dx = -\frac{3}{8}$

20. If f has a discontinuity at 0, then $\displaystyle\int_{-1}^{1} f(x)\,dx$ does not exist.

EXERCISES

1. Use the given graph of f to find the Riemann sum with six subintervals. Take the sample points to be (a) left endpoints and (b) midpoints. In each case draw a diagram and explain what the Riemann sum represents.



2. (a) Evaluate the Riemann sum for

$$f(x) = x^2 - x \qquad 0 \leqslant x \leqslant 2$$

with four subintervals, taking the sample points to be right endpoints. Explain, with the aid of a diagram, what the Riemann sum represents.

(b) Use the definition of a definite integral (with right endpoints) to calculate the value of the integral

$$\int_0^2 (x^2 - x)\,dx$$

(c) Use the Fundamental Theorem to check your answer to part (b).

(d) Draw a diagram to explain the geometric meaning of the integral in part (b).

3. Evaluate

$$\int_0^1 \left(x + \sqrt{1 - x^2}\right) dx$$

by interpreting it in terms of areas.

4. Express

$$\lim_{n \to \infty} \sum_{i=1}^{n} \sin x_i \,\Delta x$$

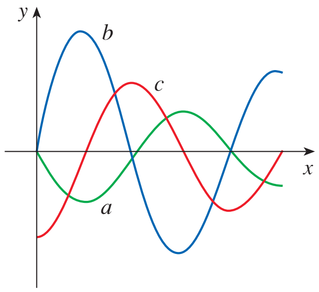
as a definite integral on the interval $[0, \pi]$ and then evaluate the integral.

5. If $\int_0^6 f(x)\,dx = 10$ and $\int_0^4 f(x)\,dx = 7$, find $\int_4^6 f(x)\,dx$.

T **6.** (a) Write $\int_1^5 (x + 2x^5)\,dx$ as a limit of Riemann sums, taking the sample points to be right endpoints. Use a computer algebra system to evaluate the sum and to compute the limit.

(b) Use the Fundamental Theorem to check your answer to part (a).

7. The figure shows the graphs of f, f', and $\int_0^x f(t)\,dt$. Identify each graph, and explain your choices.



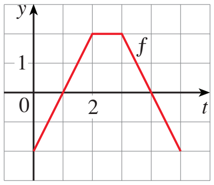
8. Evaluate:

(a) $\displaystyle\int_0^1 \frac{d}{dx}\left(e^{\arctan x}\right) dx$

(b) $\displaystyle\frac{d}{dx}\int_0^1 e^{\arctan x}\,dx$

(c) $\displaystyle\frac{d}{dx}\int_0^x e^{\arctan t}\,dt$

9. The graph of f consists of the three line segments shown. If $g(x) = \int_0^x f(t)\,dt$, find $g(4)$ and $g'(4)$.



10. If f is the function in Exercise 9, find $g''(4)$.

11–42 Evaluate the integral, if it exists.

11. $\displaystyle\int_{-1}^{0} (x^2 + 5x)\,dx$

12. $\displaystyle\int_0^T (x^4 - 8x + 7)\,dx$

13. $\displaystyle\int_0^1 (1 - x^9)\,dx$

14. $\displaystyle\int_0^1 (1 - x)^9\,dx$

15. $\displaystyle\int_1^9 \frac{\sqrt{u} - 2u^2}{u}\,du$

16. $\displaystyle\int_0^1 \left(\sqrt[4]{u} + 1\right)^2 du$

17. $\displaystyle\int_0^1 y(y^2 + 1)^5\,dy$

18. $\displaystyle\int_0^2 y^2\sqrt{1 + y^3}\,dy$

19. $\displaystyle\int_1^5 \frac{dt}{(t - 4)^2}$

20. $\displaystyle\int_0^1 \sin(3\pi t)\,dt$

21. $\displaystyle\int_0^1 v^2 \cos(v^3)\,dv$

22. $\displaystyle\int_{-1}^{1} \frac{\sin x}{1 + x^2}\,dx$

23. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{t^4 \tan t}{2 + \cos t}\,dt$

24. $\displaystyle\int_{-2}^{-1} \frac{z^2 + 1}{z}\,dz$

25. $\displaystyle\int \frac{x}{x^2 + 1}\,dx$

26. $\displaystyle\int \frac{dx}{x^2 + 1}$

27. $\displaystyle\int \frac{x+2}{\sqrt{x^2+4x}}\,dx$

28. $\displaystyle\int \frac{\csc^2 x}{1+\cot x}\,dx$

29. $\displaystyle\int \sin \pi t \cos \pi t\,dt$

30. $\displaystyle\int \sin x \cos(\cos x)\,dx$

31. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

32. $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx$

33. $\displaystyle\int \tan x \ln(\cos x)\,dx$

34. $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$

35. $\displaystyle\int \frac{x^3}{1+x^4}\,dx$

36. $\displaystyle\int \sinh(1+4x)\,dx$

37. $\displaystyle\int \frac{\sec\theta\tan\theta}{1+\sec\theta}\,d\theta$

38. $\displaystyle\int_0^{\pi/4} (1+\tan t)^3 \sec^2 t\,dt$

39. $\displaystyle\int x(1-x)^{2/3}\,dx$

40. $\displaystyle\int \frac{x}{x-3}\,dx$

41. $\displaystyle\int_0^3 |x^2-4|\,dx$

42. $\displaystyle\int_0^4 |\sqrt{x}-1|\,dx$

43–44 Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the function and its antiderivative (take $C = 0$).

43. $\displaystyle\int \frac{\cos x}{\sqrt{1+\sin x}}\,dx$

44. $\displaystyle\int \frac{x^3}{\sqrt{x^2+1}}\,dx$

---

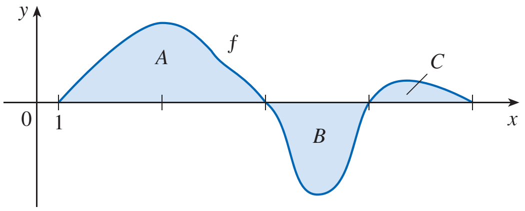
45. Use a graph to give a rough estimate of the area of the region that lies under the curve $y = x\sqrt{x}$, $0 \leqslant x \leqslant 4$. Then find the exact area.

46. Graph the function $f(x) = \cos^2 x \sin x$ and use the graph to guess the value of the integral $\int_0^{2\pi} f(x)\,dx$. Then evaluate the integral to confirm your guess.

47. Find the area under the graph of $y = x^2 + 5$ and above the x-axis, between $x = 0$ and $x = 4$.

48. Find the area under the graph of $y = \sin x$ and above the x-axis, between $x = 0$ and $x = \pi/2$.

49–50 The regions A, B, and C bounded by the graph of f and the x-axis have areas 3, 2, and 1, respectively. Evaluate the integral.



49. (a) $\displaystyle\int_1^5 f(x)\,dx$ (b) $\displaystyle\int_1^5 |f(x)|\,dx$

50. (a) $\displaystyle\int_1^4 f(x)\,dx + \int_3^5 f(x)\,dx$ (b) $\displaystyle\int_1^3 2f(x)\,dx + \int_3^5 6f(x)\,dx$

51–56 Find the derivative of the function.

51. $\displaystyle F(x) = \int_0^x \frac{t^2}{1+t^3}\,dt$

52. $\displaystyle F(x) = \int_x^1 \sqrt{t + \sin t}\,dt$

53. $\displaystyle g(x) = \int_0^{x^4} \cos(t^2)\,dt$

54. $\displaystyle g(x) = \int_1^{\sin x} \frac{1-t^2}{1+t^4}\,dt$

55. $\displaystyle y = \int_{\sqrt{x}}^{x} \frac{e^t}{t}\,dt$

56. $\displaystyle y = \int_{2x}^{3x+1} \sin(t^4)\,dt$

57–58 Use Property 8 of integrals to estimate the value of the integral.

57. $\displaystyle\int_1^3 \sqrt{x^2+3}\,dx$

58. $\displaystyle\int_2^4 \frac{1}{x^3+2}\,dx$

59–62 Use the properties of integrals to verify the inequality.

59. $\displaystyle\int_0^1 x^2 \cos x\,dx \leqslant \frac{1}{3}$

60. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\,dx \leqslant \frac{\sqrt{2}}{2}$

61. $\displaystyle\int_0^1 e^x \cos x\,dx \leqslant e - 1$

62. $\displaystyle\int_0^1 x \sin^{-1} x\,dx \leqslant \pi/4$

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63. Use the Midpoint Rule with $n = 6$ to approximate $\int_0^3 \sin(x^3)\,dx$. Round to four decimal places.

64. A particle moves along a line with velocity function $v(t) = t^2 - t$, where v is measured in meters per second. Find (a) the displacement and (b) the distance traveled by the particle during the time interval $[0, 5]$.

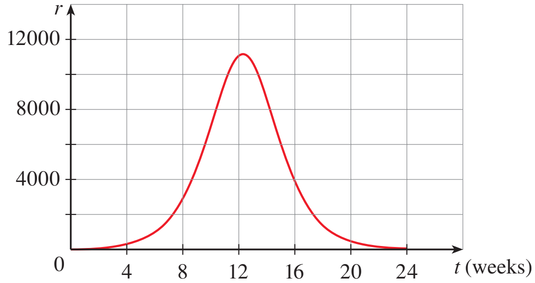
65. Let $r(t)$ be the rate at which the world's oil is consumed, where t is measured in years starting at $t = 0$ on January 1, 2000, and $r(t)$ is measured in barrels per year. What does $\int_{15}^{20} r(t)\,dt$ represent?

66. A radar gun was used to record the speed of a runner at the times given in the table. Use the Midpoint Rule to estimate the distance the runner covered during those 5 seconds.

t (s)	v (m/s)	t (s)	v (m/s)
0	0	3.0	10.51
0.5	4.67	3.5	10.67
1.0	7.34	4.0	10.76
1.5	8.86	4.5	10.81
2.0	9.73	5.0	10.81
2.5	10.22		

67. A population of honeybees increased at a rate of $r(t)$ bees per week, where the graph of r is as shown. Use the

Midpoint Rule with six subintervals to estimate the increase in the bee population during the first 24 weeks.



68. Let

$$f(x) = \begin{cases} -x - 1 & \text{if } -3 \leqslant x \leqslant 0 \\ -\sqrt{1 - x^2} & \text{if } 0 \leqslant x \leqslant 1 \end{cases}$$

Evaluate $\int_{-3}^{1} f(x)\,dx$ by interpreting the integral as a difference of areas.

69. If f is continuous and $\int_0^2 f(x)\,dx = 6$, evaluate

$$\int_0^{\pi/2} f(2 \sin \theta) \cos \theta \, d\theta$$

70. The Fresnel function $S(x) = \int_0^x \sin\left(\frac{1}{2}\pi t^2\right) dt$ was introduced in Section 5.3. Fresnel also used the function

$$C(x) = \int_0^x \cos\left(\tfrac{1}{2}\pi t^2\right) dt$$

in his theory of the diffraction of light waves.

(a) On what intervals is C increasing?

(b) On what intervals is C concave upward?

(c) Use a graph to solve the following equation correct to two decimal places:

$$\int_0^x \cos\left(\tfrac{1}{2}\pi t^2\right) dt = 0.7$$

(d) Plot the graphs of C and S on the same screen. How are these graphs related?

71. Estimate the value of the number c such that the area under the curve $y = \sinh cx$ between $x = 0$ and $x = 1$ is equal to 1.

72. Suppose that the temperature in a long, thin rod placed along the x-axis is initially $C/(2a)$ if $|x| \leqslant a$ and 0 if $|x| > a$. It can be shown that if the heat diffusivity of the rod is k, then the temperature of the rod at the point x at time t is

$$T(x, t) = \frac{C}{a\sqrt{4\pi k t}} \int_0^a e^{-(x-u)^2/(4kt)}\, du$$

To find the temperature distribution that results from an initial hot spot concentrated at the origin, we need to compute $\lim_{a \to 0} T(x, t)$. Use l'Hospital's Rule to find this limit.

73. If f is a continuous function such that

$$\int_1^x f(t)\,dt = (x - 1)e^{2x} + \int_1^x e^{-t} f(t)\,dt$$

for all x, find an explicit formula for $f(x)$.

74. Suppose h is a function such that $h(1) = -2$, $h'(1) = 2$, $h''(1) = 3$, $h(2) = 6$, $h'(2) = 5$, $h''(2) = 13$, and h'' is continuous everywhere. Evaluate $\int_1^2 h''(u)\,du$.

75. If f' is continuous on $[a, b]$, show that

$$2\int_a^b f(x) f'(x)\,dx = [f(b)]^2 - [f(a)]^2$$

76. Find

$$\lim_{h \to 0} \frac{1}{h} \int_2^{2+h} \sqrt{1 + t^3}\, dt$$

77. If f is continuous on $[0, 1]$, prove that

$$\int_0^1 f(x)\,dx = \int_0^1 f(1 - x)\,dx$$

78. Evaluate

$$\lim_{n \to \infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \left(\frac{3}{n}\right)^9 + \cdots + \left(\frac{n}{n}\right)^9\right]$$

Problems Plus

Before you look at the solution of the following example, cover it up and first try to solve the problem yourself.

EXAMPLE Evaluate $\displaystyle\lim_{x\to 3}\left(\frac{x}{x-3}\int_3^x \frac{\sin t}{t}\,dt\right)$.

SOLUTION Let's start by having a preliminary look at the ingredients of the function. What happens to the first factor, $x/(x-3)$, when x approaches 3? The numerator approaches 3 and the denominator approaches 0, so we have

$$\frac{x}{x-3}\to\infty \quad\text{as}\quad x\to 3^+ \qquad\text{and}\qquad \frac{x}{x-3}\to-\infty \quad\text{as}\quad x\to 3^-$$

The second factor approaches $\int_3^3 (\sin t)/t\,dt$, which is 0. It's not clear what happens to the function as a whole. (One factor is becoming large while the other is becoming small.) So how do we proceed?

PS Review the Principles of Problem Solving following Chapter 1.

One of the principles of problem solving is *recognizing something familiar.* Is there a part of the function that reminds us of something we've seen before? Well, the integral

$$\int_3^x \frac{\sin t}{t}\,dt$$

has x as its upper limit of integration and that type of integral occurs in Part 1 of the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

This suggests that differentiation might be involved.

Once we start thinking about differentiation, the denominator $(x-3)$ reminds us of something else that should be familiar: one of the forms of the definition of the derivative in Chapter 2 is

$$F'(a)=\lim_{x\to a}\frac{F(x)-F(a)}{x-a}$$

and with $a=3$ this becomes

$$F'(3)=\lim_{x\to 3}\frac{F(x)-F(3)}{x-3}$$

So what is the function F in our situation? Notice that if we define

$$F(x)=\int_3^x \frac{\sin t}{t}\,dt$$

then $F(3)=0$. What about the factor x in the numerator? That's just a red herring, so let's factor it out and put together the calculation:

Another approach is to use l'Hospital's Rule.

$$\lim_{x\to 3}\left(\frac{x}{x-3}\int_3^x \frac{\sin t}{t}\,dt\right)=\lim_{x\to 3}x\cdot\lim_{x\to 3}\frac{\displaystyle\int_3^x \frac{\sin t}{t}\,dt}{x-3}=3\lim_{x\to 3}\frac{F(x)-F(3)}{x-3}$$

$$=3F'(3)=3\,\frac{\sin 3}{3}=\sin 3 \qquad \text{(FTC1)} \qquad ■$$

Problems

1. If $x \sin \pi x = \int_0^{x^2} f(t)\, dt$, where f is a continuous function, find $f(4)$.

2. Suppose f is continuous, $f(0) = 0$, $f(1) = 1$, $f'(x) > 0$, and $\int_0^1 f(x)\, dx = \frac{1}{3}$. Find the value of the integral $\int_0^1 f^{-1}(y)\, dy$.

3. If $\int_0^4 e^{(x-2)^4}\, dx = k$, find the value of $\int_0^4 xe^{(x-2)^4}\, dx$.

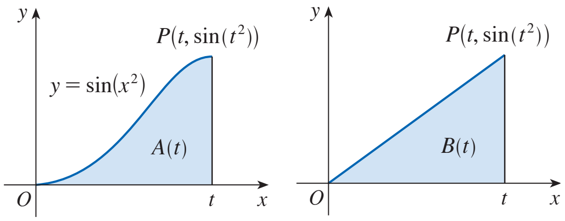
4. (a) Graph several members of the family of functions $f(x) = (2cx - x^2)/c^3$ for $c > 0$ and look at the regions enclosed by these curves and the x-axis. Make a conjecture about how the areas of these regions are related.
(b) Prove your conjecture in part (a).
(c) Take another look at the graphs in part (a) and use them to sketch the curve traced out by the vertices (highest points) of the family of functions. Can you guess what kind of curve this is?
(d) Find an equation of the curve you sketched in part (c).

5. If $f(x) = \int_0^{g(x)} \frac{1}{\sqrt{1 + t^3}}\, dt$, where $g(x) = \int_0^{\cos x} [1 + \sin(t^2)]\, dt$, find $f'(\pi/2)$.

6. If $f(x) = \int_0^x x^2 \sin(t^2)\, dt$, find $f'(x)$.

7. Evaluate $\lim_{x \to 0} (1/x) \int_0^x (1 - \tan 2t)^{1/t}\, dt$. [Assume that the integrand is defined and continuous at $t = 0$; see Exercise 5.3.82.]

8. The figure shows two regions in the first quadrant: $A(t)$ is the area under the curve $y = \sin(x^2)$ from 0 to t, and $B(t)$ is the area of the triangle with vertices O, P, and $(t, 0)$. Find $\lim\limits_{t \to 0^+} [A(t)/B(t)]$.



9. Find the interval $[a, b]$ for which the value of the integral $\int_a^b (2 + x - x^2)\, dx$ is a maximum.

10. Use an integral to estimate the sum $\sum_{i=1}^{10{,}000} \sqrt{i}$.

11. (a) Evaluate $\int_0^n [\![x]\!]\, dx$, where n is a positive integer.
(b) Evaluate $\int_a^b [\![x]\!]\, dx$, where a and b are real numbers with $0 \leqslant a < b$.

12. Find $\frac{d^2}{dx^2} \int_0^x \left(\int_1^{\sin t} \sqrt{1 + u^4}\, du \right) dt$.

13. Suppose the coefficients of the cubic polynomial $P(x) = a + bx + cx^2 + dx^3$ satisfy the equation

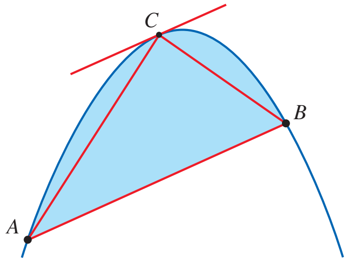
$$a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4} = 0$$

Show that the equation $P(x) = 0$ has a solution between 0 and 1. Can you generalize this result for an nth-degree polynomial?

14. A circular disk of radius r is used in an evaporator and is rotated in a vertical plane. If it is to be partially submerged in the liquid so as to maximize the exposed wetted area of the disk, show that the center of the disk should be positioned at a height $r/\sqrt{1 + \pi^2}$ above the surface of the liquid.

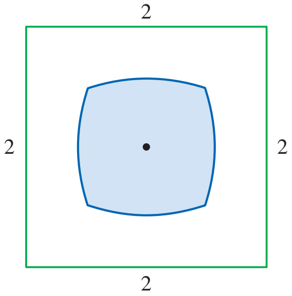
15. Prove that if f is continuous, then $\int_0^x f(u)(x-u)\,du = \int_0^x \left(\int_0^u f(t)\,dt\right) du$.

16. The figure shows a parabolic segment, that is, a portion of a parabola cut off by a chord AB. It also shows a point C on the parabola with the property that the tangent line at C is parallel to the chord AB. Archimedes proved that the area of the parabolic segment is $\frac{4}{3}$ times the area of the inscribed triangle ABC. Verify Archimedes' result for the parabola $y = 4 - x^2$ and the line $y = x + 2$.



17. Given the point (a, b) in the first quadrant, find the downward-opening parabola that passes through the point (a, b) and the origin such that the area under the parabola is a minimum.

18. The figure shows a region consisting of all points inside a square that are closer to the center than to the sides of the square. Find the area of the region.



19. Evaluate

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\,\sqrt{n+1}} + \frac{1}{\sqrt{n}\,\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}\,\sqrt{n+n}}\right)$$

20. For any number c, we let $f_c(x)$ be the smaller of the two numbers $(x-c)^2$ and $(x-c-2)^2$. Then we define $g(c) = \int_0^1 f_c(x)\,dx$. Find the maximum and minimum values of $g(c)$ if $-2 \leqslant c \leqslant 2$.

Rotation is used in many manufacturing processes. The photo shows an artist throwing a clay pot on a rotating potter's wheel. In Exercise 6.2.87 we explore the mathematics of designing a terra-cotta pot.

Rock and Wasp / Shutterstock.com

6

Applications of Integration

IN THIS CHAPTER WE EXPLORE some of the applications of the definite integral by using it to compute areas between curves, volumes of solids, and the work done by a varying force. The common theme is the following general method, which is similar to the one we used to find areas under curves: We break up a quantity Q into a large number of small parts. We next approximate each small part by a quantity of the form $f(x_i^*)\,\Delta x$ and thus approximate Q by a Riemann sum. Then we take the limit and express Q as an integral. Finally we evaluate the integral using the Fundamental Theorem of Calculus or the Midpoint Rule.

6.1 Areas Between Curves

In Chapter 5 we defined and calculated areas of regions that lie under the graphs of functions. Here we use integrals to find areas of regions that lie between the graphs of two functions.

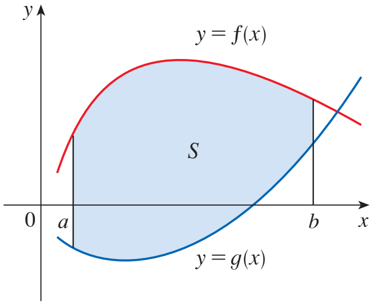


FIGURE 1
$S = \{(x, y) \mid a \leqslant x \leqslant b,$
$g(x) \leqslant y \leqslant f(x)\}$

Area Between Curves: Integrating With Respect to *x*

Consider the region S shown in Figure 1 that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geqslant g(x)$ for all x in $[a, b]$.

Just as we did for areas under curves in Section 5.1, we divide S into n strips of equal width and then we approximate the ith strip by a rectangle with base Δx and height $f(x_i^*) - g(x_i^*)$. (See Figure 2. If we like, we could take all of the sample points to be right endpoints, in which case $x_i^* = x_i$.) The Riemann sum

$$\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x$$

is therefore an approximation to what we intuitively think of as the area of S.

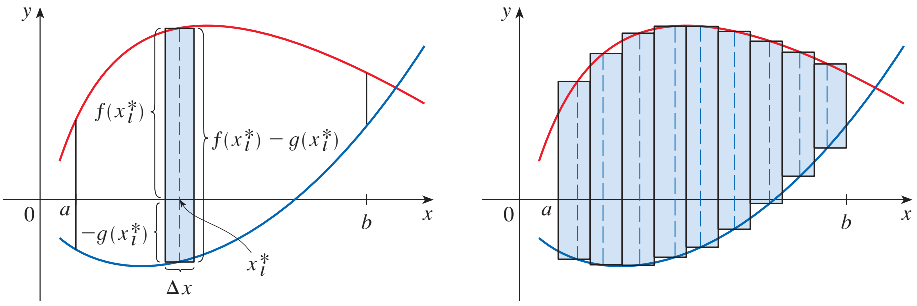


FIGURE 2

(a) Typical rectangle (b) Approximating rectangles

This approximation appears to become better and better as $n \to \infty$. Therefore we define the **area** A of the region S as the limiting value of the sum of the areas of these approximating rectangles.

1
$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x$$

We recognize the limit in (1) as the definite integral of $f - g$. Therefore we have the following formula for area.

2 The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geqslant g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

Notice that in the special case where $g(x) = 0$, S is the region under the graph of f and our general definition of area (1) reduces to our previous definition (Definition 5.1.2).

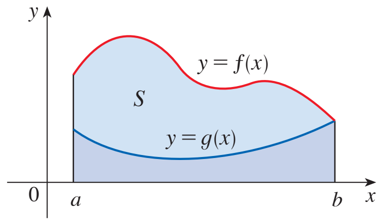


FIGURE 3

$A = \int_a^b f(x)dx - \int_a^b g(x)dx$

In the case where both f and g are positive, you can see from Figure 3 why (2) is true:

$$\begin{aligned} A &= [\text{area under } y = f(x)] - [\text{area under } y = g(x)] \\ &= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx \end{aligned}$$

EXAMPLE 1 Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

SOLUTION The region is shown in Figure 4. The upper boundary curve is $y = e^x$ and the lower boundary curve is $y = x$. So we use the area formula (2) with $f(x) = e^x$, $g(x) = x$, $a = 0$, and $b = 1$:

$$\begin{aligned} A &= \int_0^1 (e^x - x)\,dx = e^x - \tfrac{1}{2}x^2\Big]_0^1 \\ &= e - \tfrac{1}{2} - 1 = e - 1.5 \end{aligned}$$

■

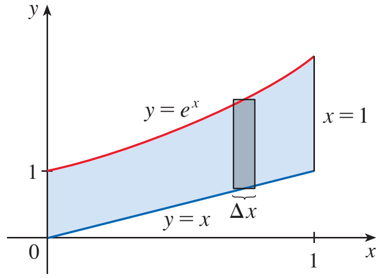


FIGURE 4

In Figure 4 we drew a typical approximating rectangle with width Δx as a reminder of the procedure by which the area is defined in (1). In general, when we set up an integral for an area, it's helpful to sketch the region to identify the top curve y_T, the bottom curve y_B, and a typical approximating rectangle as in Figure 5. Then the area of a typical rectangle is $(y_T - y_B)\,\Delta x$ and the equation

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} (y_T - y_B)\,\Delta x = \int_a^b (y_T - y_B)\,dx$$

summarizes the procedure of adding (in a limiting sense) the areas of all the typical rectangles.

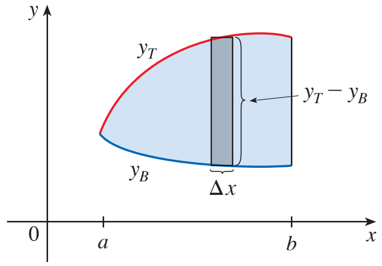


FIGURE 5

Notice that in Figure 5 the left-hand boundary reduces to a point, whereas in Figure 3 the right-hand boundary reduces to a point. In the next example both of the side boundaries reduce to a point, so the first step is to find a and b.

EXAMPLE 2 Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

SOLUTION We first find the points of intersection of the parabolas by solving their equations simultaneously. This gives $x^2 = 2x - x^2$, or $2x^2 - 2x = 0$. Thus $2x(x - 1) = 0$, so $x = 0$ or 1. The points of intersection are $(0, 0)$ and $(1, 1)$.

We see from Figure 6 that the top and bottom boundaries are

$$y_T = 2x - x^2 \qquad \text{and} \qquad y_B = x^2$$

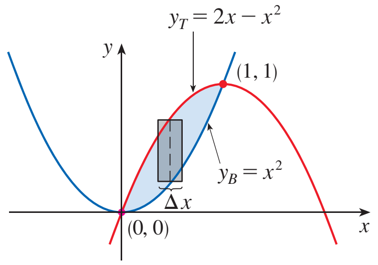


FIGURE 6

The area of a typical rectangle is

$$(y_T - y_B)\,\Delta x = (2x - x^2 - x^2)\,\Delta x = (2x - 2x^2)\,\Delta x$$

and the region lies between $x = 0$ and $x = 1$. So the total area is

$$\begin{aligned} A &= \int_0^1 (2x - 2x^2)\,dx = 2\int_0^1 (x - x^2)\,dx \\ &= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3} \end{aligned}$$

■

Sometimes it's difficult, or even impossible, to find the points of intersection of two curves exactly. As shown in the following example, we can use a graphing calculator or computer to find approximate values for the intersection points and then proceed as before.

EXAMPLE 3 Find the approximate area of the region bounded by the curves $y = x/\sqrt{x^2+1}$ and $y = x^4 - x$.

SOLUTION If we were to try to find the exact intersection points, we would have to solve the equation

$$\frac{x}{\sqrt{x^2+1}} = x^4 - x$$

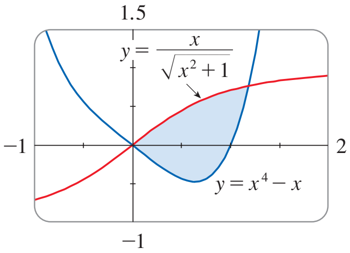


FIGURE 7

This looks like a very difficult equation to solve exactly (in fact, it's impossible), so instead we graph the two curves using a computer (see Figure 7). One intersection point is the origin, and we find that the other occurs when $x \approx 1.18$. So an approximation to the area between the curves is

$$A \approx \int_0^{1.18} \left[\frac{x}{\sqrt{x^2+1}} - (x^4 - x) \right] dx$$

To integrate the first term we use the substitution $u = x^2 + 1$. Then $du = 2x\,dx$, and when $x = 1.18$, we have $u \approx 2.39$; when $x = 0$, $u = 1$. So

$$\begin{aligned} A &\approx \tfrac{1}{2} \int_1^{2.39} \frac{du}{\sqrt{u}} - \int_0^{1.18} (x^4 - x)\,dx \\ &= \sqrt{u}\,\Big]_1^{2.39} - \left[\frac{x^5}{5} - \frac{x^2}{2} \right]_0^{1.18} \\ &= \sqrt{2.39} - 1 - \frac{(1.18)^5}{5} + \frac{(1.18)^2}{2} \\ &\approx 0.785 \end{aligned}$$

■

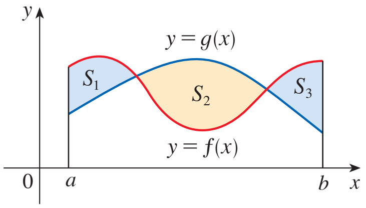


FIGURE 8

If we are asked to find the area between the curves $y = f(x)$ and $y = g(x)$ where $f(x) \geqslant g(x)$ for some values of x but $g(x) \geqslant f(x)$ for other values of x, then we split the given region S into several regions $S_1, S_2, \ldots$ with areas $A_1, A_2, \ldots$ as shown in Figure 8. We then define the area of the region S to be the sum of the areas of the smaller regions $S_1, S_2, \ldots$, that is, $A = A_1 + A_2 + \cdots$. Since

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{when } f(x) \geqslant g(x) \\ g(x) - f(x) & \text{when } g(x) \geqslant f(x) \end{cases}$$

we have the following expression for A.

3 The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)|\,dx$$

When evaluating the integral in (3), however, we must still split it into integrals corresponding to $A_1, A_2, \ldots$.

EXAMPLE 4 Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

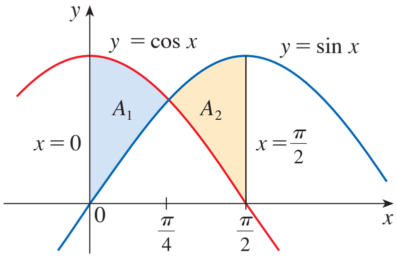


FIGURE 9

SOLUTION The points of intersection occur when $\sin x = \cos x$, that is, when $x = \pi/4$ (since $0 \leqslant x \leqslant \pi/2$). The region is sketched in Figure 9.

Observe that $\cos x \geqslant \sin x$ when $0 \leqslant x \leqslant \pi/4$ but $\sin x \geqslant \cos x$ when $\pi/4 \leqslant x \leqslant \pi/2$. Therefore the required area is

$$\begin{aligned} A &= \int_0^{\pi/2} |\cos x - \sin x|\, dx = A_1 + A_2 \\ &= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx \\ &= \Big[\sin x + \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2} \\ &= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) \\ &= 2\sqrt{2} - 2 \end{aligned}$$

In this particular example we could have saved some work by noticing that the region is symmetric about $x = \pi/4$ and so

$$A = 2A_1 = 2\int_0^{\pi/4} (\cos x - \sin x)\, dx$$ ■

■ Area Between Curves: Integrating With Respect to *y*

Some regions are best treated by regarding x as a function of y. If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where f and g are continuous and $f(y) \geqslant g(y)$ for $c \leqslant y \leqslant d$ (see Figure 10), then its area is

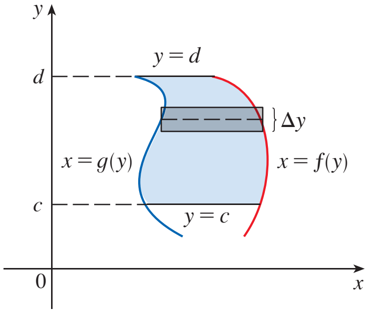


FIGURE 10

$$A = \int_c^d [f(y) - g(y)]\, dy$$

If we write x_R for the right boundary and x_L for the left boundary, then, as Figure 11 illustrates, we have

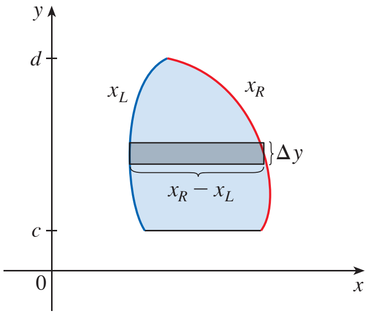


FIGURE 11

$$A = \int_c^d (x_R - x_L)\, dy$$

Here a typical approximating rectangle has dimensions $x_R - x_L$ and Δy.

EXAMPLE 5 Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

SOLUTION By solving the two equations simultaneously we find that the points of intersection are $(-1, -2)$ and $(5, 4)$. We solve the equation of the parabola for x and

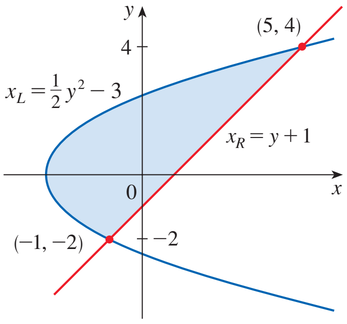


FIGURE 12

notice from Figure 12 that the left and right boundary curves are

$$x_L = \tfrac{1}{2}y^2 - 3 \qquad \text{and} \qquad x_R = y + 1$$

We must integrate between the appropriate y-values, $y = -2$ and $y = 4$. Thus

$$\begin{aligned} A &= \int_{-2}^{4} (x_R - x_L)\,dy = \int_{-2}^{4} \left[(y + 1) - \left(\tfrac{1}{2}y^2 - 3\right)\right] dy \\ &= \int_{-2}^{4} \left(-\tfrac{1}{2}y^2 + y + 4\right) dy \\ &= -\frac{1}{2}\left(\frac{y^3}{3}\right) + \frac{y^2}{2} + 4y \Bigg]_{-2}^{4} \\ &= -\tfrac{1}{6}(64) + 8 + 16 - \left(\tfrac{4}{3} + 2 - 8\right) = 18 \end{aligned}$$

■

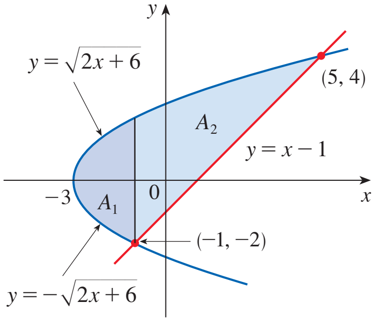


FIGURE 13

NOTE We could have found the area in Example 5 by integrating with respect to x instead of y, but the calculation is much more involved. Because the bottom boundary consists of two different curves, it would have meant splitting the region in two and computing the areas labeled A_1 and A_2 in Figure 13. The method we used in Example 5 is much easier.

EXAMPLE 6 Find the area of the region enclosed by the curves $y = 1/x$, $y = x$, and $y = \frac{1}{4}x$, using (a) x as the variable of integration and (b) y as the variable of integration.

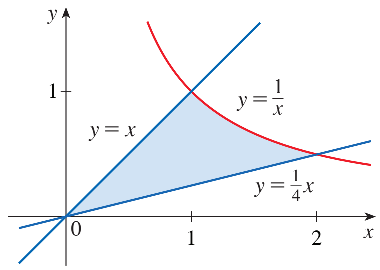


FIGURE 14

SOLUTION The region is graphed in Figure 14.

(a) If we integrate with respect to x, we must split the region into two parts because the top boundary consists of two separate curves, as shown in Figure 15(a). We compute the area as

$$\begin{aligned} A = A_1 + A_2 &= \int_0^1 \left(x - \tfrac{1}{4}x\right) dx + \int_1^2 \left(\frac{1}{x} - \frac{1}{4}x\right) dx \\ &= \left[\tfrac{3}{8}x^2\right]_0^1 + \left[\ln x - \tfrac{1}{8}x^2\right]_1^2 = \ln 2 \end{aligned}$$

(b) If we integrate with respect to y, we also need to divide the region into two parts because the right boundary consists of two separate curves, as shown in Figure 15(b). We compute the area as

$$\begin{aligned} A = A_1 + A_2 &= \int_0^{1/2} (4y - y)\,dy + \int_{1/2}^{1} \left(\frac{1}{y} - y\right) dy \\ &= \left[\tfrac{3}{2}y^2\right]_0^{1/2} + \left[\ln y - \tfrac{1}{2}y^2\right]_{1/2}^{1} = \ln 2 \end{aligned}$$

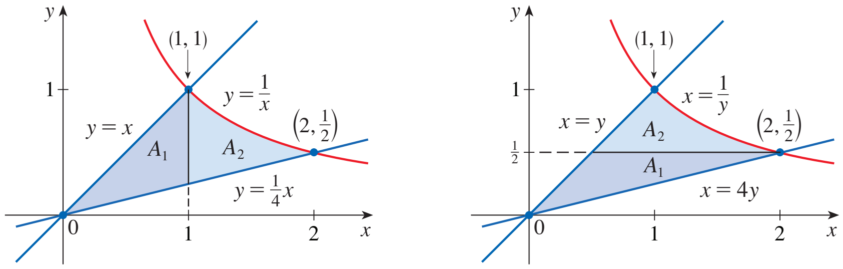


FIGURE 15

(a) (b)

■

Applications

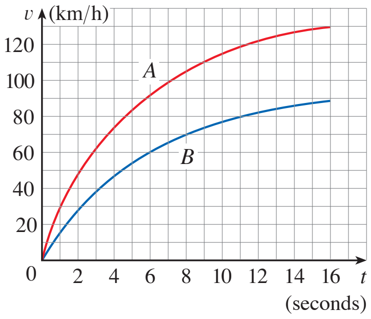


FIGURE 16

EXAMPLE 7 Figure 16 shows velocity curves for two cars, A and B, that start side by side and move along the same road. What does the area between the curves represent? Use the Midpoint Rule to estimate it.

SOLUTION We know from Section 5.4 that the area under the velocity curve A represents the distance traveled by car A during the first 16 seconds. Similarly, the area under curve B is the distance traveled by car B during that time period. So the area between these curves, which is the difference of the areas under the curves, is the distance between the cars after 16 seconds. We read the velocities from the graph and convert them to meters per second ($1 \text{ km/h} = \frac{1000}{3600} \text{ m/s}$).

t	0	2	4	6	8	10	12	14	16
v_A	0	10.4	16.5	20.4	23.2	25.6	27.1	28.0	29.0
v_B	0	6.4	10.4	13.4	15.5	17.1	18.3	19.2	19.8
$v_A - v_B$	0	4.0	6.1	7.0	7.7	8.5	8.8	8.8	9.2

We use the Midpoint Rule with $n = 4$ intervals, so that $\Delta t = 4$. The midpoints of the intervals are $\bar{t}_1 = 2$, $\bar{t}_2 = 6$, $\bar{t}_3 = 10$, and $\bar{t}_4 = 14$. We estimate the distance between the cars after 16 seconds as follows:

$$\int_0^{16} (v_A - v_B)\,dt \approx \Delta t\,[4.0 + 7.0 + 8.5 + 8.8]$$

$$= 4(28.3) = 113.2 \text{ m}$$

■

EXAMPLE 8 Figure 17 is an example of a *pathogenesis curve* for a measles infection. It shows how the disease develops in an individual with no immunity after the measles virus spreads to the bloodstream from the respiratory tract.

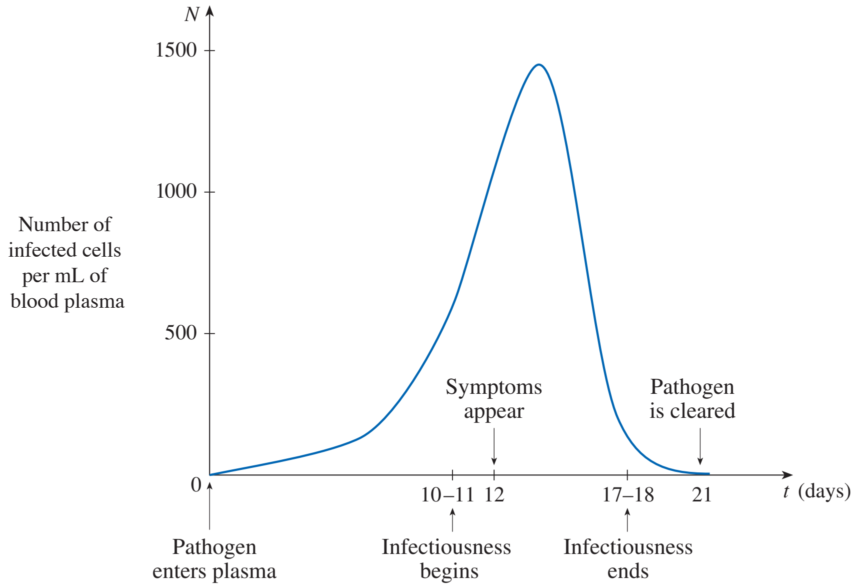


FIGURE 17

Measles pathogenesis curve

Source: J. M. Heffernan et al., "An In-Host Model of Acute Infection: Measles as a Case Study," *Theoretical Population Biology* 73 (2008): 134–47.

The patient becomes infectious to others once the concentration of infected cells becomes great enough and remains infectious until the immune system manages to prevent further transmission. However, symptoms don't develop until the "amount of infection" reaches a particular threshold. The amount of infection needed to develop symptoms depends on both the concentration of infected cells and time, and corresponds to the area under the pathogenesis curve until symptoms appear. (See Exercise 5.1.15.)

(a) The pathogenesis curve in Figure 17 has been modeled by $f(t) = -t(t - 21)(t + 1)$. If infectiousness begins on day $t_1 = 10$ and ends on day $t_2 = 18$, what are the corresponding concentration levels of infected cells?

(b) The *level of infectiousness* for an infected person is the area between $N = f(t)$ and the line through the points $P_1(t_1, f(t_1))$ and $P_2(t_2, f(t_2))$, measured in (cells/mL) · days. (See Figure 18.) Compute the level of infectiousness for this particular patient.

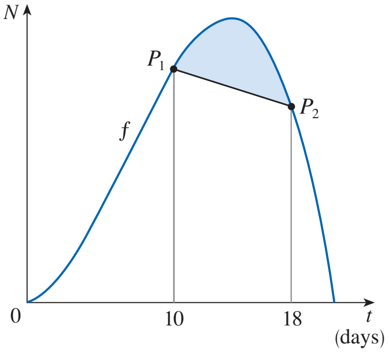


FIGURE 18

SOLUTION

(a) Infectiousness begins when the concentration reaches $f(10) = 1210$ cells/mL and ends when the concentration reduces to $f(18) = 1026$ cells/mL.

(b) The line through P_1 and P_2 has slope $\frac{1026 - 1210}{18 - 10} = -\frac{184}{8} = -23$ and equation $N - 1210 = -23(t - 10) \iff N = -23t + 1440$. The area between f and this line is

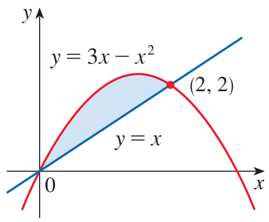
$$\begin{aligned}\int_{10}^{18} [f(t) - (-23t + 1440)]\,dt &= \int_{10}^{18} (-t^3 + 20t^2 + 21t + 23t - 1440)\,dt\\ &= \int_{10}^{18} (-t^3 + 20t^2 + 44t - 1440)\,dt\\ &= \left[-\frac{t^4}{4} + 20\,\frac{t^3}{3} + 44\,\frac{t^2}{2} - 1440t\right]_{10}^{18}\\ &= -6156 - \left(-8033\tfrac{1}{3}\right) \approx 1877\end{aligned}$$

Thus the level of infectiousness for this patient is about 1877 (cells/mL) · days. ■

6.1 Exercises

1–4
(a) Set up an integral for the area of the shaded region.
(b) Evaluate the integral to find the area.

1.



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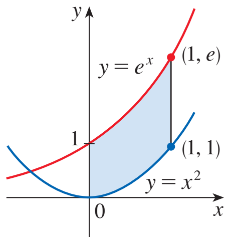

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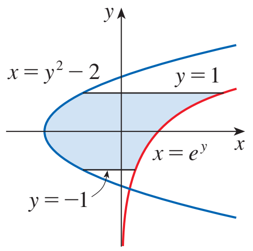

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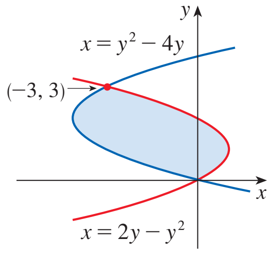
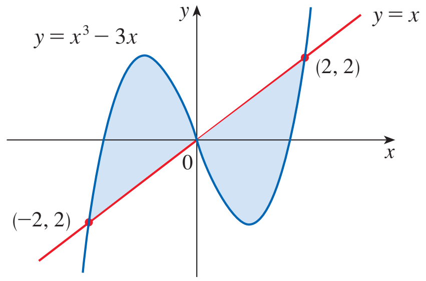
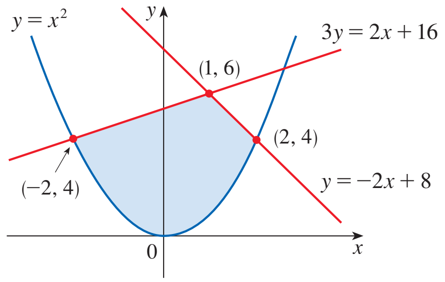


5–6 Find the area of the shaded region.

5.



6.



7–10 Set up, but do not evaluate, an integral representing the area of the region enclosed by the given curves.

7. $y = 2^x, \quad y = 3^x, \quad x = 1$

8. $y = \ln x, \quad y = \ln(x^2), \quad x = 2$

9. $y = 2 - x, \quad y = 2x - x^2$

10. $x = y^4, \quad x = 2 - y^2$

11–18 Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Draw a typical approximating rectangle and label its height and width. Then find the area of the region.

11. $y = x^2 + 2, \quad y = -x - 1, \quad x = 0, \quad x = 1$

12. $y = 1 + x^3, \quad y = 2 - x, \quad x = -1, \quad x = 0$

13. $y = 1/x, \quad y = 1/x^2, \quad x = 2$

14. $y = \cos x, \quad y = e^x, \quad x = \pi/2$

15. $y = (x - 2)^2, \quad y = x$

16. $y = x^2 - 4x, \quad y = 2x$

17. $x = 1 - y^2, \quad x = y^2 - 1$

18. $4x + y^2 = 12, \quad x = y$

19–36 Sketch the region enclosed by the given curves and find its area.

19. $y = 12 - x^2, \quad y = x^2 - 6$

20. $y = x^2, \quad y = 4x - x^2$

21. $x = 2y^2, \quad x = 4 + y^2$

22. $y = \sqrt{x - 1}, \quad x - y = 1$

23. $y = \sqrt[3]{2x}, \quad y = \frac{1}{2}x$

24. $y = x^3, \quad y = x$

25. $y = \sqrt{x}, \quad y = \frac{1}{3}x, \quad 0 \leqslant x \leqslant 16$

26. $y = \cos x, \quad y = 2 - \cos x, \quad 0 \leqslant x \leqslant 2\pi$

27. $y = \cos x, \quad y = \sin 2x, \quad 0 \leqslant x \leqslant \pi/2$

28. $y = \cos x, \quad y = 1 - \cos x, \quad 0 \leqslant x \leqslant \pi$

29. $y = \sec^2 x, \quad y = 8 \cos x, \quad -\pi/3 \leqslant x \leqslant \pi/3$

30. $y = x^4 - 3x^2, \quad y = x^2$

31. $y = x^4, \quad y = 2 - |x|$

32. $y = x^2, \quad y = \dfrac{32}{x^2 + 4}$

33. $y = \sin \dfrac{\pi x}{2}, \quad y = x^3$

34. $y = 4 - 2 \cosh x, \quad y = \frac{1}{2} \sinh x$

35. $y = 1/x, \quad y = x, \quad y = \frac{1}{4}x, \quad x > 0$

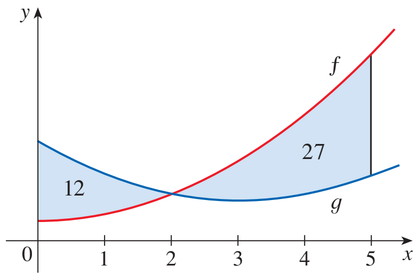
36. $y = \frac{1}{4}x^2, \quad y = 2x^2, \quad x + y = 3, \quad x \geqslant 0$

---

37. The graphs of two functions are shown with the areas of the regions between the curves indicated.

(a) What is the total area between the curves for $0 \leqslant x \leqslant 5$?

(b) What is the value of $\int_0^5 [f(x) - g(x)]\, dx$?



38–40 Sketch the region enclosed by the given curves and find its area.

38. $y = \dfrac{x}{\sqrt{1 + x^2}}, \quad y = \dfrac{x}{\sqrt{9 - x^2}}, \quad x \geqslant 0$

39. $y = \dfrac{x}{1 + x^2}, \quad y = \dfrac{x^2}{1 + x^3}$

40. $y = \dfrac{\ln x}{x}, \quad y = \dfrac{(\ln x)^2}{x}$

41–42 Use calculus to find the area of the triangle with the given vertices.

41. $(0, 0), \quad (3, 1), \quad (1, 2)$

42. $(2, 0), \quad (0, 2), \quad (-1, 1)$

43–44 Evaluate the integral and interpret it as the area of a region. Sketch the region.

43. $\displaystyle\int_0^{\pi/2} |\sin x - \cos 2x|\, dx$

44. $\displaystyle\int_{-1}^{1} |3^x - 2^x|\, dx$

45–48 Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then find (approximately) the area of the region bounded by the curves.

45. $y = x \sin(x^2), \quad y = x^4, \quad x \geqslant 0$

46. $y = \dfrac{x}{(x^2 + 1)^2}, \quad y = x^5 - x, \quad x \geqslant 0$

47. $y = 3x^2 - 2x, \quad y = x^3 - 3x + 4$

48. $y = 1.3^x, \quad y = 2\sqrt{x}$

T **49–52** Graph the region between the curves and compute the area correct to five decimal places.

49. $y = \dfrac{2}{1 + x^4}, \quad y = x^2$

50. $y = e^{1 - x^2}, \quad y = x^4$

51. $y = \tan^2 x, \quad y = \sqrt{x}$

52. $y = \cos x, \quad y = x + 2\sin^4 x$

---

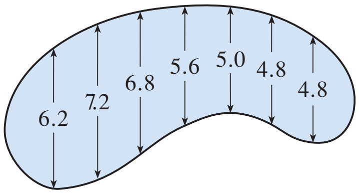
53. Use a computer algebra system to find the exact area enclosed by the curves $y = x^5 - 6x^3 + 4x$ and $y = x$.

54. Sketch the region in the xy-plane defined by the inequalities $x - 2y^2 \geqslant 0$, $1 - x - |y| \geqslant 0$ and find its area.

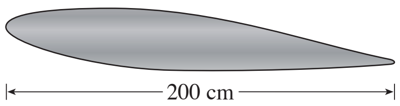
55. Racing cars driven by Chris and Kelly are side by side at the start of a race. The table shows the velocities of each car (in kilometers per hour) during the first ten seconds of the race. Use the Midpoint Rule to estimate how much farther Kelly travels than Chris does during the first ten seconds.

t	v_C	v_K	t	v_C	v_K
0	0	0	6	110	128
1	32	35	7	120	138
2	51	59	8	130	150
3	74	83	9	138	157
4	86	98	10	144	163
5	99	114			

56. The widths (in meters) of a kidney-shaped swimming pool were measured at 2-meter intervals as indicated in the figure. Use the Midpoint Rule to estimate the area of the pool.



57. A cross-section of an airplane wing is shown. Measurements of the thickness of the wing, in centimeters, at 20-centimeter intervals are 5.8, 20.3, 26.7, 29.0, 27.6, 27.3, 23.8, 20.5, 15.1, 8.7, and 2.8. Use the Midpoint Rule to estimate the area of the wing's cross-section.



58. If the birth rate of a population is $b(t) = 2200e^{0.024t}$ people per year and the death rate is $d(t) = 1460e^{0.018t}$ people per year, find the area between these curves for $0 \leqslant t \leqslant 10$. What does this area represent?

59. In Example 8, we modeled a measles pathogenesis curve by a function f. A patient infected with the measles virus who has some immunity to the virus has a pathogenesis curve that can be modeled by, for instance, $g(t) = 0.9f(t)$.

(a) If the same threshold concentration of the virus is required for infectiousness to begin as in Example 8, on what day does this occur?

(b) Let P_3 be the point on the graph of g where infectiousness begins. It has been shown that infectiousness ends at a point P_4 on the graph of g where the line through P_3, P_4 has the same slope as the line through P_1, P_2 in Example 8(b). On what day does infectiousness end?

(c) Compute the level of infectiousness for this patient.

60. The rates at which rain fell, in inches per hour, in two different locations t hours after the start of a storm were modeled by $f(t) = 0.73t^3 - 2t^2 + t + 0.6$ and $g(t) = 0.17t^2 - 0.5t + 1.1$. Compute the area between the graphs for $0 \leqslant t \leqslant 2$ and interpret your result in this context.

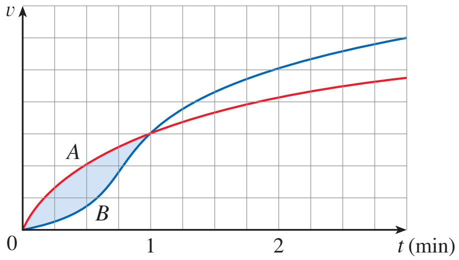
61. Two cars, A and B, start side by side and accelerate from rest. The figure shows the graphs of their velocity functions.

(a) Which car is ahead after one minute? Explain.

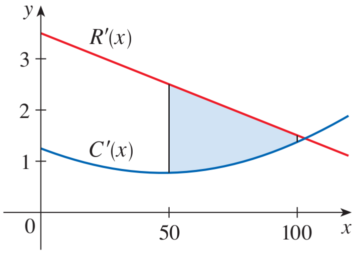
(b) What is the meaning of the area of the shaded region?

(c) Which car is ahead after two minutes? Explain.

(d) Estimate the time at which the cars are again side by side.



62. The figure shows graphs of the marginal revenue function R' and the marginal cost function C' for a manufacturer. [Recall from Section 4.7 that $R(x)$ and $C(x)$ represent the revenue and cost when x units are manufactured. Assume that R and C are measured in thousands of dollars.] What is the meaning of the area of the shaded region? Use the Midpoint Rule to estimate the value of this quantity.



63. The curve with equation $y^2 = x^2(x + 3)$ is called **Tschirnhausen's cubic**. If you graph this curve you will see that part of the curve forms a loop. Find the area enclosed by the loop.

64. Find the area of the region bounded by the parabola $y = x^2$, the tangent line to this parabola at $(1, 1)$, and the x-axis.

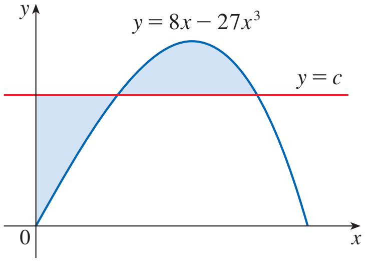
65. Find the number b such that the line $y = b$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two regions with equal area.

66. (a) Find the number a such that the line $x = a$ bisects the area under the curve $y = 1/x^2$, $1 \leqslant x \leqslant 4$.
(b) Find the number b such that the line $y = b$ bisects the area in part (a).

67. Find the values of c such that the area of the region bounded by the parabolas $y = x^2 - c^2$ and $y = c^2 - x^2$ is 576.

68. Suppose that $0 < c < \pi/2$. For what value of c is the area of the region enclosed by the curves $y = \cos x$, $y = \cos(x - c)$, and $x = 0$ equal to the area of the region enclosed by the curves $y = \cos(x - c)$, $x = \pi$, and $y = 0$?

69. The figure shows a horizontal line $y = c$ intersecting the curve $y = 8x - 27x^3$. Find the number c such that the areas of the shaded regions are equal.



70. For what values of m do the line $y = mx$ and the curve $y = x/(x^2 + 1)$ enclose a region? Find the area of the region.

APPLIED PROJECT | THE GINI INDEX

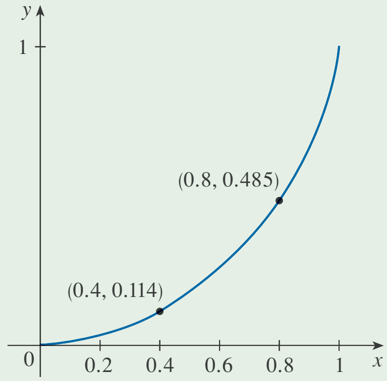


FIGURE 1
Lorenz curve for the United States in 2016

How is it possible to measure the distribution of income among the inhabitants of a given country? One such measure is the *Gini index*, named after the Italian economist Corrado Gini, who first devised it in 1912.

We first rank all households in a country by income and then we compute the percentage of households whose total income is a given percentage of the country's total income. We define a **Lorenz curve** $y = L(x)$ on the interval $[0, 1]$ by plotting the point $(a/100, b/100)$ on the curve if the bottom $a\%$ of households receive $b\%$ of the total income. For instance, in Figure 1 the point (0.4, 0.114) is on the Lorenz curve for the United States in 2016 because the poorest 40% of the population received just 11.4% of the total income. Likewise, the bottom 80% of the population received 48.5% of the total income, so the point (0.8, 0.485) lies on the Lorenz curve. (The Lorenz curve is named after the American economist Max Lorenz.)

Figure 2 shows some typical Lorenz curves. They all pass through the points $(0, 0)$ and $(1, 1)$ and are concave upward. In the extreme case $L(x) = x$, society is perfectly egalitarian: the poorest $a\%$ of the population receives $a\%$ of the total income and so everybody receives the same income. The area between a Lorenz curve $y = L(x)$ and the line $y = x$ measures how much the income distribution differs from absolute equality. The **Gini index** (sometimes called the **Gini coefficient** or the **coefficient of inequality**) is the area between the Lorenz curve and the line $y = x$ (shaded in Figure 3) divided by the area under $y = x$.

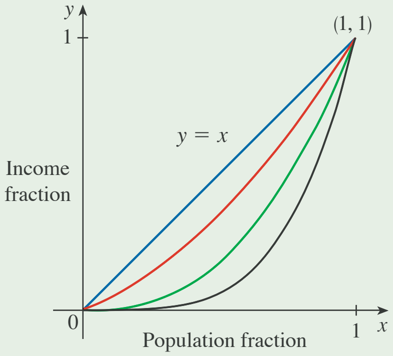


FIGURE 2

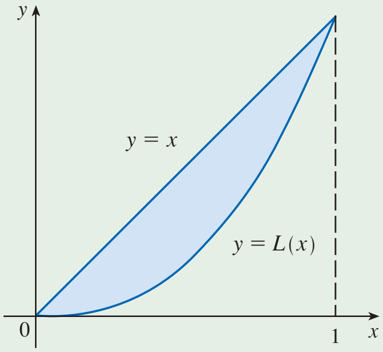


FIGURE 3

(*continued*)

1. (a) Show that the Gini index G is twice the area between the Lorenz curve and the line $y = x$, that is,

$$G = 2 \int_0^1 [x - L(x)]\, dx$$

(b) What is the value of G for a perfectly egalitarian society (everybody has the same income)? What is the value of G for a perfectly totalitarian society (a single person receives all the income)?

2. The following table (derived from data supplied by the US Census Bureau) shows values of the Lorenz function for income distribution in the United States for the year 2016.

x	0.0	0.2	0.4	0.6	0.8	1.0
$L(x)$	0.000	0.031	0.114	0.256	0.485	1.000

(a) What percentage of the total US income was received by the richest 20% of the population in 2016?

T (b) Use a calculator or computer to fit a quadratic function to the data in the table. Graph the data points and the quadratic function. Is the quadratic model a reasonable fit?

(c) Use the quadratic model for the Lorenz function to estimate the Gini index for the United States in 2016.

3. The following table gives values for the Lorenz function in the years 1980, 1990, 2000, and 2010. Use the method of Problem 2 to estimate the Gini index for the United States for those years and compare with your answer to Problem 2(c). Do you notice a trend?

x	0.0	0.2	0.4	0.6	0.8	1.0
1980	0.000	0.042	0.144	0.312	0.559	1.000
1990	0.000	0.038	0.134	0.293	0.533	1.000
2000	0.000	0.036	0.125	0.273	0.503	1.000
2010	0.000	0.033	0.118	0.264	0.498	1.000

T **4.** A power model often provides a more accurate fit than a quadratic model for a Lorenz function. Use a calculator or computer to fit a power function ($y = ax^k$) to the data in Problem 2 and use it to estimate the Gini index for the United States in 2016. Compare with your answer to parts (b) and (c) of Problem 2.

6.2 Volumes

In trying to find the volume of a solid we face the same type of problem as in finding areas. We have an intuitive idea of what volume means, but we must make this idea precise by using calculus to give an exact definition of volume.

Definition of Volume

We start with a simple type of solid called a **cylinder** (or, more precisely, a *right cylinder*). As illustrated in Figure 1(a), a cylinder is bounded by a plane region B_1, called the

base, and a congruent region B_2 in a parallel plane. The cylinder consists of all points on line segments that are perpendicular to the base and join B_1 to B_2. If the area of the base is A and the height of the cylinder (the distance from B_1 to B_2) is h, then the volume V of the cylinder is defined as

$$V = Ah$$

In particular, if the base is a circle with radius r, then the cylinder is a circular cylinder with volume $V = \pi r^2 h$ [see Figure 1(b)], and if the base is a rectangle with length l and width w, then the cylinder is a rectangular box (also called a *rectangular parallelepiped*) with volume $V = lwh$ [see Figure 1(c)].

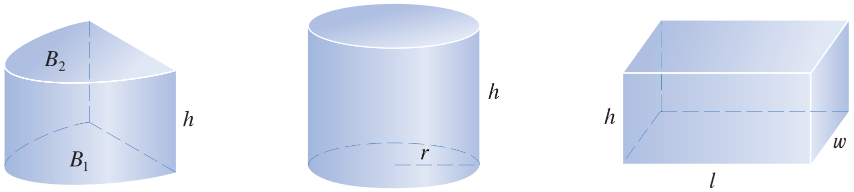


FIGURE 1 (a) Cylinder $V = Ah$ (b) Circular cylinder $V = \pi r^2 h$ (c) Rectangular box $V = lwh$

For a solid S that isn't a cylinder we first "cut" S into pieces and approximate each piece by a cylinder. We estimate the volume of S by adding the volumes of the cylinders. We arrive at the exact volume of S through a limiting process in which the number of pieces becomes large.

We start by intersecting S with a plane and obtaining a plane region that is called a **cross-section** of S. Let $A(x)$ be the area of the cross-section of S in a plane P_x perpendicular to the x-axis and passing through the point x, where $a \leqslant x \leqslant b$. (See Figure 2. Think of slicing S with a knife through x and computing the area of this slice.) The cross-sectional area $A(x)$ will vary as x increases from a to b.

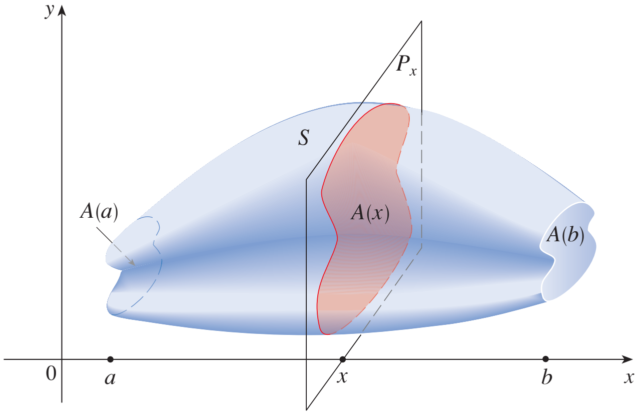


FIGURE 2

Let's divide S into n "slabs" of equal width Δx by using the planes $P_{x_1}, P_{x_2}, \ldots$ to slice the solid. (Think of slicing a loaf of bread.) If we choose sample points x_i^* in $[x_{i-1}, x_i]$, we can approximate the ith slab S_i (the part of S that lies between the planes $P_{x_{i-1}}$ and P_{x_i}) by a cylinder with base area $A(x_i^*)$ and "height" Δx. (See Figure 3.)

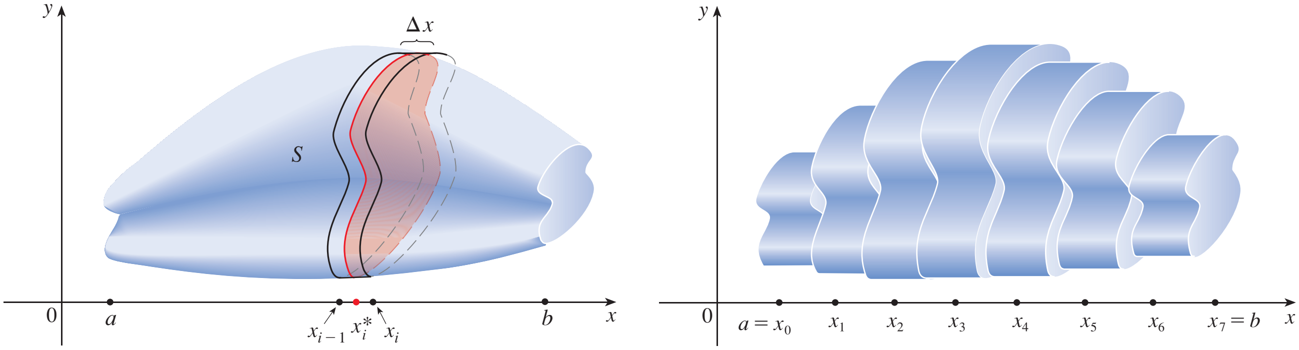


FIGURE 3

The volume of this cylinder is $A(x_i^*)\,\Delta x$, so an approximation to our intuitive conception of the volume of the ith slab S_i is

$$V(S_i) \approx A(x_i^*)\,\Delta x$$

Adding the volumes of these slabs, we get an approximation to the total volume (that is, what we think of intuitively as the volume):

$$V \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x$$

This approximation appears to become better and better as $n \to \infty$. (Think of the slices as becoming thinner and thinner.) Therefore we *define* the volume as the limit of these sums as $n \to \infty$. But we recognize the limit of Riemann sums as a definite integral and so we have the following definition.

It can be proved that this definition is independent of how S is situated with respect to the x-axis. In other words, no matter how we slice S with parallel planes, we always get the same answer for V.

Definition of Volume Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the **volume** of S is

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$

When we use the volume formula $V = \int_a^b A(x)\,dx$, it is important to remember that $A(x)$ is the area of a moving cross-section obtained by slicing through x perpendicular to the x-axis.

Notice that, for a cylinder, the cross-sectional area is constant: $A(x) = A$ for all x. So our definition of volume gives $V = \int_a^b A\,dx = A(b - a)$; this agrees with the formula $V = Ah$.

EXAMPLE 1 Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

SOLUTION If we place the sphere so that its center is at the origin, then the plane P_x intersects the sphere in a circle whose radius (from the Pythagorean Theorem) is

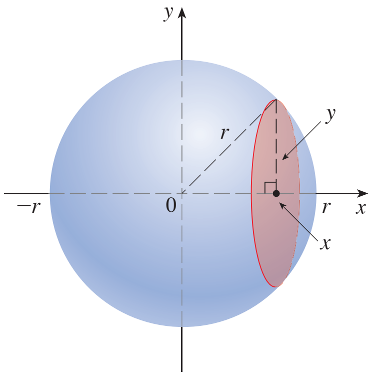


FIGURE 4

$y = \sqrt{r^2 - x^2}$. (See Figure 4.) So the cross-sectional area is

$$A(x) = \pi y^2 = \pi(r^2 - x^2)$$

Using the definition of volume with $a = -r$ and $b = r$, we have

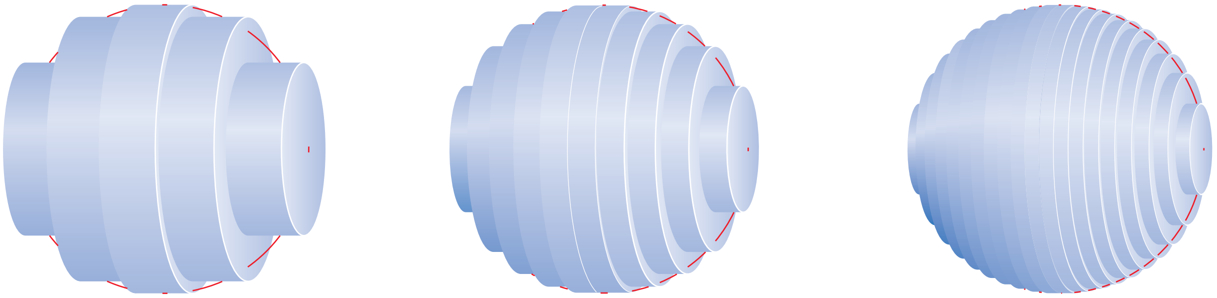
$$\begin{aligned} V &= \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx \\ &= 2\pi \int_{0}^{r} (r^2 - x^2)\,dx \qquad \text{(The integrand is even.)} \\ &= 2\pi\left[r^2x - \frac{x^3}{3}\right]_0^r = 2\pi\left(r^3 - \frac{r^3}{3}\right) = \tfrac{4}{3}\pi r^3 \end{aligned}$$

■

Figure 5 illustrates the definition of volume when the solid is a sphere with radius $r = 1$. From the result of Example 1, we know that the volume of the sphere is $\frac{4}{3}\pi$, which is approximately 4.18879. Here the slabs are circular cylinders, or *disks*, and the three parts of Figure 5 show the geometric interpretations of the Riemann sums

$$\sum_{i=1}^{n} A(\bar{x}_i)\,\Delta x = \sum_{i=1}^{n} \pi(1^2 - \bar{x}_i^2)\,\Delta x$$

when $n = 5$, 10, and 20 if we choose the sample points x_i^* to be the midpoints $\bar{x}_i$. Notice that as we increase the number of approximating cylinders, the corresponding Riemann sums become closer to the true volume.



(a) Using 5 disks, $V \approx 4.2726$ (b) Using 10 disks, $V \approx 4.2097$ (c) Using 20 disks, $V \approx 4.1940$

FIGURE 5 Approximating the volume of a sphere with radius 1

Volumes of Solids of Revolution

If we revolve a region about a line, we obtain a **solid of revolution**. In the following examples we see that for such a solid, cross-sections perpendicular to the axis of rotation are circular.

EXAMPLE 2 Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. Illustrate the definition of volume by sketching a typical approximating cylinder.

SOLUTION The region is shown in Figure 6(a) on the following page. If we rotate about the x-axis, we get the solid shown in Figure 6(b). When we slice through the point x, we get a disk with radius $\sqrt{x}$. The area of this cross-section is

$$A(x) = \pi\underbrace{(\sqrt{x})}_{\text{radius}}{}^2 = \pi x$$

and the volume of the approximating cylinder (a disk with thickness Δx) is

$$A(x)\,\Delta x = \pi x\,\Delta x$$

The solid lies between $x = 0$ and $x = 1$, so its volume is

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi x\,dx = \pi \frac{x^2}{2}\bigg]_0^1 = \frac{\pi}{2}$$

Did we get a reasonable answer in Example 2? As a check on our work, let's replace the given region by a square with base $[0, 1]$ and height 1. If we rotate this square, we get a cylinder with radius 1, height 1, and volume $\pi \cdot 1^2 \cdot 1 = \pi$. We computed that the given solid has half this volume. That seems about right.

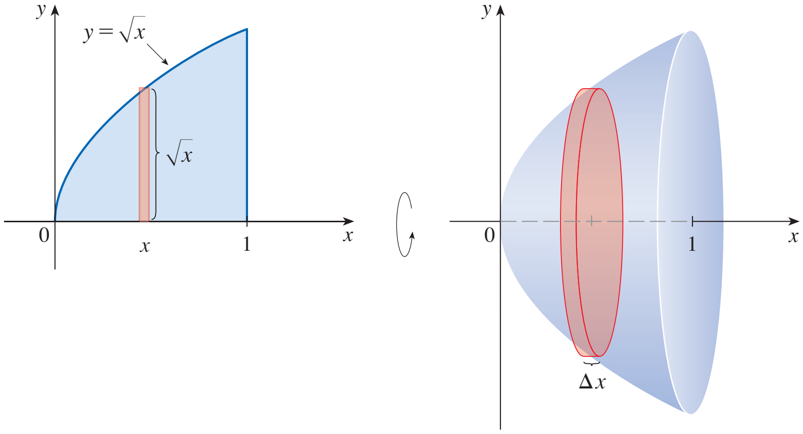


FIGURE 6

(a) (b)

EXAMPLE 3 Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

SOLUTION The region is shown in Figure 7(a) and the resulting solid is shown in Figure 7(b). Because the region is rotated about the y-axis, it makes sense to slice the solid perpendicular to the y-axis (obtaining circular cross-sections) and therefore to integrate with respect to y. If we slice at height y, we get a circular disk with radius x, where $x = \sqrt[3]{y}$. So the area of a cross-section through y is

$$A(y) = \pi \underbrace{(x)}_{\text{radius}}{}^2 = \pi \underbrace{(\sqrt[3]{y}\,)}_{\text{radius}}{}^2 = \pi y^{2/3}$$

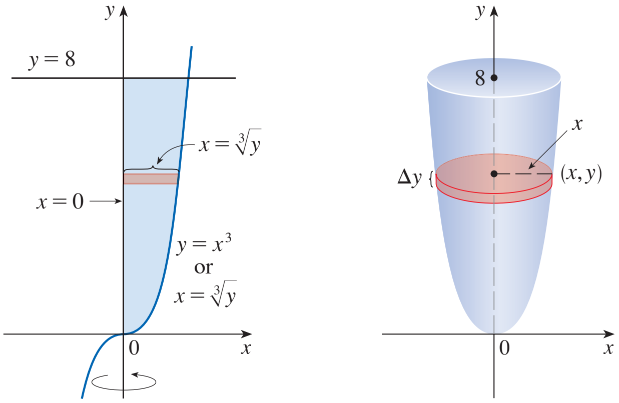


FIGURE 7

(a) (b)

and the volume of the approximating cylinder pictured in Figure 7(b) is

$$A(y)\,\Delta y = \pi y^{2/3}\,\Delta y$$

Since the solid lies between $y = 0$ and $y = 8$, its volume is

$$V = \int_0^8 A(y)\,dy = \int_0^8 \pi y^{2/3}\,dy = \pi\Big[\tfrac{3}{5}y^{5/3}\Big]_0^8 = \frac{96\pi}{5}$$ ■

In the following examples we see that some solids of revolution have a hollow core surrounding the axis of revolution.

EXAMPLE 4 The region $\mathcal{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

SOLUTION The curves $y = x$ and $y = x^2$ intersect at the points (0, 0) and (1, 1). The region between them, the solid of rotation, and a cross-section perpendicular to the x-axis are shown in Figure 8. A cross-section in the plane P_x has the shape of a *washer* (an annular ring) with inner radius x^2 and outer radius x [see Figure 8(c)], so we find the cross-sectional area by subtracting the area of the inner circle from the area of the outer circle:

$$A(x) = \pi\underbrace{(x)}_{\substack{\text{outer}\\\text{radius}}}{}^2 - \pi\underbrace{(x^2)}_{\substack{\text{inner}\\\text{radius}}}{}^2 = \pi(x^2 - x^4)$$

Therefore we have

$$\begin{aligned} V &= \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2 - x^4)\,dx \\ &= \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{2\pi}{15} \end{aligned}$$

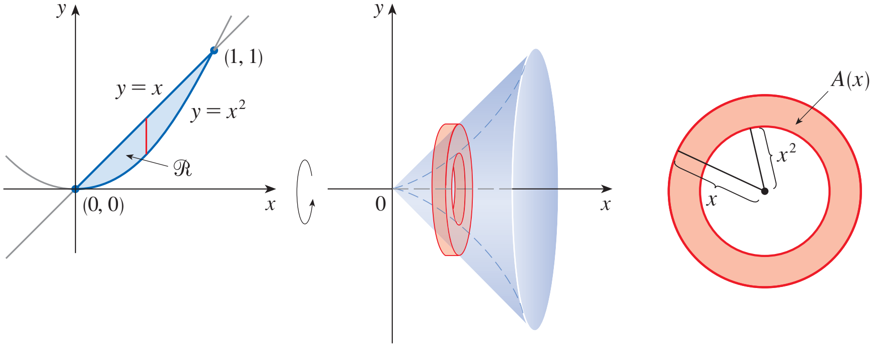


FIGURE 8

(a) (b) (c) ■

The next example shows that when a solid of revolution is created by rotating about an axis *other* than a coordinate axis, we must determine the radii of cross-sections carefully.

EXAMPLE 5 Find the volume of the solid obtained by rotating the region in Example 4 about the line $y = 2$.

SOLUTION The solid and a cross-section are shown in Figure 9. Again the cross-section is a washer, but this time the inner radius is $2 - x$ and the outer radius is $2 - x^2$.

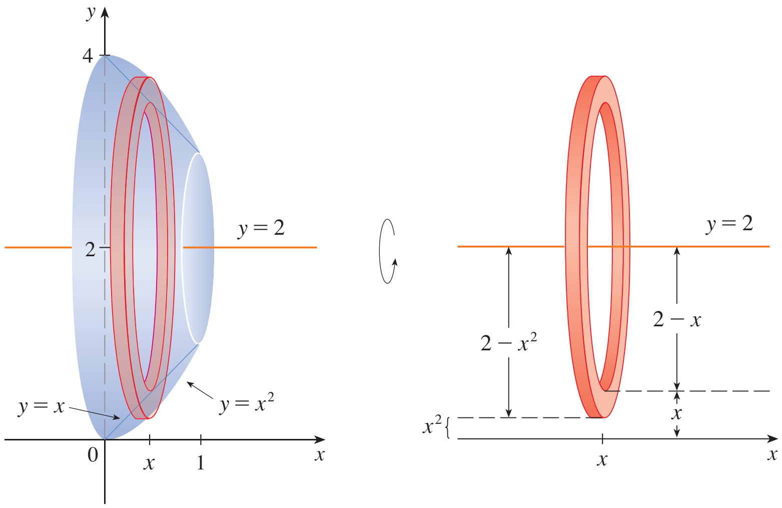


FIGURE 9

The cross-sectional area is

$$A(x) = \pi\underbrace{(2 - x^2)^2}_{\text{outer radius}} - \pi\underbrace{(2 - x)^2}_{\text{inner radius}}$$

and so the volume of S is

$$\begin{aligned} V &= \int_0^1 A(x)\,dx \\ &= \pi \int_0^1 \left[(2 - x^2)^2 - (2 - x)^2\right] dx \\ &= \pi \int_0^1 (x^4 - 5x^2 + 4x)\,dx \\ &= \pi \left[\frac{x^5}{5} - 5\frac{x^3}{3} + 4\frac{x^2}{2}\right]_0^1 = \frac{8\pi}{15} \end{aligned}$$

■

NOTE In general, we calculate the volume of a solid of revolution by using the basic defining formula

$$V = \int_a^b A(x)\,dx \qquad \text{or} \qquad V = \int_c^d A(y)\,dy$$

and we find the cross-sectional area $A(x)$ or $A(y)$ in one of the following ways:

- If the cross-section is a disk (as in Examples 1–3), we find the radius of the disk (in terms of x or y) and use

$$A = \pi(\text{radius})^2$$

- If the cross-section is a washer (as in Examples 4 and 5), we find the inner radius r_{in} and outer radius r_{out} from a sketch (as in Figures 8, 9, and 10) and compute the area of the washer by subtracting the area of the inner disk from the area of the outer disk:

$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$

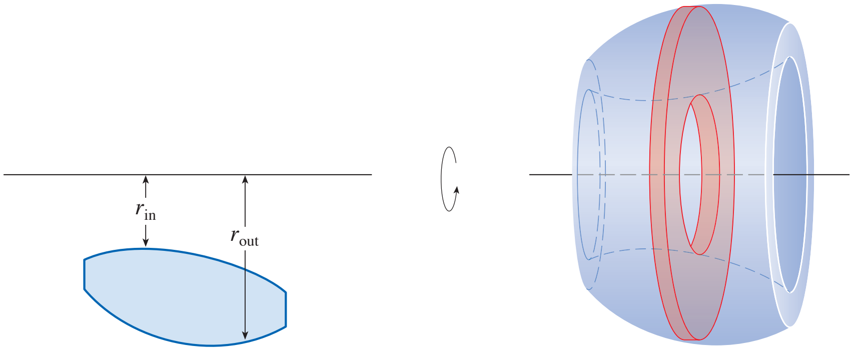


FIGURE 10

The next example gives a further illustration of the procedure.

EXAMPLE 6 Find the volume of the solid obtained by rotating the region in Example 4 about the line $x = -1$.

SOLUTION Figure 11 shows a horizontal cross-section. It is a washer with inner radius $1 + y$ and outer radius $1 + \sqrt{y}$, so the cross-sectional area is

$$\begin{aligned} A(y) &= \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 \\ &= \pi\left(1 + \sqrt{y}\right)^2 - \pi(1 + y)^2 \end{aligned}$$

The volume is

$$\begin{aligned} V &= \int_0^1 A(y)\,dy = \pi \int_0^1 \left[\left(1 + \sqrt{y}\right)^2 - (1 + y)^2\right] dy \\ &= \pi \int_0^1 \left(2\sqrt{y} - y - y^2\right) dy = \pi \left[\frac{4y^{3/2}}{3} - \frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{\pi}{2} \end{aligned}$$

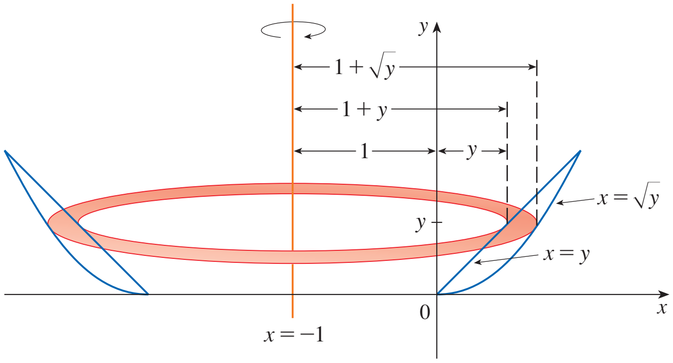


FIGURE 11

■

Finding Volume Using Cross-Sectional Area

We now find the volumes of solids that are not solids of revolution but whose cross-sections have areas that are readily computable.

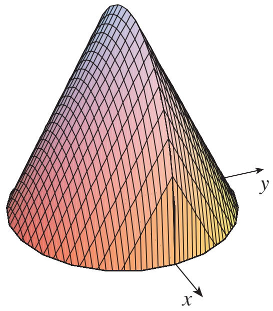


FIGURE 12
Computer-generated picture of the solid in Example 7

EXAMPLE 7 Figure 12 shows a solid with a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.

SOLUTION Let's take the circle to be $x^2 + y^2 = 1$. The solid, its base, and a typical cross-section at a distance x from the origin are shown in Figure 13.

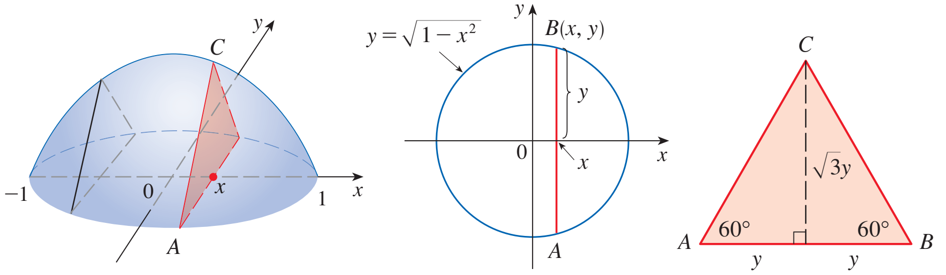


FIGURE 13

(a) The solid (b) Its base (c) A cross-section

Since B lies on the circle, we have $y = \sqrt{1 - x^2}$ and so the base of the triangle ABC is $|AB| = 2y = 2\sqrt{1 - x^2}$. Since the triangle is equilateral, we see from Figure 13(c) that its height is $\sqrt{3}\,y = \sqrt{3}\sqrt{1 - x^2}$. The cross-sectional area is therefore

$$A(x) = \tfrac{1}{2} \cdot 2\sqrt{1 - x^2} \cdot \sqrt{3}\sqrt{1 - x^2} = \sqrt{3}\,(1 - x^2)$$

and the volume of the solid is

$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \sqrt{3}\,(1 - x^2)\,dx$$

$$= 2\int_{0}^{1} \sqrt{3}\,(1 - x^2)\,dx = 2\sqrt{3}\left[x - \frac{x^3}{3}\right]_0^1 = \frac{4\sqrt{3}}{3}$$

■

EXAMPLE 8 Find the volume of a pyramid whose base is a square with side L and whose height is h.

SOLUTION We place the origin O at the vertex of the pyramid and the x-axis along its central axis as in Figure 14. Any plane P_x that passes through x and is perpendicular to

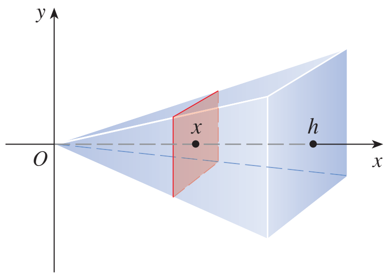


FIGURE 14

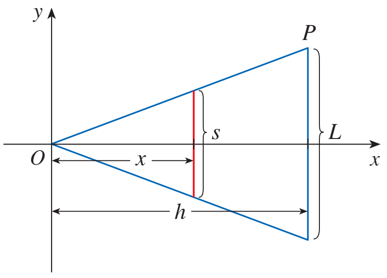


FIGURE 15

the x-axis intersects the pyramid in a square with side of length s, say. We can express s in terms of x by observing from the similar triangles in Figure 15 that

$$\frac{x}{h} = \frac{s/2}{L/2} = \frac{s}{L}$$

and so $s = Lx/h$. [Another method is to observe that the line OP has slope $L/(2h)$ and so its equation is $y = Lx/(2h)$.] Therefore the cross-sectional area is

$$A(x) = s^2 = \frac{L^2}{h^2}x^2$$

The pyramid lies between $x = 0$ and $x = h$, so its volume is

$$V = \int_0^h A(x)\,dx = \int_0^h \frac{L^2}{h^2}x^2\,dx$$

$$= \frac{L^2}{h^2}\frac{x^3}{3}\Big]_0^h = \frac{L^2h}{3}$$

■

NOTE We didn't need to place the vertex of the pyramid at the origin in Example 8; we did so merely to make the equations simple. If, instead, we had placed the center of the base at the origin and the vertex on the positive y-axis, as in Figure 16, you can verify that we would have obtained the integral

$$V = \int_0^h \frac{L^2}{h^2}(h - y)^2\,dy = \frac{L^2h}{3}$$

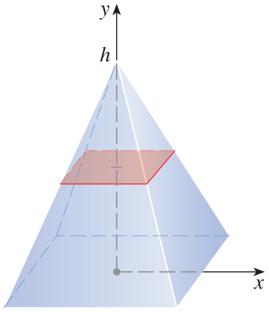


FIGURE 16

EXAMPLE 9 A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.

SOLUTION If we place the x-axis along the diameter where the planes meet, then the base of the solid is a semicircle with equation $y = \sqrt{16 - x^2}$, $-4 \leqslant x \leqslant 4$. A cross-section perpendicular to the x-axis at a distance x from the origin is a triangle ABC, as shown in Figure 17, whose base is $y = \sqrt{16 - x^2}$ and whose height is $|BC| = y \tan 30° = \sqrt{16 - x^2}/\sqrt{3}$. So the cross-sectional area is

$$A(x) = \tfrac{1}{2}\sqrt{16 - x^2} \cdot \frac{1}{\sqrt{3}}\sqrt{16 - x^2}$$

$$= \frac{16 - x^2}{2\sqrt{3}}$$

and the volume is

$$V = \int_{-4}^{4} A(x)\,dx = \int_{-4}^{4} \frac{16 - x^2}{2\sqrt{3}}\,dx$$

$$= \frac{1}{\sqrt{3}}\int_0^4 (16 - x^2)\,dx = \frac{1}{\sqrt{3}}\left[16x - \frac{x^3}{3}\right]_0^4 = \frac{128}{3\sqrt{3}}$$

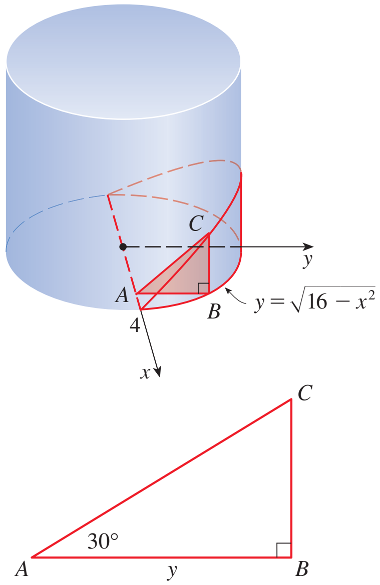


FIGURE 17

For another method see Exercise 77. ■

6.2 Exercises

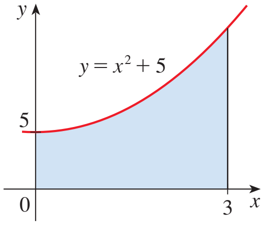
1–4 A solid is obtained by revolving the shaded region about the specified line.

(a) Sketch the solid and a typical disk or washer.

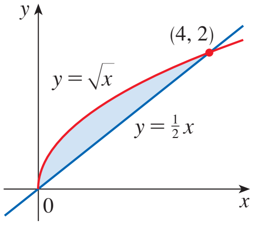
(b) Set up an integral for the volume of the solid.

(c) Evaluate the integral to find the volume of the solid.

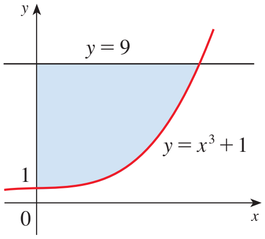
1. About the x-axis



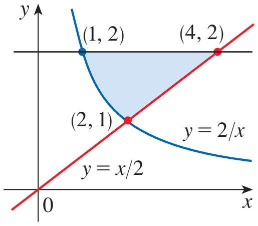
2. About the x-axis



3. About the y-axis



4. About the y-axis



5–10 Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

5. $y = \ln x, \ y = 0, \ x = 3$; about the x-axis

6. $x = \sqrt{5 - y}, \ y = 0, \ x = 0$; about the y-axis

7. $8y = x^2, \ y = \sqrt{x}$; about the y-axis

8. $y = (x - 2)^2, \ y = x + 10$; about the x-axis

9. $y = \sin x, \ y = 0, \ 0 \leqslant x \leqslant \pi$; about $y = -2$

10. $y = \sqrt{x}, \ y = 0, \ x = 4$; about $x = 6$

11–28 Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or washer.

11. $y = x + 1, \ y = 0, \ x = 0, \ x = 2$; about the x-axis

12. $y = 1/x, \ y = 0, \ x = 1, \ x = 4$; about the x-axis

13. $y = \sqrt{x - 1}, \ y = 0, \ x = 5$; about the x-axis

14. $y = e^x, \ y = 0, \ x = -1, \ x = 1$; about the x-axis

15. $x = 2\sqrt{y}, \ x = 0, \ y = 9$; about the y-axis

16. $2x = y^2, \ x = 0, \ y = 4$; about the y-axis

17. $y = x^2, \ y = 2x$; about the y-axis

18. $y = 6 - x^2, \ y = 2$; about the x-axis

19. $y = x^3, \ y = \sqrt{x}$; about the x-axis

20. $x = 2 - y^2, \ x = y^4$; about the y-axis

21. $y = x^2, \ x = y^2$; about $y = 1$

22. $y = x^3, \ y = 1, \ x = 2$; about $y = -3$

23. $y = 1 + \sec x, \ y = 3$; about $y = 1$

24. $y = \sin x, \ y = \cos x, \ 0 \leqslant x \leqslant \pi/4$; about $y = -1$

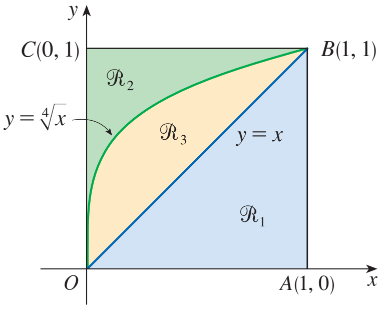
25. $y = x^3, \ y = 0, \ x = 1$; about $x = 2$

26. $xy = 1, \ y = 0, \ x = 1, \ x = 2$; about $x = -1$

27. $x = y^2, \ x = 1 - y^2$; about $x = 3$

28. $y = x, \ y = 0, \ x = 2, \ x = 4$; about $x = 1$

29–40 Refer to the figure and find the volume generated by rotating the given region about the specified line.



29. $\mathcal{R}_1$ about OA

30. $\mathcal{R}_1$ about OC

31. $\mathcal{R}_1$ about AB

32. $\mathcal{R}_1$ about BC

33. $\mathcal{R}_2$ about OA

34. $\mathcal{R}_2$ about OC

35. $\mathcal{R}_2$ about AB

36. $\mathcal{R}_2$ about BC

37. $\mathcal{R}_3$ about OA

38. $\mathcal{R}_3$ about OC

39. $\mathcal{R}_3$ about AB

40. $\mathcal{R}_3$ about BC

T **41–44** Set up an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Then use a calculator or computer to evaluate the integral correct to five decimal places.

41. $y = e^{-x^2}, \ y = 0, \ x = -1, \ x = 1$
(a) About the x-axis (b) About $y = -1$

42. $y = 0, \ y = \cos^2 x, \ -\pi/2 \leqslant x \leqslant \pi/2$
(a) About the x-axis (b) About $y = 1$

43. $x^2 + 4y^2 = 4$
(a) About $y = 2$ (b) About $x = 2$

44. $y = x^2,\ x^2 + y^2 = 1,\ y \geqslant 0$
(a) About the x-axis (b) About the y-axis

T **45–46** Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then use a calculator or computer to find (approximately) the volume of the solid obtained by rotating about the x-axis the region bounded by these curves.

45. $y = \ln(x^6 + 2), \quad y = \sqrt{3 - x^3}$

46. $y = 1 + xe^{-x^3}, \quad y = \arctan x^2$

T **47–48** Use a computer algebra system to find the exact volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

47. $y = \sin^2 x,\ y = 0,\ 0 \leqslant x \leqslant \pi;\quad$ about $y = -1$

48. $y = x,\ y = xe^{1-(x/2)};\quad$ about $y = 3$

49–54 Each integral represents the volume of a solid of revolution. Describe the solid.

49. $\pi \int_0^{\pi/2} \sin^2 x\, dx$

50. $\pi \int_0^{\ln 2} e^{2x}\, dx$

51. $\pi \int_0^1 (x^4 - x^6)\, dx$

52. $\pi \int_{-1}^1 (1 - y^2)^2\, dy$

53. $\pi \int_0^4 y\, dy$

54. $\pi \int_1^4 \left[3^2 - \left(3 - \sqrt{x}\,\right)^2\right] dx$

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55. A CAT scan produces equally spaced cross-sectional views
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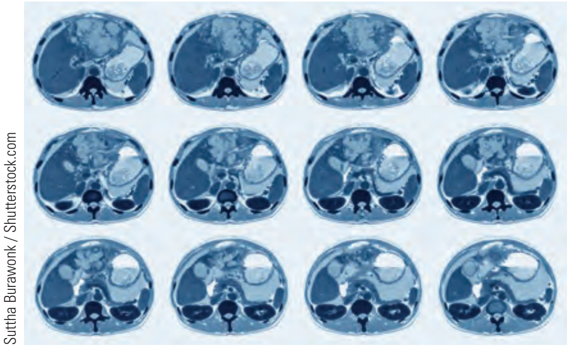
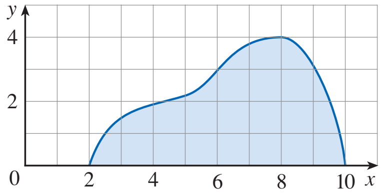

Suttha Burawonk / Shutterstock.com

56. A log 10 m long is cut at 1-meter intervals and its cross-sectional areas A (at a distance x from the end of the log) are listed in the table. Use the Midpoint Rule with $n = 5$ to estimate the volume of the log.

x (m)	A (m^2)	x (m)	A (m^2)
0	0.68	6	0.53
1	0.65	7	0.55
2	0.64	8	0.52
3	0.61	9	0.50
4	0.58	10	0.48
5	0.59		

57. (a) If the region shown in the figure is rotated about the x-axis to form a solid, use the Midpoint Rule with $n = 4$ to estimate the volume of the solid.



(b) Estimate the volume if the region is rotated about the y-axis. Again use the Midpoint Rule with $n = 4$.

T **58.** (a) A model for the shape of a bird's egg is obtained by rotating about the x-axis the region under the graph of

$$f(x) = (ax^3 + bx^2 + cx + d)\sqrt{1 - x^2}$$

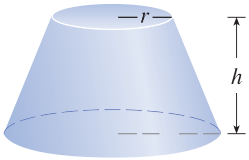
Use a computer algebra system to find the volume of such an egg.

(b) For a red-throated loon, $a = -0.06$, $b = 0.04$, $c = 0.1$, and $d = 0.54$. Graph f and find the volume of an egg of this species.

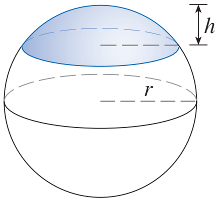
59–74 Find the volume of the described solid S.

59. A right circular cone with height h and base radius r

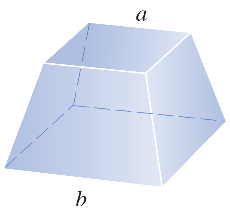
60. A frustum of a right circular cone with height h, lower base radius R, and top radius r



61. A cap of a sphere with radius r and height h



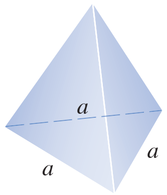
62. A frustum of a pyramid with square base of side b, square top of side a, and height h



What happens if $a = b$? What happens if $a = 0$?

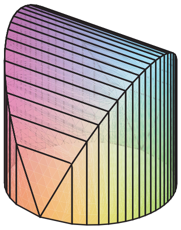
63. A pyramid with height h and rectangular base with dimensions b and $2b$

64. A pyramid with height h and base an equilateral triangle with side a (a tetrahedron)



65. A tetrahedron with three mutually perpendicular faces and three mutually perpendicular edges with lengths 3 cm, 4 cm, and 5 cm

66. The base of S is a circular disk with radius r. Parallel cross-sections perpendicular to the base are squares.



67. The base of S is an elliptical region with boundary curve $9x^2 + 4y^2 = 36$. Cross-sections perpendicular to the x-axis are isosceles right triangles with hypotenuse in the base.

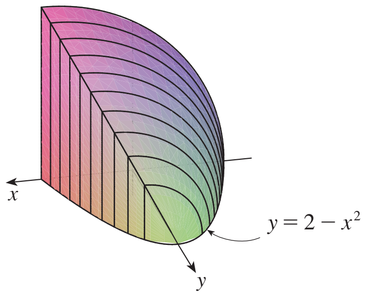
68. The base of S is the triangular region with vertices (0, 0), (1, 0), and (0, 1). Cross-sections perpendicular to the y-axis are equilateral triangles.

69. The base of S is the same base as in Exercise 68, but cross-sections perpendicular to the x-axis are squares.

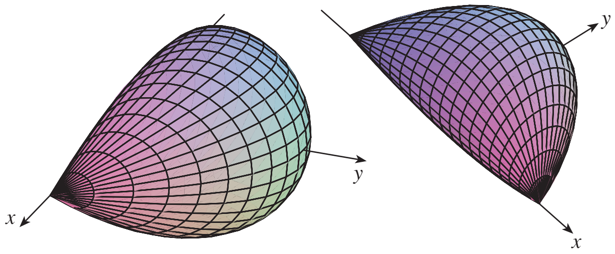
70. The base of S is the region enclosed by the parabola $y = 1 - x^2$ and the x-axis. Cross-sections perpendicular to the y-axis are squares.

71. The base of S is the same base as in Exercise 70, but cross-sections perpendicular to the x-axis are isosceles triangles with height equal to the base.

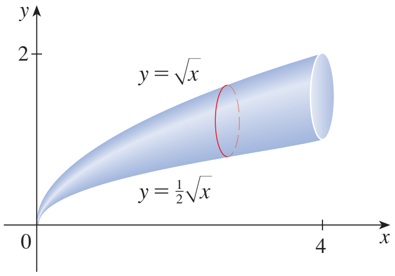
72. The base of S is the region enclosed by $y = 2 - x^2$ and the x-axis. Cross-sections perpendicular to the y-axis are quarter-circles.



73. The solid S is bounded by circles that are perpendicular to the x-axis, intersect the x-axis, and have centers on the parabola $y = \frac{1}{2}(1 - x^2)$, $-1 \leqslant x \leqslant 1$.

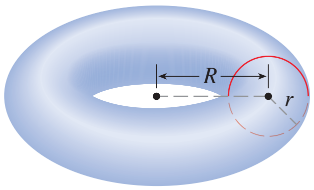


74. Cross-sections of the solid S in planes perpendicular to the x-axis are circles with diameters extending from the curve $y = \frac{1}{2}\sqrt{x}$ to the curve $y = \sqrt{x}$ for $0 \leqslant x \leqslant 4$.



75. (a) Set up an integral for the volume of a solid *torus* (the donut-shaped solid shown in the figure) with radii r and R.

(b) By interpreting the integral as an area, find the volume of the torus.

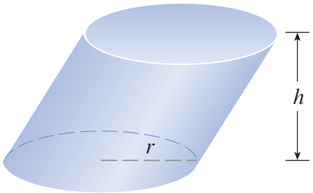


76. The base of a solid S is a circular disk with radius r. Parallel cross-sections perpendicular to the base are isosceles triangles with height h and unequal side in the base.
(a) Set up an integral for the volume of S.
(b) By interpreting the integral as an area, find the volume of S.

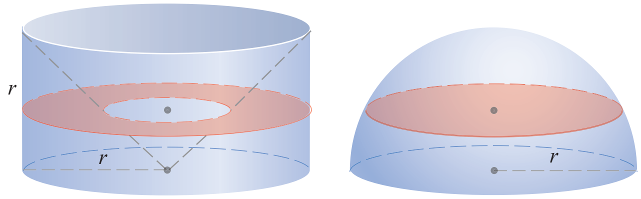
77. Solve Example 9 taking cross-sections to be parallel to the line of intersection of the two planes.

78–79 Cavalieri's Principle Cavalieri's Principle states that if a family of parallel planes gives equal cross-sectional areas for two solids S_1 and S_2, then the volumes of S_1 and S_2 are equal.

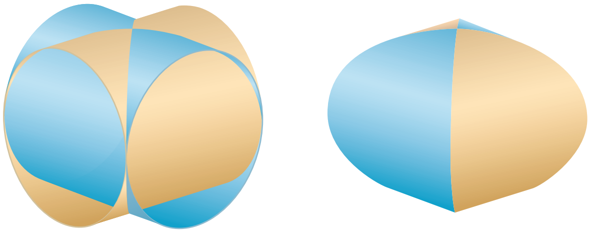
78. (a) Prove Cavalieri's Principle.
(b) Use Cavalieri's Principle to find the volume of the oblique cylinder shown in the figure.



79. Use Cavalieri's Principle to show that the volume of a solid hemisphere of radius r is equal to the volume of a cylinder of radius r and height r with a cone removed, as shown in the figure.



80. Find the volume common to two circular cylinders, each with radius r, if the axes of the cylinders intersect at right angles.



81. Find the volume common to two spheres, each with radius r, if the center of each sphere lies on the surface of the other sphere.

82. A bowl is shaped like a hemisphere with diameter 30 cm. A heavy ball with diameter 10 cm is placed in the bowl and water is poured into the bowl to a depth of h centimeters. Find the volume of water in the bowl.

83. A hole of radius r is bored through the middle of a cylinder of radius $R > r$ at right angles to the axis of the cylinder. Set up, but do not evaluate, an integral for the volume that is cut out.

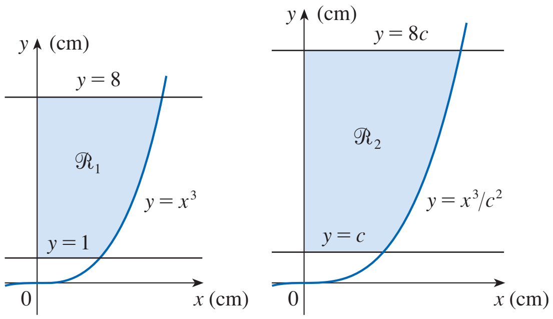
84. A hole of radius r is bored through the center of a sphere of radius $R > r$. Find the volume of the remaining portion of the sphere.

85. Some of the pioneers of calculus, such as Kepler and Newton, were inspired by the problem of finding the volumes of wine barrels. (Kepler published a book *Stereometria doliorum* in 1615 devoted to methods for finding the volumes of barrels.) They often approximated the shape of the sides by parabolas.
(a) A barrel with height h and maximum radius R is constructed by rotating about the x-axis the parabola $y = R - cx^2$, $-h/2 \leqslant x \leqslant h/2$, where c is a positive constant. Show that the radius of each end of the barrel is $r = R - d$, where $d = ch^2/4$.
(b) Show that the volume enclosed by the barrel is

$$V = \tfrac{1}{3}\pi h\left(2R^2 + r^2 - \tfrac{2}{5}d^2\right)$$

86. Suppose that a region $\mathcal{R}$ has area A and lies above the x-axis. When $\mathcal{R}$ is rotated about the x-axis, it sweeps out a solid with volume V_1. When $\mathcal{R}$ is rotated about the line $y = -k$ (where k is a positive number), it sweeps out a solid with volume V_2. Express V_2 in terms of V_1, k, and A.

87. A *dilation* of the plane with scaling factor c is a transformation that maps the point (x, y) to the point (cx, cy). Applying a dilation to a region in the plane produces a geometrically similar shape. A manufacturer wants to produce a 5-liter ($5000\ \text{cm}^3$) terra-cotta pot whose shape is geometrically similar to the solid obtained by rotating the region $\mathcal{R}_1$ shown in the figure about the y-axis.
(a) Find the volume V_1 of the pot obtained by rotating the region $\mathcal{R}_1$.
(b) Show that applying a dilation with scaling factor c transforms the region $\mathcal{R}_1$ into the region $\mathcal{R}_2$.
(c) Show that the volume V_2 of the pot obtained by rotating the region $\mathcal{R}_2$ is c^3V_1.
(d) Find the scaling factor c that produces a 5-liter pot.



6.3 Volumes by Cylindrical Shells

Some volume problems are very difficult to handle by the methods of the preceding section. For instance, let's consider the problem of finding the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$. (See Figure 1.) If we slice perpendicular to the y-axis, we get a washer. But to compute the inner radius and the outer radius of the washer, we'd have to solve the cubic equation $y = 2x^2 - x^3$ for x in terms of y; that's not easy.

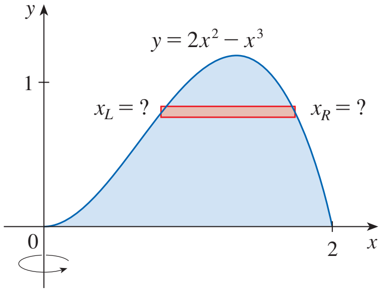


FIGURE 1

The Method of Cylindrical Shells

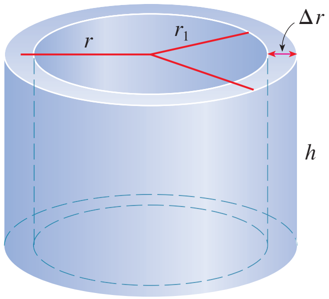


FIGURE 2

There is a method, called the *method of cylindrical shells*, that is easier to use in a case like the one shown in Figure 1. Figure 2 shows a cylindrical shell with inner radius r_1, outer radius r_2, and height h. Its volume V is calculated by subtracting the volume V_1 of the inner cylinder from the volume V_2 of the outer cylinder:

$$\begin{aligned} V &= V_2 - V_1 \\ &= \pi r_2^2 h - \pi r_1^2 h = \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi \frac{r_2 + r_1}{2} h(r_2 - r_1) \end{aligned}$$

If we let $\Delta r = r_2 - r_1$ (the thickness of the shell) and $r = \frac{1}{2}(r_2 + r_1)$ (the average radius of the shell), then this formula for the volume of a cylindrical shell becomes

$$\boxed{1} \qquad \boxed{V = 2\pi rh\, \Delta r}$$

and it can be remembered as

$$V = [\text{circumference}][\text{height}][\text{thickness}]$$

Now let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x)$ [where $f(x) \geqslant 0$], $y = 0$, $x = a$, and $x = b$, where $b > a \geqslant 0$. (See Figure 3.)

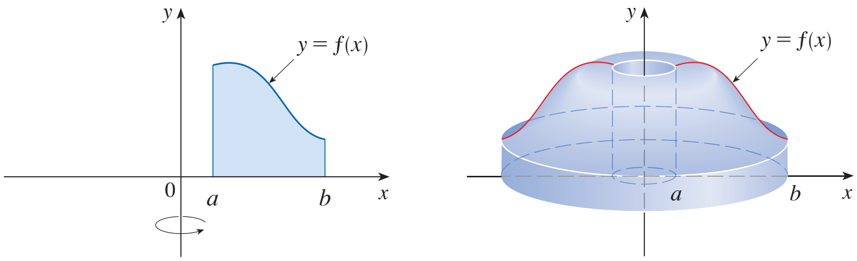


FIGURE 3

We divide the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width Δx and let $\bar{x}_i$ be the midpoint of the ith subinterval. If the rectangle with base $[x_{i-1}, x_i]$ and height $f(\bar{x}_i)$ is rotated about the y-axis, then the result is a cylindrical shell with average radius $\bar{x}_i$, height $f(\bar{x}_i)$, and thickness Δx. (See Figure 4.) So by Formula 1 its volume is

$$V_i = (2\pi\bar{x}_i)[f(\bar{x}_i)]\,\Delta x$$

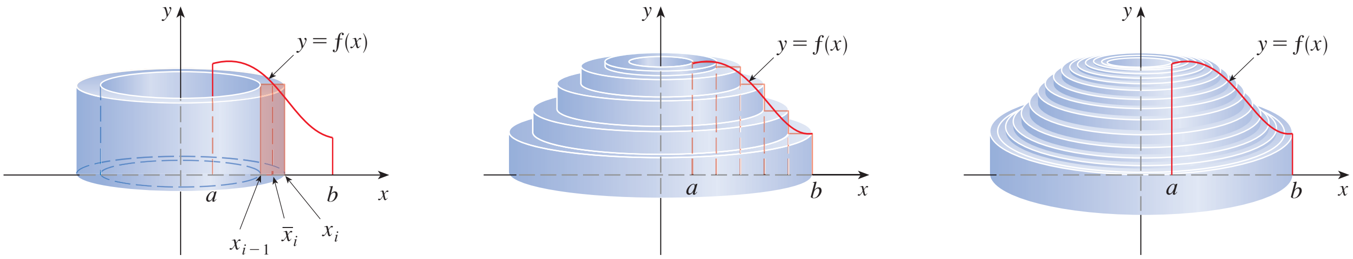


FIGURE 4

Therefore an approximation to the volume V of S is given by the sum of the volumes of these shells:

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi\bar{x}_i f(\bar{x}_i)\,\Delta x$$

This approximation appears to become better as $n \to \infty$. But, from the definition of an integral, we know that

$$\lim_{n\to\infty} \sum_{i=1}^{n} 2\pi\bar{x}_i f(\bar{x}_i)\,\Delta x = \int_a^b 2\pi x\, f(x)\, dx$$

Thus the following formula appears plausible:

2 The volume of the solid in Figure 3, obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b, is

$$V = \int_a^b 2\pi x f(x)\, dx \qquad \text{where } 0 \leqslant a < b$$

The argument using cylindrical shells makes Formula 2 seem reasonable, but later we will be able to prove it (see Exercise 7.1.81).

The best way to remember Formula 2 is to think of a typical shell, cut and flattened as in Figure 5, with radius x, circumference $2\pi x$, height $f(x)$, and thickness Δx or dx:

$$V = \int_a^b \underbrace{(2\pi x)}_{\text{circumference}}\ \underbrace{[f(x)]}_{\text{height}}\ \underbrace{dx}_{\text{thickness}}$$

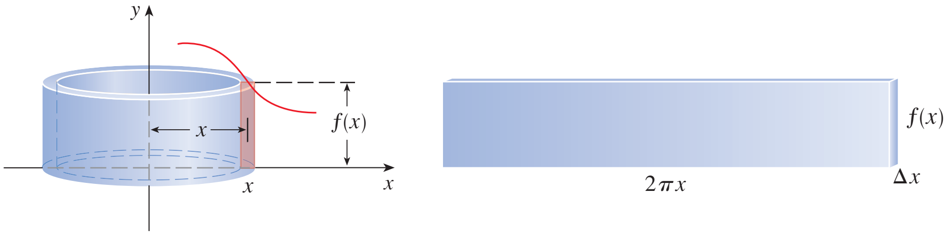


FIGURE 5

This type of reasoning will be helpful in other situations, such as when we rotate regions about lines other than the y-axis.

EXAMPLE 1 Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

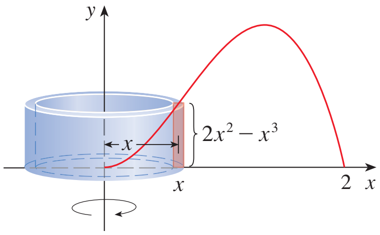


FIGURE 6

SOLUTION From the sketch in Figure 6 we see that a typical shell has radius x, circumference $2\pi x$, and height $f(x) = 2x^2 - x^3$. So, by the shell method, the volume is

$$V = \int_0^2 \underbrace{(2\pi x)}_{\text{circumference}}\ \underbrace{(2x^2 - x^3)}_{\text{height}}\ \underbrace{dx}_{\text{thickness}}$$

$$= 2\pi \int_0^2 (2x^3 - x^4)\,dx = 2\pi\Big[\tfrac{1}{2}x^4 - \tfrac{1}{5}x^5\Big]_0^2$$

$$= 2\pi\big(8 - \tfrac{32}{5}\big) = \tfrac{16}{5}\pi$$

It can be verified that the shell method gives the same answer as slicing. ■

Figure 7 shows a computer-generated picture of the solid whose volume we computed in Example 1.

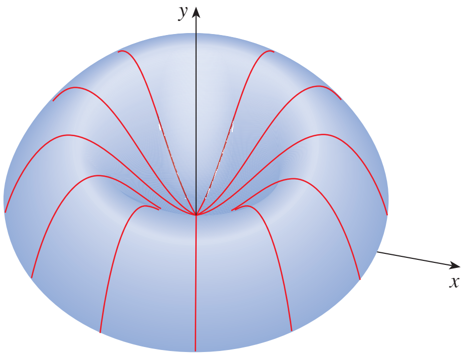


FIGURE 7

NOTE Comparing the solution of Example 1 with the remarks at the beginning of this section, we see that the method of cylindrical shells is much easier than the washer method for this problem. We did not have to find the coordinates of the local maximum and we did not have to solve the equation of the curve for x in terms of y. However, in other examples the methods of the preceding section may be easier.

EXAMPLE 2 Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.

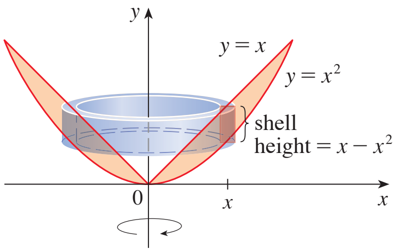


FIGURE 8

SOLUTION The region and a typical shell are shown in Figure 8. We see that the shell has radius x, circumference $2\pi x$, and height $x - x^2$. So the volume is

$$V = \int_0^1 (2\pi x)(x - x^2)\,dx = 2\pi \int_0^1 (x^2 - x^3)\,dx$$

$$= 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{6}$$ ■

As the following example shows, the shell method works just as well if we rotate a region about the x-axis. We simply have to draw a diagram to identify the radius and height of a shell.

EXAMPLE 3 Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

SOLUTION This problem was solved using disks in Example 6.2.2. To use shells we relabel the curve $y = \sqrt{x}$ (in the figure in that example) as $x = y^2$ in Figure 9. For rotation about the x-axis we see that a typical shell has radius y, circumference $2\pi y$, and height $1 - y^2$. So the volume is

$$V = \int_0^1 (2\pi y)(1 - y^2)\,dy = 2\pi \int_0^1 (y - y^3)\,dy$$

$$= 2\pi \left[\frac{y^2}{2} - \frac{y^4}{4} \right]_0^1 = \frac{\pi}{2}$$

In this problem the disk method was simpler. ■

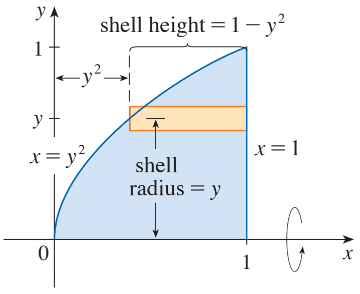


FIGURE 9

EXAMPLE 4 Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

SOLUTION Figure 10 shows the region and a cylindrical shell formed by rotation about the line $x = 2$. It has radius $2 - x$, circumference $2\pi(2 - x)$, and height $x - x^2$.

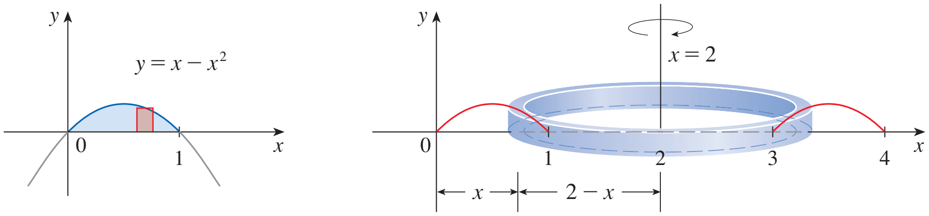


FIGURE 10

The volume of the given solid is

$$V = \int_0^1 2\pi(2 - x)(x - x^2)\,dx$$

$$= 2\pi \int_0^1 (x^3 - 3x^2 + 2x)\,dx$$

$$= 2\pi \left[\frac{x^4}{4} - x^3 + x^2 \right]_0^1 = \frac{\pi}{2}$$ ■

■ Disks and Washers versus Cylindrical Shells

When computing the volume of a solid of revolution, how do we know whether to use disks (or washers) or cylindrical shells? There are several considerations to take into account: Is the region more easily described by top and bottom boundary curves of the form $y = f(x)$, or by left and right boundaries $x = g(y)$? Which choice is easier to work with? Are the limits of integration easier to find for one variable versus the other? Does the region require two separate integrals when using x as the variable but only one integral in y? Are we able to evaluate the integral we set up with our choice of variable?

If we decide that one variable is easier to work with than the other, then this dictates which method to use. Draw a sample rectangle in the region, corresponding to a cross-section of the solid. The thickness of the rectangle, either Δx or Δy, corresponds to the integration variable. If you imagine the rectangle revolving, it becomes either a disk (washer) or a shell. Sometimes either method works, as in the next example.

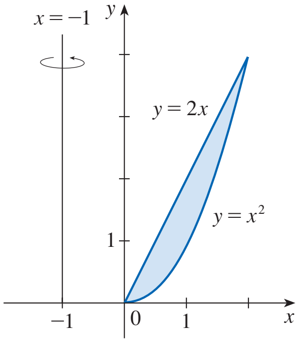


FIGURE 11

EXAMPLE 5 Figure 11 shows the region in the first quadrant bounded by the curves $y = x^2$ and $y = 2x$. A solid is formed by rotating the region about the line $x = -1$. Find the volume of the solid using (a) x as the variable of integration and (b) y as the variable of integration.

SOLUTION The solid is shown in Figure 12(a).

(a) To find the volume using x as the variable of integration, we draw the sample rectangle vertically, as in Figure 12(b). Rotating the region about the line $x = -1$ produces cylindrical shells, so the volume is

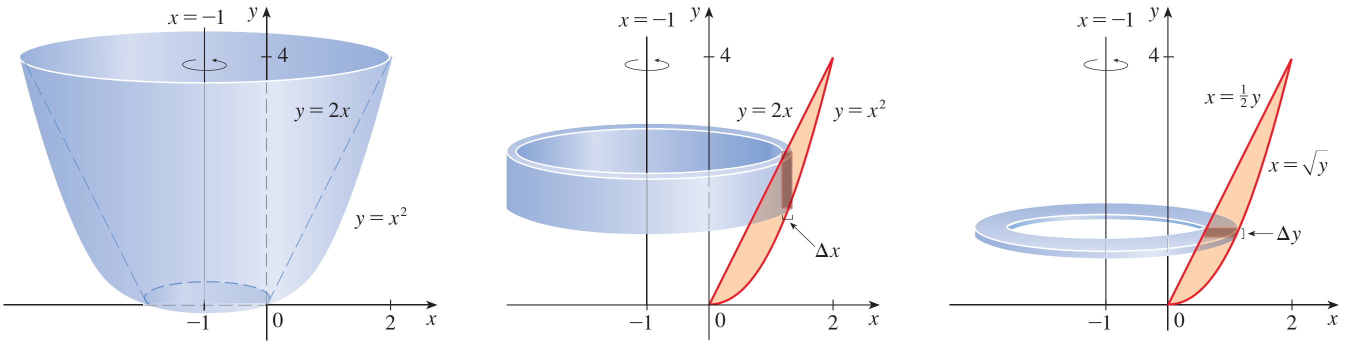
$$V = \int_0^2 2\pi(x + 1)(2x - x^2)\,dx = 2\pi \int_0^2 (x^2 + 2x - x^3)\,dx$$

$$= 2\pi\left[\frac{x^3}{3} + x^2 - \frac{x^4}{4}\right]_0^2 = \frac{16\pi}{3}$$

(b) To find the volume using y as the variable of integration, we draw the sample rectangle horizontally as in Figure 12(c). Rotating the region about the line produces washer-shaped cross-sections, so the volume is

$$V = \int_0^4 \left[\pi\left(\sqrt{y} + 1\right)^2 - \pi\left(\tfrac{1}{2}y + 1\right)^2\right] dy = \pi \int_0^4 \left(2\sqrt{y} - \tfrac{1}{4}y^2\right) dy$$

$$= \pi\left[\tfrac{4}{3}y^{3/2} - \tfrac{1}{12}y^3\right]_0^4 = \frac{16\pi}{3}$$

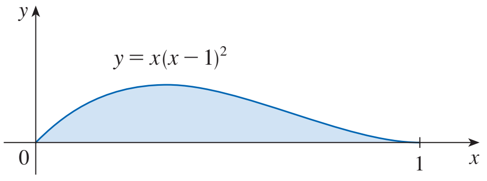


(a) (b) (c)

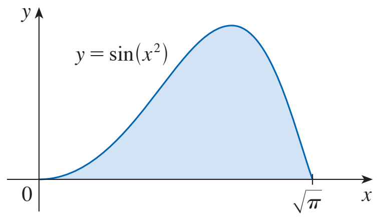
FIGURE 12 ■

6.3 Exercises

1. Let S be the solid obtained by rotating the region shown in the figure about the y-axis. Explain why it is awkward to use the washer method to find the volume V of S. Sketch a typical approximating shell. What are its circumference and height? Use shells to find V.



2. Let S be the solid obtained by rotating the region shown in the figure about the y-axis. Sketch a typical cylindrical shell and find its circumference and height. Use shells to find the volume of S. Is this method preferable to using washers?

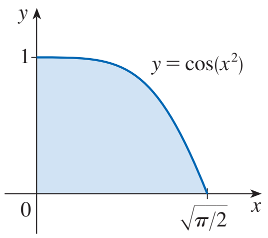


3–4 A solid is obtained by rotating the shaded region about the specified line.

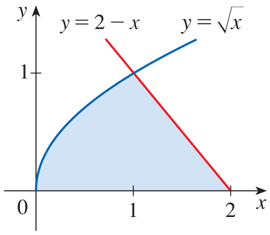
(a) Set up an integral using the method of cylindrical shells for the volume of the solid.

(b) Evaluate the integral to find the volume of the solid.

3. About the y-axis



4. About the x-axis



5–8 Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

5. $y = \ln x,\ y = 0,\ x = 2;$ about the y-axis

6. $y = x^3,\ y = 8,\ x = 0;$ about the x-axis

7. $y = \sin^{-1}x,\ y = \pi/2,\ x = 0;$ about $y = 3$

8. $y = 4x - x^2,\ y = x;$ about $x = 7$

9–14 Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the y-axis.

9. $y = \sqrt{x},\quad y = 0,\quad x = 4$

10. $y = x^3,\quad y = 0,\quad x = 1,\quad x = 2$

11. $y = 1/x,\quad y = 0,\quad x = 1,\quad x = 4$

12. $y = e^{-x^2},\quad y = 0,\quad x = 0,\quad x = 1$

13. $y = \sqrt{5 + x^2},\quad y = 0,\quad x = 0,\quad x = 2$

14. $y = 4x - x^2,\quad y = x$

15–20 Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis.

15. $xy = 1,\quad x = 0,\quad y = 1,\quad y = 3$

16. $y = \sqrt{x},\quad x = 0,\quad y = 2$

17. $y = x^{3/2},\quad y = 8,\quad x = 0$

18. $x = -3y^2 + 12y - 9,\quad x = 0$

19. $x = 1 + (y - 2)^2,\quad x = 2$

20. $x + y = 4,\quad x = y^2 - 4y + 4$

21–22 The region bounded by the given curves is rotated about the specified axis. Find the volume of the resulting solid using (a) x as the variable of integration and (b) y as the variable of integration.

21. $y = x^2,\ y = 8\sqrt{x};$ about the y-axis

22. $y = x^3,\ y = 4x^2;$ about the x-axis

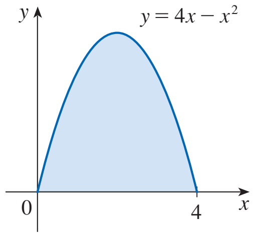
23–24 A solid is obtained by rotating the shaded region about the specified axis.

(a) Sketch the solid and a typical approximating cylindrical shell.

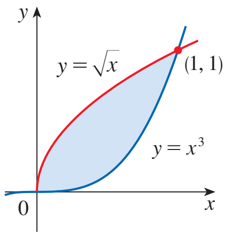
(b) Use the method of cylindrical shells to set up an integral for the volume of the solid.

(c) Evaluate the integral to find the volume.

23. About $x = -2$



24. About $y = -1$



25–30 Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified axis.

25. $y = x^3,\ y = 8,\ x = 0;$ about $x = 3$

26. $y = 4 - 2x,\ y = 0,\ x = 0;$ about $x = -1$

27. $y = 4x - x^2,\ y = 3;$ about $x = 1$

28. $y = \sqrt{x},\ x = 2y;$ about $x = 5$

29. $x = 2y^2,\ y \geqslant 0,\ x = 2;$ about $y = 2$

30. $x = 2y^2,\ x = y^2 + 1;$ about $y = -2$

31–36

(a) Set up an integral for the volume of the solid obtained by rotating the region bounded by the given curve about the specified axis.

T (b) Use a calculator or computer to evaluate the integral correct to five decimal places.

31. $y = xe^{-x},\ y = 0,\ x = 2;$ about the y-axis

32. $y = \tan x,\ y = 0,\ x = \pi/4;$ about $x = \pi/2$

33. $y = \cos^4 x,\ y = -\cos^4 x,\ -\pi/2 \leqslant x \leqslant \pi/2;$ about $x = \pi$

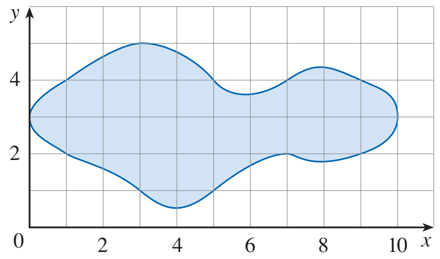
34. $y = x,\ y = 2x/(1 + x^3);$ about $x = -1$

35. $x = \sqrt{\sin y},\ 0 \leqslant y \leqslant \pi,\ x = 0;$ about $y = 4$

36. $x^2 - y^2 = 7,\ x = 4;$ about $y = 5$

37. Use the Midpoint Rule with $n = 5$ to estimate the volume obtained by rotating about the y-axis the region under the curve $y = \sqrt{1 + x^3}$, $0 \leqslant x \leqslant 1$.

38. If the region shown in the figure is rotated about the y-axis to form a solid, use the Midpoint Rule with $n = 5$ to estimate the volume of the solid.



39–42 Each integral represents the volume of a solid. Describe the solid.

39. $\int_0^3 2\pi x^5\,dx$

40. $\int_1^3 2\pi y \ln y\,dy$

41. $2\pi \int_1^4 \dfrac{y+2}{y^2}\,dy$

42. $\int_0^1 2\pi(2 - x)(3^x - 2^x)\,dx$

T **43–44** Use a graph to estimate the x-coordinates of the points of intersection of the given curves. Then use this information and a calculator or computer to estimate the volume of the solid obtained by rotating about the y-axis the region enclosed by these curves.

43. $y = x^2 - 2x$, $\quad y = \dfrac{x}{x^2 + 1}$

44. $y = e^{\sin x}$, $\quad y = x^2 - 4x + 5$

T **45–46** Use a computer algebra system to find the exact volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

45. $y = \sin^2 x$, $y = \sin^4 x$, $0 \leqslant x \leqslant \pi$; about $x = \pi/2$

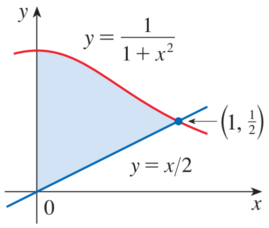
46. $y = x^3 \sin x$, $y = 0$, $0 \leqslant x \leqslant \pi$; about $x = -1$

47–52 A solid is obtained by rotating the shaded region about the specified line.

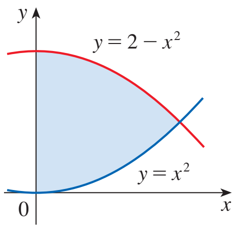
(a) Set up an integral using any method to find the volume of the solid.

(b) Evaluate the integral to find the volume of the solid.

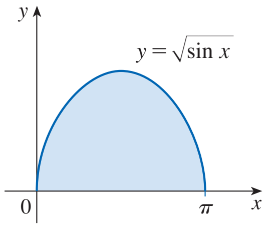
47. About the y-axis



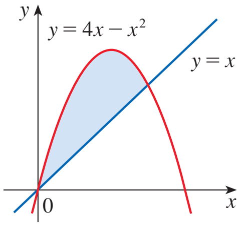
48. About the x-axis



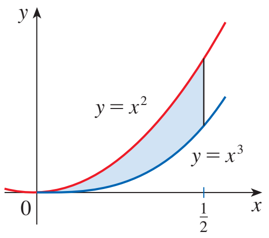
49. About the x-axis



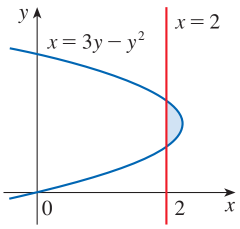
50. About the y-axis



51. About the line $x = -2$



52. About the line $y = 3$



53–59 The region bounded by the given curves is rotated about the specified axis. Find the volume of the resulting solid by any method.

53. $y = -x^2 + 6x - 8$, $y = 0$; about the y-axis

54. $y = -x^2 + 6x - 8$, $y = 0$; about the x-axis

55. $y^2 - x^2 = 1$, $y = 2$; about the x-axis

56. $y^2 - x^2 = 1$, $y = 2$; about the y-axis

57. $x^2 + (y - 1)^2 = 1$; about the y-axis

58. $x = (y - 3)^2$, $x = 4$; about $y = 1$

59. $x = (y - 1)^2$, $x - y = 1$; about $x = -1$

---

60. Let T be the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$, and let V be the volume of the solid generated when T is rotated about the line $x = a$, where $a > 1$. Express a in terms of V.

61–63 Use cylindrical shells to find the volume of the solid.

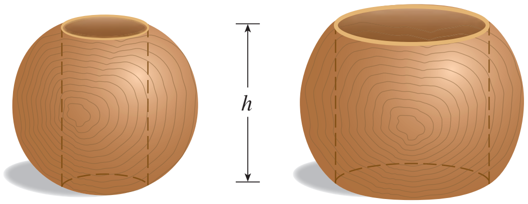
61. A sphere of radius r

62. The solid torus of Exercise 6.2.75

63. A right circular cone with height h and base radius r

---

64. Suppose you make napkin rings by drilling holes with different diameters through two wooden balls (which also have different diameters). You discover that both napkin rings have the same height h, as shown in the figure.
(a) Guess which ring has more wood in it.
(b) Check your guess: use cylindrical shells to compute the volume of a napkin ring created by drilling a hole with radius r through the center of a sphere of radius R and express the answer in terms of h.



6.4 Work

The term *work* is used in everyday language to mean the total amount of effort required to perform a task. In physics it has a technical meaning that depends on the idea of a *force*. Intuitively, you can think of a force as describing a push or pull on an object—for example, a horizontal push of a book across a table or the downward pull of the earth's gravity on a ball. In general, if an object moves along a straight line with position function $s(t)$, then the **force** F on the object (in the same direction) is given by Newton's Second Law of Motion as the product of its mass m and its acceleration a:

$$\boxed{1} \qquad F = ma = m\frac{d^2s}{dt^2}$$

In the SI metric system, the mass is measured in kilograms (kg), the displacement in meters (m), the time in seconds (s), and the force in newtons ($\text{N} = \text{kg}\cdot\text{m/s}^2$). Thus a force of 1 N acting on a mass of 1 kg produces an acceleration of 1 m/s^2. In the US Customary system the fundamental unit is chosen to be the unit of force, which is the pound.

In the case of constant acceleration, the force F is also constant and the work done is defined to be the product of the force F and the distance d that the object moves:

$$\boxed{2} \qquad W = Fd \qquad \text{work} = \text{force} \times \text{distance}$$

If F is measured in newtons and d in meters, then the unit for W is a newton-meter, which is called a joule (J). If F is measured in pounds and d in feet, then the unit for W is a foot-pound (ft-lb), which is about 1.36 J.

EXAMPLE 1

(a) How much work is done in lifting a 1.2-kg book off the floor to put it on a desk that is 0.7 m high? Use the fact that the acceleration due to gravity is $g = 9.8\ \text{m/s}^2$.
(b) How much work is done in lifting a 20-lb weight 6 ft off the ground?

SOLUTION

(a) The force exerted is equal and opposite to that exerted by gravity, so Equation 1 gives

$$F = mg = (1.2)(9.8) = 11.76\ \text{N}$$

and then Equation 2 gives the work done as

$$W = Fd = (11.76\ \text{N})(0.7\ \text{m}) \approx 8.2\ \text{J}$$

(b) Here the force is given as $F = 20$ lb, so the work done is

$$W = Fd = (20 \text{ lb})(6 \text{ ft}) = 120 \text{ ft-lb}$$

Notice that in part (b), unlike part (a), we did not have to multiply by g because we were given the *weight* (which is a force) and not the mass of the object. ■

Equation 2 defines work as long as the force is constant, but what happens if the force is variable? Let's suppose that the object moves along the x-axis in the positive direction, from $x = a$ to $x = b$, and at each point x between a and b a force $f(x)$ acts on the object, where f is a continuous function. We divide the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx. We choose a sample point x_i^* in the ith subinterval $[x_{i-1}, x_i]$. Then the force at that point is $f(x_i^*)$. If n is large, then Δx is small, and since f is continuous, the values of f don't change very much over the interval $[x_{i-1}, x_i]$. In other words, f is almost constant on the interval and so the work W_i that is done in moving the particle from x_{i-1} to x_i is approximately given by Equation 2:

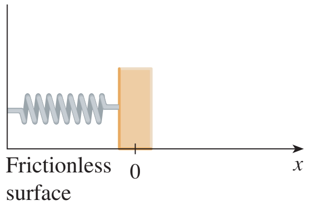
$$W_i \approx f(x_i^*)\,\Delta x$$

Thus we can approximate the total work by

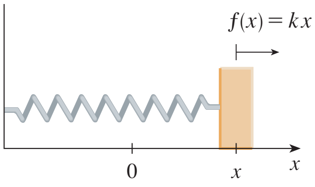
$$\boxed{3} \qquad W \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

It seems that this approximation becomes better as we make n larger. Therefore we define the **work done in moving the object from a to b** as the limit of this quantity as $n \to \infty$. Since the right side of (3) is a Riemann sum, we recognize its limit as being a definite integral and so

$$\boxed{4} \qquad \boxed{W = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \int_a^b f(x)\,dx}$$



(a) Natural position of spring



(b) Stretched position of spring

FIGURE 1
Hooke's Law

EXAMPLE 2 When a particle is located a distance x feet from the origin, a force of $x^2 + 2x$ newtons acts on it. How much work is done in moving it from $x = 1$ to $x = 3$?

SOLUTION

$$W = \int_1^3 (x^2 + 2x)\,dx = \frac{x^3}{3} + x^2 \Big]_1^3 = \frac{50}{3}$$

The work done is $16\frac{2}{3}$ J. ■

In the next example we use a law from physics. **Hooke's Law** states that the force required to maintain a spring stretched x units beyond its natural length is proportional to x:

$$f(x) = kx$$

where k is a positive constant called the **spring constant** (see Figure 1). Hooke's Law holds provided that x is not too large.

EXAMPLE 3 A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm?

SOLUTION According to Hooke's Law, the force required to hold the spring stretched x meters beyond its natural length is $f(x) = kx$. When the spring is stretched from 10 cm to 15 cm, the amount stretched is 5 cm $= 0.05$ m. This means that $f(0.05) = 40$, so

$$0.05k = 40 \qquad k = \frac{40}{0.05} = 800$$

Thus $f(x) = 800x$ and the work done in stretching the spring from 15 cm to 18 cm is

$$W = \int_{0.05}^{0.08} 800x\,dx = 800\,\frac{x^2}{2}\bigg]_{0.05}^{0.08}$$

$$= 400[(0.08)^2 - (0.05)^2] = 1.56 \text{ J}$$ ■

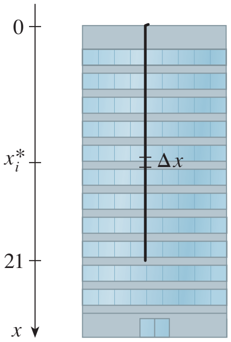


FIGURE 2

EXAMPLE 4 A 90 kg cable is 21 m long and hangs vertically from the top of a tall building.

(a) How much work is required to lift the cable to the top of the building?
(b) How much work is required to pull up only 6 meters of the cable?

SOLUTION

(a) One method is to use an argument similar to the one that led to Definition 4. [For another method, see Exercise 14(b).]

Let's place the origin at the top of the building and the x-axis pointing downward as in Figure 2. We divide the cable into small parts with length Δx. If x_i^* is a point in the ith such interval, then all points in the interval are lifted by approximately the same amount, namely x_i^*. The cable has a mass of 30/7 kg per meter, so the force acting on the ith part is $(30/7 \text{ kg/m})(9.8 \text{ m/s}^2)(\Delta x \text{ m}) = 42\,\Delta x$ N. Thus the work done on the ith part, in joules, is

$$\underbrace{(42\,\Delta x)}_{\text{force}} \cdot \underbrace{x_i^*}_{\text{distance}} = 42x_i^*\,\Delta x$$

If we had placed the origin at the bottom of the cable and the x-axis upward, we would have gotten

$$W = \int_0^{21} 42(21 - x)\,dx$$

which gives the same answer.

We get the total work done by adding all these approximations and letting the number of parts become large (so $\Delta x \to 0$):

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 42x_i^*\,\Delta x = \int_0^{21} 42x\,dx$$

$$= 21x^2\Big]_0^{21} = 18{,}900 \text{ J}$$

(b) The work required to move the top 6 m of cable to the top of the building is computed in the same manner as part (a):

$$W_1 = \int_0^6 42x\,dx = 21x^2\Big]_0^6 = 756 \text{ J}$$

Every part of the lower 15 m of cable moves the same distance, namely 6 m, so the work done is

$$W_2 = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\underbrace{6}_{\text{distance}} \cdot \underbrace{42\Delta x}_{\text{force}}\right) = \int_6^{21} 252\,dx = 3780 \text{ J}$$

(Alternatively, we can observe that the lower 15 m of cable weighs $15 \cdot 30/7 \cdot 9.8 = 630$ N and moves uniformly 6 m, so the work done is $630 \cdot 6 = 3780$ J.)

The total work done is $W_1 + W_2 = 756 + 3780 = 4536$ J. ■

EXAMPLE 5 A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m^3.)

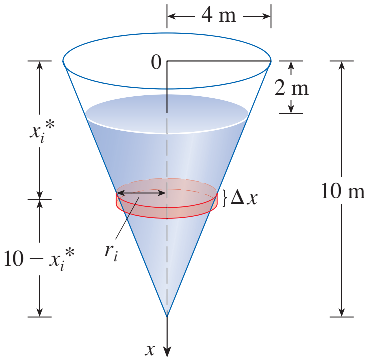


FIGURE 3

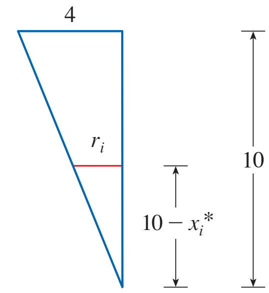


FIGURE 4

SOLUTION Let's measure depths from the top of the tank by introducing a vertical coordinate line as in Figure 3. The water extends from a depth of 2 m to a depth of 10 m and so we divide the interval $[2, 10]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and choose x_i^* in the ith subinterval. This divides the water into n layers. The ith layer is approximated by a circular cylinder with radius r_i and height Δx. We can compute r_i from similar triangles, using Figure 4, as follows:

$$\frac{r_i}{10 - x_i^*} = \frac{4}{10} \qquad r_i = \tfrac{2}{5}(10 - x_i^*)$$

Thus an approximation to the volume of the ith layer of water is

$$V_i \approx \pi r_i^2 \,\Delta x = \frac{4\pi}{25}(10 - x_i^*)^2 \,\Delta x$$

and so its mass is

$$\begin{aligned} m_i &= \text{density} \times \text{volume} \\ &\approx 1000 \cdot \frac{4\pi}{25}(10 - x_i^*)^2 \,\Delta x = 160\pi(10 - x_i^*)^2 \,\Delta x \end{aligned}$$

The force required to raise this layer must overcome the force of gravity and so

$$\begin{aligned} F_i = m_i g &\approx (9.8)160\pi(10 - x_i^*)^2 \,\Delta x \\ &= 1568\pi(10 - x_i^*)^2 \,\Delta x \end{aligned}$$

Each particle in the layer must travel a distance upward of approximately x_i^*. The work W_i done to raise this layer to the top is approximately the product of the force F_i and the distance x_i^*:

$$W_i \approx F_i x_i^* \approx 1568\pi x_i^*(10 - x_i^*)^2 \,\Delta x$$

To find the total work done in emptying the entire tank, we add the contributions of each of the n layers and then take the limit as $n \to \infty$:

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} 1568\pi x_i^*(10 - x_i^*)^2 \,\Delta x = \int_2^{10} 1568\pi x(10 - x)^2 \,dx \\ &= 1568\pi \int_2^{10} (100x - 20x^2 + x^3)\,dx = 1568\pi \left[50x^2 - \frac{20x^3}{3} + \frac{x^4}{4} \right]_2^{10} \\ &= 1568\pi\left(\tfrac{2048}{3}\right) \approx 3.4 \times 10^6 \text{ J} \end{aligned}$$

■

6.4 Exercises

1. How much work is done when a weight lifter lifts 200 kg from 1.5 m to 2.0 m above the ground?

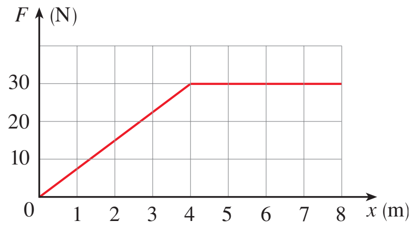
2. Compute the work done in hoisting an 500 kg grand piano from the ground up to the third floor, 10 m above the ground.

3. A variable force of $5x^{-2}$ newtons moves an object along a straight line when it is x meters from the origin. Calculate the work done in moving the object from $x = 1$ m to $x = 10$ m.

4. A variable force of $4\sqrt{x}$ newtons moves a particle along a straight path when it is x meters from the origin. Calculate the work done in moving the particle from $x = 4$ to $x = 16$.

5. Shown is the graph of a force function (in newtons) that increases to its maximum value and then remains constant.

How much work is done by the force in moving an object a distance of 8 m?



6. The table shows values of a force function $f(x)$, where x is measured in meters and $f(x)$ in newtons. Use the Midpoint Rule to estimate the work done by the force in moving an object from $x = 4$ to $x = 20$.

x	4	6	8	10	12	14	16	18	20
$f(x)$	5	5.8	7.0	8.8	9.6	8.2	6.7	5.2	4.1

7. A force of 45 N is required to hold a spring stretched 10 cm beyond its natural length. How much work is done in stretching it from its natural length to 15 cm beyond its natural length?

8. A spring has a natural length of 40 cm. If a 60-N force is required to keep the spring compressed 10 cm, how much work is done during this compression? How much work is required to compress the spring to a length of 25 cm?

9. Suppose that 2 J of work is needed to stretch a spring from its natural length of 30 cm to a length of 42 cm.
(a) How much work is needed to stretch the spring from 35 cm to 40 cm?
(b) How far beyond its natural length will a force of 30 N keep the spring stretched?

10. If the work required to stretch a spring 1 m beyond its natural length is 16 J, how much work is needed to stretch it 9 in. beyond its natural length?

11. A spring has natural length 20 cm. Compare the work W_1 done in stretching the spring from 20 cm to 30 cm with the work W_2 done in stretching it from 30 cm to 40 cm. How are W_2 and W_1 related?

12. If 6 J of work is needed to stretch a spring from 10 cm to 12 cm and another 10 J is needed to stretch it from 12 cm to 14 cm, what is the natural length of the spring?

13–22 Show how to approximate the required work by a Riemann sum. Then express the work as an integral and evaluate it.

13. A heavy rope, 15 m long, weighs 0.75 kg/m and hangs over the edge of a building 35 m high.
(a) How much work is done in pulling the rope to the top of the building?
(b) How much work is done in pulling half the rope to the top of the building?

14. A thick cable, 20 m long and weighing 80 kg, hangs from a winch on a crane. Compute in two different ways the work done if the winch winds up 7 m of the cable.
(a) Follow the method of Example 4.
(b) Write a function for the weight of the remaining cable after x meters has been wound up by the winch. Estimate the amount of work done when the winch pulls up Δx m of cable.

15. A cable that weighs 3 kg/m is used to lift 350 kg of coal up a mine shaft 150 m deep. Find the work done.

16. A chain lying on the ground is 10 m long and its mass is 80 kg. How much work is required to raise one end of the chain to a height of 6 m?

17. A 3 m chain weighs 10 kg and hangs from a ceiling. Find the work done in lifting the lower end of the chain to the ceiling so that it is level with the upper end.

18. A 0.4-kg model rocket is loaded with 0.75 kg of rocket fuel. After launch, the rocket rises at a constant rate of 4 m/s but the rocket fuel is dissipated at a rate of 0.15 kg/s. Find the work done in propelling the rocket 20 m above the ground.

19. A leaky 10-kg bucket is lifted from the ground to a height of 12 m at a constant speed with a rope that weighs 0.8 kg/m. Initially the bucket contains 36 kg of water, but the water leaks at a constant rate and finishes draining just as the bucket reaches the 12-m level. How much work is done?

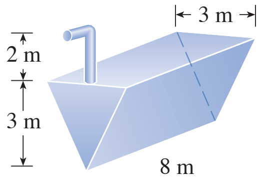
20. A circular swimming pool has a diameter of 7 m, the sides are 1.5 m high, and the depth of the water is 1.2 m. How much work is required to pump all of the water out over the side? (Use the fact that water weighs 1000 kg/m^3.)

21. An aquarium 2 m long, 1 m wide, and 1 m deep is full of water. Find the work needed to pump half of the water out of the aquarium. (Use the fact that the density of water is 1000 kg/m^3.)

22. A spherical water tank, 7 m in diameter, sits atop a 18 m-tall tower. The tank is filled by a hose attached to the bottom of the sphere. If a 1.5-horsepower pump is used to deliver water up to the tank, how long will it take to fill the tank? (One horsepower = 745.7 J of work per second.)

23–26 A tank is full of water. Find the work required to pump the water out of the spout. Use the fact that the density of water is 1000 kg/m^3.

23.



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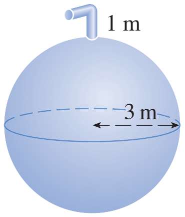

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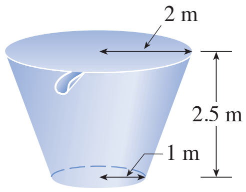
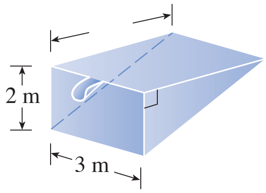


frustum of a cone

26.

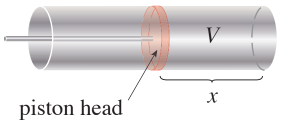


27. Suppose that for the tank in Exercise 23 the pump breaks down after 4.7×10^5 J of work has been done. What is the depth of the water remaining in the tank?

28. Solve Exercise 24 if the tank is half full of oil that has a density of 900 kg/m^3.

29. When gas expands in a cylinder with radius r, the pressure at any given time is a function of the volume: $P = P(V)$. The force exerted by the gas on the piston (see the figure) is the product of the pressure and the area: $F = \pi r^2 P$. Show that the work done by the gas when the volume expands from volume V_1 to volume V_2 is

$$W = \int_{V_1}^{V_2} P\,dV$$



30. In a steam engine the pressure P and volume V of steam satisfy the equation $PV^{1.4} = k$, where k is a constant. (This is true for adiabatic expansion, that is, expansion in which there is no heat transfer between the cylinder and its surroundings.) Use Exercise 29 to calculate the work done by the engine during a cycle when the steam starts at a pressure of 1100 kPa and a volume of 1 m^3 and expands to a volume of 8 m^3.

31–33 Work-Energy Theorem The kinetic energy KE of an object of mass m moving with velocity v is defined as $\text{KE} = \frac{1}{2}mv^2$. If a force $f(x)$ acts on the object, moving it along the x-axis from x_1 to x_2, the *Work-Energy Theorem* states that the net work done is equal to the change in kinetic energy: $\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$, where v_1 is the velocity at x_1 and v_2 is the velocity at x_2.

31. Let $x = s(t)$ be the position function of the object at time t and $v(t)$, $a(t)$ the velocity and acceleration functions. Prove the Work-Energy Theorem by first using the Substitution Rule for Definite Integrals (5.5.6) to show that

$$W = \int_{x_1}^{x_2} f(x)\,dx = \int_{t_1}^{t_2} f(s(t))\;v(t)\,dt$$

Then use Newton's Second Law of Motion (force = mass × acceleration) and the substitution $u = v(t)$ to evaluate the integral.

32. How much work (in J) is required to hurl a 5 kg bowling ball at 30 km/h?

33. Suppose that when launching an 800-kg roller coaster car an electromagnetic propulsion system exerts a force of $(5.7x^2 + 1.5x)$ newtons on the car at a distance x meters along the track. Use Exercise 31 to find the speed of the car when it has traveled 60 meters.

34. When a particle is located a distance x meters from the origin, a force of $\cos(\pi x/3)$ newtons acts on it. How much work is done in moving the particle from $x = 1$ to $x = 2$? Interpret your answer by considering the work done from $x = 1$ to $x = 1.5$ and from $x = 1.5$ to $x = 2$.

35. (a) Newton's Law of Gravitation states that two bodies with masses m_1 and m_2 attract each other with a force

$$F = G\frac{m_1 m_2}{r^2}$$

where r is the distance between the bodies and G is the gravitational constant. If one of the bodies is fixed, find the work needed to move the other from $r = a$ to $r = b$.

(b) Compute the work required to launch a 1000-kg satellite vertically to a height of 1000 km. You may assume that the earth's mass is 5.98×10^{24} kg and is concentrated at its center. Take the radius of the earth to be 6.37×10^6 m and $G = 6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2$.

36. The Great Pyramid of King Khufu was built of limestone in Egypt over a 20-year time period from 2580 BC to 2560 BC. Its base is a square with side length 230 m and its height when built was 147 m. (It was the tallest man-made structure in the world for more than 3800 years.) The density of the limestone is about 2400 kg/m^3.

(a) Estimate the total work done in building the pyramid.

(b) If each laborer worked 10 hours a day for 20 years, for 340 days a year, and did 250 J/h of work in lifting the limestone blocks into place, about how many laborers were needed to construct the pyramid?

Vladimir Korostyshevskiy / Shutterstock.com

6.5 Average Value of a Function

It is easy to calculate the average value of finitely many numbers $y_1, y_2, \ldots, y_n$:

$$y_{\text{avg}} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

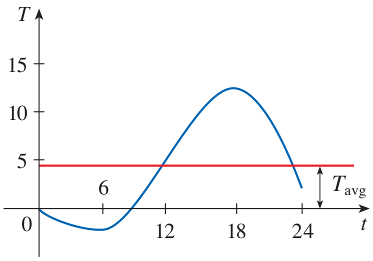


FIGURE 1

But how do we compute the average temperature during a day if infinitely many temperature readings are possible? Figure 1 shows the graph of a temperature function $T(t)$, where t is measured in hours and T in °C, and a guess at the average temperature, T_{avg}.

In general, let's try to compute the average value of a function $y = f(x)$, $a \leqslant x \leqslant b$. We start by dividing the interval $[a, b]$ into n equal subintervals, each with length $\Delta x = (b - a)/n$. Then we choose points $x_1^*, \ldots, x_n^*$ in successive subintervals and calculate the average of the numbers $f(x_1^*), \ldots, f(x_n^*)$:

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{n}$$

(For example, if f represents a temperature function and $n = 24$, this means that we take temperature readings every hour and then average them.) Since $\Delta x = (b - a)/n$, we can write $n = (b - a)/\Delta x$ and the average value becomes

$$\begin{aligned} \frac{f(x_1^*) + \cdots + f(x_n^*)}{\dfrac{b - a}{\Delta x}} &= \frac{1}{b - a}\left[f(x_1^*) + \cdots + f(x_n^*)\right]\Delta x \\ &= \frac{1}{b - a}\left[f(x_1^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x\right] \\ &= \frac{1}{b - a}\sum_{i=1}^{n} f(x_i^*)\,\Delta x \end{aligned}$$

If we let n increase, we would be computing the average value of a large number of closely spaced values. (For example, we would be averaging temperature readings taken every minute or even every second.) The limiting value is

$$\lim_{n \to \infty} \frac{1}{b - a}\sum_{i=1}^{n} f(x_i^*)\,\Delta x = \frac{1}{b - a}\int_a^b f(x)\,dx$$

by the definition of a definite integral.

Therefore we define the **average value of f** on the interval $[a, b]$ as

$$\boxed{f_{\text{avg}} = \frac{1}{b - a}\int_a^b f(x)\,dx}$$

For a positive function, we can think of this definition as saying

$$\frac{\text{area}}{\text{width}} = \text{average height}$$

EXAMPLE 1 Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

SOLUTION With $a = -1$ and $b = 2$ we have

$$f_{\text{avg}} = \frac{1}{b - a}\int_a^b f(x)\,dx = \frac{1}{2 - (-1)}\int_{-1}^{2} (1 + x^2)\,dx = \frac{1}{3}\left[x + \frac{x^3}{3}\right]_{-1}^{2} = 2 \quad ■$$

If $T(t)$ is the temperature at time t, we might wonder if there is a specific time when the temperature is the same as the average temperature. For the temperature function graphed in Figure 1, we see that there are two such times—just before noon and just before midnight. In general, is there a number c at which the value of a function f is exactly equal to the average value of the function, that is, $f(c) = f_{\text{avg}}$? The following theorem says that this is true for continuous functions.

The Mean Value Theorem for Integrals If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

that is,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

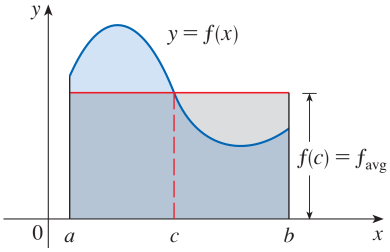


FIGURE 2

You can always chop off the top of a (two-dimensional) mountain at a certain height (namely, f_{avg}) and use it to fill in the valleys so that the mountain becomes completely flat.

The Mean Value Theorem for Integrals is a consequence of the Mean Value Theorem for derivatives and the Fundamental Theorem of Calculus. The proof is outlined in Exercise 28.

The geometric interpretation of the Mean Value Theorem for Integrals is that, for *positive* functions f, there is a number c such that the rectangle with base $[a, b]$ and height $f(c)$ has the same area as the region under the graph of f from a to b. (See Figure 2 and the more picturesque interpretation in the margin note.)

EXAMPLE 2 Since $f(x) = 1 + x^2$ is continuous on the interval $[-1, 2]$, the Mean Value Theorem for Integrals says there is a number c in $[-1, 2]$ such that

$$\int_{-1}^{2} (1 + x^2)\,dx = f(c)[2 - (-1)]$$

In this particular case we can find c explicitly. From Example 1 we know that $f_{\text{avg}} = 2$, so the value of c satisfies

$$f(c) = f_{\text{avg}} = 2$$

Therefore $\qquad 1 + c^2 = 2 \qquad \text{so} \qquad c^2 = 1$

So in this case there happen to be two numbers $c = \pm 1$ in the interval $[-1, 2]$ that work in the Mean Value Theorem for Integrals. ■

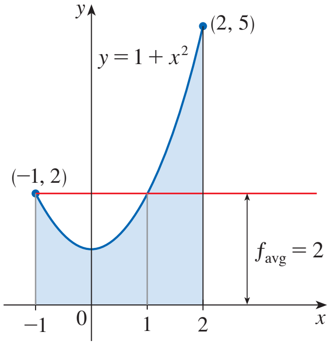


FIGURE 3

Examples 1 and 2 are illustrated by Figure 3.

EXAMPLE 3 Show that the average velocity of a car over a time interval $[t_1, t_2]$ is the same as the average of its velocities during the trip.

SOLUTION If $s(t)$ is the displacement of the car at time t, then, by definition, the average velocity of the car over the interval is

$$\frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

On the other hand, the average value of the velocity function on the interval is

$$\begin{aligned} v_{\text{avg}} &= \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} v(t)\,dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} s'(t)\,dt \\ &= \frac{1}{t_2 - t_1}[s(t_2) - s(t_1)] \qquad \text{(by the Net Change Theorem)} \\ &= \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \text{average velocity} \end{aligned}$$

■

6.5 Exercises

1–8 Find the average value of the function on the given interval.

1. $f(x) = 3x^2 + 8x, \quad [-1, 2]$

2. $f(x) = \sqrt{x}, \quad [0, 4]$

3. $g(x) = 3\cos x, \quad [-\pi/2, \pi/2]$

4. $f(z) = \dfrac{e^{1/z}}{z^2}, \quad [1, 4]$

5. $g(t) = \dfrac{9}{1 + t^2}, \quad [0, 2]$

6. $f(x) = \dfrac{x^2}{(x^3 + 3)^2}, \quad [-1, 1]$

7. $h(x) = \cos^4 x \sin x, \quad [0, \pi]$

8. $h(u) = \dfrac{\ln u}{u}, \quad [1, 5]$

9–12

(a) Find the average value of f on the given interval.

(b) Find c in the given interval such that $f_{\text{avg}} = f(c)$.

(c) Sketch the graph of f and a rectangle whose base is the given interval and whose area is the same as the area under the graph of f.

9. $f(t) = 1/t^2, \quad [1, 3]$

10. $g(x) = (x + 1)^3, \quad [0, 2]$

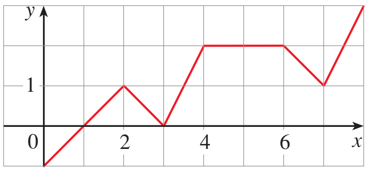
11. $f(x) = 2\sin x - \sin 2x, \quad [0, \pi]$

12. $f(x) = 2xe^{-x^2}, \quad [0, 2]$

13. If f is continuous and $\int_1^3 f(x)\,dx = 8$, show that f takes on the value 4 at least once on the interval $[1, 3]$.

14. Find the numbers b such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

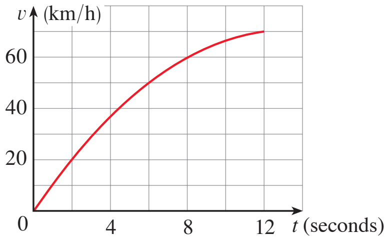
15. Find the average value of f on $[0, 8]$.



16. The velocity graph of an accelerating car is shown.

(a) Use the Midpoint Rule to estimate the average velocity of the car during the first 12 seconds.

(b) At what time was the instantaneous velocity equal to the average velocity?



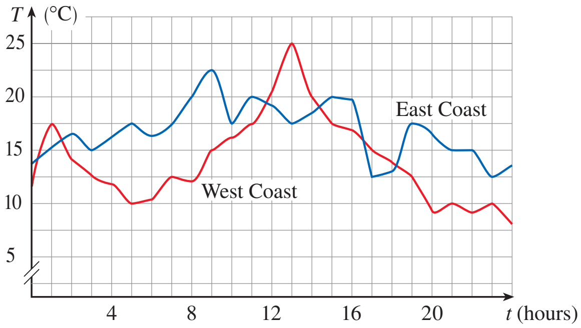
17. In a certain city the temperature (in °C) t hours after 9 AM was modeled by the function

$$T(t) = 10 + 4\sin\frac{\pi t}{12}$$

Find the average temperature during the period from 9 AM to 9 PM.

18. The figure shows graphs of the temperatures for a city on the East Coast and a city on the West Coast during a 24-hour period starting at midnight. Which city had the highest temperature that day? Find the average temperature during this time period for each city using the Midpoint Rule with

$n = 12$. Interpret your results; which city was "warmer" overall that day?



19. The linear density in a rod 8 m long is $12/\sqrt{x+1}$ kg/m, where x is measured in meters from one end of the rod. Find the average density of the rod.

20. The velocity v of blood that flows in a blood vessel with radius R and length l at a distance r from the central axis is

$$v(r) = \frac{P}{4\eta l}(R^2 - r^2)$$

where P is the pressure difference between the ends of the vessel and η is the viscosity of the blood (see Example 3.7.7). Find the average velocity (with respect to r) over the interval $0 \leqslant r \leqslant R$. Compare the average velocity with the maximum velocity.

21. In Example 3.8.1 we modeled the world population in the second half of the 20th century by the equation $P(t) = 2560e^{0.017185t}$. Use this equation to estimate the average world population during this time period (1950–2000).

22. (a) A cup of coffee has temperature 95°C and takes 30 minutes to cool to 61°C in a room with temperature 20°C. Use Newton's Law of Cooling (Section 3.8) to show that the temperature of the coffee after t minutes is

$$T(t) = 20 + 75e^{-kt}$$

where $k \approx 0.02$.

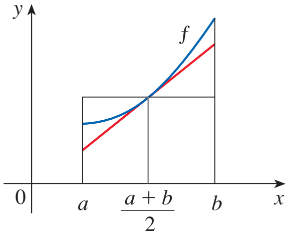
(b) What is the average temperature of the coffee during the first half hour?

23. Use the result of Exercise 5.5.89 to compute the average volume of inhaled air in the lungs in one respiratory cycle.

24. If a freely falling body starts from rest, then its displacement is given by $s = \frac{1}{2}gt^2$. Let the velocity after a time T be v_T. Show that if we compute the average of the velocities with respect to t we get $v_{\text{avg}} = \frac{1}{2}v_T$, but if we compute the average of the velocities with respect to s we get $v_{\text{avg}} = \frac{2}{3}v_T$.

25. Use the diagram to show that if f is concave upward on $[a, b]$, then

$$f_{\text{avg}} > f\left(\frac{a+b}{2}\right)$$



26–27 Let $f_{\text{avg}}[a, b]$ denote the average value of f on the interval $[a, b]$.

26. Show that if $a < c < b$, then

$$f_{\text{avg}}[a, b] = \left(\frac{c-a}{b-a}\right) f_{\text{avg}}[a, c] + \left(\frac{b-c}{b-a}\right) f_{\text{avg}}[c, b]$$

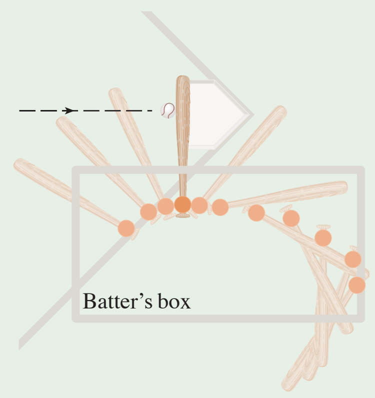
27. Show that if f is continuous, then $\lim_{t \to a^+} f_{\text{avg}}[a, t] = f(a)$.

28. Prove the Mean Value Theorem for Integrals by applying the Mean Value Theorem for derivatives (see Section 4.2) to the function $F(x) = \int_a^x f(t)\,dt$.

APPLIED PROJECT | CALCULUS AND BASEBALL

In this project we explore three of the many applications of calculus to baseball. The physical interactions of the game, especially the collision of ball and bat, are quite complex and their models are discussed in detail in a book by Robert Adair, *The Physics of Baseball,* 3d ed. (New York, 2002).

1. It may surprise you to learn that the collision of baseball and bat lasts only about a thousandth of a second. Here we calculate the average force on the bat during this collision by first computing the change in the ball's momentum.



An overhead view of the position of a baseball bat, shown every fiftieth of a second during a typical swing.
(Adapted from *The Physics of Baseball*)

The *momentum* p of an object is the product of its mass m and its velocity v, that is, $p = mv$. Suppose an object, moving along a straight line, is acted on by a force $F = F(t)$ that is a continuous function of time.

(a) Show that the change in momentum over a time interval $[t_0, t_1]$ is equal to the integral of F from t_0 to t_1; that is, show that

$$p(t_1) - p(t_0) = \int_{t_0}^{t_1} F(t)\, dt$$

This integral is called the *impulse* of the force over the time interval.

(b) A pitcher throws a 145 km/h fastball to a batter, who hits a line drive directly back to the pitcher. The ball is in contact with the bat for 0.001 s and leaves the bat with velocity 180 km/h. A baseball weighs .14 kg.
  (i) Find the change in the ball's momentum.
  (ii) Find the average force on the bat.

2. In this problem we calculate the work required for a pitcher to throw a 145 km/h fastball by first considering kinetic energy.

The *kinetic energy* K of an object of mass m and velocity v is given by $K = \frac{1}{2}mv^2$. Suppose an object of mass m, moving in a straight line, is acted on by a force $F = F(s)$ that depends on its position s. According to Newton's Second Law

$$F(s) = ma = m\frac{dv}{dt}$$

where a and v denote the acceleration and velocity of the object.

(a) Show that the work done in moving the object from a position s_0 to a position s_1 is equal to the change in the object's kinetic energy; that is, show that

$$W = \int_{s_0}^{s_1} F(s)\, ds = \tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_0^2$$

where $v_0 = v(s_0)$ and $v_1 = v(s_1)$ are the velocities of the object at the positions s_0 and s_1. *Hint:* By the Chain Rule,

$$m\frac{dv}{dt} = m\frac{dv}{ds}\frac{ds}{dt} = mv\frac{dv}{ds}$$

(b) How many joules of work does it take to throw a baseball at a speed of 145 km/h?

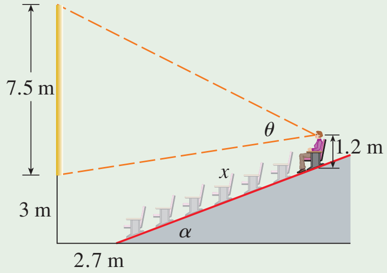
3. (a) An outfielder fields a baseball 85 m away from home plate and throws it directly to the catcher with an initial velocity of 30 m/s. Assume that the velocity $v(t)$ of the ball after t seconds satisfies the differential equation $dv/dt = -\frac{1}{10}v$ because of air resistance. How long does it take for the ball to reach home plate? (Ignore any vertical motion of the ball.)

(b) The manager of the team wonders whether the ball will reach home plate sooner if it is relayed by an infielder. The shortstop can position himself directly between the outfielder and home plate, catch the ball thrown by the outfielder, turn, and throw the ball to the catcher with an initial velocity of 32 m/s. The manager clocks the relay time of the shortstop (catching, turning, throwing) at half a second. How far from home plate should the shortstop position himself to minimize the total time for the ball to reach home plate? Should the manager encourage a direct throw or a relayed throw? What if the shortstop can throw at 35 m/s?

T (c) For what throwing velocity of the shortstop does a relayed throw take the same time as a direct throw?

APPLIED PROJECT | WHERE TO SIT AT THE MOVIES

A movie theater has a screen that is positioned 3 m off the floor and is 7.5 m high. The first row of seats is placed 2.7 m from the screen and the rows are set 0.9 m apart. The floor of the seating area is inclined at an angle of $\alpha = 20°$ above the horizontal and the distance up the incline that you sit is x. The theater has 21 rows of seats, so $0 \leqslant x \leqslant 18$. Suppose you decide that the best place to sit is in the row where the angle θ subtended by the screen at your eyes is a maximum. Let's also suppose that your eyes are 1.2 m above the floor, as shown in the figure. (In Exercise 4.7.84 we looked at a simpler version of this problem, where the floor is horizontal, but this project involves a more complicated situation and requires technology.)



1. Show that

$$\theta = \arccos\left(\frac{a^2 + b^2 - 56.25}{2ab}\right)$$

where $\qquad a^2 = (2.7 + x\cos\alpha)^2 + (9.3 - x\sin\alpha)^2$

and $\qquad b^2 = (2.7 + x\cos\alpha)^2 + (x\sin\alpha - 1.8)^2$

2. Use a graph of θ as a function of x to estimate the value of x that maximizes θ. In which row should you sit? What is the viewing angle θ in this row?

3. Use a computer algebra system to differentiate θ and find a numerical value for the root of the equation $d\theta/dx = 0$. Does this value confirm your result in Problem 2?

4. Use the graph of θ to estimate the average value of θ on the interval $0 \leqslant x \leqslant 18$. Then use a computer algebra system to compute the average value. Compare with the maximum and minimum values of θ.

6 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. (a) Draw two typical curves $y = f(x)$ and $y = g(x)$, where $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$. Show how to approximate the area between these curves by a Riemann sum and sketch the corresponding approximating rectangles. Then write an expression for the exact area.
   (b) Explain how the situation changes if the curves have equations $x = f(y)$ and $x = g(y)$, where $f(y) \geqslant g(y)$ for $c \leqslant y \leqslant d$.

2. Suppose that Sue runs faster than Kathy throughout a 1500-meter race. What is the physical meaning of the area between their velocity curves for the first minute of the race?

3. (a) Suppose S is a solid with known cross-sectional areas. Explain how to approximate the volume of S by a Riemann sum. Then write an expression for the exact volume.
   (b) If S is a solid of revolution, how do you find the cross-sectional areas?

4. (a) What is the volume of a cylindrical shell?
   (b) Explain how to use cylindrical shells to find the volume of a solid of revolution.
   (c) Why might you want to use the shell method instead of the disk or washer method?

5. Suppose that you push a book across a 6-meter-long table by exerting a force $f(x)$ at each point from $x = 0$ to $x = 6$. What does $\int_0^6 f(x)\,dx$ represent? If $f(x)$ is measured in newtons, what are the units for the integral?

6. (a) What is the average value of a function f on an interval $[a, b]$?
   (b) What does the Mean Value Theorem for Integrals say? What is its geometric interpretation?

TRUE-FALSE QUIZ

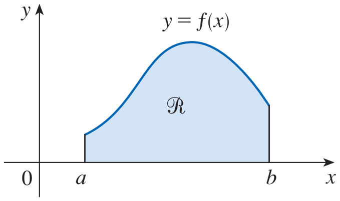
Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. The area between the curves $y = f(x)$ and $y = g(x)$ for $a \leqslant x \leqslant b$ is $A = \int_a^b [f(x) - g(x)]\,dx$.

2. A cube is a solid of revolution.

3. If the region bounded by the curves $y = \sqrt{x}$ and $y = x$ is revolved about the x-axis, then the volume of the resulting solid is $V = \int_0^1 \pi(\sqrt{x} - x)^2\,dx$.

4–9 Let $\mathcal{R}$ be the region shown.



4. If $\mathcal{R}$ is revolved about the y-axis, then the volume of the resulting solid is $V = \int_a^b 2\pi x\, f(x)\,dx$.

5. If $\mathcal{R}$ is revolved about the x-axis, then the volume of the resulting solid is $V = \int_a^b \pi[f(x)]^2\,dx$.

6. If $\mathcal{R}$ is revolved about the x-axis, then vertical cross-sections perpendicular to the x-axis of the resulting solid are disks.

7. If $\mathcal{R}$ is revolved about the y-axis, then horizontal cross-sections of the resulting solid are cylindrical shells.

8. The volume of the solid obtained by revolving $\mathcal{R}$ about the line $x = -2$ is the same as the volume of the solid obtained by revolving $\mathcal{R}$ about the y-axis.

9. If $\mathcal{R}$ is the base of a solid S and cross-sections of S perpendicular to the x-axis are squares, then the volume of S is $V = \int_a^b [f(x)]^2\,dx$.

10. A cable hangs vertically from a winch located at the top of a tall building. The work required for the winch to pull up the top half of the cable is half of the work required to pull up the entire cable.

11. If $\int_2^5 f(x)\,dx = 12$, then the average value of f on $[2, 5]$ is 4.

EXERCISES

1–6 Find the area of the region bounded by the given curves.

1. $y = x^2, \quad y = 8x - x^2$

2. $y = \sqrt{x}, \quad y = -\sqrt[3]{x}, \quad y = x - 2$

3. $y = 1 - 2x^2, \quad y = |x|$

4. $x + y = 0, \quad x = y^2 + 3y$

5. $y = \sin(\pi x/2), \quad y = x^2 - 2x$

6. $y = \sqrt{x}, \quad y = x^2, \quad x = 2$

7–11 Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.

7. $y = 2x,\ y = x^2;$ about the x-axis

8. $x = 1 + y^2,\ y = x - 3;$ about the y-axis

9. $x = 0,\ x = 9 - y^2;$ about $x = -1$

10. $y = x^2 + 1,\ y = 9 - x^2;$ about $y = -1$

11. $x^2 - y^2 = a^2,\ x = a + h$ (where $a > 0,\ h > 0$); about the y-axis

12–14 Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.

12. $y = \tan x,\ y = x,\ x = \pi/3;$ about the y-axis

13. $y = \cos^2 x,\ |x| \leqslant \pi/2,\ y = \frac{1}{4};$ about $x = \pi/2$

14. $y = \ln x,\ y = 0,\ x = 4;$ about $x = -1$

15–16 The region bounded by the given curves is rotated about the specified axis. Find the volume of the solid using (a) x as the variable of integration and (b) y as the variable of integration.

15. $y = x^3,\ y = 3x^2;$ about $x = -1$

16. $y = \sqrt{x},\ y = x^2;$ about $y = 3$

17. Find the volumes of the solids obtained by rotating the region bounded by the curves $y = x$ and $y = x^2$ about the following lines.
(a) The x-axis (b) The y-axis (c) $y = 2$

18. Let $\mathcal{R}$ be the region in the first quadrant bounded by the curves $y = x^3$ and $y = 2x - x^2$. Calculate the following quantities.
(a) The area of $\mathcal{R}$
(b) The volume obtained by rotating $\mathcal{R}$ about the x-axis
(c) The volume obtained by rotating $\mathcal{R}$ about the y-axis

19. Let $\mathcal{R}$ be the region bounded by the curves $y = \tan(x^2)$, $x = 1$, and $y = 0$. Use the Midpoint Rule with $n = 4$ to estimate the following quantities.
(a) The area of $\mathcal{R}$
(b) The volume obtained by rotating $\mathcal{R}$ about the x-axis

20. Let $\mathcal{R}$ be the region bounded by the curves $y = 1 - x^2$ and $y = x^6 - x + 1$. Estimate the following quantities.
- (a) The x-coordinates of the points of intersection of the curves
- (b) The area of $\mathcal{R}$
- (c) The volume generated when $\mathcal{R}$ is rotated about the x-axis
- (d) The volume generated when $\mathcal{R}$ is rotated about the y-axis

21–24 Each integral represents the volume of a solid. Describe the solid.

21. $\int_0^{\pi/2} 2\pi x \cos x \, dx$

22. $\int_0^{\pi/2} 2\pi \cos^2 x \, dx$

23. $\int_0^{\pi} \pi(2 - \sin x)^2 \, dx$

24. $\int_0^{4} 2\pi(6 - y)(4y - y^2) \, dy$

---

25. The base of a solid is a circular disk with radius 3. Find the volume of the solid if parallel cross-sections perpendicular to the base are isosceles right triangles with hypotenuse lying along the base.

26. The base of a solid is the region bounded by the parabolas $y = x^2$ and $y = 2 - x^2$. Find the volume of the solid if the cross-sections perpendicular to the x-axis are squares with one side lying along the base.

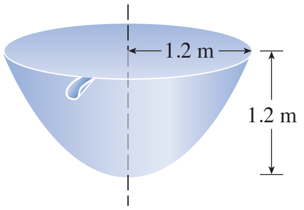
27. The height of a monument is 20 m. A horizontal cross-section at a distance x meters from the top is an equilateral triangle with side $\frac{1}{4}x$ meters. Find the volume of the monument.

28. (a) The base of a solid is a square with vertices located at $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$. Each cross-section perpendicular to the x-axis is a semicircle. Find the volume of the solid.
- (b) Show that the volume of the solid of part (a) can be computed more simply by first cutting the solid and rearranging it to form a cone.

29. A force of 30 N is required to maintain a spring stretched from its natural length of 12 cm to a length of 15 cm. How much work is done in stretching the spring from 12 cm to 20 cm?

30. A 725 kg elevator is suspended by a 60 m cable that has a mass of 15 kg/m. How much is required to raise the elevator from the basement to the third floor, a distance of 9 m?

31. A tank full of water has the shape of a paraboloid of revolution as shown in the figure; that is, its shape is obtained by rotating a parabola about a vertical axis.
- (a) If its height is 1.2 m and the radius at the top is 1.2 m, find the work required to pump the water out of the tank.
- (b) After 4000 J of work has been done, what is the depth of the water remaining in the tank?



32. A steel tank has the shape of a circular cylinder oriented vertically with diameter 4 m and height 5 m. The tank is currently filled to a level of 3 m with cooking oil that has a density of 920 kg/m^3. Compute the work required to pump the oil out through a 1-m spout at the top of the tank.

33. Find the average value of the function $f(t) = \sec^2 t$ on the interval $[0, \pi/4]$.

34. (a) Find the average value of the function $f(x) = 1/\sqrt{x}$ on the interval $[1, 4]$.
- (b) Find the value c guaranteed by the Mean Value Theorem for Integrals such that $f_{\text{avg}} = f(c)$.
- (c) Sketch the graph of f on $[1, 4]$ and a rectangle with base $[1, 4]$ whose area is the same as the area under the graph of f.

35. Let $\mathcal{R}_1$ be the region bounded by $y = x^2$, $y = 0$, and $x = b$, where $b > 0$. Let $\mathcal{R}_2$ be the region bounded by $y = x^2$, $x = 0$, and $y = b^2$.
- (a) Is there a value of b such that $\mathcal{R}_1$ and $\mathcal{R}_2$ have the same area?
- (b) Is there a value of b such that $\mathcal{R}_1$ sweeps out the same volume when rotated about the x-axis and the y-axis?
- (c) Is there a value of b such that $\mathcal{R}_1$ and $\mathcal{R}_2$ sweep out the same volume when rotated about the x-axis?
- (d) Is there a value of b such that $\mathcal{R}_1$ and $\mathcal{R}_2$ sweep out the same volume when rotated about the y-axis?

Problems Plus

1. A solid is generated by rotating about the x-axis the region under the curve $y = f(x)$, where f is a positive function and $x \geqslant 0$. The volume generated by the part of the curve from $x = 0$ to $x = b$ is b^2 for all $b > 0$. Find the function f.

2. There is a line through the origin that divides the region bounded by the parabola $y = x - x^2$ and the x-axis into two regions with equal area. What is the slope of that line?

3. The figure shows a curve C with the property that, for every point P on the middle curve $y = 2x^2$, the areas A and B are equal. Find an equation for C.

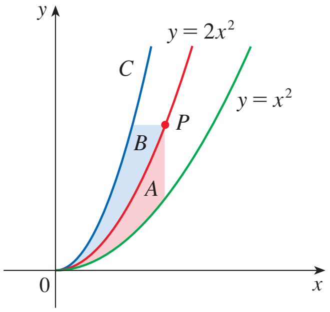


FIGURE FOR PROBLEM 3

4. A cylindrical glass of radius r and height L is filled with water and then tilted until the water remaining in the glass exactly covers its base.

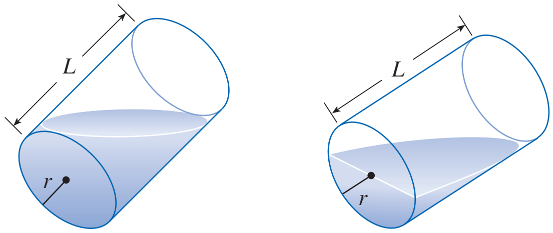
(a) Determine a way to "slice" the water into parallel rectangular cross-sections and then *set up* a definite integral for the volume of the water in the glass.

(b) Determine a way to "slice" the water into parallel cross-sections that are trapezoids and then *set up* a definite integral for the volume of the water.

(c) Find the volume of water in the glass by evaluating one of the integrals in part (a) or part (b).

(d) Find the volume of the water in the glass from purely geometric considerations.

(e) Suppose the glass is tilted until the water exactly covers half the base. In what direction can you "slice" the water into triangular cross-sections? Rectangular cross-sections? Cross-sections that are segments of circles? Find the volume of water in the glass.



5. Water in an open bowl evaporates at a rate proportional to the area of the surface of the water. (This means that the rate of decrease of the volume is proportional to the area of the surface.) Show that the depth of the water decreases at a constant rate, regardless of the shape of the bowl.

6. Archimedes' Principle states that the buoyant force on an object partially or fully submerged in a fluid is equal to the weight of the fluid that the object displaces. Thus, for an object of density ρ_0 floating partly submerged in a fluid of density ρ_f, the buoyant force is given by $F = \rho_f g \int_{-h}^{0} A(y)\,dy$, where g is the acceleration due to gravity and $A(y)$ is the area of a typical cross-section of the object (see the figure). The weight of the object is given by

$$W = \rho_0 g \int_{-h}^{L-h} A(y)\,dy$$

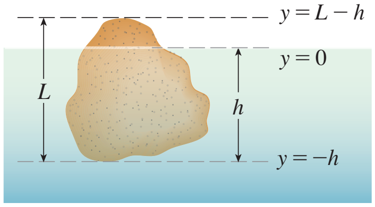


FIGURE FOR PROBLEM 6

(a) Show that the percentage of the volume of the object above the surface of the liquid is

$$100\,\frac{\rho_f - \rho_0}{\rho_f}$$

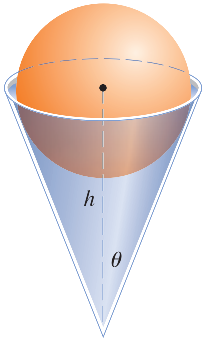
(b) The density of ice is 917 kg/m^3 and the density of seawater is 1030 kg/m^3. What percentage of the volume of an iceberg is above water?

(c) An ice cube floats in a glass filled to the brim with water. Does the water overflow when the ice melts?

(d) A sphere of radius 0.4 m and having negligible weight is floating in a large freshwater lake. How much work is required to completely submerge the sphere? The density of the water is 1000 kg/m^3.

7. A sphere of radius 1 overlaps a smaller sphere of radius r in such a way that their intersection is a circle of radius r. (In other words, they intersect in a great circle of the small sphere.) Find r so that the volume inside the small sphere and outside the large sphere is as large as possible.

8. A paper drinking cup filled with water has the shape of a cone with height h and semi-vertical angle θ. (See the figure.) A ball is placed carefully in the cup, thereby displacing some of the water and making it overflow. What is the radius of the ball that causes the greatest volume of water to spill out of the cup?



9. A *clepsydra*, or water clock, is a glass container with a small hole in the bottom through which water can flow. The "clock" is calibrated for measuring time by placing markings on the container corresponding to water levels at equally spaced times. Let $x = f(y)$ be continuous on the interval $[0, b]$ and assume that the container is formed by rotating the graph of f about the y-axis. Let V denote the volume of water and h the height of the water level at time t. (See the figure.)

(a) Determine V as a function of h.

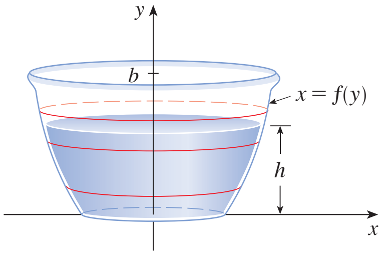
(b) Show that

$$\frac{dV}{dt} = \pi[f(h)]^2 \frac{dh}{dt}$$

(c) Suppose that A is the area of the hole in the bottom of the container. It follows from Torricelli's Law that the rate of change of the volume of the water is given by

$$\frac{dV}{dt} = kA\sqrt{h}$$

where k is a negative constant. Determine a formula for the function f such that dh/dt is a constant C. What is the advantage in having $dh/dt = C$?



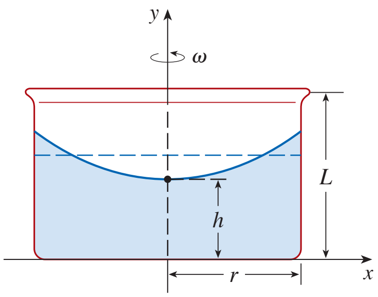


FIGURE FOR PROBLEM 10

10. A cylindrical container of radius r and height L is partially filled with a liquid whose volume is V. If the container is rotated about its axis of symmetry with constant angular speed ω, then the container will induce a rotational motion in the liquid around the same axis. Eventually, the liquid will be rotating at the same angular speed as the container. The surface of the liquid will be convex, as indicated in the figure, because the centrifugal force on the liquid particles increases with the distance from the axis of the container. It can be shown that the surface of the liquid is a paraboloid of revolution generated by rotating the parabola

$$y = h + \frac{\omega^2 x^2}{2g}$$

about the y-axis, where g is the acceleration due to gravity.

(a) Determine h as a function of ω.

(b) At what angular speed will the surface of the liquid touch the bottom? At what speed will it spill over the top?

(c) Suppose the radius of the container is 2 m, the height is 7 m, and the container and liquid are rotating at the same constant angular speed. The surface of the liquid is 5 m below the top of the tank at the central axis and 4 m below the top of the tank 1 m out from the central axis.

  (i) Determine the angular speed of the container and the volume of the fluid.

  (ii) How far below the top of the tank is the liquid at the wall of the container?

11. Suppose the graph of a cubic polynomial intersects the parabola $y = x^2$ when $x = 0$, $x = a$, and $x = b$, where $0 < a < b$. If the two regions between the curves have the same area, how is b related to a?

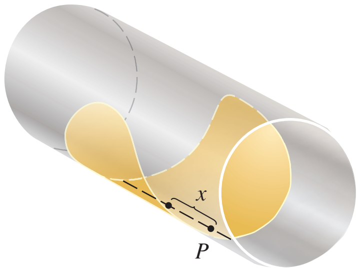
T **12.** Suppose we are planning to make a taco from a round tortilla with diameter 8 inches by bending the tortilla so that it is shaped as if it is partially wrapped around a circular cylinder. We will fill the tortilla to the edge (but no more) with meat, cheese, and other ingredients. Our problem is to decide how to curve the tortilla in order to maximize the volume of food it can hold.

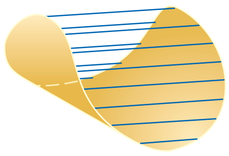
(a) We start by placing a circular cylinder of radius r along a diameter of the tortilla and folding the tortilla around the cylinder. Let x represent the distance from the center of the tortilla to a point P on the diameter (see the figure). Show that the cross-sectional area of the filled taco in the plane through P perpendicular to the axis of the cylinder is

$$A(x) = r\sqrt{16 - x^2} - \tfrac{1}{2}r^2 \sin\left(\frac{2}{r}\sqrt{16 - x^2}\right)$$

and write an expression for the volume of the filled taco.

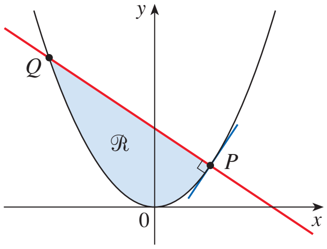
(b) Determine (approximately) the value of r that maximizes the volume of the taco. (Use a graphical approach.)





13. If the tangent at a point P on the curve $y = x^3$ intersects the curve again at Q, let A be the area of the region bounded by the curve and the line segment PQ. Let B be the area of the region defined in the same way starting with Q instead of P. What is the relationship between A and B?

14. Let $P(a, a^2)$, $a > 0$, be any point on the part of the parabola $y = x^2$ in the first quadrant, and let $\mathcal{R}$ be the region bounded by the parabola and the normal line through P. (See the figure.) Show that the area of $\mathcal{R}$ is smallest when $a = \frac{1}{2}$. (See also Problem 11 in Problems Plus following Chapter 4.)



The physical principles that govern the motion of a model rocket also apply to rockets that send spacecraft into earth orbit. In Exercise 7.1.74 you will use an integral to calculate the fuel needed to send a rocket to a specified height above the earth.

Ben Cooper / Science Faction / Getty Images; inset: Rasvan ILIESCU / Alamy Stock Photo

7

Techniques of Integration

BECAUSE OF THE FUNDAMENTAL THEOREM of Calculus, we can integrate a function if we know an antiderivative, that is, an indefinite integral. We summarize here the most important integrals that we have learned so far.

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C$$

$$\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C, \ a > 0$$

$$\int \sinh x\,dx = \cosh x + C$$

$$\int \cosh x\,dx = \sinh x + C$$

In this chapter we develop techniques for using these basic integration formulas to obtain indefinite integrals of more complicated functions. We learned the most important method of integration, the Substitution Rule, in Section 5.5. The other general technique, integration by parts, is presented in Section 7.1. Then we learn methods that are special to particular classes of functions, such as trigonometric functions and rational functions.

Integration is not as straightforward as differentiation; there are no rules that absolutely guarantee obtaining an indefinite integral of a function. Therefore we discuss a strategy for integration in Section 7.5.

7.1 Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The integration rule that corresponds to the Product Rule for differentiation is called *integration by parts*.

Integration by Parts: Indefinite Integrals

The Product Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

In the notation for indefinite integrals this equation becomes

$$\int [f(x)g'(x) + g(x)f'(x)]\,dx = f(x)g(x)$$

or

$$\int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx = f(x)g(x)$$

We can rearrange this equation as

$$\boxed{1} \qquad \boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx}$$

Formula 1 is called the **formula for integration by parts**. It is perhaps easier to remember in the following notation. Let $u = f(x)$ and $v = g(x)$. Then the differentials are $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, so, by the Substitution Rule, the formula for integration by parts becomes

$$\boxed{2} \qquad \boxed{\int u\,dv = uv - \int v\,du}$$

EXAMPLE 1 Find $\int x \sin x\,dx$.

SOLUTION USING FORMULA 1 Suppose we choose $f(x) = x$ and $g'(x) = \sin x$. Then $f'(x) = 1$ and $g(x) = -\cos x$. (For g we can choose *any* antiderivative of g'.) Thus, using Formula 1, we have

$$\begin{aligned}\int x \sin x\,dx &= f(x)g(x) - \int g(x)f'(x)\,dx \\ &= x(-\cos x) - \int (-\cos x)\,dx = -x\cos x + \int \cos x\,dx \\ &= -x\cos x + \sin x + C\end{aligned}$$

It's wise to check the answer by differentiating it. If we do so, we get $x \sin x$, as expected.

It is helpful to use the pattern:

$$\begin{array}{ll} u = \square & dv = \square \\ du = \square & v = \square \end{array}$$

SOLUTION USING FORMULA 2 Let

$$u = x \qquad dv = \sin x\, dx$$

Then $$du = dx \qquad v = -\cos x$$

and so

$$\int x \sin x\, dx = \int \overbrace{x}^{u}\ \overbrace{\sin x\, dx}^{dv} = \overbrace{x}^{u}\ \overbrace{(-\cos x)}^{v} - \int \overbrace{(-\cos x)}^{v}\ \overbrace{dx}^{du}$$

$$= -x \cos x + \int \cos x\, dx$$

$$= -x \cos x + \sin x + C$$ ■

NOTE Our aim in using integration by parts is to obtain a simpler integral than the one we started with. Thus in Example 1 we started with $\int x \sin x\, dx$ and expressed it in terms of the simpler integral $\int \cos x\, dx$. If we had instead chosen $u = \sin x$ and $dv = x\, dx$, then $du = \cos x\, dx$ and $v = x^2/2$, so integration by parts gives

$$\int x \sin x\, dx = (\sin x)\frac{x^2}{2} - \frac{1}{2}\int x^2 \cos x\, dx$$

Although this is true, $\int x^2 \cos x\, dx$ is a more difficult integral than the one we started with. In general, when deciding on a choice for u and dv, we usually try to choose $u = f(x)$ to be a function that becomes simpler when differentiated (or at least not more complicated) as long as $dv = g'(x)\, dx$ can be readily integrated to give v.

EXAMPLE 2 Evaluate $\int \ln x\, dx$.

SOLUTION Here we don't have much choice for u and dv. Let

$$u = \ln x \qquad dv = dx$$

Then $$du = \frac{1}{x}\, dx \qquad v = x$$

Integrating by parts, we get

$$\int \ln x\, dx = x \ln x - \int x \cdot \frac{1}{x}\, dx$$

It's customary to write $\int 1\, dx$ as $\int dx$.

$$= x \ln x - \int dx$$

Check the answer by differentiating it.

$$= x \ln x - x + C$$

Integration by parts is effective in this example because the derivative of the function $f(x) = \ln x$ is simpler than f. ■

EXAMPLE 3 Find $\int t^2 e^t\,dt$.

SOLUTION Notice that e^t is unchanged when differentiated or integrated whereas t^2 becomes simpler when differentiated, so we choose

$$u = t^2 \qquad dv = e^t\,dt$$

Then

$$du = 2t\,dt \qquad v = e^t$$

Integration by parts gives

$$\boxed{3} \qquad \int t^2 e^t\,dt = t^2 e^t - 2\int te^t\,dt$$

The integral that we obtained, $\int te^t\,dt$, is simpler than the original integral but is still not obvious. Therefore we use integration by parts a second time, this time with $u = t$ and $dv = e^t\,dt$. Then $du = dt$, $v = e^t$, and

$$\begin{aligned}\int te^t\,dt &= te^t - \int e^t\,dt \\ &= te^t - e^t + C\end{aligned}$$

Putting this in Equation 3, we get

$$\begin{aligned}\int t^2 e^t\,dt &= t^2 e^t - 2\int te^t\,dt \\ &= t^2 e^t - 2(te^t - e^t + C) \\ &= t^2 e^t - 2te^t + 2e^t + C_1 \qquad \text{where } C_1 = -2C\end{aligned}$$

■

EXAMPLE 4 Evaluate $\int e^x \sin x\,dx$.

An easier method, using complex numbers, is given in Exercise 50 in Appendix H.

SOLUTION Neither e^x nor $\sin x$ becomes simpler when differentiated, so let's try choosing $u = e^x$ and $dv = \sin x\,dx$. (It turns out that, in this example, choosing $u = \sin x$, $dv = e^x\,dx$ also works.) Then $du = e^x\,dx$ and $v = -\cos x$, so integration by parts gives

$$\boxed{4} \qquad \int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx$$

The integral that we have obtained, $\int e^x \cos x\,dx$, is no simpler than the original one, but at least it's no more difficult. Having had success in the preceding example integrating by parts twice, we persevere and integrate by parts again. It is important that we again choose $u = e^x$, so $dv = \cos x\,dx$. Then $du = e^x\,dx$, $v = \sin x$, and

$$\boxed{5} \qquad \int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

At first glance, it appears as if we have accomplished nothing because we have arrived at $\int e^x \sin x\,dx$, which is where we started. However, if we put the expression for $\int e^x \cos x\,dx$ from Equation 5 into Equation 4 we get

$$\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx$$

This can be regarded as an equation to be solved for the unknown integral. Adding $\int e^x \sin x\,dx$ to both sides, we obtain

$$2\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x$$

Dividing by 2 and adding the constant of integration, we get

$$\int e^x \sin x\,dx = \tfrac{1}{2}e^x(\sin x - \cos x) + C$$ ■

Figure 1 illustrates Example 4 by showing the graphs of $f(x) = e^x \sin x$ and $F(x) = \frac{1}{2}e^x(\sin x - \cos x)$. As a visual check on our work, notice that $f(x) = 0$ when F has a maximum or minimum.

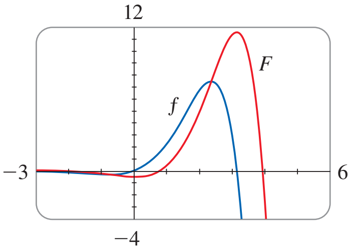


FIGURE 1

Integration by Parts: Definite Integrals

If we combine the formula for integration by parts with Part 2 of the Fundamental Theorem of Calculus, we can evaluate definite integrals by parts. Evaluating both sides of Formula 1 between a and b, assuming f' and g' are continuous, and using the Fundamental Theorem, we obtain

$$\boxed{6} \qquad \boxed{\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx}$$

EXAMPLE 5 Calculate $\int_0^1 \tan^{-1}x\,dx$.

SOLUTION Let

$$u = \tan^{-1}x \qquad dv = dx$$

Then $$du = \frac{dx}{1+x^2} \qquad v = x$$

So Formula 6 gives

$$\int_0^1 \tan^{-1}x\,dx = x\tan^{-1}x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= 1\cdot\tan^{-1}1 - 0\cdot\tan^{-1}0 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx$$

Since $\tan^{-1}x \geqslant 0$ for $x \geqslant 0$, the integral in Example 5 can be interpreted as the area of the region shown in Figure 2.

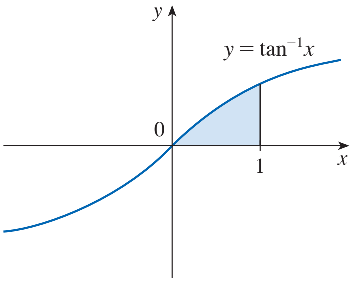


FIGURE 2

To evaluate this integral we use the substitution $t = 1 + x^2$ (since u has another meaning in this example). Then $dt = 2x\,dx$, so $x\,dx = \frac{1}{2}\,dt$. When $x = 0$, $t = 1$; when $x = 1$, $t = 2$; so

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_1^2 \frac{dt}{t} = \frac{1}{2}\ln|t|\Big]_1^2$$

$$= \tfrac{1}{2}(\ln 2 - \ln 1) = \tfrac{1}{2}\ln 2$$

Therefore $$\int_0^1 \tan^{-1}x\,dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx = \frac{\pi}{4} - \frac{\ln 2}{2}$$ ■

Reduction Formulas

The preceding examples show that integration by parts often allows us to express one integral in terms of a simpler one. If the integrand contains a power of a function, we can sometimes use integration by parts to reduce the power. In this way we can find a *reduction formula* as in the next example.

EXAMPLE 6 Prove the reduction formula

Equation 7 is called a *reduction formula* because the exponent n has been *reduced* to $n - 1$ and $n - 2$.

$$\boxed{7} \qquad \int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

where $n \geqslant 2$ is an integer.

SOLUTION Let

$$u = \sin^{n-1}x \qquad\qquad dv = \sin x\,dx$$

Then

$$du = (n-1)\sin^{n-2}x\cos x\,dx \qquad\qquad v = -\cos x$$

and integration by parts gives

$$\int \sin^n x\,dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\cos^2 x\,dx$$

Since $\cos^2 x = 1 - \sin^2 x$, we have

$$\int \sin^n x\,dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$$

As in Example 4, we solve this equation for the desired integral by taking the last term on the right side to the left side. Thus we have

$$n\int \sin^n x\,dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx$$

or

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

■

The reduction formula (7) is useful because by using it repeatedly we could eventually express $\int \sin^n x\,dx$ in terms of $\int \sin x\,dx$ (if n is odd) or $\int (\sin x)^0\,dx = \int dx$ (if n is even).

7.1 Exercises

1–4 Evaluate the integral using integration by parts with the indicated choices of u and dv.

1. $\int xe^{2x}\,dx$; $\quad u = x,\ dv = e^{2x}\,dx$

2. $\int \sqrt{x}\ln x\,dx$; $\quad u = \ln x,\ dv = \sqrt{x}\,dx$

3. $\int x\cos 4x\,dx$; $\quad u = x,\ dv = \cos 4x\,dx$

4. $\int \sin^{-1}x\,dx$; $\quad u = \sin^{-1}x,\ dv = dx$

5–42 Evaluate the integral.

5. $\int te^{2t}\,dt$

6. $\int ye^{-y}\,dy$

7. $\int x\sin 10x\,dx$

8. $\int (\pi - x)\cos \pi x\,dx$

9. $\int w\ln w\,dw$

10. $\int \frac{\ln x}{x^2}\,dx$

11. $\int (x^2 + 2x)\cos x\,dx$

12. $\int t^2\sin \beta t\,dt$

13. $\int \cos^{-1}x\,dx$

14. $\int \ln\sqrt{x}\,dx$

15. $\int t^4\ln t\,dt$

16. $\int \tan^{-1}(2y)\,dy$

17. $\int t\csc^2 t\,dt$

18. $\int x\cosh ax\,dx$

19. $\int (\ln x)^2\,dx$

20. $\int \frac{z}{10^z}\,dz$

21. $\int e^{3x}\cos x\,dx$

22. $\int e^x\sin \pi x\,dx$

23. $\int e^{2\theta}\sin 3\theta\,d\theta$

24. $\int e^{-\theta}\cos 2\theta\,d\theta$

25. $\int z^3 e^z\,dz$

26. $\int (\arcsin x)^2\,dx$

27. $\int (1 + x^2)\,e^{3x}\,dx$

28. $\int_0^{1/2} \theta \sin 3\pi\theta\,d\theta$

29. $\int_0^1 x\,3^x\,dx$

30. $\int_0^1 \dfrac{xe^x}{(1 + x)^2}\,dx$

31. $\int_0^2 y \sinh y\,dy$

32. $\int_1^2 w^2 \ln w\,dw$

33. $\int_1^5 \dfrac{\ln R}{R^2}\,dR$

34. $\int_0^{2\pi} t^2 \sin 2t\,dt$

35. $\int_0^{\pi} x \sin x \cos x\,dx$

36. $\int_1^{\sqrt{3}} \arctan(1/x)\,dx$

37. $\int_1^5 \dfrac{M}{e^M}\,dM$

38. $\int_1^2 \dfrac{(\ln x)^2}{x^3}\,dx$

39. $\int_0^{\pi/3} \sin x \ln(\cos x)\,dx$

40. $\int_0^1 \dfrac{r^3}{\sqrt{4 + r^2}}\,dr$

41. $\int_0^{\pi} \cos x \sinh x\,dx$

42. $\int_0^t e^s \sin(t - s)\,ds$

43–48 First make a substitution and then use integration by parts to evaluate the integral.

43. $\int e^{\sqrt{x}}\,dx$

44. $\int \cos(\ln x)\,dx$

45. $\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\,d\theta$

46. $\int_0^{\pi} e^{\cos t} \sin 2t\,dt$

47. $\int x \ln(1 + x)\,dx$

48. $\int \dfrac{\arcsin(\ln x)}{x}\,dx$

49–52 Evaluate the indefinite integral. Illustrate, and check that your answer is reasonable, by graphing both the function and its antiderivative (take $C = 0$).

49. $\int xe^{-2x}\,dx$

50. $\int x^{3/2} \ln x\,dx$

51. $\int x^3\sqrt{1 + x^2}\,dx$

52. $\int x^2 \sin 2x\,dx$

53. (a) Use the reduction formula in Example 6 to show that

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) Use part (a) and the reduction formula to evaluate $\int \sin^4 x\,dx$.

54. (a) Prove the reduction formula

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x \sin x + \frac{n - 1}{n}\int \cos^{n-2}x\,dx$$

(b) Use part (a) to evaluate $\int \cos^2 x\,dx$.

(c) Use parts (a) and (b) to evaluate $\int \cos^4 x\,dx$.

55. (a) Use the reduction formula in Example 6 to show that

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n - 1}{n}\int_0^{\pi/2} \sin^{n-2}x\,dx$$

where $n \geqslant 2$ is an integer.

(b) Use part (a) to evaluate $\int_0^{\pi/2} \sin^3 x\,dx$ and $\int_0^{\pi/2} \sin^5 x\,dx$.

(c) Use part (a) to show that, for odd powers of sine,

$$\int_0^{\pi/2} \sin^{2n+1}x\,dx = \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n + 1)}$$

56. Prove that, for even powers of sine,

$$\int_0^{\pi/2} \sin^{2n}x\,dx = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}\,\frac{\pi}{2}$$

57–60 Use integration by parts to prove the reduction formula.

57. $\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

58. $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

59. $\int \tan^n x\,dx = \dfrac{\tan^{n-1}x}{n - 1} - \int \tan^{n-2}x\,dx \quad (n \neq 1)$

60. $\int \sec^n x\,dx = \dfrac{\tan x \sec^{n-2}x}{n - 1} + \dfrac{n - 2}{n - 1}\int \sec^{n-2}x\,dx \quad (n \neq 1)$

61. Use Exercise 57 to find $\int (\ln x)^3\,dx$.

62. Use Exercise 58 to find $\int x^4 e^x\,dx$.

63–64 Find the area of the region bounded by the given curves.

63. $y = x^2 \ln x, \quad y = 4 \ln x$

64. $y = x^2 e^{-x}, \quad y = xe^{-x}$

65–66 Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then find (approximately) the area of the region bounded by the curves.

65. $y = \arcsin\left(\frac{1}{2}x\right), \quad y = 2 - x^2$

66. $y = x\ln(x + 1), \quad y = 3x - x^2$

67–70 Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the curves about the given axis.

67. $y = \cos(\pi x/2),\ y = 0,\ 0 \leqslant x \leqslant 1;$ about the y-axis

68. $y = e^x,\ y = e^{-x},\ x = 1;$ about the y-axis

69. $y = e^{-x},\ y = 0,\ x = -1,\ x = 0;$ about $x = 1$

70. $y = e^x,\ x = 0,\ y = 3;$ about the x-axis

71. Calculate the volume generated by rotating the region bounded by the curves $y = \ln x$, $y = 0$, and $x = 2$ about each axis.
(a) The y-axis (b) The x-axis

72. Calculate the average value of $f(x) = x \sec^2 x$ on the interval $[0, \pi/4]$.

73. The Fresnel function $S(x) = \int_0^x \sin\left(\frac{1}{2}\pi t^2\right) dt$ was discussed in Example 5.3.3 and is used extensively in the theory of optics. Find $\int S(x)\,dx$. [Your answer will involve $S(x)$.]

74. A Rocket Equation A rocket accelerates by burning its onboard fuel, so its mass decreases with time. Suppose the initial mass of the rocket at liftoff (including its fuel) is m, the fuel is consumed at rate r, and the exhaust gases are ejected with constant velocity v_e (relative to the rocket). A model for the velocity of the rocket at time t is given by the equation

$$v(t) = -gt - v_e \ln \frac{m - rt}{m}$$

where g is the acceleration due to gravity and t is not too large. If $g = 9.8 \text{ m/s}^2$, $m = 30{,}000$ kg, $r = 160$ kg/s, and $v_e = 3000$ m/s, find the height of the rocket (a) one minute after liftoff and (b) after it has consumed 6000 kg of fuel.

75. A particle that moves along a straight line has velocity $v(t) = t^2 e^{-t}$ meters per second after t seconds. How far will it travel during the first t seconds?

76. If $f(0) = g(0) = 0$ and f'' and g'' are continuous, show that

$$\int_0^a f(x)g''(x)\,dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)\,dx$$

77. Suppose that $f(1) = 2$, $f(4) = 7$, $f'(1) = 5$, $f'(4) = 3$, and f'' is continuous. Find the value of $\int_1^4 x f''(x)\,dx$.

78. (a) Use integration by parts to show that

$$\int f(x)\,dx = xf(x) - \int xf'(x)\,dx$$

(b) If f and g are inverse functions and f' is continuous, prove that

$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\,dy$$

[*Hint:* Use part (a) and make the substitution $y = f(x)$.]
(c) In the case where f and g are positive functions and $b > a > 0$, draw a diagram to give a geometric interpretation of part (b).
(d) Use part (b) to evaluate $\int_1^e \ln x\,dx$.

79. (a) Recall that the formula for integration by parts is obtained from the Product Rule. Use similar reasoning to obtain the following integration formula from the Quotient Rule.

$$\int \frac{u}{v^2}\,dv = -\frac{u}{v} + \int \frac{1}{v}\,du$$

(b) Use the formula in part (a) to evaluate $\displaystyle\int \frac{\ln x}{x^2}\,dx$.

80. The Wallis Product Formula for π Let $I_n = \int_0^{\pi/2} \sin^n x\,dx$.
(a) Show that $I_{2n+2} \leqslant I_{2n+1} \leqslant I_{2n}$.
(b) Use Exercise 56 to show that

$$\frac{I_{2n+2}}{I_{2n}} = \frac{2n+1}{2n+2}$$

(c) Use parts (a) and (b) to show that

$$\frac{2n+1}{2n+2} \leqslant \frac{I_{2n+1}}{I_{2n}} \leqslant 1$$

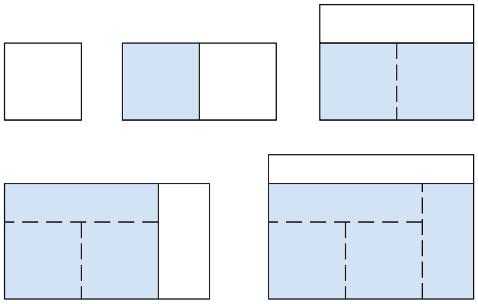
and deduce that $\lim_{n\to\infty} I_{2n+1}/I_{2n} = 1$.
(d) Use part (c) and Exercises 55 and 56 to show that

$$\lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1} = \frac{\pi}{2}$$

This formula is usually written as an infinite product:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots$$

and is called the *Wallis product*.
(e) We construct rectangles as follows. Start with a square of area 1 and attach rectangles of area 1 alternately beside or on top of the previous rectangle (see the figure). Find the limit of the ratios of width to height of these rectangles.

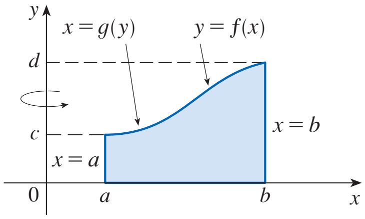


81. We arrived at Formula 6.3.2, $V = \int_a^b 2\pi x f(x)\,dx$, by using cylindrical shells, but now we can use integration by parts to prove it using the slicing method of Section 6.2, at least for the case where f is one-to-one and therefore has an inverse function g. Use the figure to show that

$$V = \pi b^2 d - \pi a^2 c - \int_c^d \pi[g(y)]^2\,dy$$

Make the substitution $y = f(x)$ and then use integration by parts on the resulting integral to prove that

$$V = \int_a^b 2\pi x f(x)\,dx$$



7.2 Trigonometric Integrals

In this section we use trigonometric identities to integrate certain combinations of trigonometric functions.

Integrals of Powers of Sine and Cosine

We begin by considering integrals in which the integrand is a power of sine, a power of cosine, or a product of these.

EXAMPLE 1 Evaluate $\int \cos^3 x\,dx$.

SOLUTION Simply substituting $u = \cos x$ isn't helpful, since then $du = -\sin x\,dx$. In order to integrate powers of cosine, we would need an extra $\sin x$ factor. Similarly, a power of sine would require an extra $\cos x$ factor. Thus here we can separate one cosine factor and convert the remaining $\cos^2 x$ factor to an expression involving sine using the identity $\sin^2 x + \cos^2 x = 1$:

$$\cos^3 x = \cos^2 x \cdot \cos x = (1 - \sin^2 x)\cos x$$

We can then evaluate the integral by substituting $u = \sin x$, so $du = \cos x\,dx$ and

$$\begin{aligned}\int \cos^3 x\,dx &= \int \cos^2 x \cdot \cos x\,dx = \int (1 - \sin^2 x)\cos x\,dx \\ &= \int (1 - u^2)\,du = u - \tfrac{1}{3}u^3 + C \\ &= \sin x - \tfrac{1}{3}\sin^3 x + C\end{aligned}$$ ■

In general, we try to write an integrand involving powers of sine and cosine in a form where we have only one sine factor (and the remainder of the expression in terms of cosine) or only one cosine factor (and the remainder of the expression in terms of sine). The identity $\sin^2 x + \cos^2 x = 1$ enables us to convert back and forth between even powers of sine and cosine.

EXAMPLE 2 Find $\int \sin^5 x \cos^2 x\,dx$.

SOLUTION We could convert $\cos^2 x$ to $1 - \sin^2 x$, but we would be left with an expression in terms of $\sin x$ with no extra $\cos x$ factor. Instead, we separate a single sine factor and rewrite the remaining $\sin^4 x$ factor in terms of $\cos x$:

$$\sin^5 x \cos^2 x = (\sin^2 x)^2 \cos^2 x \sin x = (1 - \cos^2 x)^2 \cos^2 x \sin x$$

Substituting $u = \cos x$, we have $du = -\sin x\,dx$ and so

$$\begin{aligned}\int \sin^5 x \cos^2 x\,dx &= \int (\sin^2 x)^2 \cos^2 x \sin x\,dx \\ &= \int (1 - \cos^2 x)^2 \cos^2 x \sin x\,dx \\ &= \int (1 - u^2)^2 u^2(-du) = -\int (u^2 - 2u^4 + u^6)\,du \\ &= -\left(\frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7}\right) + C \\ &= -\tfrac{1}{3}\cos^3 x + \tfrac{2}{5}\cos^5 x - \tfrac{1}{7}\cos^7 x + C\end{aligned}$$ ■

Figure 1 shows the graphs of the integrand $\sin^5 x \cos^2 x$ in Example 2 and its indefinite integral (with $C = 0$). Which is which?

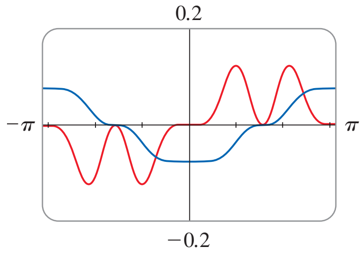


FIGURE 1

In the preceding examples, an odd power of sine or cosine enabled us to separate a single factor and convert the remaining even power. If the integrand contains even powers of both sine and cosine, this strategy fails. In this case, we can take advantage of the following half-angle identities (see Equations 18b and 18a in Appendix D):

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \text{and} \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

EXAMPLE 3 Evaluate $\int_0^\pi \sin^2 x\, dx$.

Example 3 shows that the area of the region shown in Figure 2 is $\pi/2$.

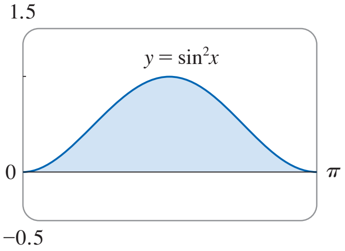


FIGURE 2

SOLUTION If we write $\sin^2 x = 1 - \cos^2 x$, the integral is no simpler to evaluate. Using the half-angle formula for $\sin^2 x$, however, we have

$$\begin{aligned} \int_0^\pi \sin^2 x\, dx &= \tfrac{1}{2}\int_0^\pi (1 - \cos 2x)\, dx \\ &= \left[\tfrac{1}{2}\left(x - \tfrac{1}{2}\sin 2x\right)\right]_0^\pi \\ &= \tfrac{1}{2}\left(\pi - \tfrac{1}{2}\sin 2\pi\right) - \tfrac{1}{2}\left(0 - \tfrac{1}{2}\sin 0\right) = \tfrac{1}{2}\pi \end{aligned}$$

Notice that we mentally made the substitution $u = 2x$ when integrating $\cos 2x$. Another method for evaluating this integral was given in Exercise 7.1.53. ■

EXAMPLE 4 Find $\int \sin^4 x\, dx$.

SOLUTION We could evaluate this integral using the reduction formula for $\int \sin^n x\, dx$ (Equation 7.1.7) together with Example 3 (as in Exercise 7.1.53), but a better method is to write $\sin^4 x = (\sin^2 x)^2$ and use a half-angle formula:

$$\begin{aligned} \int \sin^4 x\, dx &= \int (\sin^2 x)^2\, dx \\ &= \int \left[\tfrac{1}{2}(1 - \cos 2x)\right]^2 dx \\ &= \tfrac{1}{4}\int \left[1 - 2\cos 2x + \cos^2(2x)\right] dx \end{aligned}$$

Since $\cos^2(2x)$ occurs, we use the half-angle formula for cosine to write

$$\cos^2(2x) = \tfrac{1}{2}[1 + \cos(2 \cdot 2x)] = \tfrac{1}{2}(1 + \cos 4x)$$

This gives

$$\begin{aligned} \int \sin^4 x\, dx &= \tfrac{1}{4}\int \left[1 - 2\cos 2x + \tfrac{1}{2}(1 + \cos 4x)\right] dx \\ &= \tfrac{1}{4}\int \left(\tfrac{3}{2} - 2\cos 2x + \tfrac{1}{2}\cos 4x\right) dx \\ &= \tfrac{1}{4}\left(\tfrac{3}{2}x - \sin 2x + \tfrac{1}{8}\sin 4x\right) + C \end{aligned}$$ ■

To summarize, we list guidelines to follow when evaluating integrals of the form $\int \sin^m x \cos^n x\, dx$, where $m \geqslant 0$ and $n \geqslant 0$ are integers.

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\begin{aligned}\int \sin^m x \cos^{2k+1} x \, dx &= \int \sin^m x \, (\cos^2 x)^k \cos x \, dx \\ &= \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx\end{aligned}$$

Then substitute $u = \sin x$. See Example 1.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\begin{aligned}\int \sin^{2k+1} x \cos^n x \, dx &= \int (\sin^2 x)^k \cos^n x \, \sin x \, dx \\ &= \int (1 - \cos^2 x)^k \cos^n x \, \sin x \, dx\end{aligned}$$

Then substitute $u = \cos x$. See Example 2.

[Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

See Examples 3 and 4.

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2} \sin 2x$$

Integrals of Powers of Secant and Tangent

We use similar reasoning to evaluate integrals of the form $\int \tan^m x \sec^n x \, dx$. Because $(d/dx) \tan x = \sec^2 x$, we can separate a $\sec^2 x$ factor and convert the remaining (even) power of secant to an expression involving tangent using the identity $\sec^2 x = 1 + \tan^2 x$. Or, since $(d/dx) \sec x = \sec x \tan x$, we can separate a $\sec x \tan x$ factor and convert the remaining (even) power of tangent to secant.

EXAMPLE 5 Evaluate $\int \tan^6 x \sec^4 x \, dx$.

SOLUTION If we separate one $\sec^2 x$ factor, we can express the remaining $\sec^2 x$ factor in terms of tangent using the identity $\sec^2 x = 1 + \tan^2 x$. We can then evaluate the integral by substituting $u = \tan x$ so that $du = \sec^2 x \, dx$:

$$\begin{aligned}\int \tan^6 x \sec^4 x \, dx &= \int \tan^6 x \sec^2 x \sec^2 x \, dx \\ &= \int \tan^6 x \, (1 + \tan^2 x) \sec^2 x \, dx \\ &= \int u^6 (1 + u^2) \, du = \int (u^6 + u^8) \, du \\ &= \frac{u^7}{7} + \frac{u^9}{9} + C = \tfrac{1}{7} \tan^7 x + \tfrac{1}{9} \tan^9 x + C\end{aligned}$$

■

EXAMPLE 6 Find $\int \tan^5\theta \, \sec^7\theta \, d\theta$.

SOLUTION If we separate a $\sec^2\theta$ factor, as in the preceding example, we are left with a $\sec^5\theta$ factor, which isn't easily converted to tangent. However, if we separate a $\sec\theta \tan\theta$ factor, we can convert the remaining power of tangent to an expression involving only secant using the identity $\tan^2\theta = \sec^2\theta - 1$. We can then evaluate the integral by substituting $u = \sec\theta$, so $du = \sec\theta \, \tan\theta \, d\theta$:

$$\begin{aligned}
\int \tan^5\theta \, \sec^7\theta \, d\theta &= \int \tan^4\theta \, \sec^6\theta \, \sec\theta \, \tan\theta \, d\theta \\
&= \int (\sec^2\theta - 1)^2 \sec^6\theta \, \sec\theta \, \tan\theta \, d\theta \\
&= \int (u^2 - 1)^2 u^6 \, du \\
&= \int (u^{10} - 2u^8 + u^6) \, du \\
&= \frac{u^{11}}{11} - 2\frac{u^9}{9} + \frac{u^7}{7} + C \\
&= \tfrac{1}{11}\sec^{11}\theta - \tfrac{2}{9}\sec^9\theta + \tfrac{1}{7}\sec^7\theta + C
\end{aligned}$$

■

The preceding examples demonstrate strategies for evaluating integrals of the form $\int \tan^m x \, \sec^n x \, dx$ for two cases, which we summarize here.

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

(a) If the power of secant is even ($n = 2k$, $k \geqslant 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\begin{aligned}
\int \tan^m x \, \sec^{2k} x \, dx &= \int \tan^m x \, (\sec^2 x)^{k-1} \sec^2 x \, dx \\
&= \int \tan^m x \, (1 + \tan^2 x)^{k-1} \sec^2 x \, dx
\end{aligned}$$

Then substitute $u = \tan x$. See Example 5.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\begin{aligned}
\int \tan^{2k+1} x \, \sec^n x \, dx &= \int (\tan^2 x)^k \, \sec^{n-1} x \, \sec x \, \tan x \, dx \\
&= \int (\sec^2 x - 1)^k \, \sec^{n-1} x \, \sec x \, \tan x \, dx
\end{aligned}$$

Then substitute $u = \sec x$. See Example 6.

For other cases, the guidelines are not as clear-cut. We may need to use identities, integration by parts, and occasionally a little ingenuity. We will sometimes need to be able to integrate $\tan x$ by using the formula established in (5.5.5):

$$\boxed{\int \tan x \, dx = \ln|\sec x| + C}$$

We will also need the indefinite integral of secant:

Formula 1 was discovered by James Gregory in 1668. (See his biography in Section 3.4.) Gregory used this formula to solve a problem in constructing nautical tables.

1 $$\boxed{\int \sec x\,dx = \ln|\sec x + \tan x| + C}$$

We could verify Formula 1 by differentiating the right side, or as follows. First we multiply numerator and denominator by $\sec x + \tan x$:

$$\begin{aligned}\int \sec x\,dx &= \int \sec x\,\frac{\sec x + \tan x}{\sec x + \tan x}\,dx\\ &= \int \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}\,dx\end{aligned}$$

If we substitute $u = \sec x + \tan x$, then $du = (\sec x\tan x + \sec^2 x)\,dx$, so the integral becomes $\int (1/u)\,du = \ln|u| + C$. Thus we have

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

EXAMPLE 7 Find $\int \tan^3 x\,dx$.

SOLUTION Here only $\tan x$ occurs, so we use $\tan^2 x = \sec^2 x - 1$ to rewrite a $\tan^2 x$ factor in terms of $\sec^2 x$:

$$\begin{aligned}\int \tan^3 x\,dx &= \int \tan x\,\tan^2 x\,dx = \int \tan x\,(\sec^2 x - 1)\,dx\\ &= \int \tan x\,\sec^2 x\,dx - \int \tan x\,dx\\ &= \tfrac{1}{2}\tan^2 x - \ln|\sec x| + C\end{aligned}$$

In the first integral we mentally substituted $u = \tan x$ so that $du = \sec^2 x\,dx$. ■

If an even power of tangent appears with an odd power of secant, it is helpful to express the integrand completely in terms of $\sec x$. Powers of $\sec x$ may require integration by parts, as shown in the following example.

EXAMPLE 8 Find $\int \sec^3 x\,dx$.

SOLUTION Here we integrate by parts with

$$\begin{aligned} u &= \sec x & dv &= \sec^2 x\,dx\\ du &= \sec x\tan x\,dx & v &= \tan x\end{aligned}$$

Then

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x\tan x - \int \sec x\tan^2 x\,dx\\ &= \sec x\tan x - \int \sec x\,(\sec^2 x - 1)\,dx\\ &= \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx\end{aligned}$$

Using Formula 1 and solving for the required integral, we get

$$\int \sec^3 x\,dx = \tfrac{1}{2}\big(\sec x \tan x + \ln|\sec x + \tan x|\big) + C$$ ■

Integrals such as the one in the preceding example may seem very special but they occur frequently in applications of integration, as we will see in Chapter 8.

Finally, integrals of the form

$$\int \cot^m x \csc^n x\,dx$$

can be found in a similar way by using the identity $1 + \cot^2 x = \csc^2 x$.

Using Product Identities

The following product identities are useful in evaluating certain trigonometric integrals.

These product identities are discussed in Appendix D.

2 To evaluate the integrals (a) $\int \sin mx \cos nx\,dx$, (b) $\int \sin mx \sin nx\,dx$, or (c) $\int \cos mx \cos nx\,dx$, use the corresponding identity:

$$\text{(a)}\quad \sin A \cos B = \tfrac{1}{2}[\sin(A - B) + \sin(A + B)]$$

$$\text{(b)}\quad \sin A \sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\text{(c)}\quad \cos A \cos B = \tfrac{1}{2}[\cos(A - B) + \cos(A + B)]$$

EXAMPLE 9 Evaluate $\int \sin 4x \cos 5x\,dx$.

SOLUTION This integral could be evaluated using integration by parts, but it's easier to use the identity in Equation 2(a) as follows:

$$\begin{aligned}\int \sin 4x \cos 5x\,dx &= \int \tfrac{1}{2}[\sin(-x) + \sin 9x]\,dx \\ &= \tfrac{1}{2}\int (-\sin x + \sin 9x)\,dx \\ &= \tfrac{1}{2}\big(\cos x - \tfrac{1}{9}\cos 9x\big) + C\end{aligned}$$ ■

7.2 Exercises

1–56 Evaluate the integral.

1. $\int \sin^3 x \cos^2 x\,dx$

2. $\int \cos^6 y \sin^3 y\,dy$

3. $\int_0^{\pi/2} \cos^9 x \sin^5 x\,dx$

4. $\int_0^{\pi/4} \sin^5 x\,dx$

5. $\int \sin^5(2t) \cos^2(2t)\,dt$

6. $\int \cos^3(t/2) \sin^2(t/2)\,dt$

7. $\int_0^{\pi/2} \cos^2\theta\,d\theta$

8. $\int_0^{\pi/4} \sin^2(2\theta)\,d\theta$

9. $\int_0^{\pi} \cos^4(2t)\,dt$

10. $\int_0^{\pi} \sin^2 t \cos^4 t\,dt$

11. $\int_0^{\pi/2} \sin^2 x \cos^2 x\,dx$

12. $\int_0^{\pi/2} (2 - \sin\theta)^2\,d\theta$

13. $\int \sqrt{\cos\theta}\,\sin^3\theta\,d\theta$

14. $\int \big(1 + \sqrt[3]{\sin t}\big)\cos^3 t\,dt$

15. $\int \sin x \sec^5 x\,dx$

16. $\int \csc^5\theta \cos^3\theta\,d\theta$

17. $\int \cot x \cos^2 x\,dx$

18. $\int \tan^2 x \cos^3 x\,dx$

19. $\int \sin^2 x \sin 2x\,dx$

20. $\int \sin x \cos\big(\tfrac{1}{2}x\big)\,dx$

21. $\int \tan x \sec^3 x\, dx$

22. $\int \tan^2\theta \sec^4\theta\, d\theta$

23. $\int \tan^2 x\, dx$

24. $\int (\tan^2 x + \tan^4 x)\, dx$

25. $\int \tan^4 x \sec^6 x\, dx$

26. $\int_0^{\pi/4} \sec^6\theta \tan^6\theta\, d\theta$

27. $\int \tan^3 x \sec x\, dx$

28. $\int \tan^5 x \sec^3 x\, dx$

29. $\int \tan^3 x \sec^6 x\, dx$

30. $\int_0^{\pi/4} \tan^4 t\, dt$

31. $\int \tan^5 x\, dx$

32. $\int \tan^2 x \sec x\, dx$

33. $\int \dfrac{1 - \tan^2 x}{\sec^2 x}\, dx$

34. $\int \dfrac{\tan x \sec^2 x}{\cos x}\, dx$

35. $\int_0^{\pi/4} \dfrac{\sin^3 x}{\cos x}\, dx$

36. $\int \dfrac{\sin\theta + \tan\theta}{\cos^3\theta}\, d\theta$

37. $\int_{\pi/6}^{\pi/2} \cot^2 x\, dx$

38. $\int_{\pi/4}^{\pi/2} \cot^3 x\, dx$

39. $\int_{\pi/4}^{\pi/2} \cot^5\phi \csc^3\phi\, d\phi$

40. $\int_{\pi/4}^{\pi/2} \csc^4\theta \cot^4\theta\, d\theta$

41. $\int \csc x\, dx$

42. $\int_{\pi/6}^{\pi/3} \csc^3 x\, dx$

43. $\int \sin 8x \cos 5x\, dx$

44. $\int \sin 2\theta \sin 6\theta\, d\theta$

45. $\int_0^{\pi/2} \cos 5t \cos 10t\, dt$

46. $\int t \cos^5(t^2)\, dt$

47. $\int \dfrac{\sin^2(1/t)}{t^2}\, dt$

48. $\int \sec^2 y \cos^3(\tan y)\, dy$

49. $\int_0^{\pi/6} \sqrt{1 + \cos 2x}\, dx$

50. $\int_0^{\pi/4} \sqrt{1 - \cos 4\theta}\, d\theta$

51. $\int t \sin^2 t\, dt$

52. $\int x \sec x \tan x\, dx$

53. $\int x \tan^2 x\, dx$

54. $\int x \sin^3 x\, dx$

55. $\int \dfrac{dx}{\cos x - 1}$

56. $\int \dfrac{1}{\sec\theta + 1}\, d\theta$

57–60 Evaluate the indefinite integral. Illustrate, and check that your answer is reasonable, by graphing both the integrand and its antiderivative (taking $C = 0$).

57. $\int x \sin^2(x^2)\, dx$

58. $\int \sin^5 x \cos^3 x\, dx$

59. $\int \sin 3x \sin 6x\, dx$

60. $\int \sec^4\left(\tfrac{1}{2}x\right) dx$

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61. If $\int_0^{\pi/4} \tan^6 x \sec x\, dx = I$, express the value of $\int_0^{\pi/4} \tan^8 x \sec x\, dx$ in terms of I.

62. (a) Prove the reduction formula

$$\int \tan^{2n} x\, dx = \frac{\tan^{2n-1} x}{2n - 1} - \int \tan^{2n-2} x\, dx$$

(b) Use this formula to find $\int \tan^8 x\, dx$.

63. Find the average value of the function $f(x) = \sin^2 x \cos^3 x$ on the interval $[-\pi, \pi]$.

64. Evaluate $\int \sin x \cos x\, dx$ by four methods:
- (a) the substitution $u = \cos x$
- (b) the substitution $u = \sin x$
- (c) the identity $\sin 2x = 2 \sin x \cos x$
- (d) integration by parts

Explain the different appearances of the answers.

65–66 Find the area of the region bounded by the given curves.

65. $y = \sin^2 x, \quad y = \sin^3 x, \quad 0 \leqslant x \leqslant \pi$

66. $y = \tan x, \quad y = \tan^2 x, \quad 0 \leqslant x \leqslant \pi/4$

67–68 Use a graph of the integrand to guess the value of the integral. Then use the methods of this section to prove that your guess is correct.

67. $\int_0^{2\pi} \cos^3 x\, dx$

68. $\int_0^2 \sin 2\pi x \cos 5\pi x\, dx$

69–72 Find the volume obtained by rotating the region bounded by the curves about the given axis.

69. $y = \sin x,\ y = 0,\ \pi/2 \leqslant x \leqslant \pi;$ about the x-axis

70. $y = \sin^2 x,\ y = 0,\ 0 \leqslant x \leqslant \pi;$ about the x-axis

71. $y = \sin x,\ y = \cos x,\ 0 \leqslant x \leqslant \pi/4;$ about $y = 1$

72. $y = \sec x,\ y = \cos x,\ 0 \leqslant x \leqslant \pi/3;$ about $y = -1$

---

73. A particle moves on a straight line with velocity function $v(t) = \sin \omega t \cos^2 \omega t$. Find its position function $s = f(t)$ if $f(0) = 0$.

74. Household electricity is supplied in the form of alternating current that varies from 155 V to -155 V with a frequency of 60 cycles per second (Hz). The voltage is thus given by the equation

$$E(t) = 155 \sin(120\pi t)$$

where t is the time in seconds. Voltmeters read the RMS (root-mean-square) voltage, which is the square root of the average value of $[E(t)]^2$ over one cycle.
- (a) Calculate the RMS voltage of household current.
- (b) Many electric stoves require an RMS voltage of 220 V. Find the corresponding amplitude A needed for the voltage $E(t) = A \sin(120\pi t)$.

75–77 Prove the formula, where m and n are positive integers.

75. $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$

76. $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$

77. $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$

78. A *finite Fourier series* is given by the sum

$$\begin{aligned} f(x) &= \sum_{n=1}^{N} a_n \sin nx \\ &= a_1 \sin x + a_2 \sin 2x + \cdots + a_N \sin Nx \end{aligned}$$

Use the result of Exercise 76 to show that the mth coefficient a_m is given by the formula

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx$$

7.3 Trigonometric Substitution

In finding the area of a circle or an ellipse, an integral of the form $\int \sqrt{a^2 - x^2}\, dx$ arises, where $a > 0$. If the integral were $\int x\sqrt{a^2 - x^2}\, dx$, the substitution $u = a^2 - x^2$ would be effective but, as it stands, $\int \sqrt{a^2 - x^2}\, dx$ is more difficult. If we change the variable from x to θ by the substitution $x = a \sin \theta$, then the identity $1 - \sin^2\theta = \cos^2\theta$ allows us to eliminate the root sign because

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2 \cos^2\theta} = a\,|\cos \theta|$$

Notice the difference between the substitution $u = a^2 - x^2$ (in which the new variable is a function of the old one) and the substitution $x = a \sin \theta$ (the old variable is a function of the new one).

In general, we can make a substitution of the form $x = g(t)$ by using the Substitution Rule in reverse. To make our calculations simpler, we assume that g has an inverse function; that is, g is one-to-one. In this case, if we replace u by x and x by t in the Substitution Rule (Equation 5.5.4), we obtain

$$\int f(x)\, dx = \int f(g(t))g'(t)\, dt$$

This kind of substitution is called *inverse substitution.*

We can make the inverse substitution $x = a \sin \theta$ provided that it defines a one-to-one function. This can be accomplished by restricting θ to lie in the interval $[-\pi/2, \pi/2]$.

In the following table we list trigonometric substitutions that are effective for the given radical expressions because of the specified trigonometric identities. In each case the restriction on θ is imposed to ensure that the function that defines the substitution is one-to-one. (These are the same intervals used in Section 1.5 in defining the inverse functions.)

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

EXAMPLE 1 Evaluate $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$.

SOLUTION Let $x = 3\sin\theta$, where $-\pi/2 \leqslant \theta \leqslant \pi/2$. Then $dx = 3\cos\theta\,d\theta$ and

$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3\,|\cos\theta| = 3\cos\theta$$

(Note that $\cos\theta \geqslant 0$ because $-\pi/2 \leqslant \theta \leqslant \pi/2$.) Thus the Inverse Substitution Rule gives

$$\begin{aligned}\int \frac{\sqrt{9-x^2}}{x^2}\,dx &= \int \frac{3\cos\theta}{9\sin^2\theta}\,3\cos\theta\,d\theta \\ &= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \cot^2\theta\,d\theta \\ &= \int (\csc^2\theta - 1)\,d\theta \\ &= -\cot\theta - \theta + C\end{aligned}$$

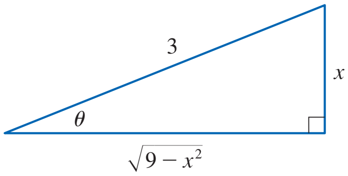


FIGURE 1

$\sin\theta = \dfrac{x}{3}$

Because this is an indefinite integral, we must return to the original variable x. This can be done either by using trigonometric identities to express $\cot\theta$ in terms of $\sin\theta = x/3$ or by drawing a diagram, as in Figure 1, where θ is interpreted as an angle of a right triangle. Since $\sin\theta = x/3$, we label the opposite side and the hypotenuse as having lengths x and 3. Then the Pythagorean Theorem gives the length of the adjacent side as $\sqrt{9-x^2}$, so we can simply read the value of $\cot\theta$ from the figure:

$$\cot\theta = \frac{\sqrt{9-x^2}}{x}$$

(Although $\theta > 0$ in the diagram, this expression for $\cot\theta$ is valid even when $\theta < 0$.) Since $\sin\theta = x/3$, we have $\theta = \sin^{-1}(x/3)$ and so

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = -\cot\theta - \theta + C = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\!\left(\frac{x}{3}\right) + C \qquad ■$$

EXAMPLE 2 Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

SOLUTION Solving the equation of the ellipse for y, we get

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2-x^2}{a^2} \qquad \text{or} \qquad y = \pm\frac{b}{a}\sqrt{a^2-x^2}$$

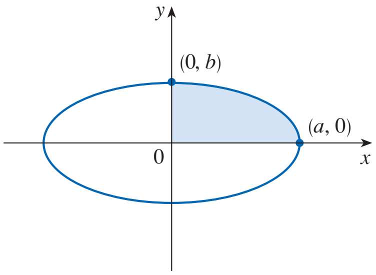


FIGURE 2

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Because the ellipse is symmetric with respect to both axes, the total area A is four times the area in the first quadrant (see Figure 2). The part of the ellipse in the first quadrant is given by the function

$$y = \frac{b}{a}\sqrt{a^2-x^2} \qquad 0 \leqslant x \leqslant a$$

and so

$$\tfrac{1}{4}A = \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

To evaluate this integral we substitute $x = a\sin\theta$. Then $dx = a\cos\theta\,d\theta$. To change the limits of integration we note that when $x = 0$, $\sin\theta = 0$, so $\theta = 0$; when $x = a$, $\sin\theta = 1$, so $\theta = \pi/2$. Also

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2\cos^2\theta} = a\,|\cos\theta| = a\cos\theta$$

since $0 \leqslant \theta \leqslant \pi/2$. Therefore

$$\begin{aligned} A &= 4\frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx = 4\frac{b}{a}\int_0^{\pi/2} a\cos\theta \cdot a\cos\theta\,d\theta \\ &= 4ab\int_0^{\pi/2}\cos^2\theta\,d\theta = 4ab\int_0^{\pi/2}\tfrac{1}{2}(1 + \cos 2\theta)\,d\theta \\ &= 2ab\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{\pi/2} = 2ab\left(\frac{\pi}{2} + 0 - 0\right) = \pi ab \end{aligned}$$

We have shown that the area of an ellipse with semiaxes a and b is πab. In particular, taking $a = b = r$, we have proved the famous formula that the area of a circle with radius r is πr^2. ■

NOTE Because the integral in Example 2 was a definite integral, we changed the limits of integration and did not have to convert back to the original variable x.

EXAMPLE 3 Find $\displaystyle\int \frac{1}{x^2\sqrt{x^2 + 4}}\,dx$.

SOLUTION Let $x = 2\tan\theta$, $-\pi/2 < \theta < \pi/2$. Then $dx = 2\sec^2\theta\,d\theta$ and

$$\sqrt{x^2 + 4} = \sqrt{4(\tan^2\theta + 1)} = \sqrt{4\sec^2\theta} = 2\,|\sec\theta| = 2\sec\theta$$

So we have

$$\int \frac{dx}{x^2\sqrt{x^2 + 4}} = \int \frac{2\sec^2\theta\,d\theta}{4\tan^2\theta \cdot 2\sec\theta} = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

To evaluate this trigonometric integral we put everything in terms of $\sin\theta$ and $\cos\theta$:

$$\frac{\sec\theta}{\tan^2\theta} = \frac{1}{\cos\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin^2\theta}$$

Therefore, making the substitution $u = \sin\theta$, we have

$$\begin{aligned} \int \frac{dx}{x^2\sqrt{x^2 + 4}} &= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{du}{u^2} \\ &= \frac{1}{4}\left(-\frac{1}{u}\right) + C = -\frac{1}{4\sin\theta} + C \\ &= -\frac{\csc\theta}{4} + C \end{aligned}$$

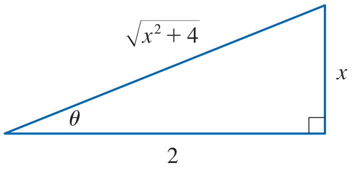


FIGURE 3

$\tan\theta = \dfrac{x}{2}$

We use Figure 3 to determine that $\csc\theta = \sqrt{x^2 + 4}/x$ and so

$$\int \frac{dx}{x^2\sqrt{x^2 + 4}} = -\frac{\sqrt{x^2 + 4}}{4x} + C$$ ■

EXAMPLE 4 Find $\displaystyle\int \frac{x}{\sqrt{x^2+4}}\,dx$.

SOLUTION It would be possible to use the trigonometric substitution $x = 2\tan\theta$ here (as in Example 3). But the direct substitution $u = x^2 + 4$ is simpler, because then $du = 2x\,dx$ and

$$\int \frac{x}{\sqrt{x^2+4}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u} + C = \sqrt{x^2+4} + C \qquad \blacksquare$$

NOTE Example 4 illustrates the fact that even when trigonometric substitutions are possible, they may not give the easiest solution. You should look for a simpler method first.

EXAMPLE 5 Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}}$, where $a > 0$.

SOLUTION 1 We let $x = a\sec\theta$, where $0 < \theta < \pi/2$ or $\pi < \theta < 3\pi/2$. Then $dx = a\sec\theta\,\tan\theta\,d\theta$ and

$$\sqrt{x^2-a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = a\,|\tan\theta| = a\tan\theta$$

Therefore

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\,\tan\theta}{a\tan\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$

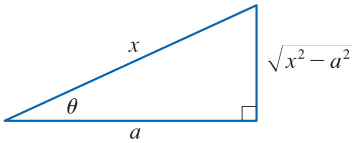


FIGURE 4

$\sec\theta = \dfrac{x}{a}$

The triangle in Figure 4 gives $\tan\theta = \sqrt{x^2-a^2}/a$, so we have

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C$$

$$= \ln\left|x + \sqrt{x^2-a^2}\right| - \ln a + C$$

Writing $C_1 = C - \ln a$, we have

$$\boxed{1} \qquad \int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x + \sqrt{x^2-a^2}\right| + C_1$$

SOLUTION 2 For $x > 0$ the hyperbolic substitution $x = a\cosh t$ can also be used. Using the identity $\cosh^2 y - \sinh^2 y = 1$, we have

$$\sqrt{x^2-a^2} = \sqrt{a^2(\cosh^2 t - 1)} = \sqrt{a^2\sinh^2 t} = a\sinh t$$

Since $dx = a\sinh t\,dt$, we obtain

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sinh t\,dt}{a\sinh t} = \int dt = t + C$$

Since $\cosh t = x/a$, we have $t = \cosh^{-1}(x/a)$ and

$$\boxed{2} \qquad \int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$$

Although Formulas 1 and 2 look quite different, they are actually equivalent by Formula 3.11.4. $\blacksquare$

NOTE As Example 5 illustrates, hyperbolic substitutions can be used in place of trigonometric substitutions and sometimes they lead to simpler answers. But we usually use trigonometric substitutions because trigonometric identities are more familiar than hyperbolic identities.

As Example 6 shows, trigonometric substitution is sometimes a good idea when $(x^2 + a^2)^{n/2}$ occurs in an integral, where n is any integer. The same is true when $(a^2 - x^2)^{n/2}$ or $(x^2 - a^2)^{n/2}$ occur.

EXAMPLE 6 Find $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2 + 9)^{3/2}}\, dx$.

SOLUTION First we note that $(4x^2 + 9)^{3/2} = (\sqrt{4x^2 + 9})^3$ so trigonometric substitution is appropriate. Although $\sqrt{4x^2 + 9}$ is not quite one of the expressions in the table of trigonometric substitutions, it becomes one of them if we make the preliminary substitution $u = 2x$, which gives $\sqrt{u^2 + 9}$. Then we substitute $u = 3\tan\theta$ or, equivalently, $x = \frac{3}{2}\tan\theta$, which gives $dx = \frac{3}{2}\sec^2\theta\, d\theta$ and

$$\sqrt{4x^2 + 9} = \sqrt{9\tan^2\theta + 9} = 3\sec\theta$$

When $x = 0$, $\tan\theta = 0$, so $\theta = 0$; when $x = 3\sqrt{3}/2$, $\tan\theta = \sqrt{3}$, so $\theta = \pi/3$.

$$\begin{aligned}
\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2 + 9)^{3/2}}\, dx &= \int_0^{\pi/3} \frac{\frac{27}{8}\tan^3\theta}{27\sec^3\theta}\, \tfrac{3}{2}\sec^2\theta\, d\theta \\
&= \tfrac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\, d\theta = \tfrac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta}\, d\theta \\
&= \tfrac{3}{16}\int_0^{\pi/3} \frac{1 - \cos^2\theta}{\cos^2\theta}\sin\theta\, d\theta
\end{aligned}$$

Now we substitute $u = \cos\theta$ so that $du = -\sin\theta\, d\theta$. When $\theta = 0$, $u = 1$; when $\theta = \pi/3$, $u = \frac{1}{2}$. Therefore

$$\begin{aligned}
\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2 + 9)^{3/2}}\, dx &= -\tfrac{3}{16}\int_1^{1/2} \frac{1 - u^2}{u^2}\, du \\
&= \tfrac{3}{16}\int_1^{1/2} (1 - u^{-2})\, du = \tfrac{3}{16}\left[u + \frac{1}{u}\right]_1^{1/2} \\
&= \tfrac{3}{16}\left[\left(\tfrac{1}{2} + 2\right) - (1 + 1)\right] = \tfrac{3}{32}
\end{aligned}$$

■

EXAMPLE 7 Evaluate $\displaystyle\int \frac{x}{\sqrt{3 - 2x - x^2}}\, dx$.

SOLUTION We can transform the integrand into a function for which trigonometric substitution is appropriate by first completing the square under the root sign:

$$\begin{aligned}
3 - 2x - x^2 &= 3 - (x^2 + 2x) = 3 + 1 - (x^2 + 2x + 1) \\
&= 4 - (x + 1)^2
\end{aligned}$$

This suggests that we make the substitution $u = x + 1$. Then $du = dx$ and $x = u - 1$, so

$$\int \frac{x}{\sqrt{3 - 2x - x^2}}\, dx = \int \frac{u - 1}{\sqrt{4 - u^2}}\, du$$

Figure 5 shows the graphs of the integrand in Example 7 and its indefinite integral (with $C = 0$). Which is which?

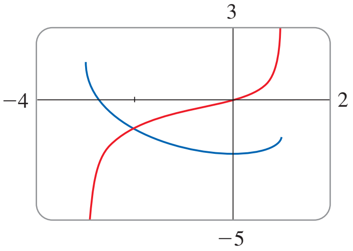


FIGURE 5

We now substitute $u = 2\sin\theta$, giving $du = 2\cos\theta\,d\theta$ and $\sqrt{4-u^2} = 2\cos\theta$, so

$$\begin{aligned}\int \frac{x}{\sqrt{3-2x-x^2}}\,dx &= \int \frac{2\sin\theta - 1}{2\cos\theta}\,2\cos\theta\,d\theta \\ &= \int (2\sin\theta - 1)\,d\theta \\ &= -2\cos\theta - \theta + C \\ &= -\sqrt{4-u^2} - \sin^{-1}\left(\frac{u}{2}\right) + C \\ &= -\sqrt{3-2x-x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C \qquad \blacksquare\end{aligned}$$

7.3 Exercises

1–4 (a) Determine an appropriate trigonometric substitution. (b) Apply the substitution to transform the integral into a trigonometric integral. Do not evaluate the integral.

1. $\displaystyle\int \frac{x^3}{\sqrt{1+x^2}}\,dx$

2. $\displaystyle\int \frac{x^3}{\sqrt{9-x^2}}\,dx$

3. $\displaystyle\int \frac{x^2}{\sqrt{x^2-2}}\,dx$

4. $\displaystyle\int \frac{x^3}{(9-4x^2)^{3/2}}\,dx$

5–8 Evaluate the integral using the indicated trigonometric substitution. Sketch and label the associated right triangle.

5. $\displaystyle\int \frac{x^3}{\sqrt{1-x^2}}\,dx \qquad x = \sin\theta$

6. $\displaystyle\int \frac{x^3}{\sqrt{9+x^2}}\,dx \qquad x = 3\tan\theta$

7. $\displaystyle\int \frac{\sqrt{4x^2-25}}{x}\,dx \qquad x = \tfrac{5}{2}\sec\theta$

8. $\displaystyle\int \frac{\sqrt{2-x^2}}{x^2}\,dx \qquad x = \sqrt{2}\sin\theta$

9–36 Evaluate the integral.

9. $\displaystyle\int x^3\sqrt{16+x^2}\,dx$

10. $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

11. $\displaystyle\int \frac{\sqrt{x^2-1}}{x^4}\,dx$

12. $\displaystyle\int_0^3 \frac{x}{\sqrt{36-x^2}}\,dx$

13. $\displaystyle\int_0^a \frac{dx}{(a^2+x^2)^{3/2}}, \quad a > 0$

14. $\displaystyle\int \frac{dt}{t^2\sqrt{t^2-16}}$

15. $\displaystyle\int_2^3 \frac{dx}{(x^2-1)^{3/2}}$

16. $\displaystyle\int_0^{2/3} \sqrt{4-9x^2}\,dx$

17. $\displaystyle\int_0^{1/2} x\sqrt{1-4x^2}\,dx$

18. $\displaystyle\int_0^2 \frac{dt}{\sqrt{4+t^2}}$

19. $\displaystyle\int \frac{\sqrt{x^2-9}}{x^3}\,dx$

20. $\displaystyle\int_0^1 \frac{dx}{(x^2+1)^2}$

21. $\displaystyle\int_0^a x^2\sqrt{a^2-x^2}\,dx$

22. $\displaystyle\int_{1/4}^{\sqrt{3}/4} \sqrt{1-4x^2}\,dx$

23. $\displaystyle\int \frac{x}{\sqrt{x^2-7}}\,dx$

24. $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx$

25. $\displaystyle\int \frac{\sqrt{1+x^2}}{x}\,dx$

26. $\displaystyle\int_0^{0.3} \frac{x}{(9-25x^2)^{3/2}}\,dx$

27. $\displaystyle\int_0^{0.6} \frac{x^2}{\sqrt{9-25x^2}}\,dx$

28. $\displaystyle\int_0^1 \sqrt{x^2+1}\,dx$

29. $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+5}}$

30. $\displaystyle\int_0^1 \sqrt{x-x^2}\,dx$

31. $\displaystyle\int x^2\sqrt{3+2x-x^2}\,dx$

32. $\displaystyle\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\,dx$

33. $\displaystyle\int \sqrt{x^2+2x}\,dx$

34. $\displaystyle\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx$

35. $\displaystyle\int x\sqrt{1-x^4}\,dx$

36. $\displaystyle\int_0^{\pi/2} \frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt$

37. (a) Use trigonometric substitution to show that

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

(b) Use the hyperbolic substitution $x = a \sinh t$ to show that

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

These formulas are connected by Formula 3.11.3.

38. Evaluate

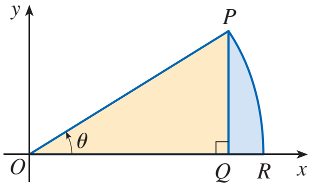
$$\int \frac{x^2}{(x^2 + a^2)^{3/2}}\, dx$$

(a) by trigonometric substitution.
(b) by the hyperbolic substitution $x = a \sinh t$.

39. Find the average value of $f(x) = \sqrt{x^2 - 1}/x$, $1 \leqslant x \leqslant 7$.

40. Find the area of the region bounded by the hyperbola $9x^2 - 4y^2 = 36$ and the line $x = 3$.

41. Prove the formula $A = \frac{1}{2}r^2\theta$ for the area of a sector of a circle with radius r and central angle θ. [*Hint:* Assume $0 < \theta < \pi/2$ and place the center of the circle at the origin so it has the equation $x^2 + y^2 = r^2$. Then A is the sum of the area of the triangle POQ and the area of the region PQR in the figure.]



42. Evaluate the integral

$$\int \frac{dx}{x^4\sqrt{x^2 - 2}}$$

Graph the integrand and its indefinite integral on the same screen and check that your answer is reasonable.

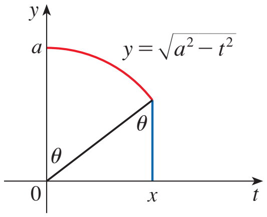
43. Find the volume of the solid obtained by rotating about the x-axis the region enclosed by the curves $y = 9/(x^2 + 9)$, $y = 0$, $x = 0$, and $x = 3$.

44. Find the volume of the solid obtained by rotating about the line $x = 1$ the region under the curve $y = x\sqrt{1 - x^2}$, $0 \leqslant x \leqslant 1$.

45. (a) Use trigonometric substitution to verify that

$$\int_0^x \sqrt{a^2 - t^2}\, dt = \tfrac{1}{2}a^2 \sin^{-1}(x/a) + \tfrac{1}{2}x\sqrt{a^2 - x^2}$$

(b) Use the figure to give trigonometric interpretations of both terms on the right side of the equation in part (a).



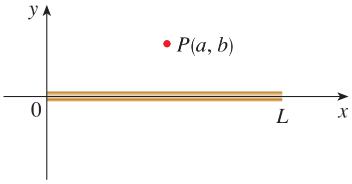
46. The parabola $y = \frac{1}{2}x^2$ divides the disk $x^2 + y^2 \leqslant 8$ into two parts. Find the areas of both parts.

47. A torus is generated by rotating the circle $x^2 + (y - R)^2 = r^2$ about the x-axis. Find the volume enclosed by the torus.

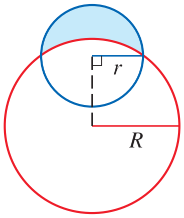
48. A charged rod of length L produces an electric field at point $P(a, b)$ given by

$$E(P) = \int_{-a}^{L-a} \frac{\lambda b}{4\pi\varepsilon_0(x^2 + b^2)^{3/2}}\, dx$$

where λ is the charge density per unit length on the rod and ε_0 is the free space permittivity (see the figure). Evaluate the integral to determine an expression for the electric field $E(P)$.



49. Find the area of the crescent-shaped region (called a *lune*) bounded by arcs of circles with radii r and R. (See the figure.)



50. A water storage tank has the shape of a cylinder with diameter 10 m. It is mounted so that the circular cross-sections are vertical. If the depth of the water is 7 m, what percentage of the total capacity is being used?

7.4 Integration of Rational Functions by Partial Fractions

In this section we show how to integrate any rational function (a ratio of polynomials) by expressing it as a sum of simpler fractions, called *partial fractions,* that we already know how to integrate. To illustrate the method, observe that by taking the fractions $2/(x-1)$ and $1/(x+2)$ to a common denominator we obtain

$$\frac{2}{x-1}-\frac{1}{x+2}=\frac{2(x+2)-(x-1)}{(x-1)(x+2)}=\frac{x+5}{x^2+x-2}$$

If we now reverse the procedure, we see how to integrate the function on the right side of this equation:

$$\begin{aligned}\int\frac{x+5}{x^2+x-2}\,dx&=\int\left(\frac{2}{x-1}-\frac{1}{x+2}\right)dx\\&=2\ln|x-1|-\ln|x+2|+C\end{aligned}$$

The Method of Partial Fractions

To see how the method of partial fractions works in general, let's consider a rational function

$$f(x)=\frac{P(x)}{Q(x)}$$

where P and Q are polynomials. It's possible to express f as a sum of simpler fractions provided that the degree of P is less than the degree of Q. Such a rational function is called *proper*. Recall that if

$$P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$$

where $a_n\neq 0$, then the degree of P is n and we write $\deg(P)=n$.

If f is *improper*, that is, $\deg(P)\geqslant\deg(Q)$, then we must take the preliminary step of dividing Q into P (by long division) until a remainder $R(x)$ is obtained such that $\deg(R)<\deg(Q)$. The result is

$$\boxed{1}\qquad f(x)=\frac{P(x)}{Q(x)}=S(x)+\frac{R(x)}{Q(x)}$$

where S and R are also polynomials.

As the following example illustrates, sometimes this preliminary step is all that is required.

EXAMPLE 1 Find $\displaystyle\int\frac{x^3+x}{x-1}\,dx$.

$$\begin{array}{r|l} & x^2+x+2 \\ \hline x-1 & x^3 \qquad +x \\ & x^3-x^2 \\ & \quad x^2+x \\ & \quad x^2-x \\ & \qquad 2x \\ & \qquad 2x-2 \\ & \qquad\quad 2\end{array}$$

SOLUTION Since the degree of the numerator is greater than the degree of the denominator, we first perform the long division. This enables us to write

$$\begin{aligned}\int\frac{x^3+x}{x-1}\,dx&=\int\left(x^2+x+2+\frac{2}{x-1}\right)dx\\&=\frac{x^3}{3}+\frac{x^2}{2}+2x+2\ln|x-1|+C\end{aligned}$$

■

If the denominator $Q(x)$ in Equation 1 is factorable, then the next step is to factor $Q(x)$ as far as possible. It can be shown that any polynomial Q can be factored as a product of linear factors (of the form $ax + b$) and irreducible quadratic factors (of the form $ax^2 + bx + c$, where $b^2 - 4ac < 0$). For instance, if $Q(x) = x^4 - 16$, we could factor it as

$$Q(x) = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$$

The third step is to express the proper rational function $R(x)/Q(x)$ (from Equation 1) as a sum of **partial fractions** of the form

$$\frac{A}{(ax + b)^i} \quad \text{or} \quad \frac{Ax + B}{(ax^2 + bx + c)^j}$$

A theorem in algebra guarantees that it is always possible to do this. We explain the details for the four cases that occur.

CASE I The denominator $Q(x)$ is a product of distinct linear factors.
This means that we can write

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$$

where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that

$$\boxed{2} \qquad \frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

These constants can be determined as in the following example.

EXAMPLE 2 Evaluate $\displaystyle\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx$.

SOLUTION Since the degree of the numerator is less than the degree of the denominator, we don't need to divide. We factor the denominator as

$$2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(2x - 1)(x + 2)$$

Since the denominator has three distinct linear factors, the partial fraction decomposition of the integrand (2) has the form

$$\boxed{3} \qquad \frac{x^2 + 2x - 1}{x(2x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2}$$

Another method for finding A, B, and C is given in the note after this example.

To determine the values of A, B, and C, we multiply both sides of this equation by the least common denominator, $x(2x - 1)(x + 2)$, obtaining

$$\boxed{4} \qquad x^2 + 2x - 1 = A(2x - 1)(x + 2) + Bx(x + 2) + Cx(2x - 1)$$

Expanding the right side of Equation 4 and writing it in the standard form for polynomials, we get

$$\boxed{5} \qquad x^2 + 2x - 1 = (2A + B + 2C)x^2 + (3A + 2B - C)x - 2A$$

The polynomials on each side of Equation 5 are identical, so the coefficients of corresponding terms must be equal. The coefficient of x^2 on the right side, $2A + B + 2C$, must equal the coefficient of x^2 on the left side—namely, 1. Likewise, the coefficients of x are equal and the constant terms are equal. This gives the following system of equations for A, B, and C:

$$\begin{aligned} 2A + B + 2C &= 1 \\ 3A + 2B - C &= 2 \\ -2A &= -1 \end{aligned}$$

Solving, we get $A = \frac{1}{2}$, $B = \frac{1}{5}$, and $C = -\frac{1}{10}$, and so

We could check our work by taking the terms to a common denominator and adding them.

$$\begin{aligned} \int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx &= \int \left(\frac{1}{2}\,\frac{1}{x} + \frac{1}{5}\,\frac{1}{2x - 1} - \frac{1}{10}\,\frac{1}{x + 2} \right) dx \\ &= \tfrac{1}{2}\ln|x| + \tfrac{1}{10}\ln|2x - 1| - \tfrac{1}{10}\ln|x + 2| + K \end{aligned}$$

In integrating the middle term we have made the mental substitution $u = 2x - 1$, which gives $du = 2\,dx$ and $dx = \frac{1}{2}\,du$. ■

NOTE We can use an alternative method to find the coefficients A, B, and C in Example 2. Equation 4 is an identity; it is true for every value of x. Let's choose values of x that simplify the equation. If we put $x = 0$ in Equation 4, then the second and third terms on the right side vanish and the equation then becomes $-2A = -1$, or $A = \frac{1}{2}$. Likewise, $x = \frac{1}{2}$ gives $5B/4 = \frac{1}{4}$ and $x = -2$ gives $10C = -1$, so $B = \frac{1}{5}$ and $C = -\frac{1}{10}$. (You may object that Equation 3 is not valid for $x = 0, \frac{1}{2}$, or -2, so why should Equation 4 be valid for those values? In fact, Equation 4 is true for all values of x, even $x = 0, \frac{1}{2}$, and -2. See Exercise 75 for the reason.)

EXAMPLE 3 Find $\displaystyle\int \frac{dx}{x^2 - a^2}$, where $a \neq 0$.

SOLUTION The method of partial fractions gives

$$\frac{1}{x^2 - a^2} = \frac{1}{(x - a)(x + a)} = \frac{A}{x - a} + \frac{B}{x + a}$$

and therefore

$$A(x + a) + B(x - a) = 1$$

Using the method of the preceding note, we put $x = a$ in this equation and get $A(2a) = 1$, so $A = 1/(2a)$. If we put $x = -a$, we get $B(-2a) = 1$, so $B = -1/(2a)$. Thus

$$\begin{aligned} \int \frac{dx}{x^2 - a^2} &= \frac{1}{2a} \int \left(\frac{1}{x - a} - \frac{1}{x + a} \right) dx \\ &= \frac{1}{2a}\left(\ln|x - a| - \ln|x + a|\right) + C \end{aligned}$$

Since $\ln x - \ln y = \ln(x/y)$, we can write the integral as

$$\boxed{6} \qquad \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

(See Exercises 61–62 for ways of using Formula 6.) ■

CASE II $Q(x)$ **is a product of linear factors, some of which are repeated.** Suppose the first linear factor $(a_1x + b_1)$ is repeated r times; that is, $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$ in Equation 2, we would use

$$\boxed{7} \qquad \frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

By way of illustration, we could write

$$\frac{x^3 - x + 1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

but we prefer to work out in detail a simpler example.

EXAMPLE 4 Find $\displaystyle\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx$.

SOLUTION The first step is to divide. The result of long division is

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}$$

The second step is to factor the denominator $Q(x) = x^3 - x^2 - x + 1$. Since $Q(1) = 0$, we know that $x - 1$ is a factor and we obtain

$$\begin{aligned} x^3 - x^2 - x + 1 &= (x-1)(x^2-1) = (x-1)(x-1)(x+1) \\ &= (x-1)^2(x+1) \end{aligned}$$

Since the linear factor $x - 1$ occurs twice, the partial fraction decomposition is

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

Multiplying by the least common denominator, $(x-1)^2(x+1)$, we get

$$\boxed{8} \qquad \begin{aligned} 4x &= A(x-1)(x+1) + B(x+1) + C(x-1)^2 \\ &= (A + C)x^2 + (B - 2C)x + (-A + B + C) \end{aligned}$$

Now we equate coefficients:

$$\begin{aligned} A \qquad + \ C &= 0 \\ B - 2C &= 4 \\ -A + B + \ C &= 0 \end{aligned}$$

Solving, we obtain $A = 1$, $B = 2$, and $C = -1$, so

$$\begin{aligned}\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx &= \int \left[x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1} \right] dx \\ &= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + K \\ &= \frac{x^2}{2} + x - \frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + K \end{aligned}$$

■

NOTE We could also determine the coefficients A, B, and C in Example 4 by following the method given after Example 2. Putting $x = 1$ in Equation 8 gives $4 = 2B$, so $B = 2$ and, likewise, putting $x = -1$ gives $-4 = 4C$, so $C = -1$. There is no value of x that makes the second and third terms on the right side of Equation 8 vanish, so we can't find the value of A as easily. But we can choose a third value for x that still gives a useful relationship between A, B, and C. For instance $x = 0$ gives $0 = -A + B + C$, so $A = 1$.

CASE III $Q(x)$ contains irreducible quadratic factors, none of which is repeated. If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions in Equations 2 and 7, the expression for $R(x)/Q(x)$ will have a term of the form

$$\boxed{9} \qquad \frac{Ax + B}{ax^2 + bx + c}$$

where A and B are constants to be determined. For instance, the function given by $f(x) = x/[(x-2)(x^2+1)(x^2+4)]$ has a partial fraction decomposition of the form

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{x^2+4}$$

The term given in (9) can be integrated by completing the square (if necessary) and using the formula

$$\boxed{10} \qquad \boxed{\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C}$$

EXAMPLE 5 Evaluate $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx$.

SOLUTION Since $x^3 + 4x = x(x^2 + 4)$ can't be factored further, we write

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Multiplying by $x(x^2 + 4)$, we have

$$\begin{aligned} 2x^2 - x + 4 &= A(x^2 + 4) + (Bx + C)x \\ &= (A + B)x^2 + Cx + 4A \end{aligned}$$

Equating coefficients, we obtain

$$A + B = 2 \qquad C = -1 \qquad 4A = 4$$

Therefore $A = 1$, $B = 1$, and $C = -1$ and so

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx = \int \left(\frac{1}{x} + \frac{x - 1}{x^2 + 4}\right) dx$$

In order to integrate the second term we split it into two parts:

$$\int \frac{x - 1}{x^2 + 4}\,dx = \int \frac{x}{x^2 + 4}\,dx - \int \frac{1}{x^2 + 4}\,dx$$

We make the substitution $u = x^2 + 4$ in the first of these integrals so that $du = 2x\,dx$. We evaluate the second integral by means of Formula 10 with $a = 2$:

$$\begin{aligned}\int \frac{2x^2 - x + 4}{x(x^2 + 4)}\,dx &= \int \frac{1}{x}\,dx + \int \frac{x}{x^2 + 4}\,dx - \int \frac{1}{x^2 + 4}\,dx \\ &= \ln|x| + \tfrac{1}{2}\ln(x^2 + 4) - \tfrac{1}{2}\tan^{-1}(x/2) + K\end{aligned}$$

■

EXAMPLE 6 Evaluate $\displaystyle\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\,dx$.

SOLUTION Since the degree of the numerator is *not less than* the degree of the denominator, we first divide and obtain

$$\frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} = 1 + \frac{x - 1}{4x^2 - 4x + 3}$$

Notice that the quadratic $4x^2 - 4x + 3$ is irreducible because its discriminant is $b^2 - 4ac = -32 < 0$. This means it can't be factored, so we don't need to use the partial fraction technique.

To integrate the given function we complete the square in the denominator:

$$4x^2 - 4x + 3 = (2x - 1)^2 + 2$$

This suggests that we make the substitution $u = 2x - 1$. Then $du = 2\,dx$ and $x = \frac{1}{2}(u + 1)$, so

$$\begin{aligned}\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\,dx &= \int \left(1 + \frac{x - 1}{4x^2 - 4x + 3}\right) dx \\ &= x + \tfrac{1}{2}\int \frac{\frac{1}{2}(u + 1) - 1}{u^2 + 2}\,du = x + \tfrac{1}{4}\int \frac{u - 1}{u^2 + 2}\,du \\ &= x + \tfrac{1}{4}\int \frac{u}{u^2 + 2}\,du - \tfrac{1}{4}\int \frac{1}{u^2 + 2}\,du \\ &= x + \tfrac{1}{8}\ln(u^2 + 2) - \frac{1}{4} \cdot \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C \\ &= x + \tfrac{1}{8}\ln(4x^2 - 4x + 3) - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x - 1}{\sqrt{2}}\right) + C\end{aligned}$$

■

NOTE Example 6 illustrates the general procedure for integrating a partial fraction of the form

$$\frac{Ax+B}{ax^2+bx+c} \qquad \text{where } b^2-4ac<0$$

We complete the square in the denominator and then make a substitution that brings the integral into the form

$$\int \frac{Cu+D}{u^2+a^2}\,du = C\int \frac{u}{u^2+a^2}\,du + D\int \frac{1}{u^2+a^2}\,du$$

Then the first integral is a logarithm and the second is expressed in terms of $\tan^{-1}$.

CASE IV ***$Q(x)$* contains a repeated irreducible quadratic factor.**
If $Q(x)$ has the factor $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then instead of the single partial fraction (9), the sum

$$\boxed{11} \qquad \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each of the terms in (11) can be integrated by using a substitution or by first completing the square if necessary.

It would be extremely tedious to work out by hand the numerical values of the coefficients in Example 7. Most computer algebra systems, however, can find the numerical values very quickly:

$$A=-1, \quad B=\tfrac{1}{8}, \quad C=D=-1,$$
$$E=\tfrac{15}{8}, \quad F=-\tfrac{1}{8}, \quad G=H=\tfrac{3}{4},$$
$$I=-\tfrac{1}{2}, \quad J=\tfrac{1}{2}$$

EXAMPLE 7 Write out the form of the partial fraction decomposition of the function

$$\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

SOLUTION

$$\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$
$$= \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3} \qquad ■$$

EXAMPLE 8 Evaluate $\displaystyle\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$.

SOLUTION The form of the partial fraction decomposition is

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiplying by $x(x^2+1)^2$, we have

$$\begin{aligned} -x^3+2x^2-x+1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A \end{aligned}$$

If we equate coefficients, we get the system

$$A+B=0 \qquad C=-1 \qquad 2A+B+D=2 \qquad C+E=-1 \qquad A=1$$

which has the solution $A = 1$, $B = -1$, $C = -1$, $D = 1$, and $E = 0$. Thus

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}\,dx = \int \left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2} \right) dx$$

$$= \int \frac{dx}{x} - \int \frac{x}{x^2+1}\,dx - \int \frac{dx}{x^2+1} + \int \frac{x\,dx}{(x^2+1)^2}$$

In the second and fourth terms we made the mental substitution $u = x^2 + 1$.

$$= \ln|x| - \tfrac{1}{2}\ln(x^2+1) - \tan^{-1}x - \frac{1}{2(x^2+1)} + K \quad ■$$

NOTE Example 8 worked out rather nicely because the coefficient E turned out to be 0. In general, we might get a term of the form $1/(x^2 + 1)^2$. One way to integrate such a term is to make the substitution $x = \tan\theta$. Another method is to use the formula in Exercise 76.

Partial fractions can sometimes be avoided when integrating a rational function. For instance, although the integral

$$\int \frac{x^2+1}{x(x^2+3)}\,dx$$

could be evaluated by using the method of Case III, it's much easier to observe that if $u = x(x^2 + 3) = x^3 + 3x$, then $du = (3x^2 + 3)\,dx$ and so

$$\int \frac{x^2+1}{x(x^2+3)}\,dx = \tfrac{1}{3}\ln|x^3 + 3x| + C$$

■ Rationalizing Substitutions

Some nonrational functions can be changed into rational functions by means of appropriate substitutions. In particular, when an integrand contains an expression of the form $\sqrt[n]{g(x)}$, then the substitution $u = \sqrt[n]{g(x)}$ may be effective. Other instances appear in the exercises.

EXAMPLE 9 Evaluate $\displaystyle\int \frac{\sqrt{x+4}}{x}\,dx$.

SOLUTION Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $dx = 2u\,du$. Therefore

$$\int \frac{\sqrt{x+4}}{x}\,dx = \int \frac{u}{u^2-4}\,2u\,du = 2\int \frac{u^2}{u^2-4}\,du = 2\int \left(1 + \frac{4}{u^2-4}\right) du$$

We can evaluate this integral either by factoring $u^2 - 4$ as $(u - 2)(u + 2)$ and using partial fractions or by using Formula 6 with $a = 2$:

$$\int \frac{\sqrt{x+4}}{x}\,dx = 2\int du + 8\int \frac{du}{u^2-4}$$

$$= 2u + 8 \cdot \frac{1}{2 \cdot 2} \ln\left|\frac{u-2}{u+2}\right| + C$$

$$= 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C \quad ■$$

7.4 Exercises

1–6 Write out the form of the partial fraction decomposition of the function (as in Example 7). Do not determine the numerical values of the coefficients.

1. (a) $\dfrac{1}{(x-3)(x+5)}$ (b) $\dfrac{2x+5}{(x-2)^2(x^2+2)}$

2. (a) $\dfrac{x-6}{x^2+x-6}$ (b) $\dfrac{1}{x^2+x^4}$

3. (a) $\dfrac{x^2+4}{x^3-3x^2+2x}$ (b) $\dfrac{x^3+x}{x(2x-1)^2(x^2+3)^2}$

4. (a) $\dfrac{5}{x^4-1}$ (b) $\dfrac{x^4+x+1}{(x^3-1)(x^2-1)}$

5. (a) $\dfrac{x^5+1}{(x^2-x)(x^4+2x^2+1)}$ (b) $\dfrac{x^2}{x^2+x-6}$

6. (a) $\dfrac{x^6}{x^2-4}$ (b) $\dfrac{x^4}{(x^2-x+1)(x^2+2)^2}$

7–40 Evaluate the integral.

7. $\displaystyle\int \frac{5}{(x-1)(x+4)}\,dx$ **8.** $\displaystyle\int \frac{x-12}{x^2-4x}\,dx$

9. $\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}\,dx$ **10.** $\displaystyle\int \frac{y}{(y+4)(2y-1)}\,dy$

11. $\displaystyle\int_0^1 \frac{2}{2x^2+3x+1}\,dx$ **12.** $\displaystyle\int_0^1 \frac{x-4}{x^2-5x+6}\,dx$

13. $\displaystyle\int \frac{1}{x(x-a)}\,dx$ **14.** $\displaystyle\int \frac{1}{(x+a)(x+b)}\,dx$

15. $\displaystyle\int \frac{x^2}{x-1}\,dx$ **16.** $\displaystyle\int \frac{3t-2}{t+1}\,dt$

17. $\displaystyle\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy$ **18.** $\displaystyle\int_1^2 \frac{3x^2+6x+2}{x^2+3x+2}\,dx$

19. $\displaystyle\int_0^1 \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx$ **20.** $\displaystyle\int_2^3 \frac{x(3-5x)}{(3x-1)(x-1)^2}\,dx$

21. $\displaystyle\int \frac{dt}{(t^2-1)^2}$ **22.** $\displaystyle\int \frac{3x^2+12x-20}{x^4-8x^2+16}\,dx$

23. $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,dx$ **24.** $\displaystyle\int \frac{3x^2-x+8}{x^3+4x}\,dx$

25. $\displaystyle\int_{-1}^0 \frac{x^3-4x+1}{x^2-3x+2}\,dx$ **26.** $\displaystyle\int_1^2 \frac{x^3+4x^2+x-1}{x^3+x^2}\,dx$

27. $\displaystyle\int \frac{4x}{x^3+x^2+x+1}\,dx$ **28.** $\displaystyle\int \frac{x^2+x+1}{(x^2+1)^2}\,dx$

29. $\displaystyle\int \frac{x^3+4x+3}{x^4+5x^2+4}\,dx$ **30.** $\displaystyle\int \frac{x^3+6x-2}{x^4+6x^2}\,dx$

31. $\displaystyle\int \frac{x+4}{x^2+2x+5}\,dx$ **32.** $\displaystyle\int_0^1 \frac{x}{x^2+4x+13}\,dx$

33. $\displaystyle\int \frac{1}{x^3-1}\,dx$ **34.** $\displaystyle\int \frac{x^3-2x^2+2x-5}{x^4+4x^2+3}\,dx$

35. $\displaystyle\int_0^1 \frac{x^3+2x}{x^4+4x^2+3}\,dx$ **36.** $\displaystyle\int \frac{x^5+x-1}{x^3+1}\,dx$

37. $\displaystyle\int \frac{5x^4+7x^2+x+2}{x(x^2+1)^2}\,dx$ **38.** $\displaystyle\int \frac{x^4+3x^2+1}{x^5+5x^3+5x}\,dx$

39. $\displaystyle\int \frac{x^2-3x+7}{(x^2-4x+6)^2}\,dx$ **40.** $\displaystyle\int \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}\,dx$

41–56 Make a substitution to express the integrand as a rational function and then evaluate the integral.

41. $\displaystyle\int \frac{dx}{x\sqrt{x-1}}$ **42.** $\displaystyle\int \frac{dx}{2\sqrt{x+3}+x}$

43. $\displaystyle\int \frac{dx}{x^2+x\sqrt{x}}$ **44.** $\displaystyle\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx$

45. $\displaystyle\int \frac{x^3}{\sqrt[3]{x^2+1}}\,dx$ **46.** $\displaystyle\int \frac{dx}{(1+\sqrt{x})^2}$

47. $\displaystyle\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$ [*Hint:* Substitute $u=\sqrt[6]{x}$.]

48. $\displaystyle\int \frac{1}{x-x^{1/5}}\,dx$

49. $\displaystyle\int \frac{1}{x-3\sqrt{x}+2}\,dx$ **50.** $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx$

51. $\displaystyle\int \frac{e^{2x}}{e^{2x}+3e^x+2}\,dx$ **52.** $\displaystyle\int \frac{\sin x}{\cos^2 x-3\cos x}\,dx$

53. $\displaystyle\int \frac{\sec^2 t}{\tan^2 t+3\tan t+2}\,dt$ **54.** $\displaystyle\int \frac{e^x}{(e^x-2)(e^{2x}+1)}\,dx$

55. $\displaystyle\int \frac{dx}{1+e^x}$ **56.** $\displaystyle\int \frac{\cosh t}{\sinh^2 t+\sinh^4 t}\,dt$

57–58 Use integration by parts, together with the techniques of this section, to evaluate the integral.

57. $\displaystyle\int \ln(x^2-x+2)\,dx$ **58.** $\displaystyle\int x\tan^{-1}x\,dx$

59. Use a graph of $f(x)=1/(x^2-2x-3)$ to decide whether $\int_0^2 f(x)\,dx$ is positive or negative. Use the graph to give a rough estimate of the value of the integral and then use partial fractions to find the exact value.

60. Evaluate

$$\int \frac{1}{x^2 + k}\,dx$$

by considering several cases for the constant k.

61–62 Evaluate the integral by completing the square and using Formula 6.

61. $\displaystyle\int \frac{dx}{x^2 - 2x}$ **62.** $\displaystyle\int \frac{2x + 1}{4x^2 + 12x - 7}\,dx$

---

63. Weierstrass Substitution The German mathematician Karl Weierstrass (1815–1897) noticed that the substitution $t = \tan(x/2)$ will convert any rational function of $\sin x$ and $\cos x$ into an ordinary rational function of t.

(a) If $t = \tan(x/2)$, $-\pi < x < \pi$, sketch a right triangle or use trigonometric identities to show that

$$\cos\frac{x}{2} = \frac{1}{\sqrt{1 + t^2}} \qquad \text{and} \qquad \sin\frac{x}{2} = \frac{t}{\sqrt{1 + t^2}}$$

(b) Show that

$$\cos x = \frac{1 - t^2}{1 + t^2} \qquad \text{and} \qquad \sin x = \frac{2t}{1 + t^2}$$

(c) Show that $dx = \dfrac{2}{1 + t^2}\,dt$.

64–67 Use the substitution in Exercise 63 to transform the integrand into a rational function of t and then evaluate the integral.

64. $\displaystyle\int \frac{dx}{1 - \cos x}$ **65.** $\displaystyle\int \frac{1}{3\sin x - 4\cos x}\,dx$

66. $\displaystyle\int_{\pi/3}^{\pi/2} \frac{1}{1 + \sin x - \cos x}\,dx$ **67.** $\displaystyle\int_{0}^{\pi/2} \frac{\sin 2x}{2 + \cos x}\,dx$

68–69 Find the area of the region under the given curve from 1 to 2.

68. $y = \dfrac{1}{x^3 + x}$ **69.** $y = \dfrac{x^2 + 1}{3x - x^2}$

---

70. Find the volume of the resulting solid if the region under the curve

$$y = \frac{1}{x^2 + 3x + 2}$$

from $x = 0$ to $x = 1$ is rotated about (a) the x-axis and (b) the y-axis.

71. One method of slowing the growth of an insect population without using pesticides is to introduce into the population a number of sterile males that mate with fertile females but produce no offspring. (The photo shows a screw-worm fly, the first pest effectively eliminated from a region by this method.) Let P represent the number of female insects in a population and S the number of sterile males introduced each generation. Let r be the per capita rate of production of females by females, provided their chosen mate is not sterile. Then the female population is related to time t by

$$t = \int \frac{P + S}{P[(r - 1)P - S]}\,dP$$

Suppose an insect population with 10,000 females grows at a rate of $r = 1.1$ and 900 sterile males are added initially. Evaluate the integral to give an equation relating the female population to time. (Note that the resulting equation can't be solved explicitly for P.)

USDA

72. Factor $x^4 + 1$ as a difference of squares by first adding and subtracting the same quantity. Use this factorization to evaluate $\int 1/(x^4 + 1)\,dx$.

T **73.** (a) Use a computer algebra system to find the partial fraction decomposition of the function

$$f(x) = \frac{4x^3 - 27x^2 + 5x - 32}{30x^5 - 13x^4 + 50x^3 - 286x^2 - 299x - 70}$$

(b) Use part (a) to find $\int f(x)\,dx$ (by hand) and compare with the result of using the CAS to integrate f directly. Comment on any discrepancy.

T **74.** (a) Use a computer algebra system to find the partial fraction decomposition of the function

$$f(x) = \frac{12x^5 - 7x^3 - 13x^2 + 8}{100x^6 - 80x^5 + 116x^4 - 80x^3 + 41x^2 - 20x + 4}$$

(b) Use part (a) to find $\int f(x)\,dx$ and graph f and its indefinite integral on the same screen.

(c) Use the graph of f to discover the main features of the graph of $\int f(x)\,dx$.

75. Suppose that F, G, and Q are polynomials and

$$\frac{F(x)}{Q(x)} = \frac{G(x)}{Q(x)}$$

for all x except when $Q(x) = 0$. Prove that $F(x) = G(x)$ for all x. [*Hint:* Use continuity.]

76. (a) Use integration by parts to show that, for any positive integer n,

$$\int \frac{dx}{(x^2 + a^2)^n} = \frac{x}{2a^2(n - 1)(x^2 + a^2)^{n-1}} + \frac{2n - 3}{2a^2(n - 1)}\int \frac{dx}{(x^2 + a^2)^{n-1}}$$

(b) Use part (a) to evaluate

$$\int \frac{dx}{(x^2+1)^2} \qquad \text{and} \qquad \int \frac{dx}{(x^2+1)^3}$$

77. If $a \neq 0$ and n is a positive integer, find the partial fraction decomposition of $f(x) = 1/(x^n(x-a))$. [*Hint:* First find the coefficient of $1/(x-a)$. Then subtract the resulting term and simplify what is left.]

78. If f is a quadratic function such that $f(0) = 1$ and

$$\int \frac{f(x)}{x^2(x+1)^3}\,dx$$

is a rational function, find the value of $f'(0)$.

7.5 Strategy for Integration

As we have seen, integration is more challenging than differentiation. In finding the derivative of a function it is obvious which differentiation formula we should apply. But it may not be obvious which technique we should use to integrate a given function.

Guidelines for Integration

Until now individual techniques have been applied in each section. For instance, we usually used substitution in Exercises 5.5, integration by parts in Exercises 7.1, and partial fractions in Exercises 7.4. But in this section we present a collection of miscellaneous integrals in random order and the main challenge is to recognize which technique or formula to use. No hard and fast rules can be given as to which method applies in a given situation, but we give some general guidelines that you may find useful.

A prerequisite for applying a strategy is a knowledge of the basic integration formulas. In the following table we have collected the integrals from our previous list together with several additional formulas that we have learned in this chapter.

Table of Integration Formulas Constants of integration have been omitted.

1. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$

2. $\displaystyle\int \frac{1}{x}\,dx = \ln|x|$

3. $\displaystyle\int e^x\,dx = e^x$

4. $\displaystyle\int b^x\,dx = \frac{b^x}{\ln b}$

5. $\displaystyle\int \sin x\,dx = -\cos x$

6. $\displaystyle\int \cos x\,dx = \sin x$

7. $\displaystyle\int \sec^2 x\,dx = \tan x$

8. $\displaystyle\int \csc^2 x\,dx = -\cot x$

9. $\displaystyle\int \sec x \tan x\,dx = \sec x$

10. $\displaystyle\int \csc x \cot x\,dx = -\csc x$

11. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x|$

12. $\displaystyle\int \csc x\,dx = \ln|\csc x - \cot x|$

13. $\displaystyle\int \tan x\,dx = \ln|\sec x|$

14. $\displaystyle\int \cot x\,dx = \ln|\sin x|$

15. $\displaystyle\int \sinh x\,dx = \cosh x$

16. $\displaystyle\int \cosh x\,dx = \sinh x$

17. $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

18. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right), \quad a > 0$

***19.** $\displaystyle\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

***20.** $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$

Most of these formulas should be memorized. It is useful to know them all, but the ones marked with an asterisk need not be memorized since they are easily derived. Formula 19 can be avoided by using partial fractions, and trigonometric substitutions can be used in place of Formula 20.

Once you are armed with these basic integration formulas, if you don't immediately see how to attack a given integral, you might try the following four-step strategy.

1. **Simplify the Integrand If Possible** Sometimes the use of algebraic manipulation or trigonometric identities will simplify the integrand and make the method of integration obvious. Here are some examples:

$$\int \sqrt{x}\,\left(1 + \sqrt{x}\right) dx = \int \left(\sqrt{x} + x\right) dx$$

$$\begin{aligned}\int \frac{\tan\theta}{\sec^2\theta}\,d\theta &= \int \frac{\sin\theta}{\cos\theta}\cos^2\theta\,d\theta\\ &= \int \sin\theta\,\cos\theta\,d\theta = \tfrac{1}{2}\int \sin 2\theta\,d\theta\end{aligned}$$

$$\begin{aligned}\int (\sin x + \cos x)^2\,dx &= \int (\sin^2 x + 2\sin x\cos x + \cos^2 x)\,dx\\ &= \int (1 + 2\sin x\cos x)\,dx\end{aligned}$$

2. **Look for an Obvious Substitution** Try to find some function $u = g(x)$ in the integrand whose differential $du = g'(x)\,dx$ also occurs, apart from a constant factor. For instance, in the integral

$$\int \frac{x}{x^2 - 1}\,dx$$

   we notice that if $u = x^2 - 1$, then $du = 2x\,dx$. Therefore we use the substitution $u = x^2 - 1$ instead of the method of partial fractions.

3. **Classify the Integrand According to Its Form** If Steps 1 and 2 have not led to the solution, then we take a look at the form of the integrand $f(x)$.
   (a) *Trigonometric functions.* If $f(x)$ is a product of powers of $\sin x$ and $\cos x$, of $\tan x$ and $\sec x$, or of $\cot x$ and $\csc x$, then we use the substitutions recommended in Section 7.2.
   (b) *Rational functions.* If f is a rational function, we use the procedure of Section 7.4 involving partial fractions.
   (c) *Integration by parts.* If $f(x)$ is a product of a power of x (or a polynomial) and a transcendental function (such as a trigonometric, exponential, or logarithmic function), then we try integration by parts, choosing u and dv according to the advice given in Section 7.1. If you look at the functions in Exercises 7.1, you will see that most of them are the type just described.
   (d) *Radicals.* Particular kinds of substitutions are recommended when certain radicals appear.
      (i) If $\sqrt{x^2 + a^2}$, $\sqrt{x^2 - a^2}$, or $\sqrt{a^2 - x^2}$ occurs, we use a trigonometric substitution according to the table in Section 7.3.
      (ii) If $\sqrt[n]{ax + b}$ occurs, we use the rationalizing substitution $u = \sqrt[n]{ax + b}$. More generally, this sometimes works for $\sqrt[n]{g(x)}$.

4. Try Again If the first three steps have not produced the answer, remember that there are basically only two methods of integration: substitution and parts.

(a) *Try substitution.* Even if no substitution is obvious (Step 2), some inspiration or ingenuity (or even desperation) may suggest an appropriate substitution.

(b) *Try parts.* Although integration by parts is used most of the time on products of the form described in Step 3(c), it is sometimes effective on single functions. Looking at Section 7.1, we see that it works on $\tan^{-1}x$, $\sin^{-1}x$, and $\ln x$, and these are all inverse functions.

(c) *Manipulate the integrand.* Algebraic manipulations (perhaps rationalizing the denominator or using trigonometric identities) may be useful in transforming the integral into an easier form. These manipulations may be more substantial than in Step 1 and may involve some ingenuity. Here is an example:

$$\int \frac{dx}{1-\cos x} = \int \frac{1}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x}\,dx = \int \frac{1+\cos x}{1-\cos^2 x}\,dx$$

$$= \int \frac{1+\cos x}{\sin^2 x}\,dx = \int \left(\csc^2 x + \frac{\cos x}{\sin^2 x}\right) dx$$

(d) *Relate the problem to previous problems.* When you have built up some experience in integration, you may be able to use a method on a given integral that is similar to a method you have already used on a previous integral. Or you may even be able to express the given integral in terms of a previous one. For instance, $\int \tan^2 x \sec x\,dx$ is a challenging integral, but if we make use of the identity $\tan^2 x = \sec^2 x - 1$, we can write

$$\int \tan^2 x \sec x\,dx = \int \sec^3 x\,dx - \int \sec x\,dx$$

and if $\int \sec^3 x\,dx$ has previously been evaluated (see Example 7.2.8), then that calculation can be used in the present problem.

(e) *Use several methods.* Sometimes two or three methods are required to evaluate an integral. The evaluation could involve several successive substitutions of different types, or it might combine integration by parts with one or more substitutions.

In the following examples we indicate a method of attack but do not fully work out the integral.

EXAMPLE 1 $\displaystyle\int \frac{\tan^3 x}{\cos^3 x}\,dx$

In Step 1 we rewrite the integral:

$$\int \frac{\tan^3 x}{\cos^3 x}\,dx = \int \tan^3 x \sec^3 x\,dx$$

The integral is now of the form $\int \tan^m x \sec^n x\,dx$ with m odd, so we can use the advice in Section 7.2.

Alternatively, if in Step 1 we had written

$$\int \frac{\tan^3 x}{\cos^3 x}\,dx = \int \frac{\sin^3 x}{\cos^3 x}\,\frac{1}{\cos^3 x}\,dx = \int \frac{\sin^3 x}{\cos^6 x}\,dx$$

then we could have continued as follows with the substitution $u = \cos x$:

$$\int \frac{\sin^3 x}{\cos^6 x}\,dx = \int \frac{1 - \cos^2 x}{\cos^6 x}\sin x\,dx = \int \frac{1 - u^2}{u^6}(-du)$$
$$= \int \frac{u^2 - 1}{u^6}\,du = \int (u^{-4} - u^{-6})\,du \qquad ■$$

EXAMPLE 2 $\int \sin\sqrt{x}\,dx$

According to (ii) in Step 3(d), we substitute $u = \sqrt{x}$. Then $x = u^2$, so $dx = 2u\,du$ and

$$\int \sin\sqrt{x}\,dx = 2\int u\sin u\,du$$

The integrand is now a product of u and the trigonometric function $\sin u$ so it can be integrated by parts. ■

EXAMPLE 3 $\int \frac{x^5 + 1}{x^3 - 3x^2 - 10x}\,dx$

No algebraic simplification or substitution is obvious, so Steps 1 and 2 don't apply here. The integrand is a rational function so we apply the procedure of Section 7.4, remembering that the first step is to divide. ■

EXAMPLE 4 $\int \frac{dx}{x\sqrt{\ln x}}$

Here Step 2 is all that is needed. We substitute $u = \ln x$ because its differential is $du = dx/x$, which occurs in the integral. ■

EXAMPLE 5 $\int \sqrt{\frac{1 - x}{1 + x}}\,dx$

Although the rationalizing substitution

$$u = \sqrt{\frac{1 - x}{1 + x}}$$

works here [(ii) in Step 3(d)], it leads to a very complicated rational function. An easier method is to do some algebraic manipulation [either as Step 1 or as Step 4(c)]. Multiplying numerator and denominator by $\sqrt{1 - x}$, we have

$$\int \sqrt{\frac{1 - x}{1 + x}}\,dx = \int \frac{1 - x}{\sqrt{1 - x^2}}\,dx$$
$$= \int \frac{1}{\sqrt{1 - x^2}}\,dx - \int \frac{x}{\sqrt{1 - x^2}}\,dx$$
$$= \sin^{-1}x + \sqrt{1 - x^2} + C \qquad ■$$

■ Can We Integrate All Continuous Functions?

The question arises: will our strategy for integration enable us to find the integral of every continuous function? For example, can we use it to evaluate $\int e^{x^2}\,dx$? The answer is no, at least not in terms of the functions that we are familiar with.

The functions that we have been dealing with in this book are called **elementary functions**. These are the polynomials, rational functions, power functions (x^n), exponential functions (b^x), logarithmic functions, trigonometric and inverse trigonometric functions, hyperbolic and inverse hyperbolic functions, and all functions that can be obtained from these by the five operations of addition, subtraction, multiplication, division, and composition. For instance, the function

$$f(x) = \sqrt{\frac{x^2 - 1}{x^3 + 2x - 1}} + \ln(\cosh x) - xe^{\sin 2x}$$

is an elementary function.

If f is an elementary function, then f' is an elementary function but $\int f(x)\,dx$ need not be an elementary function. Consider $f(x) = e^{x^2}$. Since f is continuous, its integral exists, and if we define the function F by

$$F(x) = \int_0^x e^{t^2}\,dt$$

then we know from Part 1 of the Fundamental Theorem of Calculus that

$$F'(x) = e^{x^2}$$

Thus $f(x) = e^{x^2}$ has an antiderivative F, but it has been proved that F is not an elementary function. This means that no matter how hard we try, we will never succeed in evaluating $\int e^{x^2}\,dx$ in terms of the functions we know. (In Chapter 11, however, we will see how to express $\int e^{x^2}\,dx$ as an infinite series.) The same can be said of the following integrals:

$$\int \frac{e^x}{x}\,dx \qquad \int \sin(x^2)\,dx \qquad \int \cos(e^x)\,dx$$

$$\int \sqrt{x^3 + 1}\,dx \qquad \int \frac{1}{\ln x}\,dx \qquad \int \frac{\sin x}{x}\,dx$$

In fact, the majority of elementary functions don't have elementary antiderivatives. You may be assured, though, that the integrals in the following exercises are all elementary functions.

7.5 Exercises

1–8 Three integrals are given that, although they look similar, may require different techniques of integration. Evaluate the integrals.

1. (a) $\displaystyle\int \frac{x}{1 + x^2}\,dx$ (b) $\displaystyle\int \frac{1}{1 + x^2}\,dx$

(c) $\displaystyle\int \frac{1}{1 - x^2}\,dx$

2. (a) $\displaystyle\int x\sqrt{x^2 - 1}\,dx$ (b) $\displaystyle\int \frac{1}{x\sqrt{x^2 - 1}}\,dx$

(c) $\displaystyle\int \frac{\sqrt{x^2 - 1}}{x}\,dx$

3. (a) $\displaystyle\int \frac{\ln x}{x}\,dx$ (b) $\displaystyle\int \ln(2x)\,dx$

(c) $\displaystyle\int x \ln x\,dx$

4. (a) $\displaystyle\int \sin^2 x\,dx$ (b) $\displaystyle\int \sin^3 x\,dx$

(c) $\displaystyle\int \sin 2x\,dx$

5. (a) $\displaystyle\int \frac{1}{x^2 - 4x + 3}\,dx$ (b) $\displaystyle\int \frac{1}{x^2 - 4x + 4}\,dx$

(c) $\displaystyle\int \frac{1}{x^2 - 4x + 5}\,dx$

6. (a) $\int x \cos x^2 \, dx$ (b) $\int x \cos^2 x \, dx$

(c) $\int x^2 \cos x \, dx$

7. (a) $\int x^2 e^{x^3} \, dx$ (b) $\int x^2 e^x \, dx$

(c) $\int x^3 e^{x^2} \, dx$

8. (a) $\int e^x \sqrt{e^x - 1} \, dx$ (b) $\int \frac{e^x}{\sqrt{1 - e^{2x}}} \, dx$

(c) $\int \frac{1}{\sqrt{e^x - 1}} \, dx$

9–93 Evaluate the integral.

9. $\int \frac{\cos x}{1 - \sin x} \, dx$

10. $\int_0^1 (3x + 1)^{\sqrt{2}} \, dx$

11. $\int_1^4 \sqrt{y} \ln y \, dy$

12. $\int \frac{e^{\arcsin x}}{\sqrt{1 - x^2}} \, dx$

13. $\int \frac{\ln(\ln y)}{y} \, dy$

14. $\int_0^1 \frac{x}{(2x + 1)^3} \, dx$

15. $\int \frac{x}{x^4 + 9} \, dx$

16. $\int t \sin t \cos t \, dt$

17. $\int_2^4 \frac{x + 2}{x^2 + 3x - 4} \, dx$

18. $\int \frac{\cos(1/x)}{x^3} \, dx$

19. $\int \frac{1}{x^3 \sqrt{x^2 - 1}} \, dx$

20. $\int \frac{2x - 3}{x^3 + 3x} \, dx$

21. $\int \frac{\cos^3 x}{\csc x} \, dx$

22. $\int \ln(1 + x^2) \, dx$

23. $\int x \sec x \tan x \, dx$

24. $\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1 - x^2}} \, dx$

25. $\int_0^{\pi} t \cos^2 t \, dt$

26. $\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}} \, dt$

27. $\int e^{x + e^x} \, dx$

28. $\int \frac{e^x}{1 + e^{2a}} \, dx$

29. $\int \arctan \sqrt{x} \, dx$

30. $\int \frac{\ln x}{x \sqrt{1 + (\ln x)^2}} \, dx$

31. $\int_0^1 (1 + \sqrt{x})^8 \, dx$

32. $\int (1 + \tan x)^2 \sec x \, dx$

33. $\int_0^1 \frac{1 + 12t}{1 + 3t} \, dt$

34. $\int_0^1 \frac{3x^2 + 1}{x^3 + x^2 + x + 1} \, dx$

35. $\int \frac{dx}{1 + e^x}$

36. $\int \sin \sqrt{at} \, dt$

37. $\int \ln\left(x + \sqrt{x^2 - 1}\right) dx$

38. $\int_{-1}^{2} |e^x - 1| \, dx$

39. $\int \sqrt{\frac{1 + x}{1 - x}} \, dx$

40. $\int_1^3 \frac{e^{3/x}}{x^2} \, dx$

41. $\int \sqrt{3 - 2x - x^2} \, dx$

42. $\int_{\pi/4}^{\pi/2} \frac{1 + 4 \cot x}{4 - \cot x} \, dx$

43. $\int_{-\pi/2}^{\pi/2} \frac{x}{1 + \cos^2 x} \, dx$

44. $\int \frac{1 + \sin x}{1 + \cos x} \, dx$

45. $\int_0^{\pi/4} \tan^3 \theta \sec^2 \theta \, d\theta$

46. $\int_{\pi/6}^{\pi/3} \frac{\sin \theta \cot \theta}{\sec \theta} \, d\theta$

47. $\int \frac{\sec \theta \tan \theta}{\sec^2 \theta - \sec \theta} \, d\theta$

48. $\int_0^{\pi} \sin 6x \cos 3x \, dx$

49. $\int \theta \tan^2 \theta \, d\theta$

50. $\int \frac{1}{x \sqrt{x - 1}} \, dx$

51. $\int \frac{\sqrt{x}}{1 + x^3} \, dx$

52. $\int \sqrt{1 + e^x} \, dx$

53. $\int \frac{x}{1 + \sqrt{x}} \, dx$

54. $\int \frac{(x - 1)e^x}{x^2} \, dx$

55. $\int x^3 (x - 1)^{-4} \, dx$

56. $\int_0^1 x \sqrt{2 - \sqrt{1 - x^2}} \, dx$

57. $\int \frac{1}{x \sqrt{4x + 1}} \, dx$

58. $\int \frac{1}{x^2 \sqrt{4x + 1}} \, dx$

59. $\int \frac{1}{x \sqrt{4x^2 + 1}} \, dx$

60. $\int \frac{dx}{x(x^4 + 1)}$

61. $\int x^2 \sinh mx \, dx$

62. $\int (x + \sin x)^2 \, dx$

63. $\int \frac{dx}{x + x\sqrt{x}}$

64. $\int \frac{dx}{\sqrt{x} + x\sqrt{x}}$

65. $\int x \sqrt[3]{x + c} \, dx$

66. $\int \frac{x \ln x}{\sqrt{x^2 - 1}} \, dx$

67. $\int \frac{dx}{x^4 - 16}$

68. $\int \frac{dx}{x^2 \sqrt{4x^2 - 1}}$

69. $\int \frac{d\theta}{1 + \cos \theta}$

70. $\int \frac{d\theta}{1 + \cos^2 \theta}$

71. $\int \sqrt{x} \, e^{\sqrt{x}} \, dx$

72. $\int \frac{1}{\sqrt{\sqrt{x} + 1}} \, dx$

73. $\displaystyle\int \frac{\sin 2x}{1+\cos^4 x}\,dx$

74. $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\sin x\cos x}\,dx$

75. $\displaystyle\int \frac{1}{\sqrt{x+1}+\sqrt{x}}\,dx$

76. $\displaystyle\int \frac{x^2}{x^6+3x^3+2}\,dx$

77. $\displaystyle\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2}\,dx$

78. $\displaystyle\int \frac{1}{1+2e^x-e^{-x}}\,dx$

79. $\displaystyle\int \frac{e^{2x}}{1+e^x}\,dx$

80. $\displaystyle\int \frac{\ln(x+1)}{x^2}\,dx$

81. $\displaystyle\int \frac{x+\arcsin x}{\sqrt{1-x^2}}\,dx$

82. $\displaystyle\int \frac{4^x+10^x}{2^x}\,dx$

83. $\displaystyle\int \frac{dx}{x\ln x - x}$

84. $\displaystyle\int \frac{x^2}{\sqrt{x^2+1}}\,dx$

85. $\displaystyle\int \frac{xe^x}{\sqrt{1+e^x}}\,dx$

86. $\displaystyle\int \frac{1+\sin x}{1-\sin x}\,dx$

87. $\displaystyle\int x\sin^2 x\cos x\,dx$

88. $\displaystyle\int \frac{\sec x\cos 2x}{\sin x+\sec x}\,dx$

89. $\displaystyle\int \sqrt{1-\sin x}\,dx$

90. $\displaystyle\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx$

91. $\displaystyle\int_1^3 \left(\sqrt{\frac{9-x}{x}}-\sqrt{\frac{x}{9-x}}\right)dx$

92. $\displaystyle\int \frac{1}{(\sin x+\cos x)^2}\,dx$

93. $\displaystyle\int_0^{\pi/6} \sqrt{1+\sin 2\theta}\,d\theta$

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94. We know that $F(x)=\int_0^x e^{e^t}\,dt$ is a continuous function by FTC1, though it is not an elementary function. The functions

$$\int \frac{e^x}{x}\,dx \qquad \text{and} \qquad \int \frac{1}{\ln x}\,dx$$

are not elementary either, but they can be expressed in terms of F. Evaluate the following integrals in terms of F.

(a) $\displaystyle\int_1^2 \frac{e^x}{x}\,dx$ (b) $\displaystyle\int_2^3 \frac{1}{\ln x}\,dx$

95. The functions $y=e^{x^2}$ and $y=x^2e^{x^2}$ don't have elementary antiderivatives, but $y=(2x^2+1)e^{x^2}$ does. Evaluate $\int(2x^2+1)e^{x^2}\,dx$.

7.6 Integration Using Tables and Technology

In this section we describe how to use tables and mathematical software to integrate functions that have elementary antiderivatives. You should bear in mind, though, that even the most powerful computer software can't find explicit formulas for the antiderivatives of functions like e^{x^2} or the other functions described at the end of Section 7.5.

Tables of Integrals

Tables of indefinite integrals are very useful when we are confronted by an integral that is difficult to evaluate by hand. In some cases, the results obtained are of a simpler form than those given by a computer. A relatively brief table of 120 integrals, categorized by form, is provided on Reference Pages 6–10 at the back of the book. More extensive tables, containing hundreds or thousands of entries, are available in separate publications or on the Internet. When using such tables, remember that integrals do not often occur in exactly the form listed. Usually we need to use the Substitution Rule or algebraic manipulation to transform a given integral into one of the forms in the table.

EXAMPLE 1 The region bounded by the curves $y=\arctan x$, $y=0$, and $x=1$ is rotated about the y-axis. Find the volume of the resulting solid.

SOLUTION Using the method of cylindrical shells, we see that the volume is

$$V=\int_0^1 2\pi x\arctan x\,dx$$

The Table of Integrals appears on Reference Pages 6–10 at the back of the book.

In the section of the Table of Integrals titled *Inverse Trigonometric Forms* we locate Formula 92:

$$\int u \tan^{-1}u\, du = \frac{u^2+1}{2}\tan^{-1}u - \frac{u}{2} + C$$

So the volume is

$$V = 2\pi \int_0^1 x \tan^{-1}x\, dx = 2\pi\left[\frac{x^2+1}{2}\tan^{-1}x - \frac{x}{2}\right]_0^1$$

$$= \pi\Big[(x^2+1)\tan^{-1}x - x\Big]_0^1 = \pi(2\tan^{-1}1 - 1)$$

$$= \pi[2(\pi/4) - 1] = \tfrac{1}{2}\pi^2 - \pi$$ ■

EXAMPLE 2 Use the Table of Integrals to find $\displaystyle\int \frac{x^2}{\sqrt{5-4x^2}}\,dx$.

Remember that when we make the substitution $u = 2x$ (so $x = u/2$), we must also substitute $du = 2\,dx$ (so $dx = du/2$).

SOLUTION If we look at the section of the table titled *Forms Involving* $\sqrt{a^2-u^2}$, we see that the closest entry is Formula 34:

$$\int \frac{u^2}{\sqrt{a^2-u^2}}\,du = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\!\left(\frac{u}{a}\right) + C$$

This is not exactly what we have, but we will be able to use it if we first make the substitution $u = 2x$:

$$\int \frac{x^2}{\sqrt{5-4x^2}}\,dx = \int \frac{(u/2)^2}{\sqrt{5-u^2}}\,\frac{du}{2} = \frac{1}{8}\int \frac{u^2}{\sqrt{5-u^2}}\,du$$

Then we use Formula 34 with $a^2 = 5$ (so $a = \sqrt{5}$):

$$\int \frac{x^2}{\sqrt{5-4x^2}}\,dx = \frac{1}{8}\int \frac{u^2}{\sqrt{5-u^2}}\,du = \frac{1}{8}\left(-\frac{u}{2}\sqrt{5-u^2} + \frac{5}{2}\sin^{-1}\frac{u}{\sqrt{5}}\right) + C$$

$$= -\frac{x}{8}\sqrt{5-4x^2} + \frac{5}{16}\sin^{-1}\!\left(\frac{2x}{\sqrt{5}}\right) + C$$ ■

EXAMPLE 3 Use the Table of Integrals to evaluate $\int x^3 \sin x\, dx$.

SOLUTION If we look in the section called *Trigonometric Forms*, we see that none of the entries explicitly includes a u^3 factor. However, we can use the reduction formula in entry 84 with $n = 3$:

$$\int x^3 \sin x\, dx = -x^3\cos x + 3\int x^2 \cos x\, dx$$

85. $\displaystyle\int u^n \cos u\, du = u^n \sin u - n\int u^{n-1}\sin u\, du$

We now need to evaluate $\int x^2 \cos x\, dx$. We can use the reduction formula in entry 85 with $n = 2$, followed by Formula 82:

$$\int x^2 \cos x\, dx = x^2 \sin x - 2\int x \sin x\, dx$$

$$= x^2 \sin x - 2(\sin x - x\cos x) + K$$

Combining these results, we get

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$$

where $C = 3K$. ■

EXAMPLE 4 Use the Table of Integrals to find $\int x\sqrt{x^2 + 2x + 4}\, dx$.

SOLUTION Since the table gives forms involving $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, and $\sqrt{x^2 - a^2}$, but not $\sqrt{ax^2 + bx + c}$, we first complete the square:

$$x^2 + 2x + 4 = (x + 1)^2 + 3$$

If we make the substitution $u = x + 1$ (so $x = u - 1$), the integrand will involve the pattern $\sqrt{a^2 + u^2}$:

$$\begin{aligned} \int x\sqrt{x^2 + 2x + 4}\, dx &= \int (u - 1)\sqrt{u^2 + 3}\, du \\ &= \int u\sqrt{u^2 + 3}\, du - \int \sqrt{u^2 + 3}\, du \end{aligned}$$

The first integral is evaluated using the substitution $t = u^2 + 3$:

$$\int u\sqrt{u^2 + 3}\, du = \tfrac{1}{2}\int \sqrt{t}\, dt = \tfrac{1}{2} \cdot \tfrac{2}{3} t^{3/2} = \tfrac{1}{3}(u^2 + 3)^{3/2}$$

21. $\displaystyle\int \sqrt{a^2 + u^2}\, du = \frac{u}{2}\sqrt{a^2 + u^2} + \frac{a^2}{2}\ln\left(u + \sqrt{a^2 + u^2}\right) + C$

For the second integral we use Formula 21 with $a = \sqrt{3}$:

$$\int \sqrt{u^2 + 3}\, du = \frac{u}{2}\sqrt{u^2 + 3} + \tfrac{3}{2}\ln\left(u + \sqrt{u^2 + 3}\right)$$

Therefore

$$\begin{aligned} &\int x\sqrt{x^2 + 2x + 4}\, dx \\ &= \tfrac{1}{3}(x^2 + 2x + 4)^{3/2} - \frac{x + 1}{2}\sqrt{x^2 + 2x + 4} - \tfrac{3}{2}\ln\left(x + 1 + \sqrt{x^2 + 2x + 4}\right) + C \end{aligned}$$

■

■ Integration Using Technology

We have seen that the use of tables involves matching the form of the given integrand with the forms of the integrands in the tables. Computers are particularly good at matching patterns. And just as we used substitutions in conjunction with tables, a computer algebra system (CAS) or mathematical software with similar capabilities can perform substitutions that transform a given integral into one that occurs in its stored formulas. So it isn't surprising that we have software that excels at integration. That doesn't mean that integration by hand is an obsolete skill. We will see that a hand computation sometimes produces an indefinite integral in a form that is more convenient than a machine answer.

To begin, let's see what happens when we ask a computer to integrate the relatively simple function $y = 1/(3x - 2)$. Using the substitution $u = 3x - 2$, an easy calculation by hand gives

$$\int \frac{1}{3x - 2}\, dx = \tfrac{1}{3}\ln|3x - 2| + C$$

whereas some software packages return the answer

$$\tfrac{1}{3}\ln(3x - 2)$$

The first thing to notice is that the constant of integration is missing. In other words, we were given a *particular* antiderivative, not the most general one. Therefore, when making use of a machine integration, we might have to add a constant. Second, the absolute value signs were not included in the machine answer. That is fine if our problem is concerned only with values of x greater than $\frac{2}{3}$. But if we are interested in other values of x, then we need to insert the absolute value symbol.

In the next example we reconsider the integral of Example 4, but this time we use technology to get an answer.

EXAMPLE 5 Use a computer to find $\int x\sqrt{x^2 + 2x + 4}\,dx$.

SOLUTION Different software may respond with different forms of the answer. One computer algebra system gives

$$\tfrac{1}{3}(x^2 + 2x + 4)^{3/2} - \tfrac{1}{4}(2x + 2)\sqrt{x^2 + 2x + 4} - \frac{3}{2}\operatorname{arcsinh}\frac{\sqrt{3}}{3}(1 + x)$$

This looks different from the answer we found in Example 4, but it is equivalent because the third term can be rewritten using the identity

This is equation 3.11.3.

$$\operatorname{arcsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

Thus

$$\begin{aligned}\operatorname{arcsinh}\frac{\sqrt{3}}{3}(1 + x) &= \ln\left[\frac{\sqrt{3}}{3}(1 + x) + \sqrt{\tfrac{1}{3}(1 + x)^2 + 1}\right]\\ &= \ln\frac{1}{\sqrt{3}}\left[1 + x + \sqrt{(1 + x)^2 + 3}\right]\\ &= \ln\frac{1}{\sqrt{3}} + \ln\left(x + 1 + \sqrt{x^2 + 2x + 4}\right)\end{aligned}$$

The resulting extra term $-\frac{3}{2}\ln\left(1/\sqrt{3}\right)$ can be absorbed into the constant of integration.

Another software package gives the answer

$$\left(\frac{5}{6} + \frac{x}{6} + \frac{x^2}{3}\right)\sqrt{x^2 + 2x + 4} - \frac{3}{2}\sinh^{-1}\left(\frac{1 + x}{\sqrt{3}}\right)$$

Here the first two terms of the answer to Example 4 were combined into a single term by factoring. ■

EXAMPLE 6 Use a computer to evaluate $\int x(x^2 + 5)^8\,dx$.

SOLUTION A computer may give the answer

$$\tfrac{1}{18}x^{18} + \tfrac{5}{2}x^{16} + 50x^{14} + \tfrac{1750}{3}x^{12} + 4375x^{10} + 21875x^8 + \tfrac{218750}{3}x^6 + 156250x^4 + \tfrac{390625}{2}x^2$$

The software must have expanded $(x^2 + 5)^8$ using the Binomial Theorem and then integrated each term.

If we integrate by hand instead, using the substitution $u = x^2 + 5$, we get

$$\int x(x^2 + 5)^8 \, dx = \tfrac{1}{18}(x^2 + 5)^9 + C$$

For most purposes, this is a more convenient form of the answer. ■

EXAMPLE 7 Use a computer to find $\int \sin^5 x \cos^2 x \, dx$.

SOLUTION In Example 7.2.2 we found that

$$\boxed{1} \qquad \int \sin^5 x \cos^2 x \, dx = -\tfrac{1}{3}\cos^3 x + \tfrac{2}{5}\cos^5 x - \tfrac{1}{7}\cos^7 x + C$$

Depending on the software used, you may get the answer

$$-\tfrac{1}{7}\sin^4 x \cos^3 x - \tfrac{4}{35}\sin^2 x \cos^3 x - \tfrac{8}{105}\cos^3 x$$

or you might get

$$-\tfrac{5}{64}\cos x - \tfrac{1}{192}\cos 3x + \tfrac{3}{320}\cos 5x - \tfrac{1}{448}\cos 7x$$

We suspect that there are trigonometric identities which show that these three answers are equivalent. In fact, you may be able to use the software to simplify its initial result, using trigonometric identities, to produce the same form of the answer as in Equation 1. ■

7.6 Exercises

1–6 Use the formula in the indicated entry of the Table of Integrals on Reference Pages 6–10 to evaluate the integral.

1. $\displaystyle\int_0^{\pi/2} \cos 5x \cos 2x \, dx$; entry 80

2. $\displaystyle\int_0^1 \sqrt{x - x^2} \, dx$; entry 113

3. $\displaystyle\int x \arcsin(x^2) \, dx$; entry 87

4. $\displaystyle\int \frac{\tan\theta}{\sqrt{2 + \cos\theta}} \, d\theta$; entry 57

5. $\displaystyle\int \frac{y^5}{\sqrt{4 + y^4}} \, dy$; entry 26

6. $\displaystyle\int \frac{\sqrt{t^6 - 5}}{t} \, dt$; entry 41

7–34 Use the Table of Integrals on the Reference Pages to evaluate the integral.

7. $\displaystyle\int_0^{\pi/8} \arctan 2x \, dx$

8. $\displaystyle\int_0^2 x^2\sqrt{4 - x^2} \, dx$

9. $\displaystyle\int \frac{\cos x}{\sin^2 x - 9} \, dx$

10. $\displaystyle\int \frac{e^x}{4 - e^{2x}} \, dx$

11. $\displaystyle\int \frac{\sqrt{9x^2 + 4}}{x^2} \, dx$

12. $\displaystyle\int \frac{\sqrt{2y^2 - 3}}{y^2} \, dy$

13. $\displaystyle\int_0^{\pi} \cos^6\theta \, d\theta$

14. $\displaystyle\int x\sqrt{2 + x^4} \, dx$

15. $\displaystyle\int \frac{\arctan\sqrt{x}}{\sqrt{x}} \, dx$

16. $\displaystyle\int_0^{\pi} x^3 \sin x \, dx$

17. $\displaystyle\int \frac{\coth(1/y)}{y^2} \, dy$

18. $\displaystyle\int \frac{e^{3t}}{\sqrt{e^{2t} - 1}} \, dt$

19. $\displaystyle\int y\sqrt{6 + 4y - 4y^2} \, dy$

20. $\displaystyle\int \frac{dx}{2x^3 - 3x^2}$

21. $\displaystyle\int \sin^2 x \cos x \ln(\sin x) \, dx$

22. $\displaystyle\int \frac{\sin 2\theta}{\sqrt{5 - \sin\theta}} \, d\theta$

23. $\displaystyle\int \frac{\sin 2\theta}{\sqrt{\cos^4\theta + 4}} \, d\theta$

24. $\displaystyle\int_0^2 x^3\sqrt{4x^2 - x^4} \, dx$

25. $\displaystyle\int x^3 e^{2x} \, dx$

26. $\displaystyle\int x^3 \arcsin(x^2) \, dx$

27. $\int \cos^5 y\, dy$

28. $\int \dfrac{\sqrt{(\ln x)^2 - 9}}{x \ln x}\, dx$

29. $\int \dfrac{\cos^{-1}(x^{-2})}{x^3}\, dx$

30. $\int \dfrac{dx}{\sqrt{1 - e^{2x}}}$

31. $\int \sqrt{e^{2x} - 1}\, dx$

32. $\int \sin 2\theta \arctan(\sin \theta)\, d\theta$

33. $\int \dfrac{x^4}{\sqrt{x^{10} - 2}}\, dx$

34. $\int \dfrac{\sec^2\theta \tan^2\theta}{\sqrt{9 - \tan^2\theta}}\, d\theta$

35. The region under the curve $y = \sin^2 x$ from 0 to π is rotated about the x-axis. Find the volume of the resulting solid.

36. Find the volume of the solid obtained when the region under the curve $y = \arcsin x$, $x \geqslant 0$, is rotated about the y-axis.

37. Verify Formula 53 in the Table of Integrals (a) by differentiation and (b) by using the substitution $t = a + bu$.

38. Verify Formula 31 (a) by differentiation and (b) by substituting $u = a \sin \theta$.

T **39–46** Use a computer to evaluate the integral. Compare the answer with the result of using a table of integrals. If the answers are not the same, show that they are equivalent.

39. $\int \sec^4 x\, dx$

40. $\int \csc^5 x\, dx$

41. $\int x^2 \sqrt{x^2 + 4}\, dx$

42. $\int \dfrac{dx}{e^x(3e^x + 2)}$

43. $\int \cos^4 x\, dx$

44. $\int x^2 \sqrt{1 - x^2}\, dx$

45. $\int \tan^5 x\, dx$

46. $\int \dfrac{1}{\sqrt{1 + \sqrt[3]{x}}}\, dx$

T **47.** (a) Use the table of integrals to evaluate $F(x) = \int f(x)\, dx$, where

$$f(x) = \frac{1}{x\sqrt{1 - x^2}}$$

What is the domain of f and F?

(b) Use mathematical software to evaluate $F(x)$. What is the domain of the function F that the software produces? Is there a discrepancy between this domain and the domain of the function F that you found in part (a)?

T **48.** Machines sometimes need a helping hand from human beings. Try using a computer to evaluate

$$\int (1 + \ln x)\sqrt{1 + (x \ln x)^2}\, dx$$

If it doesn't return an answer, make a substitution that changes the integral into one that the machine *can* evaluate.

DISCOVERY PROJECT T PATTERNS IN INTEGRALS

In this project mathematical software is used to investigate indefinite integrals of families of functions. By observing the patterns that occur in the integrals of several members of the family, you will first guess, and then prove, a general formula for the integral of any member of the family.

1. (a) Use a computer to evaluate the following integrals.

(i) $\int \dfrac{1}{(x + 2)(x + 3)}\, dx$ (ii) $\int \dfrac{1}{(x + 1)(x + 5)}\, dx$

(iii) $\int \dfrac{1}{(x + 2)(x - 5)}\, dx$ (iv) $\int \dfrac{1}{(x + 2)^2}\, dx$

(b) Based on the pattern of your responses in part (a), guess the value of the integral

$$\int \frac{1}{(x + a)(x + b)}\, dx$$

if $a \neq b$. What if $a = b$?

(c) Check your guess by using the software to evaluate the integral in part (b). Then prove it using partial fractions.

2. (a) Use a computer to evaluate the following integrals.

(i) $\int \sin x \cos 2x\,dx$ (ii) $\int \sin 3x \cos 7x\,dx$ (iii) $\int \sin 8x \cos 3x\,dx$

(b) Based on the pattern of your responses in part (a), guess the value of the integral

$$\int \sin ax \cos bx\,dx$$

(c) Check your guess with the computer. Then prove it using the techniques of Section 7.2. For what values of a and b is it valid?

3. (a) Use a computer to evaluate the following integrals.

(i) $\int \ln x\,dx$ (ii) $\int x \ln x\,dx$ (iii) $\int x^2 \ln x\,dx$

(iv) $\int x^3 \ln x\,dx$ (v) $\int x^7 \ln x\,dx$

(b) Based on the pattern of your responses in part (a), guess the value of

$$\int x^n \ln x\,dx$$

(c) Use integration by parts to prove the conjecture that you made in part (b). For what values of n is it valid?

4. (a) Use a computer to evaluate the following integrals.

(i) $\int xe^x\,dx$ (ii) $\int x^2e^x\,dx$ (iii) $\int x^3e^x\,dx$

(iv) $\int x^4e^x\,dx$ (v) $\int x^5e^x\,dx$

(b) Based on the pattern of your responses in part (a), guess the value of $\int x^6e^x\,dx$. Then use the computer to check your guess.

(c) Based on the patterns in parts (a) and (b), make a conjecture as to the value of the integral

$$\int x^ne^x\,dx$$

when n is a positive integer.

(d) Use mathematical induction to prove the conjecture you made in part (c).

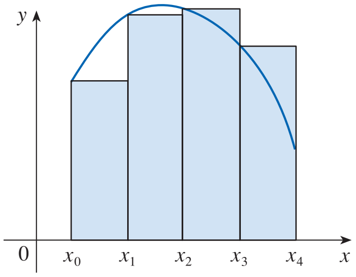
7.7 Approximate Integration

There are two situations in which it is impossible to find the exact value of a definite integral.

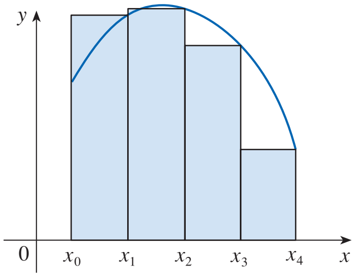
The first situation arises from the fact that in order to evaluate $\int_a^b f(x)\,dx$ using the Fundamental Theorem of Calculus we need to know an antiderivative of f. Sometimes, however, it is difficult, or even impossible, to find an antiderivative (see Section 7.5). For example, it is impossible to evaluate the following integrals exactly:

$$\int_0^1 e^{x^2}\,dx \qquad \int_{-1}^1 \sqrt{1 + x^3}\,dx$$

The second situation arises when the function is determined from a scientific experiment through instrument readings or collected data. There may be no formula for the function (see Example 5).



(a) Left endpoint approximation



(b) Right endpoint approximation

FIGURE 1

In both cases we need to find approximate values of definite integrals. We already know one such method. Recall that the definite integral is defined as a limit of Riemann sums, so any Riemann sum could be used as an approximation to the integral: if we divide $[a, b]$ into n subintervals of equal length $\Delta x = (b - a)/n$, then we have

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

where x_i^* is any point in the ith subinterval $[x_{i-1}, x_i]$. If x_i^* is chosen to be the left endpoint of the interval, then $x_i^* = x_{i-1}$ and we have

$$\boxed{1} \qquad \int_a^b f(x)\,dx \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

If $f(x) \geqslant 0$, then the integral represents an area and Equation 1 represents an approximation of this area by the rectangles shown in Figure 1(a). If we choose x_i^* to be the right endpoint, then $x_i^* = x_i$ and we have

$$\boxed{2} \qquad \int_a^b f(x)\,dx \approx R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

[See Figure 1(b).] The approximations L_n and R_n defined by Equations 1 and 2 are called the **left endpoint approximation** and **right endpoint approximation**, respectively.

The Midpoint and Trapezoidal Rules

In Section 5.2 we considered the case where x_i^* in the Riemann sum is chosen to be the midpoint $\bar{x}_i$ of the subinterval $[x_{i-1}, x_i]$. Figure 2 shows the midpoint approximation M_n for the area in Figure 1. It appears that M_n is a better approximation than either L_n or R_n.

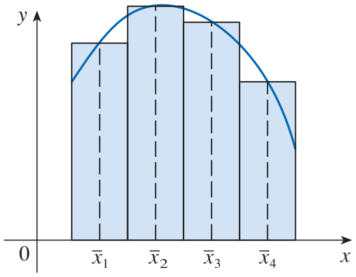


FIGURE 2
Midpoint approximation

Midpoint Rule

$$\int_a^b f(x)\,dx \approx M_n = \Delta x\,[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$$

where $\qquad \Delta x = \dfrac{b - a}{n}$

and $\qquad \bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$

Another approximation, called the Trapezoidal Rule, results from averaging the approximations in Equations 1 and 2:

$$\int_a^b f(x)\,dx \approx \frac{1}{2}\left[\sum_{i=1}^{n} f(x_{i-1})\,\Delta x + \sum_{i=1}^{n} f(x_i)\,\Delta x\right] = \frac{\Delta x}{2}\left[\sum_{i=1}^{n} \bigl(f(x_{i-1}) + f(x_i)\bigr)\right]$$

$$= \frac{\Delta x}{2}\Bigl[\bigl(f(x_0) + f(x_1)\bigr) + \bigl(f(x_1) + f(x_2)\bigr) + \cdots + \bigl(f(x_{n-1}) + f(x_n)\bigr)\Bigr]$$

$$= \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

where $\Delta x = (b - a)/n$ and $x_i = a + i\,\Delta x$.

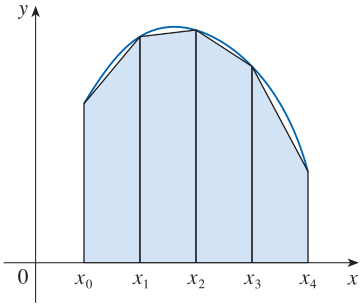


FIGURE 3
Trapezoidal approximation

The reason for the name Trapezoidal Rule can be seen from Figure 3, which illustrates the case with $f(x) \geqslant 0$ and $n = 4$. The area of the trapezoid that lies above the ith subinterval is

$$\Delta x\left(\frac{f(x_{i-1}) + f(x_i)}{2}\right) = \frac{\Delta x}{2}[f(x_{i-1}) + f(x_i)]$$

and if we add the areas of all these trapezoids, we get the right side of the Trapezoidal Rule.

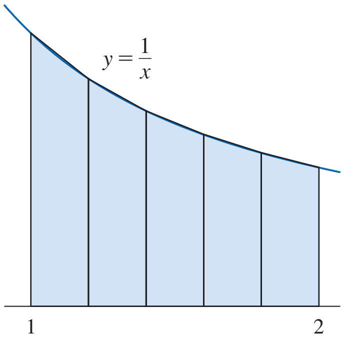


FIGURE 4

EXAMPLE 1 Use (a) the Trapezoidal Rule and (b) the Midpoint Rule with $n = 5$ to approximate the integral $\int_1^2 (1/x)\,dx$.

SOLUTION

(a) With $n = 5$, $a = 1$, and $b = 2$, we have $\Delta x = (2 - 1)/5 = 0.2$, and so the Trapezoidal Rule gives

$$\int_1^2 \frac{1}{x}\,dx \approx T_5 = \frac{0.2}{2}[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)]$$

$$= 0.1\left(\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right)$$

$$\approx 0.695635$$

This approximation is illustrated in Figure 4.

(b) The midpoints of the five subintervals are 1.1, 1.3, 1.5, 1.7, and 1.9, so the Midpoint Rule gives

$$\int_1^2 \frac{1}{x}\,dx \approx \Delta x\,[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)]$$

$$= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right)$$

$$\approx 0.691908$$

This approximation is illustrated in Figure 5. ■

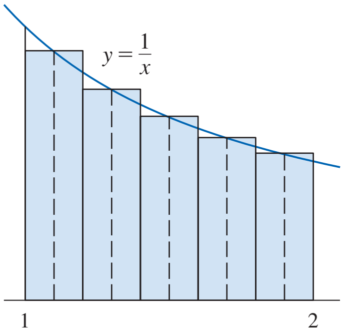


FIGURE 5

Error Bounds for the Midpoint and Trapezoidal Rules

In Example 1 we deliberately chose an integral whose value can be computed explicitly so that we can see how accurate the Trapezoidal and Midpoint Rules are. By the Fundamental Theorem of Calculus,

$$\int_1^2 \frac{1}{x}\,dx = \ln x\Big]_1^2 = \ln 2 = 0.693147\ldots$$

$\int_a^b f(x)\,dx = \text{approximation} + \text{error}$

The **error** in using an approximation is defined to be the amount that needs to be added to the approximation to make it exact. From the values in Example 1 we see that the errors in the Trapezoidal and Midpoint Rule approximations for $n = 5$ are

$$E_T \approx -0.002488 \qquad \text{and} \qquad E_M \approx 0.001239$$

In general, we have

$$E_T = \int_a^b f(x)\,dx - T_n \qquad \text{and} \qquad E_M = \int_a^b f(x)\,dx - M_n$$

The following tables show the results of calculations similar to those in Example 1, but for $n = 5$, 10, and 20 and for the left and right endpoint approximations as well as the Trapezoidal and Midpoint Rules.

Approximations to $\int_1^2 \frac{1}{x}\,dx$

n	L_n	R_n	T_n	M_n
5	0.745635	0.645635	0.695635	0.691908
10	0.718771	0.668771	0.693771	0.692835
20	0.705803	0.680803	0.693303	0.693069

Corresponding errors

n	E_L	E_R	E_T	E_M
5	-0.052488	0.047512	-0.002488	0.001239
10	-0.025624	0.024376	-0.000624	0.000312
20	-0.012656	0.012344	-0.000156	0.000078

It turns out that these observations are true in most cases.

We can make several observations from these tables:

1. In all of the methods we get more accurate approximations when we increase the value of n. (But very large values of n result in so many arithmetic operations that we have to beware of accumulated round-off error.)
2. The errors in the left and right endpoint approximations are opposite in sign and appear to decrease by a factor of about 2 when we double the value of n.
3. The Trapezoidal and Midpoint Rules are much more accurate than the endpoint approximations.
4. The errors in the Trapezoidal and Midpoint Rules are opposite in sign and appear to decrease by a factor of about 4 when we double the value of n.
5. The size of the error in the Midpoint Rule is about half the size of the error in the Trapezoidal Rule.

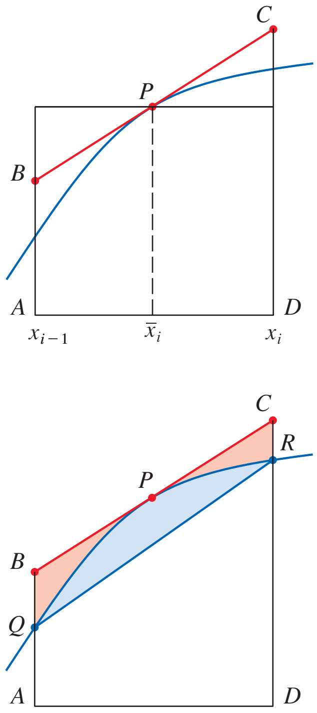


FIGURE 6

Figure 6 shows why we can usually expect the Midpoint Rule to be more accurate than the Trapezoidal Rule. The area of a typical rectangle in the Midpoint Rule is the same as the area of the trapezoid *ABCD* whose upper side is tangent to the graph at *P*. The area of this trapezoid is closer to the area under the graph than is the area of the trapezoid *AQRD* used in the Trapezoidal Rule. [The midpoint error (shaded red) is smaller than the trapezoidal error (shaded blue).]

These observations are corroborated in the following error estimates, which are proved in books on numerical analysis. Notice that Observation 4 corresponds to the n^2 in each denominator because $(2n)^2 = 4n^2$. The fact that the estimates depend on the size of the second derivative is not surprising if you look at Figure 6, because $f''(x)$ measures how much the graph is curved. [Recall that $f''(x)$ measures how fast the slope of $y = f(x)$ changes.]

3 **Error Bounds** Suppose $|f''(x)| \leqslant K$ for $a \leqslant x \leqslant b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \leqslant \frac{K(b-a)^3}{12n^2} \qquad \text{and} \qquad |E_M| \leqslant \frac{K(b-a)^3}{24n^2}$$

Let's apply this error estimate to the Trapezoidal Rule approximation in Example 1. If $f(x) = 1/x$, then $f'(x) = -1/x^2$ and $f''(x) = 2/x^3$. Because $1 \leqslant x \leqslant 2$, we have $1/x \leqslant 1$, so

$$|f''(x)| = \left|\frac{2}{x^3}\right| \leqslant \frac{2}{1^3} = 2$$

Therefore, taking $K = 2$, $a = 1$, $b = 2$, and $n = 5$ in the error estimate (3), we see that

K can be any number larger than all the values of $|f''(x)|$, but smaller values of *K* give better error bounds.

$$|E_T| \leqslant \frac{2(2-1)^3}{12(5)^2} = \frac{1}{150} \approx 0.006667$$

Comparing this error estimate of 0.006667 with the actual error of about 0.002488, we see that it can happen that the actual error is substantially less than the upper bound for the error given by (3).

EXAMPLE 2 How large should we take n in order to guarantee that the Trapezoidal and Midpoint Rule approximations for $\int_1^2 (1/x)\,dx$ are accurate to within 0.0001?

SOLUTION We saw in the preceding calculation that $|f''(x)| \leqslant 2$ for $1 \leqslant x \leqslant 2$, so we can take $K = 2$, $a = 1$, and $b = 2$ in (3). Accuracy to within 0.0001 means that the size of the error should be less than 0.0001. Therefore we choose n so that

$$\frac{2(1)^3}{12n^2} < 0.0001$$

Solving the inequality for n, we get

$$n^2 > \frac{2}{12(0.0001)}$$

It's quite possible that a lower value for n would suffice, but 41 is the smallest value for which the error bound formula can *guarantee* accuracy to within 0.0001.

and so
$$n > \frac{1}{\sqrt{0.0006}} \approx 40.8$$

Thus $n = 41$ will ensure the desired accuracy.

For the same accuracy with the Midpoint Rule we choose n so that

$$\frac{2(1)^3}{24n^2} < 0.0001 \qquad \text{and so} \qquad n > \frac{1}{\sqrt{0.0012}} \approx 29$$ ■

EXAMPLE 3

(a) Use the Midpoint Rule with $n = 10$ to approximate the integral $\int_0^1 e^{x^2}\,dx$.
(b) Give an upper bound for the error involved in this approximation.

SOLUTION

(a) Since $a = 0$, $b = 1$, and $n = 10$, the Midpoint Rule gives

$$\begin{aligned}\int_0^1 e^{x^2}dx &\approx \Delta x\,[f(0.05) + f(0.15) + \cdots + f(0.85) + f(0.95)]\\ &= 0.1[e^{0.0025} + e^{0.0225} + e^{0.0625} + e^{0.1225} + e^{0.2025} + e^{0.3025}\\ &\qquad + e^{0.4225} + e^{0.5625} + e^{0.7225} + e^{0.9025}]\\ &\approx 1.460393\end{aligned}$$

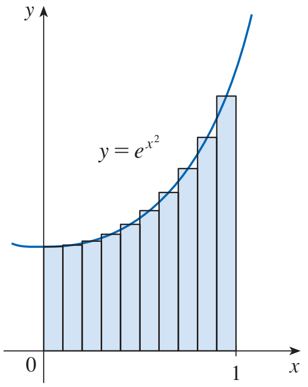


FIGURE 7

Figure 7 illustrates this approximation.

(b) Since $f(x) = e^{x^2}$, we have $f'(x) = 2xe^{x^2}$ and $f''(x) = (2 + 4x^2)e^{x^2}$. Also, since $0 \leqslant x \leqslant 1$, we have $x^2 \leqslant 1$ and so

$$0 \leqslant f''(x) = (2 + 4x^2)e^{x^2} \leqslant 6e$$

Taking $K = 6e$, $a = 0$, $b = 1$, and $n = 10$ in the error estimate (3), we see that an upper bound for the error is

$$\frac{6e(1)^3}{24(10)^2} = \frac{e}{400} \approx 0.007 \qquad ■$$

Error estimates give upper bounds for the error. They are theoretical, worst-case scenarios. The actual error in this case turns out to be about 0.0023.

■ Simpson's Rule

Another rule for approximate integration results from using parabolas instead of straight line segments to approximate a curve. As before, we divide $[a, b]$ into n subintervals of equal length $h = \Delta x = (b - a)/n$, but this time we assume that n is an *even* number. Then on each consecutive pair of intervals we approximate the curve $y = f(x) \geqslant 0$ by a parabola as shown in Figure 8. If $y_i = f(x_i)$, then $P_i(x_i, y_i)$ is the point on the curve lying above x_i. A typical parabola passes through three consecutive points P_i, P_{i+1}, and P_{i+2}.

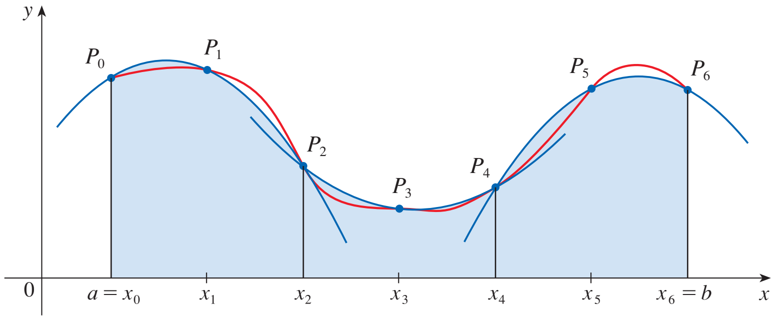


FIGURE 8

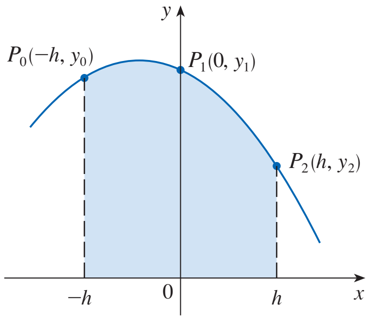


FIGURE 9

To simplify our calculations, we first consider the case where $x_0 = -h$, $x_1 = 0$, and $x_2 = h$. (See Figure 9.) We know that the equation of the parabola through P_0, P_1, and P_2

is of the form $y = Ax^2 + Bx + C$ and so the area under the parabola from $x = -h$ to $x = h$ is

Here we have used Theorem 5.5.7. Notice that $Ax^2 + C$ is an even function and Bx is odd.

$$\int_{-h}^{h} (Ax^2 + Bx + C)\,dx = 2\int_0^h (Ax^2 + C)\,dx = 2\left[A\frac{x^3}{3} + Cx\right]_0^h$$

$$= 2\left(A\frac{h^3}{3} + Ch\right) = \frac{h}{3}(2Ah^2 + 6C)$$

But, since the parabola passes through $P_0(-h, y_0)$, $P_1(0, y_1)$, and $P_2(h, y_2)$, we have

$$y_0 = A(-h)^2 + B(-h) + C = Ah^2 - Bh + C$$

$$y_1 = C$$

$$y_2 = Ah^2 + Bh + C$$

and therefore $$y_0 + 4y_1 + y_2 = 2Ah^2 + 6C$$

Thus we can rewrite the area under the parabola as

$$\frac{h}{3}(y_0 + 4y_1 + y_2)$$

Now by shifting this parabola horizontally we do not change the area under it. This means that the area under the parabola through P_0, P_1, and P_2 from $x = x_0$ to $x = x_2$ in Figure 8 is still

$$\frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly, the area under the parabola through P_2, P_3, and P_4 from $x = x_2$ to $x = x_4$ is

$$\frac{h}{3}(y_2 + 4y_3 + y_4)$$

If we compute the areas under all the parabolas in this manner and add the results, we get

$$\int_a^b f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

Although we have derived this approximation for the case in which $f(x) \geqslant 0$, it is a reasonable approximation for any continuous function f and is called Simpson's Rule after the English mathematician Thomas Simpson (1710–1761). Note the pattern of coefficients: 1, 4, 2, 4, 2, 4, 2, . . . , 4, 2, 4, 1.

Simpson

Thomas Simpson was a weaver who taught himself mathematics and went on to become one of the best English mathematicians of the 18th century. What we call Simpson's Rule was actually known to Cavalieri and Gregory in the 17th century, but Simpson popularized it in his book *Mathematical Dissertations* (1743).

Simpson's Rule

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots$$
$$+ 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

where n is even and $\Delta x = (b - a)/n$.

EXAMPLE 4 Use Simpson's Rule with $n = 10$ to approximate $\int_1^2 (1/x)\,dx$.

SOLUTION Putting $f(x) = 1/x$, $n = 10$, and $\Delta x = 0.1$ in Simpson's Rule, we obtain

$$
\begin{aligned}
\int_1^2 \frac{1}{x}\,dx &\approx S_{10} \\
&= \frac{\Delta x}{3}[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + \cdots + 2f(1.8) + 4f(1.9) + f(2)] \\
&= \frac{0.1}{3}\left(\frac{1}{1} + \frac{4}{1.1} + \frac{2}{1.2} + \frac{4}{1.3} + \frac{2}{1.4} + \frac{4}{1.5} + \frac{2}{1.6} + \frac{4}{1.7} + \frac{2}{1.8} + \frac{4}{1.9} + \frac{1}{2}\right) \\
&\approx 0.693150
\end{aligned}
$$

■

Notice that, in Example 4, Simpson's Rule gives us a *much* better approximation ($S_{10} \approx 0.693150$) to the true value of the integral ($\ln 2 \approx 0.693147\ldots$) than does the Trapezoidal Rule ($T_{10} \approx 0.693771$) or the Midpoint Rule ($M_{10} \approx 0.692835$). It turns out (see Exercise 50) that the approximations in Simpson's Rule are weighted averages of those in the Trapezoidal and Midpoint Rules:

$$S_{2n} = \tfrac{1}{3}T_n + \tfrac{2}{3}M_n$$

(Recall that E_T and E_M usually have opposite signs and that $|E_M|$ is about half the size of $|E_T|$.)

In many applications of calculus we need to evaluate an integral even if no explicit formula is known for y as a function of x. A function may be given graphically or as a table of values of collected data. If there is evidence that the values are not changing rapidly, then Simpson's Rule (or the Midpoint Rule or Trapezoidal Rule) can still be used to find an approximate value for $\int_a^b y\,dx$, the integral of y with respect to x.

EXAMPLE 5 Figure 10 shows data traffic on the link from the United States to SWITCH, the Swiss academic and research network, during one full day. $D(t)$ is the data throughput, measured in megabits per second (Mb/s). Use Simpson's Rule to estimate the total amount of data transmitted on the link from midnight to noon on that day.

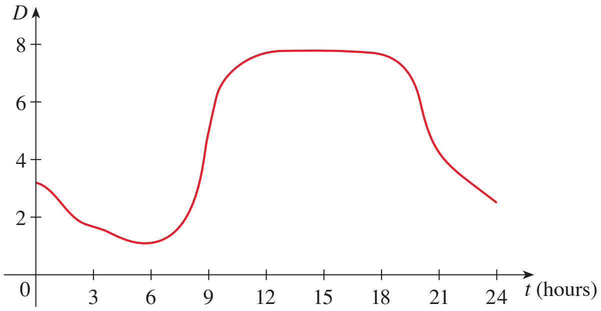


FIGURE 10

SOLUTION Because we want the units to be consistent and $D(t)$ is measured in megabits per second, we convert the units for t from hours to seconds. If we let $A(t)$ be the amount of data (in megabits) transmitted by time t, where t is measured in seconds, then $A'(t) = D(t)$. So, by the Net Change Theorem (see Section 5.4), the total amount of data transmitted by noon (when $t = 12 \times 60^2 = 43{,}200$) is

$$A(43{,}200) = \int_0^{43{,}200} D(t)\, dt$$

We estimate the values of $D(t)$ at hourly intervals from the graph and compile them in the table.

t (hours)	t (seconds)	$D(t)$	t (hours)	t (seconds)	$D(t)$
0	0	3.2	7	25,200	1.3
1	3,600	2.7	8	28,800	2.8
2	7,200	1.9	9	32,400	5.7
3	10,800	1.7	10	36,000	7.1
4	14,400	1.3	11	39,600	7.7
5	18,000	1.0	12	43,200	7.9
6	21,600	1.1			

Then we use Simpson's Rule with $n = 12$ and $\Delta t = 3600$ to estimate the integral:

$$\int_0^{43{,}200} A(t)\, dt \approx \frac{\Delta t}{3}[D(0) + 4D(3600) + 2D(7200) + \cdots + 4D(39{,}600) + D(43{,}200)]$$

$$\approx \frac{3600}{3}[3.2 + 4(2.7) + 2(1.9) + 4(1.7) + 2(1.3) + 4(1.0)$$
$$+ 2(1.1) + 4(1.3) + 2(2.8) + 4(5.7) + 2(7.1) + 4(7.7) + 7.9]$$

$$= 143{,}880$$

Thus the total amount of data transmitted from midnight to noon is about 144,000 megabits, or 144 gigabits (18 gigabytes). ■

Error Bound for Simpson's Rule

The first table in the margin shows how Simpson's Rule compares with the Midpoint Rule for the integral $\int_1^2 (1/x)\, dx$, whose value is about 0.69314718. The second table shows how the error E_S in Simpson's Rule decreases by a factor of about 16 when n is doubled. (In Exercises 27 and 28 you are asked to verify this for two additional integrals.) That is consistent with the appearance of n^4 in the denominator of the following error estimate for Simpson's Rule. It is similar to the estimates given in (3) for the Trapezoidal and Midpoint Rules, but it uses the fourth derivative of f.

n	M_n	S_n
4	0.69121989	0.69315453
8	0.69266055	0.69314765
16	0.69302521	0.69314721

n	E_M	E_S
4	0.00192729	−0.00000735
8	0.00048663	−0.00000047
16	0.00012197	−0.00000003

4 **Error Bound for Simpson's Rule** Suppose that $|f^{(4)}(x)| \leqslant K$ for $a \leqslant x \leqslant b$. If E_S is the error involved in using Simpson's Rule, then

$$|E_S| \leqslant \frac{K(b-a)^5}{180n^4}$$

EXAMPLE 6 How large should we take n in order to guarantee that the Simpson's Rule approximation for $\int_1^2 (1/x)\, dx$ is accurate to within 0.0001?

SOLUTION If $f(x) = 1/x$, then $f^{(4)}(x) = 24/x^5$. Since $x \geqslant 1$, we have $1/x \leqslant 1$ and so

$$|f^{(4)}(x)| = \left|\frac{24}{x^5}\right| \leqslant 24$$

Therefore we can take $K = 24$ in (4). Thus, for an error less than 0.0001, we should choose n so that

$$\frac{24(1)^5}{180n^4} < 0.0001$$

This gives
$$n^4 > \frac{24}{180(0.0001)}$$

and so
$$n > \frac{1}{\sqrt[4]{0.00075}} \approx 6.04$$

Therefore $n = 8$ (n must be even) gives the desired accuracy. (Compare this with Example 2, where we obtained $n = 41$ for the Trapezoidal Rule and $n = 29$ for the Midpoint Rule.) ■

Many calculators and software applications have a built-in algorithm that computes an approximation of a definite integral. Some of these algorithms use Simpson's Rule; others use more sophisticated techniques such as *adaptive* numerical integration. This means that if a function fluctuates much more on a certain part of the interval than it does elsewhere, then that part gets divided into more subintervals. This strategy reduces the number of calculations required to achieve a prescribed accuracy.

EXAMPLE 7

(a) Use Simpson's Rule with $n = 10$ to approximate the integral $\int_0^1 e^{x^2}\,dx$.
(b) Estimate the error involved in this approximation.

SOLUTION

(a) If $n = 10$, then $\Delta x = 0.1$ and Simpson's Rule gives

$$\begin{aligned}\int_0^1 e^{x^2}dx &\approx \frac{\Delta x}{3}[f(0) + 4f(0.1) + 2f(0.2) + \cdots + 2f(0.8) + 4f(0.9) + f(1)] \\ &= \frac{0.1}{3}[e^0 + 4e^{0.01} + 2e^{0.04} + 4e^{0.09} + 2e^{0.16} + 4e^{0.25} + 2e^{0.36} \\ &\qquad + 4e^{0.49} + 2e^{0.64} + 4e^{0.81} + e^1] \\ &\approx 1.462681\end{aligned}$$

Figure 11 illustrates the calculation in Example 7. Notice that the parabolic arcs are so close to the graph of $y = e^{x^2}$ that they are practically indistinguishable from it.

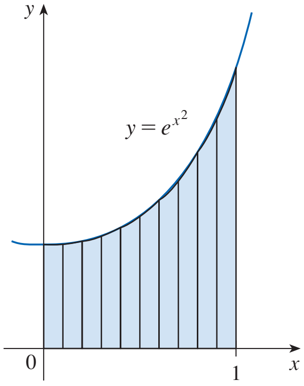


FIGURE 11

(b) The fourth derivative of $f(x) = e^{x^2}$ is

$$f^{(4)}(x) = (12 + 48x^2 + 16x^4)e^{x^2}$$

and so, since $0 \leqslant x \leqslant 1$, we have

$$0 \leqslant f^{(4)}(x) \leqslant (12 + 48 + 16)e^1 = 76e$$

Therefore, putting $K = 76e$, $a = 0$, $b = 1$, and $n = 10$ in (4), we see that the error is at most

$$\frac{76e(1)^5}{180(10)^4} \approx 0.000115$$

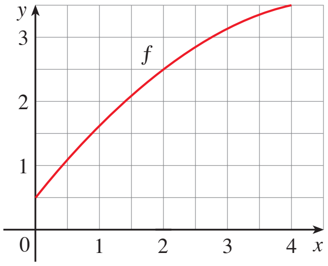
(Compare this with Example 3.) Thus, correct to three decimal places, we have

$$\int_0^1 e^{x^2}dx \approx 1.463$$ ■

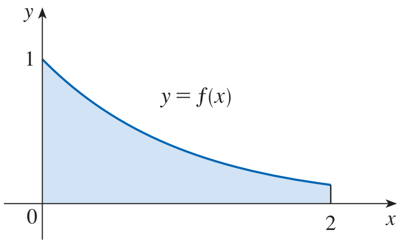
7.7 Exercises

In these exercises, round your answers to six decimal places unless otherwise instructed.

1. Let $I = \int_0^4 f(x)\,dx$, where f is the function whose graph is shown.
   (a) Use the graph to find L_2, R_2, and M_2.
   (b) Are these underestimates or overestimates of I?
   (c) Use the graph to find T_2. How does it compare with I?
   (d) For any value of n, list the numbers L_n, R_n, M_n, T_n, and I in increasing order.



2. The left, right, Trapezoidal, and Midpoint Rule approximations were used to estimate $\int_0^2 f(x)\,dx$, where f is the function whose graph is shown. The estimates were 0.7811, 0.8675, 0.8632, and 0.9540, and the same number of subintervals were used in each case.
   (a) Which rule produced which estimate?
   (b) Between which two approximations does the true value of $\int_0^2 f(x)\,dx$ lie?



3. Estimate $\int_0^1 \cos(x^2)\,dx$ using (a) the Trapezoidal Rule and (b) the Midpoint Rule, each with $n = 4$. From a graph of the integrand, decide whether your answers are underestimates or overestimates. What can you conclude about the true value of the integral?

4. Draw the graph of $f(x) = \sin\left(\frac{1}{2}x^2\right)$ in the viewing rectangle $[0, 1]$ by $[0, 0.5]$ and let $I = \int_0^1 f(x)\,dx$.
   (a) Use the graph to decide whether L_2, R_2, M_2, and T_2 underestimate or overestimate I.
   (b) For any value of n, list the numbers L_n, R_n, M_n, T_n, and I in increasing order.
   (c) Compute L_5, R_5, M_5, and T_5. From the graph, which do you think gives the best estimate of I?

5–6 Use (a) the Midpoint Rule and (b) Simpson's Rule to approximate the given integral with the specified value of n. Compare your results to the actual value to determine the error in each approximation.

5. $\int_0^{\pi} x \sin x\,dx, \quad n = 6$

6. $\int_0^2 \dfrac{x}{\sqrt{1 + x^2}}\,dx, \quad n = 8$

7–18 Use (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule to approximate the given integral with the specified value of n.

7. $\int_0^1 \sqrt{1 + x^3}\,dx, \quad n = 4$

8. $\int_1^4 \sin\sqrt{x}\,dx, \quad n = 6$

9. $\int_0^1 \sqrt{e^x - 1}\,dx, \quad n = 10$

10. $\int_0^2 \sqrt[3]{1 - x^2}\,dx, \quad n = 10$

11. $\int_{-1}^2 e^{x + \cos x}\,dx, \quad n = 6$

12. $\int_1^3 e^{1/x}\,dx, \quad n = 8$

13. $\int_0^4 \sqrt{y}\cos y\,dy, \quad n = 8$

14. $\int_2^3 \dfrac{1}{\ln t}\,dt, \quad n = 10$

15. $\int_0^1 \dfrac{x^2}{1 + x^4}\,dx, \quad n = 10$

16. $\int_1^3 \dfrac{\sin t}{t}\,dt, \quad n = 4$

17. $\int_0^4 \ln(1 + e^x)\,dx, \quad n = 8$

18. $\int_0^1 \sqrt{x + x^3}\,dx, \quad n = 10$

19. (a) Find the approximations T_8 and M_8 for the integral $\int_0^1 \cos(x^2)\,dx$.
   (b) Estimate the errors in the approximations of part (a).
   (c) How large do we have to choose n so that the approximations T_n and M_n to the integral in part (a) are accurate to within 0.0001?

20. (a) Find the approximations T_{10} and M_{10} for $\int_1^2 e^{1/x}\,dx$.
   (b) Estimate the errors in the approximations of part (a).
   (c) How large do we have to choose n so that the approximations T_n and M_n to the integral in part (a) are accurate to within 0.0001?

21. (a) Find the approximations T_{10}, M_{10}, and S_{10} for $\int_0^{\pi} \sin x\,dx$ and the corresponding errors E_T, E_M, and E_S.
   (b) Compare the actual errors in part (a) with the error estimates given by (3) and (4).
   (c) How large do we have to choose n so that the approximations T_n, M_n, and S_n to the integral in part (a) are accurate to within 0.00001?

22. How large should n be to guarantee that the Simpson's Rule approximation to $\int_0^1 e^{x^2}\,dx$ is accurate to within 0.00001?

T **23.** The trouble with the error estimates is that it is often very difficult to compute fourth derivatives and obtain a good upper bound K for $|f^{(4)}(x)|$ by hand. But mathematical software has no problem computing $f^{(4)}$ and graphing it, so we can easily find a value for K from a machine graph. This exercise deals with approximations to the integral $I = \int_0^{2\pi} f(x)\,dx$, where $f(x) = e^{\cos x}$. In parts (b), (d), and (g), round your answers to 10 decimal places.

(a) Use a graph to get a good upper bound for $|f''(x)|$.
(b) Use M_{10} to approximate I.
(c) Use part (a) to estimate the error in part (b).
(d) Use a calculator or computer to approximate I.
(e) How does the actual error compare with the error estimate in part (c)?
(f) Use a graph to get a good upper bound for $|f^{(4)}(x)|$.
(g) Use S_{10} to approximate I.
(h) Use part (f) to estimate the error in part (g).
(i) How does the actual error compare with the error estimate in part (h)?
(j) How large should n be to guarantee that the size of the error in using S_n is less than 0.0001?

T **24.** Repeat Exercise 23 for the integral $I = \int_{-1}^{1} \sqrt{4 - x^3}\,dx$.

25–26 Find the approximations L_n, R_n, T_n, and M_n for $n = 5$, 10, and 20. Then compute the corresponding errors E_L, E_R, E_T, and E_M. (You may wish to use the sum command on a computer algebra system.) What observations can you make? In particular, what happens to the errors when n is doubled?

25. $\int_0^1 xe^x\,dx$

26. $\int_1^2 \frac{1}{x^2}\,dx$

27–28 Find the approximations T_n, M_n, and S_n for $n = 6$ and $n = 12$. Then compute the corresponding errors E_T, E_M, and E_S. (You may wish to use the sum command on a computer algebra system.) What observations can you make? In particular, what happens to the errors when n is doubled?

27. $\int_0^2 x^4\,dx$

28. $\int_1^4 \frac{1}{\sqrt{x}}\,dx$

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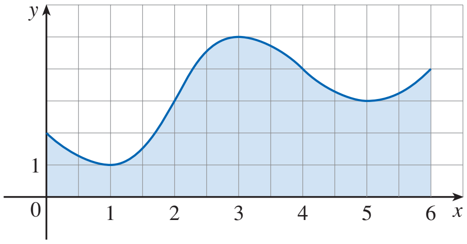
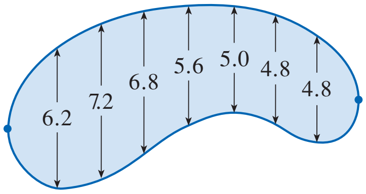
29. Estimate the area under the graph in the figure by using (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule, each with $n = 6$.



30. The widths (in meters) of a kidney-shaped swimming pool were measured at 2-meter intervals as indicated in the figure. Use Simpson's Rule with $n = 8$ to estimate the area of the pool.



31. (a) Use the Midpoint Rule and the given data to estimate the value of the integral $\int_1^5 f(x)\,dx$.

x	$f(x)$	x	$f(x)$
1.0	2.4	3.5	4.0
1.5	2.9	4.0	4.1
2.0	3.3	4.5	3.9
2.5	3.6	5.0	3.5
3.0	3.8		

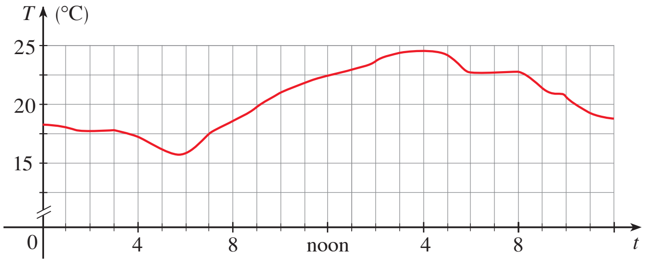
(b) If it is known that $-2 \leqslant f''(x) \leqslant 3$ for all x, estimate the error involved in the approximation in part (a).

32. (a) A table of values of a function g is given. Use Simpson's Rule to estimate $\int_0^{1.6} g(x)\,dx$.

x	$g(x)$	x	$g(x)$
0.0	12.1	1.0	12.2
0.2	11.6	1.2	12.6
0.4	11.3	1.4	13.0
0.6	11.1	1.6	13.2
0.8	11.7		

(b) If $-5 \leqslant g^{(4)}(x) \leqslant 2$ for $0 \leqslant x \leqslant 1.6$, estimate the error involved in the approximation in part (a).

33. A graph of the temperature in Boston on a summer day is shown. Use Simpson's Rule with $n = 12$ to estimate the average temperature on that day.

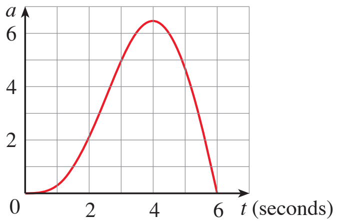


34. A radar gun was used to record the speed of a runner during the first 5 seconds of a race (see the table). Use Simpson's

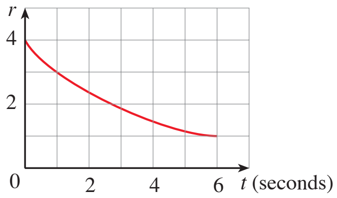
Rule to estimate the distance the runner covered during those 5 seconds.

t (s)	v (m/s)	t (s)	v (m/s)
0	0	3.0	10.51
0.5	4.67	3.5	10.67
1.0	7.34	4.0	10.76
1.5	8.86	4.5	10.81
2.0	9.73	5.0	10.81
2.5	10.22		

35. The graph of the acceleration $a(t)$ of a car, measured in m/s^2, is shown. Use Simpson's Rule to estimate the increase in the velocity of the car during the 6-second time interval.



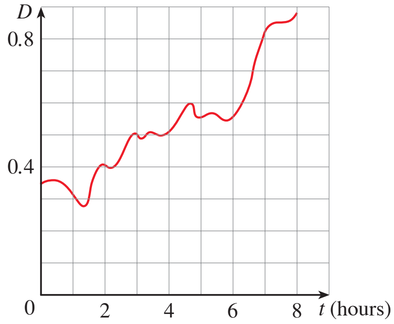
36. Water leaked from a tank at a rate of $r(t)$ liters per hour, where the graph of r is as shown. Use Simpson's Rule to estimate the total amount of water that leaked out during the first 6 hours.



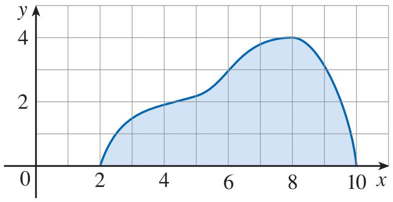
37. The table (provided by San Diego Gas and Electric) gives the power consumption P in megawatts in San Diego County from midnight to 6:00 AM on a day in December. Use Simpson's Rule to estimate the energy used during that time period. (Use the fact that power is the derivative of energy.)

t	P	t	P
0:00	1814	3:30	1611
0:30	1735	4:00	1621
1:00	1686	4:30	1666
1:30	1646	5:00	1745
2:00	1637	5:30	1886
2:30	1609	6:00	2052
3:00	1604		

38. Shown is the graph of traffic on an Internet service provider's T1 data line from midnight to 8:00 AM, where D is the data throughput, measured in megabits per second. Use Simpson's Rule to estimate the total amount of data transmitted during that time period.



39. Use Simpson's Rule with $n = 8$ to estimate the volume of the solid obtained by rotating the region shown in the figure about (a) the x-axis and (b) the y-axis.



40. The table shows values of a force function $f(x)$, where x is measured in meters and $f(x)$ in newtons. Use Simpson's Rule to estimate the work done by the force in moving an object a distance of 18 m.

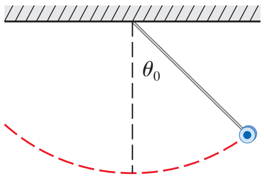
x	0	3	6	9	12	15	18
$f(x)$	9.8	9.1	8.5	8.0	7.7	7.5	7.4

41. The region bounded by the curve $y = 1/(1 + e^{-x})$, the x- and y-axes, and the line $x = 10$ is rotated about the x-axis. Use Simpson's Rule with $n = 10$ to estimate the volume of the resulting solid.

42. The figure shows a pendulum with length L that makes a maximum angle θ_0 with the vertical. Using Newton's Second Law, it can be shown that the period T (the time for one complete swing) is given by

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

where $k = \sin\left(\frac{1}{2}\theta_0\right)$ and g is the acceleration due to gravity. If $L = 1$ m and $\theta_0 = 42°$, use Simpson's Rule with $n = 10$ to find the period.



43. The intensity of light with wavelength λ traveling through a diffraction grating with N slits at an angle θ is given by $I(\theta) = (N^2 \sin^2 k)/k^2$, where $k = (\pi N d \sin\theta)/\lambda$ and d is the distance between adjacent slits. A helium-neon laser with wavelength $\lambda = 632.8 \times 10^{-9}$ m is emitting a narrow band of light, given by $-10^{-6} < \theta < 10^{-6}$, through a grating with 10,000 slits spaced 10^{-4} m apart. Use the Midpoint Rule with $n = 10$ to estimate the total light intensity $\int_{-10^{-6}}^{10^{-6}} I(\theta)\,d\theta$ emerging from the grating.

44. Use the Trapezoidal Rule with $n = 10$ to approximate $\int_0^{20} \cos(\pi x)\,dx$. Compare your result to the actual value. Can you explain the discrepancy?

45. Sketch the graph of a continuous function on $[0, 2]$ for which the Trapezoidal Rule with $n = 2$ is more accurate than the Midpoint Rule.

46. Sketch the graph of a continuous function on $[0, 2]$ for which the right endpoint approximation with $n = 2$ is more accurate than Simpson's Rule.

47. If f is a positive function and $f''(x) < 0$ for $a \leqslant x \leqslant b$, show that

$$T_n < \int_a^b f(x)\,dx < M_n$$

48. When Is Simpson's Rule Exact?

(a) Show that if f is a polynomial of degree 3 or lower, then Simpson's Rule gives the exact value of $\int_a^b f(x)\,dx$.

(b) Find the approximation S_4 for $\int_0^8 (x^3 - 6x^2 + 4x)\,dx$ and verify that S_4 is the exact value of the integral.

(c) Use the error bound given in (4) to explain why the statement in part (a) must be true.

49. Show that $\frac{1}{2}(T_n + M_n) = T_{2n}$.

50. Show that $\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}$.

7.8 Improper Integrals

In defining a definite integral $\int_a^b f(x)\,dx$ we dealt with a function f defined on a finite interval $[a, b]$ and we assumed that f does not have an infinite discontinuity (see Section 5.2). In this section we extend the concept of a definite integral to the case where the interval is infinite and also to the case where f has an infinite discontinuity in $[a, b]$. In either case the integral is called an *improper integral*. One of the most important applications of this idea, probability distributions, will be studied in Section 8.5.

Type 1: Infinite Intervals

Consider the unbounded region S that lies under the curve $y = 1/x^2$, above the x-axis, and to the right of the line $x = 1$. You might think that, since S is infinite in extent, its area must be infinite, but let's take a closer look. The area of the part of S that lies to the left of the line $x = t$ (shaded in Figure 1) is

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg]_1^t = 1 - \frac{1}{t}$$

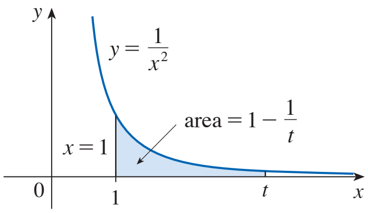


FIGURE 1

Notice that $A(t) < 1$ no matter how large t is chosen. We also observe that

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$$

The area of the shaded region approaches 1 as $t \to \infty$ (see Figure 2), so we say that the area of the infinite region S is equal to 1 and we write

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = 1$$

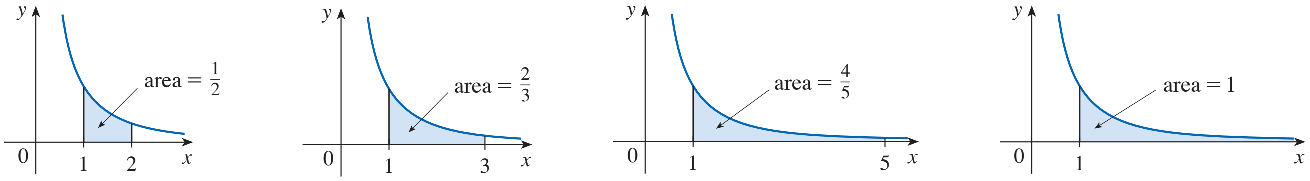


FIGURE 2

Using this example as a guide, we define the integral of f (not necessarily a positive function) over an infinite interval as the limit of integrals over finite intervals.

$\boxed{1}$ Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

In part (c) any real number a can be used (see Exercise 88).

Any of the improper integrals in Definition 1 can be interpreted as an area provided that f is a positive function. For instance, in case (a) if $f(x) \geqslant 0$ and the integral $\int_a^\infty f(x)\,dx$ is convergent, then we define the area of the region $S = \{(x, y) \mid x \geqslant a, 0 \leqslant y \leqslant f(x)\}$ in Figure 3 to be

$$A(S) = \int_a^\infty f(x)\,dx$$

This is appropriate because $\int_a^\infty f(x)\,dx$ is the limit as $t \to \infty$ of the area under the graph of f from a to t.

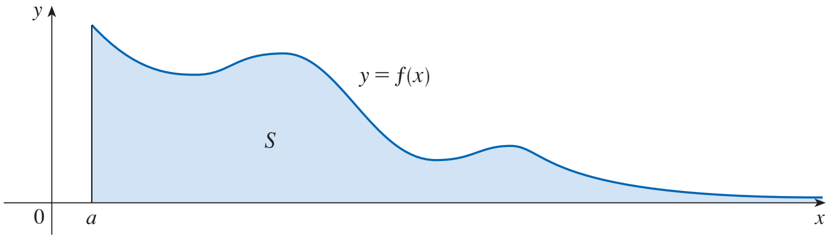


FIGURE 3

EXAMPLE 1 Determine whether the integral $\int_1^\infty (1/x)\,dx$ is convergent or divergent.

SOLUTION According to part (a) of Definition 1, we have

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big]_1^t$$

$$= \lim_{t\to\infty} (\ln t - \ln 1) = \lim_{t\to\infty} \ln t = \infty$$

The limit does not exist as a finite number and so the improper integral $\int_1^\infty (1/x)\,dx$ is divergent. ■

Let's compare the result of Example 1 with the example given at the beginning of this section:

$$\int_1^\infty \frac{1}{x^2}\,dx \;\text{ converges} \qquad\qquad \int_1^\infty \frac{1}{x}\,dx \;\text{ diverges}$$

Geometrically, this says that although the curves $y = 1/x^2$ and $y = 1/x$ look very similar for $x > 0$, the region under $y = 1/x^2$ to the right of $x = 1$ (the shaded region in Figure 4) has finite area whereas the corresponding region under $y = 1/x$ (in Figure 5) has infinite area. Note that both $1/x^2$ and $1/x$ approach 0 as $x \to \infty$ but $1/x^2$ approaches 0 faster than $1/x$. The values of $1/x$ don't decrease fast enough for its integral to have a finite value.

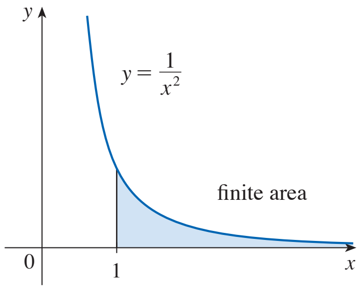


FIGURE 4
$\int_1^\infty (1/x^2)\,dx$ converges

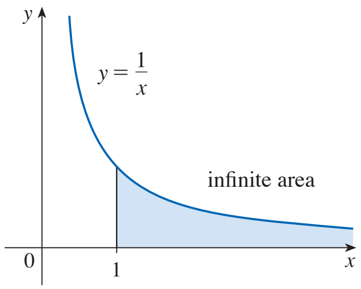


FIGURE 5
$\int_1^\infty (1/x)\,dx$ diverges

EXAMPLE 2 Evaluate $\displaystyle\int_{-\infty}^0 xe^x\,dx$.

SOLUTION Using part (b) of Definition 1, we have

$$\int_{-\infty}^0 xe^x\,dx = \lim_{t\to-\infty}\int_t^0 xe^x\,dx$$

We integrate by parts with $u = x$, $dv = e^x\,dx$ so that $du = dx$, $v = e^x$:

$$\begin{aligned}\int_t^0 xe^x\,dx &= xe^x\Big]_t^0 - \int_t^0 e^x\,dx\\ &= -te^t - 1 + e^t\end{aligned}$$

We know that $e^t \to 0$ as $t \to -\infty$, and by l'Hospital's Rule we have

$$\begin{aligned}\lim_{t\to-\infty} te^t &= \lim_{t\to-\infty}\frac{t}{e^{-t}} = \lim_{t\to-\infty}\frac{1}{-e^{-t}}\\ &= \lim_{t\to-\infty}(-e^t) = 0\end{aligned}$$

Therefore

$$\begin{aligned}\int_{-\infty}^0 xe^x\,dx &= \lim_{t\to-\infty}(-te^t - 1 + e^t)\\ &= -0 - 1 + 0 = -1\end{aligned}$$ ■

EXAMPLE 3 Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

SOLUTION It's convenient to choose $a = 0$ in Definition 1(c):

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

We must now evaluate the integrals on the right side separately:

$$\begin{aligned}\int_{0}^{\infty} \frac{1}{1+x^2}\,dx &= \lim_{t\to\infty}\int_0^t \frac{dx}{1+x^2} = \lim_{t\to\infty} \tan^{-1}x\Big]_0^t \\ &= \lim_{t\to\infty}(\tan^{-1}t - \tan^{-1}0) = \lim_{t\to\infty}\tan^{-1}t = \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx &= \lim_{t\to-\infty}\int_t^0 \frac{dx}{1+x^2} = \lim_{t\to-\infty} \tan^{-1}x\Big]_t^0 \\ &= \lim_{t\to-\infty}(\tan^{-1}0 - \tan^{-1}t) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}\end{aligned}$$

Since both of these integrals are convergent, the given integral is convergent and

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Since $1/(1 + x^2) > 0$, the given improper integral can be interpreted as the area of the infinite region that lies under the curve $y = 1/(1 + x^2)$ and above the x-axis (see Figure 6). ■

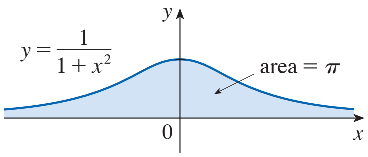


FIGURE 6

EXAMPLE 4 For what values of p is the integral

$$\int_1^{\infty} \frac{1}{x^p}\,dx$$

convergent?

SOLUTION We know from Example 1 that if $p = 1$, then the integral is divergent, so let's assume that $p \neq 1$. Then

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^p}\,dx &= \lim_{t\to\infty}\int_1^t x^{-p}\,dx = \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Bigg]_{x=1}^{x=t} \\ &= \lim_{t\to\infty} \frac{1}{1-p}\left[\frac{1}{t^{p-1}} - 1\right]\end{aligned}$$

If $p > 1$, then $p - 1 > 0$, so as $t \to \infty$, $t^{p-1} \to \infty$ and $1/t^{p-1} \to 0$. Therefore

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{p-1} \qquad \text{if } p > 1$$

and so the integral converges.

If $p < 1$, then $p - 1 < 0$ and so

$$\frac{1}{t^{p-1}} = t^{1-p} \to \infty \qquad \text{as } t \to \infty$$

and the integral diverges. ■

We summarize the result of Example 4 for future reference:

2 $$\int_1^\infty \frac{1}{x^p}\,dx \quad \text{is convergent if } p > 1 \text{ and divergent if } p \leqslant 1.$$

Type 2: Discontinuous Integrands

Suppose that f is a positive continuous function defined on a finite interval $[a, b)$ but has a vertical asymptote at b. Let S be the unbounded region under the graph of f and above the x-axis between a and b. (For Type 1 integrals, the regions extended indefinitely in a horizontal direction. Here the region is infinite in a vertical direction.) The area of the part of S between a and t (the shaded region in Figure 7) is

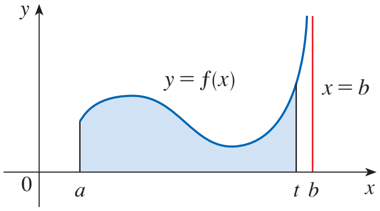


FIGURE 7

$$A(t) = \int_a^t f(x)\,dx$$

If it happens that $A(t)$ approaches a definite number A as $t \to b^-$, then we say that the area of the region S is A and we write

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

We use this equation to define an improper integral of Type 2 even when f is not a positive function, no matter what type of discontinuity f has at b.

Parts (b) and (c) of Definition 3 are illustrated in Figures 8 and 9 for the case where $f(x) \geqslant 0$ and f has vertical asymptotes at a and c, respectively.

3 Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

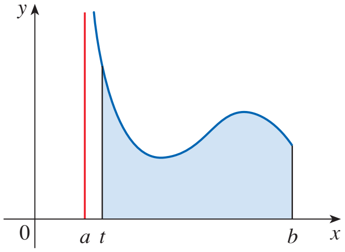


FIGURE 8

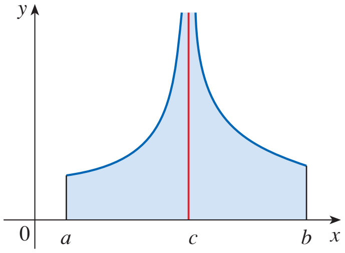


FIGURE 9

EXAMPLE 5 Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.

SOLUTION We note first that the given integral is improper because $f(x) = 1/\sqrt{x-2}$ has the vertical asymptote $x = 2$. Since the infinite discontinuity occurs at the left

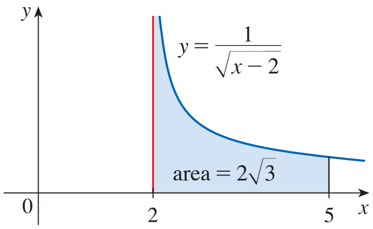


FIGURE 10

endpoint of $[2, 5]$, we use part (b) of Definition 3:

$$\int_2^5 \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2^+} 2\sqrt{x-2}\,\Big]_t^5$$

$$= \lim_{t\to 2^+} 2\left(\sqrt{3} - \sqrt{t-2}\right) = 2\sqrt{3}$$

Thus the given improper integral is convergent and, since the integrand is positive, we can interpret the value of the integral as the area of the shaded region in Figure 10. ■

EXAMPLE 6 Determine whether $\int_0^{\pi/2} \sec x\, dx$ converges or diverges.

SOLUTION Note that the given integral is improper because $\lim_{x\to(\pi/2)^-} \sec x = \infty$. Using part (a) of Definition 3 and Formula 14 from the Table of Integrals, we have

$$\int_0^{\pi/2} \sec x\, dx = \lim_{t\to(\pi/2)^-} \int_0^t \sec x\, dx = \lim_{t\to(\pi/2)^-} \ln|\sec x + \tan x|\,\Big]_0^t$$

$$= \lim_{t\to(\pi/2)^-} [\ln(\sec t + \tan t) - \ln 1] = \infty$$

because $\sec t \to \infty$ and $\tan t \to \infty$ as $t \to (\pi/2)^-$. Thus the given improper integral is divergent. ■

EXAMPLE 7 Evaluate $\int_0^3 \frac{dx}{x-1}$ if possible.

SOLUTION Observe that the line $x = 1$ is a vertical asymptote of the integrand. Since it occurs in the middle of the interval $[0, 3]$, we must use part (c) of Definition 3 with $c = 1$:

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

where $$\int_0^1 \frac{dx}{x-1} = \lim_{t\to 1^-} \int_0^t \frac{dx}{x-1} = \lim_{t\to 1^-} \ln|x-1|\,\Big]_0^t$$

$$= \lim_{t\to 1^-} \left(\ln|t-1| - \ln|-1|\right) = \lim_{t\to 1^-} \ln(1-t) = -\infty$$

because $1 - t \to 0^+$ as $t \to 1^-$. Thus $\int_0^1 dx/(x-1)$ is divergent. This implies that $\int_0^3 dx/(x-1)$ is divergent. [We do not need to evaluate $\int_1^3 dx/(x-1)$.] ■

WARNING If we had not noticed the asymptote $x = 1$ in Example 7 and had instead confused the integral with an ordinary integral, then we might have erroneously calculated $\int_0^3 dx/(x-1)$ as

$$\ln|x-1|\,\Big]_0^3 = \ln 2 - \ln 1 = \ln 2$$

This is wrong because the integral is improper and must be calculated in terms of limits.

From now on, whenever you see the symbol $\int_a^b f(x)\, dx$ you must decide, by looking at the function f on $[a, b]$, whether it is an ordinary definite integral or an improper integral.

EXAMPLE 8 Evaluate $\int_0^1 \ln x\, dx$.

SOLUTION We know that the function $f(x) = \ln x$ has a vertical asymptote at 0 because $\lim_{x \to 0^+} \ln x = -\infty$. Thus the given integral is improper and we have

$$\int_0^1 \ln x\, dx = \lim_{t \to 0^+} \int_t^1 \ln x\, dx$$

Now we integrate by parts with $u = \ln x$, $dv = dx$, $du = dx/x$, and $v = x$:

$$\int_t^1 \ln x\, dx = x \ln x\Big]_t^1 - \int_t^1 dx$$

$$= 1 \ln 1 - t \ln t - (1 - t) = -t \ln t - 1 + t$$

To find the limit of the first term we use l'Hospital's Rule:

$$\lim_{t \to 0^+} t \ln t = \lim_{t \to 0^+} \frac{\ln t}{1/t} = \lim_{t \to 0^+} \frac{1/t}{-1/t^2} = \lim_{t \to 0^+} (-t) = 0$$

Therefore $\displaystyle\int_0^1 \ln x\, dx = \lim_{t \to 0^+} (-t \ln t - 1 + t) = -0 - 1 + 0 = -1$

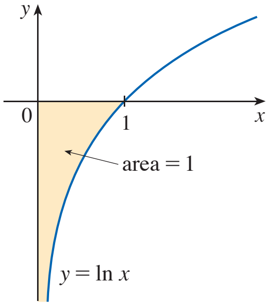


FIGURE 11

Figure 11 shows the geometric interpretation of this result. The area of the shaded region above $y = \ln x$ and below the x-axis is 1. ■

A Comparison Test for Improper Integrals

Sometimes it is impossible to find the exact value of an improper integral and yet it is important to know whether the integral is convergent or divergent. In such cases the following theorem is useful. Although we state it for Type 1 integrals, a similar theorem is true for Type 2 integrals.

Comparison Theorem Suppose that f and g are continuous functions with $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant a$.

(a) If $\int_a^\infty f(x)\, dx$ is convergent, then $\int_a^\infty g(x)\, dx$ is convergent.

(b) If $\int_a^\infty g(x)\, dx$ is divergent, then $\int_a^\infty f(x)\, dx$ is divergent.

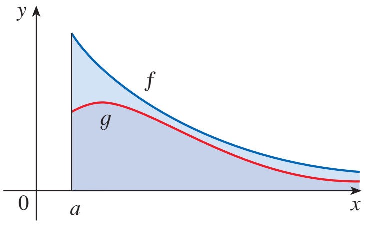


FIGURE 12

We omit the proof of the Comparison Theorem, but Figure 12 makes it seem plausible. If the area under the top curve $y = f(x)$ is finite, then so is the area under the bottom curve $y = g(x)$. And if the area under $y = g(x)$ is infinite, then so is the area under $y = f(x)$. [Note that the reverse is not necessarily true: If $\int_a^\infty g(x)\, dx$ is convergent, $\int_a^\infty f(x)\, dx$ may or may not be convergent, and if $\int_a^\infty f(x)\, dx$ is divergent, $\int_a^\infty g(x)\, dx$ may or may not be divergent.]

EXAMPLE 9 Show that $\int_0^\infty e^{-x^2} dx$ is convergent.

SOLUTION We can't evaluate the integral directly because the antiderivative of e^{-x^2} is not an elementary function (as explained in Section 7.5). We write

$$\int_0^\infty e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^\infty e^{-x^2} dx$$

and observe that the first integral on the right-hand side is just an ordinary definite integral with a finite value. In the second integral we use the fact that for $x \geqslant 1$ we have $x^2 \geqslant x$, so $-x^2 \leqslant -x$ and therefore $e^{-x^2} \leqslant e^{-x}$. (See Figure 13.) The integral of e^{-x} is easy to evaluate:

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_1^t e^{-x}\,dx = \lim_{t\to\infty}(e^{-1} - e^{-t}) = e^{-1}$$

Therefore, taking $f(x) = e^{-x}$ and $g(x) = e^{-x^2}$ in the Comparison Theorem, we see that $\int_1^\infty e^{-x^2}dx$ is convergent. It follows that $\int_0^\infty e^{-x^2}dx$ is convergent also. ■

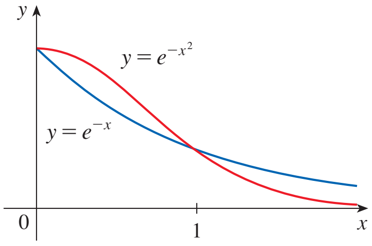


FIGURE 13

In Example 9 we showed that $\int_0^\infty e^{-x^2}dx$ is convergent without computing its value. In Exercise 84 we indicate how to show that its value is approximately 0.8862. In probability theory it is important to know the exact value of this improper integral, as we will see in Section 8.5; using the methods of multivariable calculus it can be shown that the exact value is $\sqrt{\pi}/2$. Table 1 illustrates the definition of a convergent improper integral by showing how the (computer-generated) values of $\int_0^t e^{-x^2}dx$ approach $\sqrt{\pi}/2$ as t becomes large. In fact, these values converge quite quickly because $e^{-x^2} \to 0$ very rapidly as $x \to \infty$.

Table 1

t	$\int_0^t e^{-x^2}\,dx$
1	0.7468241328
2	0.8820813908
3	0.8862073483
4	0.8862269118
5	0.8862269255
6	0.8862269255

EXAMPLE 10 The integral $\displaystyle\int_1^\infty \frac{1 + e^{-x}}{x}\,dx$ is divergent by the Comparison Theorem because

$$\frac{1 + e^{-x}}{x} > \frac{1}{x}$$

and $\int_1^\infty (1/x)\,dx$ is divergent by Example 1 [or by (2) with $p = 1$]. ■

Table 2

t	$\int_1^t [(1 + e^{-x})/x]\,dx$
2	0.8636306042
5	1.8276735512
10	2.5219648704
100	4.8245541204
1000	7.1271392134
10000	9.4297243064

Table 2 illustrates the divergence of the integral in Example 10. It appears that the values are not approaching any fixed number.

7.8 Exercises

1. Explain why each of the following integrals is improper.

(a) $\displaystyle\int_1^4 \frac{dx}{x - 3}$ (b) $\displaystyle\int_3^\infty \frac{dx}{x^2 - 4}$

(c) $\displaystyle\int_0^1 \tan \pi x\,dx$ (d) $\displaystyle\int_{-\infty}^{-1} \frac{e^x}{x}\,dx$

2. Which of the following integrals are improper? Why?

(a) $\displaystyle\int_0^\pi \sec x\,dx$ (b) $\displaystyle\int_0^4 \frac{dx}{x - 5}$

(c) $\displaystyle\int_{-1}^3 \frac{dx}{x + x^3}$ (d) $\displaystyle\int_1^\infty \frac{dx}{x + x^3}$

3. Find the area under the curve $y = 1/x^3$ from $x = 1$ to $x = t$ and evaluate it for $t = 10$, 100, and 1000. Then find the total area under this curve for $x \geqslant 1$.

4. (a) Graph the functions $f(x) = 1/x^{1.1}$ and $g(x) = 1/x^{0.9}$ in both the viewing rectangles $[0, 10]$ by $[0, 1]$ and $[0, 100]$ by $[0, 1]$.

(b) Find the areas under the graphs of f and g from $x = 1$ to $x = t$ and evaluate for $t = 10$, 100, 10^4, 10^6, 10^{10}, and 10^{20}.

(c) Find the total area under each curve for $x \geqslant 1$, if it exists.

5–48 Determine whether the integral is convergent or divergent. Evaluate integrals that are convergent.

5. $\displaystyle\int_1^\infty 2x^{-3}\,dx$ **6.** $\displaystyle\int_{-\infty}^{-1} \frac{1}{\sqrt[3]{x}}\,dx$

7. $\displaystyle\int_0^\infty e^{-2x}\,dx$ **8.** $\displaystyle\int_1^\infty \left(\tfrac{1}{3}\right)^x dx$

9. $\displaystyle\int_{-2}^\infty \frac{1}{x + 4}\,dx$ **10.** $\displaystyle\int_1^\infty \frac{1}{x^2 + 4}\,dx$

11. $\displaystyle\int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx$

12. $\displaystyle\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dx$

13. $\displaystyle\int_{-\infty}^0 \frac{x}{(x^2+1)^3}\,dx$

14. $\displaystyle\int_{-\infty}^{-3} \frac{x}{4-x^2}\,dx$

15. $\displaystyle\int_1^\infty \frac{x^2+x+1}{x^4}\,dx$

16. $\displaystyle\int_2^\infty \frac{x}{\sqrt{x^2-1}}\,dx$

17. $\displaystyle\int_0^\infty \frac{e^x}{(1+e^x)^2}\,dx$

18. $\displaystyle\int_{-\infty}^{-1} \frac{x^2+x}{x^3}\,dx$

19. $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\,dx$

20. $\displaystyle\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$

21. $\displaystyle\int_{-\infty}^\infty \cos 2t\,dt$

22. $\displaystyle\int_1^\infty \frac{e^{-1/x}}{x^2}\,dx$

23. $\displaystyle\int_0^\infty \sin^2\alpha\,d\alpha$

24. $\displaystyle\int_0^\infty \sin\theta\,e^{\cos\theta}\,d\theta$

25. $\displaystyle\int_1^\infty \frac{1}{x^2+x}\,dx$

26. $\displaystyle\int_2^\infty \frac{dv}{v^2+2v-3}$

27. $\displaystyle\int_{-\infty}^0 ze^{2z}\,dz$

28. $\displaystyle\int_2^\infty ye^{-3y}\,dy$

29. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$

30. $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$

31. $\displaystyle\int_{-\infty}^0 \frac{z}{z^4+4}\,dz$

32. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$

33. $\displaystyle\int_0^\infty e^{-\sqrt{y}}\,dy$

34. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}+x\sqrt{x}}$

35. $\displaystyle\int_0^1 \frac{1}{x}\,dx$

36. $\displaystyle\int_0^5 \frac{1}{\sqrt[3]{5-x}}\,dx$

37. $\displaystyle\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}}$

38. $\displaystyle\int_{-1}^{2} \frac{x}{(x+1)^2}\,dx$

39. $\displaystyle\int_{-2}^{3} \frac{1}{x^4}\,dx$

40. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

41. $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$

42. $\displaystyle\int_0^5 \frac{w}{w-2}\,dw$

43. $\displaystyle\int_0^{\pi/2} \tan^2\theta\,d\theta$

44. $\displaystyle\int_0^4 \frac{dx}{x^2-x-2}$

45. $\displaystyle\int_0^1 r\ln r\,dr$

46. $\displaystyle\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{\sin\theta}}\,d\theta$

47. $\displaystyle\int_{-1}^0 \frac{e^{1/x}}{x^3}\,dx$

48. $\displaystyle\int_0^1 \frac{e^{1/x}}{x^3}\,dx$

49–54 Sketch the region and find its area (if the area is finite).

49. $S = \{(x, y) \mid x \geqslant 1,\ 0 \leqslant y \leqslant e^{-x}\}$

50. $S = \{(x, y) \mid x \leqslant 0,\ 0 \leqslant y \leqslant e^{x}\}$

51. $S = \{(x, y) \mid x \geqslant 1,\ 0 \leqslant y \leqslant 1/(x^3+x)\}$

52. $S = \{(x, y) \mid x \geqslant 0,\ 0 \leqslant y \leqslant xe^{-x}\}$

53. $S = \{(x, y) \mid 0 \leqslant x < \pi/2,\ 0 \leqslant y \leqslant \sec^2 x\}$

54. $S = \left\{(x, y) \mid -2 < x \leqslant 0,\ 0 \leqslant y \leqslant 1/\sqrt{x+2}\right\}$

---

55. (a) If $g(x) = (\sin^2 x)/x^2$, use a calculator or computer to make a table of approximate values of $\int_1^t g(x)\,dx$ for $t = 2$, 5, 10, 100, 1000, and 10,000. Does it appear that $\int_1^\infty g(x)\,dx$ is convergent?

(b) Use the Comparison Theorem with $f(x) = 1/x^2$ to show that $\int_1^\infty g(x)\,dx$ is convergent.

(c) Illustrate part (b) by graphing f and g on the same screen for $1 \leqslant x \leqslant 10$. Use your graph to explain intuitively why $\int_1^\infty g(x)\,dx$ is convergent.

56. (a) If $g(x) = 1/(\sqrt{x} - 1)$, use a calculator or computer to make a table of approximate values of $\int_2^t g(x)\,dx$ for $t = 5$, 10, 100, 1000, and 10,000. Does it appear that $\int_2^\infty g(x)\,dx$ is convergent or divergent?

(b) Use the Comparison Theorem with $f(x) = 1/\sqrt{x}$ to show that $\int_2^\infty g(x)\,dx$ is divergent.

(c) Illustrate part (b) by graphing f and g on the same screen for $2 \leqslant x \leqslant 20$. Use your graph to explain intuitively why $\int_2^\infty g(x)\,dx$ is divergent.

57–64 Use the Comparison Theorem to determine whether the integral is convergent or divergent.

57. $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$

58. $\displaystyle\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\,dx$

59. $\displaystyle\int_2^\infty \frac{1}{x-\ln x}\,dx$

60. $\displaystyle\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$

61. $\displaystyle\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx$

62. $\displaystyle\int_1^\infty \frac{2+\cos x}{\sqrt{x^4+x^2}}\,dx$

63. $\displaystyle\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$

64. $\displaystyle\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx$

65–68 Improper Integrals that Are Both Type 1 and Type 2

The integral $\int_a^\infty f(x)\,dx$ is improper because the interval $[a, \infty)$ is infinite. If f has an infinite discontinuity at a, then the integral is improper for a second reason. In this case we evaluate the integral by expressing it as a sum of improper integrals of Type 2 and Type 1 as follows:

$$\int_a^\infty f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx \qquad c > a$$

Evaluate the given integral if it is convergent.

65. $\displaystyle\int_0^\infty \frac{1}{x^2}\,dx$

66. $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}}\,dx$

67. $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx$

68. $\displaystyle\int_2^\infty \frac{1}{x\sqrt{x^2-4}}\,dx$

69–71 Find the values of p for which the integral converges and evaluate the integral for those values of p.

69. $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$

70. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^p}\,dx$

71. $\displaystyle\int_0^1 x^p \ln x\,dx$

---

72. (a) Evaluate the integral $\int_0^\infty x^n e^{-x}\,dx$ for $n = 0, 1, 2$, and 3.

(b) Guess the value of $\int_0^\infty x^n e^{-x}\,dx$ when n is an arbitrary positive integer.

(c) Prove your guess using mathematical induction.

73. The *Cauchy principal value* of the integral $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ is defined by

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_{-t}^{t} f(x)\,dx$$

Show that $\int_{-\infty}^{\infty} x\,dx$ diverges but the Cauchy principal value of this integral is 0.

74. The *average speed* of molecules in an ideal gas is

$$\bar{v} = \frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{3/2}\int_0^\infty v^3 e^{-Mv^2/(2RT)}\,dv$$

where M is the molecular weight of the gas, R is the gas constant, T is the gas temperature, and v is the molecular speed. Show that

$$\bar{v} = \sqrt{\frac{8RT}{\pi M}}$$

75. We know from Example 1 that the region

$$\mathcal{R} = \{(x, y) \mid x \geqslant 1,\ 0 \leqslant y \leqslant 1/x\}$$

has infinite area. Show that by rotating $\mathcal{R}$ about the x-axis we obtain a solid (called *Gabriel's horn*) with finite volume.

76. Use the information and data in Exercise 6.4.35 to find the work required to propel a 1000-kg space vehicle out of the earth's gravitational field.

77. Find the *escape velocity* v_0 that is needed to propel a rocket of mass m out of the gravitational field of a planet with mass M and radius R. Use Newton's Law of Gravitation (see Exercise 6.4.35) and the fact that the initial kinetic energy of $\frac{1}{2}mv_0^2$ supplies the needed work.

78. Astronomers use a technique called *stellar stereography* to determine the density of stars in a star cluster from the observed (two-dimensional) density that can be analyzed from a photograph. Suppose that in a spherical cluster of radius R the density of stars depends only on the distance r from the center of the cluster. If the perceived star density is given by $y(s)$, where s is the observed planar distance from the center of the cluster, and $x(r)$ is the actual density, it can be shown that

$$y(s) = \int_s^R \frac{2r}{\sqrt{r^2 - s^2}}\,x(r)\,dr$$

If the actual density of stars in a cluster is $x(r) = \frac{1}{2}(R - r)^2$, find the perceived density $y(s)$.

79. A manufacturer of lightbulbs wants to produce bulbs that last about 700 hours but, of course, some bulbs burn out faster than others. Let $F(t)$ be the fraction of the company's bulbs that burn out before t hours, so $F(t)$ always lies between 0 and 1.

(a) Make a rough sketch of what you think the graph of F might look like.

(b) What is the meaning of the derivative $r(t) = F'(t)$?

(c) What is the value of $\int_0^\infty r(t)\,dt$? Why?

80. As we saw in Section 3.8, a radioactive substance decays exponentially: The mass at time t is $m(t) = m(0)e^{kt}$, where $m(0)$ is the initial mass and k is a negative constant. The *mean life M* of an atom in the substance is

$$M = -k\int_0^\infty te^{kt}\,dt$$

For the radioactive carbon isotope, ^{14}C, used in radiocarbon dating, the value of k is -0.000121. Find the mean life of a ^{14}C atom.

81. In a study of the spread of illicit drug use from an enthusiastic user to a population of N users, the authors model the number of expected new users by the equation

$$\gamma = \int_0^\infty \frac{cN(1 - e^{-kt})}{k}\,e^{-\lambda t}\,dt$$

where c, k, and λ are positive constants. Evaluate this integral to express γ in terms of c, N, k, and λ.

Source: F. Hoppensteadt et al., "Threshold Analysis of a Drug Use Epidemic Model," *Mathematical Biosciences* 53 (1981): 79–87.

82. Dialysis treatment removes urea and other waste products from a patient's blood by diverting some of the bloodflow externally through a machine called a dialyzer. The rate at which urea is removed from the blood (in mg/min) is often well described by the equation

$$u(t) = \frac{r}{V}C_0 e^{-rt/V}$$

where r is the rate of flow of blood through the dialyzer (in mL/min), V is the volume of the patient's blood (in mL), and C_0 is the amount of urea in the blood (in mg) at time $t = 0$. Evaluate the integral $\int_0^\infty u(t)$ and interpret it.

83. Determine how large the number a has to be so that

$$\int_a^\infty \frac{1}{x^2+1}\,dx < 0.001$$

84. Estimate the numerical value of $\int_0^\infty e^{-x^2}dx$ by writing it as the sum of $\int_0^4 e^{-x^2}dx$ and $\int_4^\infty e^{-x^2}dx$. Approximate the first integral by using Simpson's Rule with $n = 8$ and show that the second integral is smaller than $\int_4^\infty e^{-4x}dx$, which is less than 0.0000001.

85–87 The Laplace Transform If $f(t)$ is continuous for $t \geqslant 0$, the *Laplace transform* of f is the function F defined by

$$F(s) = \int_0^\infty f(t)\,e^{-st}\,dt$$

and the domain of F is the set consisting of all numbers s for which the integral converges.

85. Find the Laplace transform of each of the following functions.
(a) $f(t) = 1$ (b) $f(t) = e^t$ (c) $f(t) = t$

86. Show that if $0 \leqslant f(t) \leqslant Me^{at}$ for $t \geqslant 0$, where M and a are constants, then the Laplace transform $F(s)$ exists for $s > a$.

87. Suppose that $0 \leqslant f(t) \leqslant Me^{at}$ and $0 \leqslant f'(t) \leqslant Ke^{at}$ for $t \geqslant 0$, where f' is continuous. If the Laplace transform of $f(t)$ is $F(s)$ and the Laplace transform of $f'(t)$ is $G(s)$, show that

$$G(s) = sF(s) - f(0) \qquad s > a$$

88. If $\int_{-\infty}^\infty f(x)\,dx$ is convergent and a and b are real numbers, show that

$$\int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx = \int_{-\infty}^b f(x)\,dx + \int_b^\infty f(x)\,dx$$

89. Show that $\int_0^\infty x^2e^{-x^2}dx = \frac{1}{2}\int_0^\infty e^{-x^2}dx$.

90. Show that $\int_0^\infty e^{-x^2}dx = \int_0^1 \sqrt{-\ln y}\;dy$ by interpreting the integrals as areas.

91. Find the value of the constant C for which the integral

$$\int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2}\right)dx$$

converges. Evaluate the integral for this value of C.

92. Find the value of the constant C for which the integral

$$\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1}\right)dx$$

converges. Evaluate the integral for this value of C.

93. Suppose f is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 1$. Is it possible that $\int_0^\infty f(x)\,dx$ is convergent?

94. Show that if $a > -1$ and $b > a + 1$, then the following integral is convergent.

$$\int_0^\infty \frac{x^a}{1+x^b}\,dx$$

7 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. State the rule for integration by parts. In practice, how do you use it?

2. How do you evaluate $\int \sin^m x\,\cos^n x\,dx$ if m is odd? What if n is odd? What if m and n are both even?

3. If the expression $\sqrt{a^2 - x^2}$ occurs in an integral, what substitution might you try? What if $\sqrt{a^2 + x^2}$ occurs? What if $\sqrt{x^2 - a^2}$ occurs?

4. What is the form of the partial fraction decomposition of a rational function $P(x)/Q(x)$ if the degree of P is less than the degree of Q and $Q(x)$ has only distinct linear factors? What if a linear factor is repeated? What if $Q(x)$ has an irreducible quadratic factor (not repeated)? What if the quadratic factor is repeated?

5. State the rules for approximating the definite integral $\int_a^b f(x)\,dx$ with the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule. Which would you expect to give the best estimate? How do you approximate the error for each rule?

6. Define the following improper integrals.
(a) $\int_a^\infty f(x)\,dx$ (b) $\int_{-\infty}^b f(x)\,dx$ (c) $\int_{-\infty}^\infty f(x)\,dx$

7. Define the improper integral $\int_a^b f(x)\,dx$ for each of the following cases.
(a) f has an infinite discontinuity at a.
(b) f has an infinite discontinuity at b.
(c) f has an infinite discontinuity at c, where $a < c < b$.

8. State the Comparison Theorem for improper integrals.

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. $\int \tan^{-1}x\,dx$ can be evaluated using integration by parts.

2. $\int x^5 e^x\,dx$ can be evaluated by applying integration by parts five times.

3. To evaluate $\displaystyle\int \frac{dx}{\sqrt{25+x^2}}$ an appropriate trigonometric substitution is $x = 5\sin\theta$.

4. To evaluate $\displaystyle\int \frac{dx}{\sqrt{9+e^{2x}}}$ we can use the formula in entry 25 of the Table of Integrals to obtain $\ln\left(e^x + \sqrt{9+e^{2x}}\right) + C$.

5. $\dfrac{x(x^2+4)}{x^2-4}$ can be put in the form $\dfrac{A}{x+2} + \dfrac{B}{x-2}$.

6. $\dfrac{x^2+4}{x(x^2-4)}$ can be put in the form $\dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{x-2}$.

7. $\dfrac{x^2+4}{x^2(x-4)}$ can be put in the form $\dfrac{A}{x^2} + \dfrac{B}{x-4}$.

8. $\dfrac{x^2-4}{x(x^2+4)}$ can be put in the form $\dfrac{A}{x} + \dfrac{B}{x^2+4}$.

9. $\displaystyle\int_0^4 \frac{x}{x^2-1}\,dx = \tfrac{1}{2}\ln 15$

10. $\displaystyle\int_1^\infty \frac{1}{x^{\sqrt{2}}}\,dx$ is convergent.

11. If $\int_{-\infty}^{\infty} f(x)\,dx$ is convergent, then $\int_0^\infty f(x)\,dx$ is convergent.

12. The Midpoint Rule is always more accurate than the Trapezoidal Rule.

13. (a) Every elementary function has an elementary derivative.
(b) Every elementary function has an elementary antiderivative.

14. If f is continuous on $[0, \infty)$ and $\int_1^\infty f(x)\,dx$ is convergent, then $\int_0^\infty f(x)\,dx$ is convergent.

15. If f is a continuous, decreasing function on $[1, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_1^\infty f(x)\,dx$ is convergent.

16. If $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ are both convergent, then $\int_a^\infty [f(x) + g(x)]\,dx$ is convergent.

17. If $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ are both divergent, then $\int_a^\infty [f(x) + g(x)]\,dx$ is divergent.

18. If $f(x) \leqslant g(x)$ and $\int_0^\infty g(x)\,dx$ diverges, then $\int_0^\infty f(x)\,dx$ also diverges.

EXERCISES

Note: Additional practice in techniques of integration is provided in Exercises 7.5.

1–50 Evaluate the integral.

1. $\displaystyle\int_1^2 \frac{(x+1)^2}{x}\,dx$

2. $\displaystyle\int_1^2 \frac{x}{(x+1)^2}\,dx$

3. $\displaystyle\int \frac{e^{\sin x}}{\sec x}\,dx$

4. $\displaystyle\int_0^{\pi/6} t\sin 2t\,dt$

5. $\displaystyle\int \frac{dt}{2t^2+3t+1}$

6. $\displaystyle\int_1^2 x^5\ln x\,dx$

7. $\displaystyle\int_0^{\pi/2} \sin^3\theta\,\cos^2\theta\,d\theta$

8. $\displaystyle\int \frac{dx}{x^2\sqrt{16-x^2}}$

9. $\displaystyle\int \frac{\sin(\ln t)}{t}\,dt$

10. $\displaystyle\int_0^1 \frac{\sqrt{\arctan x}}{1+x^2}\,dx$

11. $\displaystyle\int x(\ln x)^2\,dx$

12. $\displaystyle\int \sin x\cos x\ln(\cos x)\,dx$

13. $\displaystyle\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$

14. $\displaystyle\int \frac{e^{2x}}{1+e^{4x}}\,dx$

15. $\displaystyle\int e^{\sqrt[3]{x}}\,dx$

16. $\displaystyle\int \frac{x^2+2}{x+2}\,dx$

17. $\displaystyle\int x^2\tan^{-1}x\,dx$

18. $\displaystyle\int (x+2)^2(x+1)^{20}\,dx$

19. $\displaystyle\int \frac{x-1}{x^2+2x}\,dx$

20. $\displaystyle\int \frac{\sec^6\theta}{\tan^2\theta}\,d\theta$

21. $\displaystyle\int x\cosh x\,dx$

22. $\displaystyle\int \frac{x^2+8x-3}{x^3+3x^2}\,dx$

23. $\displaystyle\int \frac{dx}{\sqrt{x^2-4x}}$

24. $\displaystyle\int \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx$

25. $\displaystyle\int \frac{x+1}{9x^2+6x+5}\,dx$

26. $\displaystyle\int \tan^5\theta\,\sec^3\theta\,d\theta$

27. $\displaystyle\int_0^2 \sqrt{x^2-2x+2}\,dx$

28. $\displaystyle\int \cos\sqrt{t}\,dt$

29. $\displaystyle\int \frac{dx}{x\sqrt{x^2+1}}$

30. $\displaystyle\int e^x\cos x\,dx$

31. $\displaystyle\int \frac{x\sin\left(\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}\,dx$ **32.** $\displaystyle\int \frac{dx}{x^{1/2}+x^{1/4}}$

33. $\displaystyle\int \frac{3x^3-x^2+6x-4}{(x^2+1)(x^2+2)}\,dx$ **34.** $\displaystyle\int x\sin x\cos x\,dx$

35. $\displaystyle\int_0^{\pi/2} \cos^3x\,\sin 2x\,dx$ **36.** $\displaystyle\int \frac{\sqrt[3]{x}+1}{\sqrt[3]{x}-1}\,dx$

37. $\displaystyle\int_{-3}^{3} \frac{x}{1+|x|}\,dx$ **38.** $\displaystyle\int \frac{dx}{e^x\sqrt{1-e^{-2x}}}$

39. $\displaystyle\int_0^{\ln 10} \frac{e^x\sqrt{e^x-1}}{e^x+8}\,dx$ **40.** $\displaystyle\int_0^{\pi/4} \frac{x\sin x}{\cos^3x}\,dx$

41. $\displaystyle\int \frac{x^2}{(4-x^2)^{3/2}}\,dx$ **42.** $\displaystyle\int (\arcsin x)^2\,dx$

43. $\displaystyle\int \frac{1}{\sqrt{x+x^{3/2}}}\,dx$ **44.** $\displaystyle\int \frac{1-\tan\theta}{1+\tan\theta}\,d\theta$

45. $\displaystyle\int (\cos x+\sin x)^2\cos 2x\,dx$

46. $\displaystyle\int x\cos^3(x^2)\sqrt{\sin(x^2)}\,dx$

47. $\displaystyle\int_0^{1/2} \frac{xe^{2x}}{(1+2x)^2}\,dx$ **48.** $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\sqrt{\tan\theta}}{\sin 2\theta}\,d\theta$

49. $\displaystyle\int \frac{1}{\sqrt{e^x-4}}\,dx$ **50.** $\displaystyle\int x\sin\left(\sqrt{1+x^2}\right)dx$

51–60 Evaluate the integral or show that it is divergent.

51. $\displaystyle\int_1^{\infty} \frac{1}{(2x+1)^3}\,dx$ **52.** $\displaystyle\int_1^{\infty} \frac{\ln x}{x^4}\,dx$

53. $\displaystyle\int_2^{\infty} \frac{dx}{x\ln x}$ **54.** $\displaystyle\int_2^{6} \frac{y}{\sqrt{y-2}}\,dy$

55. $\displaystyle\int_0^{4} \frac{\ln x}{\sqrt{x}}\,dx$ **56.** $\displaystyle\int_0^{1} \frac{1}{2-3x}\,dx$

57. $\displaystyle\int_0^{1} \frac{x-1}{\sqrt{x}}\,dx$ **58.** $\displaystyle\int_{-1}^{1} \frac{dx}{x^2-2x}$

59. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{4x^2+4x+5}$ **60.** $\displaystyle\int_1^{\infty} \frac{\tan^{-1}x}{x^2}\,dx$

61–62 Evaluate the indefinite integral. Illustrate, and check that your answer is reasonable, by graphing both the function and its antiderivative (take $C = 0$).

61. $\displaystyle\int \ln(x^2+2x+2)\,dx$ **62.** $\displaystyle\int \frac{x^3}{\sqrt{x^2+1}}\,dx$

63. Graph the function $f(x) = \cos^2x\,\sin^3x$ and use the graph to guess the value of the integral $\int_0^{2\pi} f(x)\,dx$. Then evaluate the integral to confirm your guess.

64. (a) How would you evaluate $\int x^5e^{-2x}\,dx$ by hand? (Don't actually carry out the integration.)
(b) How would you evaluate $\int x^5e^{-2x}\,dx$ using a table of integrals? (Don't actually do it.)
(c) Use a computer to evaluate $\int x^5e^{-2x}\,dx$.
(d) Graph the integrand and the indefinite integral on the same screen.

65–68 Use the Table of Integrals on Reference Pages 6–10 to evaluate the integral.

65. $\displaystyle\int \sqrt{4x^2-4x-3}\,dx$ **66.** $\displaystyle\int \csc^5t\,dt$

67. $\displaystyle\int \cos x\sqrt{4+\sin^2x}\,dx$ **68.** $\displaystyle\int \frac{\cot x}{\sqrt{1+2\sin x}}\,dx$

69. Verify Formula 33 in the Table of Integrals (a) by differentiation and (b) by using a trigonometric substitution.

70. Verify Formula 62 in the Table of Integrals.

71. Is it possible to find a number n such that $\int_0^{\infty} x^n\,dx$ is convergent?

72. For what values of a is $\int_0^{\infty} e^{ax}\cos x\,dx$ convergent? Evaluate the integral for those values of a.

73–74 Use (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule with $n = 10$ to approximate the given integral. Round your answers to six decimal places.

73. $\displaystyle\int_2^{4} \frac{1}{\ln x}\,dx$ **74.** $\displaystyle\int_1^{4} \sqrt{x}\cos x\,dx$

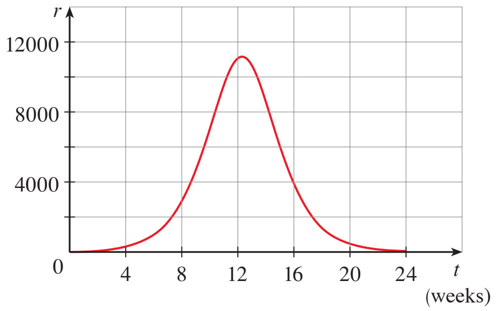
75. Estimate the errors involved in Exercise 73, parts (a) and (b). How large should n be in each case to guarantee an error of less than 0.00001?

76. Use Simpson's Rule with $n = 6$ to estimate the area under the curve $y = e^x/x$ from $x = 1$ to $x = 4$.

77. The speedometer reading (v) on a car was observed at 1-minute intervals and recorded in the chart. Use Simpson's Rule to estimate the distance traveled by the car.

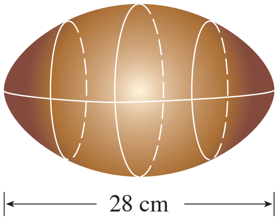
t (min)	v (km/h)	t (min)	v (km/h)
0	64	6	90
1	67	7	91
2	72	8	91
3	78	9	88
4	83	10	90
5	86		

78. A population of honeybees increased at a rate of $r(t)$ bees per week, where the graph of r is as shown. Use Simpson's Rule with six subintervals to estimate the increase in the bee population during the first 24 weeks.



T **79.** (a) If $f(x) = \sin(\sin x)$, use a computer algebra system to compute $f^{(4)}(x)$ and then use a graph to find an upper bound for $|f^{(4)}(x)|$.
(b) Use Simpson's Rule with $n = 10$ to approximate $\int_0^{\pi} f(x)\,dx$ and use part (a) to estimate the error.
(c) How large should n be to guarantee that the size of the error in using S_n is less than 0.00001?

80. Suppose you are asked to estimate the volume of a football. You measure and find that a football is 28 cm long. You use a piece of string and measure the circumference at its widest point to be 53 cm. The circumference 7 cm from each end is 45 cm. Use Simpson's Rule to make your estimate.



81. Use the Comparison Theorem to determine whether the integral is convergent or divergent.

(a) $\displaystyle\int_1^{\infty} \frac{2 + \sin x}{\sqrt{x}}\,dx$ (b) $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{1 + x^4}}\,dx$

82. Find the area of the region bounded by the hyperbola $y^2 - x^2 = 1$ and the line $y = 3$.

83. Find the area bounded by the curves $y = \cos x$ and $y = \cos^2 x$ between $x = 0$ and $x = \pi$.

84. Find the area of the region bounded by the curves $y = 1/(2 + \sqrt{x})$, $y = 1/(2 - \sqrt{x})$, and $x = 1$.

85. The region under the curve $y = \cos^2 x$, $0 \leqslant x \leqslant \pi/2$, is rotated about the x-axis. Find the volume of the resulting solid.

86. The region in Exercise 85 is rotated about the y-axis. Find the volume of the resulting solid.

87. If f' is continuous on $[0, \infty)$ and $\lim_{x \to \infty} f(x) = 0$, show that

$$\int_0^{\infty} f'(x)\,dx = -f(0)$$

88. We can extend our definition of average value of a continuous function to an infinite interval by defining the average value of f on the interval $[a, \infty)$ to be

$$f_{\text{avg}} = \lim_{t \to \infty} \frac{1}{t - a} \int_a^t f(x)\,dx$$

(a) Find the average value of $y = \tan^{-1} x$ on the interval $[0, \infty)$.
(b) If $f(x) \geqslant 0$ and $\int_a^{\infty} f(x)\,dx$ is divergent, show that the average value of f on the interval $[a, \infty)$ is $\lim_{x \to \infty} f(x)$, if this limit exists.
(c) If $\int_a^{\infty} f(x)\,dx$ is convergent, what is the average value of f on the interval $[a, \infty)$?
(d) Find the average value of $y = \sin x$ on the interval $[0, \infty)$.

89. Use the substitution $u = 1/x$ to show that

$$\int_0^{\infty} \frac{\ln x}{1 + x^2}\,dx = 0$$

90. The magnitude of the repulsive force between two point charges with the same sign, one of size 1 and the other of size q, is

$$F = \frac{q}{4\pi\varepsilon_0 r^2}$$

where r is the distance between the charges and ε_0 is a constant. The *potential* V at a point P due to the charge q is defined to be the work expended in bringing a unit charge to P from infinity along the straight line that joins q and P. Find a formula for V.

Problems Plus

Cover up the solution to the example and try it yourself first.

EXAMPLE

(a) Prove that if f is a continuous function, then

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

(b) Use part (a) to show that

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx = \frac{\pi}{4}$$

for all positive numbers n.

SOLUTION

PS You may wish to review the Principles of Problem Solving following Chapter 1.

(a) At first sight, the given equation may appear somewhat baffling. How is it possible to connect the left side to the right side? Connections can often be made through one of the principles of problem solving: *introduce something extra*. Here the extra ingredient is a new variable. We often think of introducing a new variable when we use the Substitution Rule to integrate a specific function. But that technique is still useful in the present circumstance in which we have a general function f.

Once we think of making a substitution, the form of the right side suggests that it should be $u = a - x$. Then $du = -dx$. When $x = 0$, $u = a$; when $x = a$, $u = 0$. So

$$\int_0^a f(a-x)\,dx = -\int_a^0 f(u)\,du = \int_0^a f(u)\,du$$

But this integral on the right side is just another way of writing $\int_0^a f(x)\,dx$. So the given equation is proved.

(b) If we let the given integral be I and apply part (a) with $a = \pi/2$, we get

$$I = \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx = \int_0^{\pi/2} \frac{\sin^n(\pi/2 - x)}{\sin^n(\pi/2 - x) + \cos^n(\pi/2 - x)}\,dx$$

The computer-generated graphs in Figure 1 make it seem plausible that all of the integrals in the example have the same value. The graph of each integrand is labeled with the corresponding value of n.

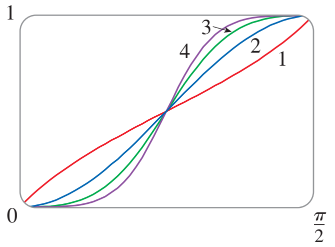


FIGURE 1

A well-known trigonometric identity tells us that $\sin(\pi/2 - x) = \cos x$ and $\cos(\pi/2 - x) = \sin x$, so we get

$$I = \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x}\,dx$$

Notice that the two expressions for I are very similar. In fact, the integrands have the same denominator. This suggests that we should add the two expressions. If we do so, we get

$$2I = \int_0^{\pi/2} \frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x}\,dx = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2}$$

Therefore $I = \pi/4$. ■

Problems

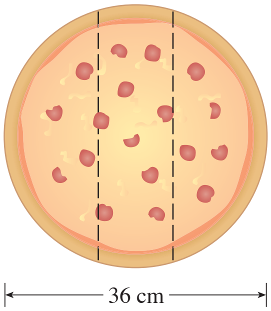


FIGURE FOR PROBLEM 1

1. Three mathematics students have ordered a 36-centimeter pizza. Instead of slicing it in the traditional way, they decide to slice it by parallel cuts, as shown in the figure. Being mathematics majors, they are able to determine where to slice so that each gets the same amount of pizza. Where are the cuts made?

2. Evaluate the integral

$$\int \frac{1}{x^7 - x}\,dx$$

The straightforward approach would be to start with partial fractions, but that would be brutal. Try a substitution.

3. Evaluate $\int_0^1 \left(\sqrt[3]{1 - x^7} - \sqrt[7]{1 - x^3}\right) dx$.

4. Suppose that f is a function that is continuous and increasing on $[0, 1]$ such that $f(0) = 0$ and $f(1) = 1$. Show that

$$\int_0^1 [f(x) + f^{-1}(x)]\,dx = 1$$

5. If f is an even function, $r > 0$, and $a > 0$, show that

$$\int_{-r}^{r} \frac{f(x)}{1 + a^x}\,dx = \int_0^r f(x)\,dx$$

Hint: $\dfrac{1}{1 + u} + \dfrac{1}{1 + u^{-1}} = 1.$

6. The centers of two disks with radius 1 are one unit apart. Find the area of the union of the two disks.

7. An ellipse is cut out of a circle with radius a. The major axis of the ellipse coincides with a diameter of the circle and the minor axis has length $2b$. Prove that the area of the remaining part of the circle is the same as the area of an ellipse with semiaxes a and $a - b$.

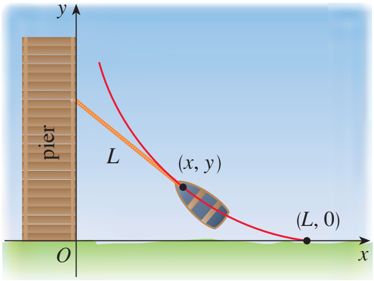


FIGURE FOR PROBLEM 8

8. A man initially standing at the point O walks along a pier pulling a rowboat by a rope of length L. The man keeps the rope straight and taut. The path followed by the boat is a curve called a *tractrix* and it has the property that the rope is always tangent to the curve (see the figure).

(a) Show that if the path followed by the boat is the graph of the function $y = f(x)$, then

$$f'(x) = \frac{dy}{dx} = \frac{-\sqrt{L^2 - x^2}}{x}$$

(b) Determine the function $y = f(x)$.

9. A function f is defined by $f(x) = \int_0^{\pi} \cos t \cos(x - t)\,dt$, $0 \leqslant x \leqslant 2\pi$. Find the minimum value of f.

10. If n is a positive integer, prove that $\int_0^1 (\ln x)^n\,dx = (-1)^n n!$.

11. Show that

$$\int_0^1 (1 - x^2)^n\,dx = \frac{2^{2n}(n!)^2}{(2n + 1)!}$$

Hint: Start by showing that if I_n denotes the integral, then

$$I_{k+1} = \frac{2k + 2}{2k + 3} I_k$$

12. Suppose that f is a positive function such that f' is continuous.

(a) How is the graph of $y = f(x) \sin nx$ related to the graph of $y = f(x)$? What happens as $n \to \infty$?

(b) Make a guess as to the value of the limit

$$\lim_{n \to \infty} \int_0^1 f(x) \sin nx \, dx$$

based on graphs of the integrand.

(c) Using integration by parts, confirm the guess that you made in part (b). [Use the fact that, since f' is continuous, there is a constant M such that $|f'(x)| \leqslant M$ for $0 \leqslant x \leqslant 1$.]

13. If $0 < a < b$, find

$$\lim_{t \to 0} \left\{ \int_0^1 [bx + a(1 - x)]^t \, dx \right\}^{1/t}$$

14. Graph $f(x) = \sin(e^x)$ and use the graph to estimate the value of t such that $\int_t^{t+1} f(x)\, dx$ is a maximum. Then find the exact value of t that maximizes this integral.

15. Evaluate $\displaystyle\int_{-1}^{\infty} \left(\frac{x^4}{1 + x^6} \right)^2 dx$.

16. Evaluate $\displaystyle\int \sqrt{\tan x}\, dx$.

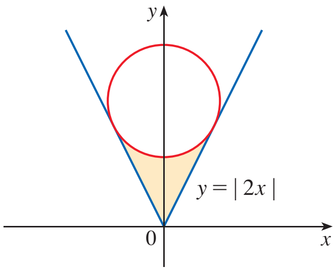


FIGURE FOR PROBLEM 17

17. The circle with radius 1 shown in the figure touches the curve $y = |2x|$ twice. Find the area of the region that lies between the two curves.

18. A rocket is fired straight up, burning fuel at the constant rate of b kilograms per second. Let $v = v(t)$ be the velocity of the rocket at time t and suppose that the velocity u of the exhaust gas is constant. Let $M = M(t)$ be the mass of the rocket at time t and note that M decreases as the fuel burns. If we neglect air resistance, it follows from Newton's Second Law that

$$F = M \frac{dv}{dt} - ub$$

where the force $F = -Mg$. Thus

$$\boxed{1} \qquad M \frac{dv}{dt} - ub = -Mg$$

Let M_1 be the mass of the rocket without fuel, M_2 the initial mass of the fuel, and $M_0 = M_1 + M_2$. Then, until the fuel runs out at time $t = M_2/b$, the mass is $M = M_0 - bt$.

(a) Substitute $M = M_0 - bt$ into Equation 1 and solve the resulting equation for v. Use the initial condition $v(0) = 0$ to evaluate the constant.

(b) Determine the velocity of the rocket at time $t = M_2/b$. This is called the *burnout velocity*.

(c) Determine the height of the rocket $y = y(t)$ at the burnout time.

(d) Find the height of the rocket at any time t.

The Gateway Arch in St. Louis, Missouri, stands 192 meters high and was completed in 1965. The arch was designed by the architect Eero Saarinen using an equation involving the hyperbolic cosine function. In Exercise 8.1.50 you are asked to compute the length of the curve that he used.

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8

Further Applications of Integration

WE LOOKED AT SOME APPLICATIONS of integrals in Chapter 6: areas, volumes, work, and average values. Here we explore some of the many other geometric applications of integration — the length of a curve, the area of a surface—as well as applications to physics, engineering, biology, economics, and statistics. For instance, we will investigate the center of gravity of a plate, the force exerted by water pressure on a dam, the flow of blood from the human heart, and the average time spent on hold during a customer support telephone call.

8.1 Arc Length

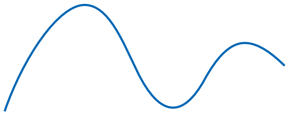


FIGURE 1

What do we mean by the length of a curve? We might think of fitting a piece of string to the curve in Figure 1 and then measuring the string against a ruler. But that might be difficult to do with much accuracy if we have a complicated curve. We need a precise definition for the length of an arc of a curve, in the same spirit as the definitions we developed for the concepts of area and volume.

Arc Length of a Curve

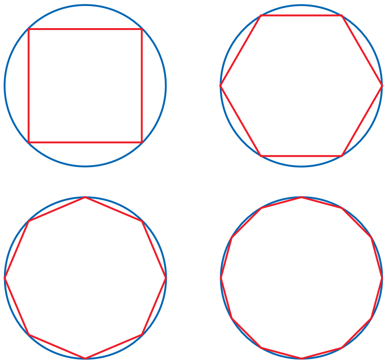


FIGURE 2

If a curve is a polygon, we can easily find its length; we just add the lengths of the line segments that form the polygon. (We can use the distance formula to find the distance between the endpoints of each segment.) We are going to define the length of a general curve by first approximating it by a polygonal path (a path consisting of connected line segments) and then taking a limit as the number of segments of the path is increased. This process is familiar for the case of a circle, where the circumference is the limit of lengths of inscribed polygons (see Figure 2).

Now suppose that a curve C is defined by the equation $y = f(x)$, where f is continuous and $a \leqslant x \leqslant b$. We obtain a polygonal approximation to C by dividing the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx. If $y_i = f(x_i)$, then the point $P_i(x_i, y_i)$ lies on C and the polygonal path with vertices $P_0, P_1, \ldots, P_n$, illustrated in Figure 3, is an approximation to C.

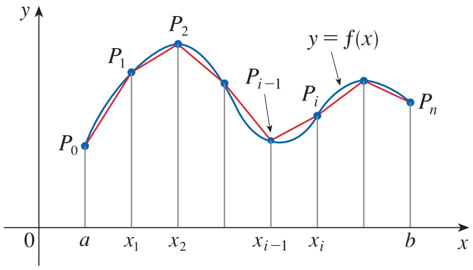


FIGURE 3

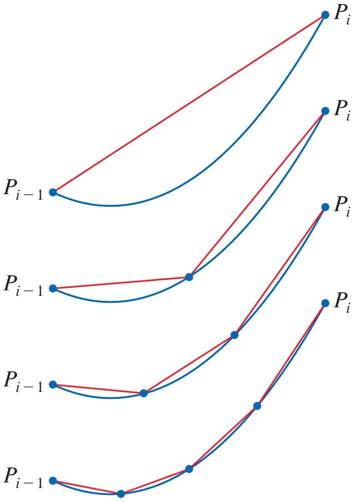


FIGURE 4

The length L of C is approximately the length of this polygonal path and the approximation gets better as we let n increase. (See Figure 4, where the arc of the curve between P_{i-1} and P_i has been magnified and approximations with successively smaller values of Δx are shown.) Therefore we define the **length** L of the curve C with equation $y = f(x)$, $a \leqslant x \leqslant b$, as the limit of the lengths of these approximating polygonal paths (if the limit exists):

1

$$\boxed{L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i|}$$

where $|P_{i-1}P_i|$ is the distance between the points P_{i-1} and P_i.

Notice that the procedure for defining arc length is very similar to the procedure we used for defining area and volume: We divided the curve into a large number of small parts. We then found the approximate lengths of the small parts and added them. Finally, we took the limit as $n \to \infty$.

The definition of arc length given by Equation 1 is not very convenient for computational purposes, but we can derive an integral formula for L in the case where f has a continuous derivative. [Such a function f is called **smooth** because a small change in x produces a small change in $f'(x)$.]

If we let $\Delta y_i = y_i - y_{i-1}$, then

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

By applying the Mean Value Theorem to f on the interval $[x_{i-1}, x_i]$, we find that there is a number x_i^* between x_{i-1} and x_i such that

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$

that is, $$\Delta y_i = f'(x_i^*)\,\Delta x$$

Thus we have

$$\begin{aligned}|P_{i-1}P_i| &= \sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \sqrt{(\Delta x)^2 + [f'(x_i^*)\,\Delta x]^2}\\ &= \sqrt{1 + [f'(x_i^*)]^2}\,\sqrt{(\Delta x)^2} = \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x \qquad \text{(since } \Delta x > 0\text{)}\end{aligned}$$

Therefore, by Definition 1,

$$L = \lim_{n\to\infty}\sum_{i=1}^{n}|P_{i-1}P_i| = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1 + [f'(x_i^*)]^2}\,\Delta x$$

We recognize this expression as being equal to

$$\int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

by the definition of a definite integral. We know that this integral exists because the function $g(x) = \sqrt{1 + [f'(x)]^2}$ is continuous. Thus we have proved the following theorem:

2 The Arc Length Formula If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leqslant x \leqslant b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

If we use Leibniz notation for derivatives, we can write the arc length formula as follows:

3 $$\boxed{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx}$$

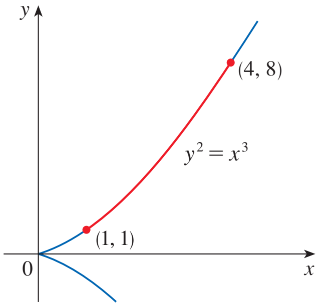


FIGURE 5

EXAMPLE 1 Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points (1, 1) and (4, 8). (See Figure 5.)

SOLUTION For the top half of the curve we have

$$y = x^{3/2} \qquad \frac{dy}{dx} = \tfrac{3}{2}x^{1/2}$$

and so the arc length formula gives

$$L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_1^4 \sqrt{1 + \tfrac{9}{4}x}\,dx$$

As a check on our answer to Example 1, notice from Figure 5 that the arc length ought to be slightly larger than the distance from (1, 1) to (4, 8), which is

$$\sqrt{58} \approx 7.615773$$

According to our calculation in Example 1, we have

$$L = \tfrac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right)$$
$$\approx 7.633705$$

Sure enough, this is a bit greater than the length of the line segment.

If we substitute $u = 1 + \frac{9}{4}x$, then $du = \frac{9}{4}\,dx$. When $x = 1$, $u = \frac{13}{4}$; when $x = 4$, $u = 10$. Therefore

$$L = \tfrac{4}{9}\int_{13/4}^{10} \sqrt{u}\,du = \tfrac{4}{9}\cdot\tfrac{2}{3}u^{3/2}\Big]_{13/4}^{10}$$

$$= \tfrac{8}{27}\left[10^{3/2} - \left(\tfrac{13}{4}\right)^{3/2}\right] = \tfrac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right)$$ ■

If a curve has the equation $x = g(y)$, $c \leqslant y \leqslant d$, and $g'(y)$ is continuous, then by interchanging the roles of x and y in Formula 2 or Equation 3, we obtain the following formula for its length:

4
$$\boxed{L = \int_c^d \sqrt{1 + [g'(y)]^2}\,dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy}$$

EXAMPLE 2 Find the length of the arc of the parabola $y^2 = x$ from (0, 0) to (1, 1).

SOLUTION Since $x = y^2$, we have $dx/dy = 2y$, and Formula 4 gives

$$L = \int_0^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_0^1 \sqrt{1 + 4y^2}\,dy$$

We make the trigonometric substitution $y = \frac{1}{2}\tan\theta$, which gives $dy = \frac{1}{2}\sec^2\theta\,d\theta$ and $\sqrt{1 + 4y^2} = \sqrt{1 + \tan^2\theta} = \sec\theta$. When $y = 0$, $\tan\theta = 0$, so $\theta = 0$; when $y = 1$, $\tan\theta = 2$, so $\theta = \tan^{-1}2 = \alpha$, say. Thus

$$L = \int_0^\alpha \sec\theta\cdot\tfrac{1}{2}\sec^2\theta\,d\theta = \tfrac{1}{2}\int_0^\alpha \sec^3\theta\,d\theta$$

$$= \tfrac{1}{2}\cdot\tfrac{1}{2}\Big[\sec\theta\,\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^\alpha \qquad \text{(from Example 7.2.8)}$$

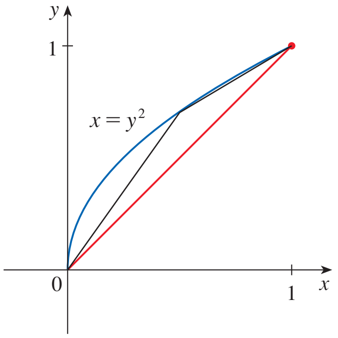
$$= \tfrac{1}{4}\left(\sec\alpha\,\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right)$$

(We could have used the formula in entry 21 of the Table of Integrals.) Since $\tan\alpha = 2$, we have $\sec^2\alpha = 1 + \tan^2\alpha = 5$, so $\sec\alpha = \sqrt{5}$ and

$$L = \frac{\sqrt{5}}{2} + \frac{\ln\left(\sqrt{5} + 2\right)}{4}$$ ■

Figure 6 shows the arc of the parabola whose length is computed in Example 2, together with polygonal approximations having $n = 1$ and $n = 2$ line segments, respectively. For $n = 1$ the approximate length is $L_1 = \sqrt{2}$, the diagonal of a square. The table shows the approximations L_n that we get by dividing [0, 1] into n equal subintervals. Notice that each time we double the number of sides of the polygonal approximation, we get closer to the exact length, which is

$$L = \frac{\sqrt{5}}{2} + \frac{\ln\left(\sqrt{5} + 2\right)}{4} \approx 1.478943$$



n	L_n
1	1.414
2	1.445
4	1.464
8	1.472
16	1.476
32	1.478
64	1.479

FIGURE 6

Because of the presence of the square root sign in Formulas 2 and 4, the calculation of an arc length often leads to an integral that is very difficult or even impossible to evaluate explicitly. Thus we sometimes have to be content with finding an approximation to the length of a curve, as in the following example.

EXAMPLE 3

(a) Set up an integral for the length of the arc of the hyperbola $xy = 1$ from the point $(1, 1)$ to the point $\left(2, \frac{1}{2}\right)$.
(b) Use Simpson's Rule with $n = 10$ to estimate the arc length.

SOLUTION

(a) We have

$$y = \frac{1}{x} \qquad \frac{dy}{dx} = -\frac{1}{x^2}$$

and so the arc length is

$$L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 \sqrt{1 + \frac{1}{x^4}}\, dx$$

(b) Using Simpson's Rule (see Section 7.7) with $a = 1$, $b = 2$, $n = 10$, $\Delta x = 0.1$, and $f(x) = \sqrt{1 + 1/x^4}$, we have

Using a computer to evaluate the definite integral numerically, we get 1.1320904. We see that the approximation using Simpson's Rule is accurate to four decimal places.

$$\begin{aligned} L &= \int_1^2 \sqrt{1 + \frac{1}{x^4}}\, dx \\ &\approx \frac{\Delta x}{3}[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + \cdots + 2f(1.8) + 4f(1.9) + f(2)] \\ &\approx 1.1321 \end{aligned}$$

■

■ The Arc Length Function

We will find it useful to have a function that measures the arc length of a curve from a particular starting point to any other point on the curve. Thus if a smooth curve C has the equation $y = f(x)$, $a \leqslant x \leqslant b$, let $s(x)$ be the distance along C from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$. Then s is a function, called the **arc length function**, and, by Formula 2,

$$\boxed{5} \qquad s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

(We have replaced the variable of integration by t so that x does not have two meanings.) We can use Part 1 of the Fundamental Theorem of Calculus to differentiate Equation 5 (since the integrand is continuous):

$$\boxed{6} \qquad \frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Equation 6 shows that the rate of change of s with respect to x is always at least 1 and is equal to 1 when $f'(x)$, the slope of the curve, is 0. The differential of arc length is

$$\boxed{7} \qquad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

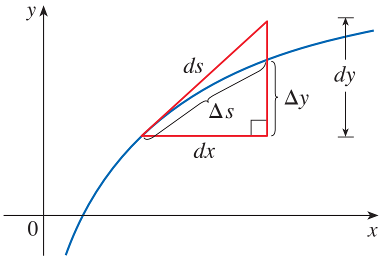


FIGURE 7

and this equation is sometimes written in the symmetric form

$$\boxed{8} \qquad (ds)^2 = (dx)^2 + (dy)^2$$

The geometric interpretation of Equation 8 is shown in Figure 7. It can be used as a mnemonic device for remembering both of the Formulas 3 and 4. If we write $L = \int ds$, then from Equation 8 either we can solve to get (7), which gives (3), or we can solve to get

$$\boxed{9} \qquad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

which gives (4).

EXAMPLE 4 Find the arc length function for the curve $y = x^2 - \frac{1}{8}\ln x$ taking $P_0(1, 1)$ as the starting point.

SOLUTION If $f(x) = x^2 - \frac{1}{8}\ln x$, then

$$f'(x) = 2x - \frac{1}{8x}$$

$$1 + [f'(x)]^2 = 1 + \left(2x - \frac{1}{8x}\right)^2 = 1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}$$

$$= 4x^2 + \frac{1}{2} + \frac{1}{64x^2} = \left(2x + \frac{1}{8x}\right)^2$$

$$\sqrt{1 + [f'(x)]^2} = 2x + \frac{1}{8x} \qquad \text{(since } x > 0\text{)}$$

Thus the arc length function is given by

$$s(x) = \int_1^x \sqrt{1 + [f'(t)]^2}\, dt$$

$$= \int_1^x \left(2t + \frac{1}{8t}\right) dt = t^2 + \tfrac{1}{8}\ln t\Big]_1^x$$

$$= x^2 + \tfrac{1}{8}\ln x - 1$$

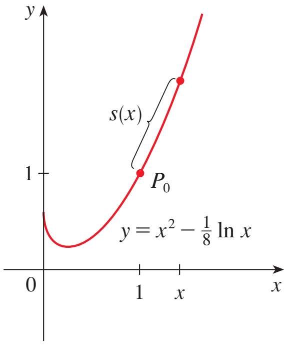


FIGURE 8

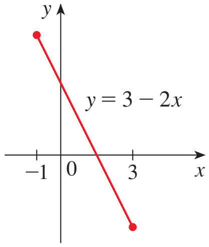
Figure 8 shows the interpretation of the arc length function in Example 4.

For instance, the arc length along the curve from (1, 1) to $(3, f(3))$ is

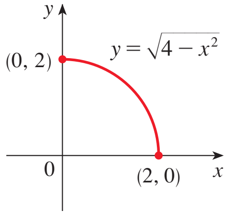
$$s(3) = 3^2 + \tfrac{1}{8}\ln 3 - 1 = 8 + \frac{\ln 3}{8} \approx 8.1373 \qquad ■$$

8.1 Exercises

1. Use the arc length formula (3) to find the length of the curve $y = 3 - 2x$, $-1 \leqslant x \leqslant 3$. Check your answer by noting that the curve is a line segment and calculating its length by the distance formula.



2. Use the arc length formula to find the length of the curve $y = \sqrt{4 - x^2}$, $0 \leqslant x \leqslant 2$. Check your answer by noting that the curve is part of a circle.



3–8 Set up, but do not evaluate, an integral for the length of the curve.

3. $y = x^3, \quad 0 \leqslant x \leqslant 2$

4. $y = e^x, \quad 1 \leqslant x \leqslant 3$

5. $y = x - \ln x, \quad 1 \leqslant x \leqslant 4$

6. $x = y^2 + y, \quad 0 \leqslant y \leqslant 3$

7. $x = \sin y, \quad 0 \leqslant y \leqslant \pi/2$

8. $y^2 = \ln x, \quad -1 \leqslant y \leqslant 1$

9–24 Find the exact length of the curve.

9. $y = \frac{2}{3}x^{3/2}, \quad 0 \leqslant x \leqslant 2$

10. $y = (x + 4)^{3/2}, \quad 0 \leqslant x \leqslant 4$

11. $y = \frac{2}{3}(1 + x^2)^{3/2}, \quad 0 \leqslant x \leqslant 1$

12. $36y^2 = (x^2 - 4)^3, \quad 2 \leqslant x \leqslant 3, \quad y \geqslant 0$

13. $y = \dfrac{x^3}{3} + \dfrac{1}{4x}, \quad 1 \leqslant x \leqslant 2$

14. $x = \dfrac{y^4}{8} + \dfrac{1}{4y^2}, \quad 1 \leqslant y \leqslant 2$

15. $y = \frac{1}{2}\ln(\sin 2x), \quad \pi/8 \leqslant x \leqslant \pi/6$

16. $y = \ln(\cos x), \quad 0 \leqslant x \leqslant \pi/3$

17. $y = \ln(\sec x), \quad 0 \leqslant x \leqslant \pi/4$

18. $x = e^y + \frac{1}{4}e^{-y}, \quad 0 \leqslant y \leqslant 1$

19. $x = \frac{1}{3}\sqrt{y}\,(y - 3), \quad 1 \leqslant y \leqslant 9$

20. $y = 3 + \frac{1}{2}\cosh 2x, \quad 0 \leqslant x \leqslant 1$

21. $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x, \quad 1 \leqslant x \leqslant 2$

22. $y = \sqrt{x - x^2} + \sin^{-1}(\sqrt{x})$

23. $y = \ln(1 - x^2), \quad 0 \leqslant x \leqslant \frac{1}{2}$

24. $y = 1 - e^{-x}, \quad 0 \leqslant x \leqslant 2$

25–26 Find the length of the arc of the curve from point P to point Q.

25. $y = \frac{1}{2}x^2, \quad P(-1, \frac{1}{2}), \quad Q(1, \frac{1}{2})$

26. $x^2 = (y - 4)^3, \quad P(1, 5), \quad Q(8, 8)$

T **27–32** Graph the curve and visually estimate its length. Then compute the length, correct to four decimal places.

27. $y = x^2 + x^3, \quad 1 \leqslant x \leqslant 2$

28. $y = x + \cos x, \quad 0 \leqslant x \leqslant \pi/2$

29. $y = \sqrt[3]{x}, \quad 1 \leqslant x \leqslant 4$

30. $y = x\tan x, \quad 0 \leqslant x \leqslant 1$

31. $y = xe^{-x}, \quad 1 \leqslant x \leqslant 2$

32. $y = \ln(x^2 + 4), \quad -2 \leqslant x \leqslant 2$

33–34 Use Simpson's Rule with $n = 10$ to estimate the arc length of the curve. Compare your answer with the value of the integral produced by a calculator or computer.

33. $y = x\sin x, \; 0 \leqslant x \leqslant 2\pi$

34. $y = e^{-x^2}, \; 0 \leqslant x \leqslant 2$

35. (a) Graph the curve $y = x\sqrt[3]{4 - x}$, $0 \leqslant x \leqslant 4$.
(b) Compute the lengths of approximating polygonal paths with $n = 1$, 2, and 4 segments. (Divide the interval into equal subintervals.) Illustrate by sketching the curve and these paths (as in Figure 6).
(c) Set up an integral for the length of the curve.
T (d) Compute the length of the curve to four decimal places. Compare with the approximations in part (b).

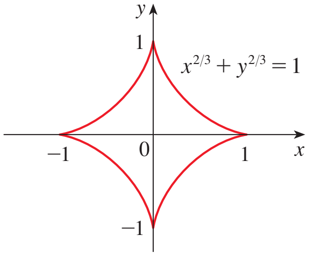
36. Repeat Exercise 35 for the curve

$$y = x + \sin x \qquad 0 \leqslant x \leqslant 2\pi$$

T **37.** Use either a computer or a table of integrals to find the *exact* length of the arc of the curve $y = e^x$ that lies between the points $(0, 1)$ and $(2, e^2)$.

T **38.** Use either a computer or a table of integrals to find the *exact* length of the arc of the curve $y = x^{4/3}$ that lies between the points $(0, 0)$ and $(1, 1)$. If your software has trouble evaluating the integral, make a substitution that changes the integral into one that the software can evaluate.

39. Find the length of the astroid $x^{2/3} + y^{2/3} = 1$.



40. (a) Sketch the curve $y^3 = x^2$.
(b) Use Formulas 3 and 4 to set up two integrals for the arc length from $(0, 0)$ to $(1, 1)$. Observe that one of these is an improper integral and evaluate both of them.
(c) Find the length of the arc of this curve from $(-1, 1)$ to $(8, 4)$.

41. Find the arc length function for the curve $y = 2x^{3/2}$ with starting point $P_0(1, 2)$.

42. (a) Find the arc length function for the curve $y = \ln(\sin x)$, $0 < x < \pi$, with starting point $(\pi/2, 0)$.
(b) Graph both the curve and its arc length function on the same screen. Why is the arc length function negative when x is less than $\pi/2$?

43. Find the arc length function for the curve $y = \sin^{-1}x + \sqrt{1 - x^2}$ with starting point $(0, 1)$.

44. The arc length function for a curve $y = f(x)$, where f is an increasing function, is $s(x) = \int_0^x \sqrt{3t + 5}\, dt$.
(a) If f has y-intercept 2, find an equation for f.
(b) What point on the graph of f is 3 units along the curve from the y-intercept? State your answer rounded to 3 decimal places.

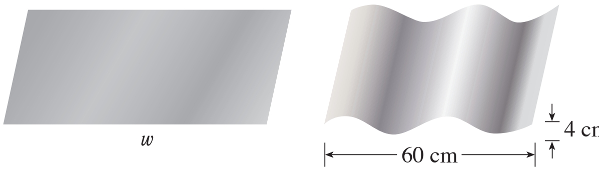
45. A hawk flying at 15 m/s at an altitude of 180 m accidentally drops its prey. The parabolic trajectory of the falling prey is described by the equation

$$y = 180 - \frac{x^2}{45}$$

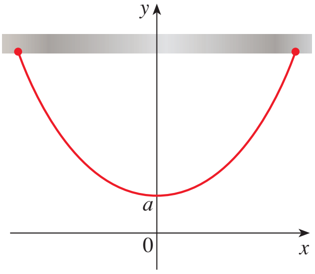
until it hits the ground, where y is its height above the ground and x is the horizontal distance traveled in meters. Calculate the distance traveled by the prey from the time it is dropped until the time it hits the ground. Express your answer correct to the nearest tenth of a meter.

46. A steady wind blows a kite due west. The kite's height above ground from horizontal position $x = 0$ to $x = 25$ m is given by $y = 50 - 0.1(x - 15)^2$. Find the distance traveled by the kite.

T **47.** A manufacturer of corrugated metal roofing wants to produce panels that are 60 cm wide and 4 cm high by processing flat sheets of metal as shown in the figure. The profile of the roofing takes the shape of a sine wave. Verify that the sine curve has equation $y = 2\sin(\pi x/15)$ and find the width w of a flat metal sheet that is needed to make a 60-cm panel. (Numerically evaluate the integral correct to four significant digits.)



48–50 Catenary Curves A chain (or cable) of uniform density that is suspended between two points, as shown in the figure, hangs in the shape of a curve called a *catenary* with equation $y = a \cosh(x/a)$. (See the Discovery Project following Section 12.2.)

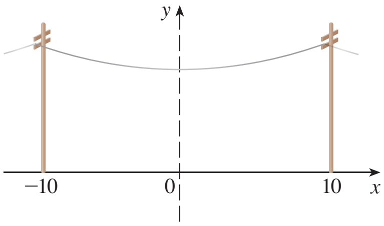


48. (a) Find the arc length of the catenary $y = a \cosh(x/a)$ on the interval $[c, d]$.
(b) Show that on *any* interval $[c, d]$, the ratio of the area under the catenary to its arc length is a.

49. The figure shows a telephone wire hanging between two poles at $x = -10$ and $x = 10$. The wire hangs in the shape of a catenary described by the equation

$$y = c + a \cosh\frac{x}{a}$$

If the length of the wire between the two poles is 20.4 m and the lowest point of the wire must be 9 m above the ground, how high up on each pole should the wire be attached?



50. The British physicist and architect Robert Hooke (1635–1703) was the first to observe that the ideal shape for a free standing arch is an inverted catenary. Hooke remarked, “As hangs the chain, so stands the arch.” The Gateway Arch in St. Louis is based on the shape of a catenary; the central curve of the arch is modeled by the equation

$$y = 211.49 - 20.96 \cosh 0.03291765x$$

where x and y are measured in meters and $|x| \leqslant 91.20$. Set up an integral for the length of the arch and evaluate the integral numerically to estimate the length correct to the nearest meter.

51. For the function $f(x) = \frac{1}{4}e^x + e^{-x}$, prove that the arc length on any interval has the same value as the area under the curve.

52. The curves with equations $x^n + y^n = 1$, $n = 4, 6, 8, \ldots,$ are called **fat circles**. Graph the curves with $n = 2, 4, 6, 8,$ and 10 to see why. Set up an integral for the length L_{2k} of the fat circle with $n = 2k$. Without attempting to evaluate this integral, state the value of $\lim_{k \to \infty} L_{2k}$.

53. Find the length of the curve

$$y = \int_1^x \sqrt{t^3 - 1}\, dt \qquad 1 \leqslant x \leqslant 4$$

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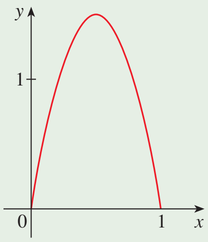
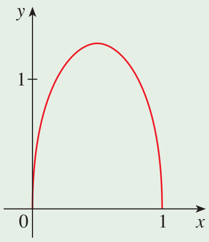
The curves shown are all examples of graphs of continuous functions f that have the following properties:

1. $f(0) = 0$ and $f(1) = 0$.
2. $f(x) \geqslant 0$ for $0 \leqslant x \leqslant 1$.
3. The area under the graph of f from 0 to 1 is equal to 1.

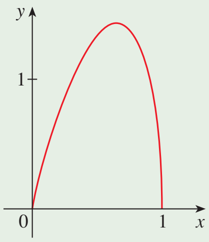
The lengths L of these curves, however, are all different.



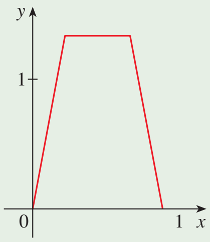
$L \approx 3.249$



$L \approx 2.919$



$L \approx 3.152$



$L \approx 3.213$

Try to discover formulas for two functions that satisfy the given conditions 1, 2, and 3. (Your graphs might be similar to the ones shown or could look quite different.) Then calculate the arc length of each graph. The winning entry will be the one with the smallest arc length.

8.2 Area of a Surface of Revolution

A surface of revolution is formed when a curve is rotated about a line. Such a surface is the lateral boundary of a solid of revolution of the type discussed in Sections 6.2 and 6.3.

We want to define the area of a surface of revolution in such a way that it corresponds to our intuition. If the surface area is A, we can imagine that painting the surface would require the same amount of paint as does a flat region with area A.

Let’s start with some simple surfaces. The lateral surface area of a circular cylinder with radius r and height h is taken to be $A = 2\pi rh$ because we can imagine cutting the

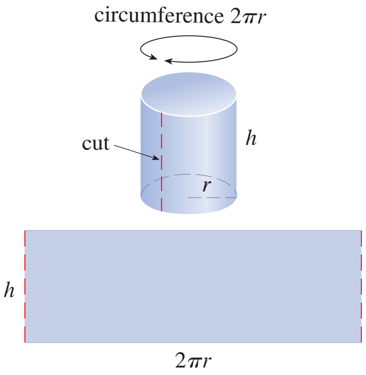


FIGURE 1

cylinder and unrolling it (as in Figure 1) to obtain a rectangle with dimensions $2\pi r$ and h.

Likewise, we can take a circular cone with base radius r and slant height l, cut it along the dashed line in Figure 2, and flatten it to form a sector of a circle with radius l and central angle $\theta = 2\pi r/l$. We know that, in general, the area of a sector of a circle with radius l and angle θ is $\frac{1}{2}l^2\theta$ (see Exercise 7.3.41) and so in this case the area is

$$A = \tfrac{1}{2}l^2\theta = \tfrac{1}{2}l^2\left(\frac{2\pi r}{l}\right) = \pi r l$$

Therefore we define the lateral surface area of a cone to be $A = \pi r l$.

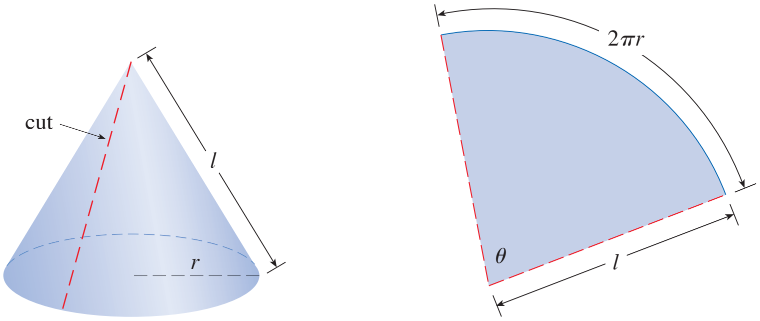


FIGURE 2

What about more complicated surfaces of revolution? If we follow the strategy we used with arc length, we can approximate the original curve by a polygonal path. When this path is rotated about an axis, it creates a simpler surface whose surface area approximates the actual surface area. By taking a limit, we can determine the exact surface area.

The approximating surface, then, consists of a number of *bands,* each formed by rotating a line segment about an axis. To find the surface area, each of these bands can be considered a portion of a circular cone, as shown in Figure 3. The area of the band (or frustum of a cone) with slant height l and upper and lower radii r_1 and r_2 is found by subtracting the areas of two cones:

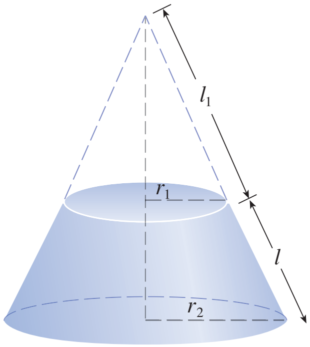


FIGURE 3

$$\boxed{1} \qquad A = \pi r_2(l_1 + l) - \pi r_1 l_1 = \pi[(r_2 - r_1)l_1 + r_2 l]$$

From similar triangles we have

$$\frac{l_1}{r_1} = \frac{l_1 + l}{r_2}$$

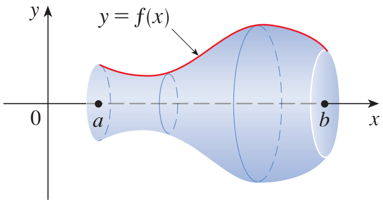
which gives

$$r_2 l_1 = r_1 l_1 + r_1 l \qquad \text{or} \qquad (r_2 - r_1)l_1 = r_1 l$$

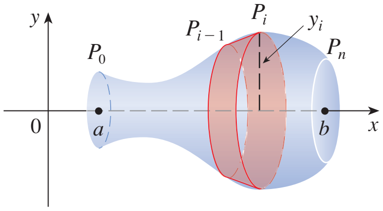
Putting this in Equation 1, we get $A = \pi(r_1 l + r_2 l)$ or

$$\boxed{2} \qquad \boxed{A = 2\pi r l}$$

where $r = \frac{1}{2}(r_1 + r_2)$ is the average radius of the band.



(a) Surface of revolution



(b) Approximating band

FIGURE 4

Now we apply Formula 2 to our strategy. Consider the surface shown in Figure 4, which is obtained by rotating the curve $y = f(x)$, $a \leqslant x \leqslant b$, about the x-axis, where f is positive and has a continuous derivative. In order to define its surface area, we divide the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx, as we did in determining arc length. If $y_i = f(x_i)$, then the point $P_i(x_i, y_i)$ lies on the curve. The part of the surface between x_{i-1} and x_i is approximated by taking the line segment $P_{i-1}P_i$ and rotating it about the x-axis. The result is a band with slant height $l = |P_{i-1}P_i|$ and average radius $r = \frac{1}{2}(y_{i-1} + y_i)$ so, by Formula 2, its surface area is

$$2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i|$$

As in the proof of Theorem 8.1.2, there is a number x_i^* between x_{i-1} and x_i, such that

$$|P_{i-1}P_i| = \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x$$

When Δx is small, we have $y_i = f(x_i) \approx f(x_i^*)$ and also $y_{i-1} = f(x_{i-1}) \approx f(x_i^*)$, since f is continuous. Therefore

$$2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i| \approx 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x$$

and so an approximation to what we think of as the area of the complete surface of revolution is

$$\sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x \tag{3}$$

This approximation appears to become better as $n \to \infty$ and, recognizing (3) as a Riemann sum for the function $g(x) = 2\pi f(x) \sqrt{1 + [f'(x)]^2}$, we have

$$\lim_{n \to \infty} \sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx$$

Therefore, in the case where f is positive and has a continuous derivative, we define the **surface area** of the surface obtained by rotating the curve $y = f(x)$, $a \leqslant x \leqslant b$, about the x-axis as

$$\boxed{S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx} \tag{4}$$

With the Leibniz notation for derivatives, this formula becomes

$$\boxed{S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx} \tag{5}$$

If the curve is described as $x = g(y)$, $c \leqslant y \leqslant d$, then the formula for surface area becomes

$$\boxed{6} \qquad \boxed{S = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy}$$

Both formulas 5 and 6 can be summarized symbolically, using the notation for arc length given in Section 8.1, as

$$\boxed{7} \qquad \boxed{S = \int 2\pi y\, ds}$$

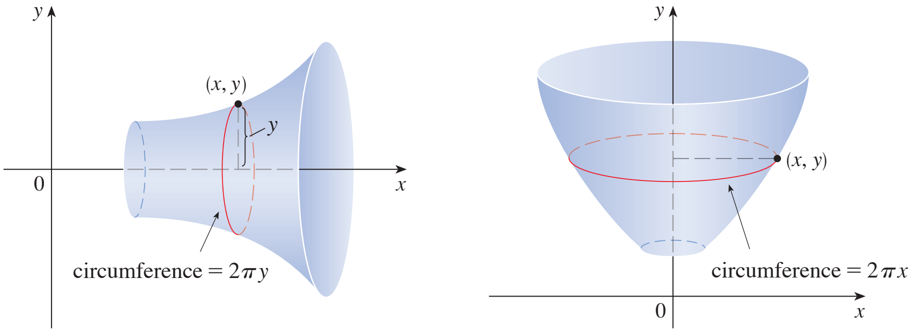
For rotation about the y-axis we can use a similar procedure to obtain the following symbolic formula for surface area:

$$\boxed{8} \qquad \boxed{S = \int 2\pi x\, ds}$$

where, as before (see Equations 8.1.7 and 8.1.9), we can use either

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{or} \qquad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

NOTE Formulas 7 and 8 can be remembered by thinking of the integrand as the circumference of a circle traced out by the point (x, y) on the curve as it is rotated about the x-axis or y-axis, respectively (see Figure 5).



Rotation about x-axis:

$$S = \int \underbrace{2\pi \overbrace{y}^{\text{radius}}}_{\text{circumference}} ds$$

Rotation about y-axis:

$$S = \int \underbrace{2\pi \overbrace{x}^{\text{radius}}}_{\text{circumference}} ds$$

FIGURE 5

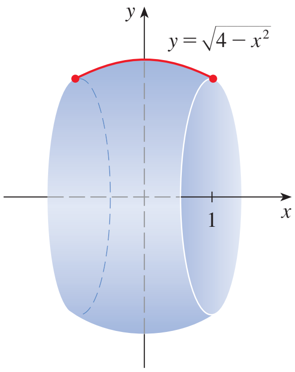


FIGURE 6
The portion of the sphere whose surface area is computed in Example 1

EXAMPLE 1 The curve $y = \sqrt{4 - x^2}$, $-1 \leqslant x \leqslant 1$, is an arc of the circle $x^2 + y^2 = 4$. Find the area of the surface obtained by rotating this arc about the x-axis. (The surface is a portion of a sphere of radius 2. See Figure 6.)

SOLUTION We have

$$\frac{dy}{dx} = \tfrac{1}{2}(4 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{4 - x^2}}$$

and so, using Formula 7 with $ds = \sqrt{1 + (dy/dx)^2}\,dx$ (or, equivalently, Formula 5), the surface area is

$$\begin{aligned} S &= \int_{-1}^{1} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\ &= 2\pi \int_{-1}^{1} \sqrt{4 - x^2}\,\sqrt{1 + \frac{x^2}{4 - x^2}}\,dx \\ &= 2\pi \int_{-1}^{1} \sqrt{4 - x^2}\,\sqrt{\frac{4 - x^2 + x^2}{4 - x^2}}\,dx \\ &= 2\pi \int_{-1}^{1} \sqrt{4 - x^2}\,\frac{2}{\sqrt{4 - x^2}}\,dx = 4\pi \int_{-1}^{1} 1\,dx = 4\pi(2) = 8\pi \end{aligned}$$

■

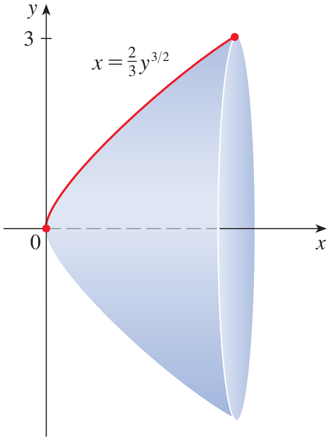


FIGURE 7
The surface of revolution whose area is computed in Example 2

EXAMPLE 2 The portion of the curve $x = \frac{2}{3}y^{3/2}$ between $y = 0$ and $y = 3$ is rotated about the x-axis (see Figure 7). Find the area of the resulting surface.

SOLUTION Since x is given as a function of y, it is natural to use y as the variable of integration. By Formula 7 with $ds = \sqrt{1 + (dx/dy)^2}\,dy$ (or Formula 6), the surface area is

$$\begin{aligned} S &= \int_{0}^{3} 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = 2\pi \int_{0}^{3} y\sqrt{1 + (y^{1/2})^2}\,dy \\ &= 2\pi \int_{0}^{3} y\sqrt{1 + y}\,dy \end{aligned}$$

Substituting $u = 1 + y$, $du = dy$, and remembering to change the limits of integration, we have

$$\begin{aligned} S &= 2\pi \int_{1}^{4} (u - 1)\sqrt{u}\,du = 2\pi \int_{1}^{4} (u^{3/2} - u^{1/2})\,du \\ &= 2\pi \left[\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right]_1^4 = \tfrac{232}{15}\pi \end{aligned}$$

■

Figure 8 shows the surface of revolution whose area is computed in Example 3.

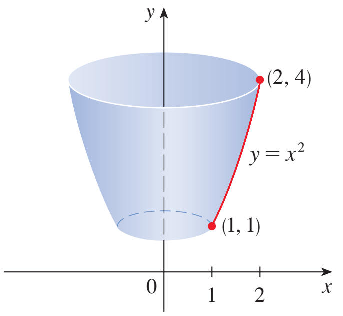


FIGURE 8

EXAMPLE 3 The arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ is rotated about the y-axis. Find the area of the resulting surface.

SOLUTION 1 Considering y as a function of x, we have

$$y = x^2 \qquad \text{and} \qquad \frac{dy}{dx} = 2x$$

Formula 8 with $ds = \sqrt{1 + (dy/dx)^2}\,dx$ gives

$$\begin{aligned} S &= \int 2\pi x\,ds \\ &= \int_1^2 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\ &= 2\pi \int_1^2 x\sqrt{1 + 4x^2}\,dx \end{aligned}$$

Substituting $u = 1 + 4x^2$, we have $du = 8x\,dx$. Remembering to change the limits of integration, we have

$$\begin{aligned} S &= 2\pi \int_5^{17} \sqrt{u} \cdot \tfrac{1}{8}\,du \\ &= \frac{\pi}{4}\int_5^{17} u^{1/2}\,du = \frac{\pi}{4}\Big[\tfrac{2}{3}u^{3/2}\Big]_5^{17} \\ &= \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) \end{aligned}$$

SOLUTION 2 Considering x as a function of y, we have

$$x = \sqrt{y} \qquad \text{and} \qquad \frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

By Formula 8 with $ds = \sqrt{1 + (dx/dy)^2}\,dy$, we have

$$\begin{aligned} S &= \int 2\pi x\,ds = \int_1^4 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy \\ &= 2\pi \int_1^4 \sqrt{y}\sqrt{1 + \frac{1}{4y}}\,dy = 2\pi \int_1^4 \sqrt{y + \tfrac{1}{4}}\,dy \\ &= 2\pi \int_1^4 \sqrt{\tfrac{1}{4}(4y+1)}\,dy = \pi \int_1^4 \sqrt{4y+1}\,dy \\ &= \frac{\pi}{4}\int_5^{17} \sqrt{u}\,du \qquad \text{(where } u = 1 + 4y\text{)} \\ &= \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) \qquad \text{(as in Solution 1)} \end{aligned}$$

■

As a check on our answer to Example 3, notice from Figure 8 that the surface area should be close to that of a circular cylinder with the same height and radius halfway between the upper and lower radius of the surface: $2\pi(1.5)(3) \approx 28.27$. We computed that the surface area was

$$\frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) \approx 30.85$$

which seems reasonable. Alternatively, the surface area should be slightly larger than the area of a frustum of a cone with the same top and bottom edges. From Equation 2, this is $2\pi(1.5)\left(\sqrt{10}\right) \approx 29.80$.

EXAMPLE 4 Set up an integral for the area of the surface generated by rotating the curve $y = e^x$, $0 \leqslant x \leqslant 1$, about the x-axis. Then evaluate the integral numerically, correct to three decimal places.

SOLUTION Using

$$y = e^x \qquad \text{and} \qquad \frac{dy}{dx} = e^x$$

and Formula 7 with $ds = \sqrt{1 + (dy/dx)^2}\, dx$ (or Formula 5), we have

$$S = \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_0^1 e^x \sqrt{1 + e^{2x}}\, dx$$

Another method: Use Formula 7 with $x = \ln y$ and $ds = \sqrt{1 + (dx/dy)^2}\, dy$ (or, equivalently, Formula 6).

Using a calculator or computer, we get

$$2\pi \int_0^1 e^x \sqrt{1 + e^{2x}}\, dx \approx 22.943$$

■

8.2 Exercises

1–4 The given curve is rotated about the x-axis. Set up, but do not evaluate, an integral for the area of the resulting surface by integrating (a) with respect to x and (b) with respect to y.

1. $y = \sqrt[3]{x}, \; 1 \leqslant x \leqslant 8$

2. $x^2 = e^y, \; 1 \leqslant x \leqslant e$

3. $x = \ln(2y + 1), \; 0 \leqslant y \leqslant 1$

4. $y = \tan^{-1} x, \; 0 \leqslant x \leqslant 1$

5–8 The given curve is rotated about the y-axis. Set up, but do not evaluate, an integral for the area of the resulting surface by integrating (a) with respect to x and (b) with respect to y.

5. $xy = 4, \; 1 \leqslant x \leqslant 8$

6. $y = (x + 1)^4, \; 0 \leqslant x \leqslant 2$

7. $y = 1 + \sin x, \; 0 \leqslant x \leqslant \pi/2$

8. $x = e^{2y}, \; 0 \leqslant y \leqslant 2$

9–16 Find the exact area of the surface obtained by rotating the curve about the x-axis.

9. $y = x^3, \; 0 \leqslant x \leqslant 2$

10. $y = \sqrt{5 - x}, \; 3 \leqslant x \leqslant 5$

11. $y^2 = x + 1, \; 0 \leqslant x \leqslant 3$

12. $y = \sqrt{1 + e^x}, \; 0 \leqslant x \leqslant 1$

13. $y = \cos\left(\tfrac{1}{2}x\right), \; 0 \leqslant x \leqslant \pi$

14. $y = \dfrac{x^3}{6} + \dfrac{1}{2x}, \; \tfrac{1}{2} \leqslant x \leqslant 1$

15. $x = \tfrac{1}{3}(y^2 + 2)^{3/2}, \; 1 \leqslant y \leqslant 2$

16. $x = 1 + 2y^2, \; 1 \leqslant y \leqslant 2$

17–20 The given curve is rotated about the y-axis. Find the area of the resulting surface.

17. $y = \tfrac{1}{3}x^{3/2}, \; 0 \leqslant x \leqslant 12$

18. $x^{2/3} + y^{2/3} = 1, \; 0 \leqslant y \leqslant 1$

19. $x = \sqrt{a^2 - y^2}, \; 0 \leqslant y \leqslant a/2$

20. $y = \tfrac{1}{4}x^2 - \tfrac{1}{2}\ln x, \; 1 \leqslant x \leqslant 2$

T **21–26** Set up an integral for the area of the surface obtained by rotating the given curve about the specified axis. Then evaluate your integral numerically, correct to four decimal places.

21. $y = e^{-x^2}, \; -1 \leqslant x \leqslant 1; \; x$-axis

22. $xy = y^2 - 1, \; 1 \leqslant y \leqslant 3; \; x$-axis

23. $x = y + y^3, \; 0 \leqslant y \leqslant 1; \; y$-axis

24. $y = x + \sin x, \; 0 \leqslant x \leqslant 2\pi/3; \; y$-axis

25. $\ln y = x - y^2, \; 1 \leqslant y \leqslant 4; \; x$-axis

26. $x = \cos^2 y, \; 0 \leqslant y \leqslant \pi/2; \; y$-axis

T **27–28** Find the exact area of the surface obtained by rotating the given curve about the x-axis.

27. $y = 1/x, \ 1 \leqslant x \leqslant 2$

28. $y = \sqrt{x^2 + 1}, \ 0 \leqslant x \leqslant 3$

T **29–30** Use a computer to find the exact area of the surface obtained by rotating the given curve about the y-axis. If your software has trouble evaluating the integral, express the surface area as an integral in the other variable.

29. $y = x^3, \ 0 \leqslant y \leqslant 1$

30. $y = \ln(x + 1), \ 0 \leqslant x \leqslant 1$

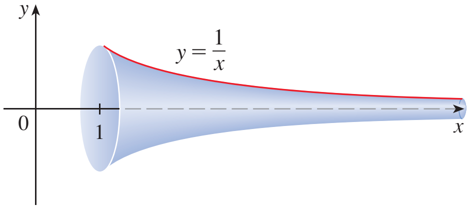
31–32 Use Simpson's Rule with $n = 10$ to approximate the area of the surface obtained by rotating the given curve about the x-axis. Compare your answer with the value of the integral produced by a calculator or computer.

31. $y = \frac{1}{5}x^5, \ 0 \leqslant x \leqslant 5$

32. $y = x \ln x, \ 1 \leqslant x \leqslant 2$

---

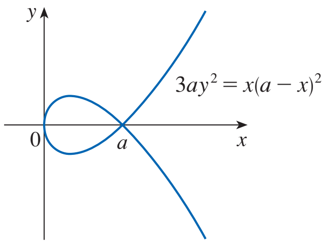
33. Gabriel's Horn The surface formed by rotating the curve $y = 1/x$, $x \geqslant 1$, about the x-axis is known as *Gabriel's horn*. Show that the surface area is infinite (although the enclosed volume is finite; see Exercise 7.8.75).



34. If the infinite curve $y = e^{-x}$, $x \geqslant 0$, is rotated about the x-axis, find the area of the resulting surface.

35. (a) If $a > 0$, find the area of the surface generated by rotating the loop of the curve $3ay^2 = x(a - x)^2$ about the x-axis.

(b) Find the surface area if the loop is rotated about the y-axis.



36. A group of engineers are building a parabolic satellite dish whose shape will be formed by rotating the curve $y = ax^2$ about the y-axis. If the dish is to have a 3-m diameter and a maximum depth of 1 m, find the value of a and the surface area of the dish.

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37. (a) The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a > b$$

is rotated about the x-axis to form a surface called an *ellipsoid*, or *prolate spheroid*. Find the surface area of this ellipsoid.

(b) If the ellipse in part (a) is rotated about its minor axis (the y-axis), the resulting ellipsoid is called an *oblate spheroid*. Find the surface area of this ellipsoid.

38. Find the surface area of the torus in Exercise 6.2.75.

39. (a) If the curve $y = f(x)$, $a \leqslant x \leqslant b$, is rotated about the horizontal line $y = c$, where $f(x) \leqslant c$, find a formula for the area of the resulting surface.

T (b) Set up an integral to find the area of the surface generated by rotating the curve $y = \sqrt{x}$, $0 \leqslant x \leqslant 4$, about the line $y = 4$. Then evaluate the integral numerically, correct to four decimal places.

T **40.** Set up an integral for the area of the surface obtained by rotating the curve $y = x^3$, $1 \leqslant x \leqslant 2$, about the given line. Then evaluate the integral numerically, correct to two decimal places.

(a) $x = -1$
(b) $x = 4$
(c) $y = \frac{1}{2}$
(d) $y = 10$

41. Find the area of the surface obtained by rotating the circle $x^2 + y^2 = r^2$ about the line $y = r$.

42–43 Zone of a Sphere A *zone of a sphere* is the portion of the sphere that lies between two parallel planes.

42. Show that the surface area of a zone of a sphere is $S = 2\pi Rh$, where R is the radius of the sphere and h is the distance between the planes. (Notice that S depends only on the distance between the planes and not on their location, provided that both planes intersect the sphere.)

43. Show that the surface area of a zone of a *cylinder* with radius R and height h is the same as the surface area of the zone of a *sphere* in Exercise 42.

---

44. Let L be the length of the curve $y = f(x)$, $a \leqslant x \leqslant b$, where f is positive and has a continuous derivative. Let S_f be the surface area generated by rotating the curve about the x-axis. If c is a positive constant, define $g(x) = f(x) + c$ and let S_g be the corresponding surface area generated by the curve $y = g(x)$, $a \leqslant x \leqslant b$. Express S_g in terms of S_f and L.

45. Show that if we rotate the curve $y = e^{x/2} + e^{-x/2}$ about the x-axis, the area of the resulting surface is the same value as the enclosed volume for any interval $a \leqslant x \leqslant b$.

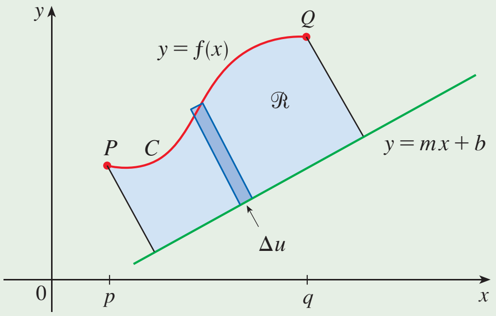
46. Formula 4 is valid only when $f(x) \geqslant 0$. Show that when $f(x)$ is not necessarily positive, the formula for surface area becomes

$$S = \int_a^b 2\pi\,|\,f(x)\,|\,\sqrt{1 + [f'(x)]^2}\,dx$$

DISCOVERY PROJECT | ROTATING ON A SLANT

We know how to find the volume of a solid of revolution obtained by rotating a region about a horizontal or vertical line (see Section 6.2). We also know how to find the surface area of a surface of revolution if we rotate a curve about a horizontal or vertical line (see Section 8.2). But what if we rotate about a slanted line, that is, a line that is neither horizontal nor vertical? In this project you are asked to discover formulas for the volume of a solid of revolution and for the area of a surface of revolution when the axis of rotation is a slanted line.

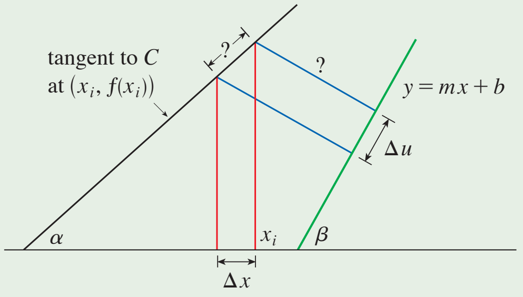
Let C be the arc of the curve $y = f(x)$ between the points $P(p, f(p))$ and $Q(q, f(q))$ and let $\mathcal{R}$ be the region bounded by C, by the line $y = mx + b$ (which lies entirely below C), and by the perpendiculars to the line from P and Q.



1. Show that the area of $\mathcal{R}$ is

$$\frac{1}{1 + m^2}\int_p^q [f(x) - mx - b][1 + mf'(x)]\,dx$$

[*Hint:* This formula can be verified by subtracting areas, but it will be helpful throughout the project to derive it by first approximating the area using rectangles perpendicular to the line, as shown in the following figure. Use the figure to help express Δu in terms of Δx.]



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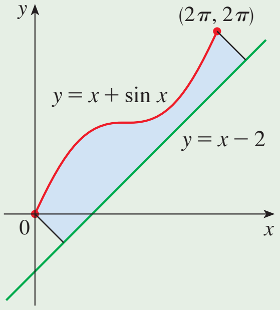


2. Find the area of the region shown in the figure at the left.

3. Find a formula (similar to the one in Problem 1) for the volume of the solid obtained by rotating $\mathcal{R}$ about the line $y = mx + b$.

4. Find the volume of the solid obtained by rotating the region of Problem 2 about the line $y = x - 2$.

5. Find a formula for the area of the surface obtained by rotating C about the line $y = mx + b$.

T **6.** Use a computer to find the exact area of the surface obtained by rotating the curve $y = \sqrt{x}$, $0 \leqslant x \leqslant 4$, about the line $y = \frac{1}{2}x$. Then approximate your result to three decimal places.

8.3 Applications to Physics and Engineering

Among the many applications of integral calculus to physics and engineering, we consider two here: force due to water pressure and centers of mass. As with our previous applications to geometry (areas, volumes, and lengths) and to work, our strategy is to break up the physical quantity into a large number of small parts, approximate each small part, add the results (giving a Riemann sum), take the limit, and then evaluate the resulting integral.

Hydrostatic Pressure and Force

Deep-sea divers know first-hand that water pressure increases as they dive deeper. This is because the weight of the water above them increases.

In general, suppose that a thin horizontal plate with area A square meters is submerged in a fluid of density ρ kilograms per cubic meter at a depth d meters below the surface of the fluid as in Figure 1. The fluid directly above the plate (think of a column of liquid) has volume $V = Ad$, so its mass is $m = \rho V = \rho Ad$. The force exerted by the fluid on the plate is therefore

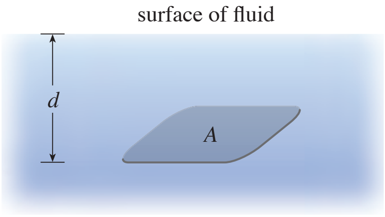


FIGURE 1

$$F = mg = \rho g A d$$

where g is the acceleration due to gravity. The **pressure** P on the plate is defined to be the force per unit area:

$$P = \frac{F}{A} = \rho g d$$

The SI unit for measuring pressure is a newton per square meter, which is called a pascal (abbreviation: $1 \text{ N/m}^2 = 1$ Pa). Since this is a small unit, the kilopascal (kPa) is often used. For instance, because the density of water is $\rho = 1000 \text{ kg/m}^3$, the pressure at the bottom of a swimming pool 2 m deep is

$$\begin{aligned} P = \rho g d &= 1000 \text{ kg/m}^3 \times 9.8 \text{ m/s}^2 \times 2 \text{ m} \\ &= 19{,}600 \text{ Pa} = 19.6 \text{ kPa} \end{aligned}$$

The pressure exerted on a submerged object by a fluid changes with depth but is independent of the volume of fluid. A fish swimming 0.5 m below the surface experiences the same water pressure whether in a small aquarium or an enormous lake.

When using US Customary units, we write $P = \rho g d = \delta d$, where $\delta = \rho g$ is the *weight density* (as opposed to ρ, which is the *mass density*). For instance, the weight density of water is $\delta = 62.5 \text{ lb/ft}^3$, so the pressure at the bottom of a swimming pool 8 ft deep is $P = \delta d = 62.5 \text{ lb/ft}^3 \times 8 \text{ ft} = 500 \text{ lb/ft}^2$.

An important principle of fluid pressure is the experimentally verified fact that *at any point in a liquid the pressure is the same in all directions.* (A diver feels the same pressure on nose and both ears.) Thus the pressure in *any* direction at a depth d in a fluid with mass density ρ is given by

$$\boxed{1} \qquad P = \rho g d$$

This helps us determine the hydrostatic force (the force exerted by a fluid at rest) against a *vertical* plate or wall or dam. This is not a straightforward problem because the pressure is not constant but increases as the depth increases.

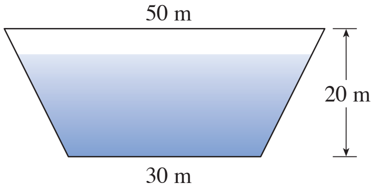


FIGURE 2

EXAMPLE 1 A dam has the shape of the trapezoid shown in Figure 2. The height is 20 m and the width is 50 m at the top and 30 m at the bottom. Find the force on the dam due to hydrostatic pressure if the water level is 4 m from the top of the dam.

SOLUTION We first assign a coordinate system to the dam. One option is to choose a vertical x-axis with origin at the surface of the water and positive direction downward as in Figure 3(a). The depth of the water is 16 m, so we divide the interval $[0, 16]$ into subintervals of equal length with endpoints x_i and we choose $x_i^* \in [x_{i-1}, x_i]$. The ith horizontal strip of the dam is approximated by a rectangle with height Δx and width w_i, where, from similar triangles in Figure 3(b),

$$\frac{a}{16 - x_i^*} = \frac{10}{20} \qquad \text{or} \qquad a = \frac{16 - x_i^*}{2} = 8 - \frac{x_i^*}{2}$$

and so
$$w_i = 2(15 + a) = 2\left(15 + 8 - \tfrac{1}{2}x_i^*\right) = 46 - x_i^*$$

If A_i is the area of the ith strip, then

$$A_i \approx w_i \, \Delta x = (46 - x_i^*)\, \Delta x$$

If Δx is small, then the pressure P_i on the ith strip is almost constant and we can use Equation 1 to write

$$P_i \approx 1000 g x_i^*$$

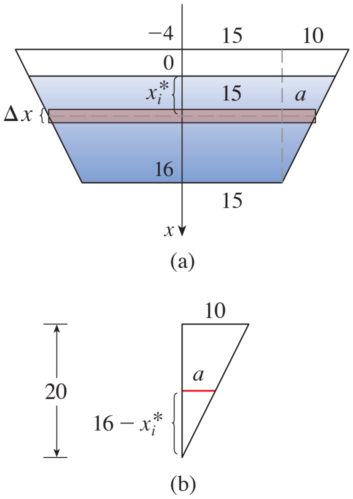
The hydrostatic force F_i acting on the ith strip is the product of the pressure and the area:

$$F_i = P_i A_i \approx 1000 g x_i^* (46 - x_i^*)\, \Delta x$$

Adding these forces and taking the limit as $n \to \infty$, we obtain the total hydrostatic force on the dam:

$$\begin{aligned} F &= \lim_{n\to\infty} \sum_{i=1}^{n} 1000 g x_i^* (46 - x_i^*)\, \Delta x = \int_0^{16} 1000 g x (46 - x)\, dx \\ &= 1000(9.8) \int_0^{16} (46x - x^2)\, dx = 9800 \left[23x^2 - \frac{x^3}{3} \right]_0^{16} \\ &\approx 4.43 \times 10^7 \text{ N} \end{aligned}$$

■



(a)

(b)

FIGURE 3

In Example 1 we could have alternatively used the usual coordinate system with the origin centered at the bottom of the dam. The equation of the right edge of the dam is $y = 2x - 30$, so the width of a horizontal strip at position $y_i{}^*$ is $2x_i{}^* = y_i{}^* + 30$. The depth there is $16 - y_i{}^*$ and so the force on the dam is given by

$$F = 1000(9.8) \int_0^{16} (y + 30)(16 - y)\, dy \approx 4.43 \times 10^7 \text{ N}$$

EXAMPLE 2 Find the hydrostatic force on one end of a cylindrical drum with radius 1 m that is submerged in water 3 m deep.

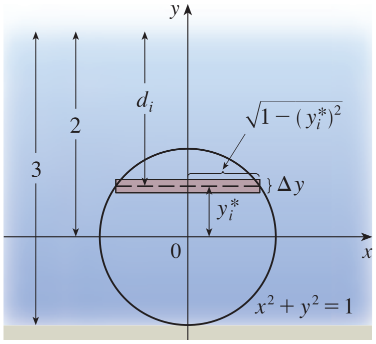


FIGURE 4

SOLUTION In this example it is convenient to choose the axes as in Figure 4 so that the origin is placed at the center of the drum. Then the circle has a simple equation, $x^2 + y^2 = 1$. As in Example 1 we divide the circular region into horizontal strips of equal width. From the equation of the circle, we see that the length of a rectangle that approximates the ith strip is $2\sqrt{1-(y_i^*)^2}$ and so the area of the ith strip is approximately

$$A_i = 2\sqrt{1-(y_i^*)^2}\,\Delta y$$

Because the mass density of water is $\rho = 1000 \text{ kg/m}^3 g$, the pressure on this strip (by Equation 1) is approximately

$$\rho \cdot g d_i = (1000)(9.8)(2 - y_i^*)$$

and so the force (pressure × area) on the strip is approximately

$$\delta d_i A_i = (1000)(9.8)(2 - y_i^*)\, 2\sqrt{1-(y_i^*)^2}\,\Delta y$$

The total force is obtained by adding the forces on all the strips and taking the limit:

$$\begin{aligned} F &= \lim_{n\to\infty} \sum_{i=1}^{n} (1000)(9.8)(2 - y_i^*)\, 2\sqrt{1-(y_i^*)^2}\,\Delta y \\ &= 19{,}600 \int_{-1}^{1} (2 - y)\sqrt{1-y^2}\,dy \\ &= 19{,}600 \cdot 2 \int_{-1}^{1} \sqrt{1-y^2}\,dy - 19{,}600 \int_{-1}^{1} y\sqrt{1-y^2}\,dy \end{aligned}$$

The second integral is 0 because the integrand is an odd function (see Theorem 5.5.7). The first integral can be evaluated using the trigonometric substitution $y = 1 \sin\theta$, but it's simpler to observe that it is the area of a semicircular disk with radius 1. Thus

$$\begin{aligned} F &= 39{,}200 \int_{-1}^{1} \sqrt{1-y^2}\,dy = 39{,}200 \cdot \tfrac{1}{2}\pi(1)^2 \\ &= \frac{39{,}200\pi}{2} \approx 50{,}270 \text{ N} \end{aligned}$$

■

Moments and Centers of Mass

Our main objective here is to find the point P on which a thin plate of any given shape balances horizontally as in Figure 5. This point is called the **center of mass** (or center of gravity) of the plate.

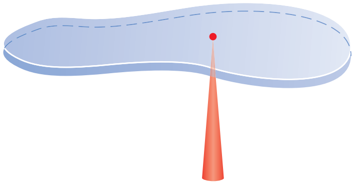


FIGURE 5

We first consider the simpler situation illustrated in Figure 6, where two masses m_1 and m_2 are attached to a rod of negligible mass on opposite sides of a fulcrum and at distances d_1 and d_2 from the fulcrum. The rod will balance if

$$\boxed{2} \qquad m_1 d_1 = m_2 d_2$$

This is an experimental fact discovered by Archimedes and called the Law of the Lever. (Think of a lighter person balancing a heavier one on a seesaw by sitting farther away from the center.)

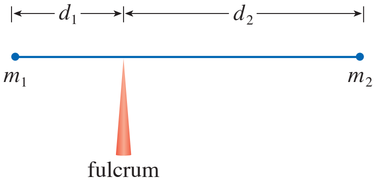


FIGURE 6

Now suppose that the rod lies along the x-axis with m_1 at x_1 and m_2 at x_2 and the center of mass at $\overline{x}$. If we compare Figures 6 and 7, we see that $d_1 = \overline{x} - x_1$ and $d_2 = x_2 - \overline{x}$ and so Equation 2 gives

$$m_1(\overline{x} - x_1) = m_2(x_2 - \overline{x})$$

$$m_1\overline{x} + m_2\overline{x} = m_1x_1 + m_2x_2$$

$$\overline{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \tag{3}$$

The numbers m_1x_1 and m_2x_2 are called the **moments** of the masses m_1 and m_2 (with respect to the origin), and Equation 3 says that the center of mass $\overline{x}$ is obtained by adding the moments of the masses and dividing by the total mass $m = m_1 + m_2$.

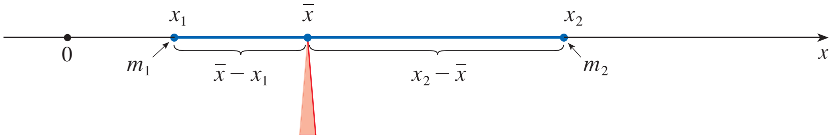


FIGURE 7

In general, if we have a system of n particles with masses $m_1, m_2, \ldots, m_n$ located at the points $x_1, x_2, \ldots, x_n$ on the x-axis, it can be shown similarly that the center of mass of the system is located at

$$\overline{x} = \frac{\sum_{i=1}^{n} m_ix_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_ix_i}{m} \tag{4}$$

where $m = \Sigma m_i$ is the total mass of the system, and the sum of the individual moments

$$M = \sum_{i=1}^{n} m_ix_i$$

is called the **moment of the system about the origin**. Then Equation 4 could be rewritten as $m\overline{x} = M$, which says that if the total mass were considered as being concentrated at the center of mass $\overline{x}$, then its moment would be the same as the moment of the system.

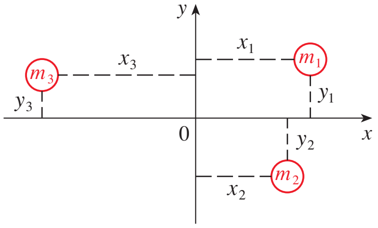


FIGURE 8

Now we consider a system of n particles with masses $m_1, m_2, \ldots, m_n$ located at the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the xy-plane as shown in Figure 8. By analogy with the one-dimensional case, we define the **moment of the system about the *y*-axis** to be

$$M_y = \sum_{i=1}^{n} m_ix_i \tag{5}$$

and the **moment of the system about the *x*-axis** as

$$M_x = \sum_{i=1}^{n} m_iy_i \tag{6}$$

Then M_y measures the tendency of the system to rotate about the y-axis and M_x measures the tendency to rotate about the x-axis.

As in the one-dimensional case, the coordinates $(\overline{x}, \overline{y})$ of the center of mass are given in terms of the moments by the formulas

$$\overline{x} = \frac{M_y}{m} \qquad \overline{y} = \frac{M_x}{m} \tag{7}$$

where $m = \sum m_i$ is the total mass. Because $m\bar{x} = M_y$ and $m\bar{y} = M_x$, the center of mass $(\bar{x}, \bar{y})$ is the point where a single particle of mass m would have the same moments as the system.

EXAMPLE 3 Find the moments and center of mass of the system of objects that have masses 3, 4, and 8 at the points $(-1, 1)$, $(2, -1)$, and $(3, 2)$, respectively.

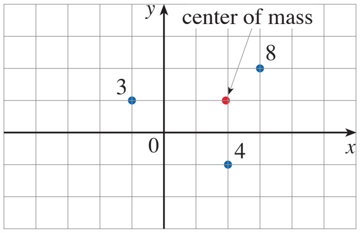


FIGURE 9

SOLUTION We use Equations 5 and 6 to compute the moments:

$$M_y = 3(-1) + 4(2) + 8(3) = 29$$

$$M_x = 3(1) + 4(-1) + 8(2) = 15$$

Since $m = 3 + 4 + 8 = 15$, we use Equations 7 to obtain

$$\bar{x} = \frac{M_y}{m} = \frac{29}{15} \qquad \bar{y} = \frac{M_x}{m} = \frac{15}{15} = 1$$

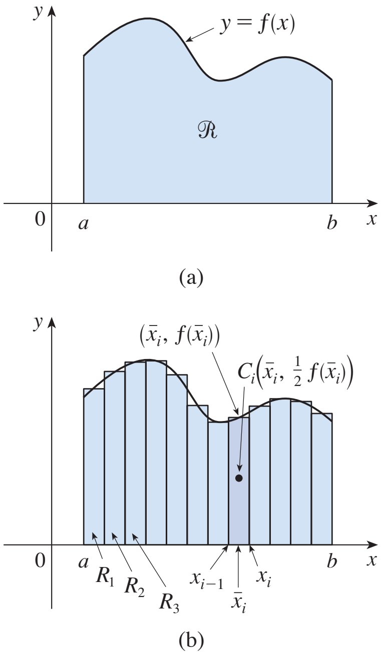
Thus the center of mass is $\left(1\frac{14}{15}, 1\right)$. (See Figure 9.) ■

The centroid of a region $\mathcal{R}$ is determined solely by the shape of the region. If a plate of *uniform* density occupies $\mathcal{R}$, then its center of mass coincides with the centroid of $\mathcal{R}$. If, however, the density is *not* uniform, then typically the center of mass is at a different location. We will examine this situation in Section 15.4.

Next we consider a flat plate (called a *lamina*) with uniform density ρ that occupies a region $\mathcal{R}$ of the plane. We wish to locate the center of mass of the plate, which is called the **centroid** of $\mathcal{R}$. In doing so we use the following physical principles: the **symmetry principle** says that if $\mathcal{R}$ is symmetric about a line l, then the centroid of $\mathcal{R}$ lies on l. (If $\mathcal{R}$ is reflected about l, then $\mathcal{R}$ remains the same so its centroid remains fixed. But the only fixed points lie on l.) Thus the centroid of a rectangle is its center. Moments should be defined so that if the entire mass of a region is concentrated at the center of mass, then its moments remain unchanged. Also, the moment of the union of two nonoverlapping regions should be the sum of the moments of the individual regions.

Suppose that the region $\mathcal{R}$ is of the type shown in Figure 10(a); that is, $\mathcal{R}$ lies between the lines $x = a$ and $x = b$, above the x-axis, and beneath the graph of f, where f is a continuous function. We divide the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx. We choose the sample point x_i^* to be the midpoint $\bar{x}_i$ of the ith subinterval, that is, $\bar{x}_i = (x_{i-1} + x_i)/2$. This determines the approximation to $\mathcal{R}$ by rectangles shown in Figure 10(b). The centroid of the ith approximating rectangle R_i is its center $C_i\left(\bar{x}_i, \frac{1}{2}f(\bar{x}_i)\right)$. Its area is $f(\bar{x}_i)\,\Delta x$, so its mass is density × area:

$$\rho f(\bar{x}_i)\,\Delta x$$


y
y = f(x)
ℛ
0
a
b
x
y
$(\bar{x}_i, f(\bar{x}_i))$
$C_i(\bar{x}_i, \frac{1}{2}f(\bar{x}_i))$
0
a
R_1
R_2
R_3
x_{i-1}
$\bar{x}_i$
x_i
b
x

(a)

(b)

FIGURE 10

The moment of R_i about the y-axis is the product of its mass and the distance from C_i to the y-axis, which is $\bar{x}_i$. Thus

$$M_y(R_i) = [\rho f(\bar{x}_i)\,\Delta x]\,\bar{x}_i = \rho \bar{x}_i f(\bar{x}_i)\,\Delta x$$

Adding these moments, we obtain the moment of the polygonal approximation to $\mathcal{R}$, and then by taking the limit as $n \to \infty$ we obtain the moment of $\mathcal{R}$ itself about the y-axis:

$$\boxed{M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \bar{x}_i f(\bar{x}_i)\,\Delta x = \rho \int_a^b x f(x)\,dx}$$

In a similar fashion we compute the moment of R_i about the x-axis as the product of its mass and the distance from C_i to the x-axis (which is half the height of R_i):

$$M_x(R_i) = [\rho f(\bar{x}_i)\,\Delta x]\,\tfrac{1}{2}f(\bar{x}_i) = \rho \cdot \tfrac{1}{2}[f(\bar{x}_i)]^2\,\Delta x$$

Again we add these moments and take the limit to obtain the moment of $\mathcal{R}$ about the x-axis:

$$\boxed{M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \cdot \tfrac{1}{2}[f(\overline{x}_i)]^2 \, \Delta x = \rho \int_a^b \tfrac{1}{2}[f(x)]^2 \, dx}$$

Just as for systems of particles, the center of mass $(\overline{x}, \overline{y})$ of the plate is defined so that $m\overline{x} = M_y$ and $m\overline{y} = M_x$. But the mass of the plate is the product of its density and its area:

$$m = \rho A = \rho \int_a^b f(x) \, dx$$

and so

$$\overline{x} = \frac{M_y}{m} = \frac{\rho \int_a^b xf(x)\,dx}{\rho \int_a^b f(x)\,dx} = \frac{\int_a^b xf(x)\,dx}{\int_a^b f(x)\,dx}$$

$$\overline{y} = \frac{M_x}{m} = \frac{\rho \int_a^b \frac{1}{2}[f(x)]^2\,dx}{\rho \int_a^b f(x)\,dx} = \frac{\int_a^b \frac{1}{2}[f(x)]^2\,dx}{\int_a^b f(x)\,dx}$$

Notice the cancellation of the ρ's. When density is constant, the location of the center of mass is independent of the density.

In summary, the center of mass of the plate (or the centroid of $\mathcal{R}$) with area A is located at the point $(\overline{x}, \overline{y})$, where

8

$$\boxed{\overline{x} = \frac{1}{A}\int_a^b xf(x)\,dx \qquad \overline{y} = \frac{1}{A}\int_a^b \tfrac{1}{2}[f(x)]^2\,dx}$$

EXAMPLE 4 Find the center of mass of a semicircular plate of radius r with uniform density.

SOLUTION In order to use (8) we place the semicircle as in Figure 11 so that $f(x) = \sqrt{r^2 - x^2}$ and $a = -r$, $b = r$. Here there is no need to use the formula to calculate $\overline{x}$ because, by the symmetry principle, the center of mass must lie on the y-axis, so $\overline{x} = 0$. The area of the semicircle is $A = \frac{1}{2}\pi r^2$, so

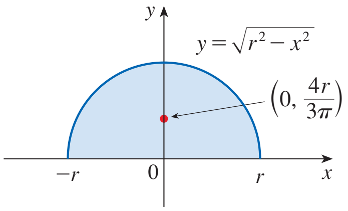


FIGURE 11

$$\begin{aligned}
\overline{y} &= \frac{1}{A}\int_{-r}^{r} \tfrac{1}{2}[f(x)]^2\,dx \\
&= \frac{1}{\frac{1}{2}\pi r^2} \cdot \frac{1}{2}\int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx \\
&= \frac{2}{\pi r^2}\int_0^r (r^2 - x^2)\,dx \qquad \text{(since the integrand is even)} \\
&= \frac{2}{\pi r^2}\left[r^2 x - \frac{x^3}{3}\right]_0^r \\
&= \frac{2}{\pi r^2}\,\frac{2r^3}{3} = \frac{4r}{3\pi}
\end{aligned}$$

The center of mass is located at the point $(0, 4r/(3\pi))$. ■

EXAMPLE 5 Find the centroid of the region in the first quadrant bounded by the curves $y = \cos x$, $y = 0$, and $x = 0$.

SOLUTION The area of the region is

$$A = \int_0^{\pi/2} \cos x \, dx = \sin x \Big]_0^{\pi/2} = 1$$

so Formulas 8 give

$$\begin{aligned}
\bar{x} &= \frac{1}{A} \int_0^{\pi/2} x f(x) \, dx = \int_0^{\pi/2} x \cos x \, dx \\
&= x \sin x \Big]_0^{\pi/2} - \int_0^{\pi/2} \sin x \, dx \qquad \text{(by integration by parts)} \\
&= \frac{\pi}{2} - 1
\end{aligned}$$

$$\begin{aligned}
\bar{y} &= \frac{1}{A} \int_0^{\pi/2} \tfrac{1}{2} [f(x)]^2 \, dx = \tfrac{1}{2} \int_0^{\pi/2} \cos^2 x \, dx \\
&= \tfrac{1}{4} \int_0^{\pi/2} (1 + \cos 2x) \, dx = \tfrac{1}{4} \Big[x + \tfrac{1}{2} \sin 2x \Big]_0^{\pi/2} = \frac{\pi}{8}
\end{aligned}$$

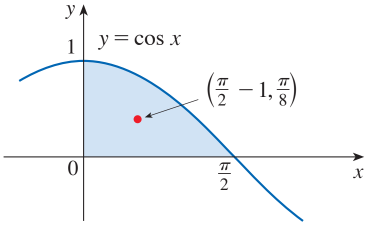


FIGURE 12

The centroid is $\left(\frac{1}{2}\pi - 1, \frac{1}{8}\pi\right) \approx (0.57, 0.39)$ and is shown in Figure 12. ■

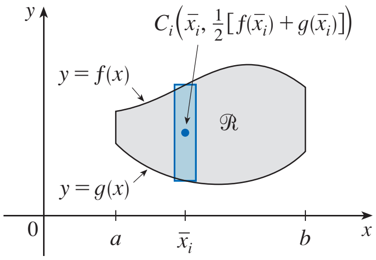

y
$C_i\left(\bar{x}_i, \frac{1}{2}[f(\bar{x}_i) + g(\bar{x}_i)]\right)$
y = f(x)
ℛ
y = g(x)
0
a
$\bar{x}_i$
b
x

FIGURE 13

If the region ℛ lies between two curves $y = f(x)$ and $y = g(x)$, where $f(x) \geqslant g(x)$, as illustrated in Figure 13, then the same sort of argument that led to Formulas 8 can be used to show that the centroid of ℛ is $(\bar{x}, \bar{y})$, where

9

$$\boxed{\begin{aligned}
\bar{x} &= \frac{1}{A} \int_a^b x[f(x) - g(x)] \, dx \\
\bar{y} &= \frac{1}{A} \int_a^b \tfrac{1}{2} \{[f(x)]^2 - [g(x)]^2\} \, dx
\end{aligned}}$$

(See Exercise 51.)

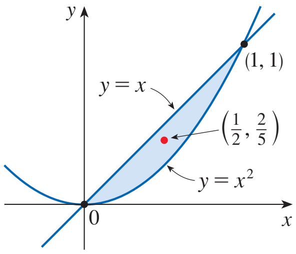


FIGURE 14

EXAMPLE 6 Find the centroid of the region bounded by the line $y = x$ and the parabola $y = x^2$.

SOLUTION The region is sketched in Figure 14. We take $f(x) = x$, $g(x) = x^2$, $a = 0$, and $b = 1$ in Formulas 9. First we note that the area of the region is

$$A = \int_0^1 (x - x^2) \, dx = \frac{x^2}{2} - \frac{x^3}{3} \Big]_0^1 = \frac{1}{6}$$

Therefore

$$\bar{x} = \frac{1}{A}\int_0^1 x[f(x) - g(x)]\,dx = \frac{1}{\frac{1}{6}}\int_0^1 x(x - x^2)\,dx$$

$$= 6\int_0^1 (x^2 - x^3)\,dx = 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2}$$

$$\bar{y} = \frac{1}{A}\int_0^1 \tfrac{1}{2}\{[f(x)]^2 - [g(x)]^2\}\,dx = \frac{1}{\frac{1}{6}}\int_0^1 \tfrac{1}{2}(x^2 - x^4)\,dx$$

$$= 3\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{2}{5}$$

The centroid is $\left(\frac{1}{2}, \frac{2}{5}\right)$. ■

Theorem of Pappus

We end this section by showing a surprising connection between centroids and volumes of revolution.

This theorem is named after the Greek mathematician Pappus of Alexandria, who lived in the fourth century AD.

Theorem of Pappus Let $\mathcal{R}$ be a plane region that lies entirely on one side of a line l in the plane. If $\mathcal{R}$ is rotated about l, then the volume of the resulting solid is the product of the area A of $\mathcal{R}$ and the distance d traveled by the centroid of $\mathcal{R}$.

PROOF We give the proof for the special case in which the region lies between $y = f(x)$ and $y = g(x)$ as in Figure 13 and the line l is the y-axis. Using the method of cylindrical shells (see Section 6.3), we have

$$V = \int_a^b 2\pi x[f(x) - g(x)]\,dx$$

$$= 2\pi\int_a^b x[f(x) - g(x)]\,dx$$

$$= 2\pi(\bar{x}A) \qquad \text{(by Formulas 9)}$$

$$= (2\pi\bar{x})A = Ad$$

where $d = 2\pi\bar{x}$ is the distance traveled by the centroid during one rotation about the y-axis. ■

EXAMPLE 7 A torus is formed by rotating a circle of radius r about a line in the plane of the circle that is a distance R ($> r$) from the center of the circle (see Figure 15). Find the volume of the torus.

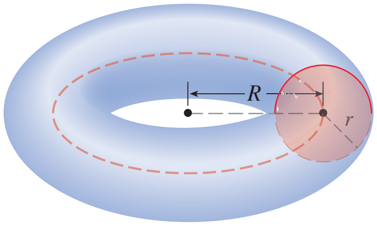


FIGURE 15

SOLUTION The circle has area $A = \pi r^2$. By the symmetry principle, its centroid is its center and so the distance traveled by the centroid during a rotation is $d = 2\pi R$. Therefore, by the Theorem of Pappus, the volume of the torus is

$$V = Ad = (2\pi R)(\pi r^2) = 2\pi^2 r^2 R$$ ■

The method of Example 7 should be compared with the method of Exercise 6.2.75.

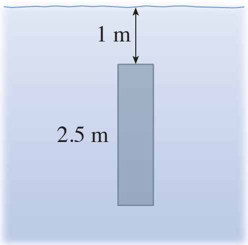
8.3 Exercises

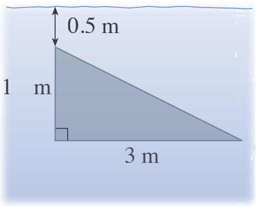
1. An aquarium 1.5 m long, 0.5 m wide, and 1 m deep is full of water. Find (a) the hydrostatic pressure on the bottom of the aquarium, (b) the hydrostatic force on the bottom, and (c) the hydrostatic force on one end of the aquarium.

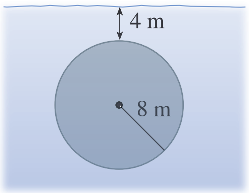
2. A tank is 8 m long, 4 m wide, 2 m high, and contains kerosene with density 820 kg/m^3 to a depth of 1.5 m. Find (a) the hydrostatic pressure on the bottom of the tank, (b) the hydrostatic force on the bottom, and (c) the hydrostatic force on one end of the tank.

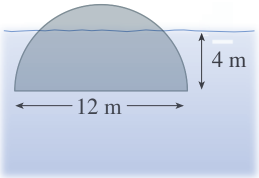
3–11 A vertical plate is submerged (or partially submerged) in

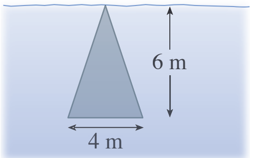
the plate by a Riemann
al and evaluate it.

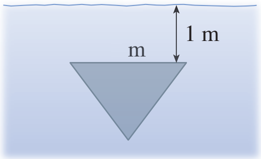


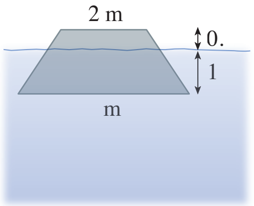


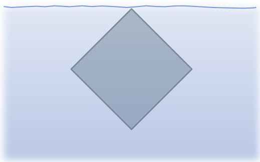


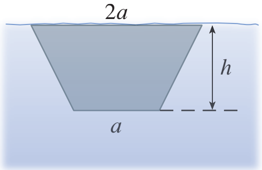




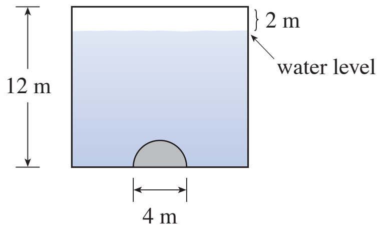








12. A vertical dam has a semicircular gate as shown in the figure. Find the hydrostatic force against the gate.



13. A tanker truck transports gasoline in a horizontal cylindrical tank with diameter 2.5 m and length 12 m. If the tank is full of gasoline with density 753 kg/m^3, compute the force exerted on one end of the tank.

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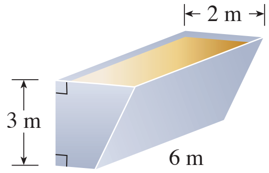
14. A trough with a trapezoidal cross-section, as shown in the figure, contains vegetable oil with density 925 kg/m^3.

(a) Find the hydrostatic force on one end of the trough if it is completely full of oil.

(b) Compute the force on one end if the trough is filled to a depth of 1.2 m.



15. A cube with 20-cm-long sides is sitting on the bottom of an aquarium in which the water is one meter deep. Find the hydrostatic force on (a) the top of the cube and (b) one of the sides of the cube.

16. A dam is inclined at an angle of 30° from the vertical and has the shape of an isosceles trapezoid 30 m wide at the top and 15 m wide at the bottom and with a slant height of 20 m. Find the hydrostatic force on the dam when the water level is at the top of the dam.

17. A swimming pool is 10 m wide and 20 m long and its bottom is an inclined plane, the shallow end having a depth of 1 m and the deep end, 3 m. If the pool is full of water, find the hydrostatic force on (a) each of the four sides and (b) the bottom of the pool.

18. Suppose that a plate is immersed vertically in a fluid with density ρ and the width of the plate is $w(x)$ at a depth of x meters beneath the surface of the fluid. If the top of the plate is at depth a and the bottom is at depth b, show that the hydrostatic force on one side of the plate is

$$F = \int_a^b \rho g x w(x)\,dx$$

19. A metal plate was found submerged vertically in seawater, which has density 1000 kg/m^3. Measurements of the width of the plate were taken at the indicated depths. Use the formula in Exercise 18 and Simpson's Rule to estimate the force of the water against the plate.

Depth (m)	2.1	2.3	2.4	2.5	2.6	2.7	2.8
Plate width (m)	0.4	0.5	1.0	1.2	1.1	1.3	1.3

20. (a) Use the formula in Exercise 18 to show that

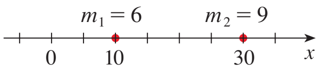
$$F = (\rho g \bar{x})A$$

where $\bar{x}$ is the x-coordinate of the centroid of the plate and A is its area. This equation shows that the hydrostatic force against a vertical plane region is the same as if the region were horizontal at the depth of the centroid of the region.

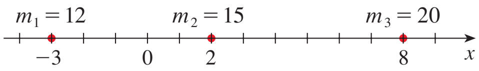
(b) Use the result of part (a) to give another solution to Exercise 10.

21–22 Point-masses m_i are located on the x-axis as shown. Find the moment M of the system about the origin and the center of mass $\bar{x}$.

21.



22.



23–24 The masses m_i are located at the points P_i. Find the moments M_x and M_y and the center of mass of the system.

23. $m_1 = 5,\ m_2 = 8,\ m_3 = 7;$

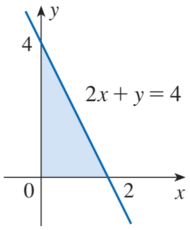
$P_1(3, 1),\ P_2(0, 4),\ P_3(-5, -2)$

24. $m_1 = 4,\ m_2 = 3,\ m_3 = 6,\ m_4 = 3;$

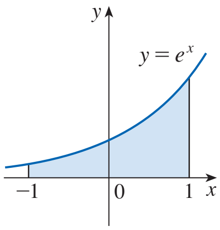
$P_1(6, 1),\ P_2(3, -1),\ P_3(-2, 2),\ P_4(-2, -5)$

25–28 Visually estimate the location of the centroid of the region shown. Then find the exact coordinates of the centroid.

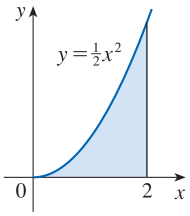
25.



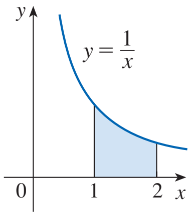
26.



27.



28.



29–33 Find the centroid of the region bounded by the given curves.

29. $y = x^2, \quad x = y^2$

30. $y = 2 - x^2, \quad y = x$

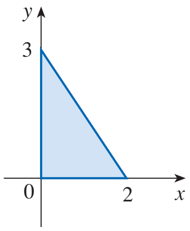
31. $y = \sin 2x, \quad y = \sin x, \quad 0 \leqslant x \leqslant \pi/3$

32. $y = x^3, \quad x + y = 2, \quad y = 0$

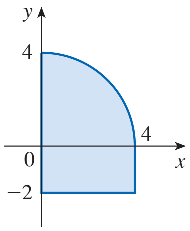
33. $x + y = 2, \quad x = y^2$

34–35 Calculate the moments M_x and M_y and the center of mass of a lamina with the given density and shape.

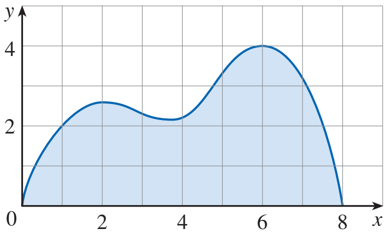
34. $\rho = 4$



35. $\rho = 6$



36. Use Simpson's Rule to estimate the centroid of the region shown.



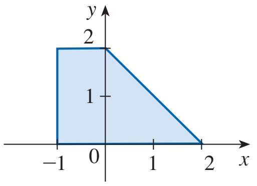
37. Find the centroid of the region bounded by the curves $y = x^3 - x$ and $y = x^2 - 1$. Sketch the region and plot the centroid to see if your answer is reasonable.

38. Use a graph to find approximate x-coordinates of the points of intersection of the curves $y = e^x$ and $y = 2 - x^2$. Then find (approximately) the centroid of the region bounded by these curves.

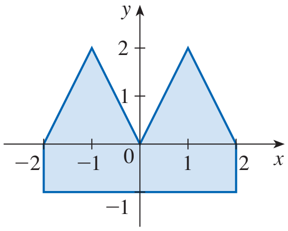
39. Prove that the centroid of any triangle is located at the point of intersection of the medians. [*Hints*: Place the axes so that the vertices are $(a, 0)$, $(0, b)$, and $(c, 0)$. Recall that a median is a line segment from a vertex to the midpoint of the opposite side. Recall also that the medians intersect at a point two-thirds of the way from each vertex (along the median) to the opposite side.]

40–41 Find the centroid of the region shown, not by integration, but by locating the centroids of the rectangles and triangles (from Exercise 39) and using additivity of moments.

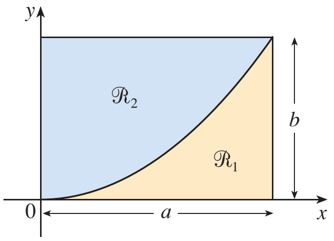
40.



41.



42. A rectangle $\mathcal{R}$ with sides a and b is divided into two parts $\mathcal{R}_1$ and $\mathcal{R}_2$ by an arc of a parabola that has its vertex at one corner of $\mathcal{R}$ and passes through the opposite corner. Find the centroids of both $\mathcal{R}_1$ and $\mathcal{R}_2$.



43. If $\bar{x}$ is the x-coordinate of the centroid of the region that lies under the graph of a continuous function f, where $a \leqslant x \leqslant b$, show that

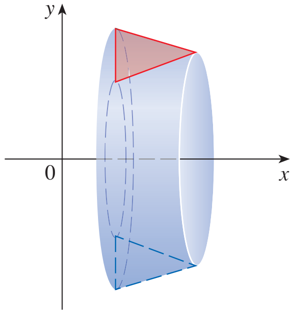
$$\int_a^b (cx + d)f(x)\,dx = (c\bar{x} + d)\int_a^b f(x)\,dx$$

44–46 Use the Theorem of Pappus to find the volume of the given solid.

44. A sphere of radius r (Use Example 4.)

45. A cone with height h and base radius r

46. The solid obtained by rotating the triangle with vertices $(2, 3)$, $(2, 5)$, and $(5, 4)$ about the x-axis



47. Centroid of a Curve The centroid of a *curve* can be found by a process similar to the one we used for finding the centroid of a region. If C is a curve with length L, then the centroid is $(\bar{x}, \bar{y})$ where $\bar{x} = (1/L)\int x\,ds$ and $\bar{y} = (1/L)\int y\,ds$. Here we assign appropriate limits of integration, and ds is as defined in Sections 8.1 and 8.2. (The centroid often doesn't lie on the curve itself. If the curve were made of wire and placed on a weightless board, the centroid would be the balance point on the board.) Find the centroid of the quarter-circle $y = \sqrt{16 - x^2}$, $0 \leqslant x \leqslant 4$.

48–49 The Second Theorem of Pappus The *Second Theorem of Pappus* is in the same spirit as Pappus's Theorem discussed in this section, but for surface area rather than volume: let C be a curve that lies entirely on one side of a line l in the plane. If C is rotated about l, then the area of the resulting surface is the product of the arc length of C and the distance traveled by the centroid of C (see Exercise 47).

48. (a) Prove the Second Theorem of Pappus for the case where C is given by $y = f(x)$, $f(x) \geqslant 0$, and C is rotated about the x-axis.

(b) Use the Second Theorem of Pappus to compute the surface area of the half-sphere obtained by rotating the curve from Exercise 47 about the x-axis. Does your answer agree with the one given by geometric formulas?

49. Use the Second Theorem of Pappus to find the surface area of the torus in Example 7.

50. Let $\mathcal{R}$ be the region that lies between the curves

$$y = x^m \qquad y = x^n \qquad 0 \leqslant x \leqslant 1$$

where m and n are integers with $0 \leqslant n < m$.

(a) Sketch the region $\mathcal{R}$.

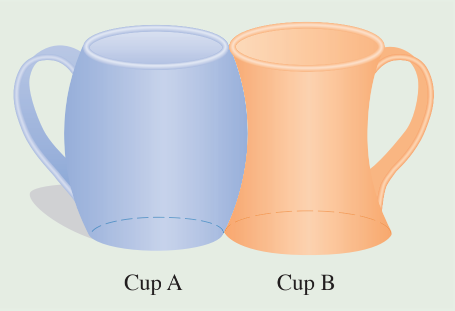
(b) Find the coordinates of the centroid of $\mathcal{R}$.

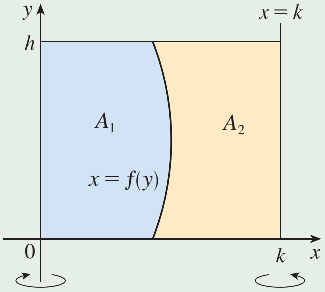
(c) Try to find values of m and n such that the centroid lies *outside* $\mathcal{R}$.

51. Prove Formulas 9.

DISCOVERY PROJECT COMPLEMENTARY COFFEE CUPS

Suppose you have a choice of two coffee cups of the type shown, one that bends outward and one inward, and you notice that they have the same height and their shapes fit together snugly. You wonder which cup holds more coffee. Of course you could fill one cup with water and pour it into the other one but, being a calculus student, you decide on a more mathematical approach. Ignoring the handles, you observe that both cups are surfaces of revolution, so you can think of the coffee as a volume of revolution.





1. Suppose the cups have height h, cup A is formed by rotating the curve $x = f(y)$ about the y-axis, and cup B is formed by rotating the same curve about the line $x = k$. Find the value of k such that the two cups hold the same amount of coffee.
2. What does your result from Problem 1 say about the areas A_1 and A_2 shown in the figure?
3. Use Pappus's Theorem to explain your result in Problems 1 and 2.
4. Based on your own measurements and observations, suggest a value for h and an equation for $x = f(y)$ and calculate the amount of coffee that each cup holds.

8.4 Applications to Economics and Biology

In this section we consider some applications of integration to economics (consumer surplus) and biology (blood flow, cardiac output). Additional applications are described in the exercises.

Consumer Surplus

Recall from Section 4.7 that the demand function $p(x)$ is the price that a company can charge in order to sell x units of a commodity. Usually, selling larger quantities requires lowering prices, so the demand function is a decreasing function. The graph of a typical demand function, called a **demand curve**, is shown in Figure 1. If X is the amount of the commodity that can currently be sold, then $P = p(X)$ is the current selling price.

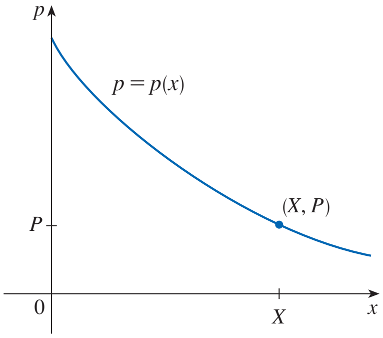


FIGURE 1
A typical demand curve

At a given price, some consumers who buy a good would be willing to pay more; they benefit by not having to. The difference between what a consumer is willing to pay and what the consumer actually pays for a good is called the **consumer surplus**. By finding the total consumer surplus among all purchasers of a good, economists can assess the overall benefit of a market to society.

To determine the total consumer surplus, we look at the demand curve and divide the interval $[0, X]$ into n subintervals, each of length $\Delta x = X/n$, and let $x_i^* = x_i$ be the right endpoint of the ith subinterval, as in Figure 2. According to the demand curve, x_{i-1} units would be purchased at a price of $p(x_{i-1})$ dollars per unit. To increase sales to x_i units, the price would have to be lowered to $p(x_i)$ dollars. In this case, an additional Δx units would be sold (but no more). In general, the consumers who would have paid $p(x_i)$ dollars placed a high value on the product; they would have paid what it was worth to them. So in paying only P dollars they have saved an amount of

$$(\text{savings per unit})(\text{number of units}) = [p(x_i) - P]\,\Delta x$$

Considering similar groups of willing consumers for each of the subintervals and adding the savings, we get the total savings:

$$\sum_{i=1}^{n} [p(x_i) - P]\,\Delta x$$

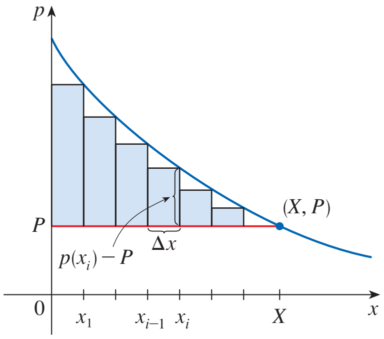


FIGURE 2

(This sum corresponds to the area enclosed by the rectangles in Figure 2.) If we let $n \to \infty$, this Riemann sum approaches the integral

$$\int_0^X [p(x) - P]\,dx \tag{1}$$

which gives the total consumer surplus for the commodity. It represents the amount of money saved by consumers in purchasing the commodity at price P, corresponding to an amount demanded of X. Figure 3 shows the interpretation of the consumer surplus as the area under the demand curve and above the line $p = P$.

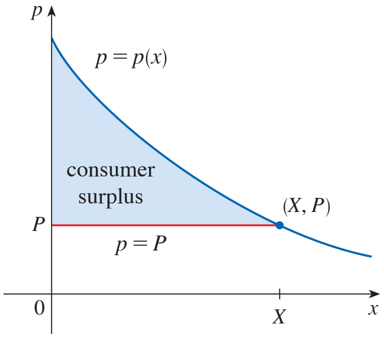


FIGURE 3

EXAMPLE 1 The demand for a product, in dollars, is

$$p = 1200 - 0.2x - 0.0001x^2$$

Find the consumer surplus when the sales level is 500.

SOLUTION Since the number of products sold is $X = 500$, the corresponding price is

$$P = 1200 - (0.2)(500) - (0.0001)(500)^2 = 1075$$

Therefore, from Definition 1, the total consumer surplus is

$$\begin{aligned}\int_0^{500} [p(x) - P]\,dx &= \int_0^{500} (1200 - 0.2x - 0.0001x^2 - 1075)\,dx \\ &= \int_0^{500} (125 - 0.2x - 0.0001x^2)\,dx \\ &= 125x - 0.1x^2 - (0.0001)\left(\frac{x^3}{3}\right)\Bigg]_0^{500} \\ &= (125)(500) - (0.1)(500)^2 - \frac{(0.0001)(500)^3}{3} \\ &= \$33{,}333.33\end{aligned}$$

■

Blood Flow

In Example 3.7.7 we discussed the law of laminar flow:

$$v(r) = \frac{P}{4\eta l}(R^2 - r^2)$$

which gives the velocity v of blood that flows along a blood vessel with radius R and length l at a distance r from the central axis, where P is the pressure difference between the ends of the vessel and η is the viscosity of the blood. Now, in order to compute the rate of blood flow, or *flux* (volume per unit time), we consider smaller, equally spaced radii $r_1, r_2, \ldots$. The approximate area of the ring (or washer) with inner radius r_{i-1} and outer radius r_i is

$$2\pi r_i\,\Delta r \qquad \text{where} \quad \Delta r = r_i - r_{i-1}$$

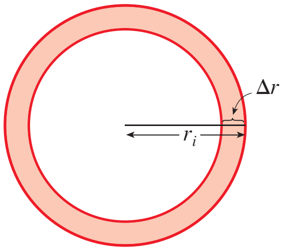


FIGURE 4

(See Figure 4.) If Δr is small, then the velocity is almost constant throughout this ring and can be approximated by $v(r_i)$. Thus the volume of blood per unit time that flows across the ring is approximately

$$(2\pi r_i\,\Delta r)\,v(r_i) = 2\pi r_i\,v(r_i)\,\Delta r$$

and the total volume of blood that flows across a cross-section per unit time is about

$$\sum_{i=1}^{n} 2\pi r_i\,v(r_i)\,\Delta r$$

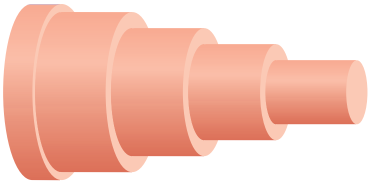


FIGURE 5

This approximation is illustrated in Figure 5. Notice that the velocity (and hence the volume per unit time) increases toward the center of the blood vessel. The approximation gets better as n increases. When we take the limit we get the exact value of the **flux** (or *discharge*), which is the volume of blood that passes a cross-section per unit time:

$$\begin{aligned} F &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi r_i\,v(r_i)\,\Delta r = \int_0^R 2\pi r\,v(r)\,dr \\ &= \int_0^R 2\pi r\,\frac{P}{4\eta l}(R^2 - r^2)\,dr \\ &= \frac{\pi P}{2\eta l}\int_0^R (R^2 r - r^3)\,dr = \frac{\pi P}{2\eta l}\left[R^2\frac{r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{r=R} \\ &= \frac{\pi P}{2\eta l}\left[\frac{R^4}{2} - \frac{R^4}{4}\right] = \frac{\pi P R^4}{8\eta l} \end{aligned}$$

The resulting equation

$$\boxed{2} \qquad F = \frac{\pi P R^4}{8\eta l}$$

is called **Poiseuille's Law**; it shows that the flux is proportional to the fourth power of the radius of the blood vessel.

Cardiac Output

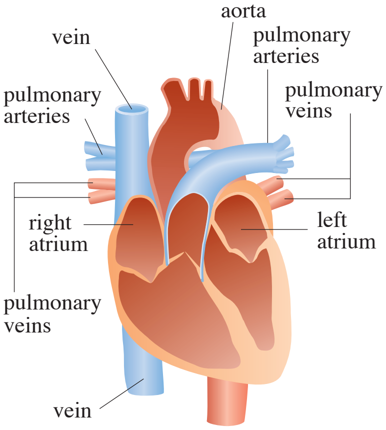


FIGURE 6

Figure 6 shows the human cardiovascular system. Blood returns from the body through the veins, enters the right atrium of the heart, and is pumped to the lungs through the pulmonary arteries for oxygenation. It then flows back into the left atrium through the pulmonary veins and then out to the rest of the body through the aorta. The **cardiac output** is the volume of blood pumped by the heart per unit time, that is, the rate of flow into the aorta.

The *dye dilution method* is used to measure the cardiac output. Dye is injected into the right atrium and flows through the heart into the aorta. A probe inserted into the aorta measures the concentration of the dye leaving the heart at equally spaced times over a time interval $[0, T]$ until the dye has cleared. Let $c(t)$ be the concentration of the dye at time t. If we divide $[0, T]$ into subintervals of equal length Δt, then the amount of dye that flows past the measuring point during the subinterval from $t = t_{i-1}$ to $t = t_i$ is approximately

$$(\text{concentration})(\text{volume}) = c(t_i)(F\,\Delta t)$$

where F is the rate of flow that we are trying to determine. Thus the total amount of dye is approximately

$$\sum_{i=1}^{n} c(t_i)F\,\Delta t = F\sum_{i=1}^{n} c(t_i)\,\Delta t$$

and, letting $n \longrightarrow \infty$, we find that the amount of dye is

$$A = F\int_0^T c(t)\,dt$$

Thus the cardiac output is given by

$$\boxed{3} \qquad F = \frac{A}{\displaystyle\int_0^T c(t)\,dt}$$

where the amount of dye A is known and the integral can be approximated from the concentration readings.

t	$c(t)$	t	$c(t)$
0	0	6	6.1
1	0.4	7	4.0
2	2.8	8	2.3
3	6.5	9	1.1
4	9.8	10	0
5	8.9		

EXAMPLE 2 A 5-mg dose (called a bolus) of dye is injected into a patient's right atrium. The concentration of the dye (in milligrams per liter) is measured in the aorta at one-second intervals as shown in the table. Estimate the cardiac output.

SOLUTION Here $A = 5$, $\Delta t = 1$, and $T = 10$. We use Simpson's Rule to approximate the integral of the concentration:

$$\begin{aligned}\int_0^{10} c(t)\,dt &\approx \tfrac{1}{3}[0 + 4(0.4) + 2(2.8) + 4(6.5) + 2(9.8) + 4(8.9)\\ &\qquad + 2(6.1) + 4(4.0) + 2(2.3) + 4(1.1) + 0]\\ &\approx 41.87\end{aligned}$$

Thus Formula 3 gives the cardiac output to be

$$F = \frac{A}{\displaystyle\int_0^{10} c(t)\,dt} \approx \frac{5}{41.87} \approx 0.12 \text{ L/s} = 7.2 \text{ L/min}$$

■

8.4 Exercises

1. The marginal cost function $C'(x)$ was defined to be the derivative of the cost function. (See Sections 3.7 and 4.7.) The marginal cost of producing x liters of orange juice is

$$C'(x) = 0.82 - 0.00003x + 0.000000003x^2$$

(measured in dollars per liter). The fixed start-up cost is $C(0) = \$18{,}000$. Use the Net Change Theorem to find the cost of producing the first 4000 liters of juice.

2. A company estimates that the marginal revenue (in dollars per unit) realized by selling x units of a product is $48 - 0.0012x$. Assuming the estimate is accurate, find the increase in revenue if sales increase from 5000 units to 10,000 units.

3. A mining company estimates that the marginal cost of extracting x tonnes of copper ore from a mine is $0.6 + 0.008x$, measured in thousands of dollars per ton. Start-up costs are \$100,000. What is the cost of extracting the first 50 tonnes of copper? What about the next 50 tonnes?

4. The demand function for a particular vacation package is $p(x) = 2000 - 46\sqrt{x}$. Find the consumer surplus when the sales level for the packages is 400. Illustrate by drawing the demand curve and identifying the consumer surplus as an area.

5. The demand function for a manufacturer's microwave oven is $p(x) = 870e^{-0.03x}$, where x is measured in thousands. Calculate the consumer surplus when the sales level for the ovens is 45,000.

6. If a demand curve is modeled by $p = 6 - (x/3500)$, find the consumer surplus when the selling price is \$2.80.

7. A concert-promoting company has been selling an average of 210 T-shirts at performances for \$18 each. The company estimates that for each dollar that the price is lowered, an additional 30 shirts will be sold. Find the demand function for the shirts and calculate the consumer surplus if the shirts are sold for \$15 each.

T **8.** A company modeled the demand curve for its product (in dollars) by the equation

$$p = \frac{800{,}000e^{-x/5000}}{x + 20{,}000}$$

Use a graph to estimate the sales level when the selling price is \$16. Then find (approximately) the consumer surplus for this sales level.

9–11 Producer Surplus The *supply function* $p_S(x)$ for a commodity gives the relation between the selling price and the number of units that manufacturers will produce at that price. For a higher price, manufacturers will produce more units, so p_S is an increasing function of x. Let X be the amount of the commodity currently produced and let $P = p_S(X)$ be the current price. Some producers would be willing to make and sell the commodity for a lower selling price and are therefore receiving more than their minimal price. The excess is called the *producer surplus*. An argument similar to that for consumer surplus shows that the surplus is given by the integral

$$\int_0^X [P - p_S(x)]\,dx$$

9. Calculate the producer surplus for the supply function $p_S(x) = 3 + 0.01x^2$ at the sales level $X = 10$. Illustrate by drawing the supply curve and identifying the producer surplus as an area.

10. If a supply curve is modeled by the equation $p = 125 + 0.002x^2$, find the producer surplus when the selling price is \$625.

T **11.** A manufacturer estimates that the supply curve for its product (in dollars) is

$$p = \sqrt{30 + 0.01xe^{0.001x}}$$

Find (approximately) the producer surplus when the selling price is \$30.

12. Market Equilibrium In a purely competitive market, the price of a good is naturally driven to the value where the quantity demanded by consumers matches the quantity made by producers, and the market is said to be in *equilibrium*. These values are the coordinates of the point of intersection of the supply and demand curves.

(a) Given the demand curve $p = 50 - \frac{1}{20}x$ and the supply curve $p = 20 + \frac{1}{10}x$ for a good, at what quantity and price is the market for the good in equilibrium?

(b) Find the consumer surplus and the producer surplus when the market is in equilibrium. Illustrate by sketching the supply and demand curves and identifying the surpluses as areas.

13–14 Total Surplus The sum of consumer surplus and producer surplus is called the *total surplus*; it is one measure economists use as an indicator of the economic health of a society. Total surplus is maximized when the market for a good is in equilibrium.

13. (a) The demand function for an electronics company's car stereos is $p(x) = 228.4 - 18x$ and the supply function is $p_S(x) = 27x + 57.4$, where x is measured in thousands. At what quantity is the market for the stereos in equilibrium?

(b) Compute the maximum total surplus for the stereos.

14. A camera company estimates that the demand function for its new digital camera is $p(x) = 312e^{-0.14x}$ and the supply function is estimated to be $p_S(x) = 26e^{0.2x}$, where x is measured in thousands. Compute the maximum total surplus.

---

15. If the amount of capital that a company has at time t is $f(t)$, then the derivative, $f'(t)$, is called the *net investment flow*. Suppose that the net investment flow is $\sqrt{t}$ million dollars per year (where t is measured in years). Find the increase in capital (the *capital formation*) from the fourth year to the eighth year.

16. If revenue flows into a company at a rate of $f(t) = 9000\sqrt{1 + 2t}$, where t is measured in years and $f(t)$ is measured in dollars per year, find the total revenue obtained in the first four years.

17. Future Value of Income If income is continuously collected at a rate of $f(t)$ dollars per year and will be invested at a constant interest rate r (compounded continuously) for a period of T years, then the *future value* of the income is given by $\int_0^T f(t)\,e^{r(T-t)}\,dt$. Compute the future value after 6 years for income received at a rate of $f(t) = 8000\,e^{0.04t}$ dollars per year and invested at 6.2% interest.

18. Present Value of Income The *present value* of an income stream is the amount that would need to be invested now to match the future value as described in Exercise 17 and is given by $\int_0^T f(t)\,e^{-rt}\,dt$. Find the present value of the income stream in Exercise 17.

19. *Pareto's Law of Income* states that the number of people with incomes between $x = a$ and $x = b$ is $N = \int_a^b Ax^{-k}\,dx$, where A and k are constants with $A > 0$ and $k > 1$. The average income of these people is

$$\bar{x} = \frac{1}{N}\int_a^b Ax^{1-k}\,dx$$

Calculate $\bar{x}$.

20. A hot, wet summer is causing a mosquito population explosion in a lake resort area. The number of mosquitoes is increasing at an estimated rate of $2200 + 10e^{0.8t}$ per week (where t is measured in weeks). By how much does the mosquito population increase between the fifth and ninth weeks of summer?

21. Use Poiseuille's Law to calculate the rate of flow in a small human artery where we can take $\eta = 0.027$, $R = 0.008$ cm, $l = 2$ cm, and $P = 4000\text{ dynes/cm}^2$.

22. High blood pressure results from constriction of the arteries. To maintain a normal flow rate (flux), the heart has to pump harder, thus increasing the blood pressure. Use Poiseuille's Law to show that if R_0 and P_0 are normal values of the radius and pressure in an artery and the constricted values are R and P, then for the flux to remain constant, P and R are related by the equation

$$\frac{P}{P_0} = \left(\frac{R_0}{R}\right)^4$$

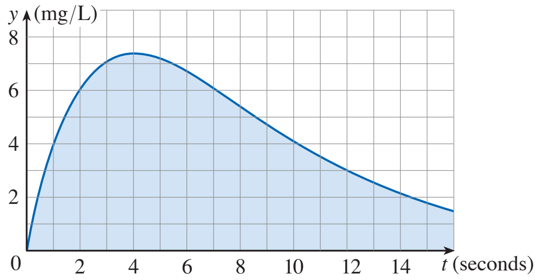
Deduce that if the radius of an artery is reduced to three-fourths of its former value, then the pressure is more than tripled.

23. A particular dye dilution method measures cardiac output with 6 mg of dye. The dye concentrations, in mg/L, are modeled by $c(t) = 20te^{-0.6t}$, $0 \leqslant t \leqslant 10$, where t is measured in seconds. Find the cardiac output.

24. After a 5.5-mg injection of dye, the readings of dye concentration, in mg/L, at two-second intervals are as shown in the table. Use Simpson's Rule to estimate the cardiac output.

t	$c(t)$	t	$c(t)$
0	0.0	10	4.3
2	4.1	12	2.5
4	8.9	14	1.2
6	8.5	16	0.2
8	6.7		

25. The graph of the concentration function $c(t)$ is shown after a 7-mg injection of dye into an atrium of a heart. Use Simpson's Rule to estimate the cardiac output.



8.5 Probability

Probability Density Functions

Calculus plays a role in the analysis of random behavior. Suppose we consider the cholesterol level of a person chosen at random from a certain age group, or the height of an adult female chosen at random, or the lifetime of a randomly chosen battery of a certain type. Such quantities are called **continuous random variables** because their values actually range over an interval of real numbers, although they might be measured or recorded only to the nearest integer. We might want to know the probability that a blood cholesterol level is greater than 250, or the probability that the height of an adult female is between 150 and 180 centimeters, or the probability that the battery we are buying lasts between 100 and 200 hours. If X represents the lifetime of that type of battery, we denote this last probability as follows:

$$P(100 \leqslant X \leqslant 200)$$

According to the frequency interpretation of probability, this number is the long-run proportion of all batteries of the specified type whose lifetimes are between 100 and 200 hours. Since it represents a proportion, the probability naturally falls between 0 and 1.

Note that we always use *intervals* of values when working with probability density functions. We wouldn't, for instance, use a density function to find the probability that X *equals* a.

Every continuous random variable X has a **probability density function** f. This means that the probability that X lies between a and b is found by integrating f from a to b:

$$P(a \leqslant X \leqslant b) = \int_a^b f(x)\,dx \tag{1}$$

For example, Figure 1 shows the graph of a model for the probability density function f for a random variable X defined to be the height in inches of an adult female in the United States (according to data from the National Health Survey). The probability that the height of a woman chosen at random from this population is between 150 and 180 centimeters is equal to the area under the graph of f from 150 to 180.

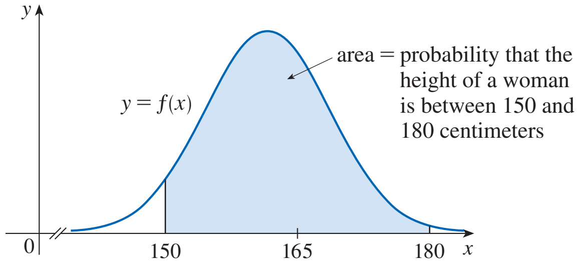


FIGURE 1
Probability density function for the height of an adult female

In general, the probability density function f of a random variable X satisfies the condition $f(x) \geqslant 0$ for all x. Because probabilities are measured on a scale from 0 to 1, it follows that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \tag{2}$$

EXAMPLE 1 Let $f(x) = 0.006x(10 - x)$ for $0 \leqslant x \leqslant 10$ and $f(x) = 0$ for all other values of x.
(a) Verify that f is a probability density function.
(b) Find $P(4 \leqslant X \leqslant 8)$.

SOLUTION
(a) For $0 \leqslant x \leqslant 10$ we have $0.006x(10 - x) \geqslant 0$, so $f(x) \geqslant 0$ for all x. We also need to check that Equation 2 is satisfied:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{10} 0.006x(10 - x)\,dx = 0.006\int_0^{10} (10x - x^2)\,dx$$

$$= 0.006\Big[5x^2 - \tfrac{1}{3}x^3\Big]_0^{10} = 0.006\big(500 - \tfrac{1000}{3}\big) = 1$$

Therefore f is a probability density function.

(b) The probability that X lies between 4 and 8 is

$$P(4 \leqslant X \leqslant 8) = \int_4^8 f(x)\,dx = 0.006\int_4^8 (10x - x^2)\,dx$$

$$= 0.006\Big[5x^2 - \tfrac{1}{3}x^3\Big]_4^8 = 0.544$$

■

EXAMPLE 2 Phenomena such as waiting times and equipment failure times are commonly modeled by exponentially decreasing probability density functions. Find the exact form of such a function.

SOLUTION Think of the random variable as being the time you wait on hold before a customer service agent answers your phone call. So instead of x, let's use t to represent time, in minutes. If f is the probability density function and you call at time $t = 0$, then, from Definition 1, $\int_0^2 f(t)\,dt$ represents the probability that an agent answers within the first two minutes and $\int_4^5 f(t)\,dt$ is the probability that your call is answered during the fifth minute.

It's clear that $f(t) = 0$ for $t < 0$ (the agent can't answer before you place the call). For $t > 0$ we are told to use an exponentially decreasing function, that is, a function of the form $f(t) = Ae^{-ct}$, where A and c are positive constants. Thus

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ Ae^{-ct} & \text{if } t \geqslant 0 \end{cases}$$

We use Equation 2 to determine the value of A:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{0} f(t)\,dt + \int_0^{\infty} f(t)\,dt \\ &= \int_0^{\infty} Ae^{-ct}\,dt = \lim_{x \to \infty} \int_0^x Ae^{-ct}\,dt \\ &= \lim_{x \to \infty} \left[-\frac{A}{c} e^{-ct} \right]_0^x = \lim_{x \to \infty} \frac{A}{c}(1 - e^{-cx}) \\ &= \frac{A}{c} \end{aligned}$$

Therefore $A/c = 1$ and so $A = c$. Thus every exponential density function has the form

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ ce^{-ct} & \text{if } t \geqslant 0 \end{cases}$$

A typical graph is shown in Figure 2. ■

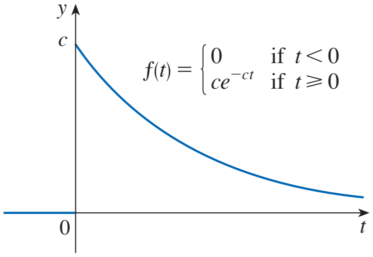


FIGURE 2
An exponential density function

Average Values

Suppose you're waiting for a company to answer your phone call and you wonder how long, on average, you can expect to wait. Let $f(t)$ be the corresponding density function, where t is measured in minutes, and think of a sample of N people who have called this company. Most likely, none of them had to wait more than an hour, so let's restrict our attention to the interval $0 \leqslant t \leqslant 60$. Let's divide that interval into n intervals of length Δt and endpoints $0, t_1, t_2, \ldots, t_n = 60$. (Think of Δt as lasting a minute, or half a minute, or 10 seconds, or even a second.) The probability that somebody's call gets answered during the time period from t_{i-1} to t_i is the area under the curve $y = f(t)$ from t_{i-1} to t_i, which is approximately equal to $f(\bar{t}_i)\,\Delta t$. (This is the area of the approximating rectangle in Figure 3, where $\bar{t}_i$ is the midpoint of the interval.)

Since the long-run proportion of calls that get answered in the time period from t_{i-1} to t_i is $f(\bar{t}_i)\,\Delta t$, we expect that, out of our sample of N callers, the number whose call was answered in that time period is approximately $Nf(\bar{t}_i)\,\Delta t$ and the time that each waited is about $\bar{t}_i$. Therefore the total time they waited is the product of these numbers:

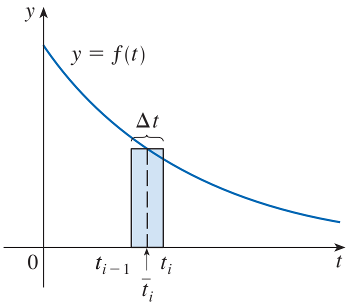


FIGURE 3

approximately $\bar{t}_i[Nf(\bar{t}_i)\,\Delta t]$. Adding over all such intervals, we get the approximate total of everybody's waiting times:

$$\sum_{i=1}^{n} N\bar{t}_i\, f(\bar{t}_i)\,\Delta t$$

If we now divide by the number of callers N, we get the approximate *average* waiting time:

$$\sum_{i=1}^{n} \bar{t}_i\, f(\bar{t}_i)\,\Delta t$$

We recognize this as a Riemann sum for the function $t\,f(t)$. As the time interval shrinks (that is, $\Delta t \to 0$ and $n \to \infty$), this Riemann sum approaches the integral

$$\int_0^{60} t\,f(t)\,dt$$

This integral is called the *mean waiting time.*

In general, the **mean** of any probability density function f is defined to be

It is traditional to denote the mean by the Greek letter μ (mu).

$$\mu = \int_{-\infty}^{\infty} x\,f(x)\,dx$$

The mean can be interpreted as the long-run average value of the random variable X. It can also be interpreted as a measure of centrality of the probability density function.

The expression for the mean resembles an integral we have seen before. If $\mathscr{R}$ is the region that lies under the graph of f, we know from Formula 8.3.8 that the x-coordinate of the centroid of $\mathscr{R}$ is

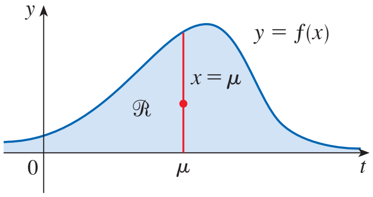


FIGURE 4
$\mathscr{R}$ balances at a point on the line $x = \mu$

$$\bar{x} = \frac{\int_{-\infty}^{\infty} x\,f(x)\,dx}{\int_{-\infty}^{\infty} f(x)\,dx} = \int_{-\infty}^{\infty} x\,f(x)\,dx = \mu$$

because of Equation 2. So a thin plate in the shape of $\mathscr{R}$ balances at a point on the vertical line $x = \mu$. (See Figure 4.)

EXAMPLE 3 Find the mean of the exponential distribution of Example 2:

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ ce^{-ct} & \text{if } t \geqslant 0 \end{cases}$$

SOLUTION According to the definition of a mean, we have

$$\mu = \int_{-\infty}^{\infty} t\,f(t)\,dt = \int_0^{\infty} tce^{-ct}\,dt$$

To evaluate this integral we use integration by parts, with $u = t$ and $dv = ce^{-ct}\,dt$, so $du = dt$ and $v = -e^{-ct}$:

$$\int_0^{\infty} tce^{-ct}\,dt = \lim_{x\to\infty}\int_0^{x} tce^{-ct}\,dt = \lim_{x\to\infty}\left(-te^{-ct}\Big]_0^x + \int_0^x e^{-ct}\,dt\right)$$

$$= \lim_{x\to\infty}\left(-xe^{-cx} + \frac{1}{c} - \frac{e^{-cx}}{c}\right) = \frac{1}{c}$$

(The limit of the first term is 0 by l'Hospital's Rule.)

The mean is $\mu = 1/c$, so we can rewrite the probability density function as

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \mu^{-1}e^{-t/\mu} & \text{if } t \geqslant 0 \end{cases}$$ ■

EXAMPLE 4 Suppose the average waiting time for a customer's call to be answered by a customer service agent is five minutes.

(a) Find the probability that a call is answered during the first minute, assuming that an exponential distribution is appropriate.
(b) Find the probability that a customer waits on hold for more than five minutes before the call is answered.

SOLUTION

(a) We are given that the mean of the exponential distribution is $\mu = 5$ min and so, from the result of Example 3, we know that the probability density function is

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ 0.2e^{-t/5} & \text{if } t \geqslant 0 \end{cases}$$

where t is measured in minutes. Thus the probability that a call is answered during the first minute is

$$\begin{aligned} P(0 \leqslant T \leqslant 1) &= \int_0^1 f(t)\,dt \\ &= \int_0^1 0.2e^{-t/5}\,dt = 0.2(-5)e^{-t/5}\Big]_0^1 \\ &= 1 - e^{-1/5} \approx 0.1813 \end{aligned}$$

So about 18% of customers' calls are answered during the first minute.

(b) The probability that a customer waits on hold more than five minutes is

$$\begin{aligned} P(T > 5) &= \int_5^\infty f(t)\,dt = \int_5^\infty 0.2e^{-t/5}\,dt \\ &= \lim_{x\to\infty}\int_5^x 0.2e^{-t/5}\,dt = \lim_{x\to\infty}\left(e^{-1} - e^{-x/5}\right) \\ &= \frac{1}{e} - 0 \approx 0.368 \end{aligned}$$

About 37% of customers wait on hold more than five minutes before their calls are answered. ■

Notice the result of Example 4(b): even though the mean waiting time is 5 minutes, only 37% of callers wait more than 5 minutes. The reason is that some callers have to wait much longer (maybe 10 or 15 minutes), and this brings up the average.

Another measure of centrality of a probability density function is the *median*. That is a number m such that half the callers have a waiting time less than m and the other callers have a waiting time longer than m. In general, the **median** of a probability density function is the number m such that

$$\int_m^\infty f(x)\,dx = \tfrac{1}{2}$$

This means that half the area under the graph of f lies to the right of m. In Exercise 9 you are asked to show that the median waiting time for the company described in Example 4 is approximately 3.5 minutes.

Normal Distributions

Many important random phenomena—such as test scores on aptitude tests, heights and weights of individuals from a homogeneous population, annual rainfall in a given location—are modeled by a **normal distribution**. This means that the probability density function of the random variable X is a member of the family of functions

$$\boxed{3} \qquad f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

The standard deviation is denoted by the lowercase Greek letter σ (sigma).

You can verify that the mean for this function is μ. The positive constant σ is called the **standard deviation**; it measures how spread out the values of X are. From the bell-shaped graphs of members of the family in Figure 5, we see that for small values of σ the values of X are clustered about the mean, whereas for larger values of σ the values of X are more spread out. Statisticians have methods for using sets of data to estimate μ and σ.

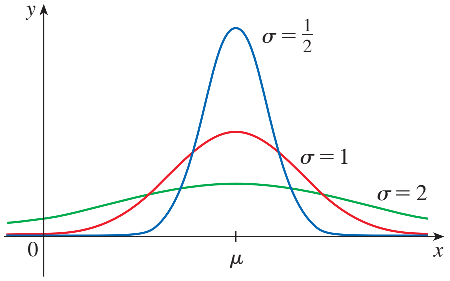


FIGURE 5
Normal distributions

The factor $1/(\sigma\sqrt{2\pi})$ is needed to make f a probability density function. In fact, it can be verified using the methods of multivariable calculus (see Exercise 15.3.48) that

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}\,dx = 1$$

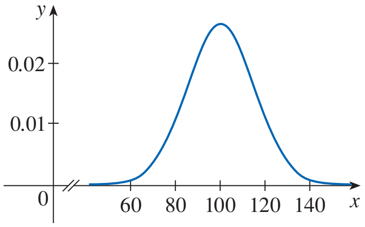


FIGURE 6

EXAMPLE 5 Intelligence Quotient (IQ) scores are distributed normally with mean 100 and standard deviation 15. (Figure 6 shows the corresponding probability density function.)

(a) What percentage of the population has an IQ score between 85 and 115?

(b) What percentage of the population has an IQ above 140?

SOLUTION

(a) Since IQ scores are normally distributed, we use the probability density function given by Equation 3 with $\mu = 100$ and $\sigma = 15$:

$$P(85 \leqslant X \leqslant 115) = \int_{85}^{115} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/(2\cdot 15^2)}\,dx$$

Recall from Section 7.5 that the function $y = e^{-x^2}$ doesn't have an elementary antiderivative, so we can't evaluate the integral exactly. But we can use the numerical integration capability of a calculator or computer (or the Midpoint Rule or Simpson's Rule) to estimate the integral. Doing so, we find that

$$P(85 \leqslant X \leqslant 115) \approx 0.68$$

So about 68% of the population has an IQ score between 85 and 115, that is, within one standard deviation of the mean.

(b) The probability that the IQ score of a person chosen at random is more than 140 is

$$P(X > 140) = \int_{140}^{\infty} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/450}\, dx$$

To avoid the improper integral we could approximate it by the integral from 140 to 200. (It's quite safe to say that people with an IQ over 200 are extremely rare.) Then

$$P(X > 140) \approx \int_{140}^{200} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/450}\, dx \approx 0.0038$$

Therefore about 0.4% of the population has an IQ score over 140. ■

8.5 Exercises

1. Let $f(x)$ be the probability density function for the lifetime of a manufacturer's highest quality car tire, where x is measured in kilometers. Explain the meaning of each integral.

(a) $\int_{50,000}^{65,000} f(x)\, dx$ (b) $\int_{40,000}^{\infty} f(x)\, dx$

2. Let $f(t)$ be the probability density function for the time it takes you to drive to school, where t is measured in minutes. Express the following probabilities as integrals.
(a) The probability that you drive to school in less than 15 minutes
(b) The probability that it takes you more than half an hour to get to school

3. Let $f(x) = 30x^2(1 - x)^2$ for $0 \leqslant x \leqslant 1$ and $f(x) = 0$ for all other values of x.
(a) Verify that f is a probability density function.
(b) Find $P\left(X \leqslant \frac{1}{3}\right)$.

4. The density function

$$f(x) = \frac{e^{3-x}}{(1 + e^{3-x})^2}$$

is an example of a *logistic distribution.*
(a) Verify that f is a probability density function.
(b) Find $P(3 \leqslant X \leqslant 4)$.
(c) Graph f. What does the mean appear to be? What about the median?

5. Let $f(x) = c/(1 + x^2)$.
(a) For what value of c is f a probability density function?
(b) For that value of c, find $P(-1 < X < 1)$.

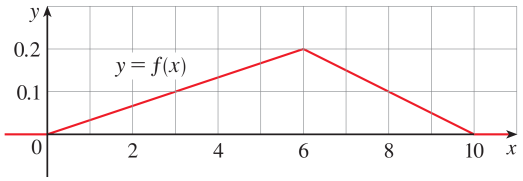
6. Let $f(x) = k(3x - x^2)$ if $0 \leqslant x \leqslant 3$ and $f(x) = 0$ if $x < 0$ or $x > 3$.
(a) For what value of k is f a probability density function?
(b) For that value of k, find $P(X > 1)$.
(c) Find the mean.

7. A spinner from a board game randomly indicates a real number between 0 and 10. The spinner is fair in the sense that it indicates a number in a given interval with the same probability as it indicates a number in any other interval of the same length.
(a) Explain why the function

$$f(x) = \begin{cases} 0.1 & \text{if } 0 \leqslant x \leqslant 10 \\ 0 & \text{if } x < 0 \text{ or } x > 10 \end{cases}$$

is a probability density function for the spinner's values.
(b) What does your intuition tell you about the value of the mean? Check your guess by evaluating an integral.

8. (a) Explain why the function whose graph is shown is a probability density function.
(b) Use the graph to find the following probabilities:
(i) $P(X < 3)$ (ii) $P(3 \leqslant X \leqslant 8)$
(c) Calculate the mean.



9. Show that the median waiting time for a phone call to the company described in Example 4 is about 3.5 minutes.

10. (a) A type of light bulb is labeled as having an average lifetime of 1000 hours. It's reasonable to model the probability of failure of these bulbs by an exponential density function with mean $\mu = 1000$. Use this model to find the probability that a bulb
(i) fails within the first 200 hours,
(ii) burns for more than 800 hours.
(b) What is the median lifetime of these light bulbs?

11. An online retailer has determined that the average time for credit card transactions to be electronically approved is 1.6 seconds.
(a) Use an exponential density function to find the probability that a customer waits less than a second for credit card approval.

(b) Find the probability that a customer waits more than 3 seconds.

(c) What is the minimum approval time for the slowest 5% of transactions?

12. The time between infection and the display of symptoms for streptococcal sore throat is a random variable whose probability density function can be approximated by $f(t) = \frac{1}{15{,}676}t^2e^{-0.05t}$ if $0 \leqslant t \leqslant 150$ and $f(t) = 0$ otherwise (t measured in hours).

(a) What is the probability that an infected patient will display symptoms within the first 48 hours?

(b) What is the probability that an infected patient will not display symptoms until after 36 hours?

Source: Adapted from P. Sartwell, "The Distribution of Incubation Periods of Infectious Disease," *American Journal of Epidemiology* 141 (1995): 386–94.

13. REM sleep is the phase of sleep when most active dreaming occurs. In a study, the amount of REM sleep during the first four hours of sleep was described by a random variable T with probability density function

$$f(t) = \begin{cases} \frac{1}{1600}t & \text{if } 0 \leqslant t \leqslant 40 \\ \frac{1}{20} - \frac{1}{1600}t & \text{if } 40 < t \leqslant 80 \\ 0 & \text{otherwise} \end{cases}$$

where t is measured in minutes.

(a) What is the probability that the amount of REM sleep is between 30 and 60 minutes?

(b) Find the mean amount of REM sleep.

14. According to the National Health Survey, the heights of adult males in the United States are normally distributed with mean 175 centimeters and standard deviation 7 centimeters.

(a) What is the probability that an adult male chosen at random is between 165 centimeters and 185 centimeters tall?

(b) What percentage of the adult male population is more than 180 centimeters tall?

15. The "Garbage Project" at the University of Arizona reports that the amount of paper discarded by households per week is normally distributed with mean 4.3 kg and standard deviation 1.9 kg. What percentage of households throw out at least 5 kg of paper a week?

16. Boxes are labeled as containing 500 g of cereal. The machine filling the boxes produces weights that are normally distributed with standard deviation 12 g.

(a) If the target weight is 500 g, what is the probability that the machine produces a box with less than 480 g of cereal?

(b) Suppose a law states that no more than 5% of a manufacturer's cereal boxes may contain less than the stated weight of 500 g. At what target weight should the manufacturer set its filling machine?

17. The speeds of vehicles on a highway with speed limit 100 km/h are normally distributed with mean 112 km/h and standard deviation 8 km/h.

(a) What is the probability that a randomly chosen vehicle is traveling at a legal speed?

(b) If police are instructed to ticket motorists driving 125 km/h or more, what percentage of motorists are targeted?

18. Show that the probability density function for a normally distributed random variable has inflection points at $x = \mu \pm \sigma$.

19. For any normal distribution, find the probability that the random variable lies within two standard deviations of the mean.

20. The standard deviation for a random variable with probability density function f and mean μ is defined by

$$\sigma = \left[\int_{-\infty}^{\infty} (x - \mu)^2 f(x)\,dx\right]^{1/2}$$

Find the standard deviation for an exponential density function with mean μ.

21. The hydrogen atom is composed of one proton in the nucleus and one electron, which moves about the nucleus. In the quantum theory of atomic structure, it is assumed that the electron does not move in a well-defined orbit. Instead, it occupies a state known as an *orbital*, which may be thought of as a "cloud" of negative charge surrounding the nucleus. At the state of lowest energy, called the *ground state*, or *1s-orbital*, the shape of this cloud is assumed to be a sphere centered at the nucleus. This sphere is described in terms of the probability density function

$$p(r) = \frac{4}{a_0^3}r^2e^{-2r/a_0} \qquad r \geqslant 0$$

where a_0 is the *Bohr radius* ($a_0 \approx 5.59 \times 10^{-11}$ m). The integral

$$P(r) = \int_0^r \frac{4}{a_0^3}s^2e^{-2s/a_0}\,ds$$

gives the probability that the electron will be found within the sphere of radius r meters centered at the nucleus.

(a) Verify that $p(r)$ is a probability density function.

(b) Find $\lim_{r\to\infty} p(r)$. For what value of r does $p(r)$ have its maximum value?

(c) Graph the density function.

(d) Find the probability that the electron will be within the sphere of radius $4a_0$ centered at the nucleus.

(e) Calculate the mean distance of the electron from the nucleus in the ground state of the hydrogen atom.

8 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. (a) How is the length of a curve defined?
(b) Write an expression for the length of a smooth curve given by $y = f(x)$, $a \leqslant x \leqslant b$.
(c) What if x is given as a function of y?

2. (a) Write an expression for the surface area of the surface obtained by rotating the curve $y = f(x)$, $a \leqslant x \leqslant b$, about the x-axis.
(b) What if x is given as a function of y?
(c) What if the curve is rotated about the y-axis?

3. Describe how we can find the hydrostatic force against a vertical wall submersed in a fluid.

4. (a) What is the physical significance of the center of mass of a thin plate?
(b) If the plate lies between $y = f(x)$ and $y = 0$, where $a \leqslant x \leqslant b$, write expressions for the coordinates of the center of mass.

5. What does the Theorem of Pappus say?

6. Given a demand function $p(x)$, explain what is meant by the consumer surplus when the amount of a commodity currently available is X and the current selling price is P. Illustrate with a sketch.

7. (a) What is cardiac output?
(b) Explain how the cardiac output can be measured by the dye dilution method.

8. What is a probability density function? What properties does such a function have?

9. Suppose $f(x)$ is the probability density function for the mass of a female college student, where x is measured in kilograms.
(a) What is the meaning of the integral $\int_0^{60} f(x)\,dx$?
(b) Write an expression for the mean of this density function.
(c) How can we find the median of this density function?

10. What is a normal distribution? What is the significance of the standard deviation?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. The arc lengths of the curves $y = f(x)$ and $y = f(x) + c$ for $a \leqslant x \leqslant b$ are equal.

2. If the curve $y = f(x)$, $a \leqslant x \leqslant b$, lies above the x-axis and if $c > 0$, then the areas of the surfaces obtained by revolving $y = f(x)$ and $y = f(x) + c$ about the x-axis are equal.

3. If $f(x) \leqslant g(x)$ for $a \leqslant x \leqslant b$, then the arc length of the curve $y = f(x)$ for $a \leqslant x \leqslant b$ is less than or equal to the arc length of the curve $y = g(x)$ for $a \leqslant x \leqslant b$.

4. The length of the curve $y = x^3$, $0 \leqslant x \leqslant 1$, is $L = \int_0^1 \sqrt{1 + x^6}\,dx$.

5. If f is continuous, $f(0) = 0$, and $f(3) = 4$, then the arc length of the curve $y = f(x)$ for $0 \leqslant x \leqslant 3$ is at least 5.

6. The center of mass of a lamina of uniform density ρ depends only on the shape of the lamina and not on ρ.

7. Hydrostatic pressure on a dam depends only on the water level at the dam and not on the size of the reservoir created by the dam.

8. If f is a probability density function, then $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

EXERCISES

1–3 Find the length of the curve.

1. $y = 4(x - 1)^{3/2}, \quad 1 \leqslant x \leqslant 4$

2. $y = 2\ln\left(\sin \frac{1}{2}x\right), \quad \pi/3 \leqslant x \leqslant \pi$

3. $12x = 4y^3 + 3y^{-1}, \quad 1 \leqslant y \leqslant 3$

4. (a) Find the length of the curve

$$y = \frac{x^4}{16} + \frac{1}{2x^2} \qquad 1 \leqslant x \leqslant 2$$

(b) Find the area of the surface obtained by rotating the curve in part (a) about the y-axis.

T **5.** Let C be the arc of the curve $y = 2/(x + 1)$ from the point $(0, 2)$ to $\left(3, \frac{1}{2}\right)$. Find a numerical approximation for each of the following, correct to four decimal places.

(a) The length of C

(b) The area of the surface obtained by rotating C about the x-axis

(c) The area of the surface obtained by rotating C about the y-axis

6. (a) The curve $y = x^2$, $0 \leqslant x \leqslant 1$, is rotated about the y-axis. Find the area of the resulting surface.

(b) Find the area of the surface obtained by rotating the curve in part (a) about the x-axis.

7. Use Simpson's Rule with $n = 10$ to estimate the length of the sine curve $y = \sin x$, $0 \leqslant x \leqslant \pi$. Round your answer to four decimal places.

8. (a) Set up, but do not evaluate, an integral for the area of the surface obtained by rotating the sine curve in Exercise 7 about the x-axis.

T (b) Evaluate your integral correct to four decimal places.

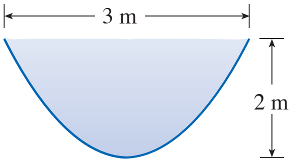
9. Find the length of the curve

$$y = \int_1^x \sqrt{\sqrt{t} - 1}\, dt \qquad 1 \leqslant x \leqslant 16$$

10. Find the area of the surface obtained by rotating the curve in Exercise 9 about the y-axis.

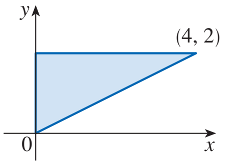
11. A gate in an irrigation canal is constructed in the form of a trapezoid 1 m wide at the bottom, 2 m wide at the top, and 1 m high. It is placed vertically in the canal so that the water just covers the gate. Find the hydrostatic force on one side of the gate.

12. A trough is filled with water and its vertical ends have the shape of the parabolic region in the figure. Find the hydrostatic force on one end of the trough.

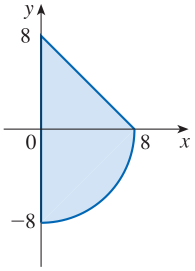


13–14 Find the centroid of the region shown.

13.



14.



15–16 Find the centroid of the region bounded by the given curves.

15. $y = \frac{1}{2}x, \quad y = \sqrt{x}$

16. $y = \sin x, \quad y = 0, \quad x = \pi/4, \quad x = 3\pi/4$

17. Find the volume obtained when the circle of radius 1 with center $(1, 0)$ is rotated about the y-axis.

18. Use the Theorem of Pappus and the fact that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ to find the centroid of the semicircular region bounded by the curve $y = \sqrt{r^2 - x^2}$ and the x-axis.

19. The demand function for a commodity is given by

$$p = 2000 - 0.1x - 0.01x^2$$

Find the consumer surplus when the sales level is 100.

20. After a 6-mg injection of dye into an atrium of a heart, the readings of dye concentration at two-second intervals are as shown in the table. Use Simpson's Rule to estimate the cardiac output.

t	$c(t)$	t	$c(t)$
0	0	14	4.7
2	1.9	16	3.3
4	3.3	18	2.1
6	5.1	20	1.1
8	7.6	22	0.5
10	7.1	24	0
12	5.8		

21. (a) Explain why the function

$$f(x) = \begin{cases} \dfrac{\pi}{20} \sin \dfrac{\pi x}{10} & \text{if } 0 \leqslant x \leqslant 10 \\ 0 & \text{if } x < 0 \text{ or } x > 10 \end{cases}$$

is a probability density function.

(b) Find $P(X < 4)$.

(c) Calculate the mean. Is the value what you would expect?

22. Lengths of human pregnancies are normally distributed with mean 268 days and standard deviation 15 days. What percentage of pregnancies last between 250 days and 280 days?

23. The length of time spent waiting in line at a certain bank is modeled by an exponential density function with mean 8 minutes.

(a) What is the probability that a customer is served in the first 3 minutes?

(b) What is the probability that a customer has to wait more than 10 minutes?

(c) What is the median waiting time?

Problems Plus

1. Find the area of the region $S = \{(x, y) \mid x \geqslant 0,\ y \leqslant 1,\ x^2 + y^2 \leqslant 4y\}$.

2. Find the centroid of the region enclosed by the loop of the curve $y^2 = x^3 - x^4$.

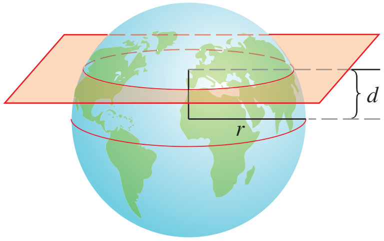
3. If a sphere of radius r is sliced by a plane whose distance from the center of the sphere is d, then the sphere is divided into two pieces called segments of one base (see the first figure). The corresponding surfaces are called *spherical zones of one base.*

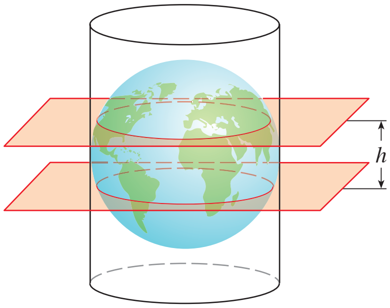
(a) Determine the surface areas of the two spherical zones indicated in the first figure.

(b) Determine the approximate area of the Arctic Ocean by assuming that it is approximately circular in shape, with center at the North Pole and "circumference" at 75° north latitude. Use $r = 6370$ km for the radius of the earth.

(c) A sphere of radius r is inscribed in a right circular cylinder of radius r. Two planes perpendicular to the central axis of the cylinder and a distance h apart cut off a *spherical zone of two bases* on the sphere (see the second figure). Show that the surface area of the spherical zone equals the surface area of the region that the two planes cut off on the cylinder.

(d) The *Torrid Zone* is the region on the surface of the earth that is between the Tropic of Cancer (23.45° north latitude) and the Tropic of Capricorn (23.45° south latitude). What is the area of the Torrid Zone?





4. (a) Show that an observer at height H above the north pole of a sphere of radius r can see a part of the sphere that has area

$$\frac{2\pi r^2 H}{r + H}$$

(b) Two spheres with radii r and R are placed so that the distance between their centers is d, where $d > r + R$. Where should a light be placed on the line joining the centers of the spheres in order to illuminate the largest total surface?

5. Suppose that the density of seawater, $\rho = \rho(z)$, varies with the depth z below the surface.

(a) Show that the hydrostatic pressure is governed by the differential equation

$$\frac{dP}{dz} = \rho(z)g$$

where g is the acceleration due to gravity. Let P_0 and ρ_0 be the pressure and density at $z = 0$. Express the pressure at depth z as an integral.

(b) Suppose the density of seawater at depth z is given by $\rho = \rho_0 e^{z/H}$, where H is a positive constant. Find the total force, expressed as an integral, exerted on a vertical circular porthole of radius r whose center is located at a distance $L > r$ below the surface.

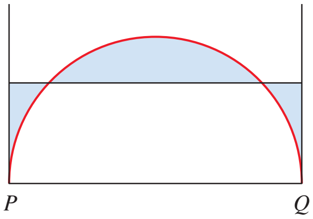


FIGURE FOR PROBLEM 6

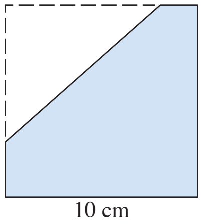
6. The figure shows a semicircle with radius 1, horizontal diameter PQ, and tangent lines at P and Q. At what height above the diameter should the horizontal line be placed so as to minimize the shaded area?

7. Let P be a pyramid with a square base of side $2b$ and suppose that S is a sphere with its center on the base of P and S is tangent to all eight edges of P. Find the height of P. Then find the volume of the intersection of S and P.

8. Consider a flat metal plate to be placed vertically underwater with its top 2 m below the surface of the water. Determine a shape for the plate so that if the plate is divided into any number of horizontal strips of equal height, the hydrostatic force on each strip is the same.

9. A uniform disk with radius 1 m is to be cut by a line so that the center of mass of the smaller piece lies halfway along a radius. How close to the center of the disk should the cut be made? (Express your answer correct to two decimal places.)

10. A triangle with area 30 cm^2 is cut from a corner of a square with side 10 cm, as shown in the figure. If the centroid of the remaining region is 4 cm from the right side of the square, how far is it from the bottom of the square?



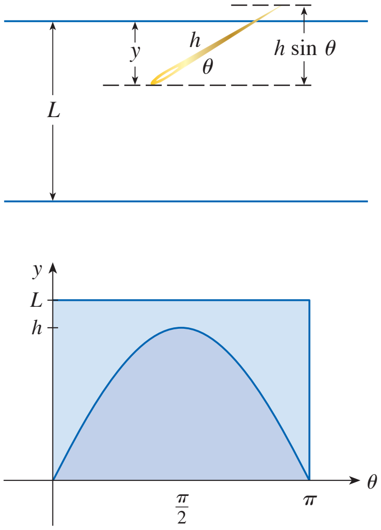


FIGURE FOR PROBLEM 11

11. In a famous 18th-century problem, known as *Buffon's needle problem*, a needle of length h is dropped onto a flat surface (for example, a table) on which parallel lines L units apart, $L \geqslant h$, have been drawn. The problem is to determine the probability that the needle will come to rest intersecting one of the lines. Assume that the lines run east-west, parallel to the x-axis in a rectangular coordinate system (as in the figure). Let y be the distance from the "southern" end of the needle to the nearest line to the north. (If the needle's southern end lies on a line, let $y = 0$. If the needle happens to lie east-west, let the "western" end be the "southern" end.) Let θ be the angle that the needle makes with a ray extending eastward from the southern end. Then $0 \leqslant y \leqslant L$ and $0 \leqslant \theta \leqslant \pi$. Note that the needle intersects one of the lines only when $y < h \sin \theta$. The total set of possibilities for the needle can be identified with the rectangular region $0 \leqslant y \leqslant L$, $0 \leqslant \theta \leqslant \pi$, and the proportion of times that the needle intersects a line is the ratio

$$\frac{\text{area under } y = h \sin \theta}{\text{area of rectangle}}$$

This ratio is the probability that the needle intersects a line. Find the probability that the needle will intersect a line if $h = L$. What if $h = \frac{1}{2}L$?

12. If the needle in Problem 11 has length $h > L$, it's possible for the needle to intersect more than one line.

(a) If $L = 4$, find the probability that a needle of length 7 will intersect at least one line. [*Hint:* Proceed as in Problem 11. Define y as before; then the total set of possibilities for the needle can be identified with the same rectangular region $0 \leqslant y \leqslant L$, $0 \leqslant \theta \leqslant \pi$. What portion of the rectangle corresponds to the needle intersecting a line?]

(b) If $L = 4$, find the probability that a needle of length 7 will intersect *two* lines.

(c) If $2L < h \leqslant 3L$, find a general formula for the probability that the needle intersects three lines.

13. Find the centroid of the region enclosed by the ellipse $x^2 + (x + y + 1)^2 = 1$.

Sea ice is an important part of the earth's ecology. In Exercise 9.3.56 you are asked to derive a differential equation that models the thickness of sea ice as it changes over time.

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9

Differential Equations

PERHAPS THE MOST IMPORTANT of all the applications of calculus is to differential equations. When physical scientists or social scientists use calculus, more often than not it is to analyze a differential equation that has arisen in the process of modeling some phenomenon that they are studying. Although it is often impossible to find an explicit formula for the solution of a differential equation, we will see that graphical and numerical approaches provide the needed information.

9.1 Modeling with Differential Equations

Now is a good time to read (or reread) the discussion of mathematical modeling in Section 1.2.

In describing the process of modeling in Chapter 1, we talked about formulating a mathematical model of a real-world problem either through intuitive reasoning about the phenomenon or from a physical law based on evidence from experiments. The mathematical model often takes the form of a *differential equation*, that is, an equation that contains an unknown function and some of its derivatives. This is not surprising because in a real-world situation we often notice that changes occur and we want to predict future behavior on the basis of how current values change. We begin by examining several examples of how differential equations arise when we model physical phenomena.

Models for Population Growth

One model for the growth of a population is based on the assumption that the population grows at a rate proportional to the size of the population. That is a reasonable assumption for a population of bacteria or animals under ideal conditions (unlimited environment, adequate nutrition, absence of predators, immunity from disease).

Let's identify and name the variables in this model:

$t =$ time (the independent variable)

$P =$ the number of individuals in the population (the dependent variable)

The rate of growth of the population is the derivative dP/dt. So our assumption that the rate of growth of the population is proportional to the population size is written as the equation

$$\boxed{1} \qquad \frac{dP}{dt} = kP$$

where k is the proportionality constant. Equation 1 is our first model for population growth; it is a differential equation because it contains an unknown function P and its derivative dP/dt.

Having formulated a model, let's look at its consequences. If we rule out a population of 0, then $P(t) > 0$ for all t. So, if $k > 0$, then Equation 1 shows that $P'(t) > 0$ for all t. This means that the population is always increasing. In fact, as $P(t)$ increases, Equation 1 shows that dP/dt becomes larger. In other words, the growth rate increases as the population increases.

Let's try to think of a solution of Equation 1. This equation asks us to find a function whose derivative is a constant multiple of itself. We know from Chapter 3 that exponential functions have that property. In fact, if we let $P(t) = Ce^{kt}$, then

$$P'(t) = C(ke^{kt}) = k(Ce^{kt}) = kP(t)$$

Thus any exponential function of the form $P(t) = Ce^{kt}$ is a solution of Equation 1. In Section 9.4, we will see that there is no other solution.

Allowing C to vary through all the real numbers, we get the *family* of solutions $P(t) = Ce^{kt}$ whose graphs are shown in Figure 1. But populations have only positive values and so we are interested only in the solutions with $C > 0$. If we are concerned

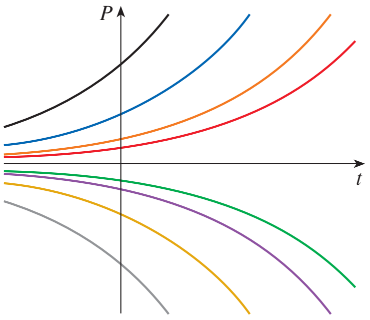

FIGURE 1
The family of solutions of $dP/dt = kP$

only with values of t greater than the initial time $t = 0$, then Figure 2 shows the physically meaningful solutions. Putting $t = 0$, we get $P(0) = Ce^{k(0)} = C$, so the constant C turns out to be the initial population, $P(0)$.

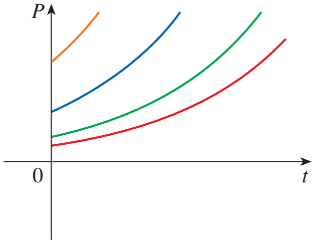

FIGURE 2
The family of solutions $P(t) = Ce^{kt}$ with $C > 0$ and $t \geqslant 0$

Equation 1 is appropriate for modeling population growth under ideal conditions, but we have to recognize that a more realistic model must reflect the fact that a given environment has limited resources. Many populations start by increasing in an exponential manner, but the population levels off when it approaches its *carrying capacity* M (or decreases toward M if it ever exceeds M). For a model to take into account both trends, it should satisfy both of the following assumptions:

- $\dfrac{dP}{dt} \approx kP$ if P is small (Initially, the growth rate is proportional to P.)
- $\dfrac{dP}{dt} < 0$ if $P > M$ (P decreases if it ever exceeds M.)

One way to incorporate both assumptions is to assume that the rate of population growth is proportional to both the population and the difference between the carrying capacity and the population. The corresponding differential equation is $dP/dt = cP(M - P)$, where c is the proportionality constant, or equivalently,

$$\boxed{2} \qquad \frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) \qquad \text{where } k = cM$$

Notice that if P is small compared with M, then P/M is close to 0 and so $dP/dt \approx kP$. If $P > M$, then $1 - P/M$ is negative and so $dP/dt < 0$.

Equation 2 is called the *logistic differential equation* and was proposed by the Dutch mathematical biologist Pierre-François Verhulst in the 1840s as a model for world population growth. We will develop techniques that enable us to find explicit solutions of the logistic equation in Section 9.4, but for now we can deduce qualitative characteristics of the solutions directly from Equation 2. We first observe that the constant functions $P(t) = 0$ and $P(t) = M$ are solutions because, in either case, one of the factors on the right side of Equation 2 is zero. (This certainly makes physical sense: if the population is ever either 0 or at the carrying capacity, it stays that way.) In general, constant solutions of a differential equation, like these two solutions, are called **equilibrium solutions**.

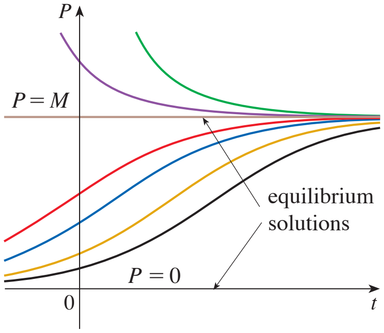

FIGURE 3
Solutions of the logistic equation

If the initial population $P(0)$ lies between 0 and M, then the right side of Equation 2 is positive, so $dP/dt > 0$ and the population increases. But if the population exceeds the carrying capacity ($P > M$), then $1 - P/M$ is negative, so $dP/dt < 0$ and the population decreases. Notice that in either case, if the population approaches the carrying capacity ($P \to M$), then $dP/dt \to 0$, which means the population levels off. So we expect that the solutions of the logistic differential equation have graphs that look something like the ones in Figure 3. Notice that the graphs move away from the equilibrium solution $P = 0$ and move toward the equilibrium solution $P = M$.

A Model for the Motion of a Spring

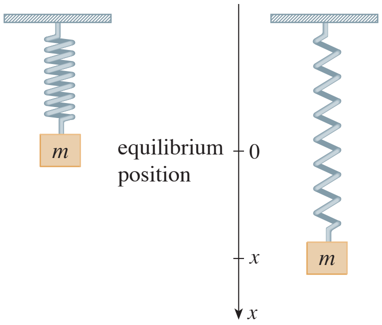

FIGURE 4

Let's now look at an example of a model from the physical sciences. We consider the motion of an object with mass m at the end of a vertical spring (as in Figure 4). In Section 6.4 we discussed Hooke's Law, which says that if the spring is stretched (or compressed) x units from its natural length, then it exerts a force that is proportional to x:

$$\text{restoring force} = -kx$$

where k is a positive constant (called the *spring constant*). If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law (force equals mass times acceleration), we have

$$\boxed{3} \qquad m\frac{d^2x}{dt^2} = -kx$$

This is an example of what is called a *second-order differential equation* because it involves second derivatives. Let's see what we can guess about the form of the solution directly from the equation. We can rewrite Equation 3 in the form

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

This says that the second derivative of x is proportional to x but has the opposite sign. We know two functions with this property, the sine and cosine functions. In fact, it turns out that all solutions of Equation 3 can be written as combinations of certain sine and cosine functions (see Exercise 16). This is not surprising; we expect the spring to oscillate about its equilibrium position and so it is natural to think that trigonometric functions are involved.

General Differential Equations

In general, a **differential equation** is an equation that contains an unknown function and one or more of its derivatives. The **order** of a differential equation is the order of the highest derivative that occurs in the equation. Thus Equations 1 and 2 are first-order equations and Equation 3 is a second-order equation. In all three of those equations the independent variable is called t and represents time, but in general the independent variable doesn't have to represent time. For example, when we consider the differential equation

$$\boxed{4} \qquad y' = xy$$

it is understood that y is an unknown function of x.

A function f is called a **solution** of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation. Thus f is a solution of Equation 4 if

$$f'(x) = xf(x)$$

for all values of x in some interval.

When we are asked to *solve* a differential equation we are expected to find all possible solutions of the equation. We have already solved some particularly simple differential equations, namely, those of the form

$$y' = f(x)$$

For instance, we know that the general solution of the differential equation

$$y' = x^3$$

is given by

$$y = \frac{x^4}{4} + C$$

where C is an arbitrary constant.

But, in general, solving a differential equation is not an easy matter. There is no systematic technique that enables us to solve all differential equations. In Section 9.2, however, we will see how to draw rough graphs of solutions even when we have no explicit formula. We will also learn how to find numerical approximations to solutions.

EXAMPLE 1 Determine whether the function $y = x + 1/x$ is a solution of the given differential equation.

(a) $xy' + y = 2x$ (b) $xy'' + 2y' = 0$

SOLUTION The first and second derivatives of $y = x + 1/x$ (with respect to x) are $y' = 1 - 1/x^2$ and $y'' = 2/x^3$.

(a) Substituting the expressions for y and y' into the left side of the differential equation, we get

$$\begin{aligned} xy' + y &= x\left(1 - \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) \\ &= x - \frac{1}{x} + x + \frac{1}{x} = 2x \end{aligned}$$

Because $2x$ is equal to the right side of the differential equation, $y = x + 1/x$ is a solution.

(b) Substituting for y' and y'', the left side becomes

$$\begin{aligned} xy'' + 2y' &= x\left(\frac{2}{x^3}\right) + 2\left(1 - \frac{1}{x^2}\right) \\ &= \frac{2}{x^2} + 2 - \frac{2}{x^2} = 2 \end{aligned}$$

which is not equal to the right side of the differential equation. Thus $y = x + 1/x$ is not a solution. ■

EXAMPLE 2 Show that every member of the family of functions

$$y = \frac{1 + ce^t}{1 - ce^t}$$

is a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$.

SOLUTION We use the Quotient Rule to differentiate the expression for y:

$$\begin{aligned} y' &= \frac{(1 - ce^t)(ce^t) - (1 + ce^t)(-ce^t)}{(1 - ce^t)^2} \\ &= \frac{ce^t - c^2e^{2t} + ce^t + c^2e^{2t}}{(1 - ce^t)^2} = \frac{2ce^t}{(1 - ce^t)^2} \end{aligned}$$

The right side of the differential equation becomes

$$\begin{aligned} \tfrac{1}{2}(y^2 - 1) &= \frac{1}{2}\left[\left(\frac{1 + ce^t}{1 - ce^t}\right)^2 - 1\right] \\ &= \frac{1}{2}\left[\frac{(1 + ce^t)^2 - (1 - ce^t)^2}{(1 - ce^t)^2}\right] \\ &= \frac{1}{2}\,\frac{4ce^t}{(1 - ce^t)^2} = \frac{2ce^t}{(1 - ce^t)^2} \end{aligned}$$

This shows that the left and right sides of the differential equation are equal. Therefore, for every value of c, the given function is a solution of the differential equation. ■

Figure 5 shows graphs of seven members of the family in Example 2. The differential equation shows that if $y \approx \pm 1$, then $y' \approx 0$. That is borne out by the flatness of the graphs near $y = 1$ and $y = -1$.

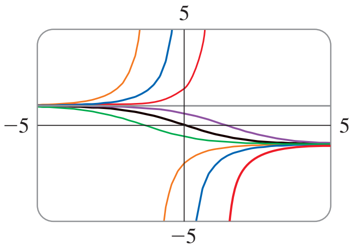

FIGURE 5

When applying differential equations, we are usually not as interested in finding a family of solutions (the *general solution*) as we are in finding a solution that satisfies some additional requirement. In many physical problems we need to find the particular solution that satisfies a condition of the form $y(t_0) = y_0$. This is called an **initial condition**, and the problem of finding a solution of the differential equation that satisfies the initial condition is called an **initial-value problem**.

Geometrically, when we impose an initial condition, we look at the family of solution curves and pick the one that passes through the point (t_0, y_0). Physically, this corresponds to measuring the state of a system at time t_0 and using the solution of the initial-value problem to predict the future behavior of the system.

EXAMPLE 3 Find a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$ that satisfies the initial condition $y(0) = 2$.

SOLUTION From Example 2 we know that for any value of c, the function

$$y = \frac{1 + ce^t}{1 - ce^t}$$

is a solution of this differential equation. Substituting the values $t = 0$ and $y = 2$, we get

$$2 = \frac{1 + ce^0}{1 - ce^0} = \frac{1 + c}{1 - c}$$

Solving this equation for c, we get $2 - 2c = 1 + c$, which gives $c = \frac{1}{3}$. So the solution of the initial-value problem is

$$y = \frac{1 + \frac{1}{3}e^t}{1 - \frac{1}{3}e^t} = \frac{3 + e^t}{3 - e^t}$$

The graph of the solution is shown in Figure 6. The curve is the one member of the family of solution curves from Figure 5 that passes through the point (0, 2). ■

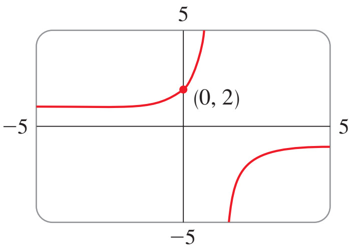

FIGURE 6

9.1 Exercises

1–5 Write a differential equation that models the given situation. In each case the stated rate of change is with respect to time t.

1. The rate of change of the radius r of a tree trunk is inversely proportional to the radius.

2. The rate of change of the velocity v of a falling body is constant.

3. For a car with maximum velocity M, the rate of change of the velocity v of the car is proportional to the difference between M and v.

4. When an infectious disease is introduced into a city of fixed population N, the rate of change of the number y of infected individuals is proportional to the product of the number of infected individuals and the number of noninfected individuals.

5. When an advertising campaign for a new product is introduced into a city of fixed population N, the rate of change of the number y of individuals who have heard about the product at time t is proportional to the number of individuals in the population who have not yet heard about the product.

6–12 Determine whether the given function is a solution of the differential equation.

6. $y = \sin x - \cos x; \quad y' + y = 2 \sin x$

7. $y = \frac{2}{3}e^x + e^{-2x}; \quad y' + 2y = 2e^x$

8. $y = \tan x; \quad y' - y^2 = 1$

9. $y = \sqrt{x}; \quad xy' - y = 0$

10. $y = \sqrt{1 - x^2}; \quad yy' - x = 0$

11. $y = x^3$; $x^2y'' - 6y = 0$

12. $y = \ln x$; $xy'' - y' = 0$

13–14 Show that the given function is a solution of the initial-value problem.

13. $y = -t\cos t - t$; $t\dfrac{dy}{dt} = y + t^2\sin t$, $y(\pi) = 0$

14. $y = 5e^{2x} + x$; $\dfrac{dy}{dx} - 2y = 1 - 2x$, $y(0) = 5$

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15. (a) For what values of r does the function $y = e^{rx}$ satisfy the differential equation $2y'' + y' - y = 0$?
(b) If r_1 and r_2 are the values of r that you found in part (a), show that every member of the family of functions $y = ae^{r_1x} + be^{r_2x}$ is also a solution.

16. (a) For what values of k does the function $y = \cos kt$ satisfy the differential equation $4y'' = -25y$?
(b) For those values of k, verify that every member of the family of functions $y = A\sin kt + B\cos kt$ is also a solution.

17. Which of the following functions are solutions of the differential equation $y'' + y = \sin x$?
(a) $y = \sin x$
(b) $y = \cos x$
(c) $y = \frac{1}{2}x\sin x$
(d) $y = -\frac{1}{2}x\cos x$

18. (a) Show that every member of the family of functions $y = (\ln x + C)/x$ is a solution of the differential equation $x^2y' + xy = 1$.
(b) Illustrate part (a) by graphing several members of the family of solutions on a common screen.
(c) Find a solution of the differential equation that satisfies the initial condition $y(1) = 2$.
(d) Find a solution of the differential equation that satisfies the initial condition $y(2) = 1$.

19. (a) What can you say about a solution of the equation $y' = -y^2$ just by looking at the differential equation?
(b) Verify that all members of the family $y = 1/(x + C)$ are solutions of the equation in part (a).
(c) Can you think of a solution of the differential equation $y' = -y^2$ that is not a member of the family in part (b)?
(d) Find a solution of the initial-value problem

$$y' = -y^2 \qquad y(0) = 0.5$$

20. (a) What can you say about the graph of a solution of the equation $y' = xy^3$ when x is close to 0? What if x is large?
(b) Verify that all members of the family $y = (c - x^2)^{-1/2}$ are solutions of the differential equation $y' = xy^3$.
(c) Graph several members of the family of solutions on a common screen. Do the graphs confirm what you predicted in part (a)?
(d) Find a solution of the initial-value problem

$$y' = xy^3 \qquad y(0) = 2$$

21. A population is modeled by the differential equation

$$\frac{dP}{dt} = 1.2P\left(1 - \frac{P}{4200}\right)$$

(a) For what values of P is the population increasing?
(b) For what values of P is the population decreasing?
(c) What are the equilibrium solutions?

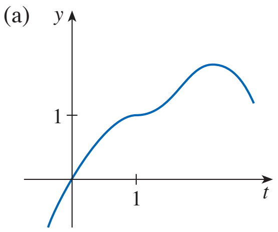
22. The Fitzhugh-Nagumo model for the electrical impulse in a neuron states that, in the absence of relaxation effects, the electrical potential in a neuron $v(t)$ obeys the differential equation

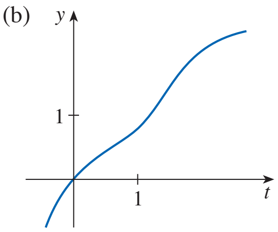
$$\frac{dv}{dt} = -v[v^2 - (1 + a)v + a]$$

where a is a positive constant such that $0 < a < 1$.
(a) For what values of v is v unchanging (that is, $dv/dt = 0$)?
(b) For what values of v is v increasing?
(c) For what values of v is v decreasing?

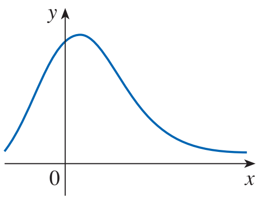
23. Explain why the functions with the given graphs *can't* be solutions of the differential equation

$$\frac{dy}{dt} = e^t(y - 1)^2$$





24. The function with the given graph is a solution of one of the following differential equations. Decide which is the correct equation and justify your answer.



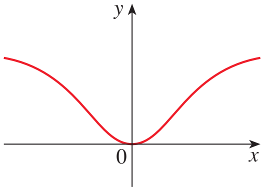
A. $y' = 1 + xy$ B. $y' = -2xy$ C. $y' = 1 - 2xy$

25. Match the differential equations with the solution graphs labeled I–IV. Give reasons for your choices.

(a) $y' = 1 + x^2 + y^2$ (b) $y' = xe^{-x^2-y^2}$

(c) $y' = \dfrac{1}{1 + e^{x^2+y^2}}$ (d) $y' = \sin(xy)\cos(xy)$

I



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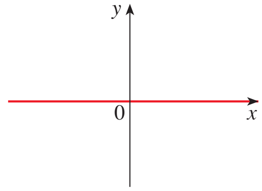

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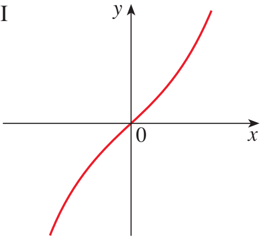
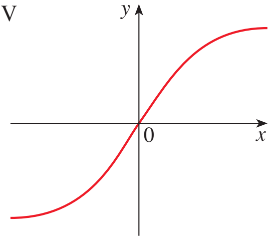


IV



26. Suppose you have just poured a cup of freshly brewed coffee with temperature 95°C in a room where the temperature is 20°C.

(a) When do you think the coffee cools most quickly? What happens to the rate of cooling as time goes by? Explain.

(b) Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. Write a differential equation that expresses Newton's Law of Cooling for this particular situation. What is the initial condition? In view of your answer to part (a), do you think this differential equation is an appropriate model for cooling?

(c) Make a rough sketch of the graph of the solution of the initial-value problem in part (b).

27. Psychologists interested in learning theory study **learning curves**. A learning curve is the graph of a function $P(t)$, the performance of someone learning a skill as a function of the training time t. The derivative dP/dt represents the rate at which performance improves.

(a) When do you think P increases most rapidly? What happens to dP/dt as t increases? Explain.

(b) If M is the maximum level of performance of which the learner is capable, explain why the differential equation

$$\frac{dP}{dt} = k(M - P) \qquad k \text{ a positive constant}$$

is a reasonable model for learning.

(c) Make a rough sketch of a possible solution of this differential equation.

28. Von Bertalanffy's equation states that the rate of growth in length of an individual fish is proportional to the difference between the current length L and the asymptotic length L_∞ (in centimeters).

(a) Write a differential equation that expresses this idea.

(b) Make a rough sketch of the graph of a solution of a typical initial-value problem for this differential equation.

29. Differential equations have been used extensively in the study of drug dissolution for patients given oral medications. One such equation is the Weibull equation for the concentration $c(t)$ of the drug:

$$\frac{dc}{dt} = \frac{k}{t^b}(c_s - c)$$

where k and c_s are positive constants and $0 < b < 1$. Verify that

$$c(t) = c_s\left(1 - e^{-\alpha t^{1-b}}\right)$$

is a solution of the Weibull equation for $t > 0$, where $\alpha = k/(1 - b)$. What does the differential equation say about how drug dissolution occurs?

9.2 Direction Fields and Euler's Method

Unfortunately, it's impossible to solve most differential equations in the sense of obtaining an explicit formula for the solution. In this section we show that, despite the absence of an explicit solution, we can still learn a lot about the solution through a graphical approach (direction fields) or a numerical approach (Euler's method).

Direction Fields

Suppose we are asked to sketch the graph of the solution of the initial-value problem

$$y' = x + y \qquad y(0) = 1$$

We don't know a formula for the solution, so how can we possibly sketch its graph? Let's think about what the differential equation means. The equation $y' = x + y$ tells us that the slope at any point (x, y) on the graph (called the *solution curve*) is equal to the sum of the x- and y-coordinates of the point (see Figure 1). In particular, because the curve passes through the point $(0, 1)$, its slope there must be $0 + 1 = 1$. So a small portion of the solution curve near the point $(0, 1)$ looks like a short line segment through $(0, 1)$ with slope 1. (See Figure 2.)

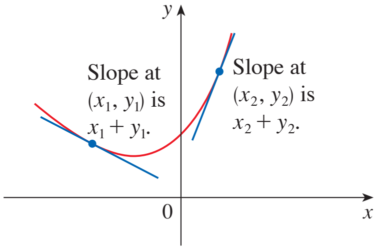

y
Slope at (x_1, y_1) is $x_1 + y_1$.
Slope at (x_2, y_2) is $x_2 + y_2$.
0
x

FIGURE 1

A solution of $y' = x + y$

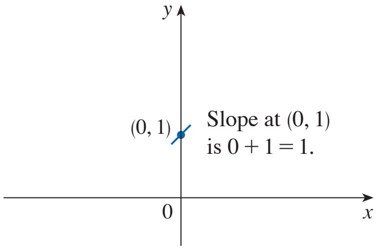


FIGURE 2

Beginning of the solution curve through (0, 1)

As a guide to sketching the rest of the curve, let's draw short line segments at a number of points (x, y) with slope $x + y$. The result is called a *direction field* and is shown in Figure 3. For instance, the line segment at the point $(1, 2)$ has slope $1 + 2 = 3$. The direction field allows us to visualize the general shape of the solution curves by indicating the direction in which the curves proceed at each point.

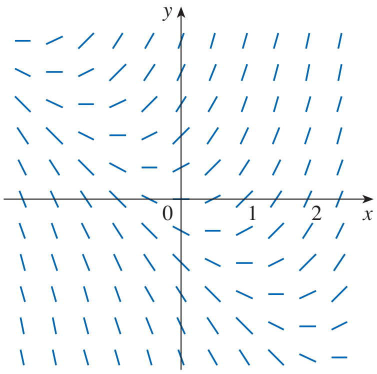


FIGURE 3

Direction field for $y' = x + y$

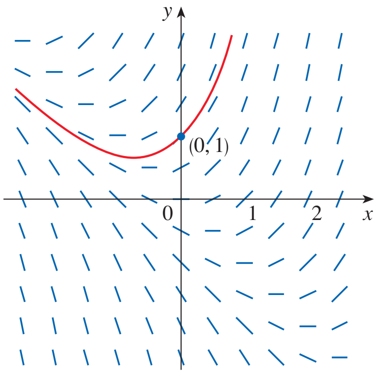


FIGURE 4

The solution curve through (0, 1)

Now we can sketch the solution curve through the point $(0, 1)$ by following the direction field as in Figure 4. Notice that we have drawn the curve so that it is parallel to nearby line segments.

In general, suppose we have a first-order differential equation of the form

$$y' = F(x, y)$$

where $F(x, y)$ is some expression in x and y. The differential equation says that the slope of a solution curve at a point (x, y) on the curve is $F(x, y)$. If we draw short line segments with slope $F(x, y)$ at several points (x, y), the result is called a **direction field** (or **slope field**). These line segments indicate the direction in which a solution curve is heading, so the direction field helps us visualize the general shape of these curves.

EXAMPLE 1

(a) Sketch the direction field for the differential equation $y' = x^2 + y^2 - 1$.
(b) Use part (a) to sketch the solution curve that passes through the origin.

SOLUTION

(a) We start by computing the slope at several points in the following table:

x	-2	-1	0	1	2	-2	-1	0	1	2	. . .
y	0	0	0	0	0	1	1	1	1	1	. . .
$y' = x^2 + y^2 - 1$	3	0	-1	0	3	4	1	0	1	4	. . .

Now we draw short line segments with these slopes at these points. The result is the direction field shown in Figure 5.

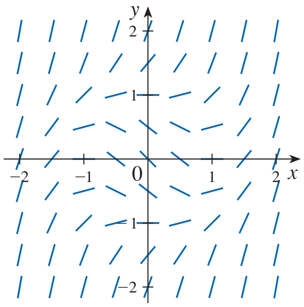


FIGURE 5

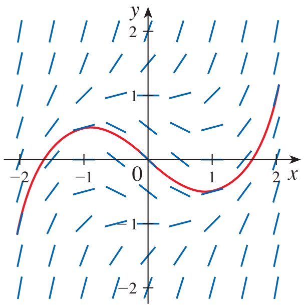


FIGURE 6

(b) We start at the origin and move to the right in the direction of the line segment (which has slope -1). We continue to draw the solution curve so that it moves parallel to the nearby line segments. The resulting solution curve is shown in Figure 6. Returning to the origin, we draw the solution curve to the left as well. ■

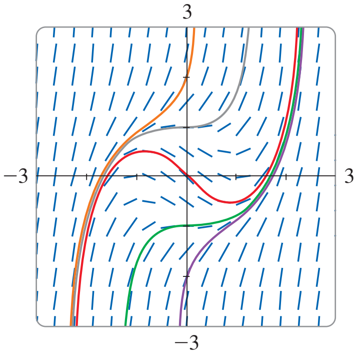


FIGURE 7

The more line segments we draw in a direction field, the clearer the picture becomes. Of course, it's tedious to compute slopes and draw line segments by hand for a huge number of points, but computers are well suited for this task. Figure 7 shows a more detailed, computer-drawn direction field for the differential equation in Example 1. It enables us to draw, with reasonable accuracy, the solution curves with y-intercepts -2, -1, 0, 1, and 2.

Now let's see how direction fields give insight into physical situations. The simple electric circuit shown in Figure 8 contains an electromotive force (usually a battery or generator) that produces a voltage of $E(t)$ volts (V) and a current of $I(t)$ amperes (A) at time t. The circuit also contains a resistor with a resistance of R ohms (Ω) and an inductor with an inductance of L henries (H).

Ohm's Law gives the drop in voltage due to the resistor as RI. The voltage drop due to the inductor is $L(dI/dt)$. One of Kirchhoff's laws says that the sum of the voltage drops is equal to the supplied voltage $E(t)$. Thus we have

$$\boxed{1} \qquad L\frac{dI}{dt} + RI = E(t)$$

which is a first-order differential equation that models the current I at time t.

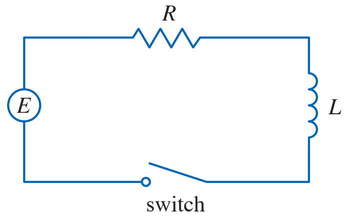


FIGURE 8

EXAMPLE 2 Suppose that in the simple circuit of Figure 8 the resistance is 12 Ω, the inductance is 4 H, and a battery gives a constant voltage of 60 V.

(a) Draw a direction field for Equation 1 with these values.

(b) What can you say about the limiting value of the current?

(c) Identify any equilibrium solutions.

(d) If the switch is closed when $t = 0$ so the current starts with $I(0) = 0$, use the direction field to sketch the solution curve.

SOLUTION

(a) If we put $L = 4$, $R = 12$, and $E(t) = 60$ in Equation 1, we get

$$4\frac{dI}{dt} + 12I = 60 \qquad \text{or} \qquad \frac{dI}{dt} = 15 - 3I$$

The direction field for this differential equation is shown in Figure 9.

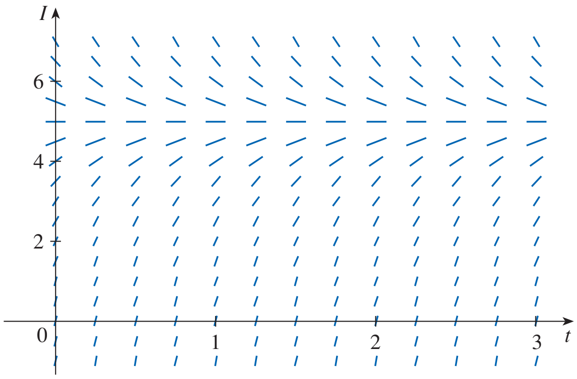


FIGURE 9

(b) It appears from the direction field that all solutions approach the value 5 A, that is,

$$\lim_{t\to\infty} I(t) = 5$$

Recall that an equilibrium solution is a constant solution (its graph is a horizontal line).

(c) From the direction field we see that the constant function $I(t) = 5$ is an equilibrium solution. Indeed, we can verify this directly from the differential equation $dI/dt = 15 - 3I$. If $I(t) = 5$, then the left side is $dI/dt = 0$ and the right side is $15 - 3(5) = 0$.

(d) We use the direction field to sketch the solution curve that passes through (0, 0), as shown in red in Figure 10.

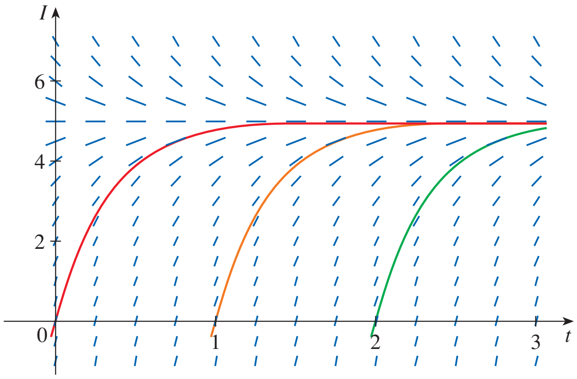


FIGURE 10

■

Notice from Figure 9 that the line segments along any horizontal line are parallel. That is because the independent variable t does not occur on the right side of the equation $I' = 15 - 3I$. In general, a differential equation of the form

$$y' = f(y)$$

in which the independent variable is missing from the right side, is called **autonomous**. For such an equation, the slopes corresponding to two different points with the same y-coordinate must be equal. This means that if we know one solution to an autonomous differential equation, then we can obtain infinitely many others just by shifting the graph of the known solution to the right or left. In Figure 10 we have shown the solutions that result from shifting the solution curve of Example 2 one and two time units (namely, seconds) to the right. They correspond to closing the switch when $t = 1$ or $t = 2$.

Euler's Method

The basic idea behind direction fields can be used to find numerical approximations to solutions of differential equations. We illustrate the method on the initial-value problem that we used to introduce direction fields:

$$y' = x + y \qquad y(0) = 1$$

The differential equation tells us that $y'(0) = 0 + 1 = 1$, so the solution curve has slope 1 at the point $(0, 1)$. As a first approximation to the solution we could use the linear approximation $L(x) = x + 1$. In other words, we could use the tangent line at $(0, 1)$ as a rough approximation to the solution curve (see Figure 11).

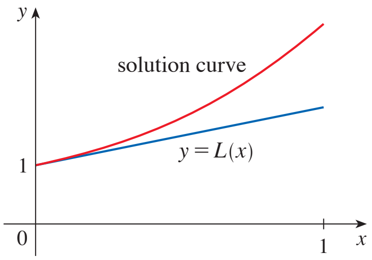


FIGURE 11
First Euler approximation

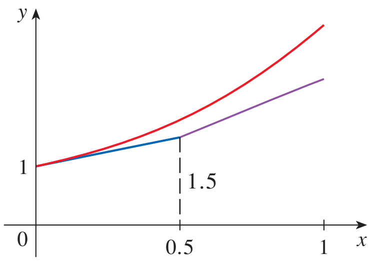


FIGURE 12
Euler approximation with step size 0.5

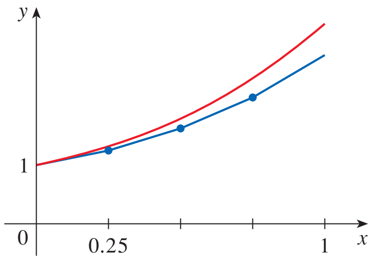


FIGURE 13
Euler approximation with step size 0.25

Euler's idea was to improve on this approximation by proceeding only a short distance along this tangent line and then making a midcourse correction by changing direction as indicated by the direction field. Figure 12 shows what happens if we start out along the tangent line but stop when $x = 0.5$. (This horizontal distance traveled is called the *step size*.) Since $L(0.5) = 1.5$, we have $y(0.5) \approx 1.5$ and we take $(0.5, 1.5)$ as the starting point for a new line segment. The differential equation tells us that $y'(0.5) = 0.5 + 1.5 = 2$, so we use the linear function

$$y = 1.5 + 2(x - 0.5) = 2x + 0.5$$

as an approximation to the solution for $x > 0.5$ (the purple segment in Figure 12). If we decrease the step size from 0.5 to 0.25, we get the better Euler approximation shown in Figure 13.

In general, Euler's method says to start at the point given by the initial value and proceed in the direction indicated by the direction field. Stop after a short distance, look at

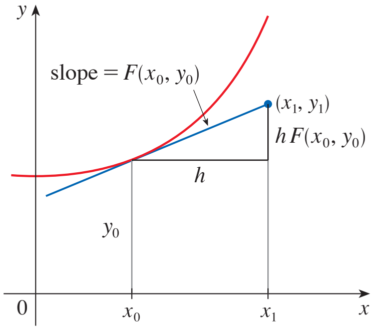


FIGURE 14

the slope at the new location, and proceed in that direction. Keep stopping and changing direction according to the direction field. Euler's method does not produce the exact solution to an initial-value problem—it gives approximations. But by decreasing the step size (and therefore increasing the number of midcourse corrections), we obtain successively better approximations to the exact solution. (Compare Figures 11, 12, and 13.)

For the general first-order initial-value problem $y' = F(x, y)$, $y(x_0) = y_0$, our aim is to find approximate values for the solution at equally spaced numbers x_0, $x_1 = x_0 + h$, $x_2 = x_1 + h, \ldots$, where h is the step size. The differential equation tells us that the slope at (x_0, y_0) is $y' = F(x_0, y_0)$, so Figure 14 shows that the approximate value of the solution when $x = x_1$ is

$$y_1 = y_0 + hF(x_0, y_0)$$

Similarly, $$y_2 = y_1 + hF(x_1, y_1)$$

In general, $$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$$

Euler's Method Approximate values for the solution of the initial-value problem $y' = F(x, y)$, $y(x_0) = y_0$, with step size h, at $x_n = x_{n-1} + h$, are

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}) \qquad n = 1, 2, 3, \ldots$$

EXAMPLE 3 Use Euler's method with step size 0.1 to construct a table of approximate values for the solution of the initial-value problem

$$y' = x + y \qquad y(0) = 1$$

SOLUTION We are given that $h = 0.1$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = x + y$. So we have

$$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1(0 + 1) = 1.1$$

$$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1(0.1 + 1.1) = 1.22$$

$$y_3 = y_2 + hF(x_2, y_2) = 1.22 + 0.1(0.2 + 1.22) = 1.362$$

This means that if $y(x)$ is the exact solution, then $y(0.3) \approx 1.362$.

Proceeding with similar calculations, we get the values in the table:

n	x_n	y_n	n	x_n	y_n
1	0.1	1.100000	6	0.6	1.943122
2	0.2	1.220000	7	0.7	2.197434
3	0.3	1.362000	8	0.8	2.487178
4	0.4	1.528200	9	0.9	2.815895
5	0.5	1.721020	10	1.0	3.187485

■

Computer software that produces numerical approximations to solutions of differential equations uses methods that are refinements of Euler's method. Although Euler's method is simple and not as accurate, it is the basic idea on which the more accurate methods are based.

For a more accurate table of values in Example 3 we could decrease the step size. But for a large number of small steps the amount of computation is considerable and so we need to program a calculator or computer to carry out these calculations. The following table shows the results of applying Euler's method with decreasing step size to the initial-value problem of Example 3.

Notice that the Euler estimates in the table below seem to be approaching limits, namely, the true values of $y(0.5)$ and $y(1)$. Figure 15 shows graphs of the Euler approximations with step sizes 0.5, 0.25, 0.1, 0.05, 0.02, 0.01, and 0.005. They are approaching the exact solution curve as the step size h approaches 0.

Step size	Euler estimate of $y(0.5)$	Euler estimate of $y(1)$
0.500	1.500000	2.500000
0.250	1.625000	2.882813
0.100	1.721020	3.187485
0.050	1.757789	3.306595
0.020	1.781212	3.383176
0.010	1.789264	3.409628
0.005	1.793337	3.423034
0.001	1.796619	3.433848

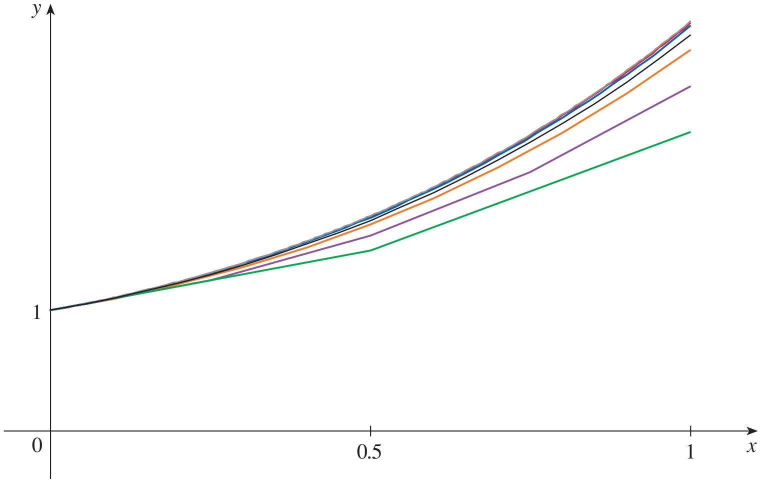


FIGURE 15 Euler approximation approaching the exact solution

Euler

Leonhard Euler (1707–1783) was the leading mathematician of the mid-18th century and the most prolific mathematician of all time. He was born in Switzerland but spent most of his career at the academies of science supported by Catherine the Great in St. Petersburg and Frederick the Great in Berlin. The collected works of Euler (pronounced *Oiler*) fill about 100 large volumes. As the French physicist Arago said, "Euler calculated without apparent effort, as men breathe or as eagles sustain themselves in the air." Euler's calculations and writings were not diminished by raising 13 children or being totally blind for the last 17 years of his life. In fact, when blind, he dictated his discoveries to his helpers from his prodigious memory and imagination. His treatises on calculus and most other mathematical subjects became the standard for mathematics instruction and the equation $e^{i\pi} + 1 = 0$ that he discovered brings together the five most famous numbers in all of mathematics.

EXAMPLE 4 In Example 2 we discussed a simple electric circuit with resistance 12 Ω, inductance 4 H, and a battery with voltage 60 V. If the switch is closed when $t = 0$, we modeled the current I at time t by the initial-value problem

$$\frac{dI}{dt} = 15 - 3I \qquad I(0) = 0$$

Estimate the current in the circuit half a second after the switch is closed.

SOLUTION We use Euler's method with $F(t, I) = 15 - 3I$, $t_0 = 0$, $I_0 = 0$, and step size $h = 0.1$ second:

$$I_1 = 0 + 0.1(15 - 3 \cdot 0) = 1.5$$

$$I_2 = 1.5 + 0.1(15 - 3 \cdot 1.5) = 2.55$$

$$I_3 = 2.55 + 0.1(15 - 3 \cdot 2.55) = 3.285$$

$$I_4 = 3.285 + 0.1(15 - 3 \cdot 3.285) = 3.7995$$

$$I_5 = 3.7995 + 0.1(15 - 3 \cdot 3.7995) = 4.15965$$

So the current after 0.5 s is

$$I(0.5) \approx 4.16 \text{ A}$$

■

9.2 Exercises

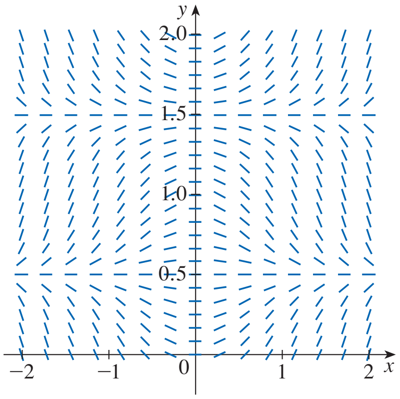
1. A direction field for the differential equation $y' = x \cos \pi y$ is shown.

(a) Sketch the graphs of the solutions that satisfy the given initial conditions.

(i) $y(0) = 0$ (ii) $y(0) = 0.5$

(iii) $y(0) = 1$ (iv) $y(0) = 1.6$

(b) Find all the equilibrium solutions.



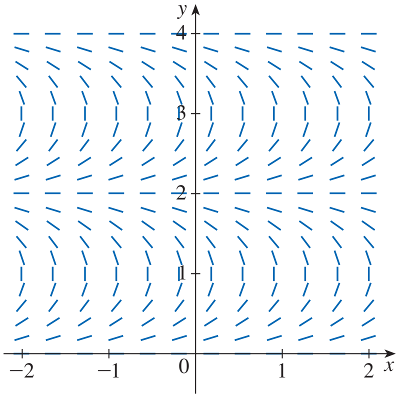
2. A direction field for the differential equation $y' = \tan\left(\frac{1}{2}\pi y\right)$ is shown.

(a) Sketch the graphs of the solutions that satisfy the given initial conditions.

(i) $y(0) = 1$ (ii) $y(0) = 0.2$

(iii) $y(0) = 2$ (iv) $y(1) = 3$

(b) Find all the equilibrium solutions.



3–6 Match the differential equation with its direction field (labeled I–IV). Give reasons for your answer.

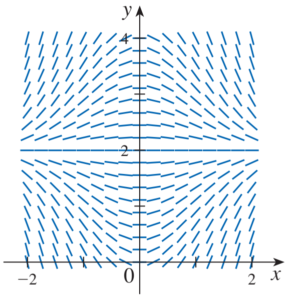
3. $y' = 2 - y$

4. $y' = x(2 - y)$

5. $y' = x + y - 1$

6. $y' = \sin x \sin y$

I



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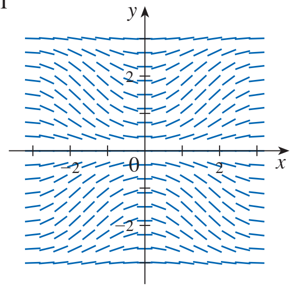
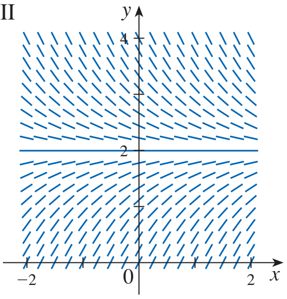


III



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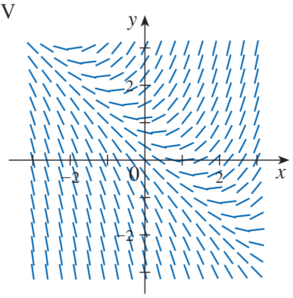


7. Use the direction field labeled I (above) to sketch the graphs of the solutions that satisfy the given initial conditions.

(a) $y(0) = 1$ (b) $y(0) = 2.5$ (c) $y(0) = 3.5$

8. Use the direction field labeled III (above) to sketch the graphs of the solutions that satisfy the given initial conditions.

(a) $y(0) = 1$ (b) $y(0) = 2.5$ (c) $y(0) = 3.5$

9–10 Sketch a direction field for the differential equation. Then use it to sketch three solution curves.

9. $y' = \frac{1}{2}y$

10. $y' = x - y + 1$

11–14 Sketch the direction field of the differential equation. Then use it to sketch a solution curve that passes through the given point.

11. $y' = y - 2x$, $(1, 0)$

12. $y' = xy - x^2$, $(0, 1)$

13. $y' = y + xy$, $(0, 1)$

14. $y' = x + y^2$, $(0, 0)$

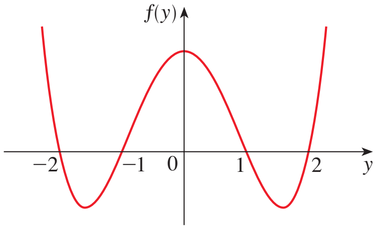
T **15–16** Use a computer to draw a direction field for the given differential equation. Get a printout and sketch on it the solution curve that passes through $(0, 1)$. Compare your sketch to a computer-drawn solution curve.

15. $y' = x^2 y - \frac{1}{2}y^2$

16. $y' = \cos(x + y)$

T **17.** Use a computer to draw a direction field for the differential equation $y' = y^3 - 4y$. Get a printout and sketch on it solutions that satisfy the initial condition $y(0) = c$ for various values of c. For what values of c does $\lim_{t\to\infty} y(t)$ exist? What are the possible values for this limit?

18. Make a rough sketch of a direction field for the autonomous differential equation $y' = f(y)$, where the graph of f is as shown. How does the limiting behavior of solutions depend on the value of $y(0)$?

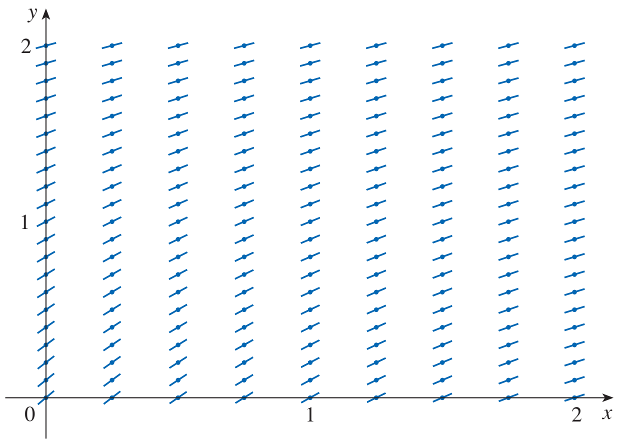


19. (a) Use Euler's method with each of the following step sizes to estimate the value of $y(0.4)$, where y is the solution of the initial-value problem $y' = y$, $y(0) = 1$.
(i) $h = 0.4$ (ii) $h = 0.2$ (iii) $h = 0.1$

(b) We know that the exact solution of the initial-value problem in part (a) is $y = e^x$. Draw, as accurately as you can, the graph of $y = e^x$, $0 \leqslant x \leqslant 0.4$, together with the Euler approximations using the step sizes in part (a). (Your sketches should resemble Figures 11, 12, and 13.) Use your sketches to decide whether your estimates in part (a) are underestimates or overestimates.

(c) The error in Euler's method is the difference between the exact value and the approximate value. Find the errors made in part (a) in using Euler's method to estimate the true value of $y(0.4)$, namely, $e^{0.4}$. What happens to the error each time the step size is halved?

20. A direction field for a differential equation is shown. Draw, with a ruler, the graphs of the Euler approximations to the solution curve that passes through the origin. Use step sizes $h = 1$ and $h = 0.5$. Will the Euler estimates be underestimates or overestimates? Explain.



21. Use Euler's method with step size 0.5 to compute the approximate y-values y_1, y_2, y_3, and y_4 of the solution of the initial-value problem $y' = y - 2x$, $y(1) = 0$.

22. Use Euler's method with step size 0.2 to estimate $y(1)$, where $y(x)$ is the solution of the initial-value problem $y' = x^2 y - \frac{1}{2}y^2$, $y(0) = 1$.

23. Use Euler's method with step size 0.1 to estimate $y(0.5)$, where $y(x)$ is the solution of the initial-value problem $y' = y + xy$, $y(0) = 1$.

24. (a) Use Euler's method with step size 0.2 to estimate $y(0.6)$, where $y(x)$ is the solution of the initial-value problem $y' = \cos(x + y)$, $y(0) = 0$.
(b) Repeat part (a) with step size 0.1.

T **25.** (a) Program a calculator or computer to use Euler's method to compute $y(1)$, where $y(x)$ is the solution of the initial-value problem

$$\frac{dy}{dx} + 3x^2 y = 6x^2 \qquad y(0) = 3$$

(i) $h = 1$ (ii) $h = 0.1$
(iii) $h = 0.01$ (iv) $h = 0.001$

(b) Verify that $y = 2 + e^{-x^3}$ is the exact solution of the differential equation.

(c) Find the errors in using Euler's method to compute $y(1)$ with the step sizes in part (a). What happens to the error when the step size is divided by 10?

T **26.** (a) Use Euler's method with step size 0.01 to calculate $y(2)$, where y is the solution of the initial-value problem

$$y' = x^3 - y^3 \qquad y(0) = 1$$

(b) Compare your answer to part (a) to the value of $y(2)$ that appears on a computer-drawn solution curve.

27. The figure shows a circuit containing an electromotive force, a capacitor with a capacitance of C farads (F), and a resistor with a resistance of R ohms (Ω). The voltage drop across the capacitor is Q/C, where Q is the charge (in coulombs, C), so in this case Kirchhoff's Law gives

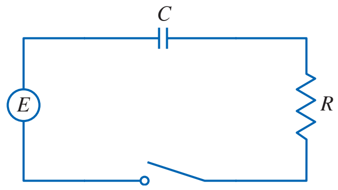
$$RI + \frac{Q}{C} = E(t)$$

But $I = dQ/dt$, so we have

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

Suppose the resistance is 5 Ω, the capacitance is 0.05 F, and a battery gives a constant voltage of 60 V.
(a) Draw a direction field for this differential equation.
(b) What is the limiting value of the charge?
(c) Is there an equilibrium solution?
(d) If the initial charge is $Q(0) = 0$ C, use the direction field to sketch the solution curve.
(e) If the initial charge is $Q(0) = 0$ C, use Euler's method

with step size 0.1 to estimate the charge after half a second.



28. In Exercise 9.1.26 we considered a 95°C cup of coffee in a 20°C room. Suppose it is known that the coffee cools at a rate of 1°C per minute when its temperature is 70°C.

(a) What does the differential equation become in this case?

(b) Sketch a direction field and use it to sketch the solution curve for the initial-value problem. What is the limiting value of the temperature?

(c) Use Euler's method with step size $h = 2$ minutes to estimate the temperature of the coffee after 10 minutes.

9.3 Separable Equations

We have looked at first-order differential equations from a geometric point of view (direction fields) and from a numerical point of view (Euler's method). What about the symbolic point of view? It would be nice to have an explicit formula for a solution of a differential equation. Unfortunately, that is not always possible. But in this section we examine a certain type of differential equation that *can* be solved explicitly.

Separable Differential Equations

A **separable equation** is a first-order differential equation in which the expression for dy/dx can be factored as a function of x times a function of y. In other words, it can be written in the form

$$\frac{dy}{dx} = g(x)f(y)$$

The name *separable* comes from the fact that the expression on the right side can be "separated" into a function of x and a function of y. Equivalently, if $f(y) \neq 0$, we could write

$$\boxed{1} \qquad \frac{dy}{dx} = \frac{g(x)}{h(y)}$$

where $h(y) = 1/f(y)$. To solve this equation we rewrite it in the differential form

$$h(y)\,dy = g(x)\,dx$$

so that all y's are on one side of the equation and all x's are on the other side. Then we integrate both sides of the equation:

$$\boxed{2} \qquad \int h(y)\,dy = \int g(x)\,dx$$

The technique for solving separable differential equations was first used by James Bernoulli (in 1690) in solving a problem about pendulums and by Leibniz (in a letter to Huygens in 1691). John Bernoulli explained the general method in a paper published in 1694.

Equation 2 defines y implicitly as a function of x. In some cases we may be able to solve for y in terms of x.

We use the Chain Rule to justify this procedure: If h and g satisfy (2), then

$$\frac{d}{dx}\left(\int h(y)\,dy\right) = \frac{d}{dx}\left(\int g(x)\,dx\right)$$

so

$$\frac{d}{dy}\left(\int h(y)\,dy\right)\frac{dy}{dx} = g(x)$$

and

$$h(y)\,\frac{dy}{dx} = g(x)$$

Thus Equation 1 is satisfied.

EXAMPLE 1

(a) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$.

(b) Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

SOLUTION

(a) We write the equation in terms of differentials and integrate both sides:

$$y^2\,dy = x^2\,dx$$

$$\int y^2\,dy = \int x^2\,dx$$

$$\tfrac{1}{3}y^3 = \tfrac{1}{3}x^3 + C$$

where C is an arbitrary constant. (We could have used a constant C_1 on the left side and another constant C_2 on the right side. But then we could combine these constants by writing $C = C_2 - C_1$.)

Solving for y, we get

$$y = \sqrt[3]{x^3 + 3C}$$

We could leave the solution like this or we could write it in the form

$$y = \sqrt[3]{x^3 + K}$$

where $K = 3C$. (Since C is an arbitrary constant, so is K.)

(b) If we put $x = 0$ in the general solution in part (a), we get $y(0) = \sqrt[3]{K}$. To satisfy the initial condition $y(0) = 2$, we must have $\sqrt[3]{K} = 2$ and so $K = 8$. Thus the solution of the initial-value problem is

$$y = \sqrt[3]{x^3 + 8}$$ ■

Figure 1 shows graphs of several members of the family of solutions of the differential equation in Example 1. The solution of the initial-value problem in part (b) is shown in red.

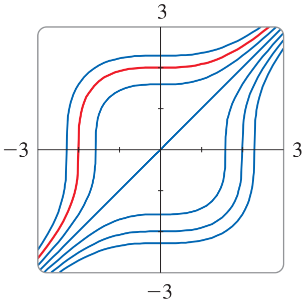


FIGURE 1

EXAMPLE 2 Solve the differential equation $\dfrac{dy}{dx} = \dfrac{6x^2}{2y + \cos y}$.

SOLUTION Writing the equation in differential form and integrating both sides, we have

$$(2y + \cos y)\,dy = 6x^2\,dx$$

$$\int (2y + \cos y)\,dy = \int 6x^2\,dx$$

$$y^2 + \sin y = 2x^3 + C \tag{3}$$

where C is a constant. Equation 3 gives the general solution implicitly. In this case it's impossible to solve the equation to express y explicitly as a function of x. ■

Some computer software can plot curves defined by implicit equations. Figure 2 shows the graphs of several members of the family of solutions of the differential equation in Example 2. As we look at the curves from left to right, the values of C are 3, 2, 1, 0, −1, −2, and −3.

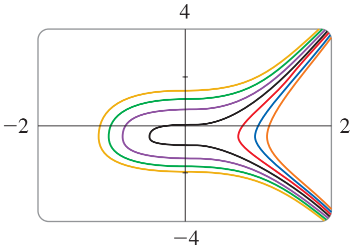


FIGURE 2

EXAMPLE 3 Solve the differential equation $y' = x^2y$.

SOLUTION First we rewrite the equation using Leibniz notation:

$$\frac{dy}{dx} = x^2y$$

It follows from a uniqueness theorem for solutions of differential equations like the equation in Example 3 that if two solutions agree at one x-value, then they must agree at all x-values. (Two solution curves are either identical or never intersect.) Because $y = 0$ is a solution of the differential equation in Example 3, we know that all other solutions must have $y(x) \neq 0$ for all x.

We can easily verify that the constant function $y = 0$ is a solution of the given differential equation. If $y \neq 0$, we can rewrite the equation in differential notation and integrate:

$$\frac{dy}{y} = x^2\,dx \qquad y \neq 0$$

$$\int \frac{dy}{y} = \int x^2\,dx$$

$$\ln|y| = \frac{x^3}{3} + C$$

This equation defines y implicitly as a function of x. But in this case we can solve explicitly for y as follows:

$$|y| = e^{\ln|y|} = e^{(x^3/3)+C} = e^C e^{x^3/3}$$

so

$$y = \pm e^C e^{x^3/3}$$

We can write the general solution in the form

$$y = Ae^{x^3/3}$$

where A is an arbitrary constant ($A = e^C$, or $A = -e^C$, or $A = 0$). ■

Figure 3 shows a direction field for the differential equation in Example 3. Compare it with Figure 4, in which we use the equation $y = Ae^{x^3/3}$ to graph solutions for several values of A. If you use the direction field to sketch solution curves with y-intercepts 5, 2, 1, -1, and -2, they will resemble the curves in Figure 4.

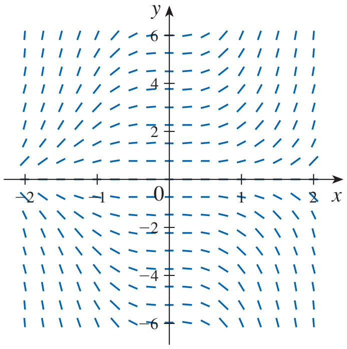


FIGURE 3

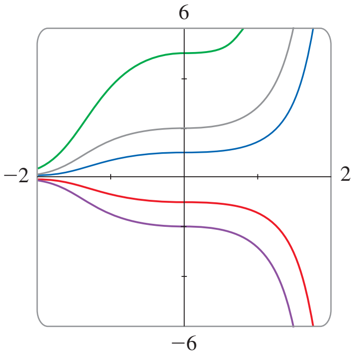


FIGURE 4

EXAMPLE 4 In Section 9.2 we modeled the current $I(t)$ in the electric circuit shown in Figure 5 by the differential equation

$$L\frac{dI}{dt} + RI = E(t)$$

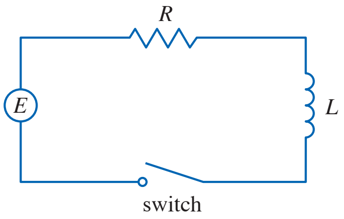


FIGURE 5

Find an expression for the current in a circuit where the resistance is 12 Ω, the inductance is 4 H, a battery gives a constant voltage of 60 V, and the switch is turned on when $t = 0$. What is the limiting value of the current?

SOLUTION With $L = 4$, $R = 12$, and $E(t) = 60$, the equation becomes

$$4\frac{dI}{dt} + 12I = 60 \qquad \text{or} \qquad \frac{dI}{dt} = 15 - 3I$$

and the initial-value problem is

$$\frac{dI}{dt} = 15 - 3I \qquad I(0) = 0$$

We recognize this equation as being separable, and we solve it as follows:

$$\int \frac{dI}{15 - 3I} = \int dt \qquad (15 - 3I \neq 0)$$

$$-\tfrac{1}{3} \ln |15 - 3I| = t + C$$

$$|15 - 3I| = e^{-3(t+C)}$$

$$15 - 3I = \pm e^{-3C} e^{-3t} = Ae^{-3t}$$

$$I = 5 - \tfrac{1}{3}Ae^{-3t}$$

Since $I(0) = 0$, we have $5 - \frac{1}{3}A = 0$, so $A = 15$ and the solution is

$$I(t) = 5 - 5e^{-3t}$$

The limiting current, in amperes, is

$$\lim_{t \to \infty} I(t) = \lim_{t \to \infty} (5 - 5e^{-3t}) = 5 - 5 \lim_{t \to \infty} e^{-3t} = 5 - 0 = 5 \qquad ■$$

Figure 6 shows how the solution in Example 4 (the current) approaches its limiting value. Comparison with Figure 9.2.10 shows that we were able to draw a fairly accurate solution curve from the direction field.

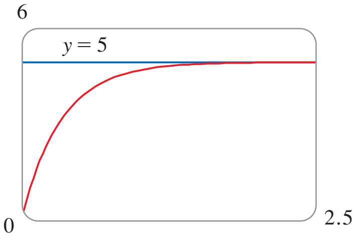


FIGURE 6

■ Orthogonal Trajectories

An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family orthogonally, that is, at right angles (see Figure 7). For instance, each member of the family $y = mx$ of straight lines through the origin is an orthogonal trajectory of the family $x^2 + y^2 = r^2$ of concentric circles with center the origin (see Figure 8). We say that the two families are orthogonal trajectories of each other.

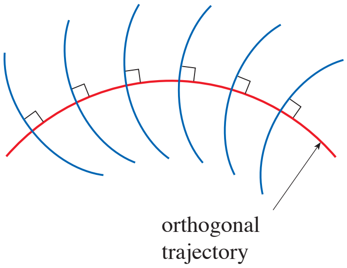


FIGURE 7

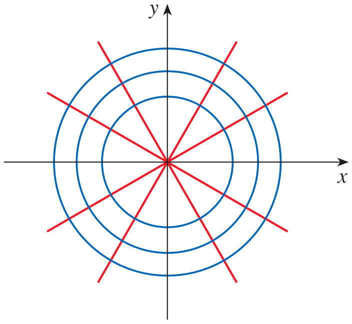


FIGURE 8

EXAMPLE 5 Find the orthogonal trajectories of the family of curves $x = ky^2$, where k is an arbitrary constant.

SOLUTION The curves $x = ky^2$ form a family of parabolas whose axis of symmetry is the x-axis. The first step is to find a single differential equation that is satisfied by all members of the family. If we differentiate $x = ky^2$, we get

$$1 = 2ky\frac{dy}{dx} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{1}{2ky}$$

This differential equation depends on k, but we need an equation that is valid for all values of k simultaneously. To eliminate k we note that, from the equation of the given general parabola $x = ky^2$, we have $k = x/y^2$ and so the differential equation can be written as

$$\frac{dy}{dx} = \frac{1}{2ky} = \frac{1}{2\dfrac{x}{y^2}y} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{y}{2x}$$

This means that the slope of the tangent line at any point (x, y) on one of the parabolas is $y' = y/(2x)$. On an orthogonal trajectory the slope of the tangent line must be the negative reciprocal of this slope. Therefore the orthogonal trajectories must satisfy the differential equation

$$\frac{dy}{dx} = -\frac{2x}{y}$$

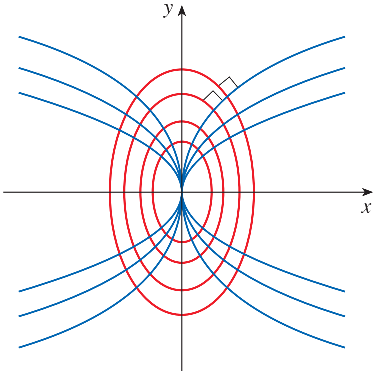


FIGURE 9

This differential equation is separable, and we solve it as follows:

$$\int y\,dy = -\int 2x\,dx$$

$$\frac{y^2}{2} = -x^2 + C$$

$$x^2 + \frac{y^2}{2} = C \tag{4}$$

where C is an arbitrary positive constant. Thus the orthogonal trajectories are the family of ellipses given by Equation 4 and sketched in Figure 9. ■

Orthogonal trajectories occur in various branches of physics. For example, in an electrostatic field the lines of force are orthogonal to the lines of constant potential. Also, the streamlines in aerodynamics are orthogonal trajectories of the velocity-equipotential curves.

■ Mixing Problems

A typical mixing problem involves a tank of fixed capacity containing a thoroughly mixed solution of some substance, such as salt. A solution of a given concentration enters the tank at a fixed rate and the mixture, thoroughly stirred, leaves at a fixed rate, which may differ from the entering rate. If $y(t)$ denotes the amount of substance in the tank at time t, then $y'(t)$ is the rate at which the substance is being added minus the rate at which it is being removed. The mathematical description of this situation often leads to a first-order separable differential equation. We can use the same type of reasoning to model a variety of phenomena: chemical reactions, discharge of pollutants into a lake, injection of a drug into the bloodstream.

EXAMPLE 6 A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt will there be in the tank after half an hour?

SOLUTION Let $y(t)$ be the amount of salt (in kilograms) after t minutes. We are given that $y(0) = 20$ and we want to find $y(30)$. We do this by finding a differential equation satisfied by $y(t)$. Note that dy/dt is the rate of change of the amount of salt, so

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out}) \tag{5}$$

where (rate in) is the rate at which salt enters the tank and (rate out) is the rate at which salt leaves the tank. We have

$$\text{rate in} = \left(0.03\,\frac{\text{kg}}{\text{L}}\right)\left(25\,\frac{\text{L}}{\text{min}}\right) = 0.75\,\frac{\text{kg}}{\text{min}}$$

The tank always contains 5000 L of liquid, so the concentration at time t is $y(t)/5000$ (measured in kilograms per liter). Since the brine flows out at a rate of 25 L/min, we have

$$\text{rate out} = \left(\frac{y(t)}{5000}\,\frac{\text{kg}}{\text{L}}\right)\left(25\,\frac{\text{L}}{\text{min}}\right) = \frac{y(t)}{200}\,\frac{\text{kg}}{\text{min}}$$

Thus, from Equation 5, we get

$$\frac{dy}{dt} = 0.75 - \frac{y(t)}{200} = \frac{150 - y(t)}{200}$$

Solving this separable differential equation, we obtain

$$\int \frac{dy}{150 - y} = \int \frac{dt}{200}$$

$$-\ln|150 - y| = \frac{t}{200} + C$$

Since $y(0) = 20$, we have $-\ln 130 = C$, so

$$-\ln|150 - y| = \frac{t}{200} - \ln 130$$

Therefore $$|150 - y| = 130e^{-t/200}$$

Since $y(t)$ is continuous and $y(0) = 20$, and the right side is never 0, we deduce that $150 - y(t)$ is always positive. Thus $|150 - y| = 150 - y$ and so

$$y(t) = 150 - 130e^{-t/200}$$

The amount of salt after 30 min is

$$y(30) = 150 - 130e^{-30/200} \approx 38.1 \text{ kg}$$ ■

Figure 10 shows the graph of the function $y(t)$ of Example 6. Notice that, as time goes by, the amount of salt approaches 150 kg.

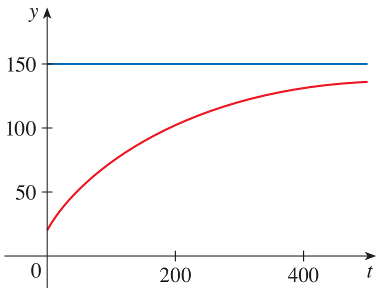


FIGURE 10

9.3 Exercises

1–12 Solve the differential equation.

1. $\dfrac{dy}{dx} = 3x^2y^2$

2. $\dfrac{dy}{dx} = \dfrac{x}{y^4}$

3. $\dfrac{dy}{dx} = x\sqrt{y}$

4. $xy' = y + 3$

5. $xyy' = x^2 + 1$

6. $y' + xe^y = 0$

7. $(e^y - 1)y' = 2 + \cos x$

8. $\dfrac{dy}{dx} = 2x(y^2 + 1)$

9. $\dfrac{dp}{dt} = t^2p - p + t^2 - 1$

10. $\dfrac{dz}{dt} + e^{t+z} = 0$

11. $\dfrac{d\theta}{dt} = \dfrac{t \sec\theta}{\theta e^{t^2}}$

12. $\dfrac{dH}{dR} = \dfrac{RH^2\sqrt{1 + R^2}}{\ln H}$

13–20 Find the solution of the differential equation that satisfies the given initial condition.

13. $\dfrac{dy}{dx} = xe^y, \quad y(0) = 0$

14. $\dfrac{dP}{dt} = \sqrt{Pt}, \quad P(1) = 2$

15. $\dfrac{dA}{dr} = Ab^2 \cos br, \quad A(0) = b^3$

16. $x^2y' = k \sec y, \quad y(1) = \pi/6$

17. $\dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u}, \quad u(0) = -5$

18. $x + 3y^2\sqrt{x^2 + 1}\,\dfrac{dy}{dx} = 0, \quad y(0) = 1$

19. $x \ln x = y\left(1 + \sqrt{3 + y^2}\right) y', \quad y(1) = 1$

20. $\dfrac{dy}{dx} = \dfrac{x \sin x}{y}, \quad y(0) = -1$

21. Find an equation of the curve that passes through the point $(0, 2)$ and whose slope at (x, y) is x/y.

22. Find the function f such that $f'(x) = xf(x) - x$ and $f(0) = 2$.

23. Solve the differential equation $y' = x + y$ by making the change of variable $u = x + y$.

24. Solve the differential equation $xy' = y + xe^{y/x}$ by making the change of variable $v = y/x$.

25. (a) Solve the differential equation $y' = 2x\sqrt{1 - y^2}$.
(b) Solve the initial-value problem $y' = 2x\sqrt{1 - y^2}$, $y(0) = 0$, and graph the solution.
(c) Does the initial-value problem $y' = 2x\sqrt{1 - y^2}$, $y(0) = 2$, have a solution? Explain.

26. Solve the differential equation $e^{-y}y' + \cos x = 0$ and graph several members of the family of solutions. How does the solution curve change as the constant C varies?

27. Solve the initial-value problem $y' = (\sin x)/\sin y$, $y(0) = \pi/2$, and graph the (implicitly defined) solution.

28. Solve the differential equation $y' = x\sqrt{x^2 + 1}/(ye^y)$ and graph several members of the family of (implicitly defined) solutions. How does the solution curve change as the constant C varies?

29–30

(a) Use a computer to draw a direction field for the differential equation. Get a printout and use it to sketch some solution curves without solving the differential equation.

(b) Solve the differential equation.

(c) Graph several members of the family of solutions obtained in part (b). Compare with the curves from part (a).

29. $y' = y^2$

30. $y' = xy$

31–34 Find the orthogonal trajectories of the family of curves. Graph several members of each family on a common screen.

31. $x^2 + 2y^2 = k^2$

32. $y^2 = kx^3$

33. $y = \dfrac{k}{x}$

34. $y = \dfrac{1}{x + k}$

35–37 Integral Equations An *integral equation* is an equation that contains an unknown function $y(x)$ and an integral that involves $y(x)$. Solve the given integral equation. [*Hint:* Use an initial condition obtained from the integral equation.]

35. $y(x) = 2 + \displaystyle\int_2^x [t - ty(t)]\,dt$

36. $y(x) = 2 + \displaystyle\int_1^x \frac{dt}{ty(t)}, \quad x > 0$

37. $y(x) = 4 + \displaystyle\int_0^x 2t\sqrt{y(t)}\,dt$

---

38. Find a function f such that $f(3) = 2$ and

$$(t^2 + 1)f'(t) + [f(t)]^2 + 1 = 0 \qquad t \neq 1$$

[*Hint:* Use the addition formula for $\tan(x + y)$ on Reference Page 2.]

39. Solve the initial-value problem in Exercise 9.2.27 to find an expression for the charge at time t. Find the limiting value of the charge.

40. In Exercise 9.2.28 we discussed a differential equation that models the temperature of a 95°C cup of coffee in a 20°C room. Solve the differential equation to find an expression for the temperature of the coffee at time t.

41. In Exercise 9.1.27 we formulated a model for learning in the form of the differential equation

$$\frac{dP}{dt} = k(M - P)$$

where $P(t)$ measures the performance of someone learning a skill after a training time t, M is the maximum level of performance, and k is a positive constant. Solve this differential equation to find an expression for $P(t)$. What is the limit of this expression?

42. In an elementary chemical reaction, single molecules of two reactants A and B form a molecule of the product C: $\mathrm{A} + \mathrm{B} \rightarrow \mathrm{C}$. The law of mass action states that the rate of reaction is proportional to the product of the concentrations of A and B:

$$\frac{d[\mathrm{C}]}{dt} = k[\mathrm{A}][\mathrm{B}]$$

(See Example 3.7.4.) Thus, if the initial concentrations are $[\mathrm{A}] = a$ moles/L and $[\mathrm{B}] = b$ moles/L and we write $x = [\mathrm{C}]$, then we have

$$\frac{dx}{dt} = k(a - x)(b - x)$$

(a) Assuming that $a \neq b$, find x as a function of t. Use the fact that the initial concentration of C is 0.

(b) Find $x(t)$ assuming that $a = b$. How does this expression for $x(t)$ simplify if it is known that $[\mathrm{C}] = \frac{1}{2}a$ after 20 seconds?

43. In contrast to the situation of Exercise 42, experiments show that the reaction $\mathrm{H_2} + \mathrm{Br_2} \longrightarrow 2\mathrm{HBr}$ satisfies the rate law

$$\frac{d[\mathrm{HBr}]}{dt} = k[\mathrm{H_2}][\mathrm{Br_2}]^{1/2}$$

and so for this reaction the differential equation becomes

$$\frac{dx}{dt} = k(a - x)(b - x)^{1/2}$$

where $x = [\mathrm{HBr}]$ and a and b are the initial concentrations of hydrogen and bromine.

(a) Find x as a function of t in the case where $a = b$. Use the fact that $x(0) = 0$.

(b) If $a > b$, find t as a function of x. [*Hint:* In performing the integration, make the substitution $u = \sqrt{b - x}$.]

44. A sphere with radius 1 m has temperature 15°C. It lies inside a concentric sphere with radius 2 m and temperature 25°C. The temperature $T(r)$ at a distance r from the common center of the spheres satisfies the differential equation

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} = 0$$

If we let $S = dT/dr$, then S satisfies a first-order differential equation. Solve it to find an expression for the temperature $T(r)$ between the spheres.

45. A glucose solution is administered intravenously into the bloodstream at a constant rate r. As the glucose is added, it is converted into other substances and removed from the bloodstream at a rate that is proportional to the concentration at that time. Thus a model for the concentration $C = C(t)$ of the glucose solution in the bloodstream is

$$\frac{dC}{dt} = r - kC$$

where k is a positive constant.

(a) Suppose that the concentration at time $t = 0$ is C_0. Determine the concentration at any time t by solving the differential equation.

(b) Assuming that $C_0 < r/k$, find $\lim_{t\to\infty} C(t)$ and interpret your answer.

46. A certain small country has \$10 billion in paper currency in circulation, and each day \$50 million comes into the country's banks. The government decides to introduce new currency by having the banks replace old bills with new ones whenever old currency comes into the banks. Let $x = x(t)$ denote the amount of new currency in circulation at time t, with $x(0) = 0$.

(a) Formulate a mathematical model in the form of an initial-value problem that represents the "flow" of the new currency into circulation.

(b) Solve the initial-value problem found in part (a).

(c) How long will it take for the new bills to account for 90% of the currency in circulation?

47. A tank contains 1000 L of brine with 15 kg of dissolved salt. Pure water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank (a) after t minutes and (b) after 20 minutes?

48. The air in a room with volume 180 m^3 contains 0.15% carbon dioxide initially. Fresher air with only 0.05% carbon dioxide flows into the room at a rate of 2 m^3/min and the mixed air flows out at the same rate. Find the percentage of carbon dioxide in the room as a function of time. What happens in the long run?

49. A vat with 2000 L of beer contains 4% alcohol (by volume). Beer with 6% alcohol is pumped into the vat at a rate of 20 L/min and the mixture is pumped out at the same rate. What is the percentage of alcohol after an hour?

50. A tank contains 1000 L of pure water. Brine that contains 0.05 kg of salt per liter of water enters the tank at a rate of 5 L/min. Brine that contains 0.04 kg of salt per liter of water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 15 L/min. How much salt is in the tank (a) after t minutes and (b) after one hour?

51. Terminal Velocity When a raindrop falls, it increases in size and so its mass at time t is a function of t, namely, $m(t)$. The rate of growth of the mass is $km(t)$ for some positive constant k. When we apply Newton's Law of Motion to the raindrop, we get $(mv)' = gm$, where v is the velocity of the raindrop (directed downward) and g is the acceleration due to gravity. The *terminal velocity* of the raindrop is $\lim_{t\to\infty} v(t)$. Find an expression for the terminal velocity in terms of g and k.

52. An object of mass m is moving horizontally through a medium which resists the motion with a force that is a function of the velocity; that is,

$$m\frac{d^2s}{dt^2} = m\frac{dv}{dt} = f(v)$$

where $v = v(t)$ and $s = s(t)$ represent the velocity and position of the object at time t, respectively. For example, think of a boat moving through the water.

(a) Suppose that the resisting force is proportional to the velocity, that is, $f(v) = -kv$, k a positive constant. (This model is appropriate for small values of v.) Let $v(0) = v_0$ and $s(0) = s_0$ be the initial values of v and s. Determine v and s at any time t. What is the total distance that the object travels from time $t = 0$?

(b) For larger values of v a better model is obtained by supposing that the resisting force is proportional to the square of the velocity, that is, $f(v) = -kv^2$, $k > 0$. (This model was first proposed by Newton.) Let v_0 and s_0 be the initial values of v and s. Determine v and s at any time t. What is the total distance that the object travels in this case?

53. Allometric Growth In biology, *allometric growth* refers to relationships between sizes of parts of an organism (skull length and body length, for instance). If $L_1(t)$ and $L_2(t)$ are the sizes of two organs in an organism of age t, then L_1 and L_2 satisfy an allometric law if their specific growth rates are proportional:

$$\frac{1}{L_1}\frac{dL_1}{dt} = k\frac{1}{L_2}\frac{dL_2}{dt}$$

where k is a constant.

(a) Use the allometric law to write a differential equation relating L_1 and L_2 and solve it to express L_1 as a function of L_2.

(b) In a study of several species of unicellular algae, the proportionality constant in the allometric law relating B (cell biomass) and V (cell volume) was found to be $k = 0.0794$. Write B as a function of V.

54. A model for tumor growth is given by the Gompertz equation

$$\frac{dV}{dt} = a(\ln b - \ln V)V$$

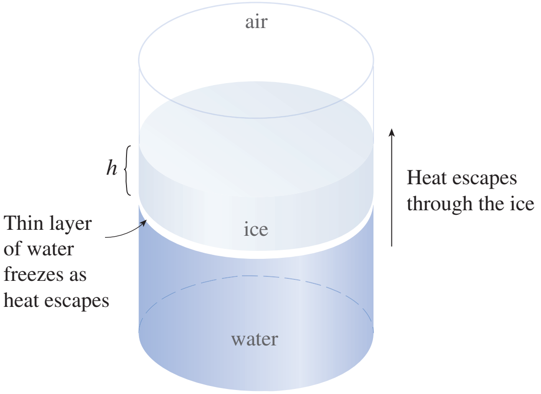
where a and b are positive constants and V is the volume of the tumor measured in mm^3.

(a) Find a family of solutions for tumor volume as a function of time.
(b) Find the solution that has an initial tumor volume of $V(0) = 1\text{ mm}^3$.

55. Let $A(t)$ be the area of a tissue culture at time t and let M be the final area of the tissue when growth is complete. Most cell divisions occur on the periphery of the tissue and the number of cells on the periphery is proportional to $\sqrt{A(t)}$. So a reasonable model for the growth of tissue is obtained by assuming that the rate of growth of the area is jointly proportional to $\sqrt{A(t)}$ and $M - A(t)$.
(a) Formulate a differential equation and use it to show that the tissue grows fastest when $A(t) = \frac{1}{3}M$.
T (b) Solve the differential equation to find an expression for $A(t)$. Use a computer to perform the integration.

56. Sea Ice Many factors influence the formation and growth of sea ice. In this exercise we develop a simplified model that describes how the thickness of sea ice is affected over time by the temperatures of the air and ocean water. As we commented in Section 1.2, a good model simplifies reality enough to permit mathematical calculations but is accurate enough to provide valuable conclusions.

Consider a column of air/ice/water as shown in the figure. Let's assume that the temperature T_a (in °C) at the ice/air interface is constant (with T_a below the freezing point of the ocean water) and that the temperature T_w at the ice/water interface also remains constant (where T_w is greater than the water's freezing point).


air
h
ice
water
Thin layer of water freezes as heat escapes
Heat escapes through the ice

Energy transfers upward through the ice from the warmer seawater to the colder air in the form of heat Q, measured in joules (J). By Fourier's law of heat conduction, the rate of heat transfer dQ/dt satisfies the differential equation

$$\frac{dQ}{dt} = \frac{kA}{h}(T_w - T_a)$$

where k is a constant called the *thermal conductivity* of the ice, A is the (horizontal) cross-sectional area (in m^2) of the column, and h is the ice thickness (in m).

(a) The loss of a small amount of heat ΔQ from the seawater causes a thin layer of thickness Δh of water at the ice/water interface to freeze. The mass density D (measured in kg/m^3) of seawater varies with temperature, but at the interface we can assume that the temperature is constant (near 0°C) and hence D is constant. Let L be the *latent heat* of seawater, defined as the amount of heat loss required to freeze 1 kg of the water. Show that $\Delta h \approx (1/LAD)\Delta Q$ and hence

$$\frac{dh}{dQ} = \frac{1}{LAD}$$

(b) Use the Chain Rule to write the differential equation

$$\frac{dh}{dt} = \frac{k}{LDh}(T_w - T_a)$$

and explain why this equation predicts the fact that thin ice grows more rapidly than thick ice, and thus a crack in ice tends to "heal" and the thickness of an ice field tends to become uniform over time.
(c) If the thickness of the ice at time $t = 0$ is h_0, find a model for the ice thickness at any time t by solving the differential equation in part (b).

Source: Adapted from M. Freiberger, "Maths and Climate Change: The Melting Arctic," *Plus* (2008): http://plus.maths.org/content/maths-and-climate-change-melting-arctic. Accessed March 9, 2019.

57. Escape Velocity According to Newton's Law of Universal Gravitation, the gravitational force on an object of mass m that has been projected vertically upward from the earth's surface is

$$F = \frac{mgR^2}{(x + R)^2}$$

where $x = x(t)$ is the object's distance above the surface at time t, R is the earth's radius, and g is the acceleration due to gravity. Also, by Newton's Second Law, $F = ma = m(dv/dt)$ and so

$$m\frac{dv}{dt} = -\frac{mgR^2}{(x + R)^2}$$

(a) Suppose a rocket is fired vertically upward with an initial velocity v_0. Let h be the maximum height above the surface reached by the object. Show that

$$v_0 = \sqrt{\frac{2gRh}{R + h}}$$

[*Hint:* By the Chain Rule, $m(dv/dt) = mv(dv/dx)$.]
(b) Calculate $v_e = \lim_{h\to\infty} v_0$. This limit is called the *escape velocity* for the earth. (Another method of finding escape velocity is given in Exercise 7.8.77.)
(c) Use $R = 6370$ km and $g = 9.8\text{ m/s}^2$ to calculate v_e in meters per second and in kilometers per second.

APPLIED PROJECT | HOW FAST DOES A TANK DRAIN?

If water (or other liquid) drains from a tank, we expect that the flow will be greatest at first (when the water depth is greatest) and will gradually decrease as the water level decreases. But we need a more precise mathematical description of how the flow decreases in order to answer the kinds of questions that engineers ask: How long does it take for a tank to drain completely? How much water should a tank hold in order to guarantee a certain minimum water pressure for a sprinkler system?

Let $h(t)$ and $V(t)$ be the height and volume of water in a tank at time t. If water drains through a hole with area a at the bottom of the tank, then Torricelli's Law says that

$$\boxed{1} \qquad \frac{dV}{dt} = -a\sqrt{2gh}$$

where g is the acceleration due to gravity. So the rate at which water flows from the tank is proportional to the square root of the water height.

1. (a) Suppose the tank is cylindrical with height 2 m and radius 1 m and the hole is circular with radius 1 inch. If we take $g = 9.8 \text{ m/s}^2$, show that h satisfies the differential equation

$$\frac{dh}{dt} = -0.0004\sqrt{20h}$$

(b) Solve this equation to find the height of the water at time t, assuming the tank is full at time $t = 0$.

(c) How long will it take for the water to drain completely?

Problem 2(b) is best done as a classroom demonstration or as a group project with three students in each group: a timekeeper to call out seconds, a bottle keeper to estimate the height every 10 seconds, and a recordkeeper to record these values.

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2. Because of the rotation and viscosity of the liquid, the theoretical model given by Equation 1 isn't quite accurate. Instead, the model

$$\boxed{2} \qquad \frac{dh}{dt} = k\sqrt{h}$$

is often used and the constant k (which depends on the physical properties of the liquid) is determined from data concerning the draining of the tank.

(a) Suppose that a hole is drilled in the side of a cylindrical bottle and the height h of the water (above the hole) decreases from 10 cm to 3 cm in 68 seconds. Use Equation 2 to find an expression for $h(t)$. Evaluate $h(t)$ for $t = 10, 20, 30, 40, 50, 60$.

(b) Drill a 4-mm hole near the bottom of the cylindrical part of a two-liter plastic soft-drink bottle. Attach a strip of masking tape marked in centimeters from 0 to 10, with 0 corresponding to the top of the hole. With one finger over the hole, fill the bottle with water to the 10-cm mark. Then take your finger off the hole and record the values of $h(t)$ for $t = 10, 20, 30, 40, 50, 60$ seconds. (You will probably find that it takes 68 seconds for the level to decrease to $h = 3$ cm.) Compare your data with the values of $h(t)$ from part (a). How well did the model predict the actual values?

3. In many parts of the world, the water for sprinkler systems in large hotels and hospitals is supplied by gravity from cylindrical tanks on or near the roofs of the buildings. Suppose such a tank has radius 3 m and the diameter of the outlet is 6 cm. An engineer has to guarantee that the water pressure at the tank outlet will be at least 104 kPa for a period of 10 minutes. (When a fire happens, the electrical system might fail and it could take up to 10 minutes for the emergency generator and fire pump to be activated.) How tall should the engineer specify the tank to be in order to make such a guarantee? (Use the fact that the water pressure at a depth of d meters is $P = 10d$ kilopascals. See Section 8.3.)

4. Not all water tanks are shaped like cylinders. Suppose a tank has cross-sectional area $A(h)$ at height h. Then the volume of water up to height h is $V = \int_0^h A(u)\,du$ and so the Fundamental Theorem of Calculus gives $dV/dh = A(h)$. It follows that

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} = A(h)\frac{dh}{dt}$$

and so Torricelli's Law becomes

$$A(h)\frac{dh}{dt} = -a\sqrt{2gh}$$

(a) Suppose the tank has the shape of a sphere with radius 2 m and is initially half full of water. If the radius of the circular hole is 1 cm and we take $g = 10\text{ m/s}^2$, show that h satisfies the differential equation

$$(4h - h^2)\frac{dh}{dt} = -0.0001\sqrt{20h}$$

(b) How long will it take for the water to drain completely?

9.4 Models for Population Growth

In Section 9.1 we developed two differential equations that describe population growth. In this section we further investigate these equations and use the techniques of Section 9.3 to obtain explicit models for a population.

The Law of Natural Growth

One of the models for population growth that we considered in Section 9.1 was based on the assumption that the population grows at a rate proportional to the size of the population:

$$\frac{dP}{dt} = kP$$

Is that a reasonable assumption? Suppose we have a population (of bacteria, for instance) with size $P = 1000$ and at a certain time it is growing at a rate of $P' = 300$ bacteria per hour. Now let's take another 1000 bacteria of the same type and put them with the first population. Each half of the combined population was previously growing at a rate of 300 bacteria per hour. We would expect the total population of 2000 to increase at a rate of 600 bacteria per hour initially (provided there's enough room and nutrition). So if we double the size, we double the growth rate. It seems reasonable that the growth rate should be proportional to the size.

In general, if $P(t)$ is the value of a quantity y at time t and if the rate of change of P with respect to t is proportional to its size $P(t)$ at any time, then

$$\boxed{1} \qquad \boxed{\frac{dP}{dt} = kP}$$

where k is a constant. Equation 1 is sometimes called the **law of natural growth**. If k is positive, then the population increases; if k is negative, it decreases.

Because Equation 1 is a separable differential equation, we can solve it by the methods of Section 9.3:

$$\int \frac{dP}{P} = \int k\,dt$$

$$\ln|P| = kt + C$$

$$|P| = e^{kt+C} = e^C e^{kt}$$

$$P = Ae^{kt}$$

where A ($= \pm e^C$ or 0) is an arbitrary constant. To see the significance of the constant A, we observe that

$$P(0) = Ae^{k \cdot 0} = A$$

Therefore A is the initial value of the function.

Examples and exercises on the use of (2) are given in Section 3.8.

2 The solution of the initial-value problem

$$\frac{dP}{dt} = kP \qquad P(0) = P_0$$

is

$$P(t) = P_0 e^{kt}$$

Another way of writing Equation 1 is

$$\frac{dP/dt}{P} = k$$

which says that the *relative growth rate* (the growth rate divided by the population size; see Section 3.8) is constant. Then (2) says that a population with constant relative growth rate must grow exponentially.

We can account for emigration (or "harvesting") from a population by modifying Equation 1: if the rate of emigration is a constant m, then the rate of change of the population is modeled by the differential equation

$$\boxed{3} \qquad \frac{dP}{dt} = kP - m$$

See Exercise 17 for the solution and consequences of Equation 3.

The Logistic Model

As we discussed in Section 9.1, a population often increases exponentially in its early stages but levels off eventually and approaches its carrying capacity because of limited resources. If $P(t)$ is the size of the population at time t, we assume that

$$\frac{dP}{dt} \approx kP \qquad \text{if } P \text{ is small}$$

This says that the growth rate is initially close to being proportional to size. In other words, the relative growth rate is almost constant when the population is small. But we also want to reflect the fact that the relative growth rate decreases as the population P increases and becomes negative if P ever exceeds its **carrying capacity** M, the maximum population that the environment is capable of sustaining in the long run. The simplest expression for the relative growth rate that incorporates these assumptions is

$$\frac{dP/dt}{P} = k\left(1 - \frac{P}{M}\right)$$

Multiplying by P, we obtain the model for population growth known as the **logistic differential equation**, which we first saw in Section 9.1:

$$\boxed{4} \qquad \boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)}$$

Notice from Equation 4 that if P is small compared with M, then P/M is close to 0 and so $dP/dt \approx kP$. However, if $P \to M$ (the population approaches its carrying capacity), then $P/M \to 1$, so $dP/dt \to 0$. We can deduce information about whether solutions increase or decrease directly from Equation 4. If the population P lies between 0 and M, then the right side of the equation is positive, so $dP/dt > 0$ and the population increases. But if the population exceeds the carrying capacity ($P > M$), then $1 - P/M$ is negative, so $dP/dt < 0$ and the population decreases.

Let's start our more detailed analysis of the logistic differential equation by looking at a direction field.

EXAMPLE 1 Draw a direction field for the logistic equation with $k = 0.08$ and carrying capacity $M = 1000$. What can you deduce about the solutions?

SOLUTION In this case the logistic differential equation is

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)$$

A direction field for this equation is shown in Figure 1. We show only the first quadrant because negative populations aren't meaningful and here we are interested only in what happens after $t = 0$.

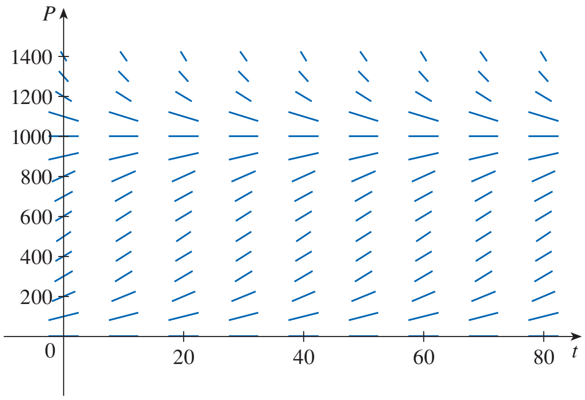


FIGURE 1
Direction field for the logistic equation in Example 1

The logistic equation is autonomous (dP/dt depends only on P, not on t), so the slopes are the same along any horizontal line. As expected, the slopes are positive for $0 < P < 1000$ and negative for $P > 1000$.

The slopes are small when P is close to 0 or 1000 (the carrying capacity). Notice that the solutions move away from the equilibrium solution $P = 0$ and move toward the equilibrium solution $P = 1000$.

In Figure 2 we use the direction field to sketch solution curves with initial populations $P(0) = 100$, $P(0) = 400$, and $P(0) = 1300$. Notice that solution curves that start below $P = 1000$ are increasing and those that start above $P = 1000$ are decreasing. The slopes are greatest when $P \approx 500$ and therefore the solution curves that start below $P = 1000$ have inflection points when $P \approx 500$. In fact we can prove that all solution curves that start below $P = 500$ have an inflection point when P is exactly 500. (See Exercise 13.)

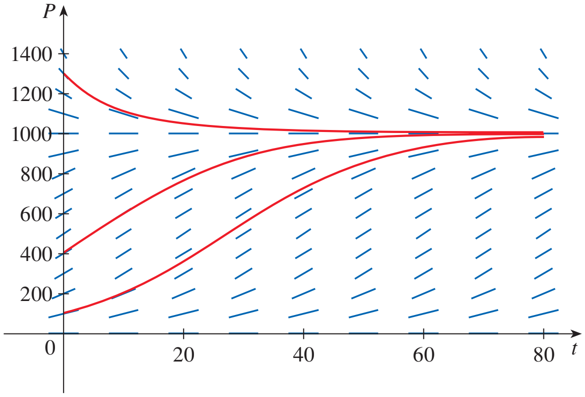


FIGURE 2
Solution curves for the logistic equation in Example 1

■

The logistic equation (4) is separable and so we can solve it explicitly using the method of Section 9.3. Since

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

we have

$$\boxed{5} \qquad \int \frac{dP}{P(1 - P/M)} = \int k\,dt$$

To evaluate the integral on the left side, we write

$$\frac{1}{P(1 - P/M)} = \frac{M}{P(M - P)}$$

Using partial fractions (see Section 7.4), we get

$$\frac{M}{P(M - P)} = \frac{1}{P} + \frac{1}{M - P}$$

This enables us to rewrite Equation 5:

$$\int \left(\frac{1}{P} + \frac{1}{M - P}\right) dP = \int k\,dt$$

$$\ln|P| - \ln|M - P| = kt + C$$

$$\ln\left|\frac{M-P}{P}\right| = -kt - C$$

$$\left|\frac{M-P}{P}\right| = e^{-kt-C} = e^{-C}e^{-kt}$$

$$\boxed{6} \qquad \frac{M-P}{P} = Ae^{-kt}$$

where $A = \pm e^{-C}$. Solving Equation 6 for P, we get

$$\frac{M}{P} - 1 = Ae^{-kt} \qquad \Rightarrow \qquad \frac{P}{M} = \frac{1}{1 + Ae^{-kt}}$$

so
$$P = \frac{M}{1 + Ae^{-kt}}$$

We find the value of A by putting $t = 0$ in Equation 6. If $t = 0$, then $P = P_0$ (the initial population), so

$$\frac{M - P_0}{P_0} = Ae^0 = A$$

Thus the solution to the logistic equation is

$$\boxed{7} \qquad \boxed{P(t) = \frac{M}{1 + Ae^{-kt}} \qquad \text{where } A = \frac{M - P_0}{P_0}}$$

Using the expression for $P(t)$ in Equation 7, we see that

$$\lim_{t \to \infty} P(t) = M$$

which is to be expected.

EXAMPLE 2 Write the solution of the initial-value problem

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) \qquad P(0) = 100$$

and use it to find the population sizes $P(40)$ and $P(80)$. At what time does the population reach 900?

SOLUTION The differential equation is a logistic equation with $k = 0.08$, carrying capacity $M = 1000$, and initial population $P_0 = 100$. So Equation 7 gives the population at time t as

$$P(t) = \frac{1000}{1 + Ae^{-0.08t}} \qquad \text{where } A = \frac{1000 - 100}{100} = 9$$

Thus
$$P(t) = \frac{1000}{1 + 9e^{-0.08t}}$$

So the population sizes when $t = 40$ and $t = 80$ are

$$P(40) = \frac{1000}{1 + 9e^{-3.2}} \approx 731.6 \qquad P(80) = \frac{1000}{1 + 9e^{-6.4}} \approx 985.3$$

The population reaches 900 when

$$\frac{1000}{1 + 9e^{-0.08t}} = 900$$

Solving this equation for t, we get

$$\begin{aligned} 1 + 9e^{-0.08t} &= \tfrac{10}{9} \\ e^{-0.08t} &= \tfrac{1}{81} \\ -0.08t &= \ln \tfrac{1}{81} = -\ln 81 \\ t &= \frac{\ln 81}{0.08} \approx 54.9 \end{aligned}$$

So the population reaches 900 when t is approximately 55. As a check on our work, we graph the population curve in Figure 3 and observe that it intersects the line $P = 900$ at $t \approx 55$. ■

Compare the solution curve in Figure 3 with the lowest solution curve we drew from the direction field in Figure 2.

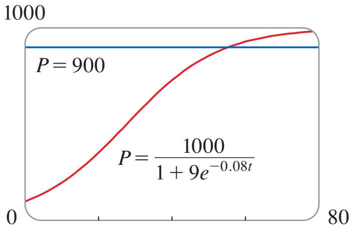


FIGURE 3

Comparison of the Natural Growth and Logistic Models

In the 1930s the biologist G. F. Gause conducted an experiment with the protozoan *Paramecium* and used a logistic equation to model his data. The table gives his daily count of the population of protozoa. He estimated the initial relative growth rate to be 0.7944 and the carrying capacity to be 64.

t (days)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
P (observed)	2	3	22	16	39	52	54	47	50	76	69	51	57	70	53	59	57

EXAMPLE 3 Find the exponential and logistic models for Gause's data. Compare the predicted values with the observed values and comment on the fit for each model.

SOLUTION Given the relative growth rate $k = 0.7944$ and the initial population $P_0 = 2$, the exponential model is

$$P(t) = P_0 e^{kt} = 2e^{0.7944t}$$

Gause used the same value of k for his logistic model. [This is reasonable because $P_0 = 2$ is small compared with the carrying capacity ($M = 64$). The equation

$$\frac{1}{P_0} \frac{dP}{dt}\bigg|_{t=0} = k\left(1 - \frac{2}{64}\right) \approx k$$

shows that the value of k for the logistic model is very close to the value for the exponential model.]

Then the solution of the logistic equation, given in Equation 7, is

$$P(t) = \frac{M}{1 + Ae^{-kt}} = \frac{64}{1 + Ae^{-0.7944t}}$$

where $$A = \frac{M - P_0}{P_0} = \frac{64 - 2}{2} = 31$$

So $$P(t) = \frac{64}{1 + 31e^{-0.7944t}}$$

We use these equations to calculate the predicted values (rounded to the nearest integer) and compare them in the following table.

t (days)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
P (observed)	2	3	22	16	39	52	54	47	50	76	69	51	57	70	53	59	57
P (logistic model)	2	4	9	17	28	40	51	57	61	62	63	64	64	64	64	64	64
P (exponential model)	2	4	10	22	48	106	. . .										

We observe from the table and from the graph in Figure 4 that for the first three or four days the exponential model gives results comparable to those of the more sophisticated logistic model. For $t \geqslant 5$, however, the exponential model is hopelessly inaccurate, but the logistic model fits the observations reasonably well.

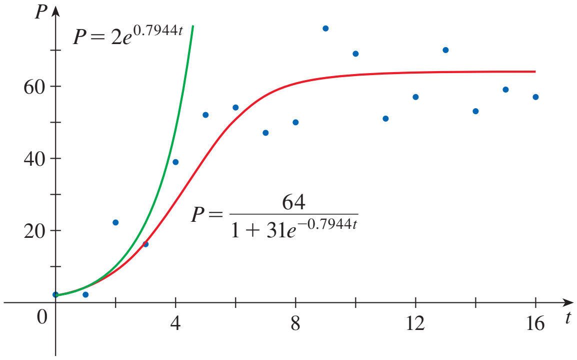


FIGURE 4
The exponential and logistic models for the *Paramecium* data

■

Many countries that formerly experienced exponential growth are now finding that their rates of population growth are declining and the logistic model provides a better model. The table in the margin shows midyear values of the population of Japan, in thousands, from 1960 to 2015. Figure 5 shows these data points, using $t = 0$ to represent 1960, together with a shifted logistic function (obtained from a calculator with the ability to fit a logistic function to data points by regression; see Exercise 15). At first the data points appear to be following an exponential curve but overall a logistic function provides a much more accurate model.

Year	Population (thousands)
1960	94,092
1965	98,883
1970	104,345
1975	111,573
1980	116,807
1985	120,754
1990	123,537
1995	125,327
2000	126,776
2005	127,715
2010	127,579
2015	126,920

Source: U.S. Census Bureau / International Programs / International Data Base. Revised Sept. 18, 2018. Version data 18.0822. Code 12.0321.

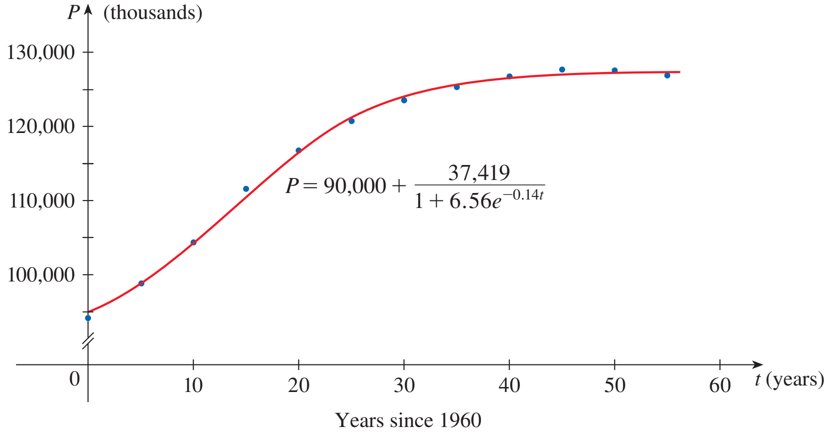


FIGURE 5
Logistic model for the population of Japan

Other Models for Population Growth

The Law of Natural Growth and the logistic differential equation are not the only equations that have been proposed to model population growth. In Exercise 22 we look at the Gompertz growth function and in Exercises 23 and 24 we investigate seasonal-growth models.

Two additional models are modifications of the logistic model. The differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - c$$

has been used to model populations that are subject to harvesting of one sort or another. (Think of a population of fish being caught at a constant rate.) This equation is explored in Exercises 19 and 20.

For some species there is a minimum population level m below which the species tends to become extinct. (Adults may not be able to find suitable mates.) Such populations have been modeled by the differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$$

where the extra factor, $1 - m/P$, takes into account the consequences of a sparse population (see Exercise 21).

9.4 Exercises

1–2 A population grows according to the given logistic equation, where t is measured in weeks.

(a) What is the carrying capacity? What is the value of k?

(b) Write the solution of the equation.

(c) What is the population after 10 weeks?

1. $\dfrac{dP}{dt} = 0.04P\left(1 - \dfrac{P}{1200}\right), \quad P(0) = 60$

2. $\dfrac{dP}{dt} = 0.02P - 0.0004P^2, \quad P(0) = 40$

---

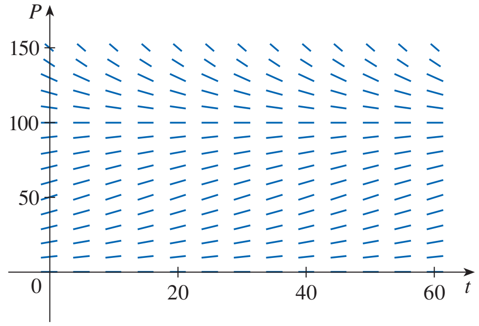
3. Suppose that a population develops according to the logistic equation

$$\frac{dP}{dt} = 0.05P - 0.0005P^2$$

where t is measured in weeks.

(a) What is the carrying capacity? What is the value of k?

(b) A direction field for this equation is shown. Where are the slopes close to 0? Where are they largest? Which solutions are increasing? Which solutions are decreasing?



(c) Use the direction field to sketch solutions for initial populations of 20, 40, 60, 80, 120, and 140. What do these solutions have in common? How do they differ? Which solutions have inflection points? At what population levels do they occur?

(d) What are the equilibrium solutions? How are the other solutions related to these solutions?

T **4.** Suppose that a population grows according to a logistic model with carrying capacity 6000 and $k = 0.0015$ per year.

(a) Write the logistic differential equation for these values.

(b) Draw a direction field (either by hand or with a computer). What does it tell you about the solution curves?

(c) Use the direction field to sketch the solution curves for initial populations of 1000, 2000, 4000, and 8000. What can you say about the concavity of these curves? What is the significance of the inflection points?
(d) Program a calculator or computer to use Euler's method with step size $h = 1$ to estimate the population after 50 years if the initial population is 1000.
(e) If the initial population is 1000, write a formula for the population after t years. Use it to find the population after 50 years and compare with your estimate in part (d).
(f) Graph the solution in part (e) and compare with the solution curve you sketched in part (c).

5. The Pacific halibut fishery has been modeled by the differential equation

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right)$$

where $y(t)$ is the biomass (the total mass of the members of the population) in kilograms at time t (measured in years), the carrying capacity is estimated to be $M = 8 \times 10^7$ kg, and $k = 0.71$ per year.
(a) If $y(0) = 2 \times 10^7$ kg, find the biomass a year later.
(b) How long will it take for the biomass to reach 4×10^7 kg?

6. Suppose a population $P(t)$ satisfies

$$\frac{dP}{dt} = 0.4P - 0.001P^2 \qquad P(0) = 50$$

where t is measured in years.
(a) What is the carrying capacity?
(b) What is $P'(0)$?
(c) When will the population reach 50% of the carrying capacity?

7. Suppose a population grows according to a logistic model with initial population 1000 and carrying capacity 10,000. If the population grows to 2500 after one year, what will the population be after another three years?

8. The table gives the number of yeast cells in a new laboratory culture.

Time (hours)	Yeast cells	Time (hours)	Yeast cells
0	18	10	509
2	39	12	597
4	80	14	640
6	171	16	664
8	336	18	672

(a) Plot the data and use the plot to estimate the carrying capacity for the yeast population.
(b) Use the data to estimate the initial relative growth rate.
(c) Find both an exponential model and a logistic model for these data.
(d) For each model, compare the predicted values with the observed values, both in a table and with graphs. Comment on how well your models fit the data.
(e) Use your logistic model to estimate the number of yeast cells after 7 hours.

9. The population of the world was about 6.1 billion in 2000. Birth rates around that time ranged from 35 to 40 million per year and death rates ranged from 15 to 20 million per year. Let's assume that the carrying capacity for world population is 20 billion.
(a) Write the logistic differential equation for these data. (Because the initial population is small compared to the carrying capacity, you can take k to be an estimate of the initial relative growth rate.)
(b) Use the logistic model to estimate the world population in the year 2010 and compare with the actual population of 6.9 billion.
(c) Use the logistic model to predict the world population in the years 2100 and 2500.

10. (a) Assume that the carrying capacity for the US population is 800 million. Use it and the fact that the population was 282 million in 2000 to formulate a logistic model for the US population.
(b) Determine the value of k in your model by using the fact that the population in 2010 was 309 million.
(c) Use your model to predict the US population in the years 2100 and 2200.
(d) Use your model to predict the year in which the US population will exceed 500 million.

11. One model for the spread of a rumor is that the rate of spread is proportional to the product of the fraction y of the population who have heard the rumor and the fraction who have not heard the rumor.
(a) Write a differential equation that is satisfied by y.
(b) Solve the differential equation.
(c) A small town has 1000 inhabitants. At 8 AM, 80 people have heard a rumor. By noon half the town has heard it. At what time will 90% of the population have heard the rumor?

12. Biologists stocked a lake with 400 fish and estimated the carrying capacity (the maximal population for the fish of that species in that lake) to be 10,000. The number of fish tripled in the first year.
(a) Assuming that the size of the fish population satisfies the logistic equation, find an expression for the size of the population after t years.
(b) How long will it take for the population to increase to 5000?

13. (a) Show that if P satisfies the logistic equation (4), then

$$\frac{d^2P}{dt^2} = k^2P\left(1 - \frac{P}{M}\right)\left(1 - \frac{2P}{M}\right)$$

(b) Deduce that a population grows fastest when it reaches half its carrying capacity.

14. For a fixed value of M (say $M = 10$), the family of logistic functions given by Equation 7 depends on the initial value P_0 and the proportionality constant k. Graph several members of this family. How does the graph change when P_0 varies? How does it change when k varies?

T **15.** **A Shifted Logistic Model** The table gives the midyear population P of Trinidad and Tobago, in thousands, from 1970 to 2015.

Year	Population (thousands)	Year	Population (thousands)
1970	955	1995	1264
1975	1007	2000	1252
1980	1091	2005	1237
1985	1189	2010	1227
1990	1255	2015	1222

Source: US Census Bureau / International Programs / International Data Base. Revised Sept. 18, 2018. Version data 18.0822. Code 12.0321.

(a) Make a scatter plot of these data. Choose $t = 0$ to correspond to the year 1970.

(b) From the scatter plot, it appears that a logistic model might be appropriate if we first shift the data points downward (so that the initial P-values are closer to 0). Subtract 900 from each value of P. Then use a calculator or computer to obtain a logistic model for the shifted data.

(c) Add 900 to your model from part (b) to obtain a shifted logistic model for the original data. Graph the model with the data points from part (a) and comment on the accuracy of the model.

(d) If the model remains accurate, what do you predict for the future population of Trinidad and Tobago?

T **16.** The table gives the number of active Twitter users worldwide, semiannually from 2010 to 2016.

Years since January 1, 2010	Twitter users (millions)	Years since January 1, 2010	Twitter users (millions)
0	30	3.5	232
0.5	49	4.0	255
1.0	68	4.5	284
1.5	101	5.0	302
2.0	138	5.5	307
2.5	167	6.0	310
3.0	204	6.5	317

Source: www.statistica.com/statistics/282087/number-of-monthly-active-twitter-users/. Accessed March 9, 2019.

Use a calculator or computer to fit both an exponential function and a logistic function to these data. Graph the data points and both functions, and comment on the accuracy of the models.

17. Consider a population $P = P(t)$ with constant relative birth and death rates α and β, respectively, and a constant emigration rate m, where α, β, and m are positive constants. Assume that $\alpha > \beta$. Then the rate of change of the population at time t is modeled by the differential equation

$$\frac{dP}{dt} = kP - m \qquad \text{where } k = \alpha - \beta$$

(a) Find the solution of this equation that satisfies the initial condition $P(0) = P_0$.

(b) What condition on m will lead to an exponential expansion of the population?

(c) What condition on m will result in a constant population? A population decline?

(d) In 1847, the population of Ireland was about 8 million and the difference between the relative birth and death rates was 1.6% of the population. Because of the potato famine in the 1840s and 1850s, about 210,000 inhabitants per year emigrated from Ireland. Was the population expanding or declining at that time?

18. **Doomsday Equation** Let c be a positive number. A differential equation of the form

$$\frac{dy}{dt} = ky^{1+c}$$

where k is a positive constant, is called a *doomsday equation* because the exponent in the expression ky^{1+c} is larger than the exponent 1 for natural growth.

(a) Determine the solution that satisfies the initial condition $y(0) = y_0$.

(b) Show that there is a finite time $t = T$ (doomsday) such that $\lim_{t \to T^-} y(t) = \infty$.

(c) An especially prolific breed of rabbits has the growth term $ky^{1.01}$. If 2 such rabbits breed initially and the warren has 16 rabbits after three months, then when is doomsday?

T **19.** Let's modify the logistic differential equation of Example 1 as follows:

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) - 15$$

(a) Suppose $P(t)$ represents a fish population at time t, where t is measured in weeks. Explain the meaning of the final term in the equation (-15).

(b) Draw a direction field for this differential equation.

(c) What are the equilibrium solutions?

(d) Use the direction field to sketch several solution curves. Describe what happens to the fish population for various initial populations.

(e) Solve this differential equation explicitly, either by using partial fractions or with a computer. Use the initial populations 200 and 300. Graph the solutions and compare with your sketches in part (d).

20. Consider the differential equation

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) - c$$

as a model for a fish population, where t is measured in weeks and c is a constant.

(a) Draw direction fields for various values of c.

(b) From your direction fields in part (a), determine the values of c for which there is at least one equilibrium solution. For what values of c does the fish population always die out?

(c) Use the differential equation to prove what you discovered graphically in part (b).

(d) What would you recommend for a limit to the weekly catch of this fish population?

21. There is considerable evidence to support the theory that for some species there is a minimum population m such that the species will become extinct if the size of the population falls below m. This condition can be incorporated into the logistic equation by introducing the factor $(1 - m/P)$. Thus the modified logistic model is given by the differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$$

(a) Use the differential equation to show that any solution is increasing if $m < P < M$ and decreasing if $0 < P < m$.

(b) For the case where $k = 0.08$, $M = 1000$, and $m = 200$, draw a direction field and use it to sketch several solution curves. Describe what happens to the population for various initial populations. What are the equilibrium solutions?

(c) Solve the differential equation explicitly, either by using partial fractions or with a computer. Use the initial population P_0.

(d) Use the solution in part (c) to show that if $P_0 < m$, then the species will become extinct. [*Hint:* Show that the numerator in your expression for $P(t)$ is 0 for some value of t.]

22. The Gompertz Function Another model for a growth function for a limited population is given by the *Gompertz function*, which is a solution of the differential equation

$$\frac{dP}{dt} = c \ln\left(\frac{M}{P}\right)P$$

where c is a constant and M is the carrying capacity.

(a) Solve this differential equation.

(b) Compute $\lim_{t\to\infty} P(t)$.

(c) Graph the Gompertz function for $M = 1000$, $P_0 = 100$, and $c = 0.05$, and compare it with the logistic function in Example 2. What are the similarities? What are the differences?

(d) We know from Exercise 13 that the logistic function grows fastest when $P = M/2$. Use the Gompertz differential equation to show that the Gompertz function grows fastest when $P = M/e$.

23. In a **seasonal-growth model**, a periodic function of time is introduced to account for seasonal variations in the rate of growth. Such variations could, for example, be caused by seasonal changes in the availability of food.

(a) Find the solution of the seasonal-growth model

$$\frac{dP}{dt} = kP\cos(rt - \phi) \qquad P(0) = P_0$$

where k, r, and ϕ are positive constants.

(b) By graphing the solution for several values of k, r, and ϕ, explain how the values of k, r, and ϕ affect the solution. What can you say about $\lim_{t\to\infty} P(t)$?

24. Suppose we alter the differential equation in Exercise 23 as follows:

$$\frac{dP}{dt} = kP\cos^2(rt - \phi) \qquad P(0) = P_0$$

(a) Solve this differential equation with the help of a table of integrals or a computer.

(b) Graph the solution for several values of k, r, and ϕ. How do the values of k, r, and ϕ affect the solution? What can you say about $\lim_{t\to\infty} P(t)$ in this case?

25. Graphs of logistic functions (Figures 2 and 3) look suspiciously similar to the graph of the hyperbolic tangent function (Figure 3.11.3). Explain the similarity by showing that the logistic function given by Equation 7 can be written as

$$P(t) = \tfrac{1}{2}M\left[1 + \tanh\left(\tfrac{1}{2}k(t - c)\right)\right]$$

where $c = (\ln A)/k$. Thus the logistic function is really just a shifted hyperbolic tangent.

9.5 Linear Equations

In Section 9.3 we learned how to solve separable first-order differential equations. In this section we investigate a method for solving a class of differential equations that are not necessarily separable.

Linear Differential Equations

A first-order **linear** differential equation is one that can be put into the form

$$\boxed{1} \qquad \frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are continuous functions on a given interval. This type of equation occurs frequently in various sciences, as we will see.

An example of a linear equation is $xy' + y = 2x$ because, for $x \neq 0$, it can be written in the form

$$\boxed{2} \qquad y' + \frac{1}{x}y = 2$$

Notice that this differential equation is not separable because it's impossible to factor the expression for y' as a function of x times a function of y. But we can still solve the equation $xy' + y = 2x$ by noticing, by the Product Rule, that

$$xy' + y = (xy)'$$

and so we can rewrite the equation as

$$(xy)' = 2x$$

If we now integrate both sides of this equation, we get

$$xy = x^2 + C \qquad \text{or} \qquad y = x + \frac{C}{x}$$

If we had been given the differential equation in the form of Equation 2, we would have had to take the preliminary step of multiplying each side of the equation by x.

It turns out that every first-order linear differential equation can be solved in a similar fashion by multiplying both sides of Equation 1 by a suitable function $I(x)$ called an *integrating factor*. We try to find I so that the left side of Equation 1, when multiplied by $I(x)$, becomes the derivative of the product $I(x)y$:

$$\boxed{3} \qquad I(x)\big(y' + P(x)y\big) = \big(I(x)y\big)'$$

If we can find such a function I, then Equation 1 becomes

$$\big(I(x)y\big)' = I(x)\,Q(x)$$

Integrating both sides, we would have

$$I(x)y = \int I(x)\,Q(x)\,dx + C$$

so the solution would be

$$\boxed{4} \qquad y(x) = \frac{1}{I(x)}\left[\int I(x)\,Q(x)\,dx + C\right]$$

To find such an I, we expand Equation 3 and cancel terms:

$$I(x)y' + I(x)\,P(x)y = \big(I(x)y\big)' = I'(x)y + I(x)y'$$

$$I(x)\,P(x) = I'(x)$$

This is a separable differential equation for I, which we solve as follows:

$$\int \frac{dI}{I} = \int P(x)\,dx$$

$$\ln|I| = \int P(x)\,dx$$

$$I = Ae^{\int P(x)\,dx}$$

where $A = \pm e^C$. We are looking for a particular integrating factor, not the most general one, so we take $A = 1$ and use

$$I(x) = e^{\int P(x)\,dx} \tag{5}$$

Thus a formula for the general solution to Equation 1 is provided by Equation 4, where I is given by Equation 5. Instead of memorizing this formula, however, we just remember the form of the integrating factor.

To solve the linear differential equation $y' + P(x)y = Q(x)$, multiply both sides by the **integrating factor** $I(x) = e^{\int P(x)\,dx}$ and then integrate both sides.

EXAMPLE 1 Solve the differential equation $\dfrac{dy}{dx} + 3x^2y = 6x^2$.

SOLUTION The given equation is linear since it has the form of Equation 1 with $P(x) = 3x^2$ and $Q(x) = 6x^2$. An integrating factor is

$$I(x) = e^{\int 3x^2\,dx} = e^{x^3}$$

Multiplying both sides of the differential equation by e^{x^3}, we get

$$e^{x^3}\frac{dy}{dx} + 3x^2e^{x^3}y = 6x^2e^{x^3}$$

or

$$\frac{d}{dx}(e^{x^3}y) = 6x^2e^{x^3} \qquad \text{(Product Rule)}$$

Integrating both sides, we have

$$e^{x^3}y = \int 6x^2e^{x^3}\,dx = 2e^{x^3} + C$$

$$y = 2 + Ce^{-x^3}$$

■

Figure 1 shows the graphs of several members of the family of solutions in Example 1. Notice that they all approach 2 as $x \to \infty$.

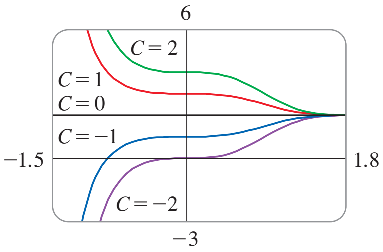


FIGURE 1

EXAMPLE 2 Find the solution of the initial-value problem

$$x^2y' + xy = 1 \qquad x > 0 \qquad y(1) = 2$$

SOLUTION We must first divide both sides by the coefficient of y' to put the differential equation into the standard form given in Equation 1:

$$y' + \frac{1}{x}y = \frac{1}{x^2} \qquad x > 0 \tag{6}$$

The integrating factor is

$$I(x) = e^{\int (1/x)\,dx} = e^{\ln x} = x$$

Multiplication of Equation 6 by x gives

$$xy' + y = \frac{1}{x} \qquad \text{or} \qquad (xy)' = \frac{1}{x}$$

The solution of the initial-value problem in Example 2 is shown in Figure 2.

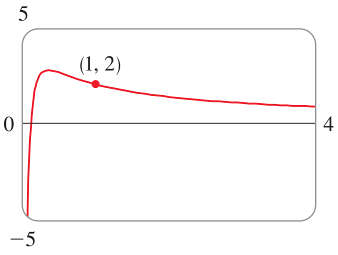


FIGURE 2

Then $$xy = \int \frac{1}{x}\,dx = \ln x + C$$

and so $$y = \frac{\ln x + C}{x}$$

Since $y(1) = 2$, we have

$$2 = \frac{\ln 1 + C}{1} = C$$

Therefore the solution to the initial-value problem is

$$y = \frac{\ln x + 2}{x}$$ ■

EXAMPLE 3 Solve $y' + 2xy = 1$.

SOLUTION The given equation is in the standard form for a linear equation. Multiplying by the integrating factor

$$e^{\int 2x\,dx} = e^{x^2}$$

we get $$e^{x^2}y' + 2xe^{x^2}y = e^{x^2}$$

or $$\left(e^{x^2}y\right)' = e^{x^2}$$

Therefore $$e^{x^2}y = \int e^{x^2}\,dx + C$$

Even though the solutions of the differential equation in Example 3 are expressed in terms of an integral, they can still be graphed by a computer (Figure 3).

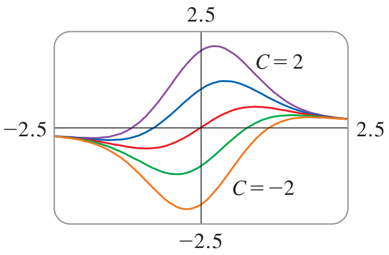


FIGURE 3

Recall from Section 7.5 that $\int e^{x^2}dx$ can't be expressed in terms of elementary functions. Nonetheless, it's a perfectly good function and we can leave the answer as

$$y = e^{-x^2}\int e^{x^2}\,dx + Ce^{-x^2}$$

Another way of writing the solution, using Part 1 of the Fundamental Theorem of Calculus, is

$$y = e^{-x^2}\int_0^x e^{t^2}\,dt + Ce^{-x^2}$$

(Any number can be chosen for the lower limit of integration.) ■

■ Application to Electric Circuits

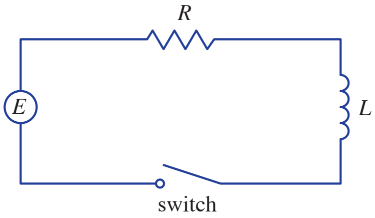


FIGURE 4

In Section 9.2 we considered the simple electric circuit shown in Figure 4: an electromotive force (usually a battery or generator) produces a voltage of $E(t)$ volts (V) and a current of $I(t)$ amperes (A) at time t. The circuit also contains a resistor with a resistance of R ohms (Ω) and an inductor with an inductance of L henries (H).

Ohm's Law gives the drop in voltage due to the resistor as RI. The voltage drop due to the inductor is $L(dI/dt)$. One of Kirchhoff's laws says that the sum of the voltage drops is equal to the supplied voltage $E(t)$. Thus we have

$$\boxed{7} \qquad L\frac{dI}{dt} + RI = E(t)$$

which is a first-order linear differential equation. The solution gives the current I at time t.

EXAMPLE 4 Suppose that in the simple circuit of Figure 4 the resistance is 12 Ω and the inductance is 4 H. If a battery gives a constant voltage of 60 V and the switch is closed when $t = 0$ so the current starts with $I(0) = 0$, find (a) $I(t)$, (b) the current after 1 second, and (c) the limiting value of the current.

The differential equation in Example 4 is both linear and separable, so an alternative method is to solve it as a separable equation (Example 9.3.4). If we replace the battery by a generator, however, we get an equation that is linear but not separable (Example 5).

SOLUTION

(a) If we put $L = 4$, $R = 12$, and $E(t) = 60$ in Equation 7, we obtain the initial-value problem

$$4\frac{dI}{dt} + 12I = 60 \qquad I(0) = 0$$

or

$$\frac{dI}{dt} + 3I = 15 \qquad I(0) = 0$$

Multiplying by the integrating factor $e^{\int 3\,dt} = e^{3t}$, we get

$$e^{3t}\frac{dI}{dt} + 3e^{3t}I = 15e^{3t}$$

$$\frac{d}{dt}(e^{3t}I) = 15e^{3t}$$

$$e^{3t}I = \int 15e^{3t}\,dt = 5e^{3t} + C$$

$$I(t) = 5 + Ce^{-3t}$$

Since $I(0) = 0$, we have $5 + C = 0$, so $C = -5$ and

$$I(t) = 5(1 - e^{-3t})$$

Figure 5 shows how the current in Example 4 approaches its limiting value.

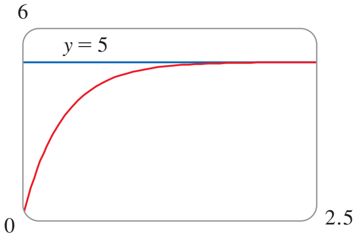


FIGURE 5

(b) After 1 second the current is

$$I(1) = 5(1 - e^{-3}) \approx 4.75 \text{ A}$$

(c) The limiting value of the current is given by

$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty} 5(1 - e^{-3t}) = 5 - 5\lim_{t\to\infty} e^{-3t} = 5 - 0 = 5 \qquad ■$$

EXAMPLE 5 Suppose that the resistance and inductance remain as in Example 4 but, instead of the battery, we use a generator that produces a variable voltage of $E(t) = 60 \sin 30t$ volts. Find $I(t)$.

Figure 6 shows the graph of the current when the battery is replaced by a generator.

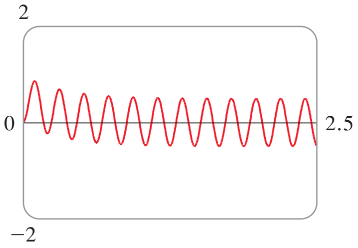


FIGURE 6

SOLUTION This time the differential equation becomes

$$4\frac{dI}{dt} + 12I = 60 \sin 30t \qquad \text{or} \qquad \frac{dI}{dt} + 3I = 15 \sin 30t$$

The same integrating factor e^{3t} gives

$$\frac{d}{dt}(e^{3t}I) = e^{3t}\frac{dI}{dt} + 3e^{3t}I = 15e^{3t} \sin 30t$$

Using the formula in entry 98 of the Table of Integrals (or a computer), we have

$$e^{3t}I = \int 15e^{3t}\sin 30t\,dt = 15\,\frac{e^{3t}}{909}(3\sin 30t - 30\cos 30t) + C$$

$$I = \tfrac{5}{101}(\sin 30t - 10\cos 30t) + Ce^{-3t}$$

Since $I(0) = 0$, we get

$$-\tfrac{50}{101} + C = 0$$

so $$I(t) = \tfrac{5}{101}(\sin 30t - 10\cos 30t) + \tfrac{50}{101}e^{-3t}$$ ■

9.5 Exercises

1–4 Determine whether the differential equation is linear. If it is linear, then write it in the form of Equation 1.

1. $y' + x\sqrt{y} = x^2$

2. $y' - x = y\tan x$

3. $ue^t = t + \sqrt{t}\,\dfrac{du}{dt}$

4. $\dfrac{dR}{dt} + t\cos R = e^{-t}$

5–16 Solve the differential equation.

5. $y' + y = 1$

6. $y' - y = e^x$

7. $y' = x - y$

8. $4x^3y + x^4y' = \sin^3 x$

9. $xy' + y = \sqrt{x}$

10. $2xy' + y = 2\sqrt{x}$

11. $xy' - 2y = x^2, \quad x > 0$

12. $y' - 3x^2y = x^2$

13. $t^2\dfrac{dy}{dt} + 3ty = \sqrt{1 + t^2}, \quad t > 0$

14. $t\ln t\,\dfrac{dr}{dt} + r = te^t$

15. $y' + y\cos x = x$

16. $y' + 2xy = x^3e^{x^2}$

17–24 Solve the initial-value problem.

17. $xy' + y = 3x^2, \quad y(1) = 4$

18. $xy' - 2y = 2x, \quad y(2) = 0$

19. $x^2y' + 2xy = \ln x, \quad y(1) = 2$

20. $t^3\dfrac{dy}{dt} + 3t^2y = \cos t, \quad y(\pi) = 0$

21. $t\dfrac{du}{dt} = t^2 + 3u, \quad t > 0, \quad u(2) = 4$

22. $xy' + y = x\ln x, \quad y(1) = 0$

23. $xy' = y + x^2\sin x, \quad y(\pi) = 0$

24. $(x^2 + 1)\dfrac{dy}{dx} + 3x(y - 1) = 0, \quad y(0) = 2$

25–26 Solve the differential equation and graph several members of the family of solutions. How does the solution curve change as C varies?

25. $xy' + 2y = e^x$

26. $xy' = x^2 + 2y$

27–29 **Bernoulli Differential Equations** A *Bernoulli differential equation* (named after James Bernoulli) is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

27. Observe that, if $n = 0$ or 1, the Bernoulli equation is linear. For other values of n, show that the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1 - n)P(x)u = (1 - n)Q(x)$$

28. Solve the differential equation $xy' + y = -xy^2$.

29. Solve the differential equation $y' + \dfrac{2}{x}y = \dfrac{y^3}{x^2}$.

30. Solve the second-order equation $xy'' + 2y' = 12x^2$ by making the substitution $u = y'$.

31. In the circuit shown in Figure 4, a battery supplies a constant voltage of 40 V, the inductance is 2 H, the resistance is 10 Ω, and $I(0) = 0$.
(a) Find $I(t)$.
(b) Find the current after 0.1 seconds.

32. In the circuit shown in Figure 4, a generator supplies a voltage of $E(t) = 40\sin 60t$ volts, the inductance is 1 H, the resistance is 20 Ω, and $I(0) = 1$ A.
(a) Find $I(t)$.
(b) Find the current after 0.1 seconds.
(c) Graph the current function.

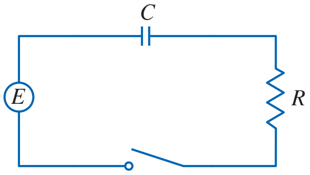
33. The figure shows a circuit containing an electromotive force, a capacitor with a capacitance of C farads (F), and a resistor with a resistance of R ohms (Ω). The voltage drop across the capacitor is Q/C, where Q is the charge (in coulombs), so in this case Kirchhoff's Law gives

$$RI + \frac{Q}{C} = E(t)$$

But $I = dQ/dt$ (see Example 3.7.3), so we have

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

Suppose the resistance is 5 Ω, the capacitance is 0.05 F, a battery gives a constant voltage of 60 V, and the initial charge is $Q(0) = 0$ C. Find the charge and the current at time t.



34. In the circuit of Exercise 33, $R = 2\ \Omega$, $C = 0.01$ F, $Q(0) = 0$, and $E(t) = 10 \sin 60t$. Find the charge and the current at time t.

35. Let $P(t)$ be the performance level of someone learning a skill as a function of the training time t. The graph of P is called a *learning curve*. In Exercise 9.1.27 we proposed the differential equation

$$\frac{dP}{dt} = k[M - P(t)]$$

as a reasonable model for learning, where k is a positive constant. Solve it as a linear differential equation and use your solution to graph the learning curve.

36. Two new workers were hired for an assembly line. Jim processed 25 units during the first hour and 45 units during the second hour. Mark processed 35 units during the first hour and 50 units the second hour. Using the model of Exercise 35 and assuming that $P(0) = 0$, estimate the maximum number of units per hour that each worker is capable of processing.

37. In Section 9.3 we looked at mixing problems in which the volume of fluid remained constant and saw that such problems give rise to separable differentiable equations. (See Example 9.3.6.) If the rates of flow into and out of the system are different, then the volume is not constant and the resulting differential equation is linear but not separable.

A tank contains 100 L of water. A solution with a salt concentration of 0.4 kg/L is added at a rate of 5 L/min. The solution is kept mixed and is drained from the tank at a rate of 3 L/min. If $y(t)$ is the amount of salt (in kilograms) after t minutes, show that y satisfies the differential equation

$$\frac{dy}{dt} = 2 - \frac{3y}{100 + 2t}$$

Solve this equation and find the concentration after 20 minutes.

38. A tank with a capacity of 400 L is full of a mixture of water and chlorine with a concentration of 0.05 g of chlorine per liter. In order to reduce the concentration of chlorine, fresh water is pumped into the tank at a rate of 4 L/s. The mixture is kept stirred and is pumped out at a rate of 10 L/s. Find the amount of chlorine in the tank as a function of time.

39. An object with mass m is dropped from rest and we assume that the air resistance is proportional to the speed of the object. If $s(t)$ is the distance dropped after t seconds, then the speed is $v = s'(t)$ and the acceleration is $a = v'(t)$. If g is the acceleration due to gravity, then the downward force on the object is $mg - cv$, where c is a positive constant, and Newton's Second Law gives

$$m\frac{dv}{dt} = mg - cv$$

(a) Solve this as a linear equation to show that

$$v = \frac{mg}{c}(1 - e^{-ct/m})$$

(b) What is the limiting velocity?
(c) Find the distance the object has fallen after t seconds.

40. If we ignore air resistance, we can conclude that heavier objects fall no faster than lighter objects. But if we take air resistance into account, our conclusion changes. Use the expression for the velocity of a falling object in Exercise 39(a) to find dv/dm and show that heavier objects *do* fall faster than lighter ones.

41. (a) Show that the substitution $z = 1/P$ transforms the logistic differential equation $P' = kP(1 - P/M)$ into the linear differential equation

$$z' + kz = \frac{k}{M}$$

(b) Solve the linear differential equation in part (a) and thus obtain an expression for $P(t)$. Compare with Equation 9.4.7.

42. To account for seasonal variation in the logistic differential equation, we could allow k and M to be functions of t:

$$\frac{dP}{dt} = k(t)P\left(1 - \frac{P}{M(t)}\right)$$

(a) Verify that the substitution $z = 1/P$ transforms this equation into the linear equation

$$\frac{dz}{dt} + k(t)z = \frac{k(t)}{M(t)}$$

(b) Write an expression for the solution of the linear equation in part (a) and use it to show that if the carrying capacity M is constant, then

$$P(t) = \frac{M}{1 + CMe^{-\int k(t)\,dt}}$$

Deduce that if $\int_0^\infty k(t)\,dt = \infty$, then $\lim_{t\to\infty} P(t) = M$. [This will be true if $k(t) = k_0 + a\cos bt$ with $k_0 > 0$, which describes a positive intrinsic growth rate with a periodic seasonal variation.]

(c) If k is constant but M varies, show that

$$z(t) = e^{-kt}\int_0^t \frac{ke^{ks}}{M(s)}\,ds + Ce^{-kt}$$

and use l'Hospital's Rule to deduce that if $M(t)$ has a limit as $t\to\infty$, then $P(t)$ has the same limit.

APPLIED PROJECT | WHICH IS FASTER, GOING UP OR COMING DOWN?

In modeling force due to air resistance, various functions have been used, depending on the physical characteristics and speed of the ball. Here we use a linear model, $-pv$, but a quadratic model ($-pv^2$ on the way up and pv^2 on the way down) is another possibility for higher speeds (see Exercise 9.3.52). For a golf ball, experiments have shown that a good model is $-pv^{1.3}$ going up and $p|v|^{1.3}$ coming down. But no matter which force function $-f(v)$ is used [where $f(v) > 0$ for $v > 0$ and $f(v) < 0$ for $v < 0$], the answer to the question remains the same. See F. Brauer, "What Goes Up Must Come Down, Eventually," *American Mathematical Monthly* 108 (2001), pp. 437–40.

Suppose you throw a ball into the air. Do you think it takes longer to reach its maximum height or to fall back to earth from its maximum height? We will solve the problem in this project, but before getting started, think about that situation and make a guess based on your physical intuition.

1. A ball with mass m is projected vertically upward from the earth's surface with a positive initial velocity v_0. We assume the forces acting on the ball are the force of gravity and a retarding force of air resistance with direction opposite to the direction of motion and with magnitude $p|v(t)|$, where p is a positive constant and $v(t)$ is the velocity of the ball at time t. In both the ascent and the descent, the total force acting on the ball is $-pv - mg$. [During ascent, $v(t)$ is positive and the resistance acts downward; during descent, $v(t)$ is negative and the resistance acts upward.] So, by Newton's Second Law, the equation of motion is

$$mv' = -pv - mg$$

Solve this linear differential equation to show that the velocity is

$$v(t) = \left(v_0 + \frac{mg}{p}\right)e^{-pt/m} - \frac{mg}{p}$$

(Note that this differential equation is also separable.)

2. Show that the height of the ball, until it hits the ground, is

$$y(t) = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}\left(1 - e^{-pt/m}\right) - \frac{mgt}{p}$$

3. Let t_1 be the time that the ball takes to reach its maximum height. Show that

$$t_1 = \frac{m}{p}\ln\left(\frac{mg + pv_0}{mg}\right)$$

Find this time for a ball with mass 1 kg and initial velocity 20 m/s. Assume the air resistance is $\frac{1}{10}$ of the speed.

4. Let t_2 be the time at which the ball falls back to earth. For the particular ball in Problem 3, estimate t_2 by using a graph of the height function $y(t)$. Which is faster, going up or coming down?

5. In general, it's not easy to find t_2 because it's impossible to solve the equation $y(t) = 0$ explicitly. We can, however, use an indirect method to determine whether ascent or descent is faster: we determine whether $y(2t_1)$ is positive or negative. Show that

$$y(2t_1) = \frac{m^2 g}{p^2}\left(x - \frac{1}{x} - 2\ln x\right)$$

where $x = e^{pt_1/m}$. Then show that $x > 1$ and the function

$$f(x) = x - \frac{1}{x} - 2 \ln x$$

is increasing for $x > 1$. Use this result to decide whether $y(2t_1)$ is positive or negative. What can you conclude? Is ascent or descent faster?

9.6 Predator-Prey Systems

We have looked at a variety of models for the growth of a single species that lives alone in an environment. In this section we consider more realistic models that take into account the interaction of two species in the same habitat. We will see that these models take the form of a pair of linked differential equations.

We first consider the situation in which one species, called the *prey*, has an ample food supply and the second species, called the *predators*, feeds on the prey. Examples of prey and predators include rabbits and wolves in an isolated forest, food-fish and sharks, aphids and ladybugs, and bacteria and amoebas. Our model will have two dependent variables and both are functions of time. We let $R(t)$ be the number of prey (using R for rabbits) and $W(t)$ be the number of predators (with W for wolves) at time t.

In the absence of predators, the ample food supply would support exponential growth of the prey, that is,

$$\frac{dR}{dt} = kR \qquad \text{where } k \text{ is a positive constant}$$

In the absence of prey, we assume that the predator population would decline through mortality at a rate proportional to itself, that is,

$$\frac{dW}{dt} = -rW \qquad \text{where } r \text{ is a positive constant}$$

With both species present, however, we assume that the principal cause of death among the prey is being eaten by a predator, and the birth and survival rates of the predators depend on their available food supply, namely, the prey. We also assume that the two species encounter each other at a rate that is proportional to both populations and is therefore proportional to the product RW. (The more there are of either population, the more encounters there are likely to be.) A system of two differential equations that incorporates these assumptions is as follows:

W represents the predators.
R represents the prey.

$$\boxed{1} \qquad \frac{dR}{dt} = kR - aRW \qquad \qquad \frac{dW}{dt} = -rW + bRW$$

where k, r, a, and b are positive constants. Notice that the term $-aRW$ decreases the natural growth rate of the prey and the term bRW increases the natural growth rate of the predators.

The Lotka-Volterra equations were proposed as a model to explain the variations in the shark and food-fish populations in the Adriatic Sea by the Italian mathematician Vito Volterra (1860–1940).

The equations in (1) are known as the **predator-prey equations**, or the **Lotka-Volterra equations**. A **solution** of this system of equations is a pair of functions $R(t)$ and $W(t)$ that describe the populations of prey and predators as functions of time. Because the system is coupled (R and W occur in both equations), we can't solve one equation and then the other; we have to solve them simultaneously. Unfortunately, it is usually impossible to find explicit formulas for R and W as functions of t. We can, however, use graphical methods to analyze the equations.

kochanowski / Shutterstock.com

EXAMPLE 1 Suppose that populations of rabbits and wolves are described by the Lotka-Volterra equations (1) with $k = 0.08$, $a = 0.001$, $r = 0.02$, and $b = 0.00002$. The time t is measured in months.

(a) Find the constant solutions (called the **equilibrium solutions**) and interpret the answer.

(b) Use the system of differential equations to find an expression for dW/dR.

(c) Draw a direction field for the resulting differential equation in the RW-plane. Then use that direction field to sketch some solution curves.

(d) Suppose that, at some point in time, there are 1000 rabbits and 40 wolves. Draw the corresponding solution curve and use it to describe the changes in both population levels.

(e) Use part (d) to make sketches of R and W as functions of t.

SOLUTION

(a) With the given values of k, a, r, and b, the Lotka-Volterra equations become

$$\frac{dR}{dt} = 0.08R - 0.001RW$$

$$\frac{dW}{dt} = -0.02W + 0.00002RW$$

Both R and W will be constant if both derivatives are 0, that is,

$$R' = R(0.08 - 0.001W) = 0$$

$$W' = W(-0.02 + 0.00002R) = 0$$

One solution is given by $R = 0$ and $W = 0$. (This makes sense: If there are no rabbits or wolves, the populations are certainly not going to increase.) The other constant solution is

$$W = \frac{0.08}{0.001} = 80$$

$$R = \frac{0.02}{0.00002} = 1000$$

So the equilibrium populations consist of 80 wolves and 1000 rabbits. This means that 1000 rabbits are just enough to support a constant wolf population of 80. There are neither too many wolves (which would result in fewer rabbits) nor too few wolves (which would result in more rabbits).

(b) We use the Chain Rule to write

$$\frac{dW}{dt} = \frac{dW}{dR}\frac{dR}{dt}$$

Solving for dW/dR gives

$$\frac{dW}{dR} = \frac{\dfrac{dW}{dt}}{\dfrac{dR}{dt}} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

(c) If we think of W as a function of R, we have the differential equation

$$\frac{dW}{dR} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

We draw the direction field for this differential equation in Figure 1 and we use it to sketch several solution curves in Figure 2. If we move along a solution curve, we observe how the relationship between R and W changes as time passes. Notice that the curves appear to be closed in the sense that if we travel along a curve, we always return to the same point. Notice also that the point (1000, 80) is inside all the solution curves. That point is called an *equilibrium point* because it corresponds to the equilibrium solution $R = 1000$, $W = 80$.

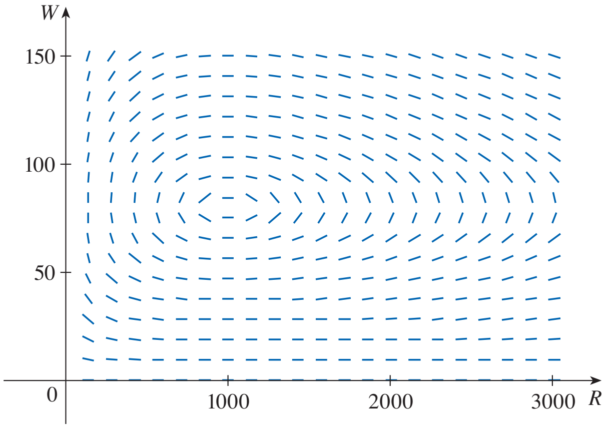


FIGURE 1
Direction field for the predator-prey system

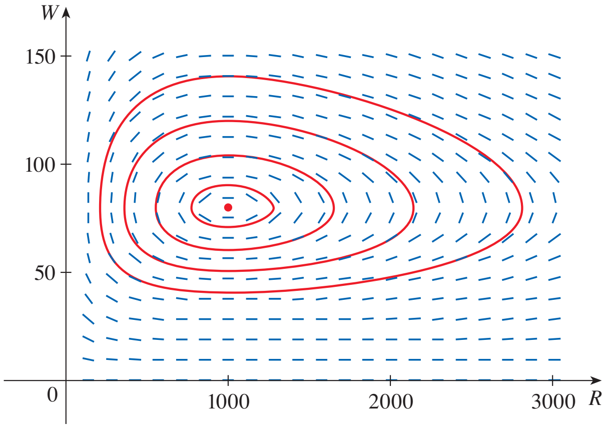


FIGURE 2
Phase portrait of the system

When we represent solutions of a system of differential equations as in Figure 2, we refer to the RW-plane as the **phase plane**, and we call the solution curves **phase trajectories**. So a phase trajectory is a path traced out by solutions (R, W) as time goes by. A **phase portrait** consists of equilibrium points and typical phase trajectories, as shown in Figure 2.

(d) Starting with 1000 rabbits and 40 wolves corresponds to drawing the solution curve through the point $P_0(1000, 40)$. Figure 3 shows this phase trajectory with the direction field removed. Starting at the point P_0 at time $t = 0$ and letting t increase, do we move

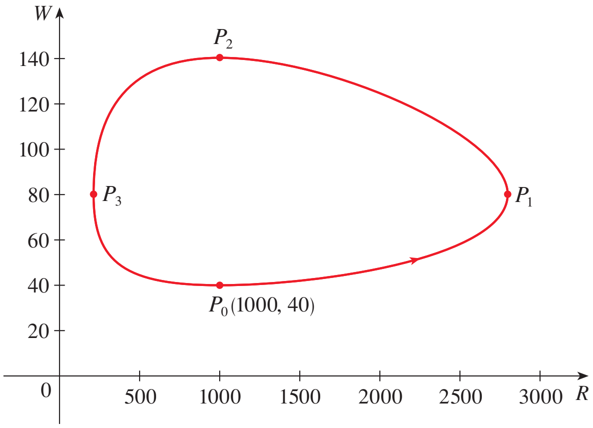


FIGURE 3
Phase trajectory through (1000, 40)

clockwise or counterclockwise around the phase trajectory? If we put $R = 1000$ and $W = 40$ in the first differential equation, we get

$$\frac{dR}{dt} = 0.08(1000) - 0.001(1000)(40) = 80 - 40 = 40$$

Since $dR/dt > 0$, we conclude that R is increasing at P_0 and so we move counterclockwise around the phase trajectory.

We see that at P_0 there aren't enough wolves to maintain a balance between the populations, so the rabbit population increases. That results in more wolves and eventually there are so many wolves that the rabbits have a hard time avoiding them. So the number of rabbits begins to decline (at P_1, where we estimate that R reaches its maximum population of about 2800). This means that at some later time the wolf population starts to decline (at P_2, where $R = 1000$ and $W \approx 140$). But this benefits the rabbits, so their population later starts to increase (at P_3, where $W = 80$ and $R \approx 210$). As a consequence, the wolf population eventually starts to increase as well. This happens when the populations return to their initial values of $R = 1000$ and $W = 40$, and the entire cycle begins again.

(e) From the description in part (d) of how the rabbit and wolf populations rise and fall, we can sketch the graphs of $R(t)$ and $W(t)$. Suppose the points P_1, P_2, and P_3 in Figure 3 are reached at times t_1, t_2, and t_3. Then we can sketch graphs of R and W as in Figure 4.

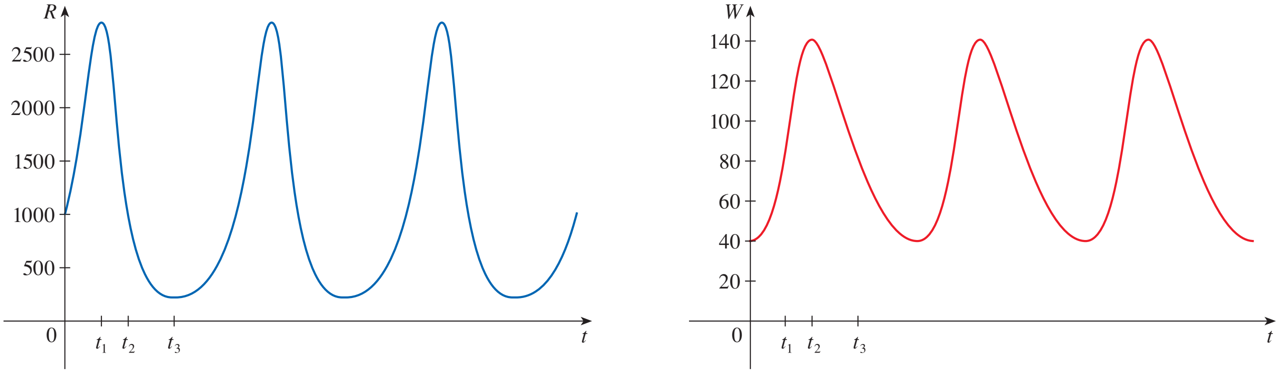


FIGURE 4 Graphs of the rabbit and wolf populations as functions of time

To make the graphs easier to compare, we draw the graphs on the same axes but with different scales for R and W, as in Figure 5. Notice that the rabbits reach their maximum populations about a quarter of a cycle before the wolves.

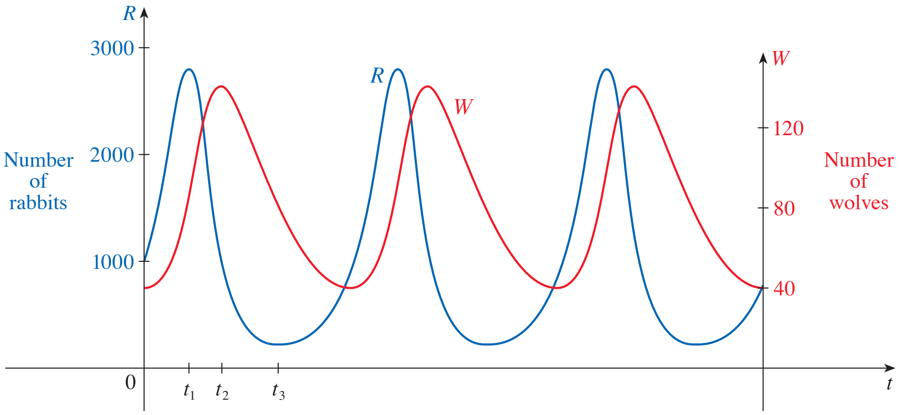


FIGURE 5
Comparison of the rabbit and wolf populations

■

Thomas Kitchin & Victoria Hurst / All Canada Photos

An important part of the modeling process, as we discussed in Section 1.2, is to interpret our mathematical conclusions as real-world predictions and to test the predictions against real data. The Hudson's Bay Company, which started trading in animal furs in Canada in 1670, has kept records that date back to the 1840s. Figure 6 shows graphs of the number of pelts of the snowshoe hare and its predator, the Canada lynx, traded by the company over a 90-year period. You can see that the coupled oscillations in the hare and lynx populations predicted by the Lotka-Volterra model do actually occur and the period of these cycles is roughly 10 years.

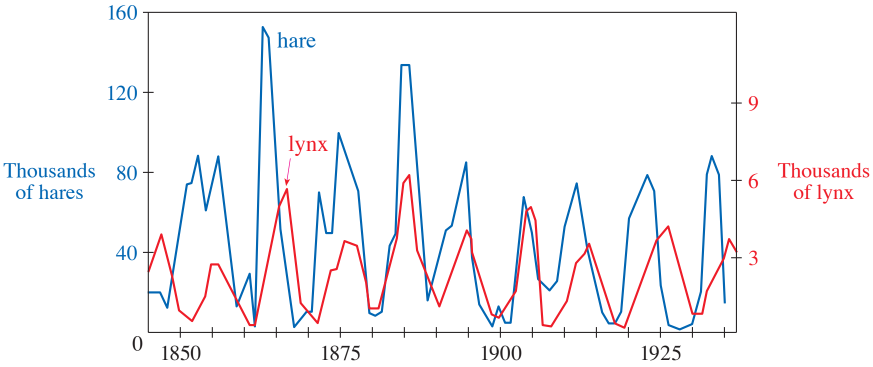


FIGURE 6
Relative abundance of hare and lynx from Hudson's Bay Company records

Although the relatively simple Lotka-Volterra model has had some success in explaining and predicting coupled populations, more sophisticated models have also been proposed. One way to modify the Lotka-Volterra equations is to assume that, in the absence of predators, the prey grow according to a logistic model with carrying capacity M. Then the Lotka-Volterra equations (1) are replaced by the system of differential equations

$$\frac{dR}{dt} = kR\left(1 - \frac{R}{M}\right) - aRW \qquad \frac{dW}{dt} = -rW + bRW$$

This model is investigated in Exercises 11 and 12.

Models have also been proposed to describe and predict population levels of two or more species that compete for the same resources or cooperate for mutual benefit. Such models are explored in Exercises 2–4.

9.6 Exercises

1. For each predator-prey system, determine which of the variables, x or y, represents the prey population and which represents the predator population. Is the growth of the prey restricted just by the predators or by other factors as well? Do the predators feed only on the prey or do they have additional food sources? Explain.

(a) $\dfrac{dx}{dt} = -0.05x + 0.0001xy$

$\dfrac{dy}{dt} = 0.1y - 0.005xy$

(b) $\dfrac{dx}{dt} = 0.2x - 0.0002x^2 - 0.006xy$

$\dfrac{dy}{dt} = -0.015y + 0.00008xy$

2. Each system of differential equations is a model for two species that either compete for the same resources or cooperate for mutual benefit (flowering plants and insect pollinators, for instance). Decide whether each system describes competition or cooperation and explain why it is a reasonable model. (Ask yourself what effect an increase in one species has on the growth rate of the other.)

(a) $\frac{dx}{dt} = 0.12x - 0.0006x^2 + 0.00001xy$

$\frac{dy}{dt} = 0.08x + 0.00004xy$

(b) $\frac{dx}{dt} = 0.15x - 0.0002x^2 - 0.0006xy$

$\frac{dy}{dt} = 0.2y - 0.00008y^2 - 0.0002xy$

3. The system of differential equations

$$\frac{dx}{dt} = 0.5x - 0.004x^2 - 0.001xy$$

$$\frac{dy}{dt} = 0.4y - 0.001y^2 - 0.002xy$$

is a model for the populations of two species.

(a) Does the model describe cooperation, or competition, or a predator-prey relationship?

(b) Find the equilibrium solutions and explain their significance.

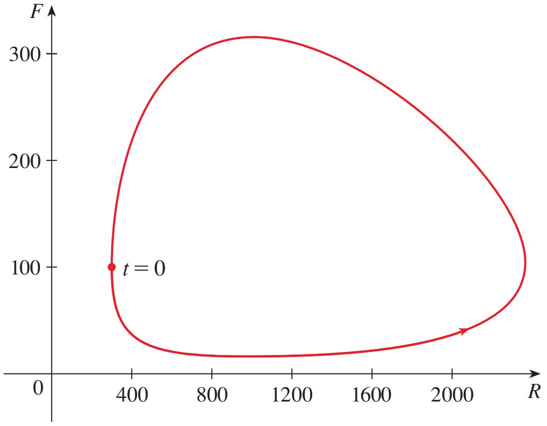
4. Lynx eat snowshoe hares and snowshoe hares eat woody plants like willows. Suppose that, in the absence of hares, the willow population will grow exponentially and the lynx population will decay exponentially. In the absence of lynx and willow, the hare population will decay exponentially. If $L(t)$, $H(t)$, and $W(t)$ represent the populations of these three species at time t, write a system of differential equations as a model for their dynamics. If the constants in your equation are all positive, explain why you have used plus or minus signs.

5–6 A phase trajectory is shown for populations of rabbits (R) and foxes (F).

(a) Describe how each population changes as time goes by.

(b) Use your description to make a rough sketch of the graphs of R and F as functions of time.

5.



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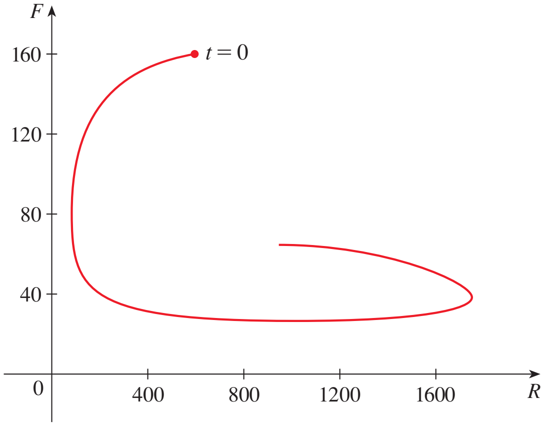
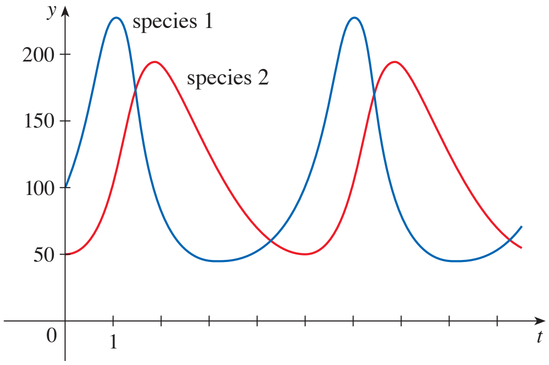


7–8 Graphs of populations of two species are shown. Use them to sketch the corresponding phase trajectory.

7.



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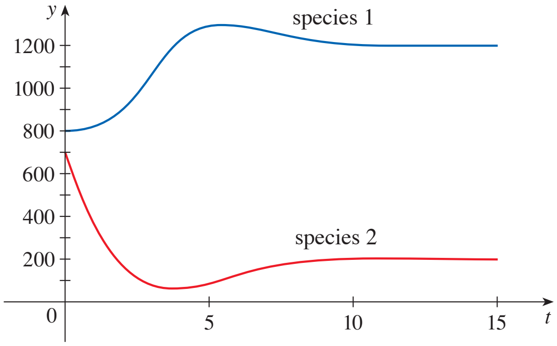


9. In Example 1(b) we showed that the rabbit and wolf populations satisfy the differential equation

$$\frac{dW}{dR} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

(a) By solving this separable differential equation, show that

$$\frac{R^{0.02}W^{0.08}}{e^{0.00002R}e^{0.001W}} = C$$

where C is a constant.

(b) It is impossible to solve this equation for W as an explicit function of R (or vice versa). Use a computer to graph the implicitly defined solution curve that passes through the point (1000, 40) and compare with Figure 3.

10. Populations of aphids and ladybugs are modeled by the equations

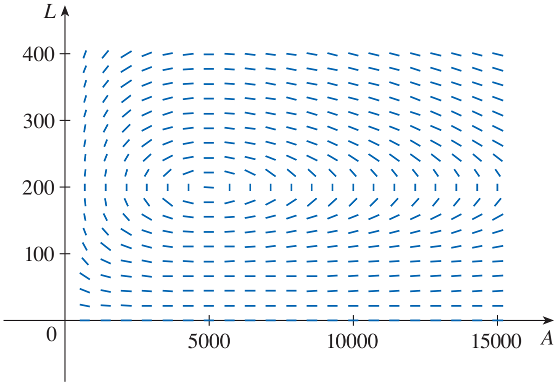
$$\frac{dA}{dt} = 2A - 0.01AL$$

$$\frac{dL}{dt} = -0.5L + 0.0001AL$$

(a) Find the equilibrium solutions and explain their significance.

(b) Find an expression for dL/dA.

(c) The direction field for the differential equation in part (b) is shown. Use it to sketch a phase portrait. What do the phase trajectories have in common?



(d) Suppose that at time $t = 0$ there are 1000 aphids and 200 ladybugs. Draw the corresponding phase trajectory and use it to describe how both populations change.

(e) Use part (d) to make rough sketches of the aphid and ladybug populations as functions of t. How are the graphs related to each other?

11. In Example 1 we used Lotka-Volterra equations to model populations of rabbits and wolves. Let's modify those equations as follows:

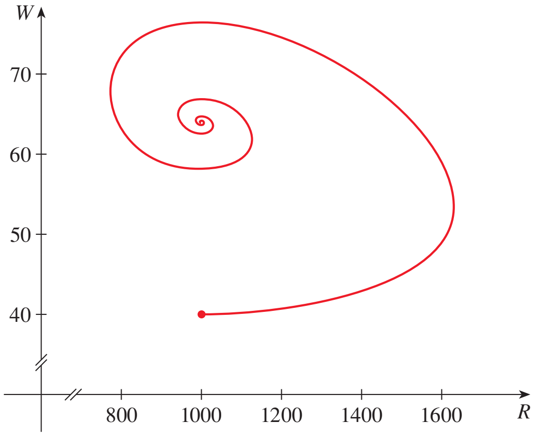
$$\frac{dR}{dt} = 0.08R(1 - 0.0002R) - 0.001RW$$

$$\frac{dW}{dt} = -0.02W + 0.00002RW$$

(a) According to these equations, what happens to the rabbit population in the absence of wolves?

(b) Find all the equilibrium solutions and explain their significance.

(c) The figure shows the phase trajectory that starts at the point (1000, 40). Describe what eventually happens to the rabbit and wolf populations.



(d) Sketch graphs of the rabbit and wolf populations as functions of time.

12. In Exercise 10 we modeled populations of aphids and ladybugs with a Lotka-Volterra system. Suppose we modify those equations as follows:

$$\frac{dA}{dt} = 2A(1 - 0.0001A) - 0.01AL$$

$$\frac{dL}{dt} = -0.5L + 0.0001AL$$

(a) In the absence of ladybugs, what does the model predict about the aphids?

(b) Find the equilibrium solutions.

(c) Find an expression for dL/dA.

(d) Use a computer to draw a direction field for the differential equation in part (c). Then use the direction field to sketch a phase portrait. What do the phase trajectories have in common?

(e) Suppose that at time $t = 0$ there are 1000 aphids and 200 ladybugs. Draw the corresponding phase trajectory and use it to describe how both populations change.

(f) Use part (e) to make rough sketches of the aphid and ladybug populations as functions of t. How are the graphs related to each other?

9 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. (a) What is a differential equation?
(b) What is the order of a differential equation?
(c) What is an initial condition?

2. What can you say about the solutions of the equation $y' = x^2 + y^2$ just by looking at the differential equation?

3. What is a direction field for the differential equation $y' = F(x, y)$?

4. Explain how Euler's method works.

5. What is a separable differential equation? How do you solve it?

6. What is a first-order linear differential equation? How do you solve it?

7. (a) Write a differential equation that expresses the law of natural growth. What does it say in terms of relative growth rate?
(b) Under what circumstances is this an appropriate model for population growth?
(c) What are the solutions of this equation?

8. (a) Write the logistic differential equation.
(b) Under what circumstances is this an appropriate model for population growth?

9. (a) Write Lotka-Volterra equations to model populations of food-fish (F) and sharks (S).
(b) What do these equations say about each population in the absence of the other?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. All solutions of the differential equation $y' = -1 - y^4$ are decreasing functions.

2. The function $f(x) = (\ln x)/x$ is a solution of the differential equation $x^2 y' + xy = 1$.

3. The function $y = 3e^{2x} - 1$ is a solution of the initial-value problem $y' - 2y = 1$, $y(0) = 2$.

4. The equation $y' = x + y$ is separable.

5. The equation $y' = 3y - 2x + 6xy - 1$ is separable.

6. The equation $e^x y' = y$ is linear.

7. The equation $y' + xy = e^y$ is linear.

8. The solution curve of the differential equation

$$(2x - y)\, y' = x + 2y$$

that passes through the point (3, 1) has slope 1 at that point.

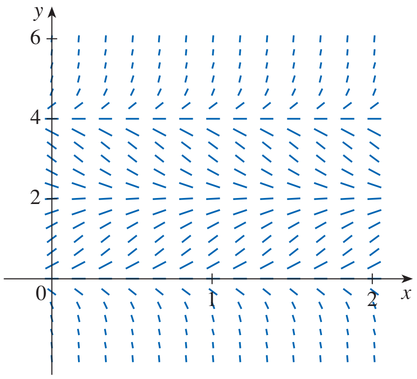
9. If y is the solution of the initial-value problem

$$\frac{dy}{dt} = 2y\left(1 - \frac{y}{5}\right) \qquad y(0) = 1$$

then $\lim_{t\to\infty} y = 5$.

EXERCISES

1. (a) A direction field for the differential equation $y' = y(y - 2)(y - 4)$ is shown. Sketch the graphs of the solutions that satisfy the given initial conditions.
(i) $y(0) = -0.3$
(ii) $y(0) = 1$
(iii) $y(0) = 3$
(iv) $y(0) = 4.3$
(b) If the initial condition is $y(0) = c$, for what values of c is $\lim_{t\to\infty} y(t)$ finite? What are the equilibrium solutions?

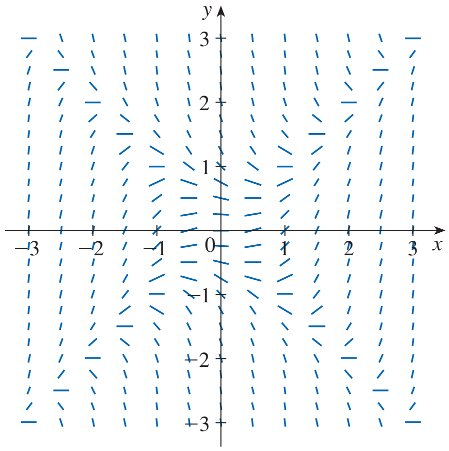


2. (a) Sketch a direction field for the differential equation $y' = x/y$. Then use it to sketch the four solutions that satisfy the initial conditions $y(0) = 1$, $y(0) = -1$, $y(2) = 1$, and $y(-2) = 1$.
(b) Check your work in part (a) by solving the differential equation explicitly. What type of curve is each solution curve?

3. (a) A direction field for the differential equation $y' = x^2 - y^2$ is shown. Sketch the solution of the initial-value problem

$$y' = x^2 - y^2 \qquad y(0) = 1$$

Use your graph to estimate the value of $y(0.3)$.



(b) Use Euler's method with step size 0.1 to estimate $y(0.3)$, where $y(x)$ is the solution of the initial-value problem in part (a). Compare with your estimate from part (a).
(c) On what lines are the centers of the horizontal line segments of the direction field in part (a) located? What happens when a solution curve crosses these lines?

4. (a) Use Euler's method with step size 0.2 to estimate $y(0.4)$, where $y(x)$ is the solution of the initial-value problem

$$y' = 2xy^2 \qquad y(0) = 1$$

(b) Repeat part (a) with step size 0.1.
(c) Find the exact solution of the differential equation and compare the value at 0.4 with the approximations in parts (a) and (b).

5–8 Solve the differential equation.

5. $y' = xe^{-\sin x} - y \cos x$

6. $\dfrac{dx}{dt} = 1 - t + x - tx$

7. $2ye^{y^2}y' = 2x + 3\sqrt{x}$

8. $x^2y' - y = 2x^3e^{-1/x}$

9–11 Solve the initial-value problem.

9. $\dfrac{dr}{dt} + 2tr = r, \quad r(0) = 5$

10. $(1 + \cos x)y' = (1 + e^{-y})\sin x, \quad y(0) = 0$

11. $xy' - y = x \ln x, \quad y(1) = 2$

---

12. Solve the initial-value problem $y' = 3x^2e^y$, $y(0) = 1$, and graph the solution.

13–14 Find the orthogonal trajectories of the family of curves.

13. $y = ke^x$

14. $y = e^{kx}$

---

15. (a) Write the solution of the initial-value problem

$$\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{2000}\right) \qquad P(0) = 100$$

and use it to find the population P when $t = 20$.
(b) When does the population reach 1200?

16. (a) The population of the world was 6.08 billion in 2000 and 7.35 billion in 2015. Find an exponential model for these data and use the model to predict the world population in the year 2030.
(b) According to the model in part (a), in what year will the world population exceed 10 billion?
(c) Use the data in part (a) to find a logistic model for the population. Assume a carrying capacity of 20 billion. Then use the logistic model to predict the population in 2030. Compare with your prediction from the exponential model.
(d) According to the logistic model, in what year will the world population exceed 10 billion? Compare with your prediction in part (b).

17. The von Bertalanffy growth model is used to predict the length $L(t)$ of a fish over a period of time. If L_∞ is the largest length for a species, then the hypothesis is that the rate of growth in length is proportional to $L_\infty - L$, the length yet to be achieved.
(a) Formulate and solve a differential equation to find an expression for $L(t)$.
(b) For the North Sea haddock it has been determined that $L_\infty = 53$ cm, $L(0) = 10$ cm, and the constant of proportionality is 0.2. What does the expression for $L(t)$ become with these data?

18. A tank contains 100 L of pure water. Brine that contains 0.1 kg of salt per liter enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank after 6 minutes?

19. One model for the spread of an epidemic is that the rate of spread is jointly proportional to the number of infected people and the number of uninfected people. In an isolated town of 5000 inhabitants, 160 people have a disease at the beginning of the week and 1200 have it at the end of the week. How many days does it take for 80% of the population to become infected?

20. The Brentano-Stevens Law in psychology models the way that a subject reacts to a stimulus. It states that if R represents the reaction to an amount S of stimulus, then the relative rates of increase are proportional:

$$\frac{dR/dt}{R} = k \cdot \frac{dS/dt}{S}$$

where k is a positive constant. Find R as a function of S.

21. The transport of a substance across a capillary wall in lung physiology has been modeled by the differential equation

$$\frac{dh}{dt} = -\frac{R}{V}\left(\frac{h}{k + h}\right)$$

where h is the hormone concentration in the bloodstream, t is time, R is the maximum transport rate, V is the volume of the capillary, and k is a positive constant that measures the affinity between the hormones and the enzymes that assist the process. Solve this differential equation to find a relationship between h and t.

22. Populations of birds and insects are modeled by the equations

$$\frac{dx}{dt} = 0.4x - 0.002xy$$

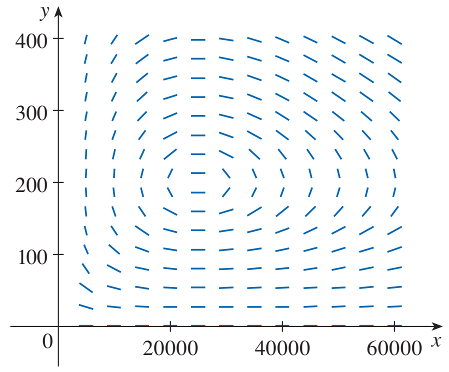
$$\frac{dy}{dt} = -0.2y + 0.000008xy$$

(a) Which of the variables, x or y, represents the bird population and which represents the insect population? Explain.

(b) Find the equilibrium solutions and explain their significance.

(c) Find an expression for dy/dx.

(d) The direction field for the differential equation in part (c) is shown. Use it to sketch the phase trajectory corresponding to initial populations of 100 birds and 40,000 insects. Then use the phase trajectory to describe how both populations change.



(e) Use part (d) to make rough sketches of the bird and insect populations as functions of time. How are these graphs related to each other?

23. Suppose the model of Exercise 22 is replaced by the equations

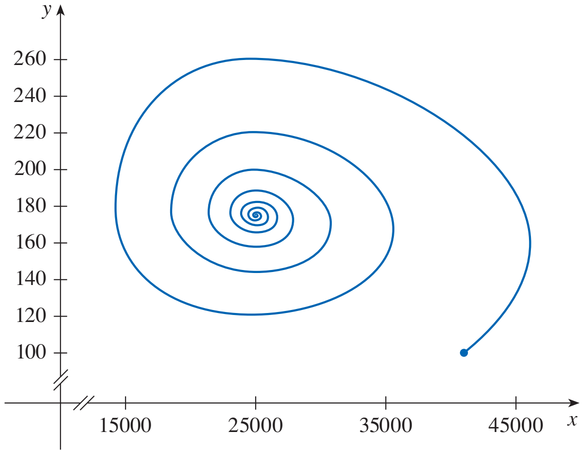
$$\frac{dx}{dt} = 0.4x(1 - 0.000005x) - 0.002xy$$

$$\frac{dy}{dt} = -0.2y + 0.000008xy$$

(a) According to these equations, what happens to the insect population in the absence of birds?

(b) Find the equilibrium solutions and explain their significance.

(c) The figure shows the phase trajectory that starts with 100 birds and 40,000 insects. Describe what eventually happens to the bird and insect populations.



(d) Sketch graphs of the bird and insect populations as functions of time.

24. Brett weighs 85 kg and is on a diet of 2200 calories per day, of which 1200 are used automatically by basal metabolism. He spends about 15 cal/kg/day times his weight doing exercise. If 1 kg of fat contains 10,000 cal and we assume that the storage of calories in the form of fat is 100% efficient, formulate a differential equation and solve it to find his weight as a function of time. Does his weight ultimately approach an equilibrium weight?

Problems Plus

1. Find all functions f such that f' is continuous and

$$[f(x)]^2 = 100 + \int_0^x \{[f(t)]^2 + [f'(t)]^2\}\,dt \qquad \text{for all real } x$$

2. A student forgot the Product Rule for differentiation and made the mistake of thinking that $(fg)' = f'g'$. However, he was lucky and got the correct answer. The function f that he used was $f(x) = e^{x^2}$ and the domain of his problem was the interval $\left(\frac{1}{2}, \infty\right)$. What was the function g?

3. Let f be a function with the property that $f(0) = 1$, $f'(0) = 1$, and $f(a + b) = f(a)f(b)$ for all real numbers a and b. Show that $f'(x) = f(x)$ for all x and deduce that $f(x) = e^x$.

4. Find all functions f that satisfy the equation

$$\left(\int f(x)\,dx\right)\left(\int \frac{1}{f(x)}\,dx\right) = -1$$

5. Find the curve $y = f(x)$ such that $f(x) \geqslant 0$, $f(0) = 0$, $f(1) = 1$, and the area under the graph of f from 0 to x is proportional to the $(n + 1)$st power of $f(x)$.

6. A *subtangent* is a portion of the x-axis that lies directly beneath the segment of a tangent line from the point of contact to the x-axis. Find the curves that pass through the point $(c, 1)$ and whose subtangents all have length c.

7. A peach pie is taken out of the oven at 5:00 PM. At that time it is piping hot, 100°C. At 5:10 PM its temperature is 80°C; at 5:20 PM it is 65°C. What is the temperature of the room?

8. Snow began to fall during the morning of February 2 and continued steadily into the afternoon. At noon a snowplow began removing snow from a road at a constant rate. The plow traveled 6 km from noon to 1 PM but only 3 km from 1 PM to 2 PM. When did the snow begin to fall? [*Hints:* To get started, let t be the time measured in hours after noon; let $x(t)$ be the distance traveled by the plow at time t; then the speed of the plow is dx/dt. Let b be the number of hours before noon that it began to snow. Find an expression for the height of the snow at time t. Then use the given information that the rate of removal R (in m^3/h) is constant.]

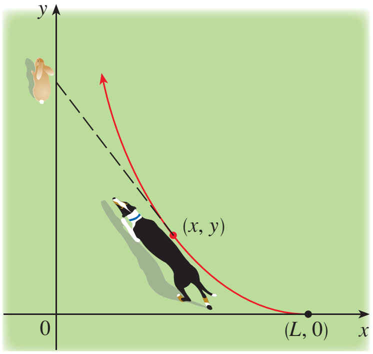


FIGURE FOR PROBLEM 9

9. A dog sees a rabbit running in a straight line across an open field and gives chase. In a rectangular coordinate system (as shown in the figure), assume:

(i) The rabbit is at the origin and the dog is at the point $(L, 0)$ at the instant the dog first sees the rabbit.

(ii) The rabbit runs up the y-axis and the dog always runs straight for the rabbit.

(iii) The dog runs at the same speed as the rabbit.

(a) Show that the dog's path is the graph of the function $y = f(x)$, where y satisfies the differential equation

$$x\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

(b) Determine the solution of the equation in part (a) that satisfies the initial conditions $y = y' = 0$ when $x = L$. [*Hint:* Let $z = dy/dx$ in the differential equation and solve the resulting first-order equation to find z; then integrate z to find y.]

(c) Does the dog ever catch the rabbit?

10. (a) Suppose that the dog in Problem 9 runs twice as fast as the rabbit. Find a differential equation for the path of the dog. Then solve it to find the point where the dog catches the rabbit.

(b) Suppose the dog runs half as fast as the rabbit. How close does the dog get to the rabbit? What are their positions when they are closest?

11. A planning engineer for a new alum plant must present some estimates to his company regarding the capacity of a silo designed to store bauxite ore until it is processed into alum. The ore resembles pink talcum powder and is poured from a conveyor at the top of the silo. The silo is a cylinder 30 m high with a radius of 60 m. The conveyor carries ore at a rate of 1500π m^3/h and the ore maintains a conical shape whose radius is 1.5 times its height.

(a) If, at a certain time t, the pile is 20 m high, how long will it take for the pile to reach the top of the silo?

(b) How much room will be left in the floor area of the silo when the pile is 20 m high? How fast is the floor area of the pile growing at that height?

(c) Suppose a loader starts removing the ore at the rate of 500π m^3/h when the height of the pile reaches 27 m. Suppose, also, that the pile continues to maintain its shape. How long will it take for the pile to reach the top of the silo under these conditions?

12. Find the curve that passes through the point $(3, 2)$ and has the property that if the tangent line is drawn at any point P on the curve, then the part of the tangent line that lies in the first quadrant is bisected at P.

13. Recall that the normal line to a curve at a point P on the curve is the line that passes through P and is perpendicular to the tangent line at P. Find the curve that passes through the point $(3, 2)$ and has the property that if the normal line is drawn at any point on the curve, then the y-intercept of the normal line is always 6.

14. Find all curves with the property that if the normal line is drawn at any point P on the curve, then the part of the normal line between P and the x-axis is bisected by the y-axis.

15. Find all curves with the property that if a line is drawn from the origin to any point (x, y) on the curve, and then a tangent is drawn to the curve at that point and extended to meet the x-axis, the result is an isosceles triangle with equal sides meeting at (x, y).

The photo shows comet Hale-Bopp as it passed the earth in 1997, due to return in 4380. One of the brightest comets of the past century, Hale-Bopp could be observed in the night sky by the naked eye for about 18 months. It was named after its discoverers Alan Hale and Thomas Bopp, who first observed it by telescope in 1995 (Hale in New Mexico and Bopp in Arizona). In Section 10.6 you will see how polar coordinates provide a convenient equation for the elliptical path of the comet's orbit.

Jeff Schneiderman / Moment Open / Getty Images

10

Parametric Equations and Polar Coordinates

SO FAR WE HAVE DESCRIBED plane curves by giving y as a function of x $[y = f(x)]$ or x as a function of y $[x = g(y)]$ or by giving a relation between x and y that defines y implicitly as a function of x $[f(x, y) = 0]$. In this chapter we discuss two new methods for describing curves.

Some curves, such as the cycloid, are best handled when both x and y are given in terms of a third variable t called a parameter $[x = f(t),\ y = g(t)]$. Other curves, such as the cardioid, have their most convenient description when we use a new coordinate system, called the polar coordinate system.

10.1 Curves Defined by Parametric Equations

Imagine that a particle moves along the curve C shown in Figure 1. It is impossible to describe C by an equation of the form $y = f(x)$ because C fails the Vertical Line Test. But the x- and y-coordinates of the particle are functions of time t and so we can write $x = f(t)$ and $y = g(t)$. Such a pair of equations is often a convenient way of describing a curve.

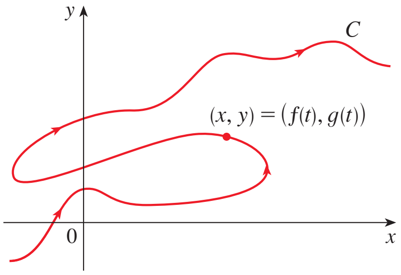


FIGURE 1

Parametric Equations

Suppose that x and y are both given as functions of a third variable t, called a **parameter**, by the equations

$$x = f(t) \qquad y = g(t)$$

which are called **parametric equations**. Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve called a **parametric curve**. The parameter t does not necessarily represent time and, in fact, we could use a letter other than t for the parameter. But in many applications of parametric curves, t does denote time and in this case we can interpret $(x, y) = (f(t), g(t))$ as the position of a moving object at time t.

EXAMPLE 1 Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1$$

SOLUTION Each value of t gives a point on the curve, as shown in the table. For instance, if $t = 1$, then $x = -1$, $y = 2$ and so the corresponding point is $(-1, 2)$. In Figure 2 we plot the points (x, y) determined by several values of the parameter and we join them to produce a curve.

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5

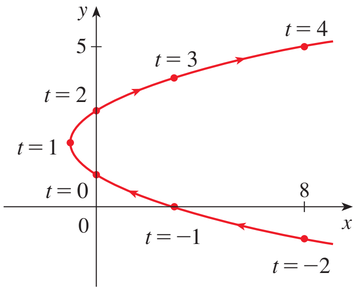


FIGURE 2

A particle whose position at time t is given by the parametric equations moves along the curve in the direction of the arrows as t increases. Notice that the consecutive points marked on the curve appear at equal time intervals but not at equal distances. That is because the particle slows down and then speeds up as t increases.

It appears from Figure 2 that the curve traced out by the particle may be a parabola. In fact, from the second equation we obtain $t = y - 1$ and substitution into the first equation gives

$$x = t^2 - 2t = (y - 1)^2 - 2(y - 1) = y^2 - 4y + 3$$

Since the equation $x = y^2 - 4y + 3$ is satisfied for all pairs of x- and y-values generated by the parametric equations, every point (x, y) on the parametric curve must lie on the parabola $x = y^2 - 4y + 3$ and so the parametric curve coincides with at least part of this parabola. Because t can be chosen to make y any real number, we know that the parametric curve is the entire parabola. ■

It is not always possible to eliminate the parameter from parametric equations. There are many parametric curves that don't have an equivalent representation as an equation in x and y.

In Example 1 we found a Cartesian equation in x and y whose graph coincided with the curve represented by parametric equations. This process is called **eliminating the parameter**; it can be helpful in identifying the shape of the parametric curve, but we lose some information in the process. The equation in x and y describes the curve the particle travels along, whereas the parametric equations have additional advantages—they tell us *where* the particle is at any given *time* and indicate the *direction* of motion. If you think of the graph of an equation in x and y as a road, then the parametric equations could track the motion of a car traveling along the road.

No restriction was placed on the parameter t in Example 1, so we assumed that t could be any real number (including negative numbers). But sometimes we restrict t to lie in a particular interval. For instance, the parametric curve

$$x = t^2 - 2t \qquad y = t + 1 \qquad 0 \leqslant t \leqslant 4$$

shown in Figure 3 is the part of the parabola in Example 1 that starts at the point $(0, 1)$ and ends at the point $(8, 5)$. The arrowhead indicates the direction in which the curve is traced as t increases from 0 to 4.

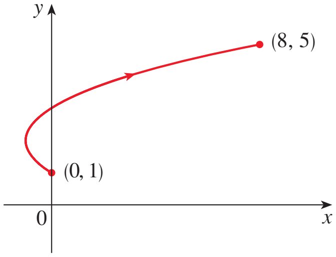


FIGURE 3

In general, the curve with parametric equations

$$x = f(t) \qquad y = g(t) \qquad a \leqslant t \leqslant b$$

has **initial point** $(f(a), g(a))$ and **terminal point** $(f(b), g(b))$.

EXAMPLE 2 What curve is represented by the following parametric equations?

$$x = \cos t \qquad y = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

SOLUTION If we plot points, it appears that the curve is a circle. We can confirm this by eliminating the parameter t. Observe that

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

Because $x^2 + y^2 = 1$ is satisfied for all pairs of x- and y-values generated by the parametric equations, the point (x, y) moves along the unit circle $x^2 + y^2 = 1$. Notice that in this example the parameter t can be interpreted as the angle (in radians) shown in Figure 4. As t increases from 0 to 2π, the point $(x, y) = (\cos t, \sin t)$ moves once around the circle in the counterclockwise direction starting from the point $(1, 0)$. ■

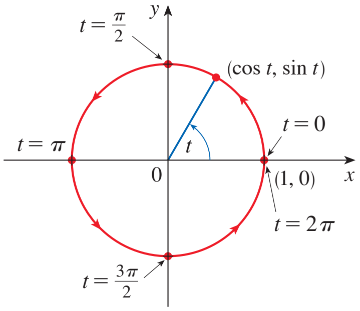


FIGURE 4

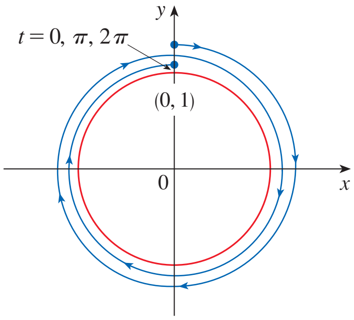


FIGURE 5

EXAMPLE 3 What curve is represented by the given parametric equations?

$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leqslant t \leqslant 2\pi$$

SOLUTION Again we have

$$x^2 + y^2 = \sin^2(2t) + \cos^2(2t) = 1$$

so the parametric equations again represent the unit circle $x^2 + y^2 = 1$. But as t increases from 0 to 2π, the point $(x, y) = (\sin 2t, \cos 2t)$ starts at $(0, 1)$ and moves *twice* around the circle in the clockwise direction as indicated in Figure 5. ■

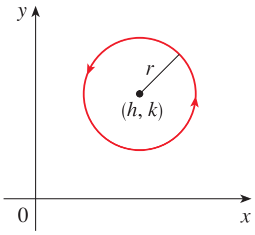


FIGURE 6
$x = h + r\cos t$, $y = k + r\sin t$

EXAMPLE 4 Find parametric equations for the circle with center (h, k) and radius r.

SOLUTION One way is to take the parametric equations of the unit circle in Example 2 and multiply the expressions for x and y by r, giving $x = r\cos t$, $y = r\sin t$. You can verify that these equations represent a circle with radius r and center the origin, traced counterclockwise. We now shift h units in the x-direction and k units in the y-direction and obtain parametric equations of the circle (Figure 6) with center (h, k) and radius r:

$$x = h + r\cos t \qquad y = k + r\sin t \qquad 0 \leqslant t \leqslant 2\pi$$ ■

NOTE Examples 2 and 3 show that different parametric equations can represent the same curve. Thus we distinguish between a *curve*, which is a set of points, and a *parametric curve*, in which the points are traced out in a particular way.

In the next example we use parametric equations to describe the motions of four different particles traveling along the same curve but in different ways.

EXAMPLE 5 Each of the following sets of parametric equations gives the position of a moving particle at time t.

(a) $x = t^3$, $y = t$ (b) $x = -t^3$, $y = -t$

(c) $x = t^{3/2}$, $y = \sqrt{t}$ (d) $x = e^{-3t}$, $y = e^{-t}$

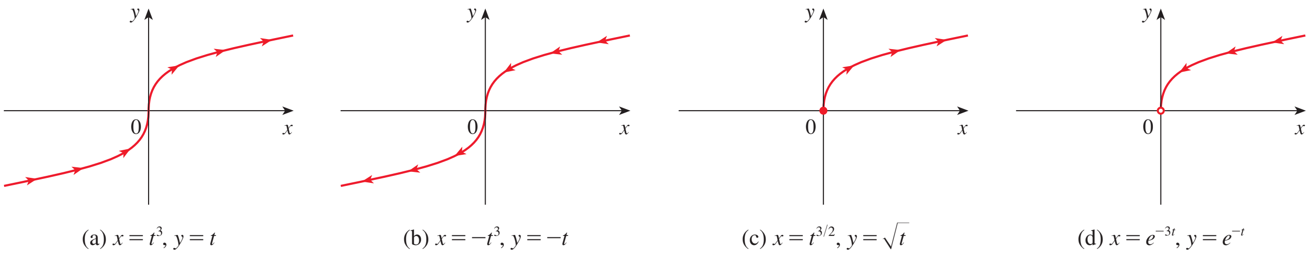
In each case, eliminating the parameter gives $x = y^3$, so each particle moves along the cubic curve $x = y^3$; however, the particles move in different ways, as illustrated in Figure 7.

(a) The particle moves from left to right as t increases.

(b) The particle moves from right to left as t increases.

(c) The equations are defined only for $t \geqslant 0$. The particle starts at the origin (where $t = 0$) and moves to the right as t increases.

(d) Here $x > 0$ and $y > 0$ for all t. The particle moves from right to left and approaches the point (1,1) as t increases (through negative values) toward 0. As t further increases, the particle approaches, but does not reach, the origin.



(a) $x = t^3$, $y = t$ (b) $x = -t^3$, $y = -t$ (c) $x = t^{3/2}$, $y = \sqrt{t}$ (d) $x = e^{-3t}$, $y = e^{-t}$

FIGURE 7 ■

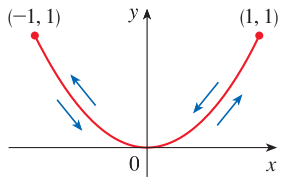
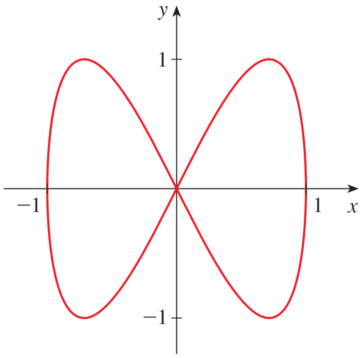


FIGURE 8

EXAMPLE 6 Sketch the curve with parametric equations $x = \sin t$, $y = \sin^2 t$.

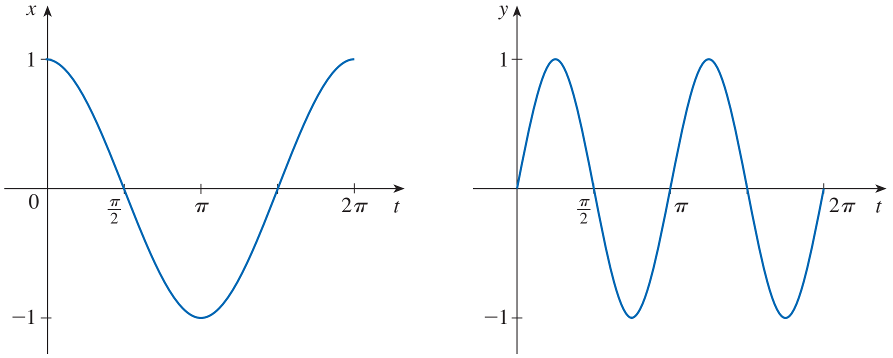
SOLUTION Observe that $y = (\sin t)^2 = x^2$ and so the point (x, y) moves on the parabola $y = x^2$. But note also that, since $-1 \leqslant \sin t \leqslant 1$, we have $-1 \leqslant x \leqslant 1$, so the parametric equations represent only the part of the parabola for which $-1 \leqslant x \leqslant 1$. Since $\sin t$ is periodic, the point $(x, y) = (\sin t, \sin^2 t)$ moves back and forth infinitely often along the parabola from $(-1, 1)$ to $(1, 1)$. (See Figure 8.) ■

EXAMPLE 7 The curve represented by the parametric equations $x = \cos t$, $y = \sin 2t$ is shown in Figure 9. It is an example of a *Lissajous figure* (see Exercise 63). It is possible to eliminate the parameter, but the resulting equation ($y^2 = 4x^2 - 4x^4$) isn't very helpful. Another way to visualize the curve is to first draw graphs of x and y individually as functions of t, as shown in Figure 10.



$x = \cos t$, $y = \sin 2t$

FIGURE 9



$x = \cos t$

$y = \sin 2t$

FIGURE 10

We see that as t increases from 0 to $\pi/2$, x decreases from 1 to 0 while y starts at 0, increases to 1, and then returns to 0. Together these descriptions produce the portion of the parametric curve that we see in the first quadrant. If we proceed similarly, we get the complete curve. (See Exercises 31–33 for practice with this technique.) ■

■ Graphing Parametric Curves with Technology

Most graphing software applications and graphing calculators can graph curves defined by parametric equations. In fact, it's instructive to watch a parametric curve being drawn by a graphing calculator because the points are plotted in order as the corresponding parameter values increase.

The next example shows that parametric equations can be used to produce the graph of a Cartesian equation where x is expressed as a function of y. (Some calculators, for instance, require y to be expressed as a function of x.)

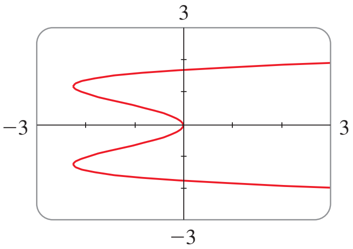


FIGURE 11

EXAMPLE 8 Use a calculator or computer to graph the curve $x = y^4 - 3y^2$.

SOLUTION If we let the parameter be $t = y$, then we have the equations

$$x = t^4 - 3t^2 \qquad y = t$$

Using these parametric equations to graph the curve, we obtain Figure 11. It would be possible to solve the given equation ($x = y^4 - 3y^2$) for y as four functions of x and graph them individually, but the parametric equations provide a much easier method. ■

In general, to graph an equation of the form $x = g(y)$, we can use the parametric equations

$$x = g(t) \qquad y = t$$

In the same spirit, notice that curves with equations $y = f(x)$ (the ones we are most familiar with—graphs of functions) can also be regarded as curves with parametric equations

$$x = t \qquad y = f(t)$$

Graphing software is particularly useful for sketching complicated parametric curves. For instance, the curves shown in Figures 12, 13, and 14 would be virtually impossible to produce by hand.

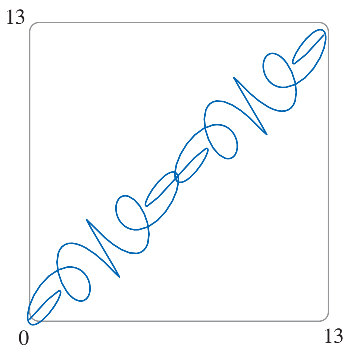


FIGURE 12
$x = t + \sin 5t$
$y = t + \sin 6t$

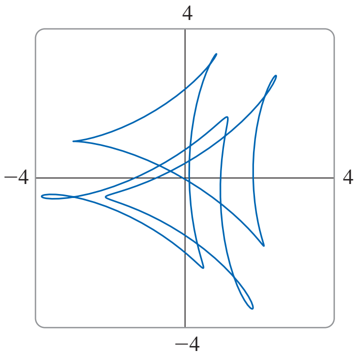


FIGURE 13
$x = \cos t + \cos 6t + 2 \sin 3t$
$y = \sin t + \sin 6t + 2 \cos 3t$

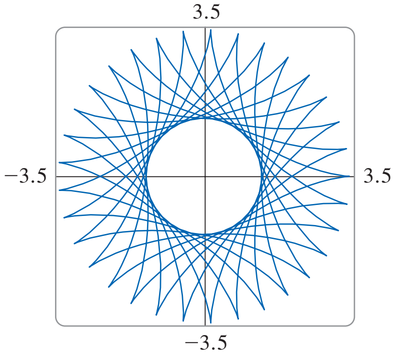


FIGURE 14
$x = 2.3 \cos 10t + \cos 23t$
$y = 2.3 \sin 10t - \sin 23t$

One of the most important uses of parametric curves is in computer-aided design (CAD). In the Discovery Project after Section 10.2 we will investigate special parametric curves, called **Bézier curves**, that are used extensively in manufacturing, especially in the automotive industry. These curves are also employed in specifying the shapes of letters and other symbols in PDF documents and laser printers.

The Cycloid

EXAMPLE 9 The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a **cycloid**. (Think of the path traced out by a pebble stuck in a car tire; see Figure 15.) If the circle has radius r and rolls along the x-axis and if one position of P is the origin, find parametric equations for the cycloid.

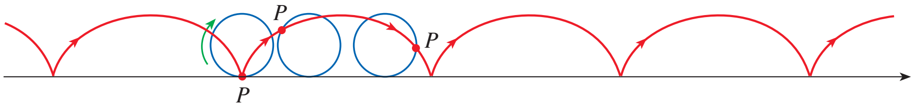


FIGURE 15

SOLUTION We choose as parameter the angle of rotation θ of the circle ($\theta = 0$ when P is at the origin). Suppose the circle has rotated through θ radians. Because the circle has been in contact with the line, we see from Figure 16 that the distance it has rolled from the origin is

$$|OT| = \text{arc } PT = r\theta$$

Therefore the center of the circle is $C(r\theta, r)$. Let the coordinates of P be (x, y). Then from Figure 16 we see that

$$x = |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta)$$

$$y = |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta)$$

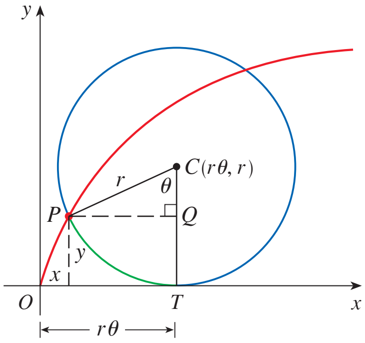


FIGURE 16

Therefore parametric equations of the cycloid are

$$\boxed{1} \qquad x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta) \qquad \theta \in \mathbb{R}$$

One arch of the cycloid comes from one rotation of the circle and so is described by $0 \leqslant \theta \leqslant 2\pi$. Although Equations 1 were derived from Figure 16, which illustrates the case where $0 < \theta < \pi/2$, it can be seen that these equations are still valid for other values of θ (see Exercise 48).

Although it is possible to eliminate the parameter θ from Equations 1, the resulting Cartesian equation in x and y is very complicated [$x = r\cos^{-1}(1 - y/r) - \sqrt{2ry - y^2}$ gives just half of one arch] and not as convenient to work with as the parametric equations. ■

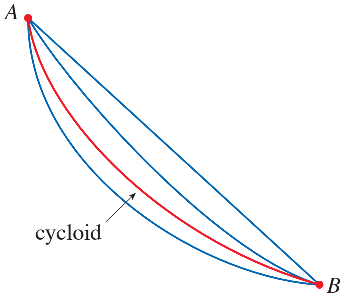


FIGURE 17

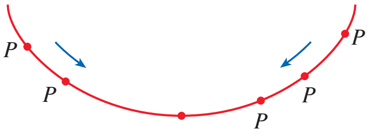


FIGURE 18

One of the first people to study the cycloid was Galileo; he proposed that bridges be built in the shape of cycloids and tried to find the area under one arch of a cycloid. Later this curve arose in connection with the **brachistochrone problem**: Find the curve along which a particle will slide in the shortest time (under the influence of gravity) from a point A to a lower point B not directly beneath A. The Swiss mathematician John Bernoulli, who posed this problem in 1696, showed that among all possible curves that join A to B, as in Figure 17, the particle will take the least time sliding from A to B if the curve is part of an inverted arch of a cycloid.

The Dutch physicist Huygens had already shown by 1673 that the cycloid is also the solution to the **tautochrone problem**; that is, no matter where a particle P is placed on an inverted cycloid, it takes the same time to slide to the bottom (see Figure 18). Huygens proposed that pendulum clocks (which he invented) should swing in cycloidal arcs because then the pendulum would take the same time to make a complete oscillation whether it swings through a wide arc or a small arc.

Families of Parametric Curves

EXAMPLE 10 Investigate the family of curves with parametric equations

$$x = a + \cos t \qquad y = a\tan t + \sin t$$

What do these curves have in common? How does the shape change as a increases?

SOLUTION We use a graphing calculator (or computer) to produce the graphs for the cases $a = -2, -1, -0.5, -0.2, 0, 0.5, 1$, and 2 shown in Figure 19. Notice that all of these curves (except the case $a = 0$) have two branches, and both branches approach the vertical asymptote $x = a$ as x approaches a from the left or right.

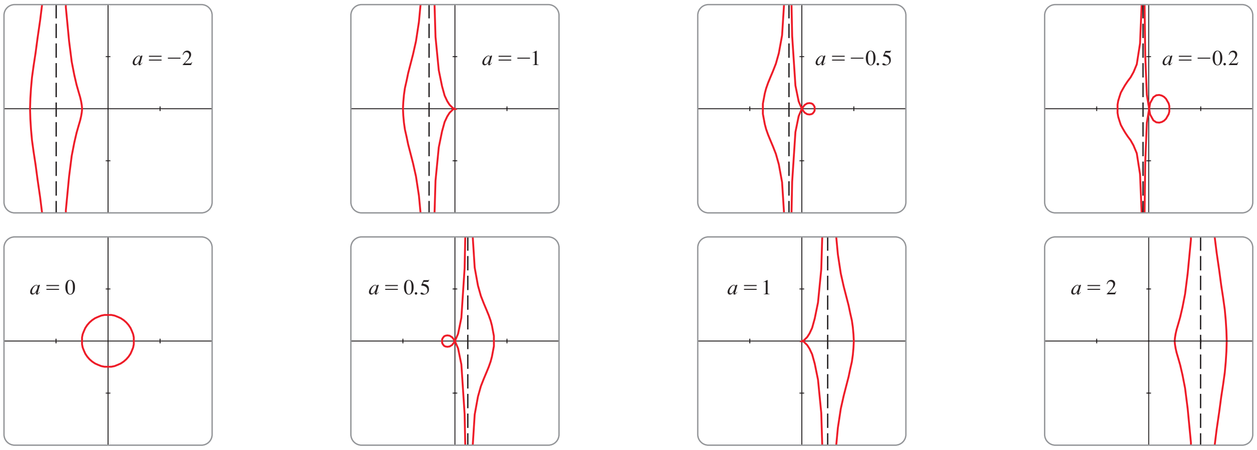


FIGURE 19
Members of the family $x = a + \cos t$, $y = a \tan t + \sin t$, all graphed in the viewing rectangle $[-4, 4]$ by $[-4, 4]$

When $a < -1$, both branches are smooth; but when a reaches -1, the right branch acquires a sharp point, called a *cusp*. For a between -1 and 0 the cusp turns into a loop, which becomes larger as a approaches 0. When $a = 0$, both branches come together and form a circle (see Example 2). For a between 0 and 1, the left branch has a loop, which shrinks to become a cusp when $a = 1$. For $a > 1$, the branches become smooth again, and as a increases further, they become less curved. Notice that the curves with a positive are reflections about the y-axis of the corresponding curves with a negative.

These curves are called **conchoids of Nicomedes** after the ancient Greek scholar Nicomedes. He called them conchoids because the shape of their outer branches resembles that of a conch shell or mussel shell. ■

10.1 Exercises

1–2 For the given parametric equations, find the points (x, y) corresponding to the parameter values $t = -2, -1, 0, 1, 2$.

1. $x = t^2 + t, \quad y = 3^{t+1}$

2. $x = \ln(t^2 + 1), \quad y = t/(t + 4)$

3–6 Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.

3. $x = 1 - t^2, \quad y = 2t - t^2, \quad -1 \leqslant t \leqslant 2$

4. $x = t^3 + t, \quad y = t^2 + 2, \quad -2 \leqslant t \leqslant 2$

5. $x = 2^t - t, \quad y = 2^{-t} + t, \quad -3 \leqslant t \leqslant 3$

6. $x = \cos^2 t, \quad y = 1 + \cos t, \quad 0 \leqslant t \leqslant \pi$

7–12
(a) Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.
(b) Eliminate the parameter to find a Cartesian equation of the curve.

7. $x = 2t - 1, \quad y = \frac{1}{2}t + 1$

8. $x = 3t + 2, \quad y = 2t + 3$

9. $x = t^2 - 3, \quad y = t + 2, \quad -3 \leqslant t \leqslant 3$

10. $x = \sin t, \quad y = 1 - \cos t, \quad 0 \leqslant t \leqslant 2\pi$

11. $x = \sqrt{t}, \quad y = 1 - t$

12. $x = t^2, \quad y = t^3$

13–22
(a) Eliminate the parameter to find a Cartesian equation of the curve.
(b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

13. $x = 3\cos t, \quad y = 3\sin t, \quad 0 \leqslant t \leqslant \pi$

14. $x = \sin 4\theta, \quad y = \cos 4\theta, \quad 0 \leqslant \theta \leqslant \pi/2$

15. $x = \cos\theta, \quad y = \sec^2\theta, \quad 0 \leqslant \theta < \pi/2$

16. $x = \csc t, \quad y = \cot t, \quad 0 < t < \pi$

17. $x = e^{-t}, \quad y = e^{t}$

18. $x = t + 2, \quad y = 1/t, \quad t > 0$

19. $x = \ln t, \quad y = \sqrt{t}, \quad t \geqslant 1$

20. $x = |t|, \quad y = \big|1 - |t|\big|$

21. $x = \sin^2 t, \quad y = \cos^2 t$

22. $x = \sinh t, \quad y = \cosh t$

23–24 The position of an object in circular motion is modeled by the given parametric equations, where t is measured in seconds. How long does it take to complete one revolution? Is the motion clockwise or counterclockwise?

23. $x = 5\cos t, \quad y = -5\sin t$

24. $x = 3\sin\left(\dfrac{\pi}{4}t\right), \quad y = 3\cos\left(\dfrac{\pi}{4}t\right)$

25–28 Describe the motion of a particle with position (x, y) as t varies in the given interval.

25. $x = 5 + 2\cos \pi t, \quad y = 3 + 2\sin \pi t, \quad 1 \leqslant t \leqslant 2$

26. $x = 2 + \sin t, \quad y = 1 + 3\cos t, \quad \pi/2 \leqslant t \leqslant 2\pi$

27. $x = 5\sin t, \quad y = 2\cos t, \quad -\pi \leqslant t \leqslant 5\pi$

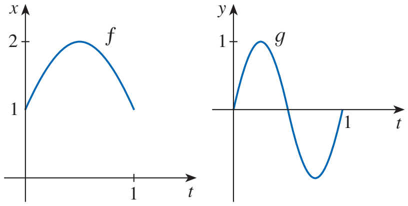
28. $x = \sin t, \quad y = \cos^2 t, \quad -2\pi \leqslant t \leqslant 2\pi$

---

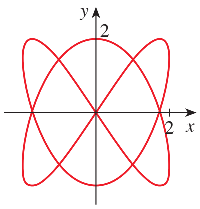
29. Suppose a curve is given by the parametric equations $x = f(t)$, $y = g(t)$, where the range of f is $[1, 4]$ and the range of g is $[2, 3]$. What can you say about the curve?

30. Match each pair of graphs of equations $x = f(t)$, $y = g(t)$ in (a)–(d) with one of the parametric curves $x = f(t)$, $y = g(t)$ labeled I–IV. Give reasons for your choices.

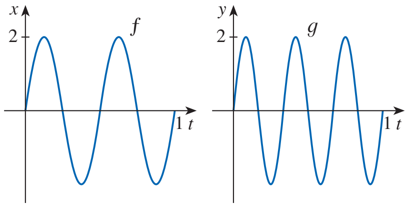
(a)



I



(b)



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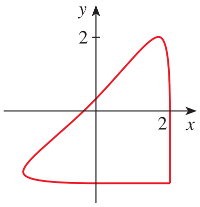
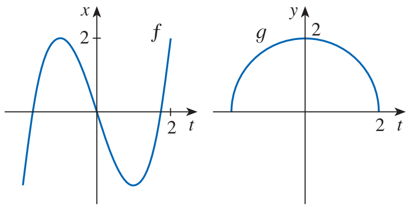
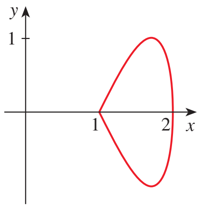


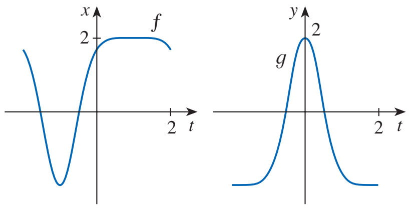
(c)



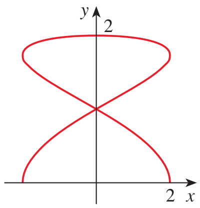
III



(d)

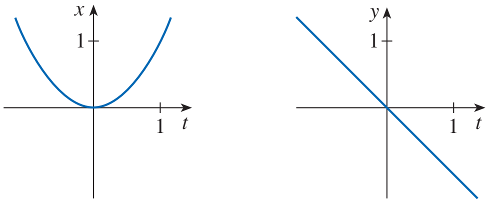


IV

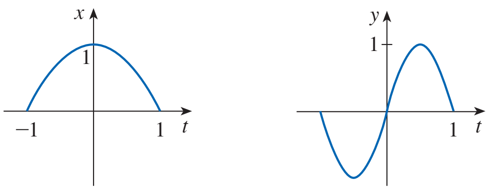


31–33 Use the graphs of $x = f(t)$ and $y = g(t)$ to sketch the parametric curve $x = f(t)$, $y = g(t)$. Indicate with arrows the direction in which the curve is traced as t increases.

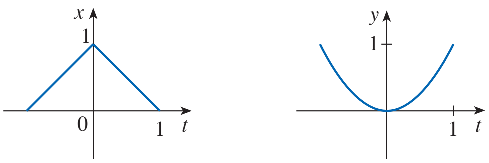
31.



32.



33.



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34. Match the parametric equations with the graphs labeled I–VI. Give reasons for your choices.

(a) $x = t^4 - t + 1, \quad y = t^2$

(b) $x = t^2 - 2t, \quad y = \sqrt{t}$

(c) $x = t^3 - 2t, \quad y = t^2 - t$

(d) $x = \cos 5t, \quad y = \sin 2t$

(e) $x = t + \sin 4t, \quad y = t^2 + \cos 3t$

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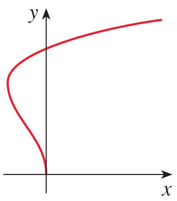
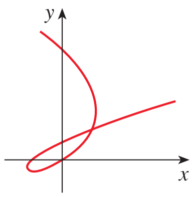
(f) $x = t + \sin 2t, \quad y = t + \sin 3t$

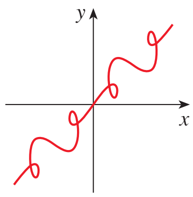
I



II



III



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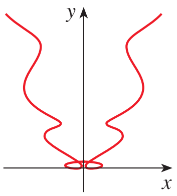
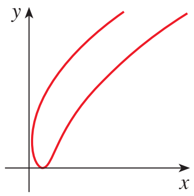
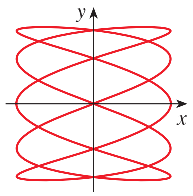


V



VI



35. Graph the curve $x = y - 2 \sin \pi y$.

36. Graph the curves $y = x^3 - 4x$ and $x = y^3 - 4y$ and find their points of intersection correct to one decimal place.

37. (a) Show that the parametric equations

$$x = x_1 + (x_2 - x_1)t \qquad y = y_1 + (y_2 - y_1)t$$

where $0 \leqslant t \leqslant 1$, describe the line segment that joins the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

(b) Find parametric equations to represent the line segment from $(-2, 7)$ to $(3, -1)$.

38. Use a graphing calculator or computer and the result of Exercise 37(a) to draw the triangle with vertices $A(1, 1)$, $B(4, 2)$, and $C(1, 5)$.

39–40 Find parametric equations for the position of a particle moving along a circle as described.

39. The particle travels clockwise around a circle centered at the origin with radius 5 and completes a revolution in 4π seconds.

40. The particle travels counterclockwise around a circle with center $(1, 3)$ and radius 1 and completes a revolution in three seconds.

---

41. Find parametric equations for the path of a particle that moves along the circle $x^2 + (y - 1)^2 = 4$ in the manner described.

(a) Once around clockwise, starting at $(2, 1)$

(b) Three times around counterclockwise, starting at $(2, 1)$

(c) Halfway around counterclockwise, starting at $(0, 3)$

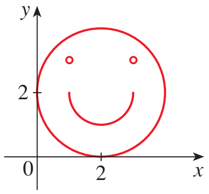
42. (a) Find parametric equations for the ellipse $x^2/a^2 + y^2/b^2 = 1$. [*Hint:* Modify the equations of the circle in Example 2.]

(b) Use these parametric equations to graph the ellipse when $a = 3$ and $b = 1, 2, 4$, and 8.

(c) How does the shape of the ellipse change as b varies?

43–44 Use a graphing calculator or computer to reproduce the picture.

43.



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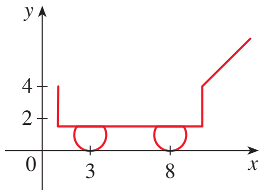

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45. (a) Show that the points on all four of the given parametric curves satisfy the same Cartesian equation.

(i) $x = t^2, \quad y = t$ (ii) $x = t, \quad y = \sqrt{t}$

(iii) $x = \cos^2 t, \quad y = \cos t$ (iv) $x = 3^{2t}, \quad y = 3^t$

(b) Sketch the graph of each curve in part (a) and explain how the curves differ from one another.

46–47 Compare the curves represented by the parametric equations. How do they differ?

46. (a) $x = t, \quad y = t^{-2}$ (b) $x = \cos t, \quad y = \sec^2 t$

(c) $x = e^t, \quad y = e^{-2t}$

47. (a) $x = t^3, \quad y = t^2$ (b) $x = t^6, \quad y = t^4$

(c) $x = e^{-3t}, \quad y = e^{-2t}$

---

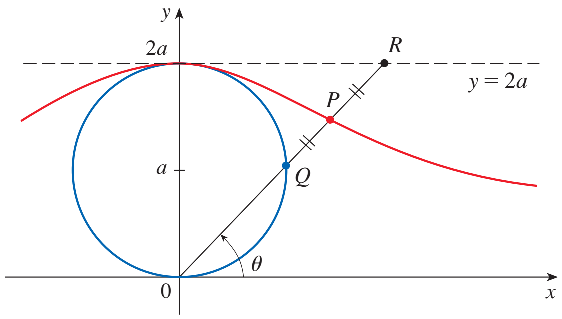
48. Derive Equations 1 for the case $\pi/2 < \theta < \pi$.

49. Let P be a point at a distance d from the center of a circle of radius r. The curve traced out by P as the circle rolls along a straight line is called a **trochoid**. (Think of the motion of a point on a spoke of a bicycle wheel.) The cycloid is the special case of a trochoid with $d = r$. Using the same parameter θ as for the cycloid, and assuming the line is the x-axis and $\theta = 0$ when P is at one of its lowest points, show that parametric equations of the trochoid are

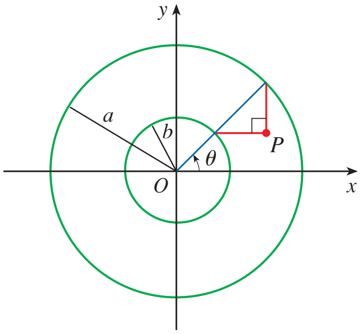
$$x = r\theta - d \sin \theta \qquad y = r - d \cos \theta$$

Sketch the trochoid for the cases $d < r$ and $d > r$.

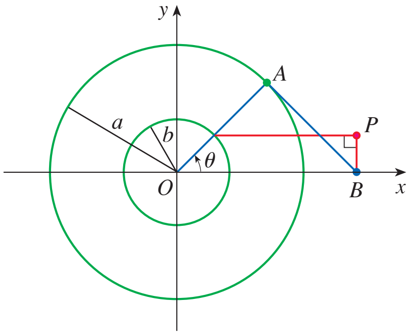
50. In the figure, the circle of radius a is stationary, and for every θ, the point P is the midpoint of the segment QR. The curve traced out by P for $0 < \theta < \pi$ is called the **longbow curve**. Find parametric equations for this curve.



51. If a and b are fixed numbers, find parametric equations for the curve that consists of all possible positions of the point P in the figure, using the angle θ as the parameter. Then eliminate the parameter and identify the curve.



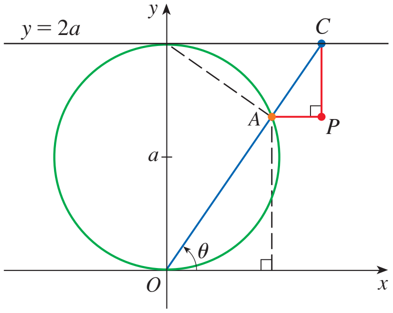
52. If a and b are fixed numbers, find parametric equations for the curve that consists of all possible positions of the point P in the figure, using the angle θ as the parameter. The line segment AB is tangent to the larger circle.



53. A curve, called a **witch of Maria Agnesi**, consists of all possible positions of the point P in the figure. Show that parametric equations for this curve can be written as

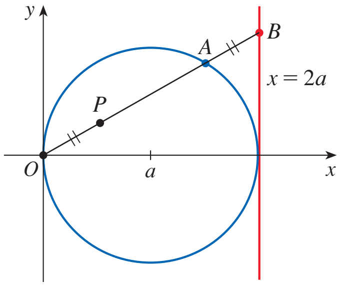
$$x = 2a\cot\theta \qquad y = 2a\sin^2\theta$$

Sketch the curve.



54. (a) Find parametric equations for the set of all points P as shown in the figure such that $|OP| = |AB|$. (This curve is called the **cissoid of Diocles** after the Greek scholar Diocles, who introduced the cissoid as a graphical method for constructing the edge of a cube whose volume is twice that of a given cube.)

(b) Use the geometric description of the curve to draw a rough sketch of the curve by hand. Check your work by using the parametric equations to graph the curve.



55–57 Intersection and Collision Suppose that the position of each of two particles is given by parametric equations. A *collision point* is a point where the particles are at the same place at the same time. If the particles pass through the same point but at different times, then the paths intersect but the particles don't collide.

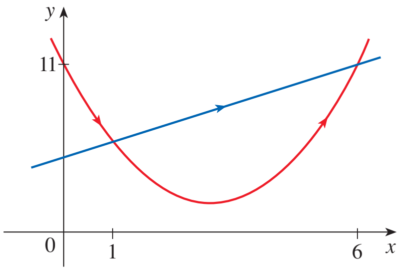
55. The position of a red particle at time t is given by

$$x = t + 5 \qquad y = t^2 + 4t + 6$$

and the position of a blue particle is given by

$$x = 2t + 1 \qquad y = 2t + 6$$

Their paths are shown in the graph.



(a) Verify that the paths of the particles intersect at the points (1, 6) and (6, 11). Is either of these points a collision point? If so, at what time do the particles collide?

(b) Suppose that the position of a green particle is given by

$$x = 2t + 4 \qquad y = 2t + 9$$

Show that this particle moves along the same path as the blue particle. Do the red and green particles collide? If so, at what point and at what time?

56. The position of one particle at time t is given by

$$x = 3\sin t \qquad y = 2\cos t \qquad 0 \leqslant t \leqslant 2\pi$$

and the position of a second particle is given by

$$x = -3 + \cos t \qquad y = 1 + \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

(a) Graph the paths of both particles. At how many points do the graphs intersect?

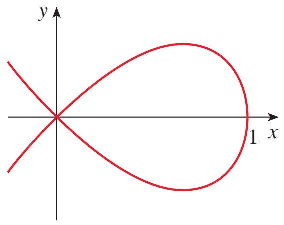
(b) Do the particles collide? If so, find the collision points.

(c) Describe what happens if the path of the second particle is given by

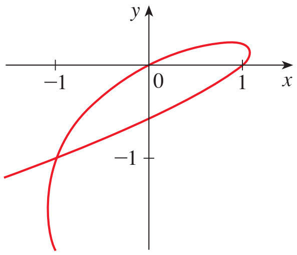
$$x = 3 + \cos t \qquad y = 1 + \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

57. Find the point at which the parametric curve intersects itself and the corresponding values of t.

(a) $x = 1 - t^2, \quad y = t - t^3$



(b) $x = 2t - t^3, \quad y = t - t^2$



58. If a projectile is fired from the origin with an initial velocity of v_0 meters per second at an angle α above the horizontal and air resistance is assumed to be negligible, then its position after t seconds is given by the parametric equations

$$x = (v_0 \cos \alpha)t \qquad y = (v_0 \sin \alpha)t - \tfrac{1}{2}gt^2$$

where g is the acceleration due to gravity (9.8 m/s^2).

(a) If a gun is fired with $\alpha = 30°$ and $v_0 = 500$ m/s, when will the bullet hit the ground? How far from the gun will it hit the ground? What is the maximum height reached by the bullet?

(b) Use a graph to check your answers to part (a). Then graph the path of the projectile for several other values of the angle α to see where it hits the ground. Summarize your findings.

(c) Show that the path is parabolic by eliminating the parameter.

59. Investigate the family of curves defined by the parametric equations $x = t^2$, $y = t^3 - ct$. How does the shape change as c increases? Illustrate by graphing several members of the family.

60. The **swallowtail catastrophe curves** are defined by the parametric equations $x = 2ct - 4t^3$, $y = -ct^2 + 3t^4$. Graph several of these curves. What features do the curves have in common? How do they change when c increases?

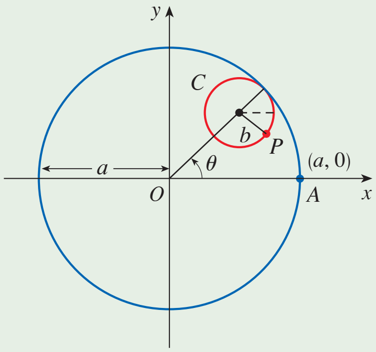
61. Graph several members of the family of curves with parametric equations $x = t + a \cos t$, $y = t + a \sin t$, where $a > 0$. How does the shape change as a increases? For what values of a does the curve have a loop?

62. Graph several members of the family of curves $x = \sin t + \sin nt$, $y = \cos t + \cos nt$, where n is a positive integer. What features do the curves have in common? What happens as n increases?

63. The curves with equations $x = a \sin nt$, $y = b \cos t$ are called **Lissajous figures**. Investigate how these curves vary when a, b, and n vary. (Take n to be a positive integer.)

64. Investigate the family of curves defined by the parametric equations $x = \cos t$, $y = \sin t - \sin ct$, where $c > 0$. Start by letting c be a positive integer and see what happens to the shape as c increases. Then explore some of the possibilities that occur when c is a fraction.

DISCOVERY PROJECT RUNNING CIRCLES AROUND CIRCLES



In this project we investigate families of curves, called *hypocycloids* and *epicycloids*, that are generated by the motion of a point on a circle that rolls inside or outside another circle.

1. A **hypocycloid** is a curve traced out by a fixed point P on a circle C of radius b as C rolls on the inside of a circle with center O and radius a. Show that if the initial position of P is $(a, 0)$ and the parameter θ is chosen as in the figure, then parametric equations of the hypocycloid are

$$x = (a - b)\cos\theta + b\cos\left(\frac{a-b}{b}\,\theta\right) \qquad y = (a - b)\sin\theta - b\sin\left(\frac{a-b}{b}\,\theta\right)$$

2. Use a graphing calculator or computer to draw the graphs of hypocycloids with a a positive integer and $b = 1$. How does the value of a affect the graph? Show that if we take $a = 4$, then the parametric equations of the hypocycloid reduce to

$$x = 4\cos^3\theta \qquad y = 4\sin^3\theta$$

This curve is called a **hypocycloid of four cusps**, or an **astroid**.

3. Now try $b = 1$ and $a = n/d$, a fraction where n and d have no common factor. First let $n = 1$ and try to determine graphically the effect of the denominator d on the shape of the graph. Then let n vary while keeping d constant. What happens when $n = d + 1$?

4. What happens if $b = 1$ and a is irrational? Experiment with an irrational number like $\sqrt{2}$ or $e - 2$. Take larger and larger values for θ and speculate on what would happen if we were to graph the hypocycloid for all real values of θ.

5. If the circle C rolls on the *outside* of the fixed circle, the curve traced out by P is called an **epicycloid**. Find parametric equations for the epicycloid.

6. Investigate the possible shapes for epicycloids. Use methods similar to Problems 2–4.

10.2 Calculus with Parametric Curves

Having seen how to represent curves by parametric equations, we now apply the methods of calculus to these parametric curves. In particular, we solve problems involving tangents, areas, arc length, speed, and surface area.

Tangents

Suppose f and g are differentiable functions and we want to find the tangent line at a point on the parametric curve $x = f(t)$, $y = g(t)$, where y is also a differentiable function of x. Then the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If $dx/dt \neq 0$, we can solve for dy/dx:

If we think of the curve as being traced out by a moving particle, then dy/dt and dx/dt are the vertical and horizontal velocities of the particle and Formula 1 says that the slope of the tangent is the ratio of these velocities.

$$\boxed{1} \qquad \boxed{\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} \qquad \text{if} \quad \frac{dx}{dt} \neq 0}$$

Equation 1 (which you can remember by thinking of canceling the dt's) enables us to find the slope dy/dx of the tangent to a parametric curve without having to eliminate the parameter t. We see from (1) that the curve has a horizontal tangent when $dy/dt = 0$, provided that $dx/dt \neq 0$, and it has a vertical tangent when $dx/dt = 0$, provided that $dy/dt \neq 0$. (If both $dx/dt = 0$ and $dy/dt = 0$, then we would need to use other methods to determine the slope of the tangent.) This information is useful for sketching parametric curves.

As we have learned, it is also often useful to consider d^2y/dx^2. This can be found by replacing y by dy/dx in Equation 1:

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}}$$

⊘ Note that $\dfrac{d^2y}{dx^2} \neq \dfrac{\dfrac{d^2y}{dt^2}}{\dfrac{d^2x}{dt^2}}$

EXAMPLE 1 A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.

(a) Show that C has two tangents at the point $(3, 0)$ and find their equations.
(b) Find the points on C where the tangent is horizontal or vertical.
(c) Determine where the curve is concave upward or downward.
(d) Sketch the curve.

SOLUTION

(a) Notice that $x = 3$ for $t = \pm\sqrt{3}$ and, in both cases, $y = t(t^2 - 3) = 0$. Therefore the point $(3, 0)$ on C arises from two values of the parameter, $t = \sqrt{3}$ and $t = -\sqrt{3}$. This indicates that C crosses itself at $(3, 0)$. Since

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t}$$

the slope of the tangent when $t = \sqrt{3}$ is $dy/dx = 6/(2\sqrt{3}) = \sqrt{3}$, and when $t = -\sqrt{3}$ the slope is $dy/dx = -6/(2\sqrt{3}) = -\sqrt{3}$. Thus we have two different tangent lines at $(3, 0)$ with equations

$$y = \sqrt{3}\,(x - 3) \qquad \text{and} \qquad y = -\sqrt{3}\,(x - 3)$$

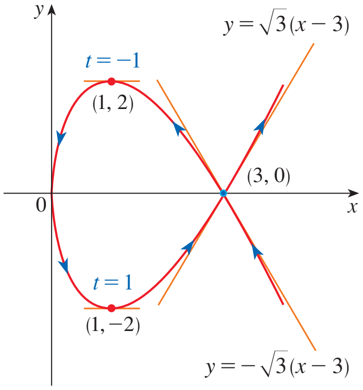


FIGURE 1

(b) C has a horizontal tangent when $dy/dx = 0$, that is, when $dy/dt = 0$ and $dx/dt \neq 0$. Since $dy/dt = 3t^2 - 3$, this happens when $t^2 = 1$, that is, $t = \pm 1$. The corresponding points on C are $(1, -2)$ and $(1, 2)$. C has a vertical tangent when $dx/dt = 2t = 0$, that is, $t = 0$. (Note that $dy/dt \neq 0$ there.) The corresponding point on C is $(0, 0)$.

(c) To determine concavity we calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}} = \frac{\dfrac{d}{dt}\left(\dfrac{3t^2 - 3}{2t}\right)}{\dfrac{dx}{dt}} = \frac{\dfrac{6t^2 + 6}{4t^2}}{2t} = \frac{3t^2 + 3}{4t^3}$$

Thus the curve is concave upward when $t > 0$ and concave downward when $t < 0$.

(d) Using the information from parts (b) and (c), we sketch C in Figure 1. ■

EXAMPLE 2

(a) Find the tangent to the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$ at the point where $\theta = \pi/3$. (See Example 10.1.9.)
(b) At what points is the tangent horizontal? When is it vertical?

SOLUTION

(a) The slope of the tangent line is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\sin\theta}{r(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$$

When $\theta = \pi/3$, we have

$$x = r\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right) = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) \qquad y = r\left(1 - \cos\frac{\pi}{3}\right) = \frac{r}{2}$$

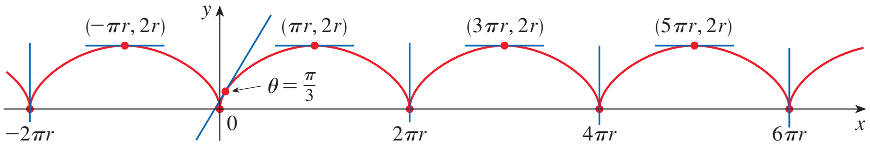
and
$$\frac{dy}{dx} = \frac{\sin(\pi/3)}{1-\cos(\pi/3)} = \frac{\sqrt{3}/2}{1-\frac{1}{2}} = \sqrt{3}$$

Therefore the slope of the tangent is $\sqrt{3}$ and its equation is

$$y - \frac{r}{2} = \sqrt{3}\left(x - \frac{r\pi}{3} + \frac{r\sqrt{3}}{2}\right) \qquad \text{or} \qquad \sqrt{3}\,x - y = r\left(\frac{\pi}{\sqrt{3}} - 2\right)$$

The tangent is sketched in Figure 2.

FIGURE 2



(b) The tangent is horizontal when $dy/dx = 0$, which occurs when $\sin\theta = 0$ and $1 - \cos\theta \neq 0$, that is, $\theta = (2n-1)\pi$, n an integer. The corresponding point on the cycloid is $((2n-1)\pi r, 2r)$.

When $\theta = 2n\pi$, both $dx/d\theta$ and $dy/d\theta$ are 0. It appears from the graph that there are vertical tangents at those points. We can verify this by using l'Hospital's Rule as follows:

$$\lim_{\theta\to 2n\pi^+}\frac{dy}{dx} = \lim_{\theta\to 2n\pi^+}\frac{\sin\theta}{1-\cos\theta} = \lim_{\theta\to 2n\pi^+}\frac{\cos\theta}{\sin\theta} = \infty$$

A similar computation shows that $dy/dx \to -\infty$ as $\theta \to 2n\pi^-$, so indeed there are vertical tangents when $\theta = 2n\pi$, that is, when $x = 2n\pi r$. (See Figure 2.) ■

■ Areas

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)\,dx$, where $F(x) \geqslant 0$. If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

The limits of integration for t are found as usual with the Substitution Rule. When $x = a$, t is either α or β. When $x = b$, t is the remaining value.

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt \qquad \left[\text{or} \quad \int_\beta^\alpha g(t)f'(t)\,dt\right]$$

EXAMPLE 3 Find the area under one arch of the cycloid

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta)$$

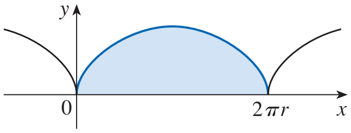


FIGURE 3

The result of Example 3 says that the area under one arch of the cycloid is three times the area of the rolling circle that generates the cycloid (see Example 10.1.9). Galileo guessed this result but it was first proved by the French mathematician Roberval and the Italian mathematician Torricelli.

SOLUTION One arch of the cycloid (shown in Figure 3) is given by $0 \leqslant \theta \leqslant 2\pi$. Using the Substitution Rule with $y = r(1 - \cos\theta)$ and $dx = r(1 - \cos\theta)\,d\theta$, we have

$$\begin{aligned}
A &= \int_0^{2\pi r} y\,dx = \int_0^{2\pi} r(1 - \cos\theta)\,r(1 - \cos\theta)\,d\theta \\
&= r^2 \int_0^{2\pi} (1 - \cos\theta)^2\,d\theta = r^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\,d\theta \\
&= r^2 \int_0^{2\pi} \left[1 - 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta \\
&= r^2 \left[\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} \\
&= r^2\left(\tfrac{3}{2} \cdot 2\pi\right) = 3\pi r^2
\end{aligned}$$

■

Arc Length

We already know how to find the length L of a curve C given in the form $y = F(x)$, $a \leqslant x \leqslant b$. Formula 8.1.3 says that if F' is continuous, then

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \tag{2}$$

Suppose that C can also be described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where $dx/dt = f'(t) > 0$. This means that C is traversed once, from left to right, as t increases from α to β and $f(\alpha) = a$, $f(\beta) = b$. Putting Formula 1 into Formula 2 and using the Substitution Rule, we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\,\frac{dx}{dt}\,dt$$

Since $dx/dt > 0$, we have

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \tag{3}$$

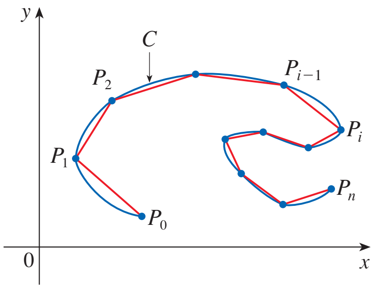


FIGURE 4

Even if C can't be expressed in the form $y = F(x)$, Formula 3 is still valid but we obtain it by polygonal approximations. We divide the parameter interval $[\alpha, \beta]$ into n subintervals of equal width Δt. If $t_0, t_1, t_2, \ldots, t_n$ are the endpoints of these subintervals, then $x_i = f(t_i)$ and $y_i = g(t_i)$ are the coordinates of points $P_i(x_i, y_i)$ that lie on C and the polygonal path with vertices $P_0, P_1, \ldots, P_n$ approximates C. (See Figure 4.)

As in Section 8.1, we define the length L of C to be the limit of the lengths of these approximating polygonal paths as $n \to \infty$:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

The Mean Value Theorem, when applied to f on the interval $[t_{i-1}, t_i]$, gives a number t_i^* in (t_{i-1}, t_i) such that

$$f(t_i) - f(t_{i-1}) = f'(t_i^*)(t_i - t_{i-1})$$

Let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$. Then the preceding equation becomes

$$\Delta x_i = f'(t_i^*)\,\Delta t$$

Similarly, when applied to g, the Mean Value Theorem gives a number t_i^{**} in (t_{i-1}, t_i) such that

$$\Delta y_i = g'(t_i^{**})\,\Delta t$$

Therefore

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{[f'(t_i^*)\,\Delta t]^2 + [g'(t_i^{**})\,\Delta t]^2} \\ &= \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\;\Delta t \end{aligned}$$

and so

4
$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\;\Delta t$$

The sum in (4) resembles a Riemann sum for the function $\sqrt{[f'(t)]^2 + [g'(t)]^2}$ but it is not exactly a Riemann sum because $t_i^* \neq t_i^{**}$ in general. Nevertheless, if f' and g' are continuous, it can be shown that the limit in (4) is the same as if t_i^* and t_i^{**} were equal, namely,

$$L = \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2}\;dt$$

Thus, using Leibniz notation, we have the following result, which has the same form as Formula 3.

5 Theorem If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\;dt$$

Notice that the formula in Theorem 5 is consistent with the general formula $L = \int ds$ of Section 8.1, where

6
$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\;dt$$

EXAMPLE 4 If we use the representation of the unit circle given in Example 10.1.2,

$$x = \cos t \qquad y = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

then $dx/dt = -\sin t$ and $dy/dt = \cos t$, so Theorem 5 gives

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\;dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\;dt = \int_0^{2\pi} dt = 2\pi$$

as expected. If, on the other hand, we use the representation given in Example 10.1.3,

$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leqslant t \leqslant 2\pi$$

then $dx/dt = 2\cos 2t$, $dy/dt = -2\sin 2t$, and the integral in Theorem 5 gives

$$\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\;dt = \int_0^{2\pi} \sqrt{4\cos^2(2t) + 4\sin^2(2t)}\;dt = \int_0^{2\pi} 2\,dt = 4\pi$$

Notice that the integral gives twice the arc length of the circle because as t increases from 0 to 2π, the point $(\sin 2t, \cos 2t)$ traverses the circle twice. In general, when finding the length of a curve C from a parametric representation, we have to be careful to ensure that C is traversed only once as t increases from α to β. ■

EXAMPLE 5 Find the length of one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.

SOLUTION From Example 3 we see that one arch is described by the parameter interval $0 \leqslant \theta \leqslant 2\pi$. Since

$$\frac{dx}{d\theta} = r(1 - \cos\theta) \qquad \text{and} \qquad \frac{dy}{d\theta} = r\sin\theta$$

we have

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{r^2(1 - \cos\theta)^2 + r^2\sin^2\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{r^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)}\, d\theta \\ &= r\int_0^{2\pi} \sqrt{2(1 - \cos\theta)}\, d\theta \end{aligned}$$

The result of Example 5 says that the length of one arch of a cycloid is eight times the radius of the generating circle (see Figure 5). This was first proved in 1658 by Sir Christopher Wren, who later became the architect of St. Paul's Cathedral in London.

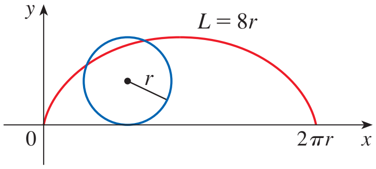


FIGURE 5

To evaluate this integral we use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ with $\theta = 2x$, which gives $1 - \cos\theta = 2\sin^2(\theta/2)$. Since $0 \leqslant \theta \leqslant 2\pi$, we have $0 \leqslant \theta/2 \leqslant \pi$ and so $\sin(\theta/2) \geqslant 0$. Therefore

$$\sqrt{2(1 - \cos\theta)} = \sqrt{4\sin^2(\theta/2)} = 2\left|\sin(\theta/2)\right| = 2\sin(\theta/2)$$

and so

$$\begin{aligned} L &= 2r\int_0^{2\pi} \sin(\theta/2)\, d\theta = 2r\Big[-2\cos(\theta/2)\Big]_0^{2\pi} \\ &= 2r[2 + 2] = 8r \end{aligned}$$

■

The arc length function and speed

Recall that the arc length function (Formula 8.1.5) gives the length of a curve from an initial point to any other point on the curve. For a parametric curve C given by $x = f(t)$, $y = g(t)$, where f' and g' are continuous, we let $s(t)$ be the arc length along C from an initial point $(f(\alpha), g(\alpha))$ to a point $(f(t), g(t))$ on C. By Theorem 5, the **arc length function** s for parametric curves is

7
$$s(t) = \int_\alpha^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du$$

(We have replaced the variable of integration by u so that t does not have two meanings.)

If parametric equations describe the position of a moving particle (with t representing time), then the **speed** of the particle at time t, $v(t)$, is the rate of change of distance traveled (arc length) with respect to time: $s'(t)$. By Equation 7 and Part 1 of the Fundamental Theorem of Calculus, we have

8
$$\boxed{v(t) = s'(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}$$

EXAMPLE 6 The position of a particle at time t is given by the parametric equations $x = 2t + 3$, $y = 4t^2$, $t \geqslant 0$. Find the speed of the particle when it is at the point $(5, 4)$.

SOLUTION By Equation 8, the speed of the particle at any time t is

$$v(t) = \sqrt{2^2 + (8t)^2} = 2\sqrt{1 + 16t^2}$$

The particle is at the point $(5, 4)$ when $t = 1$, so its speed at that point is $v(1) = 2\sqrt{17} \approx 8.25$. (If distance is measured in meters and time in seconds, then the speed is approximately 8.25 m/s.) ■

■ Surface Area

In the same way as for arc length, we can adapt Formula 8.2.5 to obtain a formula for surface area. Suppose a curve C is given by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where f', g' are continuous, $g(t) \geqslant 0$, and C is traversed exactly once as t increases from α to β. If C is rotated about the x-axis, then the area of the resulting surface is given by

$$\boxed{9} \qquad S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The general symbolic formulas $S = \int 2\pi y\, ds$ and $S = \int 2\pi x\, ds$ (Formulas 8.2.7 and 8.2.8) are still valid, where ds is given by Formula 6.

EXAMPLE 7 Show that the surface area of a sphere of radius r is $4\pi r^2$.

SOLUTION The sphere is obtained by rotating the semicircle

$$x = r\cos t \qquad y = r\sin t \qquad 0 \leqslant t \leqslant \pi$$

about the x-axis. Therefore, from Formula 9, we get

$$\begin{aligned} S &= \int_0^\pi 2\pi r\sin t\sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt \\ &= 2\pi \int_0^\pi r\sin t\sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt = 2\pi \int_0^\pi r\sin t \cdot r\, dt \\ &= 2\pi r^2 \int_0^\pi \sin t\, dt = 2\pi r^2(-\cos t)\Big]_0^\pi = 4\pi r^2 \end{aligned}$$ ■

10.2 Exercises

1–4 Find dx/dt, dy/dt, and dy/dx.

1. $x = 2t^3 + 3t, \quad y = 4t - 5t^2$

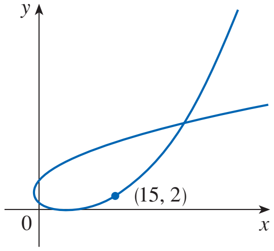
2. $x = t - \ln t, \quad y = t^2 - t^{-2}$

3. $x = te^t, \quad y = t + \sin t$

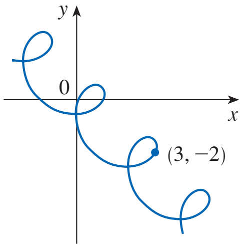
4. $x = t + \sin(t^2 + 2), \quad y = \tan(t^2 + 2)$

5–6 Find the slope of the tangent to the parametric curve at the indicated point.

5. $x = t^2 + 2t, \quad y = 2^t - 2t$



6. $x = t + \cos \pi t, \quad y = -t + \sin \pi t$



7–10 Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.

7. $x = t^3 + 1, \quad y = t^4 + t; \quad t = -1$

8. $x = \sqrt{t}, \quad y = t^2 - 2t; \quad t = 4$

9. $x = \sin 2t + \cos t, \quad y = \cos 2t - \sin t; \quad t = \pi$

10. $x = e^t \sin \pi t, \quad y = e^{2t}; \quad t = 0$

11–12 Find an equation of the tangent to the curve at the given point by two methods: (a) without eliminating the parameter and (b) by first eliminating the parameter.

11. $x = \sin t, \quad y = \cos^2 t; \quad \left(\frac{1}{2}, \frac{3}{4}\right)$

12. $x = \sqrt{t + 4}, \quad y = 1/(t + 4); \quad \left(2, \frac{1}{4}\right)$

13–14 Find an equation of the tangent to the curve at the given point. Then graph the curve and the tangent.

13. $x = t^2 - t, \quad y = t^2 + t + 1; \quad (0, 3)$

14. $x = \sin \pi t, \quad y = t^2 + t; \quad (0, 2)$

15–20 Find dy/dx and d^2y/dx^2. For which values of t is the curve concave upward?

15. $x = t^2 + 1, \quad y = t^2 + t$

16. $x = t^3 + 1, \quad y = t^2 - t$

17. $x = e^t, \quad y = te^{-t}$

18. $x = t^2 + 1, \quad y = e^t - 1$

19. $x = t - \ln t, \quad y = t + \ln t$

20. $x = \cos t, \quad y = \sin 2t, \quad 0 < t < \pi$

21–24 Find the points on the curve where the tangent is horizontal or vertical. You may want to use a graph from a calculator or computer to check your work.

21. $x = t^3 - 3t, \quad y = t^2 - 3$

22. $x = t^3 - 3t, \quad y = t^3 - 3t^2$

23. $x = \cos \theta, \quad y = \cos 3\theta$

24. $x = e^{\sin \theta}, \quad y = e^{\cos \theta}$

---

25. Use a graph to estimate the coordinates of the rightmost point on the curve $x = t - t^6$, $y = e^t$. Then use calculus to find the exact coordinates.

26. Use a graph to estimate the coordinates of the lowest point and the leftmost point on the curve $x = t^4 - 2t$, $y = t + t^4$. Then find the exact coordinates.

27–28 Graph the curve in a viewing rectangle that displays all the important aspects of the curve.

27. $x = t^4 - 2t^3 - 2t^2, \quad y = t^3 - t$

28. $x = t^4 + 4t^3 - 8t^2, \quad y = 2t^2 - t$

---

29. Show that the curve $x = \cos t$, $y = \sin t \cos t$ has two tangents at $(0, 0)$ and find their equations. Graph the curve.

30. Graph the curve $x = -2 \cos t$, $y = \sin t + \sin 2t$ to discover where it crosses itself. Then find equations of both tangents at that point.

31. (a) Find the slope of the tangent line to the trochoid $x = r\theta - d \sin \theta$, $y = r - d \cos \theta$ in terms of θ. (See Exercise 10.1.49.)
(b) Show that if $d < r$, then the trochoid does not have a vertical tangent.

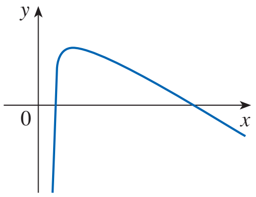
32. (a) Find the slope of the tangent to the astroid $x = a \cos^3\theta$, $y = a \sin^3\theta$ in terms of θ. (Astroids are explored in the Discovery Project following Section 10.1.)
(b) At what points is the tangent horizontal or vertical?
(c) At what points does the tangent have slope 1 or -1?

33. At what point(s) on the curve $x = 3t^2 + 1$, $y = t^3 - 1$ does the tangent line have slope $\frac{1}{2}$?

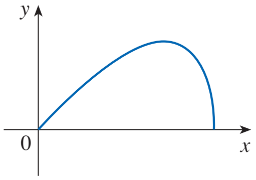
34. Find equations of the tangents to the curve $x = 3t^2 + 1$, $y = 2t^3 + 1$ that pass through the point $(4, 3)$.

35–36 Find the area enclosed by the given parametric curve and the x-axis.

35. $x = t^3 + 1, \quad y = 2t - t^2$



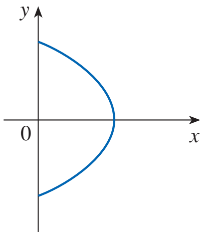
36. $x = \sin t, \quad y = \sin t \cos t, \quad 0 \leqslant t \leqslant \pi/2$

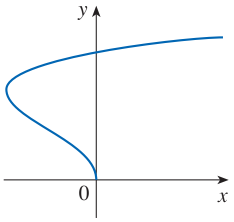


37–38 Find the area enclosed by the given parametric curve and the y-axis.

37. $x = \sin^2 t,$
$y = \cos t$

38. $x = t^2 - 2t,$
$y = \sqrt{t}$

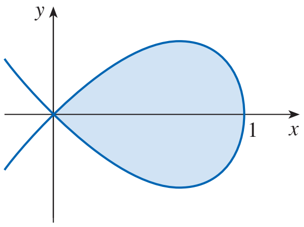




39. Use the parametric equations of an ellipse, $x = a \cos \theta$, $y = b \sin \theta$, $0 \leqslant \theta \leqslant 2\pi$, to find the area that it encloses.

40. Find the area of the region enclosed by the loop of the curve

$$x = 1 - t^2, \quad y = t - t^3$$

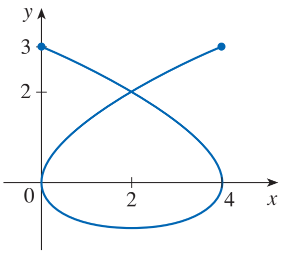


41. Find the area under one arch of the trochoid of Exercise 10.1.49 for the case $d < r$.

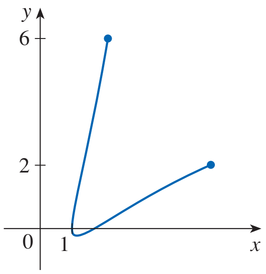
42. Let $\mathcal{R}$ be the region enclosed by the loop of the curve in Example 1.
(a) Find the area of $\mathcal{R}$.
(b) If $\mathcal{R}$ is rotated about the x-axis, find the volume of the resulting solid.
(c) Find the centroid of $\mathcal{R}$.

T **43–46** Set up an integral that represents the length of the part of the parametric curve shown in the graph. Then use a calculator (or computer) to find the length correct to four decimal places.

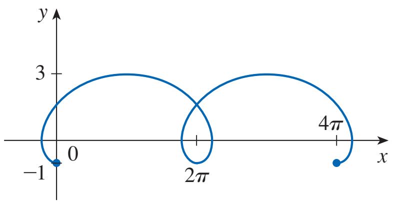
43. $x = 3t^2 - t^3, \quad y = t^2 - 2t$



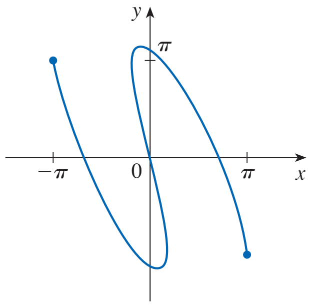
44. $x = t + e^{-t}, \quad y = t^2 + t$



45. $x = t - 2 \sin t, \quad y = 1 - 2 \cos t, \quad 0 \leqslant t \leqslant 4\pi$



46. $x = t \cos t, \quad y = t - 5 \sin t$



47–50 Find the exact length of the curve.

47. $x = \frac{2}{3}t^3, \quad y = t^2 - 2, \quad 0 \leqslant t \leqslant 3$

48. $x = e^t - t, \quad y = 4e^{t/2}, \quad 0 \leqslant t \leqslant 2$

49. $x = t \sin t, \quad y = t \cos t, \quad 0 \leqslant t \leqslant 1$

50. $x = 3 \cos t - \cos 3t, \quad y = 3 \sin t - \sin 3t, \quad 0 \leqslant t \leqslant \pi$

51–52 Graph the curve and find its exact length.

51. $x = e^t \cos t, \quad y = e^t \sin t, \quad 0 \leqslant t \leqslant \pi$

52. $x = \cos t + \ln(\tan \frac{1}{2}t), \quad y = \sin t, \quad \pi/4 \leqslant t \leqslant 3\pi/4$

53. Graph the curve $x = \sin t + \sin 1.5t$, $y = \cos t$ and find its length correct to four decimal places.

54. Find the length of the loop of the curve $x = 3t - t^3$, $y = 3t^2$.

55–56 Find the distance traveled by a particle with position (x, y) as t varies in the given time interval. Compare with the length of the curve.

55. $x = \sin^2 t, \quad y = \cos^2 t, \quad 0 \leqslant t \leqslant 3\pi$

56. $x = \cos^2 t, \quad y = \cos t, \quad 0 \leqslant t \leqslant 4\pi$

57–60 The parametric equations give the position (in meters) of a moving particle at time t (in seconds). Find the speed of the particle at the indicated time or point.

57. $x = 2t - 3, \quad y = 2t^2 - 3t + 6; \quad t = 5$

58. $x = 2 + 5 \cos\left(\dfrac{\pi}{3}t\right), \quad y = -2 + 7 \sin\left(\dfrac{\pi}{3}t\right); \quad t = 3$

59. $x = e^t, \quad y = te^t; \quad (e, e)$

60. $x = t^2 + 1, \quad y = t^4 + 2t^2 + 1; \quad (2, 4)$

61. A projectile is fired from the point $(0, 0)$ with an initial velocity of v_0 m/s at an angle α above the horizontal. (See Exercise 10.1.58.) If we assume that air resistance is negligible, then the position (in meters) of the projectile after t seconds is given by the parametric equations

$$x = (v_0 \cos \alpha)t \qquad y = (v_0 \sin \alpha)t - \tfrac{1}{2}gt^2$$

where $g = 9.8 \text{ m/s}^2$ is the acceleration due to gravity.

(a) Find the speed of the projectile when it hits the ground.
(b) Find the speed of the projectile at its highest point.

62. Show that the total length of the ellipse $x = a \sin\theta$, $y = b \cos\theta$, $a > b > 0$, is

$$L = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2\theta}\, d\theta$$

where e is the eccentricity of the ellipse $\left(e = c/a, \text{ where } c = \sqrt{a^2 - b^2}\,\right)$.

T **63.** (a) Graph the **epitrochoid** with equations

$$x = 11 \cos t - 4 \cos(11t/2)$$

$$y = 11 \sin t - 4 \sin(11t/2)$$

What parameter interval gives the complete curve?
(b) Use a calculator or computer to find the approximate length of this curve.

T **64.** A curve called **Cornu's spiral** is defined by the parametric equations

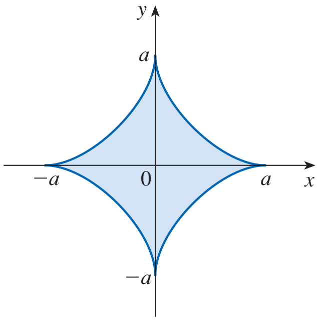
$$x = C(t) = \int_0^t \cos(\pi u^2/2)\, du$$

$$y = S(t) = \int_0^t \sin(\pi u^2/2)\, du$$

where C and S are the Fresnel functions that were introduced in Chapter 5.

(a) Graph this curve. What happens as $t \to \infty$ and as $t \to -\infty$?
(b) Find the length of Cornu's spiral from the origin to the point with parameter value t.

65–66 The curve shown in the figure is the astroid $x = a \cos^3\theta$, $y = a \sin^3\theta$. (Astroids are explored in the Discovery Project following Section 10.1.)



65. Find the area of the region enclosed by the astroid.

66. Find the perimeter of the astroid.

67–70 Set up an integral that represents the area of the surface obtained by rotating the given curve about the x-axis. Then use a calculator or computer to find the surface area correct to four decimal places.

67. $x = t \sin t, \quad y = t \cos t, \quad 0 \leqslant t \leqslant \pi/2$

68. $x = \sin t, \quad y = \sin 2t, \quad 0 \leqslant t \leqslant \pi/2$

69. $x = t + e^t, \quad y = e^{-t}, \quad 0 \leqslant t \leqslant 1$

70. $x = t^2 - t^3, \quad y = t + t^4, \quad 0 \leqslant t \leqslant 1$

71–73 Find the exact area of the surface obtained by rotating the given curve about the x-axis.

71. $x = t^3, \quad y = t^2, \quad 0 \leqslant t \leqslant 1$

72. $x = 2t^2 + 1/t, \quad y = 8\sqrt{t}, \quad 1 \leqslant t \leqslant 3$

73. $x = a \cos^3\theta, \quad y = a \sin^3\theta, \quad 0 \leqslant \theta \leqslant \pi/2$

74. Graph the curve

$$x = 2\cos\theta - \cos 2\theta$$
$$y = 2\sin\theta - \sin 2\theta$$

If this curve is rotated about the x-axis, find the exact area of the resulting surface. (Use your graph to help find the correct parameter interval.)

75–76 Find the surface area generated by rotating the given curve about the y-axis.

75. $x = 3t^2, \quad y = 2t^3, \quad 0 \leqslant t \leqslant 5$

76. $x = e^t - t, \quad y = 4e^{t/2}, \quad 0 \leqslant t \leqslant 1$

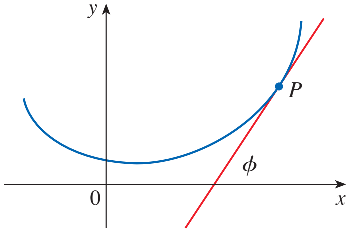
77. If f' is continuous and $f'(t) \neq 0$ for $a \leqslant t \leqslant b$, show that the parametric curve $x = f(t)$, $y = g(t)$, $a \leqslant t \leqslant b$, can be put in the form $y = F(x)$. [*Hint:* Show that f^{-1} exists.]

78. Use Formula 1 to derive Formula 9 from Formula 8.2.5 for the case in which the curve can be represented in the form $y = F(x)$, $a \leqslant x \leqslant b$.

79–83 Curvature The *curvature* at a point P of a curve is defined as

$$\kappa = \left| \frac{d\phi}{ds} \right|$$

where ϕ is the angle of inclination of the tangent line at P, as shown in the figure. Thus the curvature is the absolute value of the rate of change of ϕ with respect to arc length. It can be regarded as a measure of the rate of change of direction of the curve at P and will be studied in greater detail in Chapter 13.



79. For a parametric curve $x = x(t)$, $y = y(t)$, derive the formula

$$\kappa = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$$

where the dots indicate derivatives with respect to t, so $\dot{x} = dx/dt$. [*Hint:* Use $\phi = \tan^{-1}(dy/dx)$ and Formula 2 to find $d\phi/dt$. Then use the Chain Rule to find $d\phi/ds$.]

80. By regarding a curve $y = f(x)$ as the parametric curve $x = x$, $y = f(x)$ with parameter x, show that the formula in Exercise 79 becomes

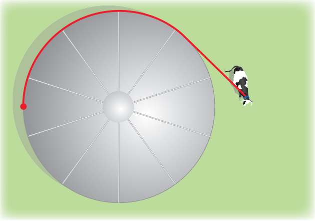
$$\kappa = \frac{|d^2y/dx^2|}{[1 + (dy/dx)^2]^{3/2}}$$

81. Use the formula in Exercise 79 to find the curvature of the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$ at the top of one of its arches.

82. (a) Use the formula in Exercise 80 to find the curvature of the parabola $y = x^2$ at the point $(1, 1)$.
(b) At what point does this parabola have maximum curvature?

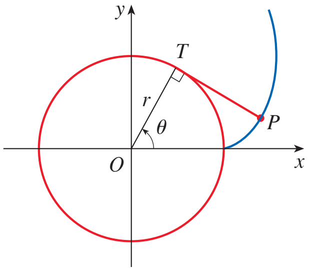
83. (a) Show that the curvature at each point of a straight line is $\kappa = 0$.
(b) Show that the curvature at each point of a circle of radius r is $\kappa = 1/r$.

84. A cow is tied to a silo with radius r by a rope just long enough to reach the opposite side of the silo. Find the grazing area available for the cow.



85. A string is wound around a circle and then unwound while being held taut. The curve traced by the point P at the end of the string is called the **involute** of the circle. If the circle has radius r and center O and the initial position of P is $(r, 0)$, and if the parameter θ is chosen as in the figure, show that parametric equations of the involute are

$$x = r(\cos\theta + \theta\sin\theta)$$
$$y = r(\sin\theta - \theta\cos\theta)$$



DISCOVERY PROJECT BÉZIER CURVES

Bézier curves are used in computer-aided design (CAD) and are named after the French mathematician Pierre Bézier (1910–1999), who worked in the automotive industry. A cubic Bézier curve is determined by four *control points*, $P_0(x_0, y_0)$, $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$, and is defined by the parametric equations

$$x = x_0(1 - t)^3 + 3x_1t(1 - t)^2 + 3x_2t^2(1 - t) + x_3t^3$$
$$y = y_0(1 - t)^3 + 3y_1t(1 - t)^2 + 3y_2t^2(1 - t) + y_3t^3$$

where $0 \leqslant t \leqslant 1$. Notice that when $t = 0$ we have $(x, y) = (x_0, y_0)$ and when $t = 1$ we have $(x, y) = (x_3, y_3)$, so the curve starts at P_0 and ends at P_3.

1. Graph the Bézier curve with control points $P_0(4, 1)$, $P_1(28, 48)$, $P_2(50, 42)$, and $P_3(40, 5)$. Then, on the same screen, graph the line segments P_0P_1, P_1P_2, and P_2P_3. (Exercise 10.1.37 shows how to do this.) Notice that the middle control points P_1 and P_2 don't lie on the curve; the curve starts at P_0, heads toward P_1 and P_2 without reaching them, and ends at P_3.
2. From the graph in Problem 1, it appears that the tangent at P_0 passes through P_1 and the tangent at P_3 passes through P_2. Prove it.
3. Try to produce a Bézier curve with a loop by changing the second control point in Problem 1.
4. Some laser printers use Bézier curves to represent letters and other symbols. Experiment with control points until you find a Bézier curve that gives a reasonable representation of the letter C.
5. More complicated shapes can be represented by piecing together two or more Bézier curves. Suppose the first Bézier curve has control points P_0, P_1, P_2, P_3 and the second one has control points P_3, P_4, P_5, P_6. If we want these two pieces to join together smoothly, then the tangents at P_3 should match and so the points P_2, P_3, and P_4 all have to lie on this common tangent line. Using this principle, find control points for a pair of Bézier curves that represent the letter S.

10.3 Polar Coordinates

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. Usually we use Cartesian coordinates, which are directed distances from two perpendicular axes. Here we describe a coordinate system introduced by Newton, called the *polar coordinate system*, which is more convenient for many purposes.

The Polar Coordinate System

We choose a point in the plane that is called the **pole** (or origin) and is labeled O. Then we draw a ray (half-line) starting at O called the **polar axis**. This axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.

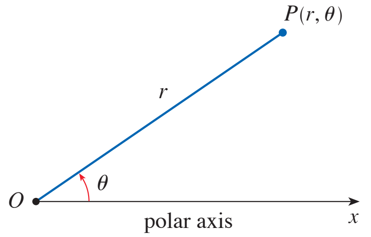

$P(r, \theta)$
r
O
θ
polar axis
x

FIGURE 1

If P is any other point in the plane, let r be the distance from O to P and let θ be the angle (usually measured in radians) between the polar axis and the line OP as in Figure 1. Then the point P is represented by the ordered pair (r, θ) and r, θ are called **polar coordinates** of P. We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If $P = O$, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of θ.

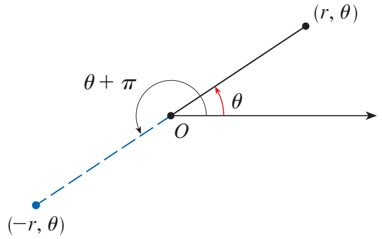

(r, θ)
$\theta + \pi$
θ
O
$(-r, \theta)$

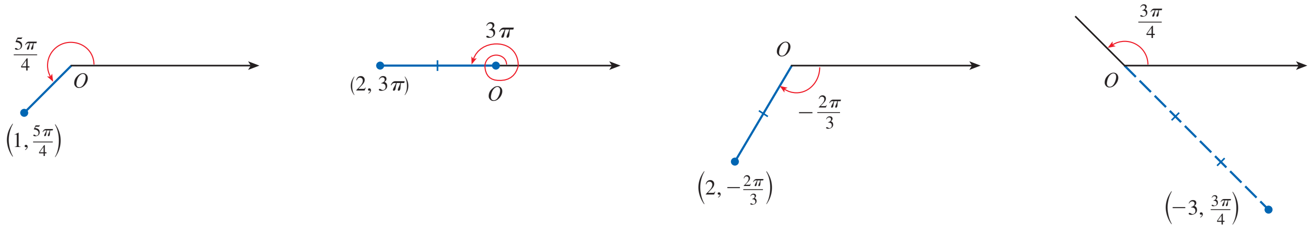
FIGURE 2

We extend the meaning of polar coordinates (r, θ) to the case in which r is negative by agreeing that, as in Figure 2, the points $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance $|r|$ from O, but on opposite sides of O. If $r > 0$, the point (r, θ) lies in the same quadrant as θ; if $r < 0$, it lies in the quadrant on the opposite side of the pole. Notice that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.

EXAMPLE 1 Plot the points whose polar coordinates are given.

(a) $(1, 5\pi/4)$ (b) $(2, 3\pi)$ (c) $(2, -2\pi/3)$ (d) $(-3, 3\pi/4)$

SOLUTION The points are plotted in Figure 3. In part (d) the point $(-3, 3\pi/4)$ is located three units from the pole in the fourth quadrant because the angle $3\pi/4$ is in the second quadrant and $r = -3$ is negative.


$\frac{5\pi}{4}$
O
$\left(1, \frac{5\pi}{4}\right)$
3π
$(2, 3\pi)$
O
O
$-\frac{2\pi}{3}$
$\left(2, -\frac{2\pi}{3}\right)$
$\frac{3\pi}{4}$
O
$\left(-3, \frac{3\pi}{4}\right)$

(a) (b) (c) (d) ■

FIGURE 3

In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. For instance, the point $(1, 5\pi/4)$ in Example 1(a) could be written as $(1, -3\pi/4)$ or $(1, 13\pi/4)$ or $(-1, \pi/4)$. (See Figure 4.)

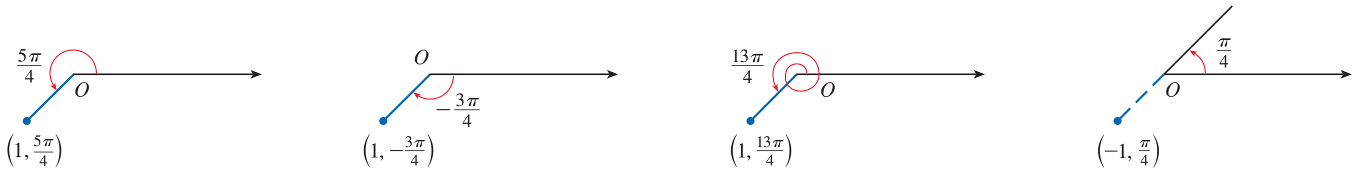

$\frac{5\pi}{4}$
O
$\left(1, \frac{5\pi}{4}\right)$
O
$-\frac{3\pi}{4}$
$\left(1, -\frac{3\pi}{4}\right)$
$\frac{13\pi}{4}$
O
$\left(1, \frac{13\pi}{4}\right)$
$\frac{\pi}{4}$
O
$\left(-1, \frac{\pi}{4}\right)$

FIGURE 4

In fact, since a complete counterclockwise rotation is given by an angle 2π, the point represented by polar coordinates (r, θ) is also represented by

$$(r, \theta + 2n\pi) \qquad \text{and} \qquad (-r, \theta + (2n + 1)\pi)$$

where n is any integer.

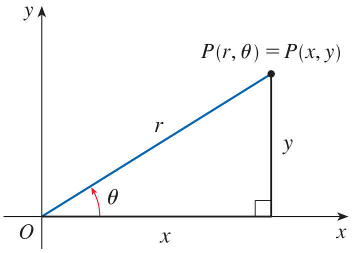


FIGURE 5

Relationship between Polar and Cartesian Coordinates

The connection between polar and Cartesian coordinates can be seen from Figure 5, in which the pole corresponds to the origin and the polar axis coincides with the positive x-axis. If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ), then, from the figure, we have $\cos\theta = x/r$ and $\sin\theta = y/r$. So to find the Cartesian coordinates (x, y) when the polar coordinates (r, θ) are known, we use the equations

1
$$\boxed{x = r\cos\theta \qquad y = r\sin\theta}$$

To find polar coordinates (r, θ) when the Cartesian coordinates (x, y) are known, we use the equations

2
$$\boxed{r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}}$$

which can be deduced from Equations 1 or simply read from Figure 5.

Although Equations 1 and 2 were deduced from Figure 5, which illustrates the case where $r > 0$ and $0 < \theta < \pi/2$, these equations are valid for all values of r and θ. (See the general definition of $\sin\theta$ and $\cos\theta$ in Appendix D.)

EXAMPLE 2 Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.

SOLUTION Since $r = 2$ and $\theta = \pi/3$, Equations 1 give

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\cdot\frac{1}{2} = 1$$

$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$$

Therefore the point is $(1, \sqrt{3})$ in Cartesian coordinates. ■

EXAMPLE 3 Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

SOLUTION If we choose r to be positive, then Equations 2 give

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\tan\theta = \frac{y}{x} = -1$$

Since the point $(1, -1)$ lies in the fourth quadrant, we can choose $\theta = -\pi/4$ or $\theta = 7\pi/4$. Thus one possible answer is $(\sqrt{2}, -\pi/4)$; another is $(\sqrt{2}, 7\pi/4)$. ■

NOTE Equations 2 do not uniquely determine θ when x and y are given because, as θ increases through the interval $0 \leqslant \theta < 2\pi$, each value of $\tan\theta$ occurs twice. Therefore, in converting from Cartesian to polar coordinates, it's not good enough just to find r and θ that satisfy Equations 2. As in Example 3, we must choose θ so that the point (r, θ) lies in the correct quadrant.

Polar Curves

The **graph of a polar equation** $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

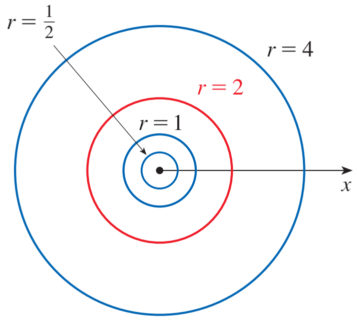


FIGURE 6

EXAMPLE 4 What curve is represented by the polar equation $r = 2$?

SOLUTION The curve consists of all points (r, θ) with $r = 2$. Since r represents the distance from the point to the pole, the curve $r = 2$ represents the circle with center O and radius 2. In general, the equation $r = a$ represents a circle with center O and radius $|a|$. (See Figure 6.) ■

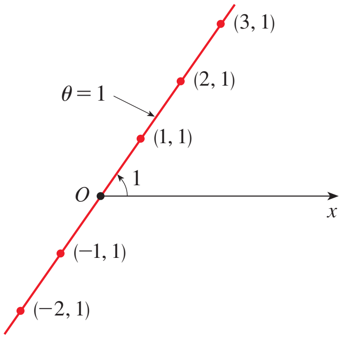


FIGURE 7

EXAMPLE 5 Sketch the polar curve $\theta = 1$.

SOLUTION This curve consists of all points (r, θ) such that the polar angle θ is 1 radian. It is the straight line that passes through O and makes an angle of 1 radian with the polar axis (see Figure 7). Notice that the points $(r, 1)$ on the line with $r > 0$ are in the first quadrant, whereas those with $r < 0$ are in the third quadrant. ■

EXAMPLE 6

(a) Sketch the curve with polar equation $r = 2\cos\theta$.

(b) Find a Cartesian equation for this curve.

SOLUTION

(a) In Figure 8 we find the values of r for some convenient values of θ and plot the corresponding points (r, θ). Then we join these points to sketch the curve, which appears to be a circle. We have used only values of θ between 0 and π, because if we let θ increase beyond π, we obtain the same points again.

θ	$r = 2\cos\theta$
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2

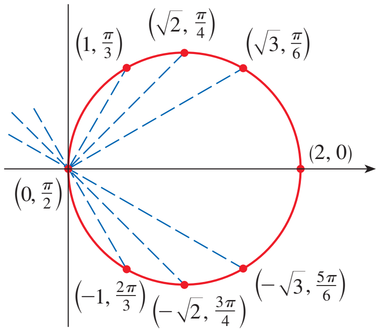


FIGURE 8
Table of values and graph of $r = 2\cos\theta$

(b) To convert the given equation to a Cartesian equation we use Equations 1 and 2. From $x = r\cos\theta$ we have $\cos\theta = x/r$, so the equation $r = 2\cos\theta$ becomes $r = 2x/r$, which gives

$$2x = r^2 = x^2 + y^2 \qquad \text{or} \qquad x^2 + y^2 - 2x = 0$$

Completing the square, we obtain

$$(x - 1)^2 + y^2 = 1$$

which is an equation of a circle with center (1, 0) and radius 1. ■

Figure 9 shows a geometric illustration that the circle in Example 6 has the equation $r = 2\cos\theta$. The angle OPQ is a right angle (why?) and so $r/2 = \cos\theta$.

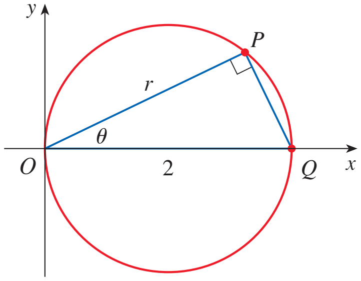


FIGURE 9

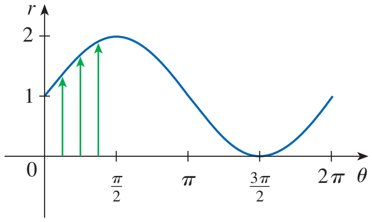


FIGURE 10
$r = 1 + \sin\theta$ in Cartesian coordinates, $0 \leqslant \theta \leqslant 2\pi$

EXAMPLE 7 Sketch the curve $r = 1 + \sin\theta$.

SOLUTION Instead of plotting points as in Example 6, we first sketch the graph of $r = 1 + \sin\theta$ in *Cartesian* coordinates in Figure 10 by shifting the sine curve up one unit. This enables us to read at a glance the values of r that correspond to increasing values of θ. For instance, we see that as θ increases from 0 to $\pi/2$, r (the distance from O) increases from 1 to 2 (see the corresponding green arrows in Figures 10 and 11), so we sketch the corresponding part of the polar curve in Figure 11(a). As θ increases from $\pi/2$ to π, Figure 10 shows that r decreases from 2 to 1, so we sketch the next part of the curve as in Figure 11(b). As θ increases from π to $3\pi/2$, r decreases from 1 to 0 as shown in part (c). Finally, as θ increases from $3\pi/2$ to 2π, r increases from 0 to 1 as shown in part (d). If we let θ increase beyond 2π or decrease beyond 0, we would simply retrace this path. Putting together the parts of the curve from Figure 11(a)–(d), we sketch the complete curve in part (e). It is called a **cardioid** because it's shaped like a heart.

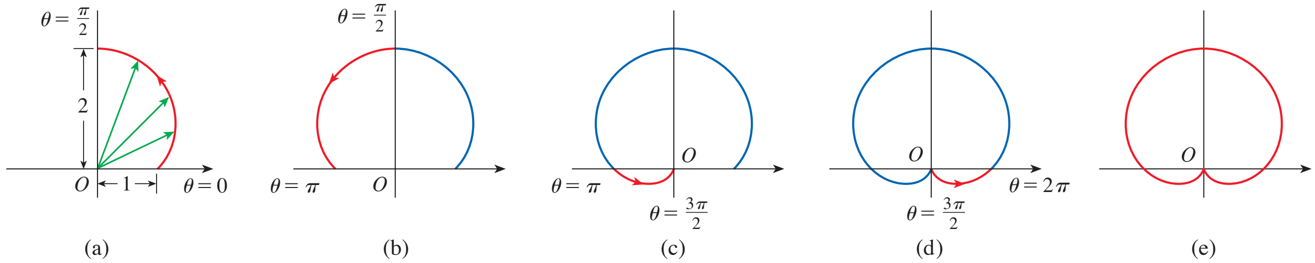


FIGURE 11 Stages in sketching the cardioid $r = 1 + \sin\theta$ ■

EXAMPLE 8 Sketch the curve $r = \cos 2\theta$.

SOLUTION As in Example 7, we first sketch $r = \cos 2\theta$, $0 \leqslant \theta \leqslant 2\pi$, in Cartesian coordinates in Figure 12. As θ increases from 0 to $\pi/4$, Figure 12 shows that r decreases from 1 to 0 and so we draw the corresponding portion of the polar curve in Figure 13 (indicated by ①). As θ increases from $\pi/4$ to $\pi/2$, r decreases from 0 to -1. This means that the distance from O increases from 0 to 1, but instead of being in the first quadrant this portion of the polar curve (indicated by ②) lies on the opposite side of the pole in the third quadrant. The remainder of the curve is drawn in a similar fashion, with the arrows and numbers indicating the order in which the portions are traced out. The resulting curve has four loops and is called a **four-leaved rose**.

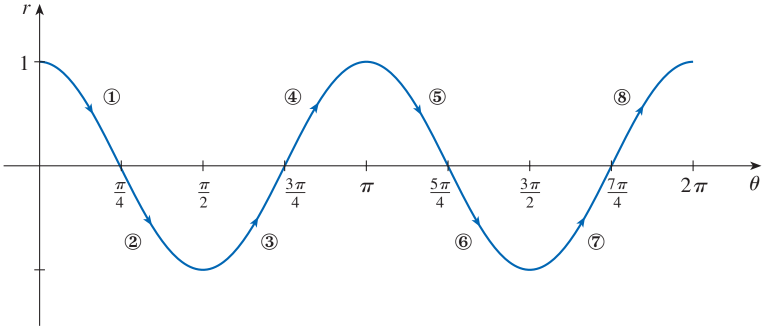


FIGURE 12
$r = \cos 2\theta$ in Cartesian coordinates

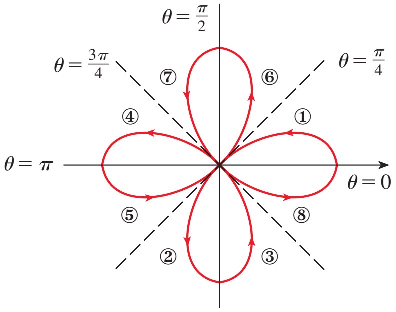


FIGURE 13
Four-leaved rose $r = \cos 2\theta$ ■

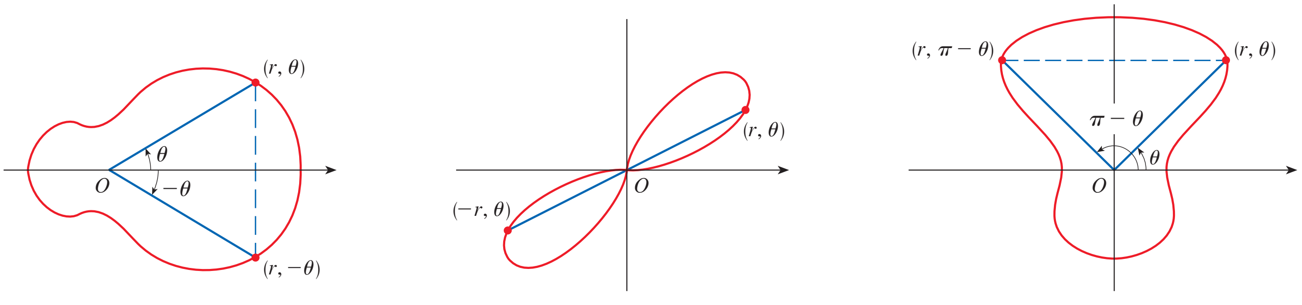
Symmetry

When we sketch polar curves it is sometimes helpful to take advantage of symmetry. The following three rules are explained by Figure 14.

(a) If a polar equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about the polar axis.

(b) If the equation is unchanged when r is replaced by $-r$, or when θ is replaced by $\theta + \pi$, the curve is symmetric about the pole. (This means that the curve remains unchanged if we rotate it through 180° about the origin.)

(c) If the equation is unchanged when θ is replaced by $\pi - \theta$, the curve is symmetric about the vertical line $\theta = \pi/2$.



(a) (b) (c)

FIGURE 14

The curves sketched in Examples 6 and 8 are symmetric about the polar axis, since $\cos(-\theta) = \cos\theta$. The curves in Examples 7 and 8 are symmetric about $\theta = \pi/2$ because $\sin(\pi - \theta) = \sin\theta$ and $\cos[2(\pi - \theta)] = \cos 2\theta$. The four-leaved rose is also symmetric about the pole. We could have used these symmetry properties in sketching the curves. For instance, in Example 6 we need only have plotted points for $0 \leqslant \theta \leqslant \pi/2$ and then reflected about the polar axis to obtain the complete circle.

Graphing Polar Curves with Technology

Although it’s useful to be able to sketch simple polar curves by hand, we need to use a graphing calculator or computer when we are faced with a curve as complicated as the ones shown in Figures 15 and 16.

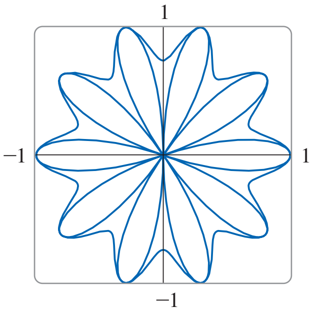


FIGURE 15
$r = \sin^3(2.5\theta) + \cos^3(2.5\theta)$

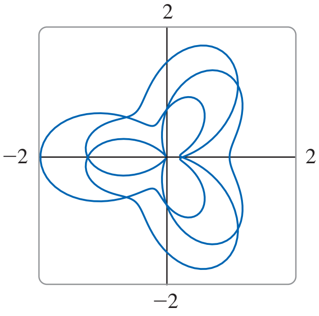


FIGURE 16
$r = \sin^2(3\theta/2) + \cos^2(2\theta/3)$

EXAMPLE 9 Graph the curve $r = \sin(8\theta/5)$.

SOLUTION First we need to determine the domain for θ. So we ask ourselves: how many complete rotations are required until the curve starts to repeat itself? If the answer is n, then

$$\sin\frac{8(\theta + 2n\pi)}{5} = \sin\left(\frac{8\theta}{5} + \frac{16n\pi}{5}\right) = \sin\frac{8\theta}{5}$$

and so we require that $16n\pi/5$ be an even multiple of π. This will first occur when $n = 5$. Therefore we will graph the entire curve if we specify that $0 \leqslant \theta \leqslant 10\pi$. Figure 17 shows the resulting curve. Notice that this curve has 16 loops. ■

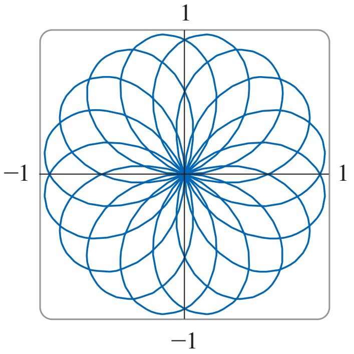


FIGURE 17
$r = \sin(8\theta/5)$

EXAMPLE 10 Investigate the family of polar curves given by $r = 1 + c\sin\theta$. How does the shape change as c changes? (These curves are called **limaçons**, after a French word for snail, because of the shape of the curves for certain values of c.)

SOLUTION Figure 18 shows computer-drawn graphs for various values of c. (Note that we obtain the complete graph for $0 \leqslant \theta \leqslant 2\pi$.) For $c > 1$ there is a loop that decreases

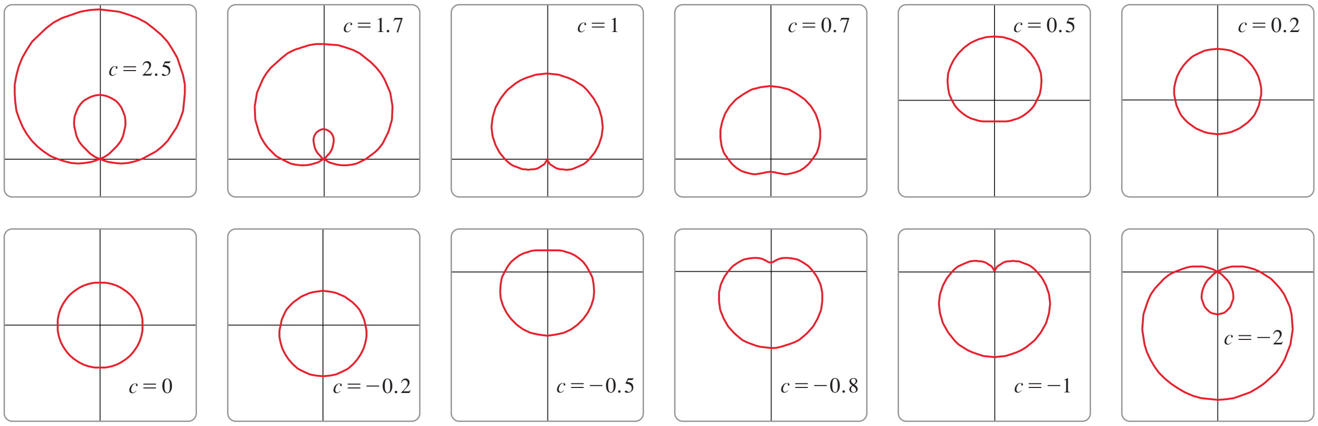


FIGURE 18 Members of the family of limaçons $r = 1 + c\sin\theta$

in size as c decreases. When $c = 1$ the loop disappears and the curve becomes the cardioid that we sketched in Example 7. For c between 1 and $\frac{1}{2}$ the cardioid's cusp is smoothed out and becomes a "dimple." When c decreases from $\frac{1}{2}$ to 0, the limaçon is shaped like an oval. This oval becomes more circular as $c \to 0$, and when $c = 0$ the curve is just the circle $r = 1$.

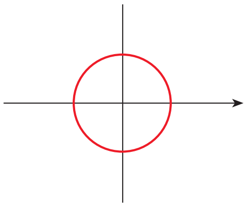
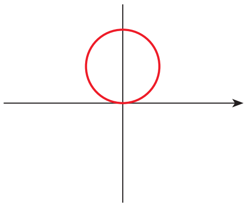
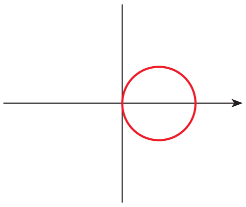
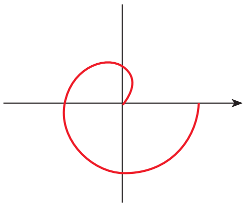
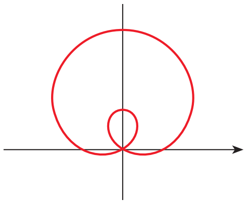
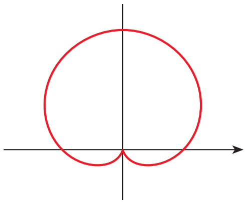
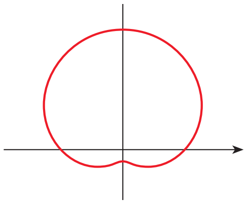
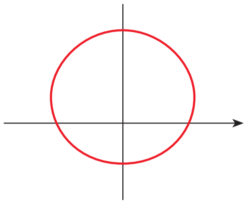
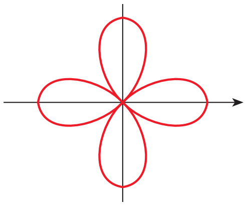
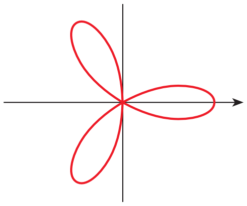
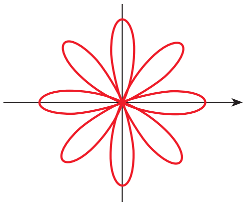
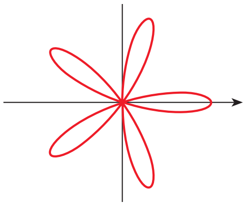
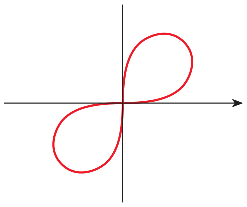
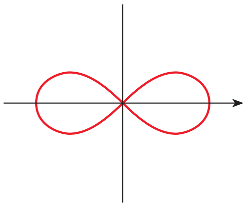
In Exercise 55 you are asked to prove analytically what we have discovered from the graphs in Figure 18.

The remaining parts of Figure 18 show that as c becomes negative, the shapes change in reverse order. In fact, these curves are reflections about the horizontal axis of the corresponding curves with positive c. ■

Limaçons arise in the study of planetary motion. In particular, the trajectory of Mars, as viewed from the planet Earth, has been modeled by a limaçon with a loop, as in the parts of Figure 18 with $|c| > 1$.

Table 1 gives a summary of some common polar curves.

Table 1 Common Polar Curves

Circles and Spiral	 $r = a$ circle	 $r = a \sin \theta$ circle	 $r = a \cos \theta$ circle	 $r = a\theta$ spiral
Limaçons $r = a \pm b \sin \theta$ $r = a \pm b \cos \theta$ $(a > 0, b > 0)$ Orientation depends on the trigonometric function (sine or cosine) and the sign of b	 $a < b$ limaçon with inner loop	 $a = b$ cardioid	 $a > b$ dimpled limaçon	 $a \geqslant 2b$ convex limaçon
Roses $r = a \sin n\theta$ $r = a \cos n\theta$ n-leaved if n is odd $2n$-leaved if n is even	 $r = a \cos 2\theta$ four-leaved rose	 $r = a \cos 3\theta$ three-leaved rose	 $r = a \cos 4\theta$ eight-leaved rose	 $r = a \cos 5\theta$ five-leaved rose
Lemniscates Figure-eight-shaped curves	 $r^2 = a^2 \sin 2\theta$ lemniscate	 $r^2 = a^2 \cos 2\theta$ lemniscate		

10.3 Exercises

1–2 Plot the point whose polar coordinates are given. Then find two other pairs of polar coordinates of this point, one with $r > 0$ and one with $r < 0$.

1. (a) $(1, \pi/4)$ (b) $(-2, 3\pi/2)$ (c) $(3, -\pi/3)$

2. (a) $(2, 5\pi/6)$ (b) $(1, -2\pi/3)$ (c) $(-1, 5\pi/4)$

3–4 Plot the point whose polar coordinates are given. Then find the Cartesian coordinates of the point.

3. (a) $(2, 3\pi/2)$ (b) $(\sqrt{2}, \pi/4)$ (c) $(-1, -\pi/6)$

4. (a) $(4, 4\pi/3)$ (b) $(-2, 3\pi/4)$ (c) $(-3, -\pi/3)$

5–6 The Cartesian coordinates of a point are given.

(i) Find polar coordinates (r, θ) of the point, where $r > 0$ and $0 \leqslant \theta < 2\pi$.

(ii) Find polar coordinates (r, θ) of the point, where $r < 0$ and $0 \leqslant \theta < 2\pi$.

5. (a) $(-4, 4)$ (b) $(3, 3\sqrt{3})$

6. (a) $(\sqrt{3}, -1)$ (b) $(-6, 0)$

7–12 Sketch the region in the plane consisting of points whose polar coordinates satisfy the given conditions.

7. $1 < r \leqslant 3$

8. $r \geqslant 2, \quad 0 \leqslant \theta \leqslant \pi$

9. $0 \leqslant r \leqslant 1, \quad -\pi/2 \leqslant \theta \leqslant \pi/2$

10. $3 < r < 5, \quad 2\pi/3 \leqslant \theta \leqslant 4\pi/3$

11. $2 \leqslant r < 4, \quad 3\pi/4 \leqslant \theta \leqslant 7\pi/4$

12. $r \geqslant 0, \quad \pi \leqslant \theta \leqslant 5\pi/2$

13. Find the distance between the points with polar coordinates $(4, 4\pi/3)$ and $(6, 5\pi/3)$.

14. Find a formula for the distance between the points with polar coordinates (r_1, θ_1) and (r_2, θ_2).

15–20 Identify the curve by finding a Cartesian equation for the curve.

15. $r^2 = 5$

16. $r = 4 \sec\theta$

17. $r = 5\cos\theta$

18. $\theta = \pi/3$

19. $r^2\cos 2\theta = 1$

20. $r^2\sin 2\theta = 1$

21–26 Find a polar equation for the curve represented by the given Cartesian equation.

21. $x^2 + y^2 = 7$

22. $x = -1$

23. $y = \sqrt{3}\,x$

24. $y = -2x^2$

25. $x^2 + y^2 = 4y$

26. $x^2 - y^2 = 4$

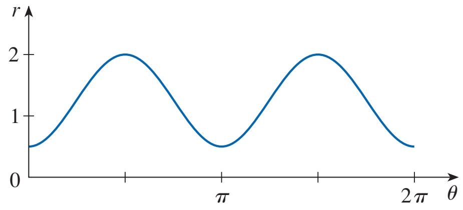
27–28 For each of the described curves, decide if the curve would be more easily given by a polar equation or a Cartesian equation. Then write an equation for the curve.

27. (a) A line through the origin that makes an angle of $\pi/6$ with the positive x-axis
(b) A vertical line through the point $(3, 3)$

28. (a) A circle with radius 5 and center $(2, 3)$
(b) A circle centered at the origin with radius 4

29–32 The figure shows a graph of r as a function of θ in Cartesian coordinates. Use it to sketch the corresponding polar curve.

29.



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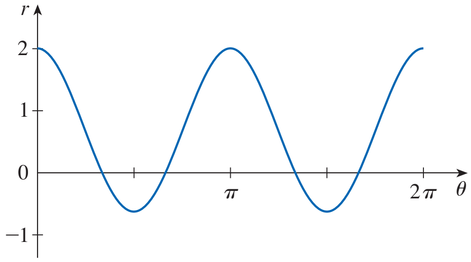

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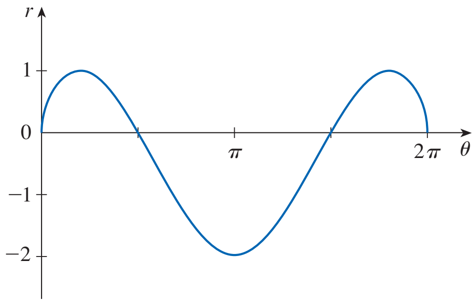
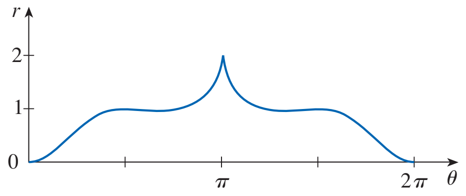


32.



33–50 Sketch the curve with the given polar equation by first sketching the graph of r as a function of θ in Cartesian coordinates.

33. $r = -2 \sin\theta$

34. $r = 1 - \cos\theta$

35. $r = 2(1 + \cos\theta)$

36. $r = 1 + 2\cos\theta$

37. $r = \theta, \ \theta \geqslant 0$

38. $r = \theta^2, \ -2\pi \leqslant \theta \leqslant 2\pi$

39. $r = 3\cos 3\theta$

40. $r = -\sin 5\theta$

41. $r = 2\cos 4\theta$

42. $r = 2\sin 6\theta$

43. $r = 1 + 3\cos\theta$

44. $r = 1 + 5\sin\theta$

45. $r^2 = 9\sin 2\theta$

46. $r^2 = \cos 4\theta$

47. $r = 2 + \sin 3\theta$

48. $r^2\theta = 1$

49. $r = \sin(\theta/2)$

50. $r = \cos(\theta/3)$

---

51. Show that the polar curve $r = 4 + 2\sec\theta$ (called a **conchoid**) has the line $x = 2$ as a vertical asymptote by showing that $\lim_{r \to \pm\infty} x = 2$. Use this fact to help sketch the conchoid.

52. Show that the curve $r = 2 - \csc\theta$ (a conchoid) has the line $y = -1$ as a horizontal asymptote by showing that $\lim_{r \to \pm\infty} y = -1$. Use this fact to help sketch the conchoid.

53. Show that the curve $r = \sin\theta \tan\theta$ (called a **cissoid of Diocles**) has the line $x = 1$ as a vertical asymptote. Show also that the curve lies entirely within the vertical strip $0 \leqslant x < 1$. Use these facts to help sketch the cissoid.

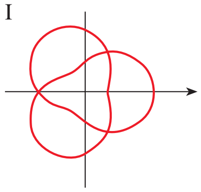
54. Sketch the curve $(x^2 + y^2)^3 = 4x^2y^2$.

55. (a) In Example 10 the graphs suggest that the limaçon $r = 1 + c\sin\theta$ has an inner loop when $|c| > 1$. Prove that this is true, and find the values of θ that correspond to the inner loop.
(b) From Figure 18 it appears that the limaçon loses its dimple when $c = \frac{1}{2}$. Prove this.

56. Match the polar equations with the graphs labeled I–IX. Give reasons for your choices.

(a) $r = \cos 3\theta$
(b) $r = \ln\theta, \quad 1 \leqslant \theta \leqslant 6\pi$
(c) $r = \cos(\theta/2)$
(d) $r = \cos(\theta/3)$
(e) $r = \sec(\theta/3)$
(f) $r = \sec\theta$
(g) $r = \theta^2, \quad 0 \leqslant \theta \leqslant 8\pi$
(h) $r = 2 + \cos 3\theta$
(i) $r = 2 + \cos(3\theta/2)$

I



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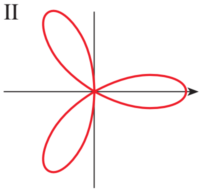
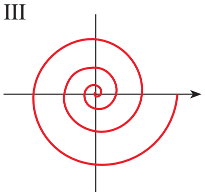
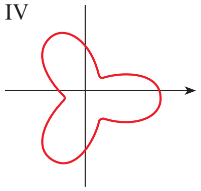


III



IV



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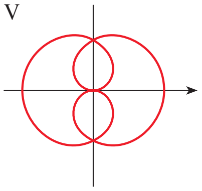

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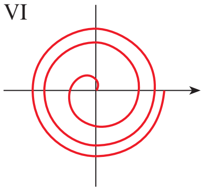

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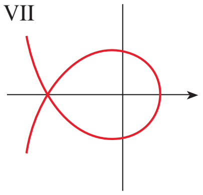

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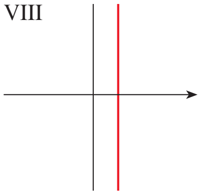

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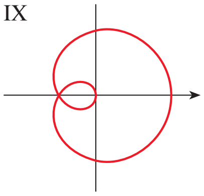


57. Show that the polar equation $r = a\sin\theta + b\cos\theta$, where $ab \neq 0$, represents a circle. Find its center and radius.

58. Show that the curves $r = a\sin\theta$ and $r = a\cos\theta$ intersect at right angles.

59–64 Graph the polar curve. Choose a parameter interval that produces the entire curve.

59. $r = 1 + 2\sin(\theta/2)$ (nephroid of Freeth)

60. $r = \sqrt{1 - 0.8\sin^2\theta}$ (hippopede)

61. $r = e^{\sin\theta} - 2\cos(4\theta)$ (butterfly curve)

62. $r = |\tan\theta|^{|\cot\theta|}$ (valentine curve)

63. $r = 1 + \cos^{999}\theta$ (Pac-Man curve)

64. $r = 2 + \cos(9\theta/4)$

---

65. How are the graphs of $r = 1 + \sin(\theta - \pi/6)$ and $r = 1 + \sin(\theta - \pi/3)$ related to the graph of $r = 1 + \sin\theta$? In general, how is the graph of $r = f(\theta - \alpha)$ related to the graph of $r = f(\theta)$?

66. Use a graph to estimate the y-coordinate of the highest points on the curve $r = \sin 2\theta$. Then use calculus to find the exact value.

67. Investigate the family of curves with polar equations $r = 1 + c\cos\theta$, where c is a real number. How does the shape change as c changes?

68. Investigate the family of polar curves $r = 1 + \cos^n\theta$, where n is a positive integer. How does the shape change as n increases? What happens as n becomes large? Explain the shape for large n by considering the graph of r as a function of θ in Cartesian coordinates.

DISCOVERY PROJECT | FAMILIES OF POLAR CURVES

In this project you will discover the interesting and beautiful shapes that members of families of polar curves can take. You will also see how the shape of the curve changes when you vary the constants.

1. (a) Investigate the family of curves defined by the polar equations $r = \sin n\theta$, where n is a positive integer. How is the number of loops related to n?

(b) What happens if the equation in part (a) is replaced by $r = |\sin n\theta|$?

2. A family of curves is given by the equations $r = 1 + c\sin n\theta$, where c is a real number and n is a positive integer. How does the graph change as n increases? How does it change as c changes? Illustrate by graphing enough members of the family to support your conclusions.

3. A family of curves has polar equations

$$r = \frac{1 - a\cos\theta}{1 + a\cos\theta}$$

Investigate how the graph changes as the number a changes. In particular, you should identify the transitional values of a for which the basic shape of the curve changes.

4. The astronomer Giovanni Cassini (1625–1712) studied the family of curves with polar equations

$$r^4 - 2c^2r^2\cos 2\theta + c^4 - a^4 = 0$$

where a and c are positive real numbers. These curves are called the **ovals of Cassini** even though they are oval shaped only for certain values of a and c. (Cassini thought that these curves might represent planetary orbits better than Kepler's ellipses.) Investigate the variety of shapes that these curves may have. In particular, how are a and c related to each other when the curve splits into two parts?

10.4 | Calculus in Polar Coordinates

In this section we apply the methods of calculus to find areas, arc lengths, and tangents involving polar curves.

Area

To develop the formula for the area of a region whose boundary is given by a polar equation, we need to use the formula for the area of a sector of a circle:

$$\boxed{1} \qquad A = \tfrac{1}{2}r^2\theta$$

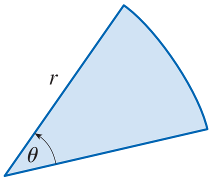


FIGURE 1

where, as in Figure 1, r is the radius and θ is the radian measure of the central angle. Formula 1 follows from the fact that the area of a sector is proportional to its central angle: $A = (\theta/2\pi)\pi r^2 = \frac{1}{2}r^2\theta$. (See also Exercise 7.3.41.)

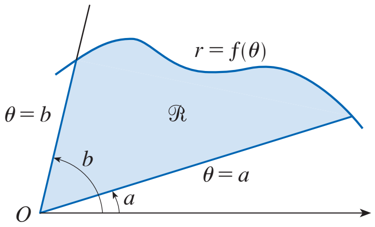


FIGURE 2

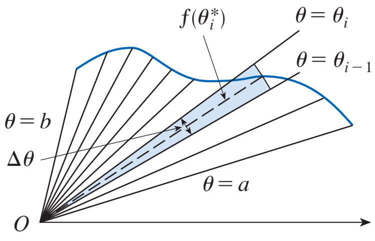


FIGURE 3

Let $\mathscr{R}$ be the region, illustrated in Figure 2, bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$, where f is a positive continuous function and where $0 < b - a \leqslant 2\pi$. We divide the interval $[a, b]$ into subintervals with endpoints $\theta_0, \theta_1, \theta_2, \ldots, \theta_n$ and equal width $\Delta\theta$. The rays $\theta = \theta_i$ then divide $\mathscr{R}$ into n smaller regions with central angle $\Delta\theta = \theta_i - \theta_{i-1}$. If we choose θ_i^* in the ith subinterval $[\theta_{i-1}, \theta_i]$, then the area ΔA_i of the ith region is approximated by the area of the sector of a circle with central angle $\Delta\theta$ and radius $f(\theta_i^*)$. (See Figure 3.)

Thus from Formula 1 we have

$$\Delta A_i \approx \tfrac{1}{2}[f(\theta_i^*)]^2\,\Delta\theta$$

and so an approximation to the total area A of $\mathscr{R}$ is

$$\boxed{A \approx \sum_{i=1}^{n} \tfrac{1}{2}[f(\theta_i^*)]^2\,\Delta\theta} \tag{2}$$

It appears from Figure 3 that the approximation in (2) improves as $n \to \infty$. But the sums in (2) are Riemann sums for the function $g(\theta) = \frac{1}{2}[f(\theta)]^2$, so

$$\lim_{n\to\infty} \sum_{i=1}^{n} \tfrac{1}{2}[f(\theta_i^*)]^2\,\Delta\theta = \int_a^b \tfrac{1}{2}[f(\theta)]^2\,d\theta$$

It therefore appears plausible (and can in fact be proved) that the formula for the area A of the polar region $\mathscr{R}$ is

$$\boxed{A = \int_a^b \tfrac{1}{2}[f(\theta)]^2\,d\theta} \tag{3}$$

Formula 3 is often written as

$$\boxed{A = \int_a^b \tfrac{1}{2}r^2\,d\theta} \tag{4}$$

with the understanding that $r = f(\theta)$. Note the similarity between Formulas 1 and 4.

When we apply Formula 3 or 4, it is helpful to think of the area as being swept out by a rotating ray through O that starts with angle a and ends with angle b.

EXAMPLE 1 Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

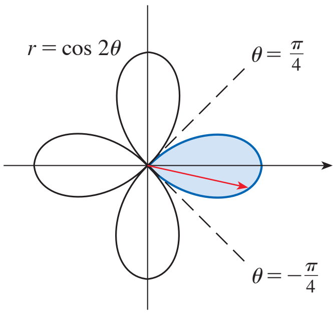


FIGURE 4

SOLUTION The curve $r = \cos 2\theta$ was sketched in Example 10.3.8. Notice from Figure 4 that the region enclosed by the right loop is swept out by a ray that rotates from $\theta = -\pi/4$ to $\theta = \pi/4$. Therefore Formula 4 gives

$$A = \int_{-\pi/4}^{\pi/4} \tfrac{1}{2}r^2\,d\theta = \tfrac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\,d\theta$$

Because the region is symmetric about the polar axis $\theta = 0$, we can write

$$\begin{aligned} A &= 2 \cdot \tfrac{1}{2}\int_0^{\pi/4} \cos^2 2\theta\,d\theta \\ &= \int_0^{\pi/4} \tfrac{1}{2}(1 + \cos 4\theta)\,d\theta \qquad \left[\text{because } \cos^2 u = \tfrac{1}{2}(1 + \cos 2u)\right] \\ &= \tfrac{1}{2}\Big[\theta + \tfrac{1}{4}\sin 4\theta\Big]_0^{\pi/4} = \frac{\pi}{8} \end{aligned}$$

■

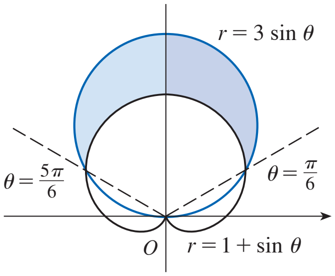


FIGURE 5

EXAMPLE 2 Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

SOLUTION The cardioid (see Example 10.3.7) and the circle are sketched in Figure 5 and the desired region is shaded. The values of a and b in Formula 4 are determined by finding the points of intersection of the two curves. They intersect when $3\sin\theta = 1 + \sin\theta$. This gives $\sin\theta = \frac{1}{2}$, so $\theta = \pi/6, 5\pi/6$. The desired area can be found by subtracting the area inside the cardioid between $\theta = \pi/6$ and $\theta = 5\pi/6$ from the area inside the circle from $\pi/6$ to $5\pi/6$. Thus

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta)^2\,d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1 + \sin\theta)^2\,d\theta$$

Since the region is symmetric about the vertical axis $\theta = \pi/2$, we can write

$$\begin{aligned} A &= 2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2} 9\sin^2\theta\,d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/2} (1 + 2\sin\theta + \sin^2\theta)\,d\theta\right] \\ &= \int_{\pi/6}^{\pi/2} (8\sin^2\theta - 1 - 2\sin\theta)\,d\theta \\ &= \int_{\pi/6}^{\pi/2} (3 - 4\cos 2\theta - 2\sin\theta)\,d\theta \qquad \left[\text{because } \sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)\right] \\ &= 3\theta - 2\sin 2\theta + 2\cos\theta\Big]_{\pi/6}^{\pi/2} = \pi \end{aligned}$$

■

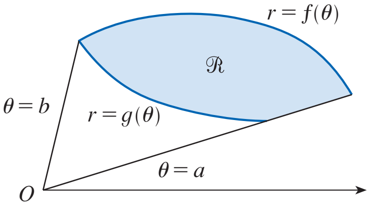


FIGURE 6

Example 2 illustrates the procedure for finding the area of the region bounded by two polar curves. In general, let $\mathcal{R}$ be a region, as illustrated in Figure 6, that is bounded by curves with polar equations $r = f(\theta)$, $r = g(\theta)$, $\theta = a$, and $\theta = b$, where $f(\theta) \geqslant g(\theta) \geqslant 0$ and $0 < b - a \leqslant 2\pi$. The area A of $\mathcal{R}$ is found by subtracting the area inside $r = g(\theta)$ from the area inside $r = f(\theta)$, so using Formula 3 we have

$$\begin{aligned} A &= \int_a^b \tfrac{1}{2}[f(\theta)]^2\,d\theta - \int_a^b \tfrac{1}{2}[g(\theta)]^2\,d\theta \\ &= \tfrac{1}{2}\int_a^b \left([f(\theta)]^2 - [g(\theta)]^2\right)d\theta \end{aligned}$$

CAUTION The fact that a single point has many representations in polar coordinates sometimes makes it difficult to find all the points of intersection of two polar curves. For instance, it is obvious from Figure 5 that the circle and the cardioid have three points of intersection; however, in Example 2 we solved the equations $r = 3\sin\theta$ and $r = 1 + \sin\theta$ and found only two such points, $\left(\frac{3}{2}, \pi/6\right)$ and $\left(\frac{3}{2}, 5\pi/6\right)$. The origin is also a point of intersection, but we can't find it by solving the equations of the curves because the origin has no single representation in polar coordinates that satisfies both equations. Notice that, when represented as $(0, 0)$ or $(0, \pi)$, the origin satisfies $r = 3\sin\theta$ and so it lies on the circle; when represented as $(0, 3\pi/2)$, it satisfies $r = 1 + \sin\theta$ and so it lies on the cardioid. Think of two points moving along the curves as the parameter value θ increases from 0 to 2π. On one curve the origin is reached at $\theta = 0$ and $\theta = \pi$; on the other curve it is reached at $\theta = 3\pi/2$. The points don't collide at the origin because they reach the origin at different times, but the curves intersect there nonetheless. (See also Exercises 10.1.55–57.)

Thus, to find *all* points of intersection of two polar curves, it is recommended that you draw the graphs of both curves. It is especially convenient to use a graphing calculator or computer to help with this task.

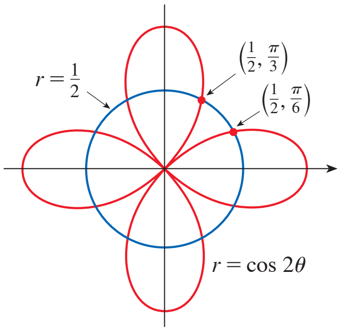

$r = \frac{1}{2}$
$\left(\frac{1}{2}, \frac{\pi}{3}\right)$
$\left(\frac{1}{2}, \frac{\pi}{6}\right)$
$r = \cos 2\theta$

FIGURE 7

EXAMPLE 3 Find all points of intersection of the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.

SOLUTION If we solve the equations $r = \cos 2\theta$ and $r = \frac{1}{2}$ simultaneously, we get $\cos 2\theta = \frac{1}{2}$ and, therefore, $2\theta = \pi/3, 5\pi/3, 7\pi/3, 11\pi/3$. Thus the values of θ between 0 and 2π that satisfy both equations are $\theta = \pi/6, 5\pi/6, 7\pi/6, 11\pi/6$. We have found four points of intersection: $\left(\frac{1}{2}, \pi/6\right)$, $\left(\frac{1}{2}, 5\pi/6\right)$, $\left(\frac{1}{2}, 7\pi/6\right)$, and $\left(\frac{1}{2}, 11\pi/6\right)$.

However, you can see from Figure 7 that the curves have four other points of intersection—namely, $\left(\frac{1}{2}, \pi/3\right)$, $\left(\frac{1}{2}, 2\pi/3\right)$, $\left(\frac{1}{2}, 4\pi/3\right)$, and $\left(\frac{1}{2}, 5\pi/3\right)$. These can be found using symmetry or by noticing that another equation of the circle is $r = -\frac{1}{2}$ and then solving the equations $r = \cos 2\theta$ and $r = -\frac{1}{2}$ simultaneously. ■

Arc Length

Recall from Section 10.3 that rectangular coordinates (x, y) and polar coordinates (r, θ) are related by the equations $x = r\cos\theta$, $y = r\sin\theta$. Regarding θ as a parameter allows us to write parametric equations for a polar curve $r = f(\theta)$ as follows.

Parametric equations for a polar curve

5 $$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

To find the length of a polar curve $r = f(\theta)$, $a \leqslant \theta \leqslant b$, we start with Equations 5 and differentiate with respect to θ (using the Product Rule):

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

Then, using $\cos^2\theta + \sin^2\theta = 1$, we have

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\ &\qquad + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta \\ &= \left(\frac{dr}{d\theta}\right)^2 + r^2\end{aligned}$$

Assuming that f' is continuous, we can use Theorem 10.2.5 to write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

Therefore the length of a curve with polar equation $r = f(\theta)$, $a \leqslant \theta \leqslant b$, is

6 $$\boxed{L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta}$$

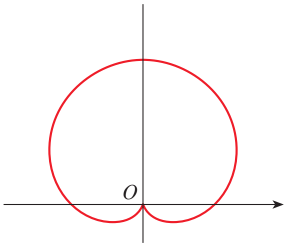


FIGURE 8
$r = 1 + \sin\theta$

EXAMPLE 4 Find the length of the cardioid $r = 1 + \sin\theta$.

SOLUTION The cardioid is shown in Figure 8. (We sketched it in Example 10.3.7.) Its full length is given by the parameter interval $0 \leqslant \theta \leqslant 2\pi$, so Formula 6 gives

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(1 + \sin\theta)^2 + \cos^2\theta}\, d\theta = \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta$$

We could evaluate this integral by multiplying and dividing the integrand by $\sqrt{2 - 2\sin\theta}$, or we could use mathematical software. In any event, we find that the length of the cardioid is $L = 8$. ■

Tangents

To find a tangent line to a polar curve $r = f(\theta)$, we again regard θ as a parameter and write parametric equations for the curve following Equations 5:

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Then, using the method for finding the slope of a parametric curve (Equation 10.2.1) and the Product Rule, we have

$$\boxed{7} \qquad \frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

We locate horizontal tangents by finding the points where $dy/d\theta = 0$ (provided that $dx/d\theta \neq 0$). Likewise, we locate vertical tangents at the points where $dx/d\theta = 0$ (provided that $dy/d\theta \neq 0$).

Notice that if we are looking for tangent lines at the pole, then $r = 0$ and Equation 7 simplifies to

$$\frac{dy}{dx} = \tan\theta \qquad \text{if } \frac{dr}{d\theta} \neq 0$$

For instance, in Example 10.3.8 we found that $r = \cos 2\theta = 0$ when $\theta = \pi/4$ or $3\pi/4$. This means that the lines $\theta = \pi/4$ and $\theta = 3\pi/4$ (or $y = x$ and $y = -x$) are tangent lines to $r = \cos 2\theta$ at the origin.

EXAMPLE 5

(a) For the cardioid $r = 1 + \sin\theta$ of Example 4, find the slope of the tangent line when $\theta = \pi/3$.
(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

SOLUTION Using Equation 7 with $r = 1 + \sin\theta$, we have

$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + (1 + \sin\theta)\cos\theta}{\cos\theta\cos\theta - (1 + \sin\theta)\sin\theta}$$

$$= \frac{\cos\theta\,(1 + 2\sin\theta)}{1 - 2\sin^2\theta - \sin\theta} = \frac{\cos\theta\,(1 + 2\sin\theta)}{(1 + \sin\theta)(1 - 2\sin\theta)}$$

(a) The slope of the tangent at the point where $\theta = \pi/3$ is

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/3} = \frac{\cos(\pi/3)[1 + 2\sin(\pi/3)]}{[1 + \sin(\pi/3)][1 - 2\sin(\pi/3)]} = \frac{\frac{1}{2}(1 + \sqrt{3})}{(1 + \sqrt{3}/2)(1 - \sqrt{3})}$$

$$= \frac{1 + \sqrt{3}}{(2 + \sqrt{3})(1 - \sqrt{3})} = \frac{1 + \sqrt{3}}{-1 - \sqrt{3}} = -1$$

(b) Observe that

$$\frac{dy}{d\theta} = \cos\theta\,(1 + 2\sin\theta) = 0 \qquad \text{when } \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\frac{dx}{d\theta} = (1 + \sin\theta)(1 - 2\sin\theta) = 0 \qquad \text{when } \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

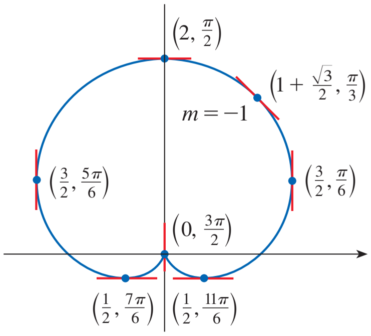
Therefore there are horizontal tangents at the points $(2, \pi/2)$, $(\frac{1}{2}, 7\pi/6)$, $(\frac{1}{2}, 11\pi/6)$ and vertical tangents at $(\frac{3}{2}, \pi/6)$ and $(\frac{3}{2}, 5\pi/6)$. When $\theta = 3\pi/2$, both $dy/d\theta$ and $dx/d\theta$ are 0, so we must be careful. Using l'Hospital's Rule, we have

$$\lim_{\theta\to(3\pi/2)^-} \frac{dy}{dx} = \left(\lim_{\theta\to(3\pi/2)^-} \frac{1 + 2\sin\theta}{1 - 2\sin\theta}\right)\left(\lim_{\theta\to(3\pi/2)^-} \frac{\cos\theta}{1 + \sin\theta}\right)$$

$$= -\frac{1}{3} \lim_{\theta\to(3\pi/2)^-} \frac{\cos\theta}{1 + \sin\theta} = -\frac{1}{3} \lim_{\theta\to(3\pi/2)^-} \frac{-\sin\theta}{\cos\theta} = \infty$$

By symmetry,
$$\lim_{\theta\to(3\pi/2)^+} \frac{dy}{dx} = -\infty$$

Thus there is a vertical tangent line at the pole (see Figure 9). ■



$(2, \frac{\pi}{2})$
$(1 + \frac{\sqrt{3}}{2}, \frac{\pi}{3})$
$m = -1$
$(\frac{3}{2}, \frac{5\pi}{6})$
$(\frac{3}{2}, \frac{\pi}{6})$
$(0, \frac{3\pi}{2})$
$(\frac{1}{2}, \frac{7\pi}{6})$
$(\frac{1}{2}, \frac{11\pi}{6})$

FIGURE 9
Tangent lines for $r = 1 + \sin\theta$

NOTE Instead of having to remember Equation 7, we could employ the method used to derive it. For instance, in Example 5 we could have written parametric equations for the curve as

$$x = r\cos\theta = (1 + \sin\theta)\cos\theta = \cos\theta + \tfrac{1}{2}\sin 2\theta$$

$$y = r\sin\theta = (1 + \sin\theta)\sin\theta = \sin\theta + \sin^2\theta$$

Then we have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta + 2\sin\theta\cos\theta}{-\sin\theta + \cos 2\theta} = \frac{\cos\theta + \sin 2\theta}{-\sin\theta + \cos 2\theta}$$

which is equivalent to our previous expression.

10.4 Exercises

1–4 Find the area of the region that is bounded by the given curve and lies in the specified sector.

1. $r = \sqrt{2\theta}, \quad 0 \leqslant \theta \leqslant \pi/2$

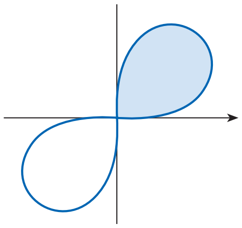
2. $r = e^{\theta}, \quad 3\pi/4 \leqslant \theta \leqslant 3\pi/2$

3. $r = \sin\theta + \cos\theta, \quad 0 \leqslant \theta \leqslant \pi$

4. $r = 1/\theta, \quad \pi/2 \leqslant \theta \leqslant 2\pi$

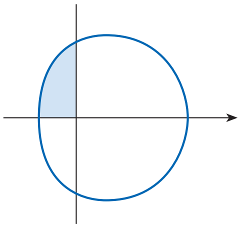
5–8 Find the area of the shaded region.

5.



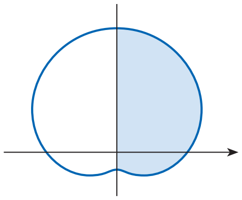
$r^2 = \sin 2\theta$

6.



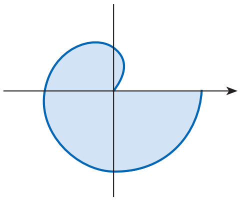
$r = 2 + \cos\theta$

7.



$r = 4 + 3\sin\theta$

8.



$r = \sqrt{\ln\theta},\ 1 \leqslant \theta \leqslant 2\pi$

9–12 Sketch the curve and find the area that it encloses.

9. $r = 4\cos\theta$

10. $r = 2 + 2\cos\theta$

11. $r = 3 - 2\sin\theta$

12. $r = 2\sin 3\theta$

13–16 Graph the curve and find the area that it encloses.

13. $r = 2 + \sin 4\theta$

14. $r = 3 - 2\cos 4\theta$

15. $r = \sqrt{1 + \cos^2(5\theta)}$

16. $r = 1 + 5\sin 6\theta$

17–21 Find the area of the region enclosed by one loop of the curve.

17. $r = 4\cos 3\theta$

18. $r^2 = 4\cos 2\theta$

19. $r = \sin 4\theta$

20. $r = 2\sin 5\theta$

21. $r = 1 + 2\sin\theta$ (inner loop)

---

22. Find the area enclosed by the loop of the **strophoid** $r = 2\cos\theta - \sec\theta$.

23–28 Find the area of the region that lies inside the first curve and outside the second curve.

23. $r = 4\sin\theta,\quad r = 2$

24. $r = 1 - \sin\theta,\quad r = 1$

25. $r^2 = 8\cos 2\theta,\quad r = 2$

26. $r = 1 + \cos\theta,\quad r = 2 - \cos\theta$

27. $r = 3\cos\theta,\quad r = 1 + \cos\theta$

28. $r = 3\sin\theta,\quad r = 2 - \sin\theta$

29–34 Find the area of the region that lies inside both curves.

29. $r = 3\sin\theta,\quad r = 3\cos\theta$

30. $r = 1 + \cos\theta,\quad r = 1 - \cos\theta$

31. $r = \sin 2\theta,\quad r = \cos 2\theta$

32. $r = 3 + 2\cos\theta,\quad r = 3 + 2\sin\theta$

33. $r^2 = 2\sin 2\theta,\quad r = 1$

34. $r = a\sin\theta,\quad r = b\cos\theta,\quad a > 0,\ b > 0$

---

35. Find the area inside the larger loop and outside the smaller loop of the limaçon $r = \frac{1}{2} + \cos\theta$.

36. Find the area between a large loop and the enclosed small loop of the curve $r = 1 + 2\cos 3\theta$.

37–42 Find all points of intersection of the given curves.

37. $r = \sin\theta,\quad r = 1 - \sin\theta$

38. $r = 1 + \cos\theta,\quad r = 1 - \sin\theta$

39. $r = 2\sin 2\theta,\quad r = 1$

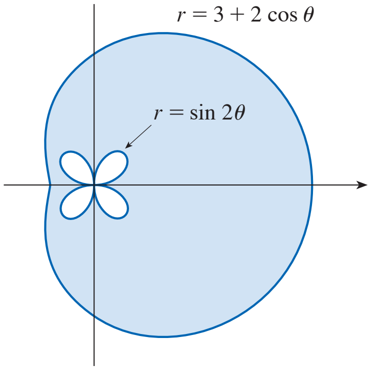
40. $r = \cos\theta,\quad r = \sin 2\theta$

41. $r^2 = 2\cos 2\theta,\quad r = 1$

42. $r^2 = \sin 2\theta,\quad r^2 = \cos 2\theta$

43–46 Find the area of the shaded region.

43.



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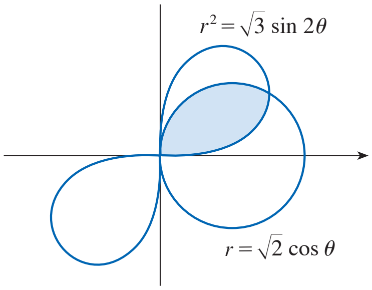

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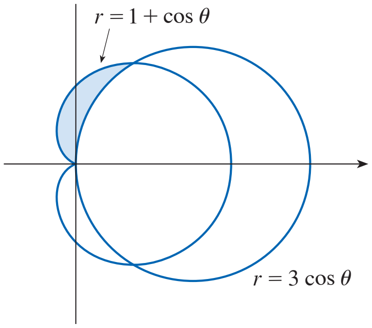

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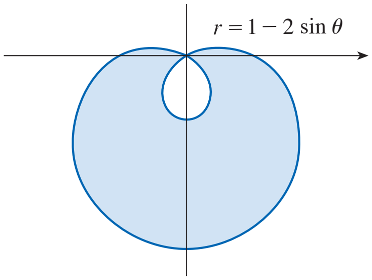
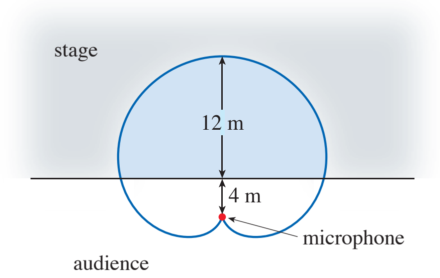


47. The points of intersection of the cardioid $r = 1 + \sin\theta$ and the spiral loop $r = 2\theta$, $-\pi/2 \leqslant \theta \leqslant \pi/2$, can't be found exactly. Use a graph to find the approximate values of θ at which the curves intersect. Then use these values to estimate the area that lies inside both curves.

48. When recording live performances, sound engineers often use a microphone with a cardioid pickup pattern because it suppresses noise from the audience. Suppose the microphone is placed 4 m from the front of the stage (as in the figure) and the boundary of the optimal pickup region is given by the cardioid $r = 8 + 8\sin\theta$, where r is measured in meters and the microphone is at the pole. The musicians want to know the area they will have on stage within the optimal



49–52 Find the exact length of the polar curve.

49. $r = 2\cos\theta$, $0 \leqslant \theta \leqslant \pi$

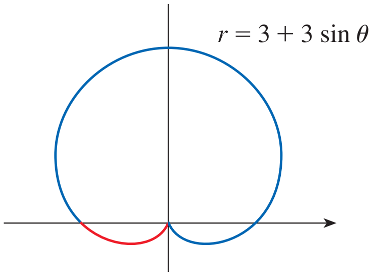
50. $r = e^{\theta/2}$, $0 \leqslant \theta \leqslant \pi/2$

51. $r = \theta^2$, $0 \leqslant \theta \leqslant 2\pi$

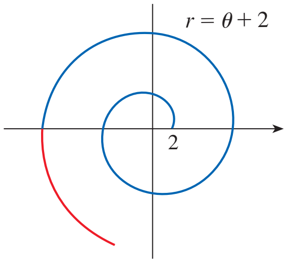
52. $r = 2(1 + \cos\theta)$

53–54 Find the exact length of the portion of the curve shown in blue.

53.



54.



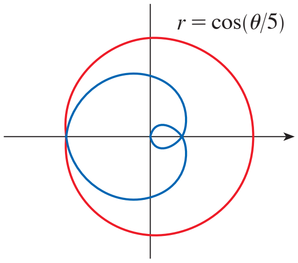
55–56 Find the exact length of the curve. Use a graph to determine the parameter interval.

55. $r = \cos^4(\theta/4)$

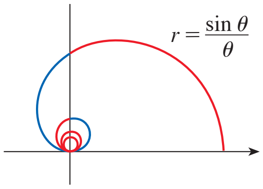
56. $r = \cos^2(\theta/2)$

57–58 Set up, but do not evaluate, an integral to find the length of the portion of the curve shown in blue.

57.



58.



59–62 Use a calculator or computer to find the length of the curve correct to four decimal places. If necessary, graph the curve to determine the parameter interval.

59. One loop of the curve $r = \cos 2\theta$

60. $r = \tan\theta$, $\pi/6 \leqslant \theta \leqslant \pi/3$

61. $r = \sin(6\sin\theta)$

62. $r = \sin(\theta/4)$

63–68 Find the slope of the tangent line to the given polar curve at the point specified by the value of θ.

63. $r = 2\cos\theta, \quad \theta = \pi/3$

64. $r = 2 + \sin 3\theta, \quad \theta = \pi/4$

65. $r = 1/\theta, \quad \theta = \pi$

66. $r = \sin\theta + 2\cos\theta, \quad \theta = \pi/2$

67. $r = \cos 2\theta, \quad \theta = \pi/4$

68. $r = 1 + 2\cos\theta, \quad \theta = \pi/3$

69–72 Find the points on the given curve where the tangent line is horizontal or vertical.

69. $r = \sin\theta$

70. $r = 1 - \sin\theta$

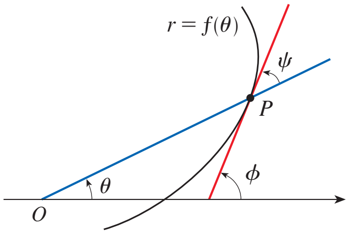
71. $r = 1 + \cos\theta$

72. $r = e^{\theta}$

73. Let P be any point (except the origin) on the curve $r = f(\theta)$. If ψ is the angle between the tangent line at P and the radial line OP, show that

$$\tan\psi = \frac{r}{dr/d\theta}$$

[*Hint:* Observe that $\psi = \phi - \theta$ in the figure.]



74. (a) Use Exercise 73 to show that the angle between the tangent line and the radial line is $\psi = \pi/4$ at every point on the curve $r = e^{\theta}$.

(b) Illustrate part (a) by graphing the curve and the tangent lines at the points where $\theta = 0$ and $\pi/2$.

(c) Prove that any polar curve $r = f(\theta)$ with the property that the angle ψ between the radial line and the tangent line is a constant must be of the form $r = Ce^{k\theta}$, where C and k are constants.

75. (a) Use Formula 10.2.9 to show that the area of the surface generated by rotating the polar curve

$$r = f(\theta) \qquad a \leqslant \theta \leqslant b$$

(where f' is continuous and $0 \leqslant a < b \leqslant \pi$) about the polar axis is

$$S = \int_a^b 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

(b) Use the formula in part (a) to find the surface area generated by rotating the lemniscate $r^2 = \cos 2\theta$ about the polar axis.

76. (a) Find a formula for the area of the surface generated by rotating the polar curve $r = f(\theta)$, $a \leqslant \theta \leqslant b$ (where f' is continuous and $0 \leqslant a < b \leqslant \pi$), about the line $\theta = \pi/2$.

(b) Find the surface area generated by rotating the lemniscate $r^2 = \cos 2\theta$ about the line $\theta = \pi/2$.

10.5 Conic Sections

In this section we give geometric definitions of parabolas, ellipses, and hyperbolas and derive their standard equations. They are called **conic sections**, or **conics**, because they result from intersecting a cone with a plane as shown in Figure 1.

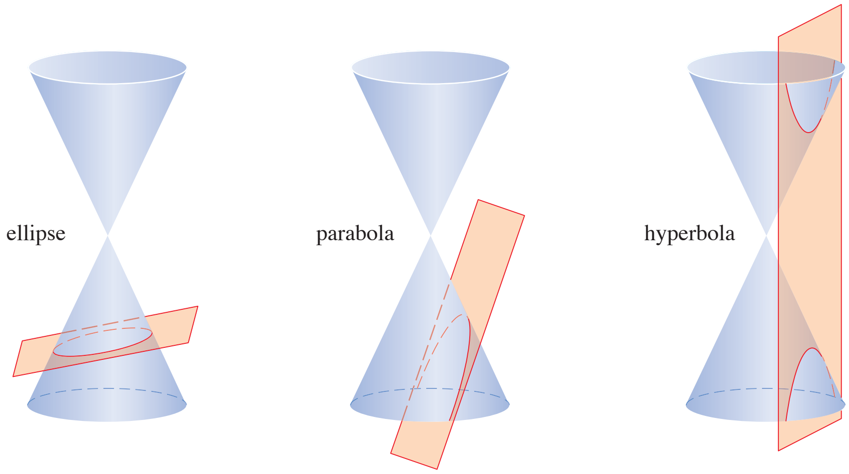


FIGURE 1
Conics

Parabolas

A **parabola** is the set of points in a plane that are equidistant from a fixed point F (called the **focus**) and a fixed line (called the **directrix**). This definition is illustrated by Figure 2. Notice that the point halfway between the focus and the directrix lies on the parabola; it is called the **vertex**. The line through the focus perpendicular to the directrix is called the **axis** of the parabola.

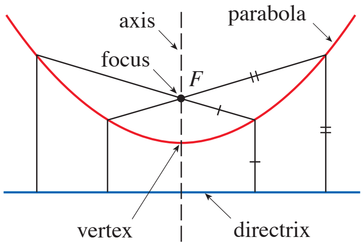


FIGURE 2

In the 16th century Galileo showed that the path of a projectile that is shot into the air at an angle to the ground is a parabola. Since then, parabolic shapes have been used in designing automobile headlights, reflecting telescopes, and suspension bridges. (See Problem 22 in Problems Plus following Chapter 3 for the reflection property of parabolas that makes them so useful.)

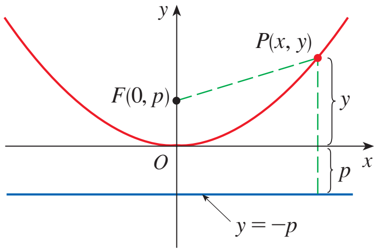


FIGURE 3

We obtain a particularly simple equation for a parabola if we place its vertex at the origin O and its directrix parallel to the x-axis as in Figure 3. If the focus is the point $(0, p)$, then the directrix has the equation $y = -p$. If $P(x, y)$ is any point on the parabola, then the distance from P to the focus is

$$|PF| = \sqrt{x^2 + (y - p)^2}$$

and the distance from P to the directrix is $|y + p|$. (Figure 3 illustrates the case where $p > 0$.) The defining property of a parabola is that these distances are equal:

$$\sqrt{x^2 + (y - p)^2} = |y + p|$$

We get an equivalent equation by squaring and simplifying:

$$x^2 + (y - p)^2 = |y + p|^2 = (y + p)^2$$

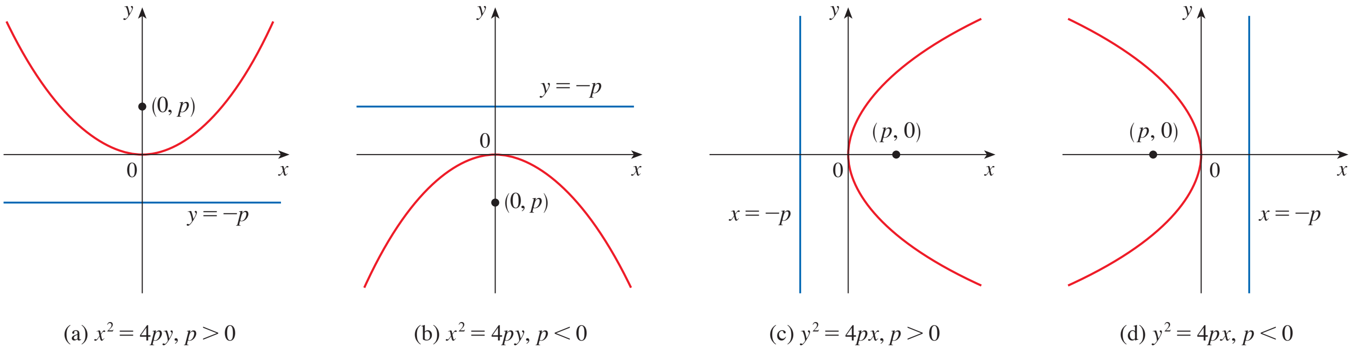
$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$

$$x^2 = 4py$$

1 An equation of the parabola with focus $(0, p)$ and directrix $y = -p$ is

$$x^2 = 4py$$

If we write $a = 1/(4p)$, then the standard equation of a parabola (1) becomes $y = ax^2$. It opens upward if $p > 0$ and downward if $p < 0$ [see Figure 4, parts (a) and (b)]. The graph is symmetric with respect to the y-axis because (1) is unchanged when x is replaced by $-x$.



(a) $x^2 = 4py,\ p > 0$ (b) $x^2 = 4py,\ p < 0$ (c) $y^2 = 4px,\ p > 0$ (d) $y^2 = 4px,\ p < 0$

FIGURE 4

If we interchange x and y in (1), we obtain the following.

2 An equation of the parabola with focus $(p, 0)$ and directrix $x = -p$ is

$$y^2 = 4px$$

(Interchanging x and y amounts to reflecting about the diagonal line $y = x$.) The parabola opens to the right if $p > 0$ and to the left if $p < 0$ [see Figure 4, parts (c) and (d)]. In both cases the graph is symmetric with respect to the x-axis, which is the axis of the parabola.

EXAMPLE 1 Find the focus and directrix of the parabola $y^2 + 10x = 0$ and sketch the graph.

SOLUTION If we write the equation as $y^2 = -10x$ and compare it with Equation 2, we see that $4p = -10$, so $p = -\frac{5}{2}$. Thus the focus is $(p, 0) = \left(-\frac{5}{2}, 0\right)$ and the directrix is $x = \frac{5}{2}$. The sketch is shown in Figure 5. ■

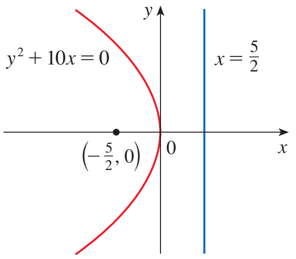


FIGURE 5

Ellipses

An **ellipse** is the set of points in a plane the sum of whose distances from two fixed points F_1 and F_2 is a constant (see Figure 6). These two fixed points are called the **foci** (plural of **focus**). One of Kepler's laws is that the orbits of the planets in the solar system are ellipses with the sun at one focus.

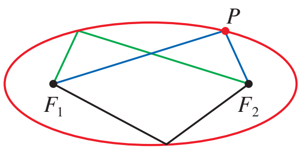


FIGURE 6

In order to obtain the simplest equation for an ellipse, we place the foci on the x-axis at the points $(-c, 0)$ and $(c, 0)$ as in Figure 7 so that the origin is halfway between the foci. Let the sum of the distances from a point on the ellipse to the foci be $2a > 0$. Then $P(x, y)$ is a point on the ellipse when

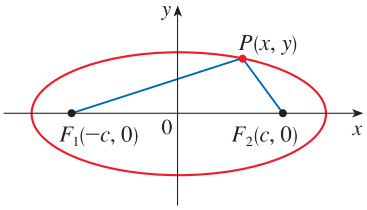


FIGURE 7
P is on the ellipse when $|PF_1| + |PF_2| = 2a$.

$$|PF_1| + |PF_2| = 2a$$

that is,
$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

or
$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

Squaring both sides, we have

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

which simplifies to
$$a\sqrt{(x+c)^2 + y^2} = a^2 + cx$$

We square again:

$$a^2(x^2 + 2cx + c^2 + y^2) = a^4 + 2a^2cx + c^2x^2$$

which becomes
$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

From triangle F_1F_2P in Figure 7 we can see that $2c < 2a$, so $c < a$ and therefore $a^2 - c^2 > 0$. For convenience, let $b^2 = a^2 - c^2$. Then the equation of the ellipse becomes $b^2x^2 + a^2y^2 = a^2b^2$ or, if both sides are divided by a^2b^2,

$$\boxed{3} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

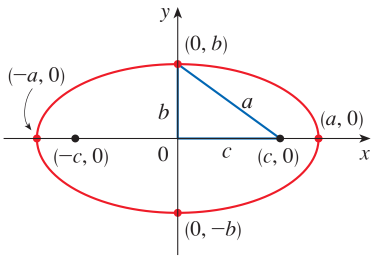


FIGURE 8

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a \geqslant b$

Since $b^2 = a^2 - c^2 < a^2$, it follows that $b < a$. The x-intercepts are found by setting $y = 0$. Then $x^2/a^2 = 1$, or $x^2 = a^2$, so $x = \pm a$. The corresponding points $(a, 0)$ and $(-a, 0)$ are called the **vertices** of the ellipse and the line segment joining the vertices is called the **major axis**. To find the y-intercepts we set $x = 0$ and obtain $y^2 = b^2$, so $y = \pm b$. The line segment joining $(0, b)$ and $(0, -b)$ is the **minor axis**. Equation 3 is unchanged if x is replaced by $-x$ or y is replaced by $-y$, so the ellipse is symmetric about both axes. Notice that if the foci coincide, then $c = 0$, so $a = b$ and the ellipse becomes a circle with radius $r = a = b$.

We summarize this discussion as follows (see also Figure 8).

4 The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a \geqslant b > 0$$

has foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$, and vertices $(\pm a, 0)$.

If the foci of an ellipse are located on the y-axis at $(0, \pm c)$, then we can find its equation by interchanging x and y in (4). (See Figure 9.)

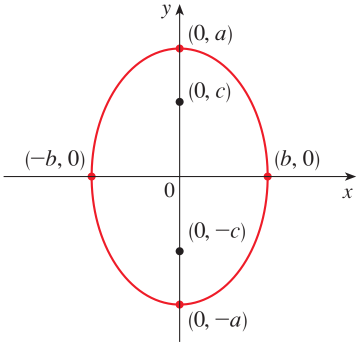


FIGURE 9

$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1,\ a \geqslant b$

5 The ellipse

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad a \geqslant b > 0$$

has foci $(0, \pm c)$, where $c^2 = a^2 - b^2$, and vertices $(0, \pm a)$.

EXAMPLE 2 Sketch the graph of $9x^2 + 16y^2 = 144$ and locate the foci.

SOLUTION Divide both sides of the equation by 144:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The equation is now in the standard form for an ellipse, so we have $a^2 = 16$, $b^2 = 9$, $a = 4$, and $b = 3$. The x-intercepts are ± 4 and the y-intercepts are ± 3. Also, $c^2 = a^2 - b^2 = 7$, so $c = \sqrt{7}$ and the foci are $\left(\pm\sqrt{7}, 0\right)$. The graph is sketched in Figure 10. ■

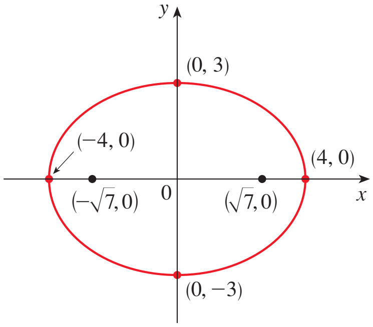


FIGURE 10

$9x^2 + 16y^2 = 144$

EXAMPLE 3 Find an equation of the ellipse with foci $(0, \pm 2)$ and vertices $(0, \pm 3)$.

SOLUTION Using the notation of (5), we have $c = 2$ and $a = 3$. Then we obtain $b^2 = a^2 - c^2 = 9 - 4 = 5$, so an equation of the ellipse is

$$\frac{x^2}{5} + \frac{y^2}{9} = 1$$

Another way of writing the equation is $9x^2 + 5y^2 = 45$. ■

Like parabolas, ellipses have an interesting reflection property that has practical consequences. If a source of light or sound is placed at one focus of a surface with elliptical cross-sections, then all the light or sound is reflected off the surface to the other focus (see Exercise 67). This principle is used in *lithotripsy,* a treatment for kidney stones. A reflector with elliptical cross-section is placed in such a way that the kidney stone is at one focus. High-intensity sound waves generated at the other focus are reflected to the stone and destroy it without damaging surrounding tissue. The patient is spared the trauma of surgery and recovers within a few days.

Hyperbolas

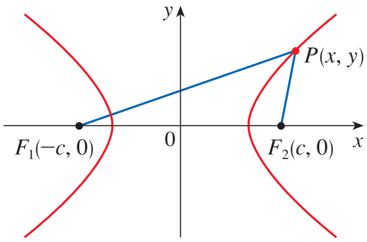


FIGURE 11 *P* is on the hyperbola when $|PF_1| - |PF_2| = \pm 2a$.

A **hyperbola** is the set of all points in a plane the difference of whose distances from two fixed points F_1 and F_2 (the **foci**) is a constant. This definition is illustrated in Figure 11.

Hyperbolas occur frequently as graphs of equations in chemistry, physics, biology, and economics (Boyle's Law, Ohm's Law, supply and demand curves). A particularly significant application of hyperbolas was found in the long-range navigation systems developed in World Wars I and II (see Exercise 53).

Notice that the definition of a hyperbola is similar to that of an ellipse; the only change is that the sum of distances has become a difference of distances. In fact, the derivation of the equation of a hyperbola is also similar to the one given earlier for an ellipse. It is left as Exercise 54 to show that when the foci are on the x-axis at $(\pm c, 0)$ and the difference of distances is $|PF_1| - |PF_2| = \pm 2a$, then the equation of the hyperbola is

$$\boxed{6} \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $c^2 = a^2 + b^2$. Notice that the x-intercepts are again $\pm a$ and the points $(a, 0)$ and $(-a, 0)$ are the **vertices** of the hyperbola. But if we put $x = 0$ in Equation 6 we get $y^2 = -b^2$, which is impossible, so there is no y-intercept. The hyperbola is symmetric with respect to both axes.

To analyze the hyperbola further, we look at Equation 6 and obtain

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geqslant 1$$

This shows that $x^2 \geqslant a^2$, so $|x| = \sqrt{x^2} \geqslant a$. Therefore we have $x \geqslant a$ or $x \leqslant -a$. This means that the hyperbola consists of two parts, called its *branches*.

When we draw a hyperbola it is useful to first draw its **asymptotes**, which are the dashed lines $y = (b/a)x$ and $y = -(b/a)x$ shown in Figure 12. Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes. (See Exercise 4.5.77, where these lines are shown to be slant asymptotes.)

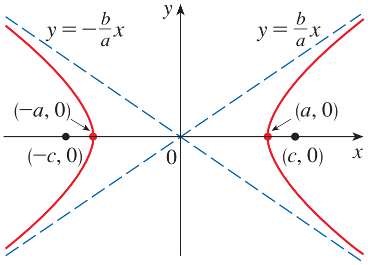


FIGURE 12
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\boxed{7}$ The hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has foci $(\pm c, 0)$, where $c^2 = a^2 + b^2$, vertices $(\pm a, 0)$, and asymptotes $y = \pm(b/a)x$.

If the foci of a hyperbola are on the y-axis, then by reversing the roles of x and y we obtain the following information, which is illustrated in Figure 13.

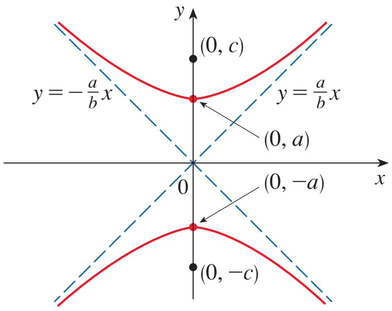


FIGURE 13

$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

8 The hyperbola

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

has foci $(0, \pm c)$, where $c^2 = a^2 + b^2$, vertices $(0, \pm a)$, and asymptotes $y = \pm(a/b)x$.

EXAMPLE 4 Find the foci and asymptotes of the hyperbola $9x^2 - 16y^2 = 144$ and sketch its graph.

SOLUTION If we divide both sides of the equation by 144, it becomes

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

which is of the form given in (7) with $a = 4$ and $b = 3$. Since $c^2 = 16 + 9 = 25$, the foci are $(\pm 5, 0)$. The asymptotes are the lines $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$. The graph is shown in Figure 14. ■

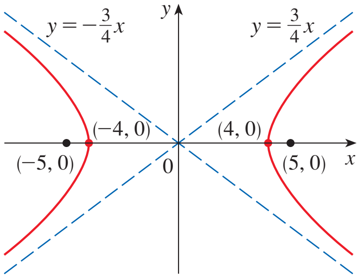


FIGURE 14

$9x^2 - 16y^2 = 144$

EXAMPLE 5 Find the foci and equation of the hyperbola with vertices $(0, \pm 1)$ and asymptote $y = 2x$.

SOLUTION From (8) and the given information, we see that $a = 1$ and $a/b = 2$. Thus $b = a/2 = \frac{1}{2}$ and $c^2 = a^2 + b^2 = \frac{5}{4}$. The foci are $\left(0, \pm\sqrt{5}/2\right)$ and the equation of the hyperbola is

$$y^2 - 4x^2 = 1$$ ■

■ Shifted Conics

As discussed in Appendix C, we shift conics by taking the standard equations (1), (2), (4), (5), (7), and (8) and replacing x and y by $x - h$ and $y - k$.

EXAMPLE 6 Find an equation of the ellipse with foci $(2, -2)$, $(4, -2)$ and vertices $(1, -2)$, $(5, -2)$.

SOLUTION The major axis is the line segment that joins the vertices $(1, -2)$, $(5, -2)$ and has length 4, so $a = 2$. The distance between the foci is 2, so $c = 1$. Thus $b^2 = a^2 - c^2 = 3$. Since the center of the ellipse is $(3, -2)$, we replace x and y in (4) by $x - 3$ and $y + 2$ to obtain

$$\frac{(x-3)^2}{4} + \frac{(y+2)^2}{3} = 1$$

as the equation of the ellipse. ■

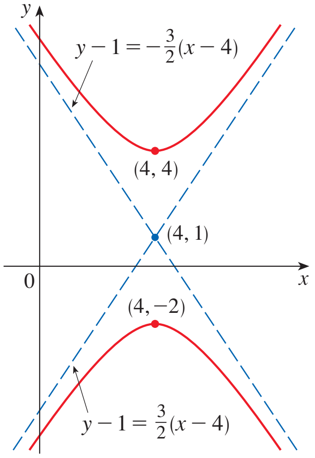


FIGURE 15
$9x^2 - 4y^2 - 72x + 8y + 176 = 0$

EXAMPLE 7 Sketch the conic $9x^2 - 4y^2 - 72x + 8y + 176 = 0$ and find its foci.

SOLUTION We complete the squares as follows:

$$4(y^2 - 2y) - 9(x^2 - 8x) = 176$$

$$4(y^2 - 2y + 1) - 9(x^2 - 8x + 16) = 176 + 4 - 144$$

$$4(y - 1)^2 - 9(x - 4)^2 = 36$$

$$\frac{(y - 1)^2}{9} - \frac{(x - 4)^2}{4} = 1$$

This is in the form (8) except that x and y are replaced by $x - 4$ and $y - 1$. Thus $a^2 = 9$, $b^2 = 4$, and $c^2 = 13$. The hyperbola is shifted four units to the right and one unit upward. The foci are $(4, 1 + \sqrt{13})$ and $(4, 1 - \sqrt{13})$ and the vertices are $(4, 4)$ and $(4, -2)$. The asymptotes are $y - 1 = \pm\frac{3}{2}(x - 4)$. The hyperbola is sketched in Figure 15. ■

10.5 Exercises

1–8 Find the vertex, focus, and directrix of the parabola and sketch its graph.

1. $x^2 = 8y$

2. $9x = y^2$

3. $5x + 3y^2 = 0$

4. $x^2 + 12y = 0$

5. $(y + 1)^2 = 16(x - 3)$

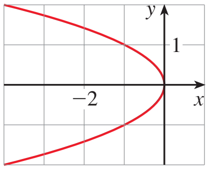
6. $(x - 3)^2 = 8(y + 1)$

7. $y^2 + 6y + 2x + 1 = 0$

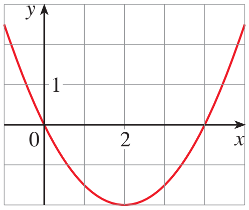
8. $2x^2 - 16x - 3y + 38 = 0$

9–10 Find an equation of the parabola. Then find the focus and directrix.

9.



10.



11–16 Find the vertices and foci of the ellipse and sketch its graph.

11. $\frac{x^2}{16} + \frac{y^2}{25} = 1$

12. $\frac{x^2}{4} + \frac{y^2}{3} = 1$

13. $x^2 + 3y^2 = 9$

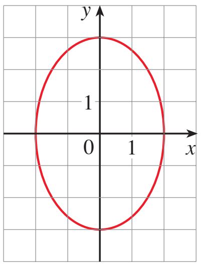
14. $x^2 = 4 - 2y^2$

15. $4x^2 + 25y^2 - 50y = 75$

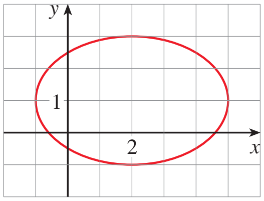
16. $9x^2 - 54x + y^2 + 2y + 46 = 0$

17–18 Find an equation of the ellipse. Then find its foci.

17.



18.



19–24 Find the vertices, foci, and asymptotes of the hyperbola and sketch its graph.

19. $\frac{y^2}{25} - \frac{x^2}{9} = 1$

20. $\frac{x^2}{36} - \frac{y^2}{64} = 1$

21. $x^2 - y^2 = 100$

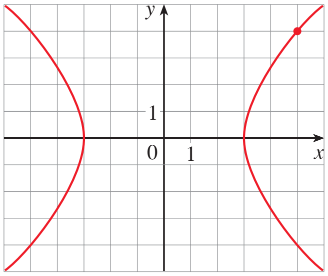
22. $y^2 - 16x^2 = 16$

23. $x^2 - y^2 + 2y = 2$

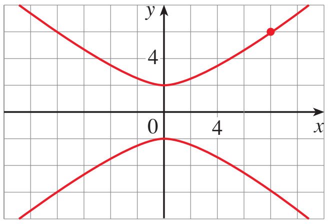
24. $9y^2 - 4x^2 - 36y - 8x = 4$

25–26 Find an equation for the hyperbola. Then find the foci and asymptotes.

25.



26.



27–32 Identify the type of conic section whose equation is given and find the vertices and foci.

27. $4x^2 = y^2 + 4$

28. $4x^2 = y + 4$

29. $x^2 = 4y - 2y^2$

30. $y^2 - 2 = x^2 - 2x$

31. $3x^2 - 6x - 2y = 1$

32. $x^2 - 2x + 2y^2 - 8y + 7 = 0$

33–50 Find an equation for the conic that satisfies the given conditions.

33. Parabola, vertex $(0, 0)$, focus $(1, 0)$

34. Parabola, focus $(0, 0)$, directrix $y = 6$

35. Parabola, focus $(-4, 0)$, directrix $x = 2$

36. Parabola, focus $(2, -1)$, vertex $(2, 3)$

37. Parabola, vertex $(3, -1)$, horizontal axis, passing through $(-15, 2)$

38. Parabola, vertical axis, passing through $(0, 4)$, $(1, 3)$, and $(-2, -6)$

39. Ellipse, foci $(\pm 2, 0)$, vertices $(\pm 5, 0)$

40. Ellipse, foci $\left(0, \pm\sqrt{2}\right)$, vertices $(0, \pm 2)$

41. Ellipse, foci $(0, 2)$, $(0, 6)$, vertices $(0, 0)$, $(0, 8)$

42. Ellipse, foci $(0, -1)$, $(8, -1)$, vertex $(9, -1)$

43. Ellipse, center $(-1, 4)$, vertex $(-1, 0)$, focus $(-1, 6)$

44. Ellipse, foci $(\pm 4, 0)$, passing through $(-4, 1.8)$

45. Hyperbola, vertices $(\pm 3, 0)$, foci $(\pm 5, 0)$

46. Hyperbola, vertices $(0, \pm 2)$, foci $(0, \pm 5)$

47. Hyperbola, vertices $(-3, -4)$, $(-3, 6)$, foci $(-3, -7)$, $(-3, 9)$

48. Hyperbola, vertices $(-1, 2)$, $(7, 2)$, foci $(-2, 2)$, $(8, 2)$

49. Hyperbola, vertices $(\pm 3, 0)$, asymptotes $y = \pm 2x$

50. Hyperbola, foci $(2, 0)$, $(2, 8)$, asymptotes $y = 3 + \frac{1}{2}x$ and $y = 5 - \frac{1}{2}x$

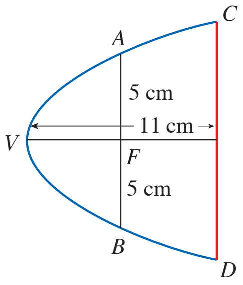
---

51. The point in a lunar orbit nearest the surface of the moon is called *perilune* and the point farthest from the surface is called *apolune*. The *Apollo 11* spacecraft was placed in an elliptical lunar orbit with perilune altitude 110 km and apolune altitude 314 km (above the moon). Find an equation of this ellipse if the radius of the moon is 1728 km and the center of the moon is at one focus.

52. A cross-section of a parabolic reflector is shown in the figure. The bulb is located at the focus and the opening at the focus is 10 cm.

(a) Find an equation of the parabola.

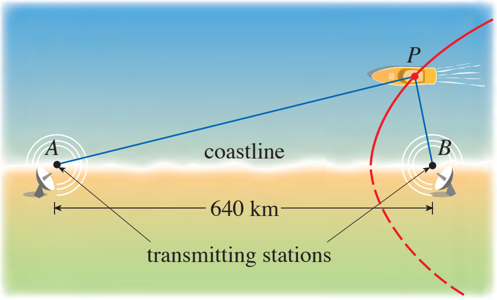
(b) Find the diameter of the opening $|CD|$, 11 cm from the vertex.



53. The LORAN (LOng RAnge Navigation) radio navigation system was widely used until the 1990s when it was superseded by the GPS system. In the LORAN system, two radio stations located at A and B transmit simultaneous signals to a ship or an aircraft located at P. The onboard computer converts the time difference in receiving these signals into a distance difference $|PA| - |PB|$, and this, according to the definition of a hyperbola, locates the ship or aircraft on one branch of a hyperbola (see the figure). Suppose that station B is located 640 km due east of station A on a coastline. A ship received the signal from station B 1200 microseconds (μs) before it received the signal from station A.

(a) Assuming that radio signals travel at a speed of 300 m/μs, find an equation of the hyperbola on which the ship lies.

(b) If the ship is due north of B, how far off the coastline is the ship?



54. Use the definition of a hyperbola to derive Equation 6 for a hyperbola with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$.

55. Show that the function defined by the upper branch of the hyperbola $y^2/a^2 - x^2/b^2 = 1$ is concave upward.

56. Find an equation for the ellipse with foci $(1, 1)$ and $(-1, -1)$ and major axis of length 4.

57. Determine the type of curve represented by the equation

$$\frac{x^2}{k} + \frac{y^2}{k - 16} = 1$$

in each of the following cases:

(a) $k > 16$ (b) $0 < k < 16$ (c) $k < 0$

(d) Show that all the curves in parts (a) and (b) have the same foci, no matter what the value of k is.

58. (a) Show that the equation of the tangent line to the parabola $y^2 = 4px$ at the point (x_0, y_0) can be written as

$$y_0 y = 2p(x + x_0)$$

(b) What is the x-intercept of this tangent line? Use this fact to draw the tangent line.

59. Show that the tangent lines to the parabola $x^2 = 4py$ drawn from any point on the directrix are perpendicular.

60. Show that if an ellipse and a hyperbola have the same foci, then their tangent lines at each point of intersection are perpendicular.

61. Use parametric equations and Simpson's Rule with $n = 8$ to estimate the circumference of the ellipse $9x^2 + 4y^2 = 36$.

62. The dwarf planet Pluto travels in an elliptical orbit around the sun (at one focus). The length of the major axis is 1.18×10^{10} km and the length of the minor axis is 1.14×10^{10} km. Use Simpson's Rule with $n = 10$ to estimate the distance traveled by the planet during one complete orbit around the sun.

63. Find the area of the region enclosed by the hyperbola $x^2/a^2 - y^2/b^2 = 1$ and the vertical line through a focus.

64. (a) If an ellipse is rotated about its major axis, find the volume of the resulting solid.

(b) If it is rotated about its minor axis, find the resulting volume.

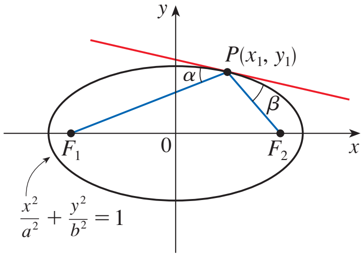
65. Find the centroid of the region enclosed by the x-axis and the top half of the ellipse $9x^2 + 4y^2 = 36$.

66. (a) Calculate the surface area of the ellipsoid that is generated by rotating an ellipse about its major axis.

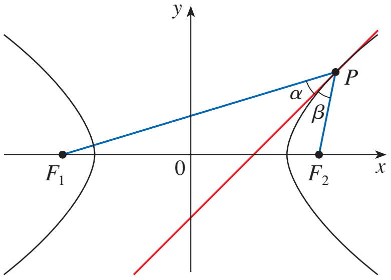
(b) What is the surface area if the ellipse is rotated about its minor axis?

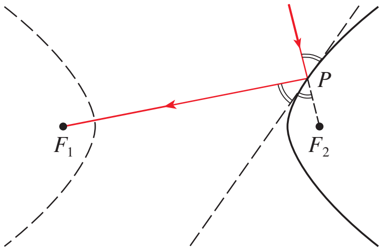
67–68 Reflection Properties of Conic Sections We saw the reflection property of parabolas in Problem 22 of Problems Plus following Chapter 3. Here we investigate the reflection properties of ellipses and hyperbolas.

67. Let $P(x_1, y_1)$ be a point on the ellipse $x^2/a^2 + y^2/b^2 = 1$ with foci F_1 and F_2 and let α and β be the angles between the lines PF_1, PF_2 and the ellipse as shown in the figure. Prove that $\alpha = \beta$. This explains how whispering galleries and lithotripsy work. Sound coming from one focus is reflected and passes through the other focus. [*Hint:* Use the formula in Problem 21 in Problems Plus following Chapter 3 to show that $\tan \alpha = \tan \beta$.]

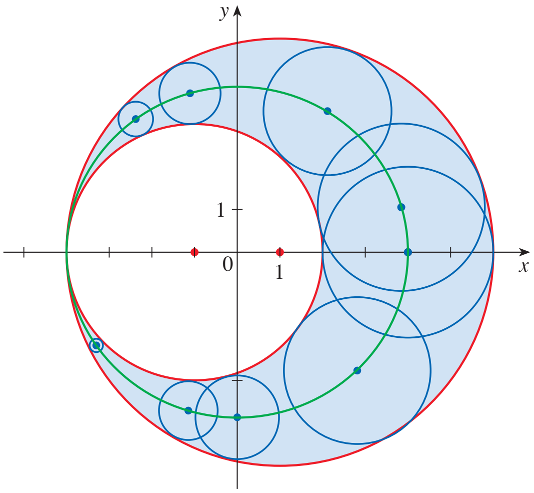


68. Let $P(x_1, y_1)$ be a point on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ with foci F_1 and F_2 and let α and β be the angles between the lines PF_1, PF_2 and the hyperbola as shown in the figure. Prove that $\alpha = \beta$. This shows that light aimed at a focus F_2 of a hyperbolic mirror is reflected toward the other focus F_1.





69. The graph shows two red circles with centers $(-1, 0)$ and $(1, 0)$ and radii 3 and 5, respectively. Consider the collection of all circles tangent to both of these circles. (Some of these are shown in blue.) Show that the centers of all such circles lie on an ellipse with foci $(\pm 1, 0)$. Find an equation of this ellipse.



10.6 Conic Sections in Polar Coordinates

In Section 10.5 we defined the parabola in terms of a focus and directrix, but we defined the ellipse and hyperbola in terms of two foci. In this section we give a more unified treatment of all three types of conic sections in terms of a focus and directrix.

A Unified Description of Conics

If we place the focus at the origin, then a conic section has a simple polar equation, which provides a convenient description of the motion of planets, satellites, and comets.

1 Theorem Let F be a fixed point (called the **focus**) and l be a fixed line (called the **directrix**) in a plane. Let e be a fixed positive number (called the **eccentricity**). The set of all points P in the plane such that

$$\frac{|PF|}{|Pl|} = e$$

(that is, the ratio of the distance from F to the distance from l is the constant e) is a conic section. The conic is

(a) an ellipse if $e < 1$
(b) a parabola if $e = 1$
(c) a hyperbola if $e > 1$

PROOF Notice that if the eccentricity is $e = 1$, then $|PF| = |Pl|$ and so the given condition simply becomes the definition of a parabola as given in Section 10.5.

Let us place the focus F at the origin and the directrix parallel to the y-axis and d units to the right. Thus the directrix has equation $x = d$ and is perpendicular to the polar axis. If the point P has polar coordinates (r, θ), we see from Figure 1 that

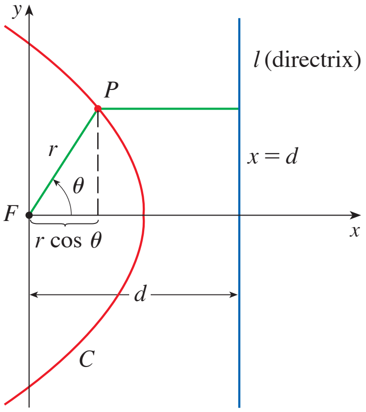


FIGURE 1

$$|PF| = r \qquad |Pl| = d - r\cos\theta$$

Thus the condition $|PF|/|Pl| = e$, or $|PF| = e|Pl|$, becomes

$$\boxed{2} \qquad r = e(d - r\cos\theta)$$

If we square both sides of this polar equation and convert to rectangular coordinates, we get

$$x^2 + y^2 = e^2(d - x)^2 = e^2(d^2 - 2dx + x^2)$$

or
$$(1 - e^2)x^2 + 2de^2x + y^2 = e^2d^2$$

After completing the square, we have

$$\boxed{3} \qquad \left(x + \frac{e^2d}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{e^2d^2}{(1 - e^2)^2}$$

If $e < 1$, we recognize Equation 3 as the equation of an ellipse. In fact, it is of the form

$$\frac{(x - h)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where

$$\boxed{4} \qquad h = -\frac{e^2d}{1 - e^2} \qquad a^2 = \frac{e^2d^2}{(1 - e^2)^2} \qquad b^2 = \frac{e^2d^2}{1 - e^2}$$

In Section 10.5 we found that the foci of an ellipse are at a distance c from the center, where

$$\boxed{5} \qquad c^2 = a^2 - b^2 = \frac{e^4d^2}{(1 - e^2)^2}$$

This shows that
$$c = \frac{e^2d}{1 - e^2} = -h$$

and confirms that the focus as defined in Theorem 1 means the same as the focus defined in Section 10.5. It also follows from Equations 4 and 5 that the eccentricity is given by

$$e = \frac{c}{a}$$

If $e > 1$, then $1 - e^2 < 0$ and we see that Equation 3 represents a hyperbola. Just as we did before, we could rewrite Equation 3 in the form

$$\frac{(x - h)^2}{a^2} - \frac{y^2}{b^2} = 1$$

and see that

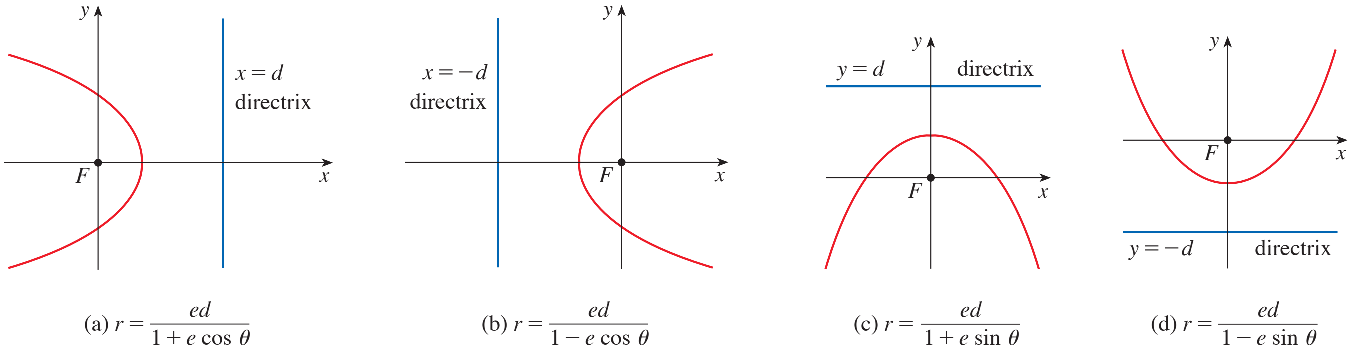
$$e = \frac{c}{a} \qquad \text{where} \quad c^2 = a^2 + b^2$$ ■

Polar Equations of Conics

In Figure 1, the focus of the conic section is located at the origin and the directrix has equation $x = d$. By solving Equation 2 for r, we see that the polar equation of this conic can be written as

$$r = \frac{ed}{1 + e\cos\theta}$$

If the directrix is chosen to be to the left of the focus as $x = -d$, or if the directrix is chosen to be parallel to the polar axis as $y = \pm d$, then the polar equation of the conic is given by the following theorem, which is illustrated by Figure 2. (See Exercises 27–29.)



(a) $r = \dfrac{ed}{1 + e\cos\theta}$ (b) $r = \dfrac{ed}{1 - e\cos\theta}$ (c) $r = \dfrac{ed}{1 + e\sin\theta}$ (d) $r = \dfrac{ed}{1 - e\sin\theta}$

FIGURE 2 Polar equations of conics

6 Theorem A polar equation of the form

$$r = \frac{ed}{1 \pm e\cos\theta} \qquad \text{or} \qquad r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic section with eccentricity e. The conic is an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$.

EXAMPLE 1 Find a polar equation for a parabola that has its focus at the origin and whose directrix is the line $y = -6$.

SOLUTION Using Theorem 6 with $e = 1$ and $d = 6$, and using part (d) of Figure 2, we see that the equation of the parabola is

$$r = \frac{6}{1 - \sin\theta}$$ ■

EXAMPLE 2 A conic is given by the polar equation

$$r = \frac{10}{3 - 2\cos\theta}$$

Find the eccentricity, identify the conic, locate the directrix, and sketch the conic.

SOLUTION Dividing numerator and denominator by 3, we write the equation as

$$r = \frac{\frac{10}{3}}{1 - \frac{2}{3}\cos\theta}$$

From Theorem 6 we see that this represents an ellipse with $e = \frac{2}{3}$. Since $ed = \frac{10}{3}$, we have

$$d = \frac{\frac{10}{3}}{e} = \frac{\frac{10}{3}}{\frac{2}{3}} = 5$$

and so the directrix has Cartesian equation $x = -5$. We find the values for r when $\theta = 0, \pi/2, \pi$, and $3\pi/2$, as shown in the table. The ellipse is sketched in Figure 3.

θ	r
0	10
$\frac{\pi}{2}$	$\frac{10}{3}$
π	2
$\frac{3\pi}{2}$	$\frac{10}{3}$

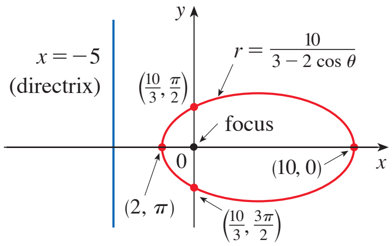

y
$x = -5$
(directrix)
$r = \dfrac{10}{3 - 2\cos\theta}$
$\left(\frac{10}{3}, \frac{\pi}{2}\right)$
focus
0
(10, 0)
x
$(2, \pi)$
$\left(\frac{10}{3}, \frac{3\pi}{2}\right)$

FIGURE 3

$$r = \frac{10}{3 - 2\cos\theta}$$

■

EXAMPLE 3 Sketch the conic $r = \dfrac{12}{2 + 4\sin\theta}$.

SOLUTION Writing the equation in the form

$$r = \frac{6}{1 + 2\sin\theta}$$

we see that the eccentricity is $e = 2$ and the equation therefore represents a hyperbola. Since $ed = 6$, we have $d = 3$ and the directrix has equation $y = 3$. We find the values for r when $\theta = 0, \pi/2, \pi$, and $3\pi/2$ as shown in the table. The vertices occur when $\theta = \pi/2$ and $3\pi/2$, so they are $(2, \pi/2)$ and $(-6, 3\pi/2) = (6, \pi/2)$. The x-intercepts occur when $\theta = 0, \pi$; in both cases $r = 6$. For additional accuracy we draw the asymptotes. Note that $r \to \pm\infty$ when $1 + 2\sin\theta \to 0^+$ or 0^- and $1 + 2\sin\theta = 0$ when $\sin\theta = -\frac{1}{2}$. Thus the asymptotes are parallel to the rays $\theta = 7\pi/6$ and $\theta = 11\pi/6$. The hyperbola is sketched in Figure 4.

θ	r
0	6
$\frac{\pi}{2}$	2
π	6
$\frac{3\pi}{2}$	-6

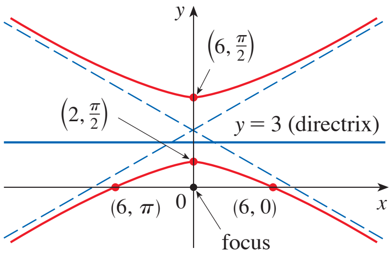

y
$\left(6, \frac{\pi}{2}\right)$
$\left(2, \frac{\pi}{2}\right)$
$y = 3$ (directrix)
$(6, \pi)$
0
(6, 0)
x
focus

FIGURE 4

$$r = \frac{12}{2 + 4\sin\theta}$$

■

When rotating conic sections, we find it much more convenient to use polar equations than Cartesian equations. We just use the fact (see Exercise 10.3.65) that the graph of

$r = f(\theta - \alpha)$ is the graph of $r = f(\theta)$ rotated counterclockwise about the origin through an angle α.

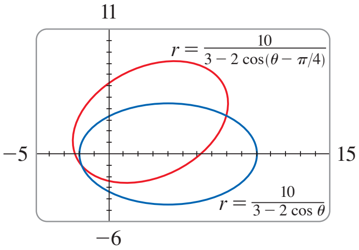


FIGURE 5

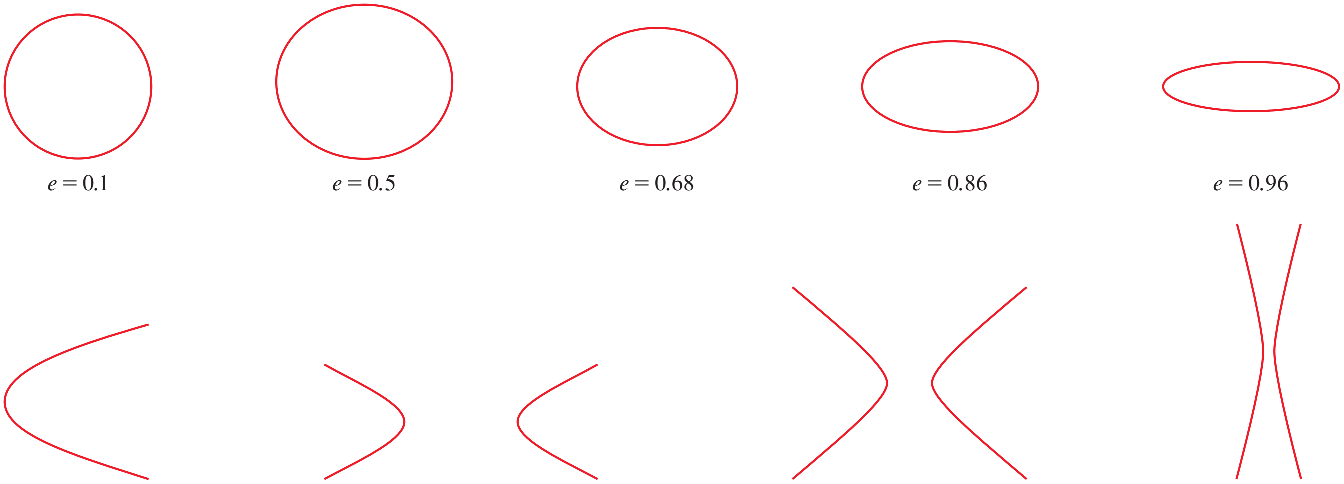
EXAMPLE 4 If the ellipse of Example 2 is rotated through an angle $\pi/4$ about the origin, find a polar equation and graph the resulting ellipse.

SOLUTION We get the equation of the rotated ellipse by replacing θ with $\theta - \pi/4$ in the equation given in Example 2. So the new equation is

$$r = \frac{10}{3 - 2\cos(\theta - \pi/4)}$$

We use this equation to graph the rotated ellipse in Figure 5. Notice that the ellipse has been rotated about its left focus. ■

In Figure 6 we use a computer to sketch a number of conics to demonstrate the effect of varying the eccentricity e. Notice that when e is close to 0 the ellipse is nearly circular, whereas it becomes more elongated as $e \to 1^-$. When $e = 1$, of course, the conic is a parabola.



$e = 0.1$ $e = 0.5$ $e = 0.68$ $e = 0.86$ $e = 0.96$

$e = 1$ $e = 1.1$ $e = 1.4$ $e = 4$

FIGURE 6

Kepler's Laws

In 1609 the German mathematician and astronomer Johannes Kepler, on the basis of huge amounts of astronomical data, published the following three laws of planetary motion.

Kepler's Laws

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.
2. The line joining the sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

Although Kepler formulated his laws in terms of the motion of planets around the sun, they apply equally well to the motion of moons, comets, satellites, and other bodies that orbit subject to a single gravitational force. In Section 13.4 we will show how to deduce Kepler's Laws from Newton's Laws. Here we use Kepler's First Law, together with the polar equation of an ellipse, to calculate quantities of interest in astronomy.

For purposes of astronomical calculations, it's useful to express the equation of an ellipse in terms of its eccentricity e and its semimajor axis a. We can write the distance d from the focus to the directrix in terms of a if we use (4):

$$a^2 = \frac{e^2d^2}{(1 - e^2)^2} \quad \Rightarrow \quad d^2 = \frac{a^2(1 - e^2)^2}{e^2} \quad \Rightarrow \quad d = \frac{a(1 - e^2)}{e}$$

So $ed = a(1 - e^2)$. If the directrix is $x = d$, then the polar equation is

$$r = \frac{ed}{1 + e\cos\theta} = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

7 The polar equation of an ellipse with focus at the origin, semimajor axis a, eccentricity e, and directrix $x = d$ can be written in the form

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

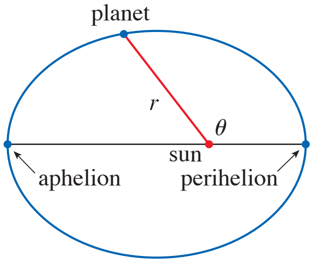


FIGURE 7

The positions of a planet that are closest to and farthest from the sun are called its **perihelion** and **aphelion**, respectively, and correspond to the vertices of the ellipse (see Figure 7). The distances from the sun to the perihelion and aphelion are called the **perihelion distance** and **aphelion distance**, respectively. In Figure 1 the sun is at the focus F, so at perihelion we have $\theta = 0$ and, from Equation 7,

$$r = \frac{a(1 - e^2)}{1 + e\cos 0} = \frac{a(1 - e)(1 + e)}{1 + e} = a(1 - e)$$

Similarly, at aphelion $\theta = \pi$ and $r = a(1 + e)$.

8 The perihelion distance from a planet to the sun is $a(1 - e)$ and the aphelion distance is $a(1 + e)$.

EXAMPLE 5

(a) Find an approximate polar equation for the elliptical orbit of the earth around the sun (at one focus) given that the eccentricity is about 0.017 and the length of the major axis is about 2.99×10^8 km.

(b) Find the distance from the earth to the sun at perihelion and at aphelion.

SOLUTION

(a) The length of the major axis is $2a = 2.99 \times 10^8$, so $a = 1.495 \times 10^8$. We are given that $e = 0.017$ and so, from Equation 7, an equation of the earth's orbit around the sun is

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} = \frac{(1.495 \times 10^8)[1 - (0.017)^2]}{1 + 0.017\cos\theta}$$

or, approximately,

$$r = \frac{1.49 \times 10^8}{1 + 0.017\cos\theta}$$

(b) From (8), the perihelion distance from the earth to the sun is

$$a(1 - e) \approx (1.495 \times 10^8)(1 - 0.017) \approx 1.47 \times 10^8 \text{ km}$$

and the aphelion distance is

$$a(1 + e) \approx (1.495 \times 10^8)(1 + 0.017) \approx 1.52 \times 10^8 \text{ km}$$ ■

10.6 Exercises

1–8 Write a polar equation of a conic with the focus at the origin and the given data.

1. Parabola, directrix $x = 2$

2. Ellipse, eccentricity $\frac{1}{3}$, directrix $y = 6$

3. Hyperbola, eccentricity 2, directrix $y = -4$

4. Hyperbola, eccentricity $\frac{5}{2}$, directrix $x = -3$

5. Ellipse, eccentricity $\frac{2}{3}$, vertex $(2, \pi)$

6. Ellipse, eccentricity 0.6, directrix $r = 4\csc\theta$

7. Parabola, vertex $(3, \pi/2)$

8. Hyperbola, eccentricity 2, directrix $r = -2\sec\theta$

9–14 Match the polar equations with the graphs labeled I–VI. Give reasons for your answer.

9. $r = \dfrac{3}{1 - \sin\theta}$

10. $r = \dfrac{9}{1 + 2\cos\theta}$

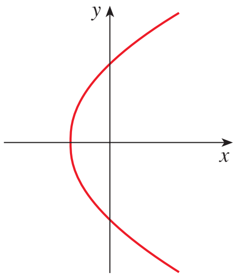
11. $r = \dfrac{12}{8 - 7\cos\theta}$

12. $r = \dfrac{12}{4 + 3\sin\theta}$

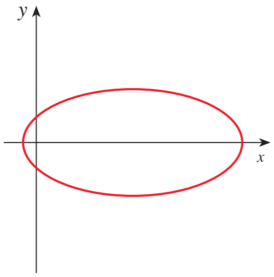
13. $r = \dfrac{5}{2 + 3\sin\theta}$

14. $r = \dfrac{3}{2 - 2\cos\theta}$

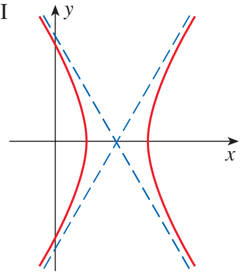
I



II



III



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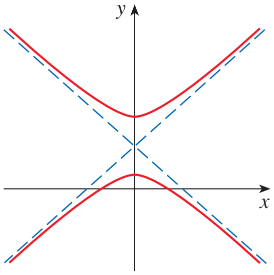
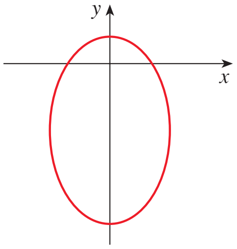
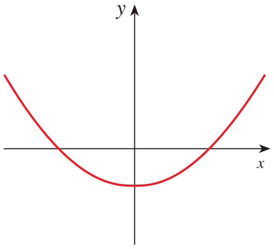


V



VI



15–22 (a) Find the eccentricity, (b) identify the conic, (c) give an equation of the directrix, and (d) sketch the conic.

15. $r = \dfrac{4}{5 - 4\sin\theta}$

16. $r = \dfrac{1}{2 + \sin\theta}$

17. $r = \dfrac{2}{3 + 3\sin\theta}$

18. $r = \dfrac{5}{2 - 4\cos\theta}$

19. $r = \dfrac{9}{6 + 2\cos\theta}$

20. $r = \dfrac{1}{3 - 3\sin\theta}$

21. $r = \dfrac{3}{4 - 8\cos\theta}$

22. $r = \dfrac{4}{2 + 3\cos\theta}$

---

23. (a) Find the eccentricity and directrix of the conic $r = 1/(1 - 2\sin\theta)$ and graph the conic and its directrix.

(b) If this conic is rotated counterclockwise about the origin through an angle $3\pi/4$, write the resulting equation and graph its curve.

24. Graph the conic

$$r = \frac{4}{5 + 6\cos\theta}$$

and its directrix. Also graph the conic obtained by rotating this curve about the origin through an angle $\pi/3$.

25. Graph the conics

$$r = \frac{e}{1 - e\cos\theta}$$

with $e = 0.4$, 0.6, 0.8, and 1.0 on a common screen. How does the value of e affect the shape of the curve?

26. (a) Graph the conics

$$r = \frac{ed}{1 + e\sin\theta}$$

for $e = 1$ and various values of d. How does the value of d affect the shape of the conic?

(b) Graph these conics for $d = 1$ and various values of e. How does the value of e affect the shape of the conic?

27. Show that a conic with focus at the origin, eccentricity e, and directrix $x = -d$ has polar equation

$$r = \frac{ed}{1 - e\cos\theta}$$

28. Show that a conic with focus at the origin, eccentricity e, and directrix $y = d$ has polar equation

$$r = \frac{ed}{1 + e\sin\theta}$$

29. Show that a conic with focus at the origin, eccentricity e, and directrix $y = -d$ has polar equation

$$r = \frac{ed}{1 - e\sin\theta}$$

30. Show that the parabolas $r = c/(1 + \cos\theta)$ and $r = d/(1 - \cos\theta)$ intersect at right angles.

31. The orbit of Mars around the sun is an ellipse with eccentricity 0.093 and semimajor axis 2.28×10^8 km. Find a polar equation for the orbit.

32. Jupiter's orbit has eccentricity 0.048 and the length of the major axis is 1.56×10^9 km. Find a polar equation for the orbit.

33. The orbit of Halley's comet, last seen in 1986 and due to return in 2061, is an ellipse with eccentricity 0.97 and one focus at the sun. The length of its major axis is 36.18 AU. [An astronomical unit (AU) is the mean distance between the earth and the sun, about 150 million kilometers.] Find a polar equation for the orbit of Halley's comet. What is the maximum distance from the comet to the sun?

34. Comet Hale-Bopp, discovered in 1995, has an elliptical orbit with eccentricity 0.9951. The length of the orbit's major axis is 356.6 AU. Find a polar equation for the orbit of this comet. How close to the sun does it come?

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35. The planet Mercury travels in an elliptical orbit with eccentricity 0.206. Its minimum distance from the sun is 4.6×10^7 km. Find its maximum distance from the sun.

36. The distance from the dwarf planet Pluto to the sun is 4.43×10^9 km at perihelion and 7.37×10^9 km at aphelion. Find the eccentricity of Pluto's orbit.

37. Using the data from Exercise 35, find the distance traveled by the planet Mercury during one complete orbit around the sun. (Evaluate the resulting definite integral numerically, using a calculator or computer, or use Simpson's Rule.)

10 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. (a) What is a parametric curve?
(b) How do you sketch a parametric curve?

2. (a) How do you find the slope of a tangent to a parametric curve?
(b) How do you find the area under a parametric curve?

3. Write an expression for each of the following:
(a) The length of a parametric curve
(b) The area of the surface obtained by rotating a parametric curve about the x-axis
(c) The speed of a particle traveling along a parametric curve

4. (a) Use a diagram to explain the meaning of the polar coordinates (r, θ) of a point.
(b) Write equations that express the Cartesian coordinates (x, y) of a point in terms of the polar coordinates.
(c) What equations would you use to find the polar coordinates of a point if you knew the Cartesian coordinates?

5. (a) How do you find the area of a region bounded by a polar curve?
(b) How do you find the length of a polar curve?
(c) How do you find the slope of a tangent line to a polar curve?

6. (a) Give a geometric definition of a parabola.
(b) Write an equation of a parabola with focus $(0, p)$ and directrix $y = -p$. What if the focus is $(p, 0)$ and the directrix is $x = -p$?

7. (a) Give a definition of an ellipse in terms of foci.
(b) Write an equation for the ellipse with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$.

8. (a) Give a definition of a hyperbola in terms of foci.
(b) Write an equation for the hyperbola with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$.
(c) Write equations for the asymptotes of the hyperbola in part (b).

9. (a) What is the eccentricity of a conic section?
(b) What can you say about the eccentricity if the conic section is an ellipse? A hyperbola? A parabola?
(c) Write a polar equation for a conic section with eccentricity e and directrix $x = d$. What if the directrix is $x = -d$? $y = d$? $y = -d$?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If the parametric curve $x = f(t)$, $y = g(t)$ satisfies $g'(1) = 0$, then it has a horizontal tangent when $t = 1$.

2. If $x = f(t)$ and $y = g(t)$ are twice differentiable, then

$$\frac{d^2y}{dx^2} = \frac{d^2y/dt^2}{d^2x/dt^2}$$

3. The length of the curve $x = f(t)$, $y = g(t)$, $a \leqslant t \leqslant b$, is

$$\int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

4. If the position of a particle at time t is given by the parametric equations $x = 3t + 1$, $y = 2t^2 + 1$, then the speed of the particle at time $t = 3$ is the value of dy/dx when $t = 3$.

5. If a point is represented by (x, y) in Cartesian coordinates (where $x \neq 0$) and (r, θ) in polar coordinates, then $\theta = \tan^{-1}(y/x)$.

6. The polar curves

$$r = 1 - \sin 2\theta \qquad r = \sin 2\theta - 1$$

have the same graph.

7. The equations $r = 2$, $x^2 + y^2 = 4$, and $x = 2 \sin 3t$, $y = 2 \cos 3t$ $(0 \leqslant t \leqslant 2\pi)$ all have the same graph.

8. The parametric equations $x = t^2$, $y = t^4$ have the same graph as $x = t^3$, $y = t^6$.

9. The graph of $y^2 = 2y + 3x$ is a parabola.

10. A tangent line to a parabola intersects the parabola only once.

11. A hyperbola never intersects its directrix.

EXERCISES

1–5 Sketch the parametric curve and eliminate the parameter to find a Cartesian equation of the curve.

1. $x = t^2 + 4t, \quad y = 2 - t, \quad -4 \leqslant t \leqslant 1$

2. $x = 1 + e^{2t}, \quad y = e^t$

3. $x = \ln t, \quad y = t^2$

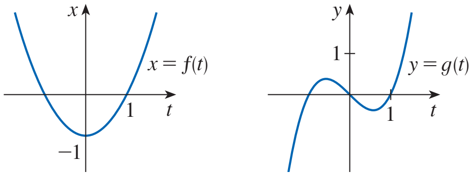
4. $x = 2\cos\theta, \quad y = 1 + \sin\theta$

5. $x = \cos\theta, \quad y = \sec\theta, \quad 0 \leqslant \theta < \pi/2$

6. Describe the motion of a particle with position (x, y), where $x = 2 + 4\cos\pi t$ and $y = -3 + 4\sin\pi t$, as t increases from 0 to 4.

7. Write three different sets of parametric equations for the curve $y = \sqrt{x}$.

8. Use the graphs of $x = f(t)$ and $y = g(t)$ to sketch the parametric curve $x = f(t)$, $y = g(t)$. Indicate with arrows the direction in which the curve is traced as t increases.



9. (a) Plot the point with polar coordinates $(4, 2\pi/3)$. Then find its Cartesian coordinates.
(b) The Cartesian coordinates of a point are $(-3, 3)$. Find two sets of polar coordinates for the point.

10. Sketch the region consisting of points whose polar coordinates satisfy $1 \leqslant r < 2$ and $\pi/6 \leqslant \theta \leqslant 5\pi/6$.

11–18 Sketch the polar curve.

11. $r = 1 + \sin\theta$

12. $r = \sin 4\theta$

13. $r = \cos 3\theta$

14. $r = 3 + \cos 3\theta$

15. $r = 1 + \cos 2\theta$

16. $r = 2\cos(\theta/2)$

17. $r = \dfrac{3}{1 + 2\sin\theta}$

18. $r = \dfrac{3}{2 - 2\cos\theta}$

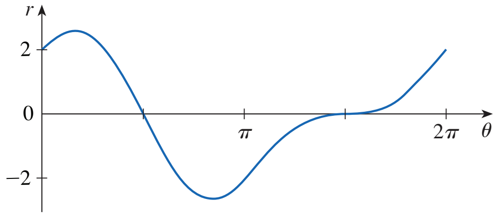
19–20 Find a polar equation for the curve represented by the given Cartesian equation.

19. $x + y = 2$

20. $x^2 + y^2 = 2$

21. The curve with polar equation $r = (\sin\theta)/\theta$ is called a **cochleoid**. Use a graph of r as a function of θ in Cartesian coordinates to sketch the cochleoid by hand. Then graph it with a calculator or computer to check your sketch.

22. The figure shows a graph of r as a function of θ in Cartesian coordinates. Use it to sketch the corresponding polar curve.



23–26 Find the slope of the tangent line to the given curve at the point corresponding to the specified value of the parameter.

23. $x = \ln t, \quad y = 1 + t^2; \quad t = 1$

24. $x = t^3 + 6t + 1, \quad y = 2t - t^2; \quad t = -1$

25. $r = e^{-\theta}; \quad \theta = \pi$

26. $r = 3 + \cos 3\theta; \quad \theta = \pi/2$

27–28 Find dy/dx and d^2y/dx^2.

27. $x = t + \sin t, \quad y = t - \cos t$

28. $x = 1 + t^2, \quad y = t - t^3$

29. Use a graph to estimate the coordinates of the lowest point on the curve $x = t^3 - 3t$, $y = t^2 + t + 1$. Then use calculus to find the exact coordinates.

30. Find the area enclosed by the loop of the curve in Exercise 29.

31. At what points does the curve

$$x = 2a\cos t - a\cos 2t \qquad y = 2a\sin t - a\sin 2t$$

have vertical or horizontal tangents? Use this information to help sketch the curve.

32. Find the area enclosed by the curve in Exercise 31.

33. Find the area enclosed by the curve $r^2 = 9\cos 5\theta$.

34. Find the area enclosed by the inner loop of the curve $r = 1 - 3\sin\theta$.

35. Find the points of intersection of the curves $r = 2$ and $r = 4\cos\theta$.

36. Find the points of intersection of the curves $r = \cot\theta$ and $r = 2\cos\theta$.

37. Find the area of the region that lies inside both of the circles $r = 2\sin\theta$ and $r = \sin\theta + \cos\theta$.

38. Find the area of the region that lies inside the curve $r = 2 + \cos 2\theta$ but outside the curve $r = 2 + \sin\theta$.

39–42 Find the length of the curve.

39. $x = 3t^2, \quad y = 2t^3, \quad 0 \leqslant t \leqslant 2$

40. $x = 2 + 3t, \quad y = \cosh 3t, \quad 0 \leqslant t \leqslant 1$

41. $r = 1/\theta, \quad \pi \leqslant \theta \leqslant 2\pi$

42. $r = \sin^3(\theta/3), \quad 0 \leqslant \theta \leqslant \pi$

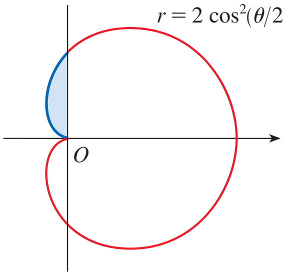
---

43. The position (in meters) of a particle at time t seconds is given by the parametric equations

$$x = \tfrac{1}{2}(t^2 + 3) \qquad y = 5 - \tfrac{1}{3}t^3$$

(a) Find the speed of the particle at the point $(6, -4)$.
(b) What is the average speed of the particle for $0 \leqslant t \leqslant 8$?

44. (a) Find the exact length of the portion of the curve shown in blue.
(b) Find the area of the shaded region.



45–46 Find the area of the surface obtained by rotating the given curve about the x-axis.

45. $x = 4\sqrt{t}, \quad y = \dfrac{t^3}{3} + \dfrac{1}{2t^2}, \quad 1 \leqslant t \leqslant 4$

46. $x = 2 + 3t, \quad y = \cosh 3t, \quad 0 \leqslant t \leqslant 1$

---

47. The curves defined by the parametric equations

$$x = \frac{t^2 - c}{t^2 + 1} \qquad y = \frac{t(t^2 - c)}{t^2 + 1}$$

are called **strophoids** (from a Greek word meaning "to turn or twist"). Investigate how these curves vary as c varies.

48. A family of curves has polar equations $r^a = |\sin 2\theta|$ where a is a positive number. Investigate how the curves change as a changes.

49–52 Find the foci and vertices and sketch the graph.

49. $\dfrac{x^2}{9} + \dfrac{y^2}{8} = 1$

50. $4x^2 - y^2 = 16$

51. $6y^2 + x - 36y + 55 = 0$

52. $25x^2 + 4y^2 + 50x - 16y = 59$

---

53. Find an equation of the ellipse with foci $(\pm 4, 0)$ and vertices $(\pm 5, 0)$.

54. Find an equation of the parabola with focus $(2, 1)$ and directrix $x = -4$.

55. Find an equation of the hyperbola with foci $(0, \pm 4)$ and asymptotes $y = \pm 3x$.

56. Find an equation of the ellipse with foci $(3, \pm 2)$ and major axis with length 8.

57. Find an equation for the ellipse that shares a vertex and a focus with the parabola $x^2 + y = 100$ and that has its other focus at the origin.

58. Show that if m is any real number, then there are exactly two lines of slope m that are tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ and their equations are

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

59. Find a polar equation for the ellipse with focus at the origin, eccentricity $\frac{1}{3}$, and directrix with equation $r = 4\sec\theta$.

60. Graph the ellipse $r = 2/(4 - 3\cos\theta)$ and its directrix. Also graph the ellipse obtained by rotation about the origin through an angle $2\pi/3$.

61. Show that the angles between the polar axis and the asymptotes of the hyperbola $r = ed/(1 - e\cos\theta)$, $e > 1$, are given by $\cos^{-1}(\pm 1/e)$.

62. A curve called the **folium of Descartes** is defined by the parametric equations

$$x = \frac{3t}{1 + t^3} \qquad y = \frac{3t^2}{1 + t^3}$$

(a) Show that if (a, b) lies on the curve, then so does (b, a); that is, the curve is symmetric with respect to the line $y = x$. Where does the curve intersect this line?
(b) Find the points on the curve where the tangent lines are horizontal or vertical.
(c) Show that the line $y = -x - 1$ is a slant asymptote.
(d) Sketch the curve.
(e) Show that a Cartesian equation of this curve is $x^3 + y^3 = 3xy$.
(f) Show that the polar equation can be written in the form

$$r = \frac{3\sec\theta\,\tan\theta}{1 + \tan^3\theta}$$

(g) Find the area enclosed by the loop of this curve.
(h) Show that the area of the loop is the same as the area that lies between the asymptote and the infinite branches of the curve. (Use a computer algebra system to evaluate the integral.)

Problems Plus

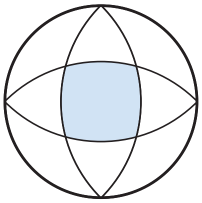


FIGURE FOR PROBLEM 1

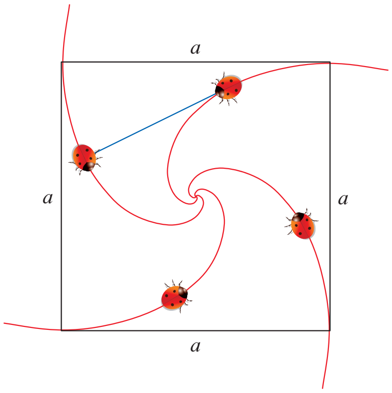


FIGURE FOR PROBLEM 4

1. The outer circle in the figure has radius 1 and the centers of the interior circular arcs lie on the outer circle. Find the area of the shaded region.

2. (a) Find the highest and lowest points on the curve $x^4 + y^4 = x^2 + y^2$.
(b) Sketch the curve. (Notice that it is symmetric with respect to both axes and both of the lines $y = \pm x$, so it suffices to consider $y \geqslant x \geqslant 0$ initially.)
(c) Use polar coordinates and a computer algebra system to find the area enclosed by the curve.

3. What is the smallest viewing rectangle that contains every member of the family of polar curves $r = 1 + c \sin\theta$, where $0 \leqslant c \leqslant 1$? Illustrate your answer by graphing several members of the family in this viewing rectangle.

4. Four bugs are placed at the four corners of a square with side length a. The bugs crawl counterclockwise at the same speed and each bug crawls directly toward the next bug at all times. They approach the center of the square along spiral paths.
(a) Find a polar equation of a bug's path assuming the pole is at the center of the square. (Use the fact that the line joining one bug to the next is tangent to the bug's path.)
(b) Find the distance traveled by a bug by the time it meets the other bugs at the center.

5. Show that any tangent line to a hyperbola touches the hyperbola halfway between the points of intersection of the tangent and the asymptotes.

6. A circle C of radius $2r$ has its center at the origin. A circle of radius r rolls without slipping in the counterclockwise direction around C. A point P is located on a fixed radius of the rolling circle at a distance b from its center, $0 < b < r$. [See parts (i) and (ii) of the figure below.] Let L be the line from the center of C to the center of the rolling circle and let θ be the angle that L makes with the positive x-axis.
(a) Using θ as a parameter, show that parametric equations of the path traced out by P are

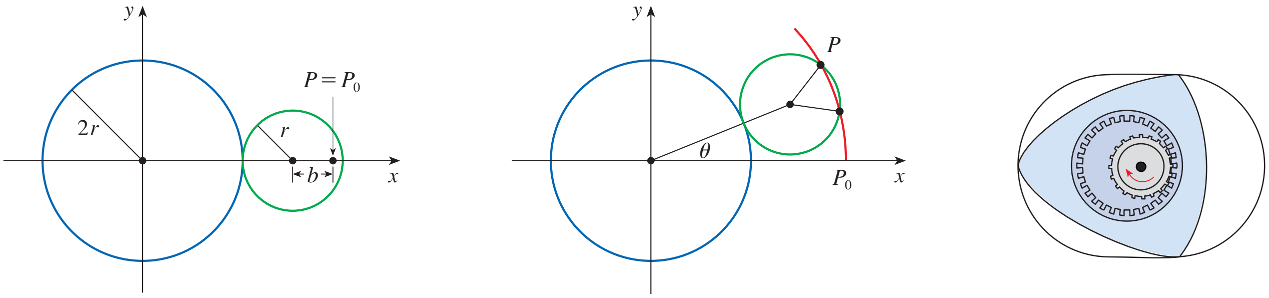
$$x = b\cos 3\theta + 3r\cos\theta \qquad y = b\sin 3\theta + 3r\sin\theta$$

Note: If $b = 0$, the path is a circle of radius $3r$; if $b = r$, the path is an *epicycloid*. The path traced out by P for $0 < b < r$ is called an *epitrochoid*.

(b) Graph the curve for various values of b between 0 and r.
(c) Show that an equilateral triangle can be inscribed in the epitrochoid and that its centroid is on the circle of radius b centered at the origin.

Note: This is the principle of the Wankel rotary engine. When the equilateral triangle rotates with its vertices on the epitrochoid, its centroid sweeps out a circle whose center is at the center of the curve.

(d) In most rotary engines the sides of the equilateral triangles are replaced by arcs of circles centered at the opposite vertices as in part (iii) of the figure. (Then the diameter of the rotor is constant.) Show that the rotor will fit in the epitrochoid if $b \leqslant \frac{3}{2}(2 - \sqrt{3})r$.



(i) (ii) (iii)

Astronomers gather information about distant celestial objects from the electromagnetic radiation that these objects emit. In the project following Section 11.11 you are asked to compare the radiation emitted by different stars, including Betelgeuse (the largest of the observable stars), Sirius, and our own Sun.

Antares StarExplorer / Shutterstock.com

11

Sequences, Series, and Power Series

IN ALL OF THE PREVIOUS CHAPTERS we studied functions that are defined on an interval. In this chapter we start by studying sequences of numbers. A sequence can be viewed as a function whose domain is a set of natural numbers. We then consider infinite series (the sum of the numbers in a sequence). Isaac Newton represented functions defined on an interval as sums of infinite series, in part because such series are readily integrated and differentiated. In Section 11.10 we will see that his idea allows us to integrate functions that we have previously been unable to find antiderivatives for, such as e^{-x^2}. Many of the functions that arise in mathematical physics and chemistry—such as Bessel functions—are defined as sums of series, so it is important to be familiar with the basic concepts of convergence of infinite sequences and series.

Physicists also use series in another way, as we will see in Section 11.11. In studying fields as diverse as optics, special relativity, electromagnetism, and cosmology, they analyze phenomena by replacing a function with the first few terms in the series that represents that function.

11.1 Sequences

Many concepts in calculus involve lists of numbers that result from applying a process in stages. For example, if we use Newton's method (Section 4.8) to approximate the zero of a function, we generate a list or *sequence* of numbers. If we compute average rates of change of a function over smaller and smaller intervals in order to approximate an instantaneous rate of change (as in Section 2.7), we also generate a sequence of numbers.

In the fifth century BC the Greek philosopher Zeno of Elea posed four problems, now known as *Zeno's paradoxes*, that were intended to challenge some of the ideas concerning space and time that were held in his day. In one of his paradoxes, Zeno argued that a man standing in a room could never walk to a wall because he would first have to walk half the distance to the wall, then half the remaining distance, and then again half of what still remains, continuing in this way indefinitely (see Figure 1). The distances that the man walks at each stage form a sequence:

$$\frac{1}{2},\quad \frac{1}{4},\quad \frac{1}{8},\quad \frac{1}{16},\quad \frac{1}{32},\quad \ldots,\quad \frac{1}{2^n},\quad \ldots$$

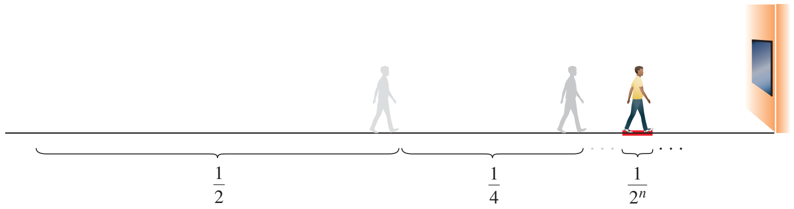


FIGURE 1
At the nth stage the man walks a distance $1/2^n$.

Infinite Sequences

An **infinite sequence**, or just a **sequence**, can be thought of as a list of numbers written in a definite order:

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_n,\ldots$$

The number a_1 is called the *first term*, a_2 is the *second term*, and in general a_n is the *nth term*. We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1}.

Notice that for every positive integer n there is a corresponding number a_n and so a sequence can be defined as a function f whose domain is the set of positive integers. But we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n.

NOTATION The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$$\{a_n\} \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}$$

Unless otherwise stated, we assume that n starts at 1.

EXAMPLE 1 Some sequences can be defined by giving a formula for the nth term.

(a) At the beginning of the section we described a sequence of distances walked by a man in a room. The following are three equivalent descriptions of this sequence:

$$\left\{\frac{1}{2^n}\right\} \qquad a_n = \frac{1}{2^n} \qquad \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots, \frac{1}{2^n}, \ldots\right\}$$

In the third description we have written out the first few terms of the sequence: $a_1 = 1/2^1$, $a_2 = 1/2^2$, and so on.

(b) The definition $\left\{\dfrac{n}{n+1}\right\}_{n=2}^{\infty}$ indicates that the formula for the nth term is $a_n = \dfrac{n}{n+1}$ and we start the sequence with $n = 2$:

$$\left\{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots\right\}$$

(c) The sequence $\{\sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \ldots\}$ can be described by $\{\sqrt{n+2}\}_{n=1}^{\infty}$ if we start with $n = 1$. Equivalently, we could start with $n = 3$ and write $\{\sqrt{n}\}_{n=3}^{\infty}$ or $a_n = \sqrt{n}$, $n \geqslant 3$.

(d) The definition $\left\{(-1)^n \dfrac{(n+1)}{3^n}\right\}_{n=0}^{\infty}$ generates the sequence

$$\left\{\frac{1}{1}, -\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \ldots\right\}$$

Here the first term corresponds to $n = 0$ and the $(-1)^n$ factor in the definition creates terms that alternate between positive and negative. ■

EXAMPLE 2 Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \ldots\right\}$$

assuming that the pattern of the first few terms continues.

SOLUTION We are given that

$$a_1 = \frac{3}{5} \qquad a_2 = -\frac{4}{25} \qquad a_3 = \frac{5}{125} \qquad a_4 = -\frac{6}{625} \qquad a_5 = \frac{7}{3125}$$

Notice that the numerators of these fractions start with 3 and increase by 1 whenever we go to the next term. The second term has numerator 4, the third term has numerator 5; in general, the nth term will have numerator $n + 2$. The denominators are the powers of 5, so a_n has denominator 5^n. The signs of the terms are alternately positive and negative, so we need to multiply by a power of -1, as in Example 1(d). Here we want a_1 to be positive and so we use $(-1)^{n-1}$ or $(-1)^{n+1}$. Therefore

$$a_n = (-1)^{n-1}\frac{n+2}{5^n}$$ ■

EXAMPLE 3 Here are some sequences that don't have a simple defining equation.

(a) The sequence $\{p_n\}$, where p_n is the population of the world as of January 1 in the year n.

(b) If we let a_n be the digit in the nth decimal place of the number e, then $\{a_n\}$ is a sequence whose first few terms are

$$\{7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \ldots\}$$

(c) **The Fibonacci sequence** $\{f_n\}$ is defined *recursively* by the conditions

$$f_1 = 1 \qquad f_2 = 1 \qquad f_n = f_{n-1} + f_{n-2} \qquad n \geqslant 3$$

Each term is the sum of the two preceding terms. The first few terms are

$$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

This sequence arose when the 13th-century Italian mathematician known as Fibonacci solved a problem concerning the breeding of rabbits (see Exercise 89). ■

■ The Limit of a Sequence

A sequence can be pictured either by plotting its terms on a number line or by plotting its graph. Figures 2 and 3 illustrate these representations for the sequence

$$\left\{\frac{n}{n+1}\right\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$$

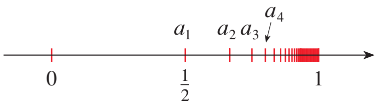


FIGURE 2

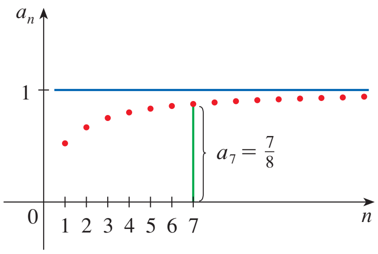


FIGURE 3

Since a sequence $\{a_n\}_{n=1}^{\infty}$ is a function whose domain is the set of positive integers, its graph consists of discrete points with coordinates

$$(1, a_1) \qquad (2, a_2) \qquad (3, a_3) \qquad \ldots \qquad (n, a_n) \qquad \ldots$$

From Figure 2 or Figure 3 it appears that the terms of the sequence $a_n = n/(n+1)$ are approaching 1 as n becomes large. In fact, the difference

$$1 - \frac{n}{n+1} = \frac{1}{n+1}$$

can be made as small as we like by taking n sufficiently large. We indicate this by writing

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

In general, the notation $\lim_{n\to\infty} a_n = L$

means that the terms of the sequence $\{a_n\}$ approach L as n becomes large. Notice that the following definition of the limit of a sequence is very similar to the definition of a limit of a function at infinity given in Section 2.6.

1 Intuitive Definition of a Limit of a Sequence A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

Figure 4 illustrates Definition 1 by showing the graphs of two convergent sequences that have the limit L.

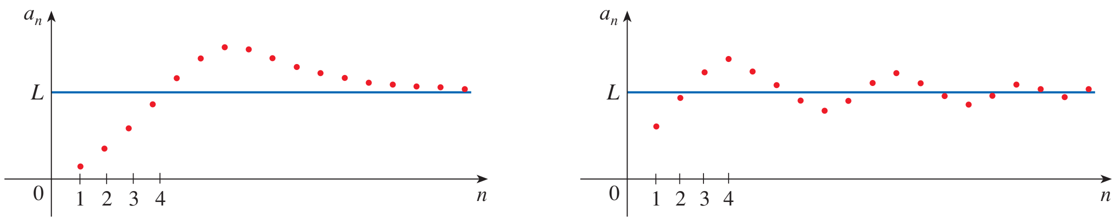


FIGURE 4 Graphs of two convergent sequences with $\lim_{n\to\infty} a_n = L$

A more precise version of Definition 1 is as follows.

Compare this definition with Definition 2.6.7.

2 **Precise Definition of a Limit of a Sequence** A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad |a_n - L| < \varepsilon$$

Definition 2 is illustrated by Figure 5, in which the terms $a_1, a_2, a_3, \ldots$ are plotted on a number line. No matter how small an interval $(L - \varepsilon, L + \varepsilon)$ is chosen, there exists an N such that all terms of the sequence from a_{N+1} onward must lie in that interval.

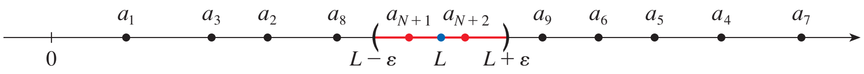


FIGURE 5

Another illustration of Definition 2 is given in Figure 6. The points on the graph of $\{a_n\}$ must lie between the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ if $n > N$. This picture must be valid no matter how small ε is chosen, but usually a smaller ε requires a larger N.

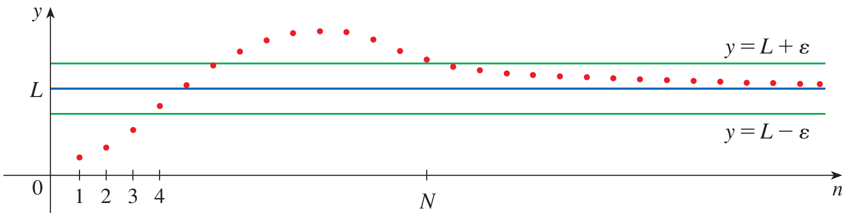
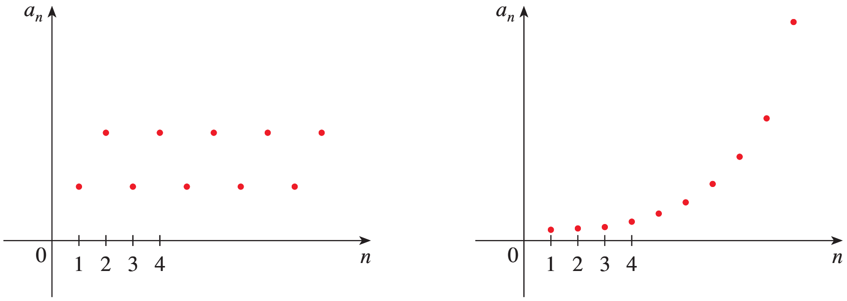


FIGURE 6

A sequence *diverges* if its terms do not approach a single number. Figure 7 illustrates two different ways in which a sequence can diverge.



(a) (b)

FIGURE 7
Graphs of two divergent sequences

The sequence graphed in Figure 7(a) diverges because it oscillates between two different numbers and does not approach a single value as $n \to \infty$. In the graph in part (b), a_n increases without bound as n becomes larger. We write $\lim_{n\to\infty} a_n = \infty$ to indicate the particular way that this sequence diverges, and we say that the sequence diverges to ∞. The following precise definition is similar to Definition 2.6.9.

3 Precise Definition of an Infinite Limit The notation $\lim_{n\to\infty} a_n = \infty$ meansthat for every positive number M there is an integer N such that

$$\text{if} \quad n > N \quad \text{then} \quad a_n > M$$

An analogous definition applies for $\lim_{n\to\infty} a_n = -\infty$.

Properties of Convergent Sequences

If you compare Definition 2 with Definition 2.6.7, you will see that the only difference between $\lim_{n\to\infty} a_n = L$ and $\lim_{x\to\infty} f(x) = L$ is that n is required to be an integer. Thus we have the following theorem, which is illustrated by Figure 8.

4 Theorem If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

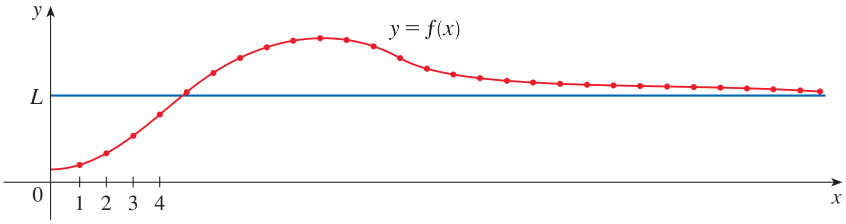


FIGURE 8

For instance, since we know that $\lim_{x\to\infty} (1/x^r) = 0$ when $r > 0$ (Theorem 2.6.5), it follows from Theorem 4 that

$$\boxed{5} \qquad \lim_{n\to\infty} \frac{1}{n^r} = 0 \qquad \text{if } r > 0$$

The Limit Laws given in Section 2.3 also hold for the limits of sequences and their proofs are similar.

Limit Laws for Sequences Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant. Then

Sum Law 1. $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

Difference Law 2. $\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$

Constant Multiple Law 3. $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$

Product Law 4. $\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

Quotient Law 5. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$

Another useful property of sequences is the following Power Law, which you are asked to prove in Exercise 94.

Power Law

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

The Squeeze Theorem can also be adapted for sequences as follows (see Figure 9).

Squeeze Theorem for Sequences

If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.

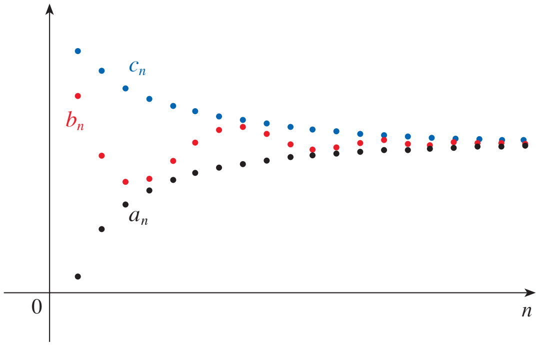


FIGURE 9
The sequence $\{b_n\}$ is squeezed between the sequences $\{a_n\}$ and $\{c_n\}$.

Another useful fact about limits of sequences is given by the following theorem; the proof is left as Exercise 93.

6 Theorem If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

EXAMPLE 4 Find $\displaystyle\lim_{n\to\infty} \frac{n}{n+1}$.

SOLUTION The method is similar to the one we used in Section 2.6: divide numerator and denominator by the highest power of n that occurs in the denominator and then use the Limit Laws for Sequences.

In general, for any constant c

$$\lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\dfrac{1}{n}} = \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \dfrac{1}{n}}$$

$$= \frac{1}{1+0} = 1$$

This shows that the guess we made earlier from Figures 2 and 3 was correct.

Here we used Equation 5 with $r = 1$. ■

EXAMPLE 5 Is the sequence $a_n = \dfrac{n}{\sqrt{10+n}}$ convergent or divergent?

SOLUTION As in Example 4, we divide numerator and denominator by n:

$$\lim_{n\to\infty}\frac{n}{\sqrt{10+n}}=\lim_{n\to\infty}\frac{1}{\sqrt{\dfrac{10}{n^2}+\dfrac{1}{n}}}=\infty$$

since the numerator is constant and the denominator (which is positive) approaches 0. So $\{a_n\}$ is divergent. ■

EXAMPLE 6 Calculate $\displaystyle\lim_{n\to\infty}\frac{\ln n}{n}$.

SOLUTION Notice that both numerator and denominator approach infinity as $n\to\infty$. We can't apply l'Hospital's Rule directly because it applies not to sequences but to functions of a real variable. However, we can apply l'Hospital's Rule to the related function $f(x)=(\ln x)/x$ and obtain

$$\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{1/x}{1}=0$$

Therefore, by Theorem 4, we have

$$\lim_{n\to\infty}\frac{\ln n}{n}=0$$ ■

EXAMPLE 7 Determine whether the sequence $a_n=(-1)^n$ is convergent or divergent.

SOLUTION If we write out the terms of the sequence, we obtain

$$\{-1,1,-1,1,-1,1,-1,\ldots\}$$

The graph of this sequence is shown in Figure 10. Since the terms oscillate between 1 and -1 infinitely often, a_n does not approach any number. Thus $\lim_{n\to\infty}(-1)^n$ does not exist; that is, the sequence $\{(-1)^n\}$ is divergent. ■

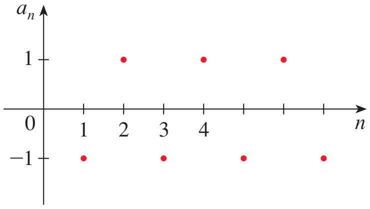


FIGURE 10 The sequence $\{(-1)^n\}$

EXAMPLE 8 Evaluate $\displaystyle\lim_{n\to\infty}\frac{(-1)^n}{n}$ if it exists.

SOLUTION We first calculate the limit of the absolute value:

$$\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right|=\lim_{n\to\infty}\frac{1}{n}=0$$

Therefore, by Theorem 6,

$$\lim_{n\to\infty}\frac{(-1)^n}{n}=0$$

The sequence is graphed in Figure 11. ■

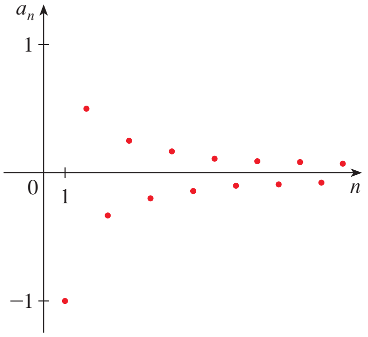


FIGURE 11 The sequence $\left\{\dfrac{(-1)^n}{n}\right\}$

The following theorem says that if we apply a continuous function to the terms of a convergent sequence, the result is also convergent. The proof is given in Appendix F.

$\boxed{7}$ **Theorem** If $\lim_{n\to\infty}a_n=L$ and the function f is continuous at L, then

$$\lim_{n\to\infty}f(a_n)=f(L)$$

EXAMPLE 9 Find $\lim_{n\to\infty} \sin\dfrac{\pi}{n}$.

SOLUTION Because the sine function is continuous at 0, Theorem 7 enables us to write

$$\lim_{n\to\infty} \sin\frac{\pi}{n} = \sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right) = \sin 0 = 0$$ ■

EXAMPLE 10 Discuss the convergence of the sequence $a_n = n!/n^n$, where $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$.

SOLUTION Both numerator and denominator approach infinity as $n \to \infty$ but here we have no corresponding function for use with l'Hospital's Rule ($x!$ is not defined when x is not an integer). Let's write out a few terms to get a feeling for what happens to a_n as n gets large:

$$a_1 = 1 \qquad a_2 = \frac{1 \cdot 2}{2 \cdot 2} \qquad a_3 = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3}$$

$$\boxed{8} \qquad a_n = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdot \cdots \cdot n}$$

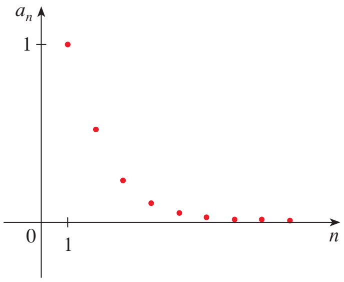


FIGURE 12 The sequence $\{n!/n^n\}$

It appears from these expressions and the graph in Figure 12 that the terms are decreasing and perhaps approach 0. To confirm this, observe from Equation 8 that

$$a_n = \frac{1}{n}\left(\frac{2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot \cdots \cdot n}\right)$$

Notice that the expression in parentheses is at most 1 because the numerator is less than (or equal to) the denominator. So

$$0 < a_n \leqslant \frac{1}{n}$$

We know that $1/n \to 0$ as $n \to \infty$. Therefore $a_n \to 0$ as $n \to \infty$ by the Squeeze Theorem. ■

EXAMPLE 11 For what values of r is the sequence $\{r^n\}$ convergent?

SOLUTION We know from Section 2.6 and the graphs of the exponential functions in Section 1.4 that $\lim_{x\to\infty} b^x = \infty$ for $b > 1$ and $\lim_{x\to\infty} b^x = 0$ for $0 < b < 1$. Therefore, putting $b = r$ and using Theorem 4, we have

$$\lim_{n\to\infty} r^n = \begin{cases} \infty & \text{if } r > 1 \\ 0 & \text{if } 0 < r < 1 \end{cases}$$

It is obvious that

$$\lim_{n\to\infty} 1^n = 1 \qquad \text{and} \qquad \lim_{n\to\infty} 0^n = 0$$

If $-1 < r < 0$, then $0 < |r| < 1$, so

$$\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0$$

and therefore $\lim_{n\to\infty} r^n = 0$ by Theorem 6. If $r \leqslant -1$, then $\{r^n\}$ diverges as in Example 7. Figure 13 shows the graphs for various values of r. (The case $r = -1$ is shown in Figure 10.)

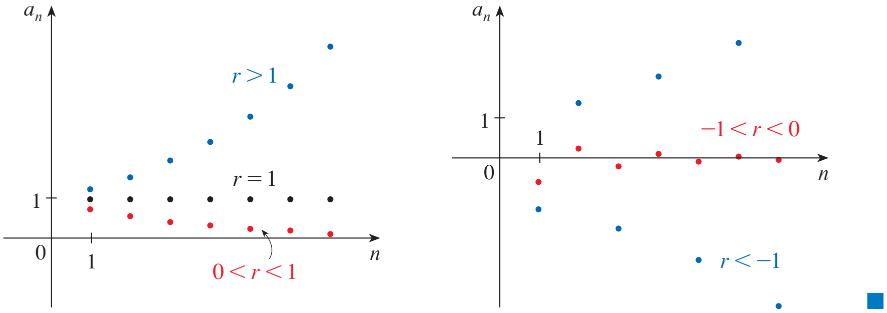


FIGURE 13
The sequence $a_n = r^n$

The results of Example 11 are summarized for future use as follows.

9 The sequence $\{r^n\}$ is convergent if $-1 < r \leqslant 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Monotonic and Bounded Sequences

Sequences for which the terms always increase (or always decrease) play a special role in the study of sequences.

10 Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$.A sequence is called **monotonic** if it is either increasing or decreasing.

EXAMPLE 12 The sequence $\left\{\dfrac{3}{n+5}\right\}$ is decreasing because

In Example 12, $3/(n+6)$ is smaller than $3/(n+5)$ because its denominator is larger.

$$a_n = \frac{3}{n+5} > \frac{3}{n+6} = \frac{3}{(n+1)+5} = a_{n+1}$$

for all $n \geqslant 1$. ■

EXAMPLE 13 Show that the sequence $a_n = \dfrac{n}{n^2+1}$ is decreasing.

SOLUTION 1 We must show that $a_n > a_{n+1}$, that is,

$$\frac{n}{n^2+1} > \frac{n+1}{(n+1)^2+1}$$

This inequality is equivalent to the one we get by cross-multiplication:

$$\frac{n}{n^2+1} > \frac{n+1}{(n+1)^2+1} \iff n[(n+1)^2+1] > (n+1)(n^2+1)$$

$$\iff n^3+2n^2+2n > n^3+n^2+n+1$$

$$\iff n^2+n > 1$$

Since $n \geqslant 1$, we know that the inequality $n^2+n>1$ is true. Therefore $a_n > a_{n+1}$ and so $\{a_n\}$ is decreasing.

SOLUTION 2 Consider the function $f(x) = \dfrac{x}{x^2+1}$:

$$f'(x) = \frac{x^2+1-x\cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0 \qquad \text{whenever } x^2>1$$

Thus f is decreasing on $(1, \infty)$ and so $f(n) > f(n+1)$. Therefore $\{a_n\}$ is decreasing. ■

11 Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leqslant M \qquad \text{for all } n \geqslant 1$$

A sequence is **bounded below** if there is a number m such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1$$

If a sequence is bounded above and below, then it is called a **bounded sequence**.

For instance, the sequence $a_n = n$ is bounded below $(a_n > 0)$ but not above. The sequence $a_n = n/(n+1)$ is bounded because $0 < a_n < 1$ for all n.

We know that not every bounded sequence is convergent [for instance, the sequence $a_n = (-1)^n$ satisfies $-1 \leqslant a_n \leqslant 1$ but is divergent from Example 7] and not every monotonic sequence is convergent $(a_n = n \to \infty)$. But if a sequence is both bounded *and* monotonic, then it must be convergent. This fact is proved as Theorem 12, but intuitively you can understand why it is true by looking at Figure 14. If $\{a_n\}$ is increasing and $a_n \leqslant M$ for all n, then the terms are forced to crowd together and approach some number L.

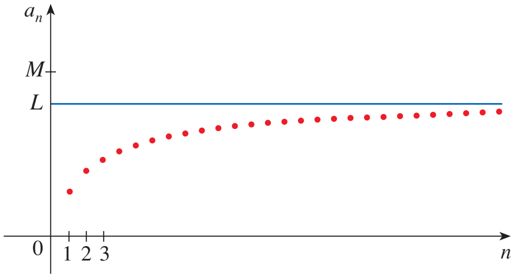


FIGURE 14

12 Monotonic Sequence Theorem Every bounded, monotonic sequence is convergent.

In particular, a sequence that is increasing and bounded above converges, and a sequence that is decreasing and bounded below converges.

The proof of Theorem 12 is based on the **Completeness Axiom** for the set $\mathbb{R}$ of real numbers, which says that if S is a nonempty set of real numbers that has an upper bound M ($x \leqslant M$ for all x in S), then S has a **least upper bound** b. (This means that b is an upper bound for S, but if M is any other upper bound, then $b \leqslant M$.) The Completeness Axiom is an expression of the fact that there is no gap or hole in the real number line.

PROOF OF THEOREM 12 Suppose $\{a_n\}$ is an increasing sequence. Since $\{a_n\}$ is bounded, the set $S = \{a_n \mid n \geqslant 1\}$ has an upper bound. By the Completeness Axiom it has a least upper bound L. Given $\varepsilon > 0$, $L - \varepsilon$ is *not* an upper bound for S (since L is the *least* upper bound). Therefore

$$a_N > L - \varepsilon \qquad \text{for some integer } N$$

But the sequence is increasing so $a_n \geqslant a_N$ for every $n > N$. Thus if $n > N$, we have

$$a_n > L - \varepsilon$$

so

$$0 \leqslant L - a_n < \varepsilon$$

since $a_n \leqslant L$. Thus

$$|L - a_n| < \varepsilon \qquad \text{whenever } \; n > N$$

so $\lim_{n\to\infty} a_n = L$.

A similar proof (using the greatest lower bound) holds if $\{a_n\}$ is decreasing. ■

EXAMPLE 14 Investigate the sequence $\{a_n\}$ defined by the *recurrence relation*

$$a_1 = 2 \qquad a_{n+1} = \tfrac{1}{2}(a_n + 6) \qquad \text{for } \; n = 1, 2, 3, \ldots$$

SOLUTION We begin by computing the first several terms:

$$a_1 = 2 \qquad a_2 = \tfrac{1}{2}(2 + 6) = 4 \qquad a_3 = \tfrac{1}{2}(4 + 6) = 5$$

$$a_4 = \tfrac{1}{2}(5 + 6) = 5.5 \qquad a_5 = 5.75 \qquad a_6 = 5.875$$

$$a_7 = 5.9375 \qquad a_8 = 5.96875 \qquad a_9 = 5.984375$$

Mathematical induction is often used in dealing with recursive sequences. See Principles of Problem Solving following Chapter 1 for a discussion of the Principle of Mathematical Induction.

These initial terms suggest that the sequence is increasing and the terms are approaching 6. To confirm that the sequence is increasing, we use mathematical induction to show that $a_{n+1} > a_n$ for all $n \geqslant 1$. This is true for $n = 1$ because $a_2 = 4 > a_1$. If we assume that it is true for $n = k$, then we have

$$a_{k+1} > a_k$$

so

$$a_{k+1} + 6 > a_k + 6$$

and

$$\tfrac{1}{2}(a_{k+1} + 6) > \tfrac{1}{2}(a_k + 6)$$

Thus

$$a_{k+2} > a_{k+1}$$

We have deduced that $a_{n+1} > a_n$ is true for $n = k + 1$. Therefore the inequality is true for all n by induction.

Next we verify that $\{a_n\}$ is bounded by showing that $a_n < 6$ for all n. (Since the sequence is increasing, we already know that it has a lower bound: $a_n \geqslant a_1 = 2$ for all n.) We know that $a_1 < 6$, so the assertion is true for $n = 1$. Suppose it is true for $n = k$. Then

$$a_k < 6$$

so $$a_k + 6 < 12$$

and $$\tfrac{1}{2}(a_k + 6) < \tfrac{1}{2}(12) = 6$$

Thus $$a_{k+1} < 6$$

This shows, by mathematical induction, that $a_n < 6$ for all n.

Since the sequence $\{a_n\}$ is increasing and bounded, Theorem 12 guarantees that it has a limit. The theorem doesn't tell us what the value of the limit is. But now that we know $L = \lim_{n\to\infty} a_n$ exists, we can use the given recurrence relation to write

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \tfrac{1}{2}(a_n + 6) = \tfrac{1}{2}\left(\lim_{n\to\infty} a_n + 6\right) = \tfrac{1}{2}(L + 6)$$

A proof of this fact is requested in Exercise 76.

Since $a_n \to L$, it follows that $a_{n+1} \to L$ too (as $n \to \infty$, $n + 1 \to \infty$ also). So we have

$$L = \tfrac{1}{2}(L + 6)$$

Solving this equation for L, we get $L = 6$, as we predicted. ■

11.1 EXERCISES

1. (a) What is a sequence?
   (b) What does it mean to say that $\lim_{n\to\infty} a_n = 8$?
   (c) What does it mean to say that $\lim_{n\to\infty} a_n = \infty$?

2. (a) What is a convergent sequence? Give two examples.
   (b) What is a divergent sequence? Give two examples.

3–16 List the first five terms of the sequence.

3. $a_n = n^3 - 1$

4. $a_n = \dfrac{1}{3^n + 1}$

5. $\{2^n + n\}_{n=2}^{\infty}$

6. $\left\{\dfrac{n^2 - 1}{n^2 + 1}\right\}_{n=3}^{\infty}$

7. $a_n = \dfrac{(-1)^{n-1}}{n^2}$

8. $a_n = \dfrac{(-1)^n}{4^n}$

9. $a_n = \cos n\pi$

10. $a_n = 1 + (-1)^n$

11. $a_n = \dfrac{(-2)^n}{(n + 1)!}$

12. $a_n = \dfrac{2n + 1}{n! + 1}$

13. $a_1 = 1, \quad a_{n+1} = 2a_n + 1$

14. $a_1 = 6, \quad a_{n+1} = \dfrac{a_n}{n}$

15. $a_1 = 2, \quad a_{n+1} = \dfrac{a_n}{1 + a_n}$

16. $a_1 = 2, \quad a_2 = 1, \quad a_{n+1} = a_n - a_{n-1}$

17–22 Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

17. $\left\{\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{6}, \tfrac{1}{8}, \tfrac{1}{10}, \ldots\right\}$

18. $\left\{4, -1, \tfrac{1}{4}, -\tfrac{1}{16}, \tfrac{1}{64}, \ldots\right\}$

19. $\left\{-3, 2, -\tfrac{4}{3}, \tfrac{8}{9}, -\tfrac{16}{27}, \ldots\right\}$

20. $\{5, 8, 11, 14, 17, \ldots\}$

21. $\left\{\tfrac{1}{2}, -\tfrac{4}{3}, \tfrac{9}{4}, -\tfrac{16}{5}, \tfrac{25}{6}, \ldots\right\}$

22. $\{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$

23–26 Calculate, to four decimal places, the first ten terms of the sequence and use them to plot the graph of the sequence by hand. Does the sequence appear to have a limit? If so, calculate it. If not, explain why.

23. $a_n = \dfrac{3n}{1 + 6n}$

24. $a_n = 2 + \dfrac{(-1)^n}{n}$

25. $a_n = 1 + \left(-\tfrac{1}{2}\right)^n$

26. $a_n = 1 + \dfrac{10^n}{9^n}$

27–62 Determine whether the sequence converges or diverges. If it converges, find the limit.

27. $a_n = \dfrac{5}{n + 2}$

28. $a_n = 5\sqrt{n + 2}$

29. $a_n = \dfrac{4n^2 - 3n}{2n^2 + 1}$

30. $a_n = \dfrac{4n^2 - 3n}{2n + 1}$

31. $a_n = \dfrac{n^4}{n^3 - 2n}$

32. $a_n = 2 + (0.86)^n$

33. $a_n = 3^n 7^{-n}$

34. $a_n = \dfrac{3\sqrt{n}}{\sqrt{n} + 2}$

35. $a_n = e^{-1/\sqrt{n}}$

36. $a_n = \dfrac{4^n}{1 + 9^n}$

37. $a_n = \sqrt{\dfrac{1 + 4n^2}{1 + n^2}}$

38. $a_n = \cos\left(\dfrac{n\pi}{n + 1}\right)$

39. $a_n = \dfrac{n^2}{\sqrt{n^3 + 4n}}$

40. $a_n = e^{2n/(n+2)}$

41. $a_n = \dfrac{(-1)^n}{2\sqrt{n}}$

42. $a_n = \dfrac{(-1)^{n+1} n}{n + \sqrt{n}}$

43. $\left\{\dfrac{(2n - 1)!}{(2n + 1)!}\right\}$

44. $\left\{\dfrac{\ln n}{\ln(2n)}\right\}$

45. $\{\sin n\}$

46. $a_n = \dfrac{\tan^{-1} n}{n}$

47. $\{n^2 e^{-n}\}$

48. $a_n = \ln(n + 1) - \ln n$

49. $a_n = \dfrac{\cos^2 n}{2^n}$

50. $a_n = \sqrt[n]{2^{1+3n}}$

51. $a_n = n \sin(1/n)$

52. $a_n = 2^{-n} \cos n\pi$

53. $a_n = \left(1 + \dfrac{2}{n}\right)^n$

54. $a_n = n^{1/n}$

55. $a_n = \ln(2n^2 + 1) - \ln(n^2 + 1)$

56. $a_n = \dfrac{(\ln n)^2}{n}$

57. $a_n = \arctan(\ln n)$

58. $a_n = n - \sqrt{n + 1}\sqrt{n + 3}$

59. $\{0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots\}$

60. $\left\{\frac{1}{1}, \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{6}, \ldots\right\}$

61. $a_n = \dfrac{n!}{2^n}$

62. $a_n = \dfrac{(-3)^n}{n!}$

63–69 Use a graph of the sequence to decide whether the sequence is convergent or divergent. If the sequence is convergent, guess the value of the limit from the graph and then prove your guess.

63. $a_n = (-1)^n \dfrac{n}{n + 1}$

64. $a_n = \dfrac{\sin n}{n}$

65. $a_n = \arctan\left(\dfrac{n^2}{n^2 + 4}\right)$

66. $a_n = \sqrt[n]{3^n + 5^n}$

67. $a_n = \dfrac{n^2 \cos n}{1 + n^2}$

68. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{n!}$

69. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{(2n)^n}$

---

70. (a) Determine whether the sequence defined as follows is convergent or divergent:

$$a_1 = 1 \qquad a_{n+1} = 4 - a_n \qquad \text{for } n \geqslant 1$$

(b) What happens if the first term is $a_1 = 2$?

71. If \$1000 is invested at 6% interest, compounded annually, then after n years the investment is worth $a_n = 1000(1.06)^n$ dollars.

(a) Find the first five terms of the sequence $\{a_n\}$.

(b) Is the sequence convergent or divergent? Explain.

72. If you deposit \$100 at the end of every month into an account that pays 3% interest per year compounded monthly, the amount of interest accumulated after n months is given by the sequence

$$I_n = 100\left(\frac{1.0025^n - 1}{0.0025} - n\right)$$

(a) Find the first six terms of the sequence.

(b) How much interest will you have earned after two years?

73. A fish farmer has 5000 catfish in his pond. The number of catfish increases by 8% per month and the farmer harvests 300 catfish per month.

(a) Show that the catfish population P_n after n months is given recursively by

$$P_n = 1.08P_{n-1} - 300 \qquad P_0 = 5000$$

(b) Find the number of catfish in the pond after six months.

74. Find the first 40 terms of the sequence defined by

$$a_{n+1} = \begin{cases} \frac{1}{2}a_n & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$$

and $a_1 = 11$. Do the same if $a_1 = 25$. Make a conjecture about this type of sequence.

75. For what values of r is the sequence $\{nr^n\}$ convergent?

76. (a) If $\{a_n\}$ is convergent, show that

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$$

(b) A sequence $\{a_n\}$ is defined by $a_1 = 1$ and $a_{n+1} = 1/(1 + a_n)$ for $n \geqslant 1$. Assuming that $\{a_n\}$ is convergent, find its limit.

77. Suppose you know that $\{a_n\}$ is a decreasing sequence and all its terms lie between the numbers 5 and 8. Explain why the sequence has a limit. What can you say about the value of the limit?

78–84 Determine whether the sequence is increasing, decreasing, or not monotonic. Is the sequence bounded?

78. $a_n = \cos n$

79. $a_n = \dfrac{1}{2n + 3}$

80. $a_n = \dfrac{1 - n}{2 + n}$

81. $a_n = n(-1)^n$

82. $a_n = 2 + \dfrac{(-1)^n}{n}$

83. $a_n = 3 - 2ne^{-n}$

84. $a_n = n^3 - 3n + 3$

---

85. Find the limit of the sequence

$$\left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots\right\}$$

86. A sequence $\{a_n\}$ is given by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$.

(a) By induction or otherwise, show that $\{a_n\}$ is increasing and bounded above by 3. Apply the Monotonic Sequence Theorem to show that $\lim_{n\to\infty} a_n$ exists.

(b) Find $\lim_{n\to\infty} a_n$.

87. Show that the sequence defined by

$$a_1 = 1 \qquad a_{n+1} = 3 - \frac{1}{a_n}$$

is increasing and $a_n < 3$ for all n. Deduce that $\{a_n\}$ is convergent and find its limit.

88. Show that the sequence defined by

$$a_1 = 2 \qquad a_{n+1} = \frac{1}{3 - a_n}$$

satisfies $0 < a_n \leqslant 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.

89. (a) Fibonacci posed the following problem:

> Suppose that rabbits live forever and that every month each pair produces a new pair which becomes productive at age 2 months. If we start with one newborn pair, how many pairs of rabbits will we have in the nth month?

Show that the answer is f_n, where $\{f_n\}$ is the Fibonacci sequence defined in Example 3(c).

(b) Let $a_n = f_{n+1}/f_n$ and show that $a_{n-1} = 1 + 1/a_{n-2}$. Assuming that $\{a_n\}$ is convergent, find its limit.

90. (a) Let $a_1 = a$, $a_2 = f(a)$, $a_3 = f(a_2) = f(f(a))$, . . . , $a_{n+1} = f(a_n)$, where f is a continuous function. If $\lim_{n\to\infty} a_n = L$, show that $f(L) = L$.

(b) Illustrate part (a) by taking $f(x) = \cos x$, $a = 1$, and estimating the value of L to five decimal places.

91. (a) Use a graph to guess the value of the limit

$$\lim_{n\to\infty} \frac{n^5}{n!}$$

(b) Use a graph of the sequence in part (a) to find the smallest values of N that correspond to $\varepsilon = 0.1$ and $\varepsilon = 0.001$ in Definition 2.

92. Use Definition 2 directly to prove that $\lim_{n\to\infty} r^n = 0$ when $|r| < 1$.

93. Prove Theorem 6.
[*Hint:* Use either Definition 2 or the Squeeze Theorem.]

94. Use Theorem 7 to prove the Power Law:

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \qquad \text{if } p > 0 \text{ and } a_n > 0$$

95. Prove that if $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}$ is bounded, then $\lim_{n\to\infty} (a_n b_n) = 0$.

96. Let $a_n = (1 + 1/n)^n$.

(a) Show that if $0 \leqslant a < b$, then

$$\frac{b^{n+1} - a^{n+1}}{b - a} < (n + 1)b^n$$

(b) Deduce that $b^n[(n + 1)a - nb] < a^{n+1}$.

(c) Use $a = 1 + 1/(n + 1)$ and $b = 1 + 1/n$ in part (b) to show that $\{a_n\}$ is increasing.

(d) Use $a = 1$ and $b = 1 + 1/(2n)$ in part (b) to show that $a_{2n} < 4$.

(e) Use parts (c) and (d) to show that $a_n < 4$ for all n.

(f) Use Theorem 12 to show that $\lim_{n\to\infty} (1 + 1/n)^n$ exists. (The limit is e. See Equation 3.6.6.)

97. Let a and b be positive numbers with $a > b$. Let a_1 be their arithmetic mean and b_1 their geometric mean:

$$a_1 = \frac{a + b}{2} \qquad b_1 = \sqrt{ab}$$

Repeat this process so that, in general,

$$a_{n+1} = \frac{a_n + b_n}{2} \qquad b_{n+1} = \sqrt{a_n b_n}$$

(a) Use mathematical induction to show that

$$a_n > a_{n+1} > b_{n+1} > b_n$$

(b) Deduce that both $\{a_n\}$ and $\{b_n\}$ are convergent.

(c) Show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Gauss called the common value of these limits the **arithmetic-geometric mean** of the numbers a and b.

98. (a) Show that if $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$, then $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.

(b) If $a_1 = 1$ and

$$a_{n+1} = 1 + \frac{1}{1 + a_n}$$

find the first eight terms of the sequence $\{a_n\}$. Then use part (a) to show that $\lim_{n\to\infty} a_n = \sqrt{2}$. This gives the **continued fraction expansion**

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}$$

99. The size of an undisturbed fish population has been modeled by the formula

$$p_{n+1} = \frac{bp_n}{a + p_n}$$

where p_n is the fish population after n years and a and b are positive constants that depend on the species and its environment. Suppose that the population in year 0 is $p_0 > 0$.

(a) Show that if $\{p_n\}$ is convergent, then the only possible values for its limit are 0 and $b - a$.

(b) Show that $p_{n+1} < (b/a)p_n$.

(c) Use part (b) to show that if $a > b$, then $\lim_{n\to\infty} p_n = 0$; in other words, the population dies out.

(d) Now assume that $a < b$. Show that if $p_0 < b - a$, then $\{p_n\}$ is increasing and $0 < p_n < b - a$. Also show that if $p_0 > b - a$, then $\{p_n\}$ is decreasing and $p_n > b - a$. Deduce that if $a < b$, then $\lim_{n\to\infty} p_n = b - a$.

DISCOVERY PROJECT T LOGISTIC SEQUENCES

A sequence that arises in ecology as a model for population growth is defined by the **logistic difference equation**

$$p_{n+1} = kp_n(1 - p_n)$$

where p_n measures the size of the population of the nth generation of a single species. To keep the numbers manageable, we take p_n to be a fraction of the maximal size of the population, so $0 \leqslant p_n \leqslant 1$. Notice that the form of this equation is similar to the logistic differential equation in Section 9.4. The discrete model—with sequences instead of continuous functions—is preferable for modeling insect populations, where mating and death occur in a periodic fashion.

An ecologist is interested in predicting the size of the population as time goes on, and asks these questions: Will it stabilize at a limiting value? Will it change in a cyclical fashion? Or will it exhibit random behavior?

Write a program to compute the first n terms of this sequence starting with an initial population p_0, where $0 < p_0 < 1$. Use this program to do the following.

1. Calculate 20 or 30 terms of the sequence for $p_0 = \frac{1}{2}$ and for two values of k such that $1 < k < 3$. Graph each sequence. Do the sequences appear to converge? Repeat for a different value of p_0 between 0 and 1. Does the limit depend on the choice of p_0? Does it depend on the choice of k?

2. Calculate terms of the sequence for a value of k between 3 and 3.4 and plot them. What do you notice about the behavior of the terms?

3. Experiment with values of k between 3.4 and 3.5. What happens to the terms?

4. For values of k between 3.6 and 4, compute and plot at least 100 terms and comment on the behavior of the sequence. What happens if you change p_0 by 0.001? This type of behavior is called *chaotic* and is exhibited by insect populations under certain conditions.

11.2 Series

Recall from Section 11.1 that Zeno, in one of his paradoxes, observed that in order for a man standing in a room to walk to a wall, he would first have to walk half the distance to the wall, then half the remaining distance $\left(\frac{1}{4}\text{ of the total}\right)$, and then again half of what still remains $\left(\frac{1}{8}\right)$, and so on (see Figure 1). Because this process can always be continued, Zeno argued that the man can never reach the wall.

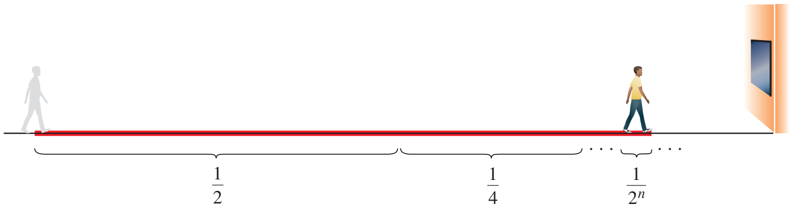


FIGURE 1
At the nth stage, the man has walked a total distance of $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$.

Of course, we know that the man can actually reach the wall, so this suggests that perhaps the total distance the man walks can be expressed as the sum of infinitely many smaller distances as follows:

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots$$

Zeno was arguing that it doesn't make sense to add infinitely many numbers together. But there are other situations in which we implicitly use infinite sums. For instance, in decimal notation, the value of π is

$$\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ldots$$

With the help of computers, researchers have found decimal approximations for π accurate to tens of trillions of decimal places.

The convention behind our decimal notation is that this number can be written as the infinite sum

$$\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \cdots$$

We can't literally add an infinite number of terms, but the more terms we add, the closer we get to the actual value of π.

Infinite Series

If we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form

$$\boxed{1} \qquad a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an **infinite series** (or just a **series**) and is denoted by the symbol

$$\sum_{n=1}^{\infty} a_n \qquad \text{or} \qquad \sum a_n$$

In general, does it make sense to talk about the sum of infinitely many numbers? For example, it would be impossible to find a finite sum for the series

$$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$$

because if we start adding the terms, then we get cumulative sums that grow increasingly larger.

However, consider the series of distances from Zeno's paradox:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots + \frac{1}{2^n} + \cdots$$

n	Sum of first n terms
1	0.50000000
2	0.75000000
3	0.87500000
4	0.93750000
5	0.96875000
6	0.98437500
7	0.99218750
10	0.99902344
15	0.99996948
20	0.99999905
25	0.99999997

If we start adding the terms, and we keep track of the subtotals as we go, we get $\frac{1}{2}$, $\frac{3}{4}$ (the sum of the first two terms), $\frac{7}{8}$ (first three terms), $\frac{15}{16}$, $\frac{31}{32}$, $\frac{63}{64}$, and so on. The table in the margin shows that as we add more and more terms, these *partial sums* become closer and closer to 1. In fact, you can verify that the nth partial sum is given by

$$\frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$$

and we can see that by adding sufficiently many terms of the series (making n sufficiently large), the partial sums can be made as close to 1 as we like. So it seems reasonable to say that the sum of this infinite series is 1 and to write

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1$$

We use a similar idea to determine whether or not a general series Σa_n has a sum. We consider the **partial sums**

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

and, in general,

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit. If $\lim_{n\to\infty} s_n$ exists (as a finite number), then we call it the sum of the infinite series Σa_n.

2 Definition Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series Σa_n is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series.

If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

Compare with the improper integral

$$\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_1^t f(x)\,dx$$

To find this integral we integrate from 1 to t and then let $t \to \infty$. For a series, we sum from 1 to n and then let $n \to \infty$.

Thus the sum of a series is the limit of the sequence of partial sums. So when we write $\sum_{n=1}^{\infty} a_n = s$, we mean that by adding sufficiently many terms of the series we can get as close as we like to the number s. Notice that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

EXAMPLE 1 Suppose we know that the sum of the first n terms of the series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = a_1 + a_2 + \cdots + a_n = \frac{2n}{3n+5}$$

Then the sum of the series is the limit of the sequence $\{s_n\}$:

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2n}{3n+5} = \lim_{n\to\infty} \frac{2}{3+\dfrac{5}{n}} = \frac{2}{3} \qquad ■$$

In Example 1 we were *given* an expression for the sum of the first n terms. In the following example we will *find* an expression for the nth partial sum.

EXAMPLE 2 Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

SOLUTION We use the definition of a convergent series and compute the partial sums.

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$$

We can simplify this expression if we use the partial fraction decomposition

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

(see Section 7.4). Thus we have

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1} \right) \\ &= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

Notice that the terms cancel in pairs. This is an example of a **telescoping sum**: because of all the cancellations, the sum collapses (like a pirate's collapsing telescope) into just two terms.

and so $$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$$

Therefore the given series is convergent and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \qquad ■$$

Figure 2 illustrates Example 2 by showing the graphs of the sequence of terms $a_n = 1/[n(n+1)]$ and the sequence $\{s_n\}$ of partial sums. Notice that $a_n \to 0$ and $s_n \to 1$. See Exercises 82 and 83 for two geometric interpretations of Example 2.

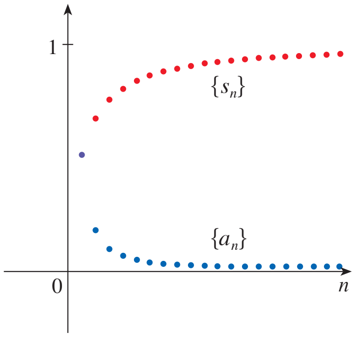


FIGURE 2

Sum of a Geometric Series

An important example of an infinite series is the **geometric series**

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

Each term is obtained from the preceding one by multiplying it by the **common ratio** r. (The series that arises from Zeno's paradox is the special case where $a = \frac{1}{2}$ and $r = \frac{1}{2}$.)

If $r = 1$, then $s_n = a + a + \cdots + a = na \to \pm\infty$. Since $\lim_{n\to\infty} s_n$ doesn't exist, the geometric series diverges in this case.

Figure 3 provides a geometric demonstration of the formula for the sum of a geometric series. If the triangles are constructed as shown and s is the sum of the series, then, by similar triangles,

$$\frac{s}{a} = \frac{a}{a - ar} \quad \text{so} \quad s = \frac{a}{1 - r}$$

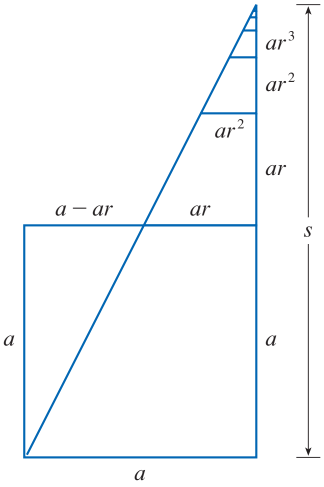


FIGURE 3

If $r \neq 1$, we have

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

and

$$rs_n = \phantom{a +{}} ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

Subtracting these equations, we get

$$s_n - rs_n = a - ar^n$$

$$\boxed{s_n = \frac{a(1 - r^n)}{1 - r}} \tag{3}$$

If $-1 < r < 1$, we know from (11.1.9) that $r^n \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{a}{1 - r} \cdot \lim_{n\to\infty} r^n = \frac{a}{1 - r}$$

Thus when $|r| < 1$ the geometric series is convergent and its sum is $a/(1 - r)$.

If $r \leqslant -1$ or $r > 1$, the sequence $\{r^n\}$ is divergent by (11.1.9) and so, by Equation 3, $\lim_{n\to\infty} s_n$ does not exist. Therefore the geometric series diverges in those cases. We summarize these results as follows.

4 The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \qquad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

In words: the sum of a convergent geometric series is

$$\frac{\text{first term}}{1 - \text{common ratio}}$$

EXAMPLE 3 Find the sum of the geometric series

$$5 - \tfrac{10}{3} + \tfrac{20}{9} - \tfrac{40}{27} + \cdots$$

SOLUTION The first term is $a = 5$ and the common ratio is $r = -\frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series is convergent by (4) and its sum is

$$\frac{5}{1 - \left(-\frac{2}{3}\right)} = \frac{5}{\frac{5}{3}} = 3$$

■

What do we really mean when we say that the sum of the series in Example 3 is 3? Of course, we can't literally add an infinite number of terms, one by one. But, according to Definition 2, the total sum is the limit of the sequence of partial sums. So, by taking the sum of sufficiently many terms, we can get as close as we like to the number 3. The table shows the first 10 partial sums s_n and the graph in Figure 4 shows how the sequence of partial sums approaches 3.

n	s_n
1	5.000000
2	1.666667
3	3.888889
4	2.407407
5	3.395062
6	2.736626
7	3.175583
8	2.882945
9	3.078037
10	2.947975

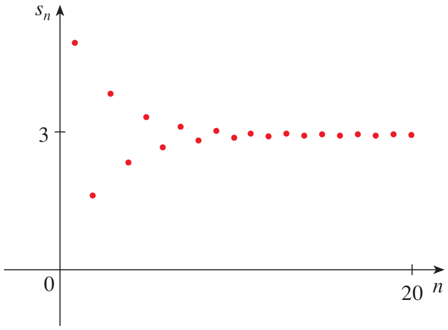


FIGURE 4

EXAMPLE 4 Is the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?

SOLUTION Let's rewrite the nth term of the series in the form ar^{n-1}:

Another way to identify a and r is to write out the first few terms:

$$4 + \tfrac{16}{3} + \tfrac{64}{9} + \cdots$$

$$\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{\infty} (2^2)^n 3^{-(n-1)} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4\left(\tfrac{4}{3}\right)^{n-1}$$

We recognize this series as a geometric series with $a = 4$ and $r = \frac{4}{3}$. Since $r > 1$, the series diverges by (4). ■

EXAMPLE 5 A drug is administered to a patient at the same time every day. Suppose the concentration of the drug is C_n (measured in mg/mL) after the injection on the nth day. Before the injection the next day, only 30% of the drug remains in the bloodstream and the daily dose raises the concentration by 0.2 mg/mL.

(a) Find the concentration just after the third injection.
(b) What is the concentration just after the nth dose?
(c) What is the limiting concentration?

SOLUTION

(a) Just before the daily dose of medication is administered, the concentration is reduced to 30% of the preceding day's concentration, that is, $0.3C_n$. With the new dose, the concentration is increased by 0.2 mg/mL and so

$$C_{n+1} = 0.2 + 0.3C_n$$

Starting with $C_0 = 0$ and putting $n = 0, 1, 2$ into this equation, we get

$$C_1 = 0.2 + 0.3C_0 = 0.2$$

$$C_2 = 0.2 + 0.3C_1 = 0.2 + 0.2(0.3) = 0.26$$

$$C_3 = 0.2 + 0.3C_2 = 0.2 + 0.2(0.3) + 0.2(0.3)^2 = 0.278$$

The concentration after three days is 0.278 mg/mL.

(b) After the nth dose the concentration is

$$C_n = 0.2 + 0.2(0.3) + 0.2(0.3)^2 + \cdots + 0.2(0.3)^{n-1}$$

This is a finite geometric series with $a = 0.2$ and $r = 0.3$, so by Formula 3 we have

$$C_n = \frac{0.2[1 - (0.3)^n]}{1 - 0.3} = \frac{2}{7}[1 - (0.3)^n] \text{ mg/mL}$$

(c) Because $0.3 < 1$, we know that $\lim_{n\to\infty} (0.3)^n = 0$. So the limiting concentration is

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty} \frac{2}{7}[1 - (0.3)^n] = \frac{2}{7}(1 - 0) = \frac{2}{7} \text{ mg/mL} \qquad \blacksquare$$

EXAMPLE 6 Write the number $2.3\overline{17} = 2.3171717\ldots$ as a ratio of integers.

SOLUTION

$$2.3171717\ldots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

After the first term we have a geometric series with $a = 17/10^3$ and $r = 1/10^2$. Therefore

$$2.3\overline{17} = 2.3 + \frac{\dfrac{17}{10^3}}{1 - \dfrac{1}{10^2}} = 2.3 + \frac{\dfrac{17}{1000}}{\dfrac{99}{100}}$$

$$= \frac{23}{10} + \frac{17}{990} = \frac{1147}{495} \qquad \blacksquare$$

EXAMPLE 7 Find the sum of the series $\sum_{n=0}^{\infty} x^n$, where $|x| < 1$.

SOLUTION Notice that this series starts with $n = 0$ and so the first term is $x^0 = 1$. (With series, we adopt the convention that $x^0 = 1$ even when $x = 0$.) Thus

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

This is a geometric series with $a = 1$ and $r = x$. Since $|r| = |x| < 1$, it converges and (4) gives

$$\boxed{5} \qquad \sum_{n=0}^{\infty} x^n = \frac{1}{1 - x} \qquad \blacksquare$$

■ Test for Divergence

Recall that a series is divergent if its sequence of partial sums is a divergent sequence.

EXAMPLE 8 Show that the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

SOLUTION For this particular series it's convenient to consider the partial sums s_2, s_4, s_8, s_{16}, s_{32}, . . . and show that they become large.

$$s_2 = 1 + \tfrac{1}{2}$$

$$s_4 = 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) > 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) = 1 + \tfrac{2}{2}$$

$$\begin{aligned} s_8 &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) \\ &> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\right) \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{3}{2} \end{aligned}$$

$$\begin{aligned} s_{16} &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \cdots + \tfrac{1}{16}\right) \\ &> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{16} + \cdots + \tfrac{1}{16}\right) \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{4}{2} \end{aligned}$$

Similarly, $s_{32} > 1 + \tfrac{5}{2}$, $s_{64} > 1 + \tfrac{6}{2}$, and in general

$$s_{2^n} > 1 + \frac{n}{2}$$

This shows that $s_{2^n} \to \infty$ as $n \to \infty$ and so $\{s_n\}$ is divergent. Therefore the harmonic series diverges. ■

The method used in Example 8 for showing that the harmonic series diverges was developed by the French scholar Nicole Oresme (1323–1382).

6 Theorem If the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

PROOF Let $s_n = a_1 + a_2 + \cdots + a_n$. Then $a_n = s_n - s_{n-1}$. Since $\Sigma\, a_n$ is convergent, the sequence $\{s_n\}$ is convergent. Let $\lim_{n\to\infty} s_n = s$. Since $n - 1 \to \infty$ as $n \to \infty$, we also have $\lim_{n\to\infty} s_{n-1} = s$. Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0 \qquad ■$$

NOTE With any *series* $\Sigma\, a_n$ we associate two *sequences:* the sequence $\{s_n\}$ of its partial sums and the sequence $\{a_n\}$ of its terms. If $\Sigma\, a_n$ is convergent, then the limit of the sequence $\{s_n\}$ is s (the sum of the series) and, as Theorem 6 asserts, the limit of the sequence $\{a_n\}$ is 0.

WARNING The converse of Theorem 6 is not true in general. If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\Sigma\, a_n$ is convergent. Observe that for the harmonic series $\Sigma\, 1/n$ we have $a_n = 1/n \to 0$ as $n \to \infty$, but we showed in Example 8 that $\Sigma\, 1/n$ is divergent.

7 Test for Divergence If $\displaystyle\lim_{n\to\infty} a_n$ does not exist or if $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

The Test for Divergence follows from Theorem 6 because, if the series is not divergent, then it is convergent, and so $\lim_{n\to\infty} a_n = 0$.

EXAMPLE 9 Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

SOLUTION

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5+4/n^2} = \frac{1}{5} \neq 0$$

So the series diverges by the Test for Divergence. ■

NOTE If we find that $\lim_{n\to\infty} a_n \neq 0$, we know that Σa_n is divergent. If we find that $\lim_{n\to\infty} a_n = 0$, this fact tells us *nothing* about the convergence or divergence of Σa_n. Remember the warning given after Theorem 6: if $\lim_{n\to\infty} a_n = 0$, the series Σa_n might converge or it might diverge.

Properties of Convergent Series

The following properties of convergent series follow from the corresponding Limit Laws for Sequences in Section 11.1.

8 **Theorem** If Σa_n and Σb_n are convergent series, then so are the series Σca_n (where c is a constant), $\Sigma(a_n + b_n)$, and $\Sigma(a_n - b_n)$, and

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

We prove part (ii); the other parts are left as exercises.

PROOF OF PART (ii) Let

$$s_n = \sum_{i=1}^{n} a_i \qquad s = \sum_{n=1}^{\infty} a_n \qquad t_n = \sum_{i=1}^{n} b_i \qquad t = \sum_{n=1}^{\infty} b_n$$

The nth partial sum for the series $\Sigma(a_n + b_n)$ is

$$u_n = \sum_{i=1}^{n} (a_i + b_i)$$

and, using Equation 5.2.10, we have

$$\begin{aligned}\lim_{n\to\infty} u_n &= \lim_{n\to\infty} \sum_{i=1}^{n} (a_i + b_i) = \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i\right)\\ &= \lim_{n\to\infty} \sum_{i=1}^{n} a_i + \lim_{n\to\infty} \sum_{i=1}^{n} b_i\\ &= \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n = s + t\end{aligned}$$

Therefore $\Sigma\,(a_n + b_n)$ is convergent and its sum is

$$\sum_{n=1}^{\infty} (a_n + b_n) = s + t = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$ ■

EXAMPLE 10 Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right)$.

SOLUTION The series $\Sigma\, 1/2^n$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$, so

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

In Example 2 we found that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

So, by Theorem 8, the given series is convergent and

$$\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right) = 3 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$= 3 \cdot 1 + 1 = 4$$ ■

NOTE A finite number of terms doesn't affect the convergence or divergence of a series. For instance, suppose that we were able to show that the series

$$\sum_{n=4}^{\infty} \frac{n}{n^3 + 1}$$

is convergent. Since

$$\sum_{n=1}^{\infty} \frac{n}{n^3 + 1} = \frac{1}{2} + \frac{2}{9} + \frac{3}{28} + \sum_{n=4}^{\infty} \frac{n}{n^3 + 1}$$

it follows that the entire series $\Sigma_{n=1}^{\infty}\, n/(n^3 + 1)$ is convergent. Similarly, if it is known that the series $\Sigma_{n=N+1}^{\infty}\, a_n$ converges, then the full series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

is also convergent.

11.2 Exercises

1. (a) What is the difference between a sequence and a series?
(b) What is a convergent series? What is a divergent series?

2. Explain what it means to say that $\Sigma_{n=1}^{\infty}\, a_n = 5$.

3–4 Calculate the sum of the series $\Sigma_{n=1}^{\infty}\, a_n$ whose partial sums are given.

3. $s_n = 2 - 3(0.8)^n$

4. $s_n = \dfrac{n^2 - 1}{4n^2 + 1}$

5–10 Calculate the first eight terms of the sequence of partial sums correct to four decimal places. Does it appear that the series is convergent or divergent?

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

7. $\displaystyle\sum_{n=1}^{\infty} \sin n$

8. $\displaystyle\sum_{n=1}^{\infty} (-1)^n n$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4 + n^2}$ **10.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$

11–14 Find at least 10 partial sums of the series. Graph both the sequence of terms and the sequence of partial sums on the same screen. Does it appear that the series is convergent or divergent? If it is convergent, find the sum. If it is divergent, explain why.

11. $\displaystyle\sum_{n=1}^{\infty} \frac{6}{(-3)^n}$ **12.** $\displaystyle\sum_{n=1}^{\infty} \cos n$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+4}}$ **14.** $\displaystyle\sum_{n=1}^{\infty} \frac{7^{n+1}}{10^n}$

---

15. Let $a_n = \dfrac{2n}{3n+1}$.

(a) Determine whether $\{a_n\}$ is convergent.

(b) Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

16. (a) Explain the difference between

$$\sum_{i=1}^{n} a_i \quad \text{and} \quad \sum_{j=1}^{n} a_j$$

(b) Explain the difference between

$$\sum_{i=1}^{n} a_i \quad \text{and} \quad \sum_{i=1}^{n} a_j$$

17–22 Determine whether the series is convergent or divergent by expressing s_n as a telescoping sum (as in Example 2). If it is convergent, find its sum.

17. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n+2} - \frac{1}{n}\right)$ **18.** $\displaystyle\sum_{n=4}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$ **20.** $\displaystyle\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$

21. $\displaystyle\sum_{n=1}^{\infty} \left(e^{1/n} - e^{1/(n+1)}\right)$ **22.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3 - n}$

23–32 Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

23. $3 - 4 + \frac{16}{3} - \frac{64}{9} + \cdots$ **24.** $4 + 3 + \frac{9}{4} + \frac{27}{16} + \cdots$

25. $10 - 2 + 0.4 - 0.08 + \cdots$

26. $2 + 0.5 + 0.125 + 0.03125 + \cdots$

27. $\displaystyle\sum_{n=1}^{\infty} 12(0.73)^{n-1}$ **28.** $\displaystyle\sum_{n=1}^{\infty} \frac{5}{\pi^n}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$ **30.** $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+1}}{(-2)^n}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}}$ **32.** $\displaystyle\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n}$

33–50 Determine whether the series is convergent or divergent. If it is convergent, find its sum.

33. $\dfrac{1}{3} + \dfrac{1}{6} + \dfrac{1}{9} + \dfrac{1}{12} + \dfrac{1}{15} + \cdots$

34. $\dfrac{1}{2} + \dfrac{2}{3} + \dfrac{3}{4} + \dfrac{4}{5} + \dfrac{5}{6} + \dfrac{6}{7} + \cdots$

35. $\dfrac{2}{5} + \dfrac{4}{25} + \dfrac{8}{125} + \dfrac{16}{625} + \dfrac{32}{3125} + \cdots$

36. $\dfrac{1}{3} + \dfrac{2}{9} + \dfrac{1}{27} + \dfrac{2}{81} + \dfrac{1}{243} + \dfrac{2}{729} + \cdots$

37. $\displaystyle\sum_{n=1}^{\infty} \frac{2+n}{1-2n}$ **38.** $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{k^2 - 2k + 5}$

39. $\displaystyle\sum_{n=1}^{\infty} 3^{n+1} 4^{-n}$ **40.** $\displaystyle\sum_{n=1}^{\infty} [(-0.2)^n + (0.6)^{n-1}]$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4 + e^{-n}}$ **42.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 4^n}{e^n}$

43. $\displaystyle\sum_{k=1}^{\infty} (\sin 100)^k$ **44.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1 + \left(\frac{2}{3}\right)^n}$

45. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$ **46.** $\displaystyle\sum_{k=0}^{\infty} \left(\sqrt{2}\right)^{-k}$

47. $\displaystyle\sum_{n=1}^{\infty} \arctan n$ **48.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right)$

49. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{e^n} + \frac{1}{n(n+1)}\right)$ **50.** $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

---

51. Let $x = 0.99999\ldots$.

(a) Do you think that $x < 1$ or $x = 1$?

(b) Sum a geometric series to find the value of x.

(c) How many decimal representations does the number 1 have?

(d) Which numbers have more than one decimal representation?

52. A sequence of terms is defined by

$$a_1 = 1 \qquad a_n = (5 - n)a_{n-1}$$

Calculate $\sum_{n=1}^{\infty} a_n$.

53–58 Express the number as a ratio of integers.

53. $0.\overline{8} = 0.8888\ldots$

54. $0.\overline{46} = 0.46464646\ldots$

55. $2.\overline{516} = 2.516516516\ldots$

56. $10.1\overline{35} = 10.135353535\ldots$

57. $1.234\overline{567}$

58. $5.\overline{71358}$

59–66 Find the values of x for which the series converges. Find the sum of the series for those values of x.

59. $\displaystyle\sum_{n=1}^{\infty} (-5)^n x^n$

60. $\displaystyle\sum_{n=1}^{\infty} (x+2)^n$

61. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$

62. $\displaystyle\sum_{n=0}^{\infty} (-4)^n (x-5)^n$

63. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{x^n}$

64. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n}$

65. $\displaystyle\sum_{n=0}^{\infty} e^{nx}$

66. $\displaystyle\sum_{n=0}^{\infty} \frac{\sin^n x}{3^n}$

T **67–68** Use the partial fraction command on a computer algebra system to find a convenient expression for the partial sum, and then use this expression to find the sum of the series. Check your answer by using the CAS to sum the series directly.

67. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+3n+1}{(n^2+n)^3}$

68. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^5 - 5n^3 + 4n}$

---

69. If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = \frac{n-1}{n+1}$$

find a_n and $\sum_{n=1}^{\infty} a_n$.

70. If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is $s_n = 3 - n2^{-n}$, find a_n and $\sum_{n=1}^{\infty} a_n$.

71. A doctor prescribes a 100-mg antibiotic tablet to be taken every eight hours. It is known that the body eliminates 75% of the drug in eight hours.

(a) How much of the drug is in the body just after the second tablet is taken? After the third tablet?

(b) If Q_n is the quantity of the antibiotic in the body just after the nth tablet is taken, find an equation that expresses Q_{n+1} in terms of Q_n.

(c) What quantity of the antibiotic remains in the body in the long run?

72. A patient is injected with a drug every 12 hours. Immediately before each injection the concentration of the drug has been reduced by 90% and the new dose increases the concentration by 1.5 mg/L.

(a) What is the concentration after three doses?

(b) If C_n is the concentration after the nth dose, find a formula for C_n as a function of n.

(c) What is the limiting value of the concentration?

73. A patient takes 150 mg of a drug at the same time every day. It is known that the body eliminates 95% of the drug in 24 hours.

(a) What quantity of the drug is in the body after the third tablet? After the nth tablet?

(b) What quantity of the drug remains in the body in the long run?

74. After injection of a dose D of insulin, the concentration of insulin in a patient's system decays exponentially and so it can be written as De^{-at}, where t represents time in hours and a is a positive constant.

(a) If a dose D is injected every T hours, write an expression for the sum of the residual concentrations just before the $(n+1)$st injection.

(b) Determine the limiting pre-injection concentration.

(c) If the concentration of insulin must always remain at or above a critical value C, determine a minimal dosage D in terms of C, a, and T.

75. When money is spent on goods and services, those who receive the money also spend some of it. The people receiving some of the twice-spent money will spend some of that, and so on. Economists call this chain reaction the *multiplier effect*. In a hypothetical isolated community, the local government begins the process by spending D dollars. Suppose that each recipient of spent money spends $100c\%$ and saves $100s\%$ of the money that he or she receives. The values c and s are called the *marginal propensity to consume* and the *marginal propensity to save* and, of course, $c + s = 1$.

(a) Let S_n be the total spending that has been generated after n transactions. Find an equation for S_n.

(b) Show that $\lim_{n\to\infty} S_n = kD$, where $k = 1/s$. The number k is called the *multiplier*. What is the multiplier if the marginal propensity to consume is 80%?

Note: The federal government uses this principle to justify deficit spending. Banks use this principle to justify lending a large percentage of the money that they receive in deposits.

76. A certain ball has the property that each time it falls from a height h onto a hard, level surface, it rebounds to a height rh, where $0 < r < 1$. Suppose that the ball is dropped from an initial height of H meters.

(a) Assuming that the ball continues to bounce indefinitely, find the total distance that it travels.

(b) Calculate the total time that the ball travels. (Use the fact that the ball falls $\frac{1}{2}gt^2$ meters in t seconds.)

(c) Suppose that each time the ball strikes the surface with velocity v it rebounds with velocity $-kv$, where $0 < k < 1$. How long will it take for the ball to come to rest?

77. Find the value of c if $\displaystyle\sum_{n=2}^{\infty} (1+c)^{-n} = 2$.

78. Find the value of c such that $\displaystyle\sum_{n=0}^{\infty} e^{nc} = 10$.

79–81 The Harmonic Series Diverges In Example 8 we proved that the harmonic series diverges. Here we outline additional methods of proving this fact. In each case, assume that the series converges with sum S, and show that this assumption leads to a contradiction.

79. $$S = \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) + \cdots$$
$$> \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \cdots = S$$

80. $$S = 1 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) +$$
$$\left(\frac{1}{8} + \frac{1}{9} + \frac{1}{10}\right) + \cdots > 1 + \frac{3}{3} + \frac{3}{6} + \frac{3}{9} + \cdots = 1 + S$$

Hint: First show that $\dfrac{1}{n-1} + \dfrac{1}{n+1} > \dfrac{2}{n}$.

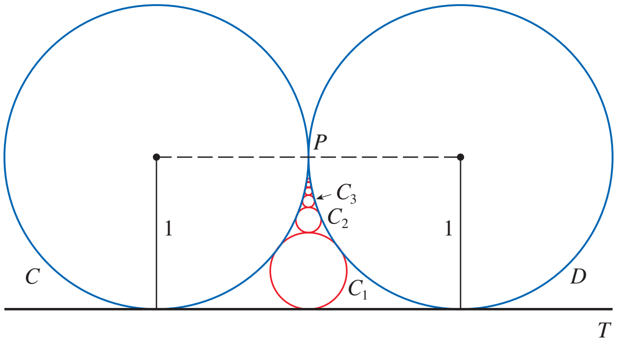
81. $$e^{1+(1/2)+(1/3)+\cdots+(1/n)} = e^1 \cdot e^{1/2} \cdot e^{1/3} \cdot \cdots \cdot e^{1/n}$$
$$> (1+1)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\cdots\left(1 + \frac{1}{n}\right) = n + 1$$

Hint: First show that $e^x > 1 + x$.

82. Graph the curves $y = x^n$, $0 \leqslant x \leqslant 1$, for $n = 0, 1, 2, 3, 4, \ldots$ on a common screen. By finding the areas between successive curves, give a geometric demonstration of the fact, shown in Example 2, that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

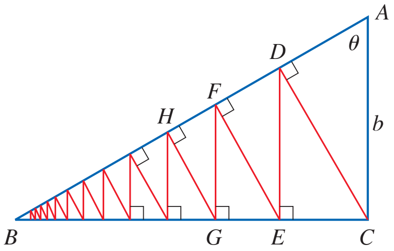
83. The figure shows two circles C and D of radius 1 that touch at P. The line T is a common tangent line; C_1 is the circle that touches C, D, and T; C_2 is the circle that touches C, D, and C_1; C_3 is the circle that touches C, D, and C_2. This procedure can be continued indefinitely and produces an infinite sequence of circles $\{C_n\}$. Find an expression for the diameter of C_n and thus provide another geometric demonstration of Example 2.



84. A right triangle ABC is given with $\angle A = \theta$ and $|AC| = b$. CD is drawn perpendicular to AB, DE is drawn perpendicular to BC, $EF \perp AB$, and this process is continued indefinitely, as shown in the figure. Find the total length of all the perpendiculars

$$|CD| + |DE| + |EF| + |FG| + \cdots$$

in terms of b and θ.



85. What is wrong with the following calculation?

$$\begin{aligned} 0 &= 0 + 0 + 0 + \cdots \\ &= (1-1) + (1-1) + (1-1) + \cdots \\ &= 1 - 1 + 1 - 1 + 1 - 1 + \cdots \\ &= 1 + (-1+1) + (-1+1) + (-1+1) + \cdots \\ &= 1 + 0 + 0 + 0 + \cdots = 1 \end{aligned}$$

(Guido Ubaldus thought that this proved the existence of God because "something has been created out of nothing.")

86. Suppose that $\sum_{n=1}^{\infty} a_n$ $(a_n \neq 0)$ is known to be a convergent series. Prove that $\sum_{n=1}^{\infty} 1/a_n$ is a divergent series.

87. (a) Prove part (i) of Theorem 8.
(b) Prove part (iii) of Theorem 8.

88. If $\sum a_n$ is divergent and $c \neq 0$, show that $\sum ca_n$ is divergent.

89. If $\sum a_n$ is convergent and $\sum b_n$ is divergent, show that the series $\sum (a_n + b_n)$ is divergent. [*Hint:* Argue by contradiction.]

90. If $\sum a_n$ and $\sum b_n$ are both divergent, is $\sum (a_n + b_n)$ necessarily divergent?

91. Suppose that a series $\sum a_n$ has positive terms and its partial sums s_n satisfy the inequality $s_n \leqslant 1000$ for all n. Explain why $\sum a_n$ must be convergent.

92. The Fibonacci sequence was defined in Section 11.1 by the equations

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2} \qquad n \geqslant 3$$

Show that each of the following statements is true.

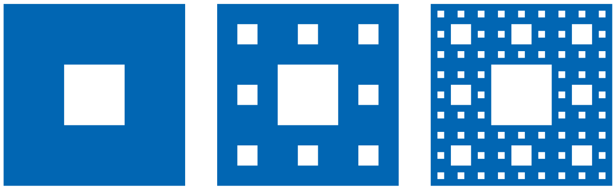
(a) $$\frac{1}{f_{n-1} f_{n+1}} = \frac{1}{f_{n-1} f_n} - \frac{1}{f_n f_{n+1}}$$

(b) $$\sum_{n=2}^{\infty} \frac{1}{f_{n-1} f_{n+1}} = 1$$ (c) $$\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1} f_{n+1}} = 2$$

93. The **Cantor set**, named after the German mathematician Georg Cantor (1845–1918), is constructed as follows. We start with the closed interval $[0, 1]$ and remove the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$. That leaves the two intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$ and we remove the open middle third of each. Four intervals remain and again we remove the open middle third of each of them. We continue this procedure indefinitely, at each step removing the open middle third of every interval that remains from the preceding step. The Cantor set consists of the numbers that remain in $[0, 1]$ after all those intervals have been removed.

(a) Show that the total length of all the intervals that are removed is 1. Despite that, the Cantor set contains infinitely many numbers. Give examples of some numbers in the Cantor set.

(b) The **Sierpinski carpet** is a two-dimensional counterpart of the Cantor set. It is constructed by removing the center one-ninth of a square of side 1, then removing the centers of the eight smaller remaining squares, and so on. (The figure shows the first three steps of the construction.) Show that the sum of the areas of the removed squares is 1. This implies that the Sierpinski carpet has area 0.



94. (a) A sequence $\{a_n\}$ is defined recursively by the equation $a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$ for $n \geqslant 3$, where a_1 and a_2 can be any real numbers. Experiment with various values of a_1 and a_2 and use a calculator to guess the limit of the sequence.

(b) Find $\lim_{n\to\infty} a_n$ in terms of a_1 and a_2 by expressing $a_{n+1} - a_n$ in terms of $a_2 - a_1$ and summing a series.

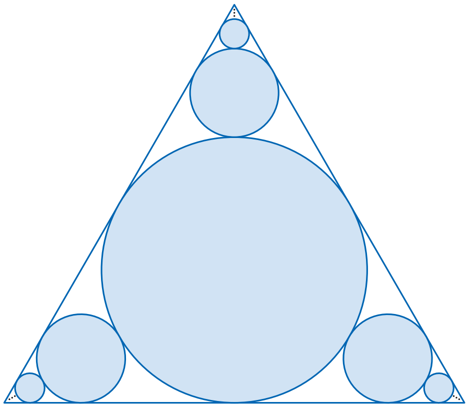
95. Consider the series $\sum_{n=1}^{\infty} n/(n+1)!$.

(a) Find the partial sums s_1, s_2, s_3, and s_4. Do you recognize the denominators? Use the pattern to guess a formula for s_n.

(b) Use mathematical induction to prove your guess.

(c) Show that the given infinite series is convergent, and find its sum.

96. The figure shows infinitely many circles approaching the vertices of an equilateral triangle, each circle touching other circles and sides of the triangle. If the triangle has sides of length 1, find the total area occupied by the circles.



11.3 The Integral Test and Estimates of Sums

In general, it is difficult to find the exact sum of a series. We were able to accomplish this for geometric series and for some telescoping series because in each of those cases we could find a simple formula for the nth partial sum s_n. But usually it isn't easy to discover such a formula. Therefore, in the next few sections, we develop several tests that enable us to determine whether a series is convergent or divergent without explicitly finding its sum. (In some cases, however, our methods will enable us to find good estimates of the sum.) Our first test involves improper integrals.

The Integral Test

We begin by investigating the series whose terms are the reciprocals of the squares of the positive integers:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

There's no simple formula for the sum s_n of the first n terms, but the computer-generated

n	$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

table of approximate values given in the margin suggests that the partial sums are approaching a number near 1.64 as $n \to \infty$ and so it looks as if the series is convergent.

We can confirm this impression with a geometric argument. Figure 1 shows the curve $y = 1/x^2$ and rectangles that lie below the curve. The base of each rectangle is an interval of length 1; the height is equal to the value of the function $y = 1/x^2$ at the right endpoint of the interval.

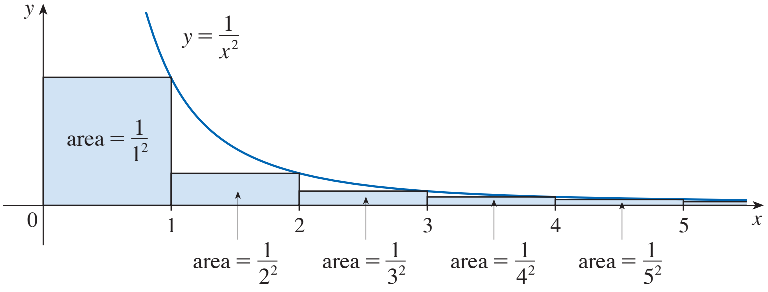


FIGURE 1

So the sum of the areas of the rectangles is

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

If we exclude the first rectangle, the total area of the remaining rectangles is smaller than the area under the curve $y = 1/x^2$ for $x \geqslant 1$, which is the value of the integral $\int_1^\infty (1/x^2)\,dx$. In Section 7.8 we discovered that this improper integral is convergent and has value 1. So the picture shows that all the partial sums are less than

$$\frac{1}{1^2} + \int_1^\infty \frac{1}{x^2}\,dx = 2$$

Thus the partial sums are bounded. We also know that the partial sums are increasing (because all the terms are positive). Therefore the partial sums converge (by the Monotonic Sequence Theorem) and so the series is convergent. The sum of the series (the limit of the partial sums) is also less than 2:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots < 2$$

[The exact sum of this series was found by the Swiss mathematician Leonhard Euler (1707–1783) to be $\pi^2/6$, but the proof of this fact is quite difficult. (See Problem 6 in the Problems Plus following Chapter 15.)]

Now let's look at the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots$$

n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

The table of values of s_n suggests that the partial sums aren't approaching a finite number, so we suspect that the given series may be divergent. Again we use a picture for confirmation. Figure 2 shows the curve $y = 1/\sqrt{x}$, but this time we use rectangles whose tops lie *above* the curve.

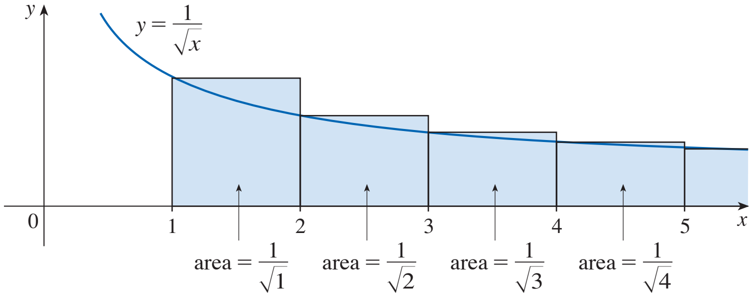


FIGURE 2

The base of each rectangle is an interval of length 1. The height is equal to the value of the function $y = 1/\sqrt{x}$ at the *left* endpoint of the interval. So the sum of the areas of all the rectangles is

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

This total area is greater than the area under the curve $y = 1/\sqrt{x}$ for $x \geqslant 1$, which is equal to the integral $\int_1^\infty \left(1/\sqrt{x}\right) dx$. But we know from Example 7.8.4 that this improper integral is divergent. In other words, the area under the curve is infinite. So the sum of the series must be infinite; that is, the series is divergent.

The same sort of geometric reasoning that we used for these two series can be used to prove the following test. (The proof is given at the end of this section.)

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^\infty f(x)\,dx$ is convergent. In other words:

(i) If $\displaystyle\int_1^\infty f(x)\,dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\displaystyle\int_1^\infty f(x)\,dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

NOTE When we use the Integral Test, it is not necessary to start the series or the integral at $n = 1$. For instance, in testing the series

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2} \qquad \text{we use} \qquad \int_4^\infty \frac{1}{(x-3)^2}\,dx$$

Also, it is not necessary that f be *always* decreasing. What is important is that f be *ultimately* decreasing, that is, decreasing for x larger than some number N. Then $\sum_{n=N}^{\infty} a_n$ is convergent, so $\sum_{n=1}^{\infty} a_n$ is convergent (see the note at the end of Section 11.2).

In order to use the Integral Test we need to be able to evaluate $\int_1^\infty f(x)\,dx$ and therefore we have to be able to find an antiderivative of f. Frequently this is difficult or impossible, so in the next three sections we develop other tests for convergence.

EXAMPLE 1 Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

SOLUTION The function $f(x) = 1/(x^2+1)$ is continuous, positive, and decreasing on $[1, \infty)$ so we use the Integral Test:

$$\int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \tan^{-1}x\Big]_1^t$$
$$= \lim_{t\to\infty}\left(\tan^{-1}t - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Thus $\int_1^\infty 1/(x^2+1)\,dx$ is a convergent integral and so, by the Integral Test, the series $\Sigma\, 1/(n^2+1)$ is convergent. ■

EXAMPLE 2 For what values of p is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

SOLUTION If $p < 0$, then $\lim_{n\to\infty} (1/n^p) = \infty$. If $p = 0$, then $\lim_{n\to\infty} (1/n^p) = 1$. In either case $\lim_{n\to\infty} (1/n^p) \neq 0$, so the given series diverges by the Test for Divergence (11.2.7).

If $p > 0$, then the function $f(x) = 1/x^p$ is clearly continuous, positive, and decreasing on $[1, \infty)$. We found in Section 7.8 [see (7.8.2)] that

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \text{converges if } p > 1 \text{ and diverges if } p \leqslant 1$$

It follows from the Integral Test that the series $\Sigma\, 1/n^p$ converges if $p > 1$ and diverges if $0 < p \leqslant 1$. (For $p = 1$, this series is the harmonic series discussed in Example 11.2.8.) ■

The series in Example 2 is called the ***p*-series**. It is important in the rest of this chapter, so we summarize the results of Example 2 for future reference as follows.

1 The p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

EXAMPLE 3

(a) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$$

is convergent because it is a p-series with $p = 3 > 1$.

(b) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$$

is divergent because it is a p-series with $p = \frac{1}{3} < 1$. ■

We can think of the convergence of a series of positive terms as depending on how "rapidly" the terms of the series approach zero. For any p-series (with $p > 0$) the terms $a_n = 1/n^p$ tend toward zero, but they do so more rapidly for larger values of p.

⊘ **NOTE** We should *not* infer from the Integral Test that the sum of the series is equal to the value of the integral. In fact,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \text{whereas} \qquad \int_1^{\infty} \frac{1}{x^2}\,dx = 1$$

Therefore, in general,

$$\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x)\,dx$$

EXAMPLE 4 Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

SOLUTION The function $f(x) = (\ln x)/x$ is positive and continuous for $x > 1$ because the logarithm function is continuous. But it is not obvious whether or not f is decreasing, so we compute its derivative:

$$f'(x) = \frac{(1/x)x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

Thus $f'(x) < 0$ when $\ln x > 1$, that is, when $x > e$. It follows that f is decreasing when $x > e$ and so we can apply the Integral Test:

$$\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \frac{(\ln x)^2}{2}\bigg]_1^t$$

$$= \lim_{t\to\infty} \frac{(\ln t)^2}{2} = \infty$$

Since this improper integral is divergent, the series $\Sigma\,(\ln n)/n$ is also divergent by the Integral Test. ■

Estimating the Sum of a Series

Suppose we have been able to use the Integral Test to show that a series $\Sigma\,a_n$ is convergent and we now want to find an approximation to the sum s of the series. Of course, any partial sum s_n is an approximation to s because $\lim_{n\to\infty} s_n = s$. But how good is such an approximation? To find out, we need to estimate the size of the **remainder**

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

The remainder R_n is the error made when s_n, the sum of the first n terms, is used as an approximation to the total sum.

We use the same notation and ideas as in the Integral Test, assuming that f is decreasing on $[n, \infty)$. Comparing the areas of the rectangles with the area under $y = f(x)$ for $x \geqslant n$ in Figure 3, we see that

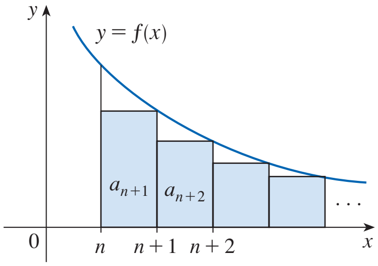


FIGURE 3

$$R_n = a_{n+1} + a_{n+2} + \cdots \leqslant \int_n^{\infty} f(x)\,dx$$

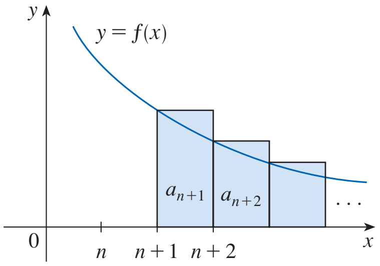


FIGURE 4

Similarly, we see from Figure 4 that

$$R_n = a_{n+1} + a_{n+2} + \cdots \geqslant \int_{n+1}^{\infty} f(x)\,dx$$

So we have proved the following error estimate.

$\boxed{2}$ **Remainder Estimate for the Integral Test** Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geqslant n$ and $\Sigma\, a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leqslant R_n \leqslant \int_{n}^{\infty} f(x)\,dx$$

EXAMPLE 5

(a) Approximate the sum of the series $\Sigma\, 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.
(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

SOLUTION In both parts (a) and (b) we need to know $\int_n^\infty f(x)\,dx$. With $f(x) = 1/x^3$, which satisfies the conditions of the Integral Test, we have

$$\int_n^\infty \frac{1}{x^3}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_n^t = \lim_{t\to\infty}\left(-\frac{1}{2t^2} + \frac{1}{2n^2}\right) = \frac{1}{2n^2}$$

(a) Approximating the sum of the series by the 10th partial sum, we have

$$\sum_{n=1}^{\infty}\frac{1}{n^3} \approx s_{10} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{10^3} \approx 1.1975$$

According to the remainder estimate in (2), we have

$$R_{10} \leqslant \int_{10}^\infty \frac{1}{x^3}\,dx = \frac{1}{2(10)^2} = \frac{1}{200}$$

So the size of the error is at most 0.005.

(b) Accuracy to within 0.0005 means that we have to find a value of n such that $R_n \leqslant 0.0005$. Since

$$R_n \leqslant \int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$$

we want

$$\frac{1}{2n^2} < 0.0005$$

Solving this inequality, we get

$$n^2 > \frac{1}{0.001} = 1000 \qquad \text{or} \qquad n > \sqrt{1000} \approx 31.6$$

We need 32 terms to ensure accuracy to within 0.0005. ■

If we add s_n to each side of the inequalities in (2), we get

$$\boxed{3} \qquad \boxed{s_n + \int_{n+1}^{\infty} f(x)\,dx \leqslant s \leqslant s_n + \int_n^{\infty} f(x)\,dx}$$

because $s_n + R_n = s$. The inequalities in (3) give a lower bound and an upper bound for s. They provide a more accurate approximation to the sum of the series than the partial sum s_n does.

Although Euler was able to calculate the exact sum of the p-series for $p = 2$, nobody has been able to find the exact sum for $p = 3$. In Example 6, however, we show how to *estimate* this sum.

EXAMPLE 6 Use (3) with $n = 10$ to estimate the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$.

SOLUTION The inequalities in (3) become

$$s_{10} + \int_{11}^{\infty} \frac{1}{x^3}\,dx \leqslant s \leqslant s_{10} + \int_{10}^{\infty} \frac{1}{x^3}\,dx$$

From Example 5 we know that

$$\int_n^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2n^2}$$

so

$$s_{10} + \frac{1}{2(11)^2} \leqslant s \leqslant s_{10} + \frac{1}{2(10)^2}$$

Using $s_{10} \approx 1.197532$, we get

$$1.201664 \leqslant s \leqslant 1.202532$$

If we approximate s by the midpoint of this interval, then the error is at most half the length of the interval. So

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2021 \qquad \text{with error} < 0.0005$$

■

If we compare Example 6 with Example 5, we see that the improved estimate in (3) can be much better than the estimate $s \approx s_n$. To make the error smaller than 0.0005 we had to use 32 terms in Example 5 but only 10 terms in Example 6.

■ Proof of the Integral Test

We have already seen the basic idea behind the proof of the Integral Test in Figures 1 and 2 for the series $\Sigma\, 1/n^2$ and $\Sigma\, 1/\sqrt{n}$. For the general series $\Sigma\, a_n$, look at Figures 5 and 6. The area of the first shaded rectangle in Figure 5 is the value of f at the right endpoint of $[1, 2]$, that is, $f(2) = a_2$. So, comparing the areas of the shaded rectangles with the area under $y = f(x)$ from 1 to n, we see that

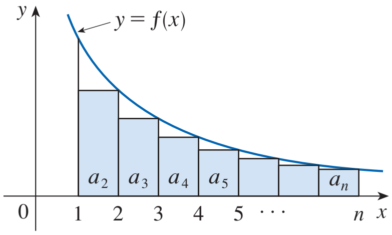


FIGURE 5

$$\boxed{4} \qquad a_2 + a_3 + \cdots + a_n \leqslant \int_1^n f(x)\,dx$$

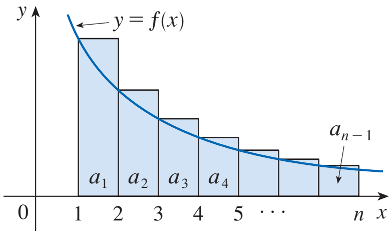


FIGURE 6

(Notice that this inequality depends on the fact that f is decreasing.) Likewise, Figure 6 shows that

$$\boxed{5} \qquad \int_1^n f(x)\,dx \leqslant a_1 + a_2 + \cdots + a_{n-1}$$

(i) If $\int_1^\infty f(x)\,dx$ is convergent, then (4) gives

$$\sum_{i=2}^{n} a_i \leqslant \int_1^n f(x)\,dx \leqslant \int_1^\infty f(x)\,dx$$

since $f(x) \geqslant 0$. Therefore

$$s_n = a_1 + \sum_{i=2}^{n} a_i \leqslant a_1 + \int_1^\infty f(x)\,dx = M, \text{ say}$$

Since $s_n \leqslant M$ for all n, the sequence $\{s_n\}$ is bounded above. Also

$$s_{n+1} = s_n + a_{n+1} \geqslant s_n$$

since $a_{n+1} = f(n+1) \geqslant 0$. Thus $\{s_n\}$ is an increasing bounded sequence and so it is convergent by the Monotonic Sequence Theorem (11.1.12). This means that $\Sigma\, a_n$ is convergent.

(ii) If $\int_1^\infty f(x)\,dx$ is divergent, then $\int_1^n f(x)\,dx \to \infty$ as $n \to \infty$ because $f(x) \geqslant 0$. But (5) gives

$$\int_1^n f(x)\,dx \leqslant \sum_{i=1}^{n-1} a_i = s_{n-1}$$

and so $s_{n-1} \to \infty$. This implies that $s_n \to \infty$ and so $\Sigma\, a_n$ diverges. ■

11.3 Exercises

1. Draw a picture to show that

$$\sum_{n=2}^{\infty} \frac{1}{n^{1.5}} < \int_1^\infty \frac{1}{x^{1.5}}\,dx$$

What can you conclude about the series?

2. Suppose f is a continuous positive decreasing function for $x \geqslant 1$ and $a_n = f(n)$. By drawing a picture, rank the following three quantities in increasing order:

$$\int_1^6 f(x)\,dx \qquad \sum_{i=1}^{5} a_i \qquad \sum_{i=2}^{6} a_i$$

3–10 Use the Integral Test to determine whether the series is convergent or divergent.

3. $\displaystyle\sum_{n=1}^{\infty} n^{-3}$

4. $\displaystyle\sum_{n=1}^{\infty} n^{-0.3}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{5n-1}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n-1)^4}$

7. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^3+1}$

8. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$

9. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^2}$

11–28 Determine whether the series is convergent or divergent.

11. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$

12. $\displaystyle\sum_{n=3}^{\infty} n^{-0.9999}$

13. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots$

14. $\frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \cdots$

15. $\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \frac{1}{19} + \cdots$

16. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+4}{n^2}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^{3/2}}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+2n+2}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^4+4}$

22. $\displaystyle\sum_{n=3}^{\infty} \frac{3n-4}{n^2-2n}$

23. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

24. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

25. $\displaystyle\sum_{k=1}^{\infty} ke^{-k}$

26. $\displaystyle\sum_{k=1}^{\infty} ke^{-k^2}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n^3}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$

29–30 Explain why the Integral Test can't be used to determine whether the series is convergent.

29. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos \pi n}{\sqrt{n}}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{1+n^2}$

31–34 Find the values of p for which the series is convergent.

31. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

32. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n\,[\ln(\ln n)]^p}$

33. $\displaystyle\sum_{n=1}^{\infty} n(1+n^2)^p$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$

35–37 The Riemann Zeta Function The function ζ defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where s is a complex number, is called the *Riemann zeta function*.

35. For which real numbers x is $\zeta(x)$ defined?

36. Leonhard Euler was able to calculate the exact sum of the p-series with $p = 2$:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Use this fact to find the sum of each series.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(n+1)^2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)^2}$

37. Euler also found the sum of the p-series with $p = 4$:

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Use Euler's result to find the sum of the series.

(a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^4$

(b) $\displaystyle\sum_{k=5}^{\infty} \frac{1}{(k-2)^4}$

38. (a) Find the partial sum s_{10} of the series $\sum_{n=1}^{\infty} 1/n^4$. Estimate the error in using s_{10} as an approximation to the sum of the series.

(b) Use (3) with $n = 10$ to give an improved estimate of the sum.

(c) Compare your estimate in part (b) with the exact value given in Exercise 37.

(d) Find a value of n so that s_n is within 0.00001 of the sum.

39. (a) Use the sum of the first 10 terms to estimate the sum of the series $\sum_{n=1}^{\infty} 1/n^2$. How good is this estimate?

(b) Improve this estimate using (3) with $n = 10$.

(c) Compare your estimate in part (b) with the exact value given in Exercise 36.

(d) Find a value of n that will ensure that the error in the approximation $s \approx s_n$ is less than 0.001.

40. Find the sum of the series $\sum_{n=1}^{\infty} ne^{-2n}$ correct to four decimal places.

41. Estimate $\sum_{n=1}^{\infty} (2n+1)^{-6}$ correct to five decimal places.

42. How many terms of the series $\sum_{n=2}^{\infty} 1/[n(\ln n)^2]$ would you need to add to find its sum to within 0.01?

43. Show that if we want to approximate the sum of the series $\sum_{n=1}^{\infty} n^{-1.001}$ so that the error is less than 5 in the ninth decimal place, then we need to add more than $10^{11,301}$ terms!

T **44.** (a) Show that the series $\sum_{n=1}^{\infty} (\ln n)^2/n^2$ is convergent.

(b) Find an upper bound for the error in the approximation $s \approx s_n$.

(c) What is the smallest value of n such that this upper bound is less than 0.05?

(d) Find s_n for this value of n.

45. (a) Use (4) to show that if s_n is the nth partial sum of the harmonic series, then

$$s_n \leqslant 1 + \ln n$$

(b) The harmonic series diverges, but very slowly. Use part (a) to show that the sum of the first million terms is less than 15 and the sum of the first billion terms is less than 22.

46. Use the following steps to show that the sequence

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n$$

has a limit. (The value of the limit is denoted by γ and is called Euler's constant.)

(a) Draw a picture like Figure 6 with $f(x) = 1/x$ and interpret t_n as an area [or use (5)] to show that $t_n > 0$ for all n.

(b) Interpret

$$t_n - t_{n+1} = [\ln(n+1) - \ln n] - \frac{1}{n+1}$$

as a difference of areas to show that $t_n - t_{n+1} > 0$. Therefore $\{t_n\}$ is a decreasing sequence.

(c) Use the Monotonic Sequence Theorem to show that $\{t_n\}$ is convergent.

47. Find all positive values of b for which the series $\Sigma_{n=1}^{\infty} b^{\ln n}$ converges.

48. Find all values of c for which the following series converges.

$$\sum_{n=1}^{\infty} \left(\frac{c}{n} - \frac{1}{n+1} \right)$$

11.4 The Comparison Tests

In the comparison tests the idea is to compare a given series with a series that is known to be convergent or divergent. If two series have only positive terms, we can compare corresponding terms directly to see which are larger (the Direct Comparison Test) or we can investigate the limit of the ratios of corresponding terms (the Limit Comparison Test).

The Direct Comparison Test

Let's consider the two series

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{2^n}$$

The second series $\Sigma_{n=1}^{\infty} 1/2^n$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$ and is therefore convergent. Since these series are so similar, we may have the feeling that the first series must converge also. In fact, it does. The inequality

$$\frac{1}{2^n + 1} < \frac{1}{2^n}$$

shows that the series $\Sigma\, 1/(2^n + 1)$ has smaller terms than those of the geometric series $\Sigma\, 1/2^n$ and therefore all its partial sums are also smaller than 1 (the sum of the geometric series). This means that its partial sums form a bounded increasing sequence, which is convergent. It also follows that the sum of the series is less than the sum of the geometric series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1} < 1$$

Similar reasoning can be used to prove the following test, which applies only to series whose terms are positive. The first part says that if we have a series whose terms are *smaller* than those of a known *convergent* series, then our series is also convergent. The second part says that if we start with a series whose terms are *larger* than those of a known *divergent* series, then it too is divergent.

The Direct Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms.

(i) If $\Sigma\, b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\Sigma\, a_n$ is also convergent.

(ii) If $\Sigma\, b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\Sigma\, a_n$ is also divergent.

PROOF

It is important to keep in mind the distinction between a sequence and a series. A sequence is a list of numbers, whereas a series is a sum. With every series $\sum a_n$ there are two associated sequences: the sequence $\{a_n\}$ of terms and the sequence $\{s_n\}$ of partial sums.

(i) Let $$s_n = \sum_{i=1}^{n} a_i \qquad t_n = \sum_{i=1}^{n} b_i \qquad t = \sum_{n=1}^{\infty} b_n$$

Since both series have positive terms, the sequences $\{s_n\}$ and $\{t_n\}$ are increasing ($s_{n+1} = s_n + a_{n+1} \geqslant s_n$). Also $t_n \to t$, so $t_n \leqslant t$ for all n. Since $a_i \leqslant b_i$, we have $s_n \leqslant t_n$. Thus $s_n \leqslant t$ for all n. This means that $\{s_n\}$ is increasing and bounded above and therefore converges by the Monotonic Sequence Theorem. Thus $\sum a_n$ converges.

(ii) If $\sum b_n$ is divergent, then $t_n \to \infty$ (since $\{t_n\}$ is increasing). But $a_i \geqslant b_i$ so $s_n \geqslant t_n$. Thus $s_n \to \infty$. Therefore $\sum a_n$ diverges. ■

Standard series for use with the comparison tests

In using the Direct Comparison Test we must, of course, have some known series $\sum b_n$ for the purpose of comparison. Most of the time we use one of these series:

- A p-series [$\sum 1/n^p$ converges if $p > 1$ and diverges if $p \leqslant 1$; see (11.3.1)]
- A geometric series [$\sum ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geqslant 1$; see (11.2.4)]

EXAMPLE 1 Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ converges or diverges.

SOLUTION For large n the dominant term in the denominator is $2n^2$, so we compare the given series with the series $\sum 5/(2n^2)$. Observe that

$$\frac{5}{2n^2 + 4n + 3} < \frac{5}{2n^2}$$

because the left side has a bigger denominator. (In the notation of the Direct Comparison Test, a_n is the left side and b_n is the right side.) We know that

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent because it's a constant times a p-series with $p = 2 > 1$. Therefore

$$\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$$

is convergent by part (i) of the Direct Comparison Test. ■

NOTE Although the condition $a_n \leqslant b_n$ or $a_n \geqslant b_n$ in the Direct Comparison Test is given for all n, we need verify only that it holds for $n \geqslant N$, where N is some fixed integer, because the convergence of a series is not affected by a finite number of terms. This is illustrated in the next example.

EXAMPLE 2 Test the series $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k}$ for convergence or divergence.

SOLUTION We used the Integral Test to test this series in Example 11.3.4, but we can also test it by comparing it with the harmonic series. Observe that $\ln k > 1$ for $k \geqslant 3$ and so

$$\frac{\ln k}{k} > \frac{1}{k} \qquad k \geqslant 3$$

We know that $\Sigma\, 1/k$ is divergent (p-series with $p = 1$). Thus the given series is divergent by the Direct Comparison Test. ■

■ Limit Comparison Test

The Direct Comparison Test is conclusive only if the terms of the series being tested are smaller than those of a convergent series or larger than those of a divergent series. If the terms are larger than the terms of a convergent series or smaller than those of a divergent series, then the Direct Comparison Test doesn't apply. Consider, for instance, the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

The inequality

$$\frac{1}{2^n - 1} > \frac{1}{2^n}$$

is useless as far as the Direct Comparison Test is concerned because $\Sigma\, b_n = \Sigma \left(\frac{1}{2}\right)^n$ is convergent and $a_n > b_n$. Nonetheless, we have the feeling that $\Sigma\, 1/(2^n - 1)$ ought to be convergent because it is very similar to the convergent geometric series $\Sigma \left(\frac{1}{2}\right)^n$. In such cases the following test can be used.

Exercises 48 and 49 deal with the cases $c = 0$ and $c = \infty$.

The Limit Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

PROOF Let m and M be positive numbers such that $m < c < M$. Because a_n/b_n is close to c for large n, there is an integer N such that

$$m < \frac{a_n}{b_n} < M \qquad \text{when } n > N$$

and so

$$mb_n < a_n < Mb_n \qquad \text{when } n > N$$

If $\Sigma\, b_n$ converges, so does $\Sigma\, Mb_n$. Thus $\Sigma\, a_n$ converges by part (i) of the Direct Comparison Test. If $\Sigma\, b_n$ diverges, so does $\Sigma\, mb_n$ and part (ii) of the Direct Comparison Test shows that $\Sigma\, a_n$ diverges. ■

EXAMPLE 3 Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

SOLUTION We use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1} \qquad b_n = \frac{1}{2^n}$$

and obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2^n - 1)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - 1/2^n} = 1 > 0$$

Since this limit exists and $\Sigma\, 1/2^n$ is a convergent geometric series, the given series converges by the Limit Comparison Test. ■

EXAMPLE 4 Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ converges or diverges.

SOLUTION The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$. This suggests taking

$$a_n = \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \qquad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n\to\infty} \frac{2n^{5/2} + 3n^{3/2}}{2\sqrt{5 + n^5}}$$

$$= \lim_{n\to\infty} \frac{2 + \dfrac{3}{n}}{2\sqrt{\dfrac{5}{n^5} + 1}} = \frac{2 + 0}{2\sqrt{0 + 1}} = 1$$

Since $\Sigma\, b_n = 2\Sigma\, 1/n^{1/2}$ is divergent $\left(p\text{-series with } p = \frac{1}{2} < 1\right)$, the given series diverges by the Limit Comparison Test. ■

Notice that in testing many series we find a suitable comparison series $\Sigma\, b_n$ by keeping only the highest powers in the numerator and denominator.

■ Estimating Sums

If we have used the Direct Comparison Test to show that a series $\Sigma\, a_n$ converges by comparison with a series $\Sigma\, b_n$, then we may be able to estimate the sum $\Sigma\, a_n$ by comparing remainders. As in Section 11.3, we consider the remainder

$$R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$$

For the comparison series $\Sigma\, b_n$ we consider the corresponding remainder

$$T_n = t - t_n = b_{n+1} + b_{n+2} + \cdots$$

Since $a_n \leqslant b_n$ for all n, we have $R_n \leqslant T_n$. If $\Sigma\, b_n$ is a p-series, we can estimate its remainder T_n as in Section 11.3. If $\Sigma\, b_n$ is a geometric series, then T_n is the sum of a geometric series and we can sum it exactly (see Exercises 43 and 44). In either case we know that R_n is smaller than T_n.

EXAMPLE 5 Use the sum of the first 100 terms to approximate the sum of the series $\Sigma\, 1/(n^3 + 1)$. Estimate the error involved in this approximation.

SOLUTION Since

$$\frac{1}{n^3 + 1} < \frac{1}{n^3}$$

the given series is convergent by the Direct Comparison Test. The remainder T_n for the comparison series $\Sigma\, 1/n^3$ was estimated in Example 11.3.5 using the Remainder Estimate for the Integral Test. There we found that

$$T_n \leqslant \int_n^{\infty} \frac{1}{x^3}\, dx = \frac{1}{2n^2}$$

Therefore the remainder R_n for the given series satisfies

$$R_n \leqslant T_n \leqslant \frac{1}{2n^2}$$

With $n = 100$ we have

$$R_{100} \leqslant \frac{1}{2(100)^2} = 0.00005$$

Using a calculator or a computer, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^3+1} \approx \sum_{n=1}^{100} \frac{1}{n^3+1} \approx 0.6864538$$

with error less than 0.00005. ■

11.4 Exercises

1. Suppose Σa_n and Σb_n are series with positive terms and Σb_n is known to be convergent.
(a) If $a_n > b_n$ for all n, what can you say about Σa_n? Why?
(b) If $a_n < b_n$ for all n, what can you say about Σa_n? Why?

2. Suppose Σa_n and Σb_n are series with positive terms and Σb_n is known to be divergent.
(a) If $a_n > b_n$ for all n, what can you say about Σa_n? Why?
(b) If $a_n < b_n$ for all n, what can you say about Σa_n? Why?

3. (a) Use the Direct Comparison Test to show that the first series converges by comparing it to the second series.

$$\sum_{n=2}^{\infty} \frac{n}{n^3+5} \qquad \sum_{n=2}^{\infty} \frac{1}{n^2}$$

(b) Use the Limit Comparison Test to show that that the first series converges by comparing it to the second series.

$$\sum_{n=2}^{\infty} \frac{n}{n^3-5} \qquad \sum_{n=2}^{\infty} \frac{1}{n^2}$$

4. (a) Use the Direct Comparison Test to show that the first series diverges by comparing it to the second series.

$$\sum_{n=2}^{\infty} \frac{n^2+n}{n^3-2} \qquad \sum_{n=2}^{\infty} \frac{1}{n}$$

(b) Use the Limit Comparison Test to show that that the first series diverges by comparing it to the second series.

$$\sum_{n=2}^{\infty} \frac{n^2-n}{n^3+2} \qquad \sum_{n=2}^{\infty} \frac{1}{n}$$

5. Which of the following inequalities can be used to show that $\sum_{n=1}^{\infty} n/(n^3+1)$ converges?

(a) $\dfrac{n}{n^3+1} \geqslant \dfrac{1}{n^3+1}$ (b) $\dfrac{n}{n^3+1} \leqslant \dfrac{1}{n}$

(c) $\dfrac{n}{n^3+1} \leqslant \dfrac{1}{n^2}$

6. Which of the following inequalities can be used to show that $\sum_{n=1}^{\infty} n/(n^2+1)$ diverges?

(a) $\dfrac{n}{n^2+1} \geqslant \dfrac{1}{n^2+1}$ (b) $\dfrac{n}{n^2+1} \leqslant \dfrac{1}{n}$

(c) $\dfrac{n}{n^2+1} \geqslant \dfrac{1}{2n}$

7–40 Determine whether the series converges or diverges.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+8}$

8. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$

13. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

14. $\displaystyle\sum_{k=1}^{\infty} \frac{k\sin^2 k}{1+k^3}$

15. $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}}$

16. $\displaystyle\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{1+\cos n}{e^n}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n-2}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}+2}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^3+n}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{1+n}}{2+n}$

26. $\displaystyle\sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{n+3^n}{n+2^n}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n+1}{ne^n+1}$

30. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{2+\sin n}{n^2}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+\cos^2 n}{n^3}$

33. $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

37. $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

38. $\displaystyle\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$

39. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\tan\frac{1}{n}$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

41–44 Use the sum of the first 10 terms to approximate the sum of the series. Estimate the error.

41. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5+n^5}$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^4}$

43. $\displaystyle\sum_{n=1}^{\infty} 5^{-n}\cos^2 n$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n+4^n}$

45. The meaning of the decimal representation of a number $0.d_1d_2d_3\ldots$ (where the digit d_i is one of the numbers 0, 1, 2, . . . , 9) is that

$$0.d_1d_2d_3d_4\ldots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} + \cdots$$

Show that this series converges for all choices of $d_1, d_2, \ldots$.

46. For what values of p does the series $\sum_{n=2}^{\infty} 1/(n^p \ln n)$ converge?

47. Show that

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln \ln n}}$$

diverges. [*Hint:* Use Formula 1.5.10 ($x^r = e^{r\ln x}$) and the fact that $\ln x < \sqrt{x}$ for $x \geqslant 1$.]

48. (a) Suppose that Σa_n and Σb_n are series with positive terms and Σb_n is convergent. Prove that if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

then Σa_n is also convergent.

(b) Use part (a) to show that the series converges.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \left(1-\cos\frac{1}{n^2}\right)$

49. (a) Suppose that Σa_n and Σb_n are series with positive terms and Σb_n is divergent. Prove that if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$$

then Σa_n is also divergent.

(b) Use part (a) to show that the series diverges.

(i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

50. Give an example of a pair of series Σa_n and Σb_n with positive terms where $\lim_{n\to\infty}(a_n/b_n) = 0$ and Σb_n diverges, but Σa_n converges. (Compare with Exercise 48.)

51. Show that if $a_n > 0$ and $\lim_{n\to\infty} na_n \neq 0$, then Σa_n is divergent.

52. Show that if $a_n > 0$ and Σa_n is convergent, then $\Sigma \ln(1+a_n)$ is convergent.

53. If Σa_n is a convergent series with positive terms, is it true that $\Sigma \sin(a_n)$ is also convergent?

54. Prove that if $a_n \geqslant 0$ and Σa_n converges, then Σa_n^2 also converges.

55. Let Σa_n and Σb_n be series with positive terms. Is each of the following statements true or false? If the statement is false, give an example that disproves the statement.

(a) If Σa_n and Σb_n are divergent, then $\Sigma a_n b_n$ is divergent.

(b) If Σa_n converges and Σb_n diverges, then $\Sigma a_n b_n$ diverges.

(c) If Σa_n and Σb_n are convergent, then $\Sigma a_n b_n$ is convergent.

11.5 Alternating Series and Absolute Convergence

The convergence tests that we have looked at so far apply only to series with positive terms. In this section and the next we learn how to deal with series whose terms are not necessarily positive. Of particular importance are *alternating series,* whose terms alternate in sign.

Alternating Series

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n+1}$$

We see from these examples that the nth term of an alternating series is of the form

$$a_n = (-1)^{n-1} b_n \qquad \text{or} \qquad a_n = (-1)^n b_n$$

where b_n is a positive number. (In fact, $b_n = |a_n|$.)

The following test says that if the terms of an alternating series decrease toward 0 in absolute value, then the series converges.

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad (b_n > 0)$$

satisfies the conditions

$$\text{(i)} \quad b_{n+1} \leqslant b_n \qquad \text{for all } n$$

$$\text{(ii)} \quad \lim_{n \to \infty} b_n = 0$$

then the series is convergent.

Before giving the proof let's look at Figure 1, which gives a picture of the idea behind the proof. We first plot $s_1 = b_1$ on a number line. To find s_2 we subtract b_2, so s_2 is to the left of s_1. Then to find s_3 we add b_3, so s_3 is to the right of s_2. But, since $b_3 < b_2$, s_3 is to the left of s_1. Continuing in this manner, we see that the partial sums oscillate back and forth. Since $b_n \to 0$, the successive steps are becoming smaller and smaller. The even partial sums $s_2, s_4, s_6, \ldots$ are increasing and the odd partial sums $s_1, s_3, s_5, \ldots$ are decreasing. Thus it seems plausible that both are converging to some number s, which is the sum of the series. Therefore we consider the even and odd partial sums separately in the following proof.

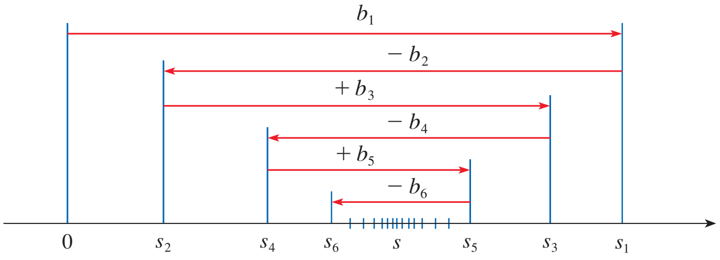


FIGURE 1

PROOF OF THE ALTERNATING SERIES TEST We first consider the even partial sums:

$$s_2 = b_1 - b_2 \geqslant 0 \qquad \text{since } b_2 \leqslant b_1$$

$$s_4 = s_2 + (b_3 - b_4) \geqslant s_2 \qquad \text{since } b_4 \leqslant b_3$$

In general $\quad s_{2n} = s_{2n-2} + (b_{2n-1} - b_{2n}) \geqslant s_{2n-2} \qquad \text{since } b_{2n} \leqslant b_{2n-1}$

Thus $\quad 0 \leqslant s_2 \leqslant s_4 \leqslant s_6 \leqslant \cdots \leqslant s_{2n} \leqslant \cdots$

But we can also write

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}$$

Every term in parentheses is positive, so $s_{2n} \leqslant b_1$ for all n. Therefore the sequence $\{s_{2n}\}$ of even partial sums is increasing and bounded above. It is therefore convergent by the Monotonic Sequence Theorem. Let's call its limit s, that is,

$$\lim_{n\to\infty} s_{2n} = s$$

Now we compute the limit of the odd partial sums:

$$\begin{aligned} \lim_{n\to\infty} s_{2n+1} &= \lim_{n\to\infty} (s_{2n} + b_{2n+1}) \\ &= \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} \\ &= s + 0 \qquad \text{[by condition (ii)]} \\ &= s \end{aligned}$$

Since both the even and odd partial sums converge to s, we have $\lim_{n\to\infty} s_n = s$ [see Exercise 11.1.98(a)] and so the series is convergent. ■

Figure 2 illustrates Example 1 by showing the graphs of the terms $a_n = (-1)^{n-1}/n$ and the partial sums s_n. Notice how the values of s_n zig-zag across the limiting value, which appears to be about 0.7. In fact, it can be proved that the exact sum of the series is $\ln 2 \approx 0.693$ (see Exercise 50).

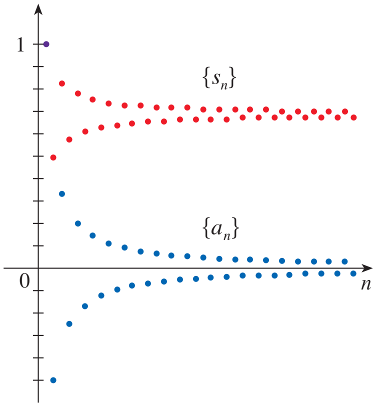


FIGURE 2

EXAMPLE 1 The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies the conditions

$$\text{(i)}\ \ b_{n+1} < b_n \qquad \text{because} \qquad \frac{1}{n+1} < \frac{1}{n}$$

$$\text{(ii)}\ \ \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

so the series is convergent by the Alternating Series Test. ■

EXAMPLE 2 The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is alternating, but

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \lim_{n\to\infty} \frac{3}{4 - \dfrac{1}{n}} = \frac{3}{4}$$

so condition (ii) is not satisfied. Thus the Alternating Series Test doesn't apply. Instead, we look at the limit of the nth term of the series:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n 3n}{4n - 1}$$

This limit does not exist, so the series diverges by the Test for Divergence. ■

EXAMPLE 3 Test the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3 + 1}$ for convergence or divergence.

SOLUTION The given series is alternating so we try to verify conditions (i) and (ii) of the Alternating Series Test.

Condition (i): Unlike the situation in Example 1, it is not obvious that the sequence given by $b_n = n^2/(n^3 + 1)$ is decreasing. However, if we consider the related function $f(x) = x^2/(x^3 + 1)$, we find that

$$f'(x) = \frac{x(2 - x^3)}{(x^3 + 1)^2}$$

Since we are considering only positive x, we see that $f'(x) < 0$ if $2 - x^3 < 0$, that is, $x > \sqrt[3]{2}$. Thus f is decreasing on the interval $\left(\sqrt[3]{2}, \infty\right)$. This means that $f(n + 1) < f(n)$ and therefore $b_{n+1} < b_n$ when $n \geqslant 2$. (The inequality $b_2 < b_1$ can be verified directly but all that really matters is that the sequence $\{b_n\}$ is eventually decreasing.)

Instead of verifying condition (i) of the Alternating Series Test by computing a derivative, we could verify that $b_{n+1} < b_n$ directly by using the technique of Solution 1 in Example 11.1.13.

Condition (ii) is readily verified:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3 + 1} = \lim_{n\to\infty} \frac{1/n}{1 + 1/n^3} = 0$$

Thus the given series is convergent by the Alternating Series Test. ■

Estimating Sums of Alternating Series

A partial sum s_n of any convergent series can be used as an approximation to the total sum s, but this is not of much use unless we can estimate the accuracy of the approximation. The error involved in using $s \approx s_n$ is the remainder $R_n = s - s_n$. The next theorem says that for series that satisfy the conditions of the Alternating Series Test, the size of the error is smaller than b_{n+1}, which is the absolute value of the first neglected term.

You can see geometrically why the Alternating Series Estimation Theorem is true by looking at Figure 1. Notice that $s - s_4 < b_5$, $|s - s_5| < b_6$, and so on. Notice also that s lies between any two consecutive partial sums.

Alternating Series Estimation Theorem If $s = \sum (-1)^{n-1} b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies

$$\text{(i)}\ \ b_{n+1} \leqslant b_n \qquad \text{and} \qquad \text{(ii)}\ \ \lim_{n\to\infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \leqslant b_{n+1}$$

PROOF We know from the proof of the Alternating Series Test that s lies between any two consecutive partial sums s_n and s_{n+1}. (There we showed that s is larger than all the even partial sums. A similar argument shows that s is smaller than all the odd sums.) It follows that

$$|s - s_n| \leqslant |s_{n+1} - s_n| = b_{n+1}$$ ■

By definition, $0! = 1$.

EXAMPLE 4 Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

SOLUTION We first observe that the series is convergent by the Alternating Series Test because

$$\text{(i)} \quad b_{n+1} = \frac{1}{(n+1)!} = \frac{1}{n!\,(n+1)} < \frac{1}{n!} = b_n$$

$$\text{(ii)} \quad 0 < \frac{1}{n!} < \frac{1}{n} \to 0 \quad \text{so} \quad b_n = \frac{1}{n!} \to 0 \quad \text{as} \quad n \to \infty$$

To get a feel for how many terms we need to use in our approximation, let's write out the first few terms of the series:

$$\begin{aligned} s &= \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots \\ &= 1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \tfrac{1}{120} + \tfrac{1}{720} - \tfrac{1}{5040} + \cdots \end{aligned}$$

Notice that $$b_7 = \tfrac{1}{5040} < \tfrac{1}{5000} = 0.0002$$

and $$s_6 = 1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \tfrac{1}{120} + \tfrac{1}{720} \approx 0.368056$$

In Section 11.10 we will prove that $e^x = \sum_{n=0}^{\infty} x^n/n!$ for all x, so what we have obtained in Example 4 is actually an approximation to the number e^{-1}.

By the Alternating Series Estimation Theorem we know that

$$|s - s_6| \leqslant b_7 < 0.0002$$

This error of less than 0.0002 does not affect the third decimal place, so we have $s \approx 0.368$ correct to three decimal places. ■

NOTE The rule that the error (in using s_n to approximate s) is smaller than the first neglected term is, in general, valid only for alternating series that satisfy the conditions of the Alternating Series Estimation Theorem. The rule does not apply to other types of series.

■ Absolute Convergence and Conditional Convergence

Given any series $\sum a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute values of the terms of the original series.

We have discussed convergence tests for series with positive terms and for alternating series. But what if the signs of the terms switch back and forth irregularly? We will see in Example 7 that the idea of absolute convergence sometimes helps in such cases.

1 Definition A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

Notice that if $\sum a_n$ is a series with positive terms, then $|a_n| = a_n$ and so absolute convergence is the same as convergence in this case.

EXAMPLE 5 The alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is a convergent p-series ($p = 2$). ■

2 Definition A series Σa_n is called **conditionally convergent** if it is convergent but not absolutely convergent; that is, if Σa_n converges but $\Sigma |a_n|$ diverges.

EXAMPLE 6 We know from Example 1 that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is convergent, but it is not absolutely convergent because the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

which is the harmonic series (p-series with $p = 1$) and is therefore divergent. Thus the alternating harmonic series is conditionally convergent. ■

Example 6 shows that it is possible for a series to be convergent but not absolutely convergent. However, the following theorem states that absolute convergence implies convergence.

3 Theorem If a series Σa_n is absolutely convergent, then it is convergent.

You can think of absolute convergence as a stronger type of convergence. An absolutely convergent series, like the one in Example 5, will converge regardless of the signs of its terms, whereas the series in Example 6 will not converge if we change all of its negative terms to positive.

PROOF Observe that the inequality

$$0 \leqslant a_n + |a_n| \leqslant 2|a_n|$$

is true because $|a_n|$ is either a_n or $-a_n$. If Σa_n is absolutely convergent, then $\Sigma |a_n|$ is convergent, so $\Sigma 2|a_n|$ is convergent. Therefore, by the Direct Comparison Test, $\Sigma(a_n + |a_n|)$ is convergent. Then

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$$

is the difference of two convergent series and is therefore convergent. ■

EXAMPLE 7 Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

is convergent or divergent.

Figure 3 shows the graphs of the terms a_n and partial sums s_n of the series in Example 7. Notice that the series is not alternating but has positive and negative terms.

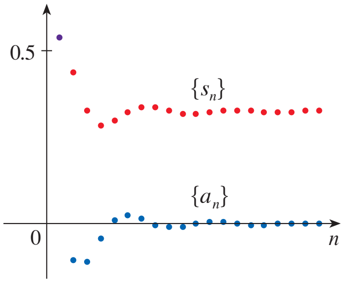


FIGURE 3

SOLUTION This series has both positive and negative terms, but it is not alternating. (The first term is positive, the next three are negative, and the following three are positive: the signs change irregularly.) We can apply the Direct Comparison Test to the series of absolute values

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

Since $|\cos n| \leqslant 1$ for all n, we have

$$\frac{|\cos n|}{n^2} \leqslant \frac{1}{n^2}$$

We know that $\Sigma\, 1/n^2$ is convergent (p-series with $p = 2$) and therefore $\Sigma\, |\cos n|/n^2$ is convergent by the Direct Comparison Test. Thus the given series $\Sigma\, (\cos n)/n^2$ is absolutely convergent and therefore convergent by Theorem 3. ■

EXAMPLE 8 Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{2n+1}$

SOLUTION

(a) Because the series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges (p-series with $p = 3$), the given series is absolutely convergent.

(b) We first test for absolute convergence. The series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt[3]{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$$

diverges $\left(p\text{-series with } p = \frac{1}{3}\right)$, so the given series is *not* absolutely convergent. The given series converges by the Alternating Series Test ($b_{n+1} \leqslant b_n$ and $\lim_{n\to\infty} b_n = 0$). Since the series converges but is not absolutely convergent, it is conditionally convergent.

(c) This series is alternating but

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \frac{n}{2n+1}$$

does not exist (see Figure 4), so the series diverges by the Test for Divergence. ■

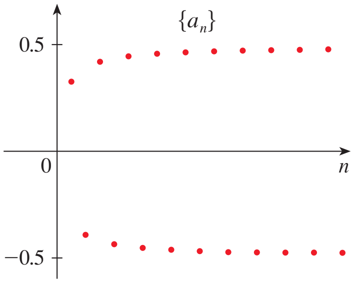


FIGURE 4
The terms of $\{a_n\}$ are alternately close to 0.5 and −0.5.

Rearrangements

The question of whether a given convergent series is absolutely convergent or conditionally convergent has a bearing on the question of whether infinite sums behave like finite sums.

If we rearrange the order of the terms in a finite sum, then of course the value of the sum remains unchanged. But this is not always the case for an infinite series. By a **rearrangement** of an infinite series $\Sigma\, a_n$ we mean a series obtained by simply changing the order of the terms. For instance, a rearrangement of $\Sigma\, a_n$ could start as follows:

$$a_1 + a_2 + a_5 + a_3 + a_4 + a_{15} + a_6 + a_7 + a_{20} + \cdots$$

It turns out that

> if $\Sigma\, a_n$ is an absolutely convergent series with sum s, then any rearrangement of $\Sigma\, a_n$ has the same sum s.

However, any conditionally convergent series can be rearranged to give a different sum. To illustrate this fact let's consider the alternating harmonic series from Example 1. In Exercise 50 you are asked to show that

$$\boxed{4} \qquad 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8} + \cdots = \ln 2$$

If we multiply this series by $\frac{1}{2}$, we get

$$\tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{6} - \tfrac{1}{8} + \cdots = \tfrac{1}{2}\ln 2$$

Adding these zeros does not affect the sum of the series; each term in the sequence of partial sums is repeated, but the limit is the same.

Inserting zeros between the terms of this series, we have

$$\boxed{5} \qquad 0 + \tfrac{1}{2} + 0 - \tfrac{1}{4} + 0 + \tfrac{1}{6} + 0 - \tfrac{1}{8} + \cdots = \tfrac{1}{2}\ln 2$$

Now we add the series in Equations 4 and 5 using Theorem 11.2.8:

$$\boxed{6} \qquad 1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \cdots = \tfrac{3}{2}\ln 2$$

Notice that the series in (6) contains the same terms as in (4) but rearranged so that one negative term occurs after each pair of positive terms. The sums of these series, however, are different. In fact, Riemann proved that

> if $\Sigma\, a_n$ is a conditionally convergent series and r is any real number whatsoever, then there is a rearrangement of $\Sigma\, a_n$ that has a sum equal to r.

A proof of this fact is outlined in Exercise 52.

11.5 Exercises

1. (a) What is an alternating series?
(b) Under what conditions does an alternating series converge?
(c) If these conditions are satisfied, what can you say about the remainder after n terms?

2–20 Test the series for convergence or divergence.

2. $\frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \frac{2}{11} - \cdots$

3. $-\frac{2}{5} + \frac{4}{6} - \frac{6}{7} + \frac{8}{8} - \frac{10}{9} + \cdots$

4. $\dfrac{1}{\ln 3} - \dfrac{1}{\ln 4} + \dfrac{1}{\ln 5} - \dfrac{1}{\ln 6} + \dfrac{1}{\ln 7} - \cdots$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3 + 5n}$

6. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$

7. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3n - 1}{2n + 1}$

8. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2 + n + 1}$

9. $\displaystyle\sum_{n=1}^{\infty} (-1)^n e^{-n}$

10. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2n + 3}$

11. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3 + 4}$

12. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n}$

13. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} e^{2/n}$

14. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \arctan n$

15. $\displaystyle\sum_{n=0}^{\infty} \frac{\sin\left(n + \frac{1}{2}\right)\pi}{1 + \sqrt{n}}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n \cos n\pi}{2^n}$

17. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\frac{\pi}{n}$

18. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos\frac{\pi}{n}$

19. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$

20. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right)$

21. (a) What does it mean for a series to be absolutely convergent?

(b) What does it mean for a series to be conditionally convergent?

(c) If the series of positive terms $\sum_{n=1}^{\infty} b_n$ converges, then what can you say about the series $\sum_{n=1}^{\infty} (-1)^n b_n$?

22–34 Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

22. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt[3]{n^2}}$

24. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} \frac{n^2}{n^2+1}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{-n}{n^2+1}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{2^n}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{1+2\sin n}{n^3}$

30. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$

31. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

32. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{3n+2}$

34. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

35–36 Graph both the sequence of terms and the sequence of partial sums on the same screen. Use the graph to make a rough estimate of the sum of the series. Then use the Alternating Series Estimation Theorem to estimate the sum correct to four decimal places.

35. $\displaystyle\sum_{n=1}^{\infty} \frac{(-0.8)^n}{n!}$

36. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{8^n}$

37–40 Show that the series is convergent. How many terms of the series do we need to add in order to find the sum to the indicated accuracy?

37. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6} \quad (|\text{error}| < 0.00005)$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{\left(-\frac{1}{3}\right)^n}{n} \quad (|\text{error}| < 0.0005)$

39. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 2^n} \quad (|\text{error}| < 0.0005)$

40. $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{1}{n}\right)^n \quad (|\text{error}| < 0.00005)$

41–44 Approximate the sum of the series correct to four decimal places.

41. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$

43. $\displaystyle\sum_{n=1}^{\infty} (-1)^n n e^{-2n}$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 4^n}$

45. Is the 50th partial sum s_{50} of the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1}/n$ an overestimate or an underestimate of the total sum? Explain.

46–48 For what values of p is each series convergent?

46. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$

47. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+p}$

48. $\displaystyle\sum_{n=2}^{\infty} (-1)^{n-1} \frac{(\ln n)^p}{n}$

49. Show that the series $\sum (-1)^{n-1} b_n$, where $b_n = 1/n$ if n is odd and $b_n = 1/n^2$ if n is even, is divergent. Why does the Alternating Series Test not apply?

50. Use the following steps to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2$$

Let h_n and s_n be the partial sums of the harmonic and alternating harmonic series.

(a) Show that $s_{2n} = h_{2n} - h_n$.

(b) From Exercise 11.3.46 we have

$$h_n - \ln n \to \gamma \qquad \text{as } n \to \infty$$

and therefore

$$h_{2n} - \ln(2n) \to \gamma \qquad \text{as } n \to \infty$$

Use these facts together with part (a) to show that $s_{2n} \to \ln 2$ as $n \to \infty$.

51. Given any series $\sum a_n$, we define a series $\sum a_n^+$ whose terms are all the positive terms of $\sum a_n$ and a series $\sum a_n^-$ whose terms are all the negative terms of $\sum a_n$. To be specific, we let

$$a_n^+ = \frac{a_n + |a_n|}{2} \qquad a_n^- = \frac{a_n - |a_n|}{2}$$

Notice that if $a_n > 0$, then $a_n^+ = a_n$ and $a_n^- = 0$, whereas if $a_n < 0$, then $a_n^- = a_n$ and $a_n^+ = 0$.

(a) If $\sum a_n$ is absolutely convergent, show that both of the series $\sum a_n^+$ and $\sum a_n^-$ are convergent.

(b) If $\sum a_n$ is conditionally convergent, show that both of the series $\sum a_n^+$ and $\sum a_n^-$ are divergent.

52. Prove that if Σa_n is a conditionally convergent series and r is any real number, then there is a rearrangement of Σa_n whose sum is r. [*Hints:* Use the notation of Exercise 51. Take just enough positive terms a_n^+ so that their sum is greater than r. Then add just enough negative terms a_n^- so that the cumulative sum is less than r. Continue in this manner and use Theorem 11.2.6.]

53. Suppose the series Σa_n is conditionally convergent.
(a) Prove that the series $\Sigma n^2 a_n$ is divergent.
(b) Conditional convergence of Σa_n is not enough to determine whether $\Sigma n a_n$ is convergent. Show this by giving an example of a conditionally convergent series such that $\Sigma n a_n$ converges and an example where $\Sigma n a_n$ diverges.

11.6 The Ratio and Root Tests

One way to determine how quickly the terms of a series are decreasing (or increasing) is to calculate the ratios of consecutive terms. For a geometric series Σar^{n-1} we have $|a_{n+1}/a_n| = |r|$ for all n, and the series converges if $|r| < 1$. The Ratio Test tells us that for any series, if the ratios $|a_{n+1}/a_n|$ approach a number less than 1 as $n \to \infty$, then the series converges. The proofs of both the Ratio Test and the Root Test involve comparing a series with a geometric series.

The Ratio Test

The following test is very useful in determining whether a given series is absolutely convergent.

The Ratio Test

(i) If $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of Σa_n.

PROOF

(i) The idea is to compare the given series with a convergent geometric series. Since $L < 1$, we can choose a number r such that $L < r < 1$. Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \qquad \text{and} \qquad L < r$$

the ratio $|a_{n+1}/a_n|$ will eventually be less than r; that is, there exists an integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| < r \qquad \text{whenever } n \geqslant N$$

or, equivalently,

$$\boxed{1} \qquad |a_{n+1}| < |a_n| r \qquad \text{whenever } n \geqslant N$$

Putting n successively equal to $N, N+1, N+2, \dots$ in (1), we obtain

$$|a_{N+1}| < |a_N| r$$

$$|a_{N+2}| < |a_{N+1}| r < |a_N| r^2$$

$$|a_{N+3}| < |a_{N+2}| r < |a_N| r^3$$

and, in general,

$$\boxed{2} \qquad |a_{N+k}| < |a_N| r^k \qquad \text{for all } k \geqslant 1$$

Now the series

$$\sum_{k=1}^{\infty} |a_N| r^k = |a_N| r + |a_N| r^2 + |a_N| r^3 + \cdots$$

is convergent because it is a geometric series with $0 < r < 1$. So the inequality (2), together with the Direct Comparison Test, shows that the series

$$\sum_{n=N+1}^{\infty} |a_n| = \sum_{k=1}^{\infty} |a_{N+k}| = |a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \cdots$$

is also convergent. It follows that the series $\sum_{n=1}^{\infty} |a_n|$ is convergent. (Recall that a finite number of terms doesn't affect convergence.) Therefore $\sum a_n$ is absolutely convergent.

(ii) If $|a_{n+1}/a_n| \to L > 1$ or $|a_{n+1}/a_n| \to \infty$, then the ratio $|a_{n+1}/a_n|$ will eventually be greater than 1; that is, there exists an integer N such that

$$\left| \frac{a_{n+1}}{a_n} \right| > 1 \qquad \text{whenever } n \geqslant N$$

This means that $|a_{n+1}| > |a_n|$ whenever $n \geqslant N$ and so

$$\lim_{n \to \infty} a_n \neq 0$$

Therefore $\sum a_n$ diverges by the Test for Divergence. ■

EXAMPLE 1 Test the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

SOLUTION We use the Ratio Test with $a_n = (-1)^n n^3/3^n$:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\dfrac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\dfrac{(-1)^n n^3}{3^n}} \right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$

$$= \frac{1}{3} \left(\frac{n+1}{n} \right)^3 = \frac{1}{3} \left(1 + \frac{1}{n} \right)^3 \to \frac{1}{3} < 1$$

Thus, by the Ratio Test, the given series is absolutely convergent. ■

Estimating Sums

In the preceding three sections we used various methods for estimating the sum of a series—the method depended on which test was used to prove convergence. What about series for which the Ratio Test works? There are two possibilities: If the series happens to be an alternating series, as in Example 1, then it is best to use the methods of Section 11.5. If the terms are all positive, then use the special methods explained in Exercise 42.

EXAMPLE 2 Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

SOLUTION Since the terms $a_n = n^n/n!$ are positive, we don't need the absolute value signs.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)(n+1)^n}{(n+1)\,n!} \cdot \frac{n!}{n^n}$$

$$= \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \to e \qquad \text{as } n \to \infty$$

(see Equation 3.6.6). Since $e > 1$, the given series is divergent by the Ratio Test. ■

NOTE Although the Ratio Test works in Example 2, an easier method is to use the Test for Divergence. Since

$$a_n = \frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n} \geqslant n$$

it follows that a_n does not approach 0 as $n \to \infty$. Therefore the given series is divergent by the Test for Divergence.

EXAMPLE 3 Part (iii) of the Ratio Test says that if $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$, then the test gives no information. For instance, let's apply the Ratio Test to each of the following series:

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad\qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

In the first series $a_n = 1/n$ and

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1/(n+1)}{1/n} = \frac{n}{n+1} \to 1 \qquad \text{as } n \to \infty$$

In the second series $a_n = 1/n^2$ and

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1/(n+1)^2}{1/n^2} = \left(\frac{n}{n+1}\right)^2 \to 1 \qquad \text{as } n \to \infty$$

The Ratio Test is usually conclusive if the nth term of the series contains an exponential or a factorial, as in Examples 1 and 2. The test will always fail for p-series, as in Example 3.

In both cases the Ratio Test fails to determine whether the series converges or diverges, so we must try another test. Here the first series is the harmonic series, which we know diverges; the second series is a p-series with $p > 1$, so it converges. ■

■ The Root Test

The following test is convenient to apply when nth powers occur. Its proof is similar to the proof of the Ratio Test and is left as Exercise 45.

The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then part (iii) of the Root Test says that the test gives no information. The series $\Sigma\, a_n$ could converge or diverge. (If $L = 1$ in the Ratio Test, don't try the Root Test because L will again be 1. And if $L = 1$ in the Root Test, don't try the Ratio Test because it will fail too.)

EXAMPLE 4 Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

SOLUTION

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2+\dfrac{3}{n}}{3+\dfrac{2}{n}} \to \frac{2}{3} < 1$$

Thus the given series is absolutely convergent (and therefore convergent) by the Root Test. ■

EXAMPLE 5 Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$ converges or diverges.

SOLUTION Here it seems natural to apply the Root Test:

$$\sqrt[n]{|a_n|} = \frac{n}{n+1} \to 1 \qquad \text{as } n \to \infty$$

Since this limit is 1, the Root Test is inconclusive. However, using Equation 3.6.6 we see that

$$a_n = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(\dfrac{n+1}{n}\right)^n} \to \frac{1}{e} \qquad \text{as } n \to \infty$$

Since this limit is different from zero, the series diverges by the Test for Divergence. ■

Example 5 serves as a reminder that when testing a series for convergence or divergence it is often helpful to apply the Test for Divergence before attempting other tests.

11.6 Exercises

1. What can you say about the series $\sum a_n$ in each of the following cases?

(a) $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 8$

(b) $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 0.8$

(c) $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 1$

2. Suppose that for the series $\sum a_n$ we have $\lim_{n\to\infty}\left|a_n/a_{n+1}\right| = 2$. What is $\lim_{n\to\infty}\left|a_{n+1}/a_n\right|$? Does the series converge?

3–20 Use the Ratio Test to determine whether the series is convergent or divergent.

3. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{5^n}$

4. $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}$

5. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n}{2^n n^3}$

6. $\displaystyle\sum_{n=0}^{\infty}\frac{(-3)^n}{(2n+1)!}$

7. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k!}$

8. $\displaystyle\sum_{k=1}^{\infty}ke^{-k}$

9. $\displaystyle\sum_{n=1}^{\infty}\frac{10^n}{(n+1)\,4^{2n+1}}$

10. $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{100^n}$

11. $\displaystyle\sum_{n=1}^{\infty}\frac{n\pi^n}{(-3)^{n-1}}$

12. $\displaystyle\sum_{n=1}^{\infty}\frac{n^{10}}{(-10)^{n+1}}$

13. $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n\pi/3)}{n!}$

14. $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{n^n}$

15. $\displaystyle\sum_{n=1}^{\infty}\frac{n^{100}\,100^n}{n!}$

16. $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$

17. $$1 - \frac{2!}{1\cdot3} + \frac{3!}{1\cdot3\cdot5} - \frac{4!}{1\cdot3\cdot5\cdot7} + \cdots + (-1)^{n-1}\frac{n!}{1\cdot3\cdot5\cdot\,\cdots\,\cdot(2n-1)} + \cdots$$

18. $$\frac{2}{3} + \frac{2\cdot5}{3\cdot5} + \frac{2\cdot5\cdot8}{3\cdot5\cdot7} + \frac{2\cdot5\cdot8\cdot11}{3\cdot5\cdot7\cdot9} + \cdots$$

19. $\displaystyle\sum_{n=1}^{\infty}\frac{2\cdot4\cdot6\cdot\,\cdots\,\cdot(2n)}{n!}$

20. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2^n n!}{5\cdot8\cdot11\cdot\,\cdots\,\cdot(3n+2)}$

21–26 Use the Root Test to determine whether the series is convergent or divergent.

21. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$

22. $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{n^n}$

23. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(\ln n)^n}$

24. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{-2n}{n+1}\right)^{5n}$

25. $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$

26. $\displaystyle\sum_{n=0}^{\infty}(\arctan n)^n$

27–34 Use any test to determine whether the series is absolutely convergent, conditionally convergent, or divergent.

27. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n\ln n}{n}$

28. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1-n}{2+3n}\right)^n$

29. $\displaystyle\sum_{n=1}^{\infty}\frac{(-9)^n}{n10^{n+1}}$

30. $\displaystyle\sum_{n=1}^{\infty}\frac{n5^{2n}}{10^{n+1}}$

31. $\displaystyle\sum_{n=2}^{\infty}\left(\frac{n}{\ln n}\right)^n$

32. $\displaystyle\sum_{n=1}^{\infty}\frac{\sin(n\pi/6)}{1+n\sqrt{n}}$

33. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\arctan n}{n^2}$

34. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n}\ln n}$ [*Hint*: $\ln x < \sqrt{x}$.]

35. The terms of a series are defined recursively by the equations

$$a_1 = 2 \qquad a_{n+1} = \frac{5n+1}{4n+3}a_n$$

Determine whether $\sum a_n$ converges or diverges.

36. A series $\sum a_n$ is defined by the equations

$$a_1 = 1 \qquad a_{n+1} = \frac{2+\cos n}{\sqrt{n}}a_n$$

Determine whether $\sum a_n$ converges or diverges.

37–38 Let $\{b_n\}$ be a sequence of positive numbers that converges to $\frac{1}{2}$. Determine whether the given series is absolutely convergent.

37. $\displaystyle\sum_{n=1}^{\infty}\frac{b_n^n\cos n\pi}{n}$

38. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n!}{n^n b_1 b_2 b_3\cdots b_n}$

39. For which of the following series is the Ratio Test inconclusive (that is, it fails to give a definite answer)?

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{2^n}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{\sqrt{n}}$

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{1+n^2}$

40. For which positive integers k is the following series convergent?

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$$

41. (a) Show that $\sum_{n=0}^{\infty} x^n/n!$ converges for all x.
(b) Deduce that $\lim_{n\to\infty} x^n/n! = 0$ for all x.

42. Let $\sum a_n$ be a series with positive terms and let $r_n = a_{n+1}/a_n$. Suppose that $\lim_{n\to\infty} r_n = L < 1$, so $\sum a_n$ converges by the Ratio Test. As usual, we let R_n be the remainder after n terms, that is,

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

(a) If $\{r_n\}$ is a decreasing sequence and $r_{n+1} < 1$, show, by summing a geometric series, that

$$R_n \leqslant \frac{a_{n+1}}{1 - r_{n+1}}$$

(b) If $\{r_n\}$ is an increasing sequence, show that

$$R_n \leqslant \frac{a_{n+1}}{1 - L}$$

43. (a) Find the partial sum s_5 of the series $\sum_{n=1}^{\infty} 1/(n2^n)$. Use Exercise 42 to estimate the error in using s_5 as an approximation to the sum of the series.
(b) Find a value of n so that s_n is within 0.00005 of the sum. Use this value of n to approximate the sum of the series.

44. Use the sum of the first 10 terms to approximate the sum of the series

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

Use Exercise 42 to estimate the error.

45. Prove the Root Test. [*Hint for part (i):* Take any number r such that $L < r < 1$ and use the fact that there is an integer N such that $\sqrt[n]{|a_n|} < r$ whenever $n \geqslant N$.]

46. Around 1910, the Indian mathematician Srinivasa Ramanujan discovered the formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}$$

William Gosper used this series in 1985 to compute the first 17 million digits of π.
(a) Verify that the series is convergent.
(b) How many correct decimal places of π do you get if you use just the first term of the series? What if you use two terms?

11.7 Strategy for Testing Series

We now have several ways of testing a series for convergence or divergence; the problem is to decide which test to use on which series. In this respect, testing series is similar to integrating functions. Again, there are no hard and fast rules about which test to apply to a given series, but you may find the following advice of some use.

It is not wise to apply a list of the tests in a specific order until one finally works. That would be a waste of time and effort. Instead, as with integration, the main strategy is to classify the series according to its *form*.

1. **Test for Divergence** If you can see that $\lim_{n\to\infty} a_n$ may be different from 0, then apply the Test for Divergence.
2. ***p*-Series** If the series is of the form $\sum 1/n^p$, then it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leqslant 1$.
3. **Geometric Series** If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, then it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geqslant 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.
4. **Comparison Tests** If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. Notice that most of the series in Exercises 11.4 have this form. (The value of p should be chosen as in Section 11.4 by keeping only the highest powers of n in the numerator and denominator.) The comparison tests apply only to series with positive terms, but if $\sum a_n$ has some negative terms, then we can apply a comparison test to $\sum |a_n|$ and test for absolute convergence.

5. **Alternating Series Test** If the series is of the form $\Sigma(-1)^{n-1}b_n$ or $\Sigma(-1)^n b_n$, then the Alternating Series Test is an obvious possibility. Note that if Σb_n converges, then the given series is absolutely convergent and therefore convergent.
6. **Ratio Test** Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.
7. **Root Test** If a_n is of the form $(b_n)^n$, then the Root Test may be useful.
8. **Integral Test** If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

In the following examples we don't work out all the details but simply indicate which tests should be used.

EXAMPLE 1 $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

Since $a_n \to \frac{1}{2} \neq 0$ as $n \to \infty$, we should use the Test for Divergence. ■

EXAMPLE 2 $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

Since a_n is an algebraic function of n, we compare the given series with a p-series. The comparison series for the Limit Comparison Test is Σb_n, where

$$b_n = \frac{\sqrt{n^3}}{3n^3} = \frac{n^{3/2}}{3n^3} = \frac{1}{3n^{3/2}}$$ ■

EXAMPLE 3 $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

Since the integral $\int_1^\infty xe^{-x^2}\,dx$ is easily evaluated, we use the Integral Test. The Ratio Test also works. ■

EXAMPLE 4 $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^4+1}$

Since the series is alternating, we use the Alternating Series Test. We can also observe that $\Sigma|a_n|$ converges (compare to $\Sigma 1/n^2$) so the given series converges absolutely and hence converges. ■

EXAMPLE 5 $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k!}$

Since the series involves $k!$, we use the Ratio Test. ■

EXAMPLE 6 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

Since the series is closely related to the geometric series $\Sigma 1/3^n$, we use the Direct Comparison Test or the Limit Comparison Test. ■

11.7 Exercises

1–8 Two similar-looking series are given. Test each one for convergence or divergence.

1. (a) $\sum_{n=1}^{\infty} \frac{1}{5^n}$ (b) $\sum_{n=1}^{\infty} \frac{1}{5^n + n}$

2. (a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

3. (a) $\sum_{n=1}^{\infty} \frac{n}{3^n}$ (b) $\sum_{n=1}^{\infty} \frac{3^n}{n}$

4. (a) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ (b) $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n}$

5. (a) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ (b) $\sum_{n=1}^{\infty} \left(\frac{n}{n^2+1}\right)^n$

6. (a) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ (b) $\sum_{n=10}^{\infty} \frac{1}{n \ln n}$

7. (a) $\sum_{n=1}^{\infty} \frac{1}{n + n!}$ (b) $\sum_{n=1}^{\infty} \left(\frac{1}{n} + \frac{1}{n!}\right)$

8. (a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$

9–48 Test the series for convergence or divergence.

9. $\sum_{n=1}^{\infty} \frac{n^2-1}{n^3+1}$ **10.** $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$

11. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^3+1}$ **12.** $\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1}$

13. $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ **14.** $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$

15. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ **16.** $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^4}{4^n}$

17. $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$ **18.** $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

19. $\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{3^n}\right)$ **20.** $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}}$

21. $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$ **22.** $\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$

23. $\sum_{k=1}^{\infty} \frac{2^{k-1}3^{k+1}}{k^k}$ **24.** $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n}$

25. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-1)}$

26. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$

27. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$ **28.** $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}-1}{k(\sqrt{k}+1)}$

29. $\sum_{n=1}^{\infty} (-1)^n \cos(1/n^2)$ **30.** $\sum_{k=1}^{\infty} \frac{1}{2+\sin k}$

31. $\sum_{n=1}^{\infty} \tan(1/n)$ **32.** $\sum_{n=1}^{\infty} n \sin(1/n)$

33. $\sum_{n=1}^{\infty} \frac{4-\cos n}{\sqrt{n}}$ **34.** $\sum_{n=1}^{\infty} \frac{8+(-1)^n n}{n}$

35. $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$ **36.** $\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$

37. $\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$ **38.** $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

39. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\cosh n}$ **40.** $\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+5}$

41. $\sum_{k=1}^{\infty} \frac{5^k}{3^k+4^k}$ **42.** $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$

43. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ **44.** $\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2 n}$

45. $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ **46.** $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$

47. $\sum_{n=1}^{\infty} (\sqrt[n]{2}-1)^n$ **48.** $\sum_{n=1}^{\infty} (\sqrt[n]{2}-1)$

11.8 Power Series

So far we have studied series of numbers: Σa_n. Here we consider series, called *power series*, in which each term includes a power of the variable x: $\Sigma c_n x^n$.

Power Series

A **power series** is a series of the form

$$\boxed{1} \qquad \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the **coefficients** of the series. For each number that we substitute for x, the series (1) is a series of constants that we can test for convergence or divergence. A power series may converge for some values of x and diverge for other values of x. The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

whose domain is the set of all x for which the series converges. Notice that f resembles a polynomial. The only difference is that f has infinitely many terms.

For instance, if we take $c_n = 1$ for all n, the power series becomes the geometric series

2
$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

which converges when $-1 < x < 1$ and diverges when $|x| \geqslant 1$. (See Equation 11.2.5.) In fact if we put $x = \frac{1}{2}$ in the geometric series (2) we get the convergent series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

but if we put $x = 2$ in (2) we get the divergent series

$$\sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + 16 + \cdots$$

More generally, a series of the form

3
$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a **power series in $(x - a)$** or a **power series centered at a** or a **power series about a**. Notice that in writing out the term corresponding to $n = 0$ in Equations 1 and 3 we have adopted the convention that $(x-a)^0 = 1$ even when $x = a$. Notice also that when $x = a$, all of the terms are 0 for $n \geqslant 1$ and so the power series (3) always converges when $x = a$.

To determine the values of x for which a power series converges, we normally use the Ratio (or Root) Test.

Trigonometric Series
A power series is a series in which each term is a power function. A **trigonometric series**

$$\sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is a series whose terms are trigonometric functions. This type of series is discussed on the website

www.StewartCalculus.com

Click on *Additional Topics* and then on *Fourier Series*.

EXAMPLE 1 For what values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

SOLUTION If we let a_n denote the nth term of the series, as usual, then $a_n = (x-3)^n/n$, and

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right|$$

$$= \frac{1}{1 + \dfrac{1}{n}} |x-3| \to |x-3| \qquad \text{as } n \to \infty$$

By the Ratio Test, the given series is absolutely convergent, and therefore convergent, when $|x-3| < 1$ and divergent when $|x-3| > 1$. Now

$$|x-3| < 1 \iff -1 < x - 3 < 1 \iff 2 < x < 4$$

so the series converges when $2 < x < 4$ and diverges when $x < 2$ or $x > 4$.

The Ratio Test gives no information when $|x - 3| = 1$ so we must consider $x = 2$ and $x = 4$ separately. If we put $x = 4$ in the series, it becomes $\Sigma\, 1/n$, the harmonic series, which is divergent. If $x = 2$, the series is $\Sigma\, (-1)^n/n$, which converges by the Alternating Series Test. Thus the given power series converges for $2 \leqslant x < 4$. ■

EXAMPLE 2 For what values of x is the series $\displaystyle\sum_{n=0}^{\infty} n!x^n$ convergent?

SOLUTION Again we use the Ratio Test. Let $a_n = n!x^n$. If $x \neq 0$, we have

Notice that

$$(n + 1)! = (n + 1)n(n - 1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$
$$= (n + 1)n!$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n + 1)!x^{n+1}}{n!x^n} \right| = \lim_{n\to\infty} (n + 1)|x| = \infty$$

By the Ratio Test, the series diverges when $x \neq 0$. Thus the given series converges only when $x = 0$. ■

EXAMPLE 3 For what values of x does the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$ converge?

SOLUTION Here $a_n = x^n/(2n)!$ and, as $n \to \infty$,

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{[2(n + 1)]!} \cdot \frac{(2n)!}{x^n} \right| = \frac{(2n)!}{(2n + 2)!} |x|$$

$$= \frac{(2n)!}{(2n)!(2n + 1)(2n + 2)} |x| = \frac{|x|}{(2n + 1)(2n + 2)} \to 0 < 1$$

for all x. Thus, by the Ratio Test, the given series converges for all values of x. ■

Interval of Convergence

For the power series that we have looked at so far, the set of values of x for which the series is convergent has always turned out to be an interval [a finite interval for the geometric series and the series in Example 1, the infinite interval $(-\infty, \infty)$ in Example 3, and a collapsed interval $[0, 0] = \{0\}$ in Example 2]. The following theorem, proved in Appendix F, says that this is true in general.

4 Theorem For a power series $\displaystyle\sum_{n=0}^{\infty} c_n(x - a)^n$, there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

The number R in case (iii) is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii). The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges. In case (i) the interval consists of just a single point a. In case (ii) the interval is $(-\infty, \infty)$. In case (iii) note that the inequality $|x - a| < R$ can be rewritten as $a - R < x < a + R$. When x is an *endpoint* of the interval, that is,

$x = a \pm R$, anything can happen—the series might converge at one or both endpoints or it might diverge at both endpoints. Thus in case (iii) there are four possibilities for the interval of convergence:

$$(a - R, a + R) \qquad (a - R, a + R] \qquad [a - R, a + R) \qquad [a - R, a + R]$$

The situation is illustrated in Figure 1.

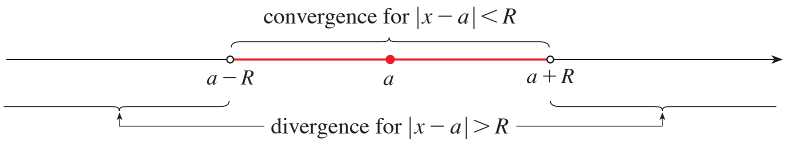


FIGURE 1

We summarize here the radius and interval of convergence for each of the examples already considered in this section.

	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 1	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
Example 2	$\sum_{n=0}^{\infty} n!\, x^n$	$R = 0$	$\{0\}$
Example 3	$\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$	$R = \infty$	$(-\infty, \infty)$

NOTE In general, the Ratio Test (or sometimes the Root Test) should be used to determine the radius of convergence R. The Ratio and Root Tests always fail when x is an endpoint of the interval of convergence, so the endpoints must be checked with some other test.

EXAMPLE 4 Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

SOLUTION Let $a_n = (-3)^n x^n/\sqrt{n+1}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n}\right| = \left|-3x\sqrt{\frac{n+1}{n+2}}\right|$$

$$= 3\sqrt{\frac{1+(1/n)}{1+(2/n)}}\,|x| \to 3|x| \qquad \text{as } n \to \infty$$

By the Ratio Test, the given series converges if $3|x| < 1$ and diverges if $3|x| > 1$. Thus it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$. This means that the radius of convergence is $R = \frac{1}{3}$.

We know the series converges in the interval $\left(-\frac{1}{3}, \frac{1}{3}\right)$, but we must now test for convergence at the endpoints of this interval. If $x = -\frac{1}{3}$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-3)^n\left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

which diverges. (It is a p-series with $p = \frac{1}{2} < 1$.) If $x = \frac{1}{3}$, the series is

$$\sum_{n=0}^{\infty} \frac{(-3)^n\left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

which converges by the Alternating Series Test. Therefore the given power series converges when $-\frac{1}{3} < x \leqslant \frac{1}{3}$, so the interval of convergence is $\left(-\frac{1}{3}, \frac{1}{3}\right]$. ■

EXAMPLE 5 Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

SOLUTION If $a_n = n(x+2)^n/3^{n+1}$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n}\right|$$

$$= \left(1 + \frac{1}{n}\right)\frac{|x+2|}{3} \to \frac{|x+2|}{3} \qquad \text{as } n \to \infty$$

Using the Ratio Test, we see that the series converges if $|x+2|/3 < 1$ and it diverges if $|x+2|/3 > 1$. So it converges if $|x+2| < 3$ and diverges if $|x+2| > 3$. Thus the radius of convergence is $R = 3$.

The inequality $|x+2| < 3$ can be written as $-5 < x < 1$, so we test the series at the endpoints -5 and 1. When $x = -5$, the series is

$$\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty} (-1)^n n$$

which diverges by the Test for Divergence [$(-1)^n n$ doesn't converge to 0]. When $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty} n$$

which also diverges by the Test for Divergence. Thus the series converges only when $-5 < x < 1$, so the interval of convergence is $(-5, 1)$. ■

11.8 Exercises

1. What is a power series?

2. (a) What is the radius of convergence of a power series? How do you find it?
(b) What is the interval of convergence of a power series? How do you find it?

3–36 Find the radius of convergence and interval of convergence of the power series.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^n n x^n$

5. $\displaystyle\sum_{n=1}^{\infty} \sqrt{n}\, x^n$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{5^n} x^n$

8. $\displaystyle\sum_{n=2}^{\infty} \frac{5^n}{n} x^n$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1} x^n$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$

13. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

14. $\displaystyle\sum_{n=1}^{\infty} n^n x^n$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^4 4^n}$

16. $\displaystyle\sum_{n=1}^{\infty} 2^n n^2 x^n$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{\sqrt{n}} x^n$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n} x^n$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n(n^2+1)} x^n$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$

21. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n} (x-1)^n$

23. $\displaystyle\sum_{n=2}^{\infty} \frac{(x+2)^n}{2^n \ln n}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{8^n} (x+6)^n$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$

27. $\displaystyle\sum_{n=4}^{\infty} \frac{\ln n}{n} x^n$

28. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} x^n$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{b^n} (x-a)^n, \quad b > 0$

30. $\displaystyle\sum_{n=2}^{\infty} \frac{b^n}{\ln n} (x-a)^n, \quad b > 0$

31. $\displaystyle\sum_{n=1}^{\infty} n!(2x-1)^n$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 x^n}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$

34. $\displaystyle\sum_{n=2}^{\infty} \frac{x^{2n}}{n(\ln n)^2}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{n!\,x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$

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37. If $\sum_{n=0}^{\infty} c_n 4^n$ is convergent, can we conclude that each of the following series is convergent?

(a) $\displaystyle\sum_{n=0}^{\infty} c_n(-2)^n$ (b) $\displaystyle\sum_{n=0}^{\infty} c_n(-4)^n$

38. Suppose that $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$ and diverges when $x = 6$. What can be said about the convergence or divergence of the following series?

(a) $\displaystyle\sum_{n=0}^{\infty} c_n$ (b) $\displaystyle\sum_{n=0}^{\infty} c_n 8^n$

(c) $\displaystyle\sum_{n=0}^{\infty} c_n(-3)^n$ (d) $\displaystyle\sum_{n=0}^{\infty} (-1)^n c_n 9^n$

39. If k is a positive integer, find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$$

40. Let p and q be real numbers with $p < q$. Find a power series whose interval of convergence is

(a) (p, q) (b) $(p, q]$ (c) $[p, q)$ (d) $[p, q]$

41. Is it possible to find a power series whose interval of convergence is $[0, \infty)$? Explain.

42. Graph the first several partial sums $s_n(x)$ of the series $\sum_{n=0}^{\infty} x^n$, together with the sum function $f(x) = 1/(1-x)$, on a common screen. On what interval do these partial sums appear to be converging to $f(x)$?

43. Show that if $\lim_{n\to\infty} \sqrt[n]{|c_n|} = c$, where $c \neq 0$, then the radius of convergence of the power series $\sum c_n x^n$ is $R = 1/c$.

44. Suppose that the power series $\sum c_n(x-a)^n$ satisfies $c_n \neq 0$ for all n. Show that if $\lim_{n\to\infty} |c_n/c_{n+1}|$ exists, then it is equal to the radius of convergence of the power series.

45. Suppose the series $\sum c_n x^n$ has radius of convergence 2 and the series $\sum d_n x^n$ has radius of convergence 3. What is the radius of convergence of the series $\sum (c_n + d_n)x^n$?

46. Suppose that the radius of convergence of the power series $\sum c_n x^n$ is R. What is the radius of convergence of the power series $\sum c_n x^{2n}$?

11.9 Representations of Functions as Power Series

In this section we learn how to represent some familiar functions as sums of power series. You might wonder why we would ever want to express a known function as a sum of infinitely many terms. We will see later that this strategy is useful for integrating functions that don't have elementary antiderivatives and for approximating functions by polynomials. (Scientists do this to simplify the expressions they deal with; computer scientists do this to evaluate functions on calculators and computers.)

Representations of Functions using Geometric Series

We will obtain power series representations for several functions by manipulating geometric series. We start with an equation that we have seen before.

$$\boxed{1} \qquad \boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1}$$

We first encountered this equation in Example 11.2.7, where we obtained it by observing that the series is a geometric series with $a = 1$ and $r = x$. Here our point of view is different: we now regard Equation 1 as expressing the function $f(x) = 1/(1-x)$ as a sum of a power series. We say that $\Sigma_{n=0}^{\infty} x^n$, $|x| < 1$, is a *power series representation* of $1/(1-x)$ on the interval $(-1, 1)$.

A geometric illustration of Equation 1 is shown in Figure 1. Because the sum of a series is the limit of the sequence of partial sums, we have

$$\frac{1}{1-x} = \lim_{n\to\infty} s_n(x)$$

where

$$s_n(x) = 1 + x + x^2 + \cdots + x^n$$

is the nth partial sum. Notice that as n increases, $s_n(x)$ becomes a better approximation to $f(x)$ for $-1 < x < 1$.

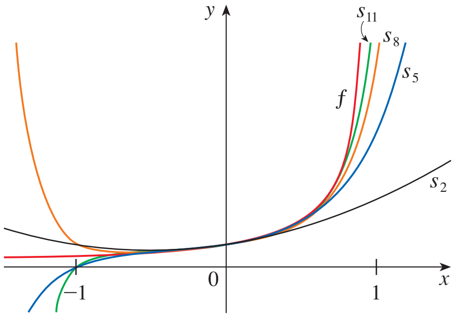


FIGURE 1
$f(x) = \dfrac{1}{1-x}$ and some of its partial sums

The power series (1) that represents the function $f(x) = 1/(1-x)$ can be used to obtain power series representations of many other functions, as we see in the following examples.

EXAMPLE 1 Express $1/(1 + x^2)$ as the sum of a power series and find the interval of convergence.

SOLUTION Replacing x by $-x^2$ in Equation 1, we have

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$
$$= \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

Because this is a geometric series, it converges when $|-x^2| < 1$, that is, $x^2 < 1$, or $|x| < 1$. Therefore the interval of convergence is $(-1, 1)$. (Of course, we could have determined the radius of convergence by applying the Ratio Test, but that much work is unnecessary here.) ■

EXAMPLE 2 Find a power series representation for $1/(x + 2)$.

SOLUTION In order to put this function in the form of the left side of Equation 1, we first factor a 2 from the denominator:

$$\frac{1}{2+x} = \frac{1}{2\left(1+\dfrac{x}{2}\right)} = \frac{1}{2\left[1-\left(-\dfrac{x}{2}\right)\right]}$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

This series converges when $|-x/2| < 1$, that is, $|x| < 2$. So the interval of convergence is $(-2, 2)$. ■

EXAMPLE 3 Find a power series representation of $x^3/(x + 2)$.

SOLUTION Since this function is just x^3 times the function in Example 2, all we have to do is to multiply that series by x^3:

It's legitimate to move x^3 across the sigma sign because it doesn't depend on n. [Use Theorem 11.2.8(i) with $c = x^3$.]

$$\frac{x^3}{x+2} = x^3 \cdot \frac{1}{x+2} = x^3 \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{n+3}$$
$$= \tfrac{1}{2}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{8}x^5 - \tfrac{1}{16}x^6 + \cdots$$

Another way of writing this series is as follows:

$$\frac{x^3}{x+2} = \sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{2^{n-2}}x^n$$

As in Example 2, the interval of convergence is $(-2, 2)$. ■

■ Differentiation and Integration of Power Series

The sum of a power series is a function $f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n$ whose domain is the interval of convergence of the series. We would like to be able to differentiate and integrate such functions, and the following theorem (which we won't prove) says that we can do so by differentiating or integrating each individual term in the series, just as we would for a polynomial. This is called **term-by-term differentiation and integration**.

2 Theorem If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\displaystyle\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

In part (i), the sum starts at $n = 1$ because the derivative of c_0, the constant term of f, is 0.

In part (ii), $\int c_0\,dx = c_0x + C_1$ is written as $c_0(x-a) + C$, where $C = C_1 + ac_0$, so all the terms of the series have the same form.

NOTE 1 Equations (i) and (ii) in Theorem 2 can be rewritten in the form

$$\text{(iii)}\quad \frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty}\frac{d}{dx}\left[c_n(x-a)^n\right]$$

$$\text{(iv)}\quad \int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right]dx = \sum_{n=0}^{\infty}\int c_n(x-a)^n\,dx$$

We know that, for finite sums, the derivative of a sum is the sum of the derivatives and the integral of a sum is the sum of the integrals. Equations (iii) and (iv) assert that the same is true for infinite sums, provided we are dealing with *power series*. (For other types of series of functions the situation is not as simple; see Exercise 44.)

NOTE 2 Although Theorem 2 says that the radius of convergence remains the same when a power series is differentiated or integrated, this does not mean that the *interval* of convergence remains the same. It may happen that the original series converges at an endpoint, whereas the differentiated series diverges there. (See Exercise 45.)

The idea of differentiating a power series term by term is the basis for a powerful method for solving differential equations. In Exercises 37–40 you will see how a function expressed as a power series can be a solution to a differential equation.

EXAMPLE 4 Express $1/(1-x)^2$ as a power series by differentiating Equation 1. What is the radius of convergence?

SOLUTION We start with

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

Differentiating each side, we get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}$$

If we wish, we can replace n by $n + 1$ and write the answer as

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$$

According to Theorem 2, the radius of convergence of the differentiated series is the same as the radius of convergence of the original series, namely, $R = 1$. ■

EXAMPLE 5 Find a power series representation for $\ln(1 + x)$ and its radius of convergence.

SOLUTION We notice that the derivative of this function is $1/(1 + x)$. From Equation 1 we have

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \cdots \qquad |x| < 1$$

Integrating both sides of this equation, we get

$$\begin{aligned} \ln(1+x) &= \int \frac{1}{1+x}\,dx = \int (1 - x + x^2 - x^3 + \cdots)\,dx \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + C \\ &= \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n} + C \qquad |x| < 1 \end{aligned}$$

To determine the value of C we put $x = 0$ in this equation and obtain $\ln(1 + 0) = C$. Thus $C = 0$ and

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n} \qquad |x| < 1$$

The radius of convergence is the same as for the original series: $R = 1$. ■

EXAMPLE 6 Find a power series representation for $f(x) = \tan^{-1}x$.

SOLUTION We observe that $f'(x) = 1/(1 + x^2)$ and find the required series by integrating the power series for $1/(1 + x^2)$ found in Example 1.

$$\begin{aligned} \tan^{-1}x &= \int \frac{1}{1+x^2}\,dx = \int (1 - x^2 + x^4 - x^6 + \cdots)\,dx \\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \end{aligned}$$

To find C we put $x = 0$ and obtain $C = \tan^{-1}0 = 0$. Therefore

$$\begin{aligned} \tan^{-1}x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \end{aligned}$$

Since the radius of convergence of the series for $1/(1 + x^2)$ is 1, the radius of convergence of this series for $\tan^{-1}x$ is also 1. ■

The power series for $\tan^{-1}x$ obtained in Example 6 is called *Gregory's series* after the Scottish mathematician James Gregory (1638–1675), who had anticipated some of Newton's discoveries. We have shown that Gregory's series is valid when $-1 < x < 1$, but it turns out (although it isn't easy to prove) that it is also valid when $x = \pm 1$. Notice that when $x = 1$ the series becomes

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

This beautiful result is known as the Leibniz formula for π.

EXAMPLE 7

(a) Evaluate $\int [1/(1+x^7)]\,dx$ as a power series.

(b) Use part (a) to approximate $\int_0^{0.5} [1/(1+x^7)]\,dx$ correct to within 10^{-7}.

SOLUTION

(a) The first step is to express the integrand, $1/(1+x^7)$, as the sum of a power series. As in Example 1, we start with Equation 1 and replace x by $-x^7$:

$$\frac{1}{1+x^7} = \frac{1}{1-(-x^7)} = \sum_{n=0}^{\infty} (-x^7)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{7n} = 1 - x^7 + x^{14} - \cdots$$

This example demonstrates one way in which power series representations are useful. Integrating $1/(1+x^7)$ by hand is incredibly difficult. Different computer algebra systems return different forms of the answer, but they are all extremely complicated. The infinite series answer that we obtain in Example 7(a) is actually much easier to deal with than the finite answer provided by a computer.

Now we integrate term by term:

$$\int \frac{1}{1+x^7}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{7n}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1}$$

$$= C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots$$

This series converges for $|-x^7| < 1$, that is, for $|x| < 1$.

(b) In applying the Fundamental Theorem of Calculus, it doesn't matter which antiderivative we use, so let's use the antiderivative from part (a) with $C = 0$:

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx = \left[x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots \right]_0^{1/2}$$

$$= \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} + \cdots + \frac{(-1)^n}{(7n+1)2^{7n+1}} + \cdots$$

This infinite series is the exact value of the definite integral, but since it is an alternating series, we can approximate the sum using the Alternating Series Estimation Theorem. If we stop adding after the term with $n = 3$, the error is smaller than the term with $n = 4$:

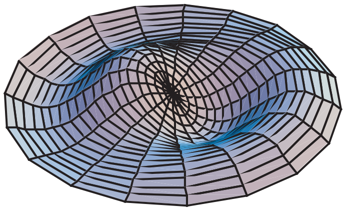
$$\frac{1}{29 \cdot 2^{29}} \approx 6.4 \times 10^{-11}$$

So we have

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx \approx \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} \approx 0.49951374 \qquad ■$$

■ Functions Defined by Power Series

Some of the most important functions in the sciences are defined by power series and are not expressible in terms of elementary functions (as described in Section 7.5). Many of these arise naturally as solutions of differential equations. One such class of functions is the **Bessel functions**, named after the German astronomer Friedrich Bessel (1784–1846). These functions first arose when Bessel solved Kepler's equation for describing planetary motion. Since that time, Bessel functions have been applied in many different physical situations, including the temperature distribution in a circular plate and the shape of a vibrating drumhead. Bessel functions appear in the next example as well as in Exercises 39 and 40. Other examples of functions defined by power series are given in Exercises 38 and 41.



A computer-generated model, involving Bessel functions and cosine functions, of a vibrating drumhead.

EXAMPLE 8 The Bessel function of order 0 is defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

(a) Find the domain of J_0.

(b) Find the derivative of J_0.

SOLUTION

(a) Let $a_n = (-1)^n x^{2n}/[2^{2n}(n!)^2]$. Then

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right| \\ &= \frac{x^{2n+2}}{2^{2n+2}(n+1)^2(n!)^2} \cdot \frac{2^{2n}(n!)^2}{x^{2n}} \\ &= \frac{x^2}{4(n+1)^2} \to 0 < 1 \qquad \text{for all } x\end{aligned}$$

Thus, by the Ratio Test, the given series converges for all values of x. In other words, the domain of the Bessel function J_0 is $(-\infty, \infty) = \mathbb{R}$.

(b) By Theorem 2, J_0 is differentiable for all x and its derivative is found by term-by-term differentiation as follows:

$$J_0'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2} \qquad ■$$

Recall that the sum of a series is equal to the limit of the sequence of partial sums. So when we define the Bessel function in Example 8 as the sum of a series we mean that, for every real number x,

$$J_0(x) = \lim_{n\to\infty} s_n(x) \qquad \text{where} \qquad s_n(x) = \sum_{i=0}^{n} \frac{(-1)^i x^{2i}}{2^{2i}(i!)^2}$$

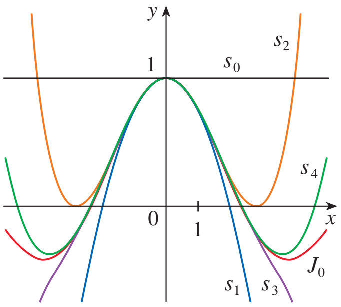


FIGURE 2
Partial sums of the Bessel function J_0

The first few partial sums are

$$s_0(x) = 1$$

$$s_1(x) = 1 - \frac{x^2}{4}$$

$$s_2(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64}$$

$$s_3(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304}$$

$$s_4(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147{,}456}$$

Figure 2 shows the graphs of these partial sums, which are polynomials. They are all approximations to the function J_0, but the approximations become better when more terms are included. Figure 3 shows a more complete graph of the Bessel function.

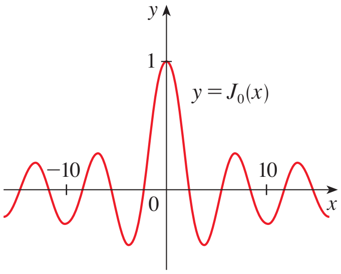


FIGURE 3

11.9 Exercises

1. If the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$ is 10, what is the radius of convergence of the series $\sum_{n=1}^{\infty} n c_n x^{n-1}$? Why?

2. Suppose you know that the series $\sum_{n=0}^{\infty} b_n x^n$ converges for $|x| < 2$. What can you say about the following series? Why?

$$\sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$$

3–12 Find a power series representation for the function and determine the interval of convergence.

3. $f(x) = \dfrac{1}{1+x}$

4. $f(x) = \dfrac{x}{1+x}$

5. $f(x) = \dfrac{1}{1-x^2}$

6. $f(x) = \dfrac{5}{1-4x^2}$

7. $f(x) = \dfrac{2}{3-x}$

8. $f(x) = \dfrac{4}{2x+3}$

9. $f(x) = \dfrac{x^2}{x^4+16}$

10. $f(x) = \dfrac{x}{2x^2+1}$

11. $f(x) = \dfrac{x-1}{x+2}$

12. $f(x) = \dfrac{x+a}{x^2+a^2}, \quad a > 0$

13–14 Express the function as the sum of a power series by first using partial fractions. Find the interval of convergence.

13. $f(x) = \dfrac{2x-4}{x^2-4x+3}$

14. $f(x) = \dfrac{2x+3}{x^2+3x+2}$

15. (a) Use differentiation to find a power series representation for

$$f(x) = \frac{1}{(1+x)^2}$$

What is the radius of convergence?

(b) Use part (a) to find a power series for

$$f(x) = \frac{1}{(1+x)^3}$$

(c) Use part (b) to find a power series for

$$f(x) = \frac{x^2}{(1+x)^3}$$

16. (a) Use Equation 1 to find a power series representation for $f(x) = \ln(1-x)$. What is the radius of convergence?

(b) Use part (a) to find a power series for $f(x) = x\ln(1-x)$.

(c) By putting $x = \frac{1}{2}$ in your result from part (a), express ln 2 as the sum of an infinite series.

17–22 Find a power series representation for the function and determine the radius of convergence.

17. $f(x) = \dfrac{x}{(1+4x)^2}$

18. $f(x) = \left(\dfrac{x}{2-x}\right)^3$

19. $f(x) = \dfrac{1+x}{(1-x)^2}$

20. $f(x) = \dfrac{x^2+x}{(1-x)^3}$

21. $f(x) = \ln(5-x)$

22. $f(x) = x^2 \tan^{-1}(x^3)$

23–26 Find a power series representation for f, and graph f and several partial sums $s_n(x)$ on the same screen. What happens as n increases?

23. $f(x) = \dfrac{x^2}{x^2+1}$

24. $f(x) = \ln(1+x^4)$

25. $f(x) = \ln\left(\dfrac{1+x}{1-x}\right)$

26. $f(x) = \tan^{-1}(2x)$

27–30 Evaluate the indefinite integral as a power series. What is the radius of convergence?

27. $\displaystyle\int \frac{t}{1-t^8}\,dt$

28. $\displaystyle\int \frac{t}{1+t^3}\,dt$

29. $\displaystyle\int x^2 \ln(1+x)\,dx$

30. $\displaystyle\int \frac{\tan^{-1}x}{x}\,dx$

31–34 Use a power series to approximate the definite integral to six decimal places.

31. $\displaystyle\int_0^{0.3} \frac{x}{1+x^3}\,dx$

32. $\displaystyle\int_0^{1/2} \arctan\frac{x}{2}\,dx$

33. $\displaystyle\int_0^{0.2} x\ln(1+x^2)\,dx$

34. $\displaystyle\int_0^{0.3} \frac{x^2}{1+x^4}\,dx$

35. Use the result of Example 6 to compute arctan 0.2 correct to five decimal places.

36. Use the result of Example 5 to compute ln 1.1 correct to four decimal places.

37. (a) Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is a solution of the differential equation

$$f'(x) = f(x)$$

(b) Show that $f(x) = e^x$.

38. Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

is a solution of the differential equation

$$f''(x) + f(x) = 0$$

39. (a) Show that J_0 (the Bessel function of order 0 given in Example 8) satisfies the differential equation

$$x^2 J_0''(x) + x J_0'(x) + x^2 J_0(x) = 0$$

(b) Evaluate $\int_0^1 J_0(x)\,dx$ correct to three decimal places.

40. The Bessel function of order 1 is defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}}$$

(a) Find the domain of J_1.
(b) Show that J_1 satisfies the differential equation

$$x^2 J_1''(x) + x J_1'(x) + (x^2 - 1) J_1(x) = 0$$

(c) Show that $J_0'(x) = -J_1(x)$.

41. The function A defined by

$$A(x) = 1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \frac{x^9}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} + \cdots$$

is called an *Airy function* after the English mathematician and astronomer Sir George Airy (1801–1892).
(a) Find the domain of the Airy function.
(b) Graph the first several partial sums on a common screen.
(c) Use a computer algebra system that has built-in Airy functions to graph A on the same screen as the partial sums in part (b) and observe how the partial sums approximate A.

42. If $f(x) = \sum_{n=0}^{\infty} c_n x^n$, where $c_{n+4} = c_n$ for all $n \geqslant 0$, find the interval of convergence of the series and a formula for $f(x)$.

43. A function f is defined by

$$f(x) = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots$$

that is, its coefficients are $c_{2n} = 1$ and $c_{2n+1} = 2$ for all $n \geqslant 0$. Find the interval of convergence of the series and find an explicit formula for $f(x)$.

44. Let $f_n(x) = (\sin nx)/n^2$. Show that the series $\sum f_n(x)$ converges for all values of x but the series of derivatives $\sum f_n'(x)$ diverges when $x = 2n\pi$, n an integer. For what values of x does the series $\sum f_n''(x)$ converge?

45. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Find the intervals of convergence for f, f', and f''.

46. (a) Starting with the geometric series $\sum_{n=0}^{\infty} x^n$, find the sum of the series

$$\sum_{n=1}^{\infty} n x^{n-1} \qquad |x| < 1$$

(b) Find the sum of each of the following series.

$$\text{(i)}\ \sum_{n=1}^{\infty} n x^n, \quad |x| < 1 \qquad \text{(ii)}\ \sum_{n=1}^{\infty} \frac{n}{2^n}$$

(c) Find the sum of each of the following series.

$$\text{(i)}\ \sum_{n=2}^{\infty} n(n-1)x^n, \quad |x| < 1$$

$$\text{(ii)}\ \sum_{n=2}^{\infty} \frac{n^2 - n}{2^n} \qquad \text{(iii)}\ \sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

47. If $f(x) = 1/(1 - x)$, find a power series representation for $h(x) = xf'(x) + x^2 f''(x)$ and determine the radius of convergence. Use this to show that

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = 6$$

48. Use the power series representation of $f(x) = 1/(1 - x)^2$ and the fact that $9801 = 99^2$ to show that $1/9801$ is a repeating decimal that contains every two digit number in order, except for 98, as shown.

$$\frac{1}{9801} = 0.\overline{00\ 01\ 02\ 03 \ldots\ 96\ 97\ 99}$$

$\left[\textit{Hint:} \text{ Consider } x = \frac{1}{100}.\right]$

49. Use the power series for $\tan^{-1}x$ to prove the following expression for π as the sum of an infinite series:

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$$

50. (a) By completing the square, show that

$$\int_0^{1/2} \frac{dx}{x^2 - x + 1} = \frac{\pi}{3\sqrt{3}}$$

(b) By factoring $x^3 + 1$ as a sum of cubes, rewrite the integral in part (a). Then express $1/(x^3 + 1)$ as the sum of a power series and use it to prove the following formula for π:

$$\pi = \frac{3\sqrt{3}}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{8^n} \left(\frac{2}{3n+1} + \frac{1}{3n+2} \right)$$

51. Use the Ratio Test to show that if the series $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence R, then each of the series

$$\sum_{n=1}^{\infty} n c_n x^{n-1} \qquad \text{and} \qquad \sum_{n=0}^{\infty} c_n \frac{x^{n+1}}{n+1}$$

also has radius of convergence R.

11.10 Taylor and Maclaurin Series

In Section 11.9 we were able to find power series representations for a certain restricted class of functions, namely, those that can be obtained from geometric series. Here we investigate more general problems: Which functions have power series representations? How can we find such representations? We will see that some of the most important functions in calculus, such as e^x and $\sin x$, can be represented as power series.

Definitions of Taylor Series and Maclaurin Series

We start by supposing that f is a function that can be represented by a power series

$$\boxed{1} \quad f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + c_4(x - a)^4 + \cdots \qquad |x - a| < R$$

Let's try to determine what the coefficients c_n must be in terms of f. To begin, notice that if we put $x = a$ in Equation 1, then all terms after the first one are 0 and we get

$$f(a) = c_0$$

By Theorem 11.9.2, we can differentiate the series in Equation 1 term by term:

$$\boxed{2} \quad f'(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + 4c_4(x - a)^3 + \cdots \qquad |x - a| < R$$

and substitution of $x = a$ in Equation 2 gives

$$f'(a) = c_1$$

Now we differentiate both sides of Equation 2 and obtain

$$\boxed{3} \quad f''(x) = 2c_2 + 2 \cdot 3c_3(x - a) + 3 \cdot 4c_4(x - a)^2 + \cdots \qquad |x - a| < R$$

Again we put $x = a$ in Equation 3. The result is

$$f''(a) = 2c_2$$

Let's apply the procedure one more time. Differentiation of the series in Equation 3 gives

$$\boxed{4} \quad f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x - a) + 3 \cdot 4 \cdot 5c_5(x - a)^2 + \cdots \qquad |x - a| < R$$

and substitution of $x = a$ in Equation 4 gives

$$f'''(a) = 2 \cdot 3c_3 = 3!c_3$$

By now you can see the pattern. If we continue to differentiate and substitute $x = a$, we obtain

$$f^{(n)}(a) = 2 \cdot 3 \cdot 4 \cdot \cdots \cdot nc_n = n!c_n$$

Solving this equation for the nth coefficient c_n, we get

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This formula remains valid even for $n = 0$ if we adopt the conventions that $0! = 1$ and $f^{(0)} = f$. Thus we have proved the following theorem.

5 Theorem If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n \qquad |x - a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Substituting this formula for c_n back into the series, we see that *if* f has a power series expansion at a, then it must be of the following form.

$$\boxed{6} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$$
$$= f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots$$

Taylor and Maclaurin

The Taylor series is named after the English mathematician Brook Taylor (1685–1731) and the Maclaurin series is named in honor of the Scottish mathematician Colin Maclaurin (1698–1746) despite the fact that the Maclaurin series is really just a special case of the Taylor series. But the idea of representing particular functions as sums of power series goes back to Newton, and the general Taylor series was known to the Scottish mathematician James Gregory in 1668 and to the Swiss mathematician John Bernoulli in the 1690s. Taylor was apparently unaware of the work of Gregory and Bernoulli when he published his discoveries on series in 1715 in his book *Methodus incrementorum directa et inversa*. Maclaurin series are named after Colin Maclaurin because he popularized them in his calculus textbook *Treatise of Fluxions* published in 1742.

The series in Equation 6 is called the **Taylor series of the function *f* at *a*** (or **about *a*** or **centered at *a***). For the special case $a = 0$ the Taylor series becomes

$$\boxed{7} \qquad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

This case arises frequently enough that it is given the special name **Maclaurin series**.

NOTE 1 When we find a Taylor series for a function f, there is no guarantee that the sum of the Taylor series is equal to f. Theorem 5 says that *if* f has a power series representation about a, then that power series must be the Taylor series of f. There exist functions that are not equal to the sum of their Taylor series, such as the function given in Exercise 96.

NOTE 2 The power series representation at a of a function is unique, regardless of how it is found, because Theorem 5 states that if f has a power series representation $f(x) = \Sigma c_n(x - a)^n$, then c_n must be $f^{(n)}(a)/n!$. Thus all the power series representations we developed in Section 11.9 are in fact the Taylor series of the functions they represent.

EXAMPLE 1 We know from Equation 11.9.1 that the function $f(x) = 1/(1 - x)$ has a power series representation

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1$$

According to Theorem 5, this series must be the Maclaurin series of f with coefficients c_n given by $f^{(n)}(0)/n!$. To confirm this, we compute

$$\begin{aligned} f(x) &= \frac{1}{1-x} & f(0) &= 1 \\ f'(x) &= \frac{1}{(1-x)^2} & f'(0) &= 1 \\ f''(x) &= \frac{1 \cdot 2}{(1-x)^3} & f''(0) &= 1 \cdot 2 \\ f'''(x) &= \frac{1 \cdot 2 \cdot 3}{(1-x)^4} & f'''(0) &= 1 \cdot 2 \cdot 3 \end{aligned}$$

and, in general,

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \qquad f^{(n)}(0) = n!$$

Thus

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{n!}{n!} = 1$$

and, from Equation 7,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} x^n$$ ■

EXAMPLE 2 For the function $f(x) = e^x$, find the Maclaurin series and its radius of convergence.

SOLUTION If $f(x) = e^x$, then $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = e^0 = 1$ for all n. Therefore the Taylor series for f at 0 (that is, the Maclaurin series) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

To find the radius of convergence we let $a_n = x^n/n!$. Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \to 0 < 1$$

so, by the Ratio Test, the series converges for all x and the radius of convergence is $R = \infty$. ■

When Is a Function Represented by Its Taylor Series?

From Theorem 5 and Example 2 we can conclude that *if we know* that e^x has a power series representation at 0, then this power series must be its Maclaurin series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

So how can we determine whether e^x *does* have a power series representation?

Let's investigate the more general question: under what circumstances is a function equal to the sum of its Taylor series? In other words, if f has derivatives of all orders, when is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

As with any convergent series, this means that $f(x)$ is the limit of the sequence of partial sums. In the case of the Taylor series, the partial sums are

$$\begin{aligned} T_n(x) &= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

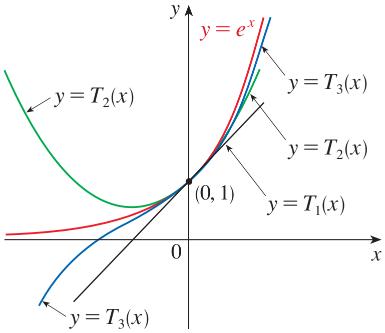


FIGURE 1

As n increases, $T_n(x)$ appears to approach e^x in Figure 1. This suggests that e^x is equal to the sum of its Taylor series.

Notice that T_n is a polynomial of degree n called the ***n*th-degree Taylor polynomial of *f* at *a***. For instance, for the exponential function $f(x) = e^x$, the result of Example 2 shows that the Taylor polynomials at 0 (or Maclaurin polynomials) with $n = 1$, 2, and 3 are

$$T_1(x) = 1 + x \qquad T_2(x) = 1 + x + \frac{x^2}{2!} \qquad T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

The graphs of the exponential function and these three Taylor polynomials are drawn in Figure 1.

In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n\to\infty} T_n(x)$$

If we let

$$R_n(x) = f(x) - T_n(x) \qquad \text{so that} \qquad f(x) = T_n(x) + R_n(x)$$

then $R_n(x)$ is called the **remainder** of the Taylor series. If we can somehow show that $\lim_{n\to\infty} R_n(x) = 0$, then it follows that

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} [f(x) - R_n(x)] = f(x) - \lim_{n\to\infty} R_n(x) = f(x)$$

We have therefore proved the following theorem.

8 Theorem If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a, and if

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

In trying to show that $\lim_{n\to\infty} R_n(x) = 0$ for a specific function f, we usually use the following theorem.

9 Taylor's Inequality If $|f^{(n+1)}(x)| \leqslant M$ for $|x - a| \leqslant d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \leqslant d$$

PROOF We first prove Taylor's Inequality for $n = 1$. Assume that $|f''(x)| \leqslant M$. In particular, we have $f''(x) \leqslant M$, so for $a \leqslant x \leqslant a + d$ we have

$$\int_a^x f''(t)\,dt \leqslant \int_a^x M\,dt$$

An antiderivative of f'' is f', so by Part 2 of the Fundamental Theorem of Calculus, we have

$$f'(x) - f'(a) \leqslant M(x - a) \qquad \text{or} \qquad f'(x) \leqslant f'(a) + M(x - a)$$

Thus

$$\int_a^x f'(t)\,dt \leqslant \int_a^x [f'(a) + M(t - a)]\,dt$$

$$f(x) - f(a) \leqslant f'(a)(x - a) + M\frac{(x - a)^2}{2}$$

$$f(x) - f(a) - f'(a)(x - a) \leqslant \frac{M}{2}(x - a)^2$$

But $R_1(x) = f(x) - T_1(x) = f(x) - f(a) - f'(a)(x - a)$. So

$$R_1(x) \leqslant \frac{M}{2}(x - a)^2$$

A similar argument, using $f''(x) \geqslant -M$, shows that

$$R_1(x) \geqslant -\frac{M}{2}(x - a)^2$$

So

$$|R_1(x)| \leqslant \frac{M}{2}|x - a|^2$$

Although we have assumed that $x > a$, similar calculations show that this inequality is also true for $x < a$.

This proves Taylor's Inequality for the case where $n = 1$. The result for any n is proved in a similar way by integrating $n + 1$ times. (See Exercise 95 for the case $n = 2$.) ■

Formulas for the Taylor Remainder Term

As alternatives to Taylor's Inequality, we have the following formulas for the remainder term. If $f^{(n+1)}$ is continuous on an interval I and $x \in I$, then

$$R_n(x) = \frac{1}{n!}\int_a^x (x - t)^n f^{(n+1)}(t)\,dt$$

This is called the *integral form of the remainder term*. Another formula, called *Lagrange's form of the remainder term,* states that there is a number z between x and a such that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n + 1)!}(x - a)^{n+1}$$

This version is an extension of the Mean Value Theorem (which is the case $n = 0$).

Proofs of these formulas, together with discussions of how to use them to solve the examples of Sections 11.10 and 11.11, are given on the website

www.StewartCalculus.com

Click on *Additional Topics* and then on *Formulas for the Remainder Term in Taylor series*.

NOTE In Section 11.11 we will explore the use of Taylor's Inequality in approximating functions. Our immediate use of it is in conjunction with Theorem 8.

When we apply Theorems 8 and 9 it is often helpful to make use of the following fact.

10
$$\boxed{\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad \text{for every real number } x}$$

This is true because we know from Example 2 that the series $\Sigma x^n/n!$ converges for all x and so its nth term approaches 0.

EXAMPLE 3 Prove that e^x is equal to the sum of its Maclaurin series.

SOLUTION If $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$ for all n. If d is any positive number and $|x| \leqslant d$, then $|f^{(n+1)}(x)| = e^x \leqslant e^d$. So Taylor's Inequality, with $a = 0$ and $M = e^d$, says that

$$|R_n(x)| \leqslant \frac{e^d}{(n+1)!}|x|^{n+1} \qquad \text{for } |x| \leqslant d$$

Notice that the same constant $M = e^d$ works for every value of n. But, from Equation 10, we have

$$\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

It follows from the Squeeze Theorem that $\lim_{n\to\infty}|R_n(x)| = 0$ and therefore $\lim_{n\to\infty} R_n(x) = 0$ for all values of x. By Theorem 8, e^x is equal to the sum of its Maclaurin series, that is,

11
$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{for all } x}$$
■

In particular, if we put $x = 1$ in Equation 11, we obtain the following expression for the number e as a sum of an infinite series:

With the help of computers, researchers have now accurately computed the value of e to trillions of decimal places.

12
$$\boxed{e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots}$$

EXAMPLE 4 Find the Taylor series for $f(x) = e^x$ at $a = 2$.

SOLUTION We have $f^{(n)}(2) = e^2$ and so, putting $a = 2$ in the definition of a Taylor series (6), we get

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$$

Again it can be verified, as in Example 2, that the radius of convergence is $R = \infty$. As in Example 3 we can verify that $\lim_{n\to\infty} R_n(x) = 0$, so

13
$$e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n \qquad \text{for all } x$$
■

We have two power series expansions for e^x, the Maclaurin series in Equation 11 and the Taylor series in Equation 13. The first is better if we are interested in values of x near 0 and the second is better if x is near 2.

■ Taylor Series of Important Functions

In Examples 2 and 4 we developed power series representations of the function e^x, and in Section 11.9 we found power series representations for several other functions, including $\ln(1 + x)$ and $\tan^{-1}x$. We now find representations for some additional important functions, including $\sin x$ and $\cos x$.

EXAMPLE 5 Find the Maclaurin series for sin x and prove that it represents sin x for all x.

SOLUTION We arrange our computation in two columns:

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

Since the derivatives repeat in a cycle of four, we can write the Maclaurin series as follows:

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Since $f^{(n+1)}(x)$ is ±sin x or ±cos x, we know that $|f^{(n+1)}(x)| \leqslant 1$ for all x. So we can take $M = 1$ in Taylor's Inequality:

$$\boxed{14} \qquad |R_n(x)| \leqslant \frac{M}{(n+1)!}|x^{n+1}| = \frac{|x|^{n+1}}{(n+1)!}$$

By Equation 10 the right side of this inequality approaches 0 as $n \to \infty$, so $|R_n(x)| \to 0$ by the Squeeze Theorem. It follows that $R_n(x) \to 0$ as $n \to \infty$, so sin x is equal to the sum of its Maclaurin series by Theorem 8. ■

Figure 2 shows the graph of sin x together with its Taylor (or Maclaurin) polynomials

$$T_1(x) = x$$

$$T_3(x) = x - \frac{x^3}{3!}$$

$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Notice that, as n increases, $T_n(x)$ becomes a better approximation to sin x.

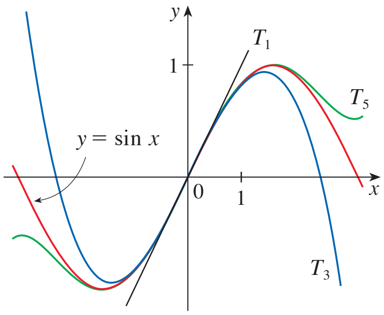


FIGURE 2

We state the result of Example 5 for future reference.

$$\boxed{15} \qquad \boxed{\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{for all } x \end{aligned}}$$

EXAMPLE 6 Find the Maclaurin series for cos x.

SOLUTION We could proceed directly as in Example 5, but it's easier to use Theorem 11.9.2 to differentiate the Maclaurin series for sin x given by Equation 15:

$$\cos x = \frac{d}{dx}(\sin x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Theorem 11.9.2 tells us that the differentiated series for sin x converges to the derivative of sin x, namely cos x, and the radius of convergence remains unchanged, so the series converges for all x. ■

The Maclaurin series for e^x, sin x, and cos x that we found in Examples 3, 5, and 6 were discovered, using different methods, by Newton. These equations are remarkable because they say we know everything about each of these functions if we know all its derivatives at the single number 0.

We state the result of Example 6 for future reference.

$$\boxed{\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{for all } x \end{aligned}} \tag{16}$$

EXAMPLE 7 Represent $f(x) = \sin x$ as the sum of its Taylor series centered at $\pi/3$.

SOLUTION Arranging our work in columns, we have

$$\begin{aligned} f(x) &= \sin x & f\left(\frac{\pi}{3}\right) &= \frac{\sqrt{3}}{2} \\ f'(x) &= \cos x & f'\left(\frac{\pi}{3}\right) &= \frac{1}{2} \\ f''(x) &= -\sin x & f''\left(\frac{\pi}{3}\right) &= -\frac{\sqrt{3}}{2} \\ f'''(x) &= -\cos x & f'''\left(\frac{\pi}{3}\right) &= -\frac{1}{2} \end{aligned}$$

and this pattern repeats indefinitely. Therefore the Taylor series at $\pi/3$ is

$$f\left(\frac{\pi}{3}\right) + \frac{f'\left(\frac{\pi}{3}\right)}{1!}\left(x - \frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{f'''\left(\frac{\pi}{3}\right)}{3!}\left(x - \frac{\pi}{3}\right)^3 + \cdots$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2 \cdot 1!}\left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2 \cdot 2!}\left(x - \frac{\pi}{3}\right)^2 - \frac{1}{2 \cdot 3!}\left(x - \frac{\pi}{3}\right)^3 + \cdots$$

We have obtained two different series representations for sin x, the Maclaurin series in Example 5 and the Taylor series in Example 7. It is best to use the Maclaurin series for values of x near 0 and the Taylor series for x near $\pi/3$. Notice that the third Taylor polynomial T_3 in Figure 3 is a good approximation to sin x near $\pi/3$ but not as accurate near 0. Compare it with the third Maclaurin polynomial T_3 in Figure 2, where the opposite is true.

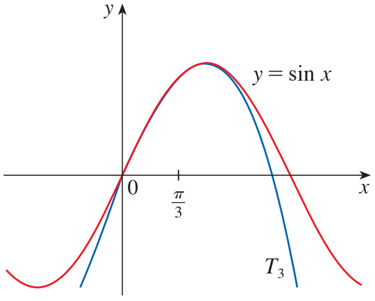


FIGURE 3

The proof that this series represents sin x for all x is very similar to that in Example 5. [Just replace x by $x - \pi/3$ in (14).] We can write the series in sigma notation if we separate the terms that contain $\sqrt{3}$:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{3}}{2(2n)!}\left(x - \frac{\pi}{3}\right)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n+1)!}\left(x - \frac{\pi}{3}\right)^{2n+1}$$ ■

EXAMPLE 8 Find the Maclaurin series for $f(x) = (1 + x)^k$, where k is any real number.

SOLUTION We start by computing derivatives:

$$\begin{aligned}
f(x) &= (1 + x)^k & f(0) &= 1 \\
f'(x) &= k(1 + x)^{k-1} & f'(0) &= k \\
f''(x) &= k(k - 1)(1 + x)^{k-2} & f''(0) &= k(k - 1) \\
f'''(x) &= k(k - 1)(k - 2)(1 + x)^{k-3} & f'''(0) &= k(k - 1)(k - 2) \\
&\vdots & &\vdots \\
f^{(n)}(x) &= k(k - 1) \cdots (k - n + 1)(1 + x)^{k-n} & f^{(n)}(0) &= k(k - 1) \cdots (k - n + 1)
\end{aligned}$$

Therefore the Maclaurin series of $f(x) = (1 + x)^k$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k - 1) \cdots (k - n + 1)}{n!} x^n$$

This series is called the **binomial series**. Notice that if k is a nonnegative integer, then the terms are eventually 0 and so the series is finite. For other values of k none of the terms is 0 and so we can investigate the convergence of the series by using the Ratio Test. If the nth term is a_n, then

$$\begin{aligned}
\left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{k(k - 1) \cdots (k - n + 1)(k - n)x^{n+1}}{(n + 1)!} \cdot \frac{n!}{k(k - 1) \cdots (k - n + 1)x^n} \right| \\
&= \frac{|k - n|}{n + 1} |x| = \frac{\left| 1 - \dfrac{k}{n} \right|}{1 + \dfrac{1}{n}} |x| \to |x| \qquad \text{as } n \to \infty
\end{aligned}$$

Thus, by the Ratio Test, the binomial series converges if $|x| < 1$ and diverges if $|x| > 1$. ■

The traditional notation for the coefficients in the binomial series is

$$\binom{k}{n} = \frac{k(k - 1)(k - 2) \cdots (k - n + 1)}{n!}$$

and these numbers are called the **binomial coefficients**.

The following theorem states that $(1 + x)^k$ is equal to the sum of its Maclaurin series. It is possible to prove this by showing that the remainder term $R_n(x)$ approaches 0, but that turns out to be quite difficult. The proof outlined in Exercise 97 is much easier.

17 **The Binomial Series** If k is any real number and $|x| < 1$, then

$$(1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k - 1)}{2!} x^2 + \frac{k(k - 1)(k - 2)}{3!} x^3 + \cdots$$

Although the binomial series always converges when $|x| < 1$, the question of whether or not it converges at the endpoints, ±1, depends on the value of k. It turns out that the series converges at 1 if $-1 < k \leqslant 0$ and at both endpoints if $k \geqslant 0$. Notice that if k is a positive integer and $n > k$, then the expression for $\binom{k}{n}$ contains a factor $(k - k)$, so $\binom{k}{n} = 0$

for $n > k$. This means that the series terminates and reduces to the ordinary Binomial Theorem when k is a positive integer. (See Reference Page 1.)

EXAMPLE 9 For the function $f(x) = \dfrac{1}{\sqrt{4 - x}}$, find the Maclaurin series and its radius of convergence.

SOLUTION We rewrite $f(x)$ in a form where we can use the binomial series:

$$\frac{1}{\sqrt{4 - x}} = \frac{1}{\sqrt{4\left(1 - \dfrac{x}{4}\right)}} = \frac{1}{2\sqrt{1 - \dfrac{x}{4}}} = \frac{1}{2}\left(1 - \frac{x}{4}\right)^{-1/2}$$

Using the binomial series with $k = -\frac{1}{2}$ and with x replaced by $-x/4$, we have

$$\frac{1}{\sqrt{4 - x}} = \frac{1}{2}\left(1 - \frac{x}{4}\right)^{-1/2} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^n$$

$$= \frac{1}{2}\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(-\frac{x}{4}\right)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\left(-\frac{x}{4}\right)^3\right.$$

$$\left. + \cdots + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{1}{2} - n + 1)}{n!}\left(-\frac{x}{4}\right)^n + \cdots\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{8}x + \frac{1 \cdot 3}{2!8^2}x^2 + \frac{1 \cdot 3 \cdot 5}{3!8^3}x^3 + \cdots + \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{n!8^n}x^n + \cdots\right]$$

We know from (17) that this series converges when $|-x/4| < 1$, that is, $|x| < 4$, so the radius of convergence is $R = 4$. ■

For future reference we collect in the following table some important Maclaurin series that we have derived in this section and in Section 11.9.

Table 1
Important Maclaurin Series and Their Radii of Convergence

$\dfrac{1}{1 - x} = \displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$R = 1$
$e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$R = \infty$
$\sin x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n + 1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$R = \infty$
$\cos x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$R = \infty$
$\tan^{-1}x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$R = 1$
$\ln(1 + x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$R = 1$
$(1 + x)^k = \displaystyle\sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k - 1)}{2!}x^2 + \frac{k(k - 1)(k - 2)}{3!}x^3 + \cdots$	$R = 1$

New Taylor Series from Old

As we observed in Note 2, if a function has a power series representation at a, then the series is uniquely determined. That is, no matter how a power series representation for a function f is obtained, it must be the Taylor series of f. So, we can obtain new Taylor series representations by manipulating series from Table 1, rather than using the coefficient formula given in Theorem 5.

As we saw in the examples of Section 11.9, we can replace x in a given Taylor series by an expression of the form cx^m, we can multiply (or divide) the series by such an expression, and we can differentiate or integrate term by term (Theorem 11.9.2). It can be shown that we can also obtain new Taylor series by adding, subtracting, multiplying, or dividing Taylor series.

EXAMPLE 10 Find the Maclaurin series for (a) $f(x) = x\cos x$ and (b) $f(x) = \ln(1 + 3x^2)$.

SOLUTION
(a) We multiply the Maclaurin series for $\cos x$ (see Table 1) by x:

$$x\cos x = x\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n)!} \qquad \text{for all } x$$

(b) Replacing x by $3x^2$ in the Maclaurin series for $\ln(1 + x)$ gives

$$\ln(1 + 3x^2) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(3x^2)^n}{n} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n x^{2n}}{n}$$

We know from Table 1 that this series converges for $|3x^2| < 1$, that is $|x| < 1/\sqrt{3}$, so the radius of convergence is $R = 1/\sqrt{3}$. ■

EXAMPLE 11 Find the function represented by the power series $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{2^n x^n}{n!}$.

SOLUTION By writing

$$\sum_{n=0}^{\infty}(-1)^n\frac{2^n x^n}{n!} = \sum_{n=0}^{\infty}\frac{(-2x)^n}{n!}$$

we see that this series is obtained by replacing x with $-2x$ in the series for e^x (in Table 1). Thus the series represents the function e^{-2x}. ■

EXAMPLE 12 Find the sum of the series $\dfrac{1}{1\cdot 2} - \dfrac{1}{2\cdot 2^2} + \dfrac{1}{3\cdot 2^3} - \dfrac{1}{4\cdot 2^4} + \cdots$.

SOLUTION With sigma notation we can write the given series as

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n\cdot 2^n} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\left(\frac{1}{2}\right)^n}{n}$$

Then from Table 1 we see that this series matches the entry for $\ln(1 + x)$ with $x = \frac{1}{2}$. So

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n\cdot 2^n} = \ln\left(1 + \tfrac{1}{2}\right) = \ln\tfrac{3}{2}$$ ■

One reason that Taylor series are important is that they enable us to integrate functions that we couldn't previously handle. In fact, in the introduction to this chapter we mentioned that Newton often integrated functions by first expressing them as power series and then integrating the series term by term. The function $f(x) = e^{-x^2}$ can't be integrated by techniques discussed so far because its antiderivative is not an elementary function (see Section 7.5). In the following example we use Newton's idea to integrate this function.

EXAMPLE 13

(a) Evaluate $\int e^{-x^2}\,dx$ as an infinite series.

(b) Evaluate $\int_0^1 e^{-x^2}\,dx$ correct to within an error of 0.001.

SOLUTION

(a) First we find the Maclaurin series for $f(x) = e^{-x^2}$. Although it's possible to use the direct method, let's find it by simply replacing x with $-x^2$ in the series for e^x given in Table 1. Thus, for all values of x,

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

Now we integrate term by term:

$$\int e^{-x^2}\,dx = \int \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots\right) dx$$

$$= C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots$$

This series converges for all x because the original series for e^{-x^2} converges for all x.

(b) The Fundamental Theorem of Calculus gives

We can take $C = 0$ in the antiderivative in part (a).

$$\int_0^1 e^{-x^2}\,dx = \left[x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \cdots\right]_0^1$$

$$= 1 - \tfrac{1}{3} + \tfrac{1}{10} - \tfrac{1}{42} + \tfrac{1}{216} - \cdots \approx 1 - \tfrac{1}{3} + \tfrac{1}{10} - \tfrac{1}{42} + \tfrac{1}{216} \approx 0.7475$$

The Alternating Series Estimation Theorem shows that the error involved in this approximation is less than

$$\frac{1}{11 \cdot 5!} = \frac{1}{1320} < 0.001 \qquad ■$$

Taylor series can also be used to evaluate limits, as illustrated in the next example. (Some mathematical software computes limits in this way.)

EXAMPLE 14 Evaluate $\displaystyle\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$.

SOLUTION Using the Maclaurin series for e^x from Table 1, we see that the Maclaurin series for $(e^x - 1 - x)/x^2$ is

$$\frac{e^x - 1 - x}{x^2} = \left[\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - 1 - x\right] \Big/ x^2$$

The limit in Example 14 could also be computed using l'Hospital's Rule.

$$= \frac{1}{x^2}\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) = \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots$$

Thus

$$\begin{aligned}\lim_{x\to 0}\frac{e^x-1-x}{x^2}&=\lim_{x\to 0}\left(\frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{4!}+\cdots\right)\\&=\frac{1}{2!}+0+0+\cdots=\frac{1}{2}\end{aligned}$$

because power series are continuous functions. ■

Multiplication and Division of Power Series

If power series are added or subtracted, they behave like polynomials (Theorem 11.2.8 shows this). In fact, as the following example illustrates, they can also be multiplied and divided like polynomials. We find only the first few terms because the calculations for the later terms become tedious and the initial terms are the most important ones.

EXAMPLE 15 Find the first three nonzero terms in the Maclaurin series for (a) $e^x \sin x$ and (b) $\tan x$.

SOLUTION

(a) Using the Maclaurin series for e^x and $\sin x$ in Table 1, we have

$$e^x\sin x=\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\left(x-\frac{x^3}{3!}+\cdots\right)$$

We multiply these expressions, collecting like terms just as for polynomials:

$$\begin{array}{rrrrrl}
 & 1\,+ & x\,+ & \frac{1}{2}x^2\,+ & \frac{1}{6}x^3\,+ & \cdots \\
\times & & x & & -\frac{1}{6}x^3\,+ & \cdots \\
\hline
 & & x\,+ & x^2\,+ & \frac{1}{2}x^3\,+ & \frac{1}{6}x^4+\cdots \\
+ & & & & -\frac{1}{6}x^3\,- & \frac{1}{6}x^4-\cdots \\
\hline
 & & x\,+ & x^2\,+ & \frac{1}{3}x^3\,+ & \cdots
\end{array}$$

Thus $$e^x\sin x=x+x^2+\tfrac{1}{3}x^3+\cdots$$

(b) Using the Maclaurin series in Table 1, we have

$$\tan x=\frac{\sin x}{\cos x}=\frac{x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\cdots}{1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\cdots}$$

We use a procedure like long division:

$$\begin{array}{r|rrr}
 & x+\frac{1}{3}x^3 & +\;\frac{2}{15}x^5 & +\cdots \\
\cline{2-4}
1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots & x-\frac{1}{6}x^3 & +\;\frac{1}{120}x^5 & -\cdots \\
 & x-\frac{1}{2}x^3 & +\;\frac{1}{24}x^5 & -\cdots \\
\cline{2-4}
 & \frac{1}{3}x^3 & -\;\frac{1}{30}x^5 & +\cdots \\
 & \frac{1}{3}x^3 & -\;\frac{1}{6}x^5 & +\cdots \\
\cline{2-4}
 & & \frac{2}{15}x^5 & +\cdots
\end{array}$$

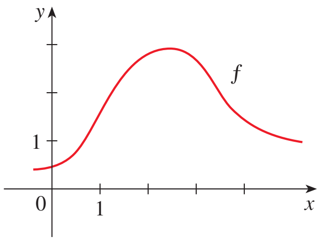
Thus $$\tan x=x+\tfrac{1}{3}x^3+\tfrac{2}{15}x^5+\cdots$$ ■

Although we have not attempted to justify the formal manipulations that were used in Example 15, they are legitimate. There is a theorem which states that if both $f(x) = \Sigma c_n x^n$ and $g(x) = \Sigma b_n x^n$ converge for $|x| < R$ and the series are multiplied as if they were polynomials, then the resulting series also converges for $|x| < R$ and represents $f(x)g(x)$. For division we require $b_0 \neq 0$; the resulting series converges for sufficiently small $|x|$.

11.10 Exercises

1. If $f(x) = \sum_{n=0}^{\infty} b_n(x-5)^n$ for all x, write a formula for b_8.

2. The graph of f is shown.



(a) Explain why the series $1.1 + 0.7x^2 + 2.2x^3 + \cdots$ is *not* the Maclaurin series of f.

(b) Explain why the series

$$1.6 - 0.8(x-1) + 0.4(x-1)^2 - 0.1(x-1)^3 + \cdots$$

is *not* the Taylor series of f centered at 1.

(c) Explain why the series

$$2.8 + 0.5(x-2) + 1.5(x-2)^2 - 0.1(x-2)^3 + \cdots$$

is *not* the Taylor series of f centered at 2.

3. If $f^{(n)}(0) = (n+1)!$ for $n = 0, 1, 2, \ldots$, find the Maclaurin series for f and its radius of convergence.

4. Find the Taylor series for f centered at 4 if

$$f^{(n)}(4) = \frac{(-1)^n n!}{3^n(n+1)}$$

What is the radius of convergence of the Taylor series?

5–10 Use the definition of a Taylor series to find the first four nonzero terms of the series for $f(x)$ centered at the given value of a.

5. $f(x) = xe^x, \quad a = 0$

6. $f(x) = \dfrac{1}{1+x}, \quad a = 2$

7. $f(x) = \sqrt[3]{x}, \quad a = 8$

8. $f(x) = \ln x, \quad a = 1$

9. $f(x) = \sin x, \quad a = \pi/6$

10. $f(x) = \cos^2 x, \quad a = 0$

11–20 Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.

11. $f(x) = (1-x)^{-2}$

12. $f(x) = \ln(1+x)$

13. $f(x) = \cos x$

14. $f(x) = e^{-2x}$

15. $f(x) = 2x^4 - 3x^2 + 3$

16. $f(x) = \sin 3x$

17. $f(x) = 2^x$

18. $f(x) = x\cos x$

19. $f(x) = \sinh x$

20. $f(x) = \cosh x$

21–30 Find the Taylor series for $f(x)$ centered at the given value of a. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.

21. $f(x) = x^5 + 2x^3 + x, \quad a = 2$

22. $f(x) = x^6 - x^4 + 2, \quad a = -2$

23. $f(x) = \ln x, \quad a = 2$

24. $f(x) = 1/x, \quad a = -3$

25. $f(x) = e^{2x}, \quad a = 3$

26. $f(x) = 1/x^2, \quad a = 1$

27. $f(x) = \sin x, \quad a = \pi$

28. $f(x) = \cos x, \quad a = \pi/2$

29. $f(x) = \sin 2x, \quad a = \pi$

30. $f(x) = \sqrt{x}, \quad a = 16$

---

31. Prove that the series obtained in Exercise 13 represents $\cos x$ for all x.

32. Prove that the series obtained in Exercise 27 represents $\sin x$ for all x.

33. Prove that the series obtained in Exercise 19 represents $\sinh x$ for all x.

34. Prove that the series obtained in Exercise 20 represents $\cosh x$ for all x.

35–38 Use the binomial series to expand the given function as a power series. State the radius of convergence.

35. $\sqrt[4]{1-x}$

36. $\sqrt[3]{8+x}$

37. $\dfrac{1}{(2+x)^3}$

38. $(1-x)^{3/4}$

39–48 Use a Maclaurin series in Table 1 to obtain the Maclaurin series for the given function.

39. $f(x) = \arctan(x^2)$

40. $f(x) = \sin(\pi x/4)$

41. $f(x) = x\cos 2x$

42. $f(x) = e^{3x} - e^{2x}$

43. $f(x) = x\cos\left(\frac{1}{2}x^2\right)$

44. $f(x) = x^2\ln(1+x^3)$

45. $f(x) = \dfrac{x}{\sqrt{4 + x^2}}$

46. $f(x) = \dfrac{x^2}{\sqrt{2 + x}}$

47. $f(x) = \sin^2 x$ $\left[\textit{Hint: } \text{Use } \sin^2 x = \frac{1}{2}(1 - \cos 2x).\right]$

48. $f(x) = \begin{cases} \dfrac{x - \sin x}{x^3} & \text{if } x \neq 0 \\ \frac{1}{6} & \text{if } x = 0 \end{cases}$

---

49. Use the definitions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

and the Maclaurin series for e^x to show that

(a) $\sinh x = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n + 1)!}$

(b) $\cosh x = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

50. Use the formula

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right) \qquad -1 < x < 1$$

and the Maclaurin series for $\ln(1 + x)$ to show that

$$\tanh^{-1} x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n + 1}$$

51–54 Find the Maclaurin series of f (by any method) and the associated radius of convergence. Graph f and its first few Taylor polynomials on the same screen. What do you notice about the relationship between these polynomials and f?

51. $f(x) = \cos(x^2)$

52. $f(x) = \ln(1 + x^2)$

53. $f(x) = xe^{-x}$

54. $f(x) = \tan^{-1}(x^3)$

---

55. Use the Maclaurin series for $\cos x$ to compute $\cos 5°$ correct to five decimal places.

56. Use the Maclaurin series for e^x to calculate $1/\sqrt[10]{e}$ correct to five decimal places.

57. (a) Use the binomial series to expand $1/\sqrt{1 - x^2}$.
(b) Use part (a) to find the Maclaurin series for $\sin^{-1} x$.

58. (a) Expand $1/\sqrt[4]{1 + x}$ as a power series.
(b) Use part (a) to estimate $1/\sqrt[4]{1.1}$ correct to three decimal places.

59–62 Evaluate the indefinite integral as an infinite series.

59. $\displaystyle\int \sqrt{1 + x^3}\, dx$

60. $\displaystyle\int x^2 \sin(x^2)\, dx$

61. $\displaystyle\int \frac{\cos x - 1}{x}\, dx$

62. $\displaystyle\int \arctan(x^2)\, dx$

63–66 Use series to approximate the definite integral to within the indicated accuracy.

63. $\displaystyle\int_0^{1/2} x^3 \arctan x\, dx$ (four decimal places)

64. $\displaystyle\int_0^1 \sin(x^4)\, dx$ (four decimal places)

65. $\displaystyle\int_0^{0.4} \sqrt{1 + x^4}\, dx$ $\left(|\text{error}| < 5 \times 10^{-6}\right)$

66. $\displaystyle\int_0^{0.5} x^2 e^{-x^2}\, dx$ $\left(|\text{error}| < 0.001\right)$

67–71 Use series to evaluate the limit.

67. $\displaystyle\lim_{x \to 0} \frac{x - \ln(1 + x)}{x^2}$

68. $\displaystyle\lim_{x \to 0} \frac{1 - \cos x}{1 + x - e^x}$

69. $\displaystyle\lim_{x \to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$

70. $\displaystyle\lim_{x \to 0} \frac{\sqrt{1 + x} - 1 - \frac{1}{2}x}{x^2}$

71. $\displaystyle\lim_{x \to 0} \frac{x^3 - 3x + 3\tan^{-1} x}{x^5}$

---

72. Use the series in Example 15(b) to evaluate

$$\lim_{x \to 0} \frac{\tan x - x}{x^3}$$

We found this limit in Example 4.4.4 using l'Hospital's Rule three times. Which method do you prefer?

73–78 Use multiplication or division of power series to find the first three nonzero terms in the Maclaurin series for each function.

73. $y = e^{-x^2} \cos x$

74. $y = \sec x$

75. $y = \dfrac{x}{\sin x}$

76. $y = e^x \ln(1 + x)$

77. $y = (\arctan x)^2$

78. $y = e^x \sin^2 x$

79–82 Find the function represented by the given power series.

79. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!}$

80. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{4n}}{n}$

81. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2^{2n+1}(2n + 1)}$

82. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2^{2n+1}(2n + 1)!}$

83–90 Find the sum of the series.

83. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

84. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$

85. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{n\, 5^n}$

86. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{5^n n!}$

87. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!}$

88. $1 - \ln 2 + \dfrac{(\ln 2)^2}{2!} - \dfrac{(\ln 2)^3}{3!} + \cdots$

89. $3 + \dfrac{9}{2!} + \dfrac{27}{3!} + \dfrac{81}{4!} + \cdots$

90. $\dfrac{1}{1 \cdot 2} - \dfrac{1}{3 \cdot 2^3} + \dfrac{1}{5 \cdot 2^5} - \dfrac{1}{7 \cdot 2^7} + \cdots$

---

91. Show that if p is an nth-degree polynomial, then

$$p(x+1) = \sum_{i=0}^{n} \frac{p^{(i)}(x)}{i!}$$

92. Use the Maclaurin series for $f(x) = x/(1 + x^2)$ to find $f^{(101)}(0)$.

93. Use the Maclaurin series for $f(x) = x \sin(x^2)$ to find $f^{(203)}(0)$.

94. If $f(x) = (1 + x^3)^{30}$, what is $f^{(58)}(0)$?

95. Prove Taylor's Inequality for $n = 2$, that is, prove that if $|f'''(x)| \leqslant M$ for $|x - a| \leqslant d$, then

$$|R_2(x)| \leqslant \frac{M}{6}|x - a|^3 \qquad \text{for } |x - a| \leqslant d$$

96. (a) Show that the function defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not equal to its Maclaurin series.

(b) Graph the function in part (a) and comment on its behavior near the origin.

97. Use the following steps to prove Theorem 17.

(a) Let $g(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n$. Differentiate this series to show that

$$g'(x) = \frac{kg(x)}{1 + x} \qquad -1 < x < 1$$

(b) Let $h(x) = (1 + x)^{-k} g(x)$ and show that $h'(x) = 0$.

(c) Deduce that $g(x) = (1 + x)^k$.

98. In Exercise 10.2.62 it was shown that the length of the ellipse $x = a \sin\theta$, $y = b \cos\theta$, where $a > b > 0$, is

$$L = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2\theta}\, d\theta$$

where $e = \sqrt{a^2 - b^2}/a$ is the eccentricity of the ellipse.

Expand the integrand as a binomial series and use the result of Exercise 7.1.56 to express L as a series in powers of the eccentricity up to the term in e^6.

DISCOVERY PROJECT | T AN ELUSIVE LIMIT

This project deals with the function

$$f(x) = \frac{\sin(\tan x) - \tan(\sin x)}{\arcsin(\arctan x) - \arctan(\arcsin x)}$$

1. Use a computer algebra system to evaluate $f(x)$ for $x = 1$, 0.1, 0.01, 0.001, and 0.0001. (A calculator may not provide accurate values.) Does it appear that f has a limit as $x \to 0$?

2. Use the CAS to graph f near $x = 0$. Does it appear that f has a limit as $x \to 0$?

3. Try to evaluate $\lim_{x \to 0} f(x)$ with l'Hospital's Rule, using the CAS to find derivatives of the numerator and denominator. What do you discover? How many applications of l'Hospital's Rule are required?

4. Evaluate $\lim_{x \to 0} f(x)$ by using the CAS to find sufficiently many terms in the Taylor series of the numerator and denominator.

5. Use the limit command on the CAS to find $\lim_{x \to 0} f(x)$ directly. (Most computer algebra systems use the method of Problem 4 to compute limits.)

6. In view of the answers to Problems 4 and 5, how do you explain the results of Problems 1 and 2?

WRITING PROJECT | HOW NEWTON DISCOVERED THE BINOMIAL SERIES

The Binomial Theorem, which gives the expansion of $(a + b)^k$, was known to Chinese mathematicians many centuries before the time of Newton for the case where the exponent k is a positive integer. In 1665, when he was 22, Newton was the first to discover the infinite series expansion of $(a + b)^k$ when k is a fractional exponent (positive or negative). He didn't publish his discovery, but he stated it and gave examples of how to use it in a letter (now called the *epistola prior*) dated June 13, 1676, that he sent to Henry Oldenburg, secretary of the Royal Society of London, to transmit to Leibniz. When Leibniz replied, he asked how Newton had discovered the binomial series. Newton wrote a second letter, the *epistola posterior* of October 24, 1676, in which he explained in great detail how he arrived at his discovery by a very indirect route. He was investigating the areas under the curves $y = (1 - x^2)^{n/2}$ from 0 to x for $n = 0, 1, 2, 3, 4, \ldots$. These are easy to calculate if n is even. By observing patterns and interpolating, Newton was able to guess the answers for odd values of n. Then he realized he could get the same answers by expressing $(1 - x^2)^{n/2}$ as an infinite series.

Write an essay on Newton's discovery of the binomial series. Start by giving the statement of the binomial series in Newton's notation (see the *epistola prior* on page 285 of [4] or page 402 of [2]). Explain why Newton's version is equivalent to Theorem 11.10.17. Then read Newton's *epistola posterior* (page 287 in [4] or page 404 in [2]) and explain the patterns that Newton discovered in the areas under the curves $y = (1 - x^2)^{n/2}$. Show how he was able to guess the areas under the remaining curves and how he verified his answers. Finally, explain how these discoveries led to the binomial series. The books by Edwards [1] and Katz [3] contain commentaries on Newton's letters.

1. C. H. Edwards, Jr., *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979), pp. 178–87.
2. Jahn Fauvel and Jeremy Gray, eds., *The History of Mathematics: A Reader* (Basingstoke, UK: MacMillan Education, 1987).
3. Victor Katz, *A History of Mathematics: An Introduction*, 3rd ed. (Boston: Addison-Wesley, 2009), pp. 543–82.
4. D. J. Struik, ed., *A Source Book in Mathematics, 1200–1800* (Cambridge, MA: Harvard University Press, 1969).

11.11 Applications of Taylor Polynomials

In this section we explore two types of applications of Taylor polynomials. First we look at how they are used to approximate functions—computer scientists employ them because polynomials are the simplest of functions. Then we investigate how physicists and engineers use them in such fields as relativity, optics, blackbody radiation, electric dipoles, the velocity of water waves, and building highways across a desert.

Approximating Functions by Polynomials

Suppose that $f(x)$ is equal to the sum of its Taylor series at a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

In Section 11.10 we introduced the notation $T_n(x)$ for the nth partial sum of this series and called it the nth-degree Taylor polynomial of f at a. Thus

$$\begin{aligned} T_n(x) &= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

Since f is the sum of its Taylor series, we know that $T_n(x) \to f(x)$ as $n \to \infty$ and so T_n can be used as an approximation to f: $f(x) \approx T_n(x)$.

Notice that the first-degree Taylor polynomial

$$T_1(x) = f(a) + f'(a)(x-a)$$

is the same as the linearization of f at a that we discussed in Section 3.10. Notice also that T_1 and its derivative have the same values at a that f and f' have. In general, it can be shown that the derivatives of T_n at a agree with those of f up to and including derivatives of order n.

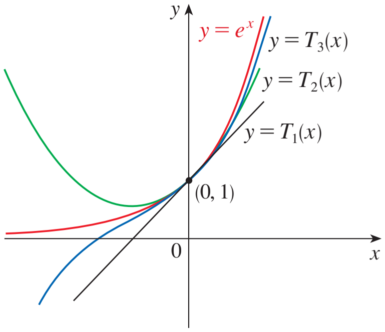


FIGURE 1

	$x = 0.2$	$x = 3.0$
$T_2(x)$	1.220000	8.500000
$T_4(x)$	1.221400	16.375000
$T_6(x)$	1.221403	19.412500
$T_8(x)$	1.221403	20.009152
$T_{10}(x)$	1.221403	20.079665
e^x	1.221403	20.085537

To illustrate these ideas let's take another look at the graphs of $y = e^x$ and its first few Taylor polynomials, as shown in Figure 1. The graph of T_1 is the tangent line to $y = e^x$ at (0, 1); this tangent line is the best linear approximation to e^x near (0, 1). The graph of T_2 is the parabola $y = 1 + x + x^2/2$, and the graph of T_3 is the cubic curve $y = 1 + x + x^2/2 + x^3/6$, which is a closer fit to the exponential curve $y = e^x$ than T_2. The next Taylor polynomial T_4 would be an even better approximation, and so on.

The values in the table give a numerical demonstration of the convergence of the Taylor polynomials $T_n(x)$ to the function $y = e^x$. We see that when $x = 0.2$ the convergence is very rapid, but when $x = 3$ it is somewhat slower. In fact, the farther x is from 0, the more slowly $T_n(x)$ converges to e^x.

When using a Taylor polynomial T_n to approximate a function f, we have to ask the questions: How good an approximation is it? How large should we take n to be in order to achieve a desired accuracy? To answer these questions we need to look at the absolute value of the remainder:

$$|R_n(x)| = |f(x) - T_n(x)|$$

There are three possible methods for estimating the size of the error:

1. We can use a calculator or computer to graph $|R_n(x)| = |f(x) - T_n(x)|$ and thereby estimate the error.
2. If the series happens to be an alternating series, we can use the Alternating Series Estimation Theorem.
3. In all cases we can use Taylor's Inequality (Theorem 11.10.9), which says that if $|f^{(n+1)}(x)| \leqslant M$, then

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x-a|^{n+1}$$

EXAMPLE 1

(a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.
(b) How accurate is this approximation when $7 \leqslant x \leqslant 9$?

SOLUTION

(a)

$$\begin{aligned} f(x) &= \sqrt[3]{x} = x^{1/3} & f(8) &= 2 \\ f'(x) &= \tfrac{1}{3}x^{-2/3} & f'(8) &= \tfrac{1}{12} \\ f''(x) &= -\tfrac{2}{9}x^{-5/3} & f''(8) &= \tfrac{1}{144} \\ f'''(x) &= \tfrac{10}{27}x^{-8/3} & & \end{aligned}$$

Thus the second-degree Taylor polynomial is

$$\begin{aligned} T_2(x) &= f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2 \\ &= 2 + \tfrac{1}{12}(x-8) - \tfrac{1}{288}(x-8)^2 \end{aligned}$$

The desired approximation is

$$\sqrt[3]{x} \approx T_2(x) = 2 + \tfrac{1}{12}(x-8) - \tfrac{1}{288}(x-8)^2$$

(b) The Taylor series is not alternating when $x < 8$, so we can't use the Alternating Series Estimation Theorem in this example. But we can use Taylor's Inequality with $n = 2$ and $a = 8$:

$$|R_2(x)| \leqslant \frac{M}{3!}|x-8|^3$$

where $|f'''(x)| \leqslant M$. Because $x \geqslant 7$, we have $x^{8/3} \geqslant 7^{8/3}$ and so

$$f'''(x) = \frac{10}{27} \cdot \frac{1}{x^{8/3}} \leqslant \frac{10}{27} \cdot \frac{1}{7^{8/3}} < 0.0021$$

Therefore we can take $M = 0.0021$. Also $7 \leqslant x \leqslant 9$, so $-1 \leqslant x - 8 \leqslant 1$ and $|x-8| \leqslant 1$. Then Taylor's Inequality gives

$$|R_2(x)| \leqslant \frac{0.0021}{3!} \cdot 1^3 = \frac{0.0021}{6} < 0.0004$$

Thus, if $7 \leqslant x \leqslant 9$, the approximation in part (a) is accurate to within 0.0004. ■

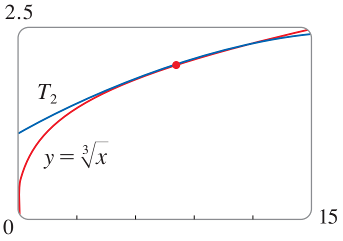


FIGURE 2

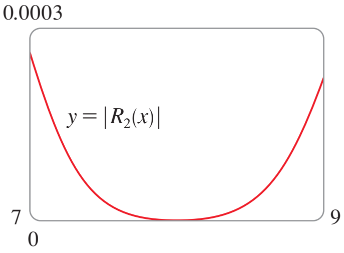


FIGURE 3

Let's check the calculation in Example 1 graphically. Figure 2 shows that the graphs of $y = \sqrt[3]{x}$ and $y = T_2(x)$ are very close to each other when x is near 8. Figure 3 shows the graph of $|R_2(x)|$ computed from the expression

$$|R_2(x)| = |\sqrt[3]{x} - T_2(x)|$$

We see from the graph that

$$|R_2(x)| < 0.0003$$

when $7 \leqslant x \leqslant 9$. Thus the error estimate from graphical methods is slightly better than the error estimate from Taylor's Inequality in this case.

EXAMPLE 2

(a) What is the maximum error possible in using the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

when $-0.3 \leqslant x \leqslant 0.3$? Use this approximation to find sin 12° correct to six decimal places.

(b) For what values of x is this approximation accurate to within 0.00005?

SOLUTION

(a) Notice that the Maclaurin series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

is alternating for all nonzero values of x, and the successive terms decrease in size because $|x| < 1$, so we can use the Alternating Series Estimation Theorem. The error in approximating $\sin x$ by the first three terms of its Maclaurin series is at most

$$\left|\frac{x^7}{7!}\right| = \frac{|x|^7}{5040}$$

If $-0.3 \leqslant x \leqslant 0.3$, then $|x| \leqslant 0.3$, so the error is smaller than

$$\frac{(0.3)^7}{5040} \approx 4.3 \times 10^{-8}$$

To find sin 12° we first convert to radian measure:

$$\sin 12° = \sin\left(\frac{12\pi}{180}\right) = \sin\left(\frac{\pi}{15}\right)$$

$$\approx \frac{\pi}{15} - \left(\frac{\pi}{15}\right)^3 \frac{1}{3!} + \left(\frac{\pi}{15}\right)^5 \frac{1}{5!} \approx 0.20791169$$

Thus, correct to six decimal places, $\sin 12° \approx 0.207912$.

(b) The error will be smaller than 0.00005 if

$$\frac{|x|^7}{5040} < 0.00005$$

Solving this inequality for x, we get

$$|x|^7 < 0.252 \qquad \text{or} \qquad |x| < (0.252)^{1/7} \approx 0.821$$

So the given approximation is accurate to within 0.00005 when $|x| < 0.82$. ■

What if we use Taylor's Inequality to solve Example 2? Since $f^{(7)}(x) = -\cos x$, we have $|f^{(7)}(x)| \leqslant 1$ and so

$$|R_6(x)| \leqslant \frac{1}{7!}|x|^7$$

So we get the same estimates as with the Alternating Series Estimation Theorem.

What about graphical methods? Figure 4 shows the graph of

$$|R_6(x)| = \left|\sin x - \left(x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5\right)\right|$$

and we see from it that $|R_6(x)| < 4.3 \times 10^{-8}$ when $|x| \leqslant 0.3$. This is the same estimate that we obtained in Example 2. For part (b) we want $|R_6(x)| < 0.00005$, so we graph both $y = |R_6(x)|$ and $y = 0.00005$ in Figure 5. From the coordinates of the right intersection point we find that the inequality is satisfied when $|x| < 0.82$. Again this is the same estimate that we obtained in the solution to Example 2.

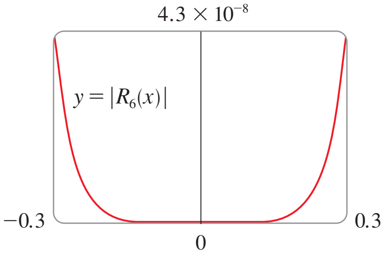


FIGURE 4

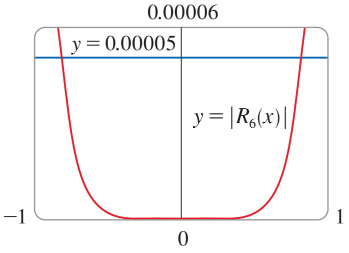


FIGURE 5

If we had been asked to approximate sin 72° instead of sin 12° in Example 2, it would have been wise to use the Taylor polynomials at $a = \pi/3$ (instead of $a = 0$) because they are better approximations to $\sin x$ for values of x close to $\pi/3$. Notice that 72° is close to 60° (or $\pi/3$ radians) and the derivatives of $\sin x$ are easy to compute at $\pi/3$.

Figure 6 shows the graphs of the Maclaurin polynomial approximations

$$T_1(x) = x \qquad\qquad T_3(x) = x - \frac{x^3}{3!}$$

$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \qquad\qquad T_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

to the sine curve. You can see that as n increases, $T_n(x)$ is a good approximation to $\sin x$ on a larger and larger interval.

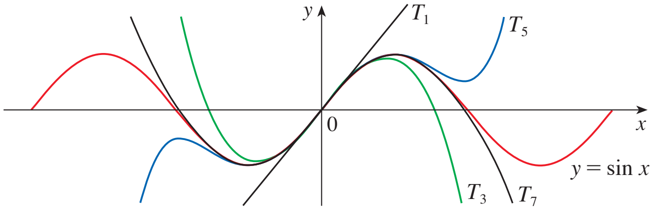


FIGURE 6

One use of the type of calculation done in Examples 1 and 2 occurs in calculators and computers. For instance, when you press the sin or e^x key on your calculator, or when a computer programmer uses a subroutine for a trigonometric or exponential or Bessel function, in many machines a polynomial approximation is calculated. The polynomial is often a Taylor polynomial that has been modified so that the error is spread more evenly throughout an interval.

Applications to Physics

Taylor polynomials are also used frequently in physics. In order to gain insight into an equation, a physicist often simplifies a function by considering only the first two or three terms in its Taylor series. In other words, the physicist uses a Taylor polynomial as an approximation to the function. Taylor's Inequality can then be used to gauge the accuracy of the approximation. The following example shows one way in which this idea is used in special relativity.

EXAMPLE 3 In Einstein's theory of special relativity the mass m of an object moving with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of the object when at rest and c is the speed of light. The kinetic energy k of the object is the difference between its total energy and its energy at rest:

$$K = mc^2 - m_0c^2$$

(a) Show that when v is very small compared with c, this expression for K agrees with classical Newtonian physics: $K = \frac{1}{2}m_0v^2$.

(b) Use Taylor's Inequality to estimate the difference in these expressions for K when $|v| \leqslant 100$ m/s.

SOLUTION

(a) Using the expressions given for K and m, we get

$$K = mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} - m_0c^2 = m_0c^2\left[\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right]$$

With $x = -v^2/c^2$, the Maclaurin series for $(1 + x)^{-1/2}$ is most easily computed as a binomial series with $k = -\frac{1}{2}$. (Notice that $|x| < 1$ because $v < c$.) Therefore we have

The upper curve in Figure 7 is the graph of the expression for the kinetic energy K of an object with velocity v in special relativity. The lower curve shows the function used for K in classical Newtonian physics. When v is much smaller than the speed of light, the curves are practically identical.

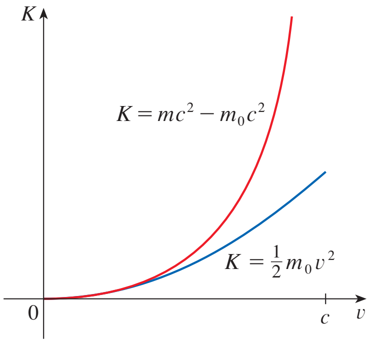


FIGURE 7

$$\begin{aligned}(1 + x)^{-1/2} &= 1 - \tfrac{1}{2}x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}x^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}x^3 + \cdots \\ &= 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 + \cdots\end{aligned}$$

and

$$\begin{aligned}K &= m_0c^2\left[\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right) - 1\right] \\ &= m_0c^2\left(\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right)\end{aligned}$$

If v is much smaller than c, then all terms after the first are very small when compared with the first term. If we omit them, we get

$$K \approx m_0c^2\left(\frac{1}{2}\frac{v^2}{c^2}\right) = \tfrac{1}{2}m_0v^2$$

(b) If $x = -v^2/c^2$, $f(x) = m_0c^2[(1 + x)^{-1/2} - 1]$, and M is a number such that $|f''(x)| \leqslant M$, then we can use Taylor's Inequality to write

$$|R_1(x)| \leqslant \frac{M}{2!}x^2$$

We have $f''(x) = \frac{3}{4}m_0c^2(1 + x)^{-5/2}$ and we are given that $|v| \leqslant 100$ m/s, so

$$|f''(x)| = \frac{3m_0c^2}{4(1 - v^2/c^2)^{5/2}} \leqslant \frac{3m_0c^2}{4(1 - 100^2/c^2)^{5/2}} \quad (=M)$$

Thus, with $c = 3 \times 10^8$ m/s,

$$|R_1(x)| \leqslant \frac{1}{2} \cdot \frac{3m_0c^2}{4(1 - 100^2/c^2)^{5/2}} \cdot \frac{100^4}{c^4} < (4.17 \times 10^{-10})m_0$$

So when $|v| \leqslant 100$ m/s, the magnitude of the error in using the Newtonian expression for kinetic energy is at most $(4.2 \times 10^{-10})m_0$. ■

Another application to physics occurs in optics. Figure 8 depicts a wave from the point source S meeting a spherical interface of radius R centered at C. The ray SA is refracted toward P.

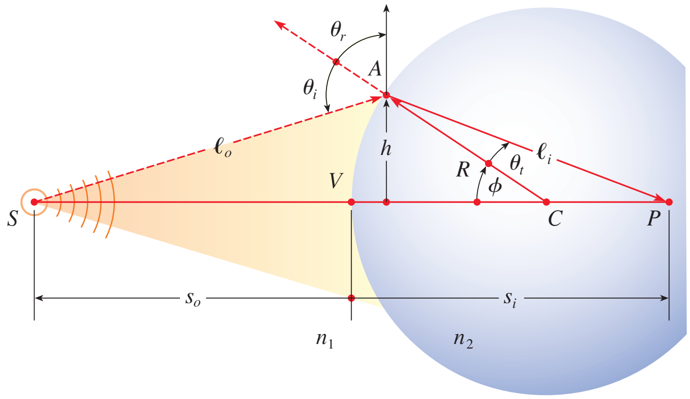


FIGURE 8
Refraction at a spherical interface

Using Fermat's principle that light travels so as to minimize the time taken, one can derive the equation

$$\boxed{1} \qquad \frac{n_1}{\ell_o} + \frac{n_2}{\ell_i} = \frac{1}{R}\left(\frac{n_2 s_i}{\ell_i} - \frac{n_1 s_o}{\ell_o}\right)$$

where n_1 and n_2 are indexes of refraction and ℓ_o, ℓ_i, s_o, and s_i are the distances indicated in Figure 8. By the Law of Cosines, applied to triangles ACS and ACP, we have

Here we use the identity

$$\cos(\pi - \phi) = -\cos\phi$$

$$\boxed{2} \qquad \ell_o = \sqrt{R^2 + (s_o + R)^2 - 2R(s_o + R)\cos\phi}$$

$$\ell_i = \sqrt{R^2 + (s_i - R)^2 + 2R(s_i - R)\cos\phi}$$

Because Equation 1 is cumbersome to work with, Gauss, in 1841, simplified it by using the linear approximation $\cos\phi \approx 1$ for small values of ϕ. (This amounts to using the Taylor polynomial of degree 1.) Then Equation 1 becomes the following simpler equation [as you are asked to show in Exercise 34(a)]:

$$\boxed{3} \qquad \frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R}$$

The resulting optical theory is known as *Gaussian optics*, or *first-order optics*, and has become the basic theoretical tool used to design lenses.

A more accurate theory is obtained by approximating $\cos\phi$ by its Taylor polynomial of degree 3 (which is the same as the Taylor polynomial of degree 2). This takes into account rays for which ϕ is not so small, that is, rays that strike the surface at greater distances h above the axis. In Exercise 34(b) you are asked to use this approximation to derive the more accurate equation

$$\boxed{4} \qquad \frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} + h^2\left[\frac{n_1}{2s_o}\left(\frac{1}{s_o} + \frac{1}{R}\right)^2 + \frac{n_2}{2s_i}\left(\frac{1}{R} - \frac{1}{s_i}\right)^2\right]$$

The resulting optical theory is known as *third-order optics.*

Other applications of Taylor polynomials to physics and engineering are explored in Exercises 32, 33, 35, 36, 37, and 38, and in the Applied Project following this section.

11.11 Exercises

1. (a) Find the Taylor polynomials up to degree 5 for $f(x) = \sin x$ centered at $a = 0$. Graph f and these polynomials on a common screen.
(b) Evaluate f and these polynomials at $x = \pi/4$, $\pi/2$, and π.
(c) Comment on how the Taylor polynomials converge to $f(x)$.

2. (a) Find the Taylor polynomials up to degree 3 for $f(x) = \tan x$ centered at $a = 0$. Graph f and these polynomials on a common screen.
(b) Evaluate f and these polynomials at $x = \pi/6$, $\pi/4$, and $\pi/3$.
(c) Comment on how the Taylor polynomials converge to $f(x)$.

3–10 Find the Taylor polynomial $T_3(x)$ for the function f centered at the number a. Graph f and T_3 on the same screen.

3. $f(x) = e^x$, $a = 1$

4. $f(x) = \sin x$, $a = \pi/6$

5. $f(x) = \cos x$, $a = \pi/2$

6. $f(x) = e^{-x} \sin x$, $a = 0$

7. $f(x) = \ln x$, $a = 1$

8. $f(x) = x \cos x$, $a = 0$

9. $f(x) = xe^{-2x}$, $a = 0$

10. $f(x) = \tan^{-1}x$, $a = 1$

11–12 Use a computer algebra system to find the Taylor polynomials T_n centered at a for $n = 2, 3, 4, 5$. Then graph these polynomials and f on the same screen.

11. $f(x) = \cot x$, $a = \pi/4$

12. $f(x) = \sqrt[3]{1 + x^2}$, $a = 0$

13–22
(a) Approximate f by a Taylor polynomial with degree n at the number a.
(b) Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval.
(c) Check your result in part (b) by graphing $|R_n(x)|$.

13. $f(x) = 1/x$, $a = 1$, $n = 2$, $0.7 \leqslant x \leqslant 1.3$

14. $f(x) = x^{-1/2}$, $a = 4$, $n = 2$, $3.5 \leqslant x \leqslant 4.5$

15. $f(x) = x^{2/3}$, $a = 1$, $n = 3$, $0.8 \leqslant x \leqslant 1.2$

16. $f(x) = \sin x$, $a = \pi/6$, $n = 4$, $0 \leqslant x \leqslant \pi/3$

17. $f(x) = \sec x$, $a = 0$, $n = 2$, $-0.2 \leqslant x \leqslant 0.2$

18. $f(x) = \ln(1 + 2x)$, $a = 1$, $n = 3$, $0.5 \leqslant x \leqslant 1.5$

19. $f(x) = e^{x^2}$, $a = 0$, $n = 3$, $0 \leqslant x \leqslant 0.1$

20. $f(x) = x \ln x$, $a = 1$, $n = 3$, $0.5 \leqslant x \leqslant 1.5$

21. $f(x) = x \sin x$, $a = 0$, $n = 4$, $-1 \leqslant x \leqslant 1$

22. $f(x) = \sinh 2x$, $a = 0$, $n = 5$, $-1 \leqslant x \leqslant 1$

23. Use the information from Exercise 5 to estimate cos 80° correct to five decimal places.

24. Use the information from Exercise 16 to estimate sin 38° correct to five decimal places.

25. Use Taylor's Inequality to determine the number of terms of the Maclaurin series for e^x that should be used to estimate $e^{0.1}$ to within 0.00001.

26. How many terms of the Maclaurin series for $\ln(1 + x)$ do you need to use to estimate ln 1.4 to within 0.001?

27–29 Use the Alternating Series Estimation Theorem or Taylor's Inequality to estimate the range of values of x for which the given approximation is accurate to within the stated error. Check your answer graphically.

27. $\sin x \approx x - \dfrac{x^3}{6}$ $\quad (|\text{error}| < 0.01)$

28. $\cos x \approx 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$ $\quad (|\text{error}| < 0.005)$

29. $\arctan x \approx x - \dfrac{x^3}{3} + \dfrac{x^5}{5}$ $\quad (|\text{error}| < 0.05)$

30. Suppose you know that

$$f^{(n)}(4) = \frac{(-1)^n n!}{3^n(n + 1)}$$

and the Taylor series of f centered at 4 converges to $f(x)$ for all x in the interval of convergence. Show that the fifth-degree Taylor polynomial approximates $f(5)$ with error less than 0.0002.

31. A car is moving with speed 20 m/s and acceleration 2 m/s^2 at a given instant. Using a second-degree Taylor polynomial, estimate how far the car moves in the next second. Would it be reasonable to use this polynomial to estimate the distance traveled during the next minute?

32. The resistivity ρ of a conducting wire is the reciprocal of the conductivity and is measured in units of ohm-meters (Ω-m). The resistivity of a given metal depends on the temperature according to the equation

$$\rho(t) = \rho_{20} e^{\alpha(t-20)}$$

where t is the temperature in °C. There are tables that list the values of α (called the temperature coefficient) and ρ_{20} (the resistivity at 20°C) for various metals. Except at very low temperatures, the resistivity varies almost linearly with temperature and so it is common to approximate the expression for $\rho(t)$ by its first- or second-degree Taylor polynomial at $t = 20$.
(a) Find expressions for these linear and quadratic approximations.

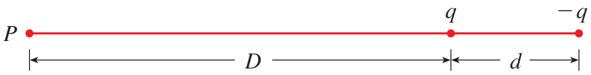
(b) For copper, the tables give $\alpha = 0.0039/°\text{C}$ and $\rho_{20} = 1.7 \times 10^{-8}\ \Omega$-m. Graph the resistivity of copper and the linear and quadratic approximations for $-250°\text{C} \leqslant t \leqslant 1000°\text{C}$.

(c) For what values of t does the linear approximation agree with the exponential expression to within one percent?

33. An electric dipole consists of two electric charges of equal magnitude and opposite sign. If the charges are q and $-q$ and are located at a distance d from each other, then the electric field E at the point P in the figure is

$$E = \frac{q}{D^2} - \frac{q}{(D + d)^2}$$

By expanding this expression for E as a series in powers of d/D, show that E is approximately proportional to $1/D^3$ when P is far away from the dipole.



34. (a) Derive Equation 3 for Gaussian optics from Equation 1 by approximating $\cos\phi$ in Equation 2 by its first-degree Taylor polynomial.

(b) Show that if $\cos\phi$ is replaced by its third-degree Taylor polynomial in Equation 2, then Equation 1 becomes Equation 4 for third-order optics. [Hint: Use the first two terms in the binomial series for ℓ_o^{-1} and ℓ_i^{-1}. Also, use $\phi \approx \sin\phi$.]

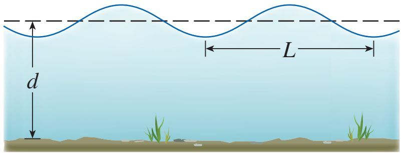
35. If a water wave with length L moves with velocity v across a body of water with depth d, as shown in the figure, then

$$v^2 = \frac{gL}{2\pi}\tanh\frac{2\pi d}{L}$$

(a) If the water is deep, show that $v \approx \sqrt{gL/(2\pi)}$.

(b) If the water is shallow, use the Maclaurin series for tanh to show that $v \approx \sqrt{gd}$. (Thus in shallow water the velocity of a wave tends to be independent of the length of the wave.)

(c) Use the Alternating Series Estimation Theorem to show that if $L > 10d$, then the estimate $v^2 \approx gd$ is accurate to within $0.014gL$.

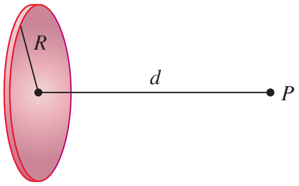


36. A uniformly charged disk has radius R and surface charge density σ as in the figure. The electric potential V at a point P at a distance d along the perpendicular central axis of the disk is

$$V = 2\pi k_e\sigma\left(\sqrt{d^2 + R^2} - d\right)$$

where k_e is a constant (called Coulomb's constant). Show that

$$V \approx \frac{\pi k_e R^2\sigma}{d} \qquad \text{for large } d$$



37. If a surveyor measures differences in elevation when making plans for a highway across a desert, corrections must be made for the curvature of the earth.

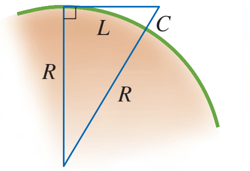
(a) If R is the radius of the earth and L is the length of the highway, show that the correction is

$$C = R\sec(L/R) - R$$

(b) Use a Taylor polynomial to show that

$$C \approx \frac{L^2}{2R} + \frac{5L^4}{24R^3}$$

(c)
pa m long.
(T)



38. The period of a pendulum with length L that makes a maximum angle θ_0 with the vertical is

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{dx}{\sqrt{1 - k^2\sin^2 x}}$$

where $k = \sin\left(\frac{1}{2}\theta_0\right)$ and g is the acceleration due to gravity. (In Exercise 7.7.42 we approximated this integral using Simpson's Rule.)

(a) Expand the integrand as a binomial series and use the result of Exercise 7.1.56 to show that

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1^2}{2^2}k^2 + \frac{1^2 3^2}{2^2 4^2}k^4 + \frac{1^2 3^2 5^2}{2^2 4^2 6^2}k^6 + \cdots\right]$$

If θ_0 is not too large, the approximation $T \approx 2\pi\sqrt{L/g}$, obtained by using only the first term in the series, is often used. A better approximation is obtained by using two terms:

$$T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \tfrac{1}{4}k^2\right)$$

(b) Notice that all the terms in the series after the first one have coefficients that are at most $\frac{1}{4}$. Use this fact to compare this series with a geometric series and show that

$$2\pi\sqrt{\frac{L}{g}}\left(1+\tfrac{1}{4}k^2\right) \leqslant T \leqslant 2\pi\sqrt{\frac{L}{g}}\,\frac{4-3k^2}{4-4k^2}$$

(c) Use the inequalities in part (b) to estimate the period of a pendulum with $L = 1$ meter and $\theta_0 = 10°$. How does it compare with the estimate $T \approx 2\pi\sqrt{L/g}$? What if $\theta_0 = 42°$?

39. In Section 4.8 we considered Newton's method for approximating a solution r of the equation $f(x) = 0$, and from an initial approximation x_1 we obtained successive approximations $x_2, x_3, \ldots$, where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Use Taylor's Inequality with $n = 1$, $a = x_n$, and $x = r$ to show that if $f''(x)$ exists on an interval I containing r, x_n, and x_{n+1}, and $|f''(x)| \leqslant M$, $|f'(x)| \geqslant K$ for all $x \in I$, then

$$|x_{n+1} - r| \leqslant \frac{M}{2K}|x_n - r|^2$$

[This means that if x_n is accurate to d decimal places, then x_{n+1} is accurate to about $2d$ decimal places. More precisely, if the error at stage n is at most 10^{-m}, then the error at stage $n + 1$ is at most $(M/2K)10^{-2m}$.]

APPLIED PROJECT | RADIATION FROM THE STARS

Luke Dodd / SCIENCE PHOTO LIBRARY / Getty Images

Any object emits radiation when heated. A *blackbody* is a system that absorbs all the radiation that falls on it. For instance, a matte black surface or a large cavity with a small hole in its wall (like a blast furnace) is a blackbody and emits blackbody radiation. Even the radiation from the sun is close to being blackbody radiation.

Proposed in the late 19th century, the Rayleigh-Jeans Law expresses the energy density of blackbody radiation of wavelength λ as

$$f(\lambda) = \frac{8\pi kT}{\lambda^4}$$

where λ is measured in meters, T is the temperature in kelvins (K), and k is Boltzmann's constant. The Rayleigh-Jeans Law agrees with experimental measurements for long wavelengths but disagrees drastically for short wavelengths. [The law predicts that $f(\lambda) \to \infty$ as $\lambda \to 0^+$ but experiments have shown that $f(\lambda) \to 0$.] This fact is known as the *ultraviolet catastrophe.*

In 1900 Max Planck found a better model (known now as Planck's Law) for blackbody radiation:

$$f(\lambda) = \frac{8\pi hc\lambda^{-5}}{e^{hc/(\lambda kT)} - 1}$$

where λ is measured in meters, T is the temperature (in kelvins), and

h = Planck's constant = 6.6262×10^{-34} J·s

c = speed of light = 2.997925×10^{8} m/s

k = Boltzmann's constant = 1.3807×10^{-23} J/K

1. Use l'Hospital's Rule to show that

$$\lim_{\lambda \to 0^+} f(\lambda) = 0 \qquad \text{and} \qquad \lim_{\lambda \to \infty} f(\lambda) = 0$$

for Planck's Law. So this law models blackbody radiation better than the Rayleigh-Jeans Law for short wavelengths.

2. Use a Taylor polynomial to show that, for large wavelengths, Planck's Law gives approximately the same values as the Rayleigh-Jeans Law.

3. Graph f as given by both laws on the same screen and comment on the similarities and differences. Use $T = 5700$ K (the temperature of the sun). (You may want to change from meters to the more convenient unit of micrometers: $1\ \mu\text{m} = 10^{-6}$ m.)

4. Use your graph in Problem 3 to estimate the value of λ for which $f(\lambda)$ is a maximum under Planck's Law.

5. Investigate how the graph of f changes as T varies. (Use Planck's Law.) In particular, graph f for the stars Betelgeuse ($T = 3400$ K), Procyon ($T = 6400$ K), and Sirius ($T = 9200$ K), as well as the sun. How does the total radiation emitted (the area under the curve) vary with T? Use the graph to comment on why Sirius is known as a blue star and Betelgeuse as a red star.

11 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. (a) What is a convergent sequence?
(b) What is a convergent series?
(c) What does $\lim_{n\to\infty} a_n = 3$ mean?
(d) What does $\sum_{n=1}^{\infty} a_n = 3$ mean?

2. (a) What is a bounded sequence?
(b) What is a monotonic sequence?
(c) What can you say about a bounded monotonic sequence?

3. (a) What is a geometric series? Under what circumstances is it convergent? What is its sum?
(b) What is a p-series? Under what circumstances is it convergent?

4. Suppose $\Sigma\, a_n = 3$ and s_n is the nth partial sum of the series. What is $\lim_{n\to\infty} a_n$? What is $\lim_{n\to\infty} s_n$?

5. State the following.
(a) The Test for Divergence
(b) The Integral Test
(c) The Direct Comparison Test
(d) The Limit Comparison Test
(e) The Alternating Series Test
(f) The Ratio Test
(g) The Root Test

6. (a) What is an absolutely convergent series?
(b) What can you say about such a series?
(c) What is a conditionally convergent series?

7. (a) If a series is convergent by the Integral Test, how do you estimate its sum?
(b) If a series is convergent by the Direct Comparison Test, how do you estimate its sum?
(c) If a series is convergent by the Alternating Series Test, how do you estimate its sum?

8. (a) Write the general form of a power series.
(b) What is the radius of convergence of a power series?
(c) What is the interval of convergence of a power series?

9. Suppose $f(x)$ is the sum of a power series with radius of convergence R.
(a) How do you differentiate f? What is the radius of convergence of the series for f'?
(b) How do you integrate f? What is the radius of convergence of the series for $\int f(x)\,dx$?

10. (a) Write an expression for the nth-degree Taylor polynomial of f centered at a.
(b) Write an expression for the Taylor series of f centered at a.
(c) Write an expression for the Maclaurin series of f.
(d) How do you show that $f(x)$ is equal to the sum of its Taylor series?
(e) State Taylor's Inequality.

11. Write the Maclaurin series and the interval of convergence for each of the following functions.
(a) $1/(1 - x)$ (b) e^x (c) $\sin x$
(d) $\cos x$ (e) $\tan^{-1}x$ (f) $\ln(1 + x)$

12. Write the binomial series expansion of $(1 + x)^k$. What is the radius of convergence of this series?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ is convergent.
2. The series $\sum_{n=1}^{\infty} n^{-\sin 1}$ is convergent.
3. If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} a_{2n+1} = L$.
4. If $\sum c_n 6^n$ is convergent, then $\sum c_n(-2)^n$ is convergent.
5. If $\sum c_n 6^n$ is convergent, then $\sum c_n(-6)^n$ is convergent.
6. If $\sum c_n x^n$ diverges when $x = 6$, then it diverges when $x = 10$.
7. The Ratio Test can be used to determine whether $\sum 1/n^3$ converges.
8. The Ratio Test can be used to determine whether $\sum 1/n!$ converges.
9. If $0 \leqslant a_n \leqslant b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.
10. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$
11. If $-1 < \alpha < 1$, then $\lim_{n\to\infty} \alpha^n = 0$.
12. If $\sum a_n$ is divergent, then $\sum |a_n|$ is divergent.
13. If $f(x) = 2x - x^2 + \frac{1}{3}x^3 - \cdots$ converges for all x, then $f'''(0) = 2$.
14. If $\{a_n\}$ and $\{b_n\}$ are divergent, then $\{a_n + b_n\}$ is divergent.
15. If $\{a_n\}$ and $\{b_n\}$ are divergent, then $\{a_n b_n\}$ is divergent.
16. If $\{a_n\}$ is decreasing and $a_n > 0$ for all n, then $\{a_n\}$ is convergent.
17. If $a_n > 0$ and $\sum a_n$ converges, then $\sum (-1)^n a_n$ converges.
18. If $a_n > 0$ and $\lim_{n\to\infty} (a_{n+1}/a_n) < 1$, then $\lim_{n\to\infty} a_n = 0$.
19. $0.99999\ldots = 1$
20. If $\displaystyle\lim_{n\to\infty} a_n = 2$, then $\displaystyle\lim_{n\to\infty} (a_{n+3} - a_n) = 0$.
21. If a finite number of terms are added to a convergent series, then the new series is still convergent.
22. If $\displaystyle\sum_{n=1}^{\infty} a_n = A$ and $\displaystyle\sum_{n=1}^{\infty} b_n = B$, then $\displaystyle\sum_{n=1}^{\infty} a_n b_n = AB$.

EXERCISES

1–8 Determine whether the sequence is convergent or divergent. If it is convergent, find its limit.

1. $a_n = \dfrac{2 + n^3}{1 + 2n^3}$
2. $a_n = \dfrac{9^{n+1}}{10^n}$
3. $a_n = \dfrac{n^3}{1 + n^2}$
4. $a_n = \cos(n\pi/2)$
5. $a_n = \dfrac{n \sin n}{n^2 + 1}$
6. $a_n = \dfrac{\ln n}{\sqrt{n}}$
7. $\{(1 + 3/n)^{4n}\}$
8. $\{(-10)^n/n!\}$

9. A sequence is defined recursively by the equations $a_1 = 1$, $a_{n+1} = \frac{1}{3}(a_n + 4)$. Show that $\{a_n\}$ is increasing and $a_n < 2$ for all n. Deduce that $\{a_n\}$ is convergent and find its limit.

10. Show that $\lim_{n\to\infty} n^4 e^{-n} = 0$ and use a graph to find the smallest value of N that corresponds to $\varepsilon = 0.1$ in the precise definition of a limit.

11–22 Determine whether the series is convergent or divergent.

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$
12. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 + 1}$
13. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{5^n}$
14. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n + 1}}$
15. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$
16. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{3n + 1}\right)$
17. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos 3n}{1 + (1.2)^n}$
18. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2n}}{(1 + 2n^2)^n}$
19. $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{5^n n!}$
20. $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^{2n}}{n^2 9^n}$
21. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n + 1}$
22. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n + 1} - \sqrt{n - 1}}{n}$

23–26 Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

23. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} n^{-1/3}$
24. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} n^{-3}$
25. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (n + 1) 3^n}{2^{2n+1}}$
26. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \sqrt{n}}{\ln n}$

27–31 Find the sum of the series.

27. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{2^{3n}}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$

29. $\displaystyle\sum_{n=1}^{\infty} [\tan^{-1}(n+1) - \tan^{-1}n]$

30. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^n}{3^{2n}(2n)!}$

31. $1 - e + \dfrac{e^2}{2!} - \dfrac{e^3}{3!} + \dfrac{e^4}{4!} - \cdots$

32. Express the repeating decimal 4.17326326326 . . . as a fraction.

33. Show that $\cosh x \geqslant 1 + \frac{1}{2}x^2$ for all x.

34. For what values of x does the series $\sum_{n=1}^{\infty} (\ln x)^n$ converge?

35. Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$$

correct to four decimal places.

36. (a) Find the partial sum s_5 of the series $\sum_{n=1}^{\infty} 1/n^6$ and estimate the error in using it as an approximation to the sum of the series.

(b) Find the sum of this series correct to five decimal places.

37. Use the sum of the first eight terms to approximate the sum of the series $\sum_{n=1}^{\infty} (2 + 5^n)^{-1}$. Estimate the error involved in this approximation.

38. (a) Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$ is convergent.

(b) Deduce that $\displaystyle\lim_{n\to\infty} \frac{n^n}{(2n)!} = 0$.

39. Prove that if the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then the series

$$\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right) a_n$$

is also absolutely convergent.

40–43 Find the radius of convergence and interval of convergence of the series.

40. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n^2 5^n}$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{n\,4^n}$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n (x-2)^n}{(n+2)!}$

43. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n (x-3)^n}{\sqrt{n+3}}$

44. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$$

45. Find the Taylor series of $f(x) = \sin x$ at $a = \pi/6$.

46. Find the Taylor series of $f(x) = \cos x$ at $a = \pi/3$.

47–54 Find the Maclaurin series for f and the associated radius of convergence. You may use either the direct method (definition of a Maclaurin series) or the Maclaurin series listed in Table 11.10.1.

47. $f(x) = \dfrac{x^2}{1+x}$

48. $f(x) = \tan^{-1}(x^2)$

49. $f(x) = \ln(4 - x)$

50. $f(x) = xe^{2x}$

51. $f(x) = \sin(x^4)$

52. $f(x) = 10^x$

53. $f(x) = 1/\sqrt[4]{16 - x}$

54. $f(x) = (1 - 3x)^{-5}$

55. Evaluate $\displaystyle\int \frac{e^x}{x}\,dx$ as an infinite series.

56. Use series to approximate $\int_0^1 \sqrt{1 + x^4}\,dx$ correct to two decimal places.

57–58

(a) Approximate f by a Taylor polynomial with degree n at the number a.

(b) Graph f and T_n on a common screen.

(c) Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval.

(d) Check your result in part (c) by graphing $|R_n(x)|$.

57. $f(x) = \sqrt{x}, \quad a = 1, \quad n = 3, \quad 0.9 \leqslant x \leqslant 1.1$

58. $f(x) = \sec x, \quad a = 0, \quad n = 2, \quad 0 \leqslant x \leqslant \pi/6$

59. Use series to evaluate the following limit.

$$\lim_{x\to 0} \frac{\sin x - x}{x^3}$$

60. The force due to gravity on an object with mass m at a height h above the surface of the earth is

$$F = \frac{mgR^2}{(R+h)^2}$$

where R is the radius of the earth and g is the acceleration due to gravity for an object on the surface of the earth.

(a) Express F as a series in powers of h/R.

(b) Observe that if we approximate F by the first term in the series, we get the expression $F \approx mg$ that is usually used when h is much smaller than R. Use the Alternating Series Estimation Theorem to estimate the range of values of h for which the approximation $F \approx mg$ is accurate to within one percent. (Use $R = 6400$ km.)

61. Suppose that $f(x) = \Sigma_{n=0}^{\infty} c_n x^n$ for all x.

(a) If f is an odd function, show that

$$c_0 = c_2 = c_4 = \cdots = 0$$

(b) If f is an even function, show that

$$c_1 = c_3 = c_5 = \cdots = 0$$

62. If $f(x) = e^{x^2}$, show that $f^{(2n)}(0) = \dfrac{(2n)!}{n!}$.

Problems Plus

Before you look at the solution of the example, cover it up and first try to solve the problem yourself.

EXAMPLE Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!}$.

SOLUTION The problem-solving principle that is relevant here is *recognizing something familiar*. Does the given series look anything like a series that we already know? Well, it does have some ingredients in common with the Maclaurin series for the exponential function:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

We can make this series look more like our given series by replacing x by $x + 2$:

$$e^{x+2} = \sum_{n=0}^{\infty} \frac{(x+2)^n}{n!} = 1 + (x+2) + \frac{(x+2)^2}{2!} + \frac{(x+2)^3}{3!} + \cdots$$

But here the exponent in the numerator matches the number in the denominator whose factorial is taken. To make that happen in the given series, let's multiply and divide by $(x+2)^3$:

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!} = \frac{1}{(x+2)^3} \sum_{n=0}^{\infty} \frac{(x+2)^{n+3}}{(n+3)!}$$

$$= (x+2)^{-3} \left[\frac{(x+2)^3}{3!} + \frac{(x+2)^4}{4!} + \cdots \right]$$

We see that the series between brackets is just the series for e^{x+2} with the first three terms missing. So

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!} = (x+2)^{-3} \left[e^{x+2} - 1 - (x+2) - \frac{(x+2)^2}{2!} \right]$$ ■

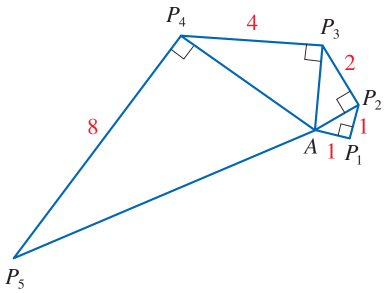
Problems

1. (a) Show that $\tan \frac{1}{2}x = \cot \frac{1}{2}x - 2 \cot x$.

(b) Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{x}{2^n}$$

2. Let $\{P_n\}$ be a sequence of points determined as in the figure. Thus $|AP_1| = 1$, $|P_nP_{n+1}| = 2^{n-1}$, and angle AP_nP_{n+1} is a right angle. Find $\lim_{n\to\infty} \angle P_nAP_{n+1}$.



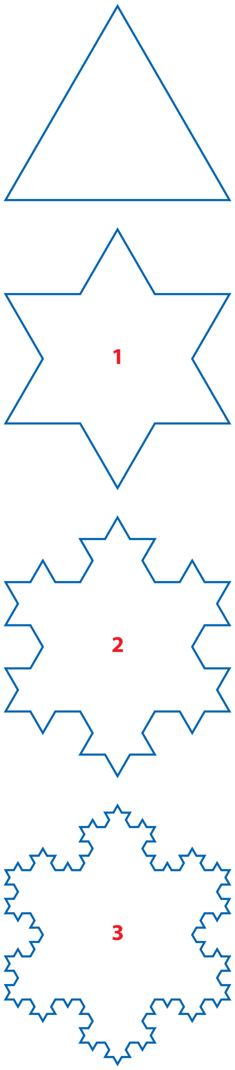


FIGURE FOR PROBLEM 3

3. To construct the **snowflake curve**, start with an equilateral triangle with sides of length 1. Step 1 in the construction is to divide each side into three equal parts, construct an equilateral triangle on the middle part, and then delete the middle part (see the figure). Step 2 is to repeat step 1 for each side of the resulting polygon. This process is repeated at each succeeding step. The snowflake curve is the curve that results from repeating this process indefinitely.

(a) Let s_n, l_n, and p_n represent the number of sides, the length of a side, and the total length of the nth approximating curve (the curve obtained after step n of the construction), respectively. Find formulas for s_n, l_n, and p_n.

(b) Show that $p_n \to \infty$ as $n \to \infty$.

(c) Sum an infinite series to find the area enclosed by the snowflake curve.

Note: Parts (b) and (c) show that the snowflake curve is infinitely long but encloses only a finite area.

4. Find the sum of the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots$$

where the terms are the reciprocals of the positive integers whose only prime factors are 2s and 3s.

5. (a) Show that for $xy \neq -1$,

$$\arctan x - \arctan y = \arctan \frac{x - y}{1 + xy}$$

if the left side lies between $-\pi/2$ and $\pi/2$.

(b) Show that $\arctan \frac{120}{119} - \arctan \frac{1}{239} = \pi/4$.

(c) Deduce the following formula of John Machin (1680–1751):

$$4 \arctan \tfrac{1}{5} - \arctan \tfrac{1}{239} = \frac{\pi}{4}$$

(d) Use the Maclaurin series for arctan to show that

$$0.1973955597 < \arctan \tfrac{1}{5} < 0.1973955616$$

(e) Show that

$$0.004184075 < \arctan \tfrac{1}{239} < 0.004184077$$

(f) Deduce that, correct to seven decimal places, $\pi \approx 3.1415927$.

Machin used this method in 1706 to find π correct to 100 decimal places. Recently, with the aid of computers, the value of π has been computed to increasingly greater accuracy, well into the trillions of decimal places.

6. (a) Prove a formula similar to the one in Problem 5(a) but involving arccot instead of arctan.

(b) Find the sum of the series $\sum_{n=0}^{\infty} \operatorname{arccot}(n^2 + n + 1)$.

7. Use the result of Problem 5(a) to find the sum of the series $\sum_{n=1}^{\infty} \arctan(2/n^2)$.

8. If $a_0 + a_1 + a_2 + \cdots + a_k = 0$, show that

$$\lim_{n\to\infty} \left(a_0\sqrt{n} + a_1\sqrt{n+1} + a_2\sqrt{n+2} + \cdots + a_k\sqrt{n+k}\right) = 0$$

If you don't see how to prove this, try the problem-solving strategy of *using analogy*. Try the special cases $k = 1$ and $k = 2$ first. If you can see how to prove the assertion for these cases, then you will probably see how to prove it in general.

PS See Principles of Problem Solving following Chapter 1.

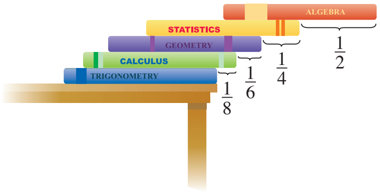


FIGURE FOR PROBLEM 10

9. Find the interval of convergence of $\sum_{n=1}^{\infty} n^3 x^n$ and find its sum.

10. Suppose you have a large supply of books, all the same size, and you stack them at the edge of a table, with each book extending farther beyond the edge of the table than the one beneath it. Show that it is possible to do this so that the top book extends entirely beyond the table. In fact, show that the top book can extend any distance at all beyond the edge of the table if the stack is high enough. Use the following method of stacking: The top book extends half its length beyond the second book. The second book extends a quarter of its length beyond the third. The third extends one-sixth of its length beyond the fourth, and so on. (Try it yourself with a deck of cards.) Consider centers of mass.

11. Find the sum of the series $\displaystyle\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$.

12. If $p > 1$, evaluate the expression

$$\frac{1 + \dfrac{1}{2^p} + \dfrac{1}{3^p} + \dfrac{1}{4^p} + \cdots}{1 - \dfrac{1}{2^p} + \dfrac{1}{3^p} - \dfrac{1}{4^p} + \cdots}$$

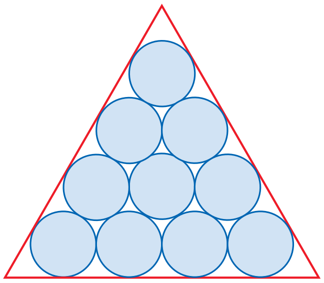


FIGURE FOR PROBLEM 13

13. Suppose that circles of equal diameter are packed tightly in n rows inside an equilateral triangle. (The figure illustrates the case $n = 4$.) If A is the area of the triangle and A_n is the total area occupied by the n rows of circles, show that

$$\lim_{n \to \infty} \frac{A_n}{A} = \frac{\pi}{2\sqrt{3}}$$

14. A sequence $\{a_n\}$ is defined recursively by the equations

$$a_0 = a_1 = 1 \qquad n(n-1)a_n = (n-1)(n-2)a_{n-1} - (n-3)a_{n-2}$$

Find the sum of the series $\sum_{n=0}^{\infty} a_n$.

15. If the curve $y = e^{-x/10} \sin x$, $x \geqslant 0$, is rotated about the x-axis, the resulting solid looks like an infinite decreasing string of beads.

(a) Find the exact volume of the nth bead. (Use either a table of integrals or a computer algebra system.)

(b) Find the total volume of the beads.

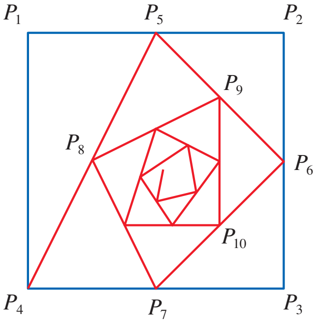


FIGURE FOR PROBLEM 16

16. Starting with the vertices $P_1(0, 1)$, $P_2(1, 1)$, $P_3(1, 0)$, $P_4(0, 0)$ of a square, we construct further points as shown in the figure: P_5 is the midpoint of P_1P_2, P_6 is the midpoint of P_2P_3, P_7 is the midpoint of P_3P_4, and so on. The polygonal spiral path $P_1P_2P_3P_4P_5P_6P_7 \ldots$ approaches a point P inside the square.

(a) If the coordinates of P_n are (x_n, y_n), show that $\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3} = 2$ and find a similar equation for the y-coordinates.

(b) Find the coordinates of P.

17. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$.

18. Carry out the following steps to show that

$$\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \frac{1}{7 \cdot 8} + \cdots = \ln 2$$

(a) Use the formula for the sum of a finite geometric series (11.2.3) to get an expression for

$$1 - x + x^2 - x^3 + \cdots + x^{2n-2} - x^{2n-1}$$

(b) Integrate the result of part (a) from 0 to 1 to get an expression for

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

as an integral.

(c) Deduce from part (b) that

$$\left| \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \cdots + \frac{1}{(2n-1)(2n)} - \int_0^1 \frac{dx}{1+x} \right| < \int_0^1 x^{2n}\, dx$$

(d) Use part (c) to show that the sum of the given series is ln 2.

19. Find all the solutions of the equation

$$1 + \frac{x}{2!} + \frac{x^2}{4!} + \frac{x^3}{6!} + \frac{x^4}{8!} + \cdots = 0$$

[*Hint:* Consider the cases $x \geqslant 0$ and $x < 0$ separately.]

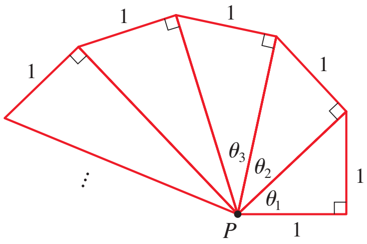


FIGURE FOR PROBLEM 20

20. Right-angled triangles are constructed as in the figure. Each triangle has height 1 and its base is the hypotenuse of the preceding triangle. Show that this sequence of triangles makes infinitely many turns around P by showing that $\Sigma\, \theta_n$ is a divergent series.

21. Consider the series whose terms are the reciprocals of the positive integers that can be written in base 10 notation without using the digit 0. Show that this series is convergent and the sum is less than 90.

22. (a) Show that the Maclaurin series of the function

$$f(x) = \frac{x}{1 - x - x^2} \qquad \text{is} \qquad \sum_{n=1}^{\infty} f_n x^n$$

where f_n is the nth Fibonacci number, that is, $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geqslant 3$. Find the radius of convergence of the series. [*Hint:* Write $x/(1 - x - x^2) = c_0 + c_1 x + c_2 x^2 + \cdots$ and multiply both sides of this equation by $1 - x - x^2$.]

(b) By writing $f(x)$ as a sum of partial fractions and thereby obtaining the Maclaurin series in a different way, find an explicit formula for the nth Fibonacci number.

23. Let

$$u = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots$$

$$v = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \cdots$$

$$w = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots$$

Show that $u^3 + v^3 + w^3 - 3uvw = 1$.

24. Prove that if $n > 1$, the nth partial sum of the harmonic series is not an integer.

Hint: Let 2^k be the largest power of 2 that is less than or equal to n and let M be the product of all odd integers that are less than or equal to n. Suppose that $s_n = m$, an integer. Then $M2^k s_n = M2^k m$. The right side of this equation is even. Prove that the left side is odd by showing that each of its terms is an even integer, except for one.

The forces created by wind and water on the sails and keel of a sailboat determine the direction in which the boat travels. Forces such as these are conveniently represented by vectors because they have both magnitude and direction. In Exercise 12.3.52 you are asked to compute the work done by the wind in moving a sailboat along a specified path.

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12

Vectors and the Geometry of Space

IN THIS CHAPTER WE INTRODUCE vectors and coordinate systems for three-dimensional space. This will be the setting for our study of the calculus of curves in space and of functions of two variables (whose graphs are surfaces in space) in Chapters 13–16. Here we will also see that vectors provide particularly simple descriptions of lines and planes in space.

12.1 Three-Dimensional Coordinate Systems

To locate a point in a plane, we need two numbers. We know that any point in the plane can be represented as an ordered pair (a, b) of real numbers, where a is the x-coordinate and b is the y-coordinate. For this reason, a plane is called two-dimensional. To locate a point in space, three numbers are required. We represent any point in space by an ordered triple (a, b, c) of real numbers.

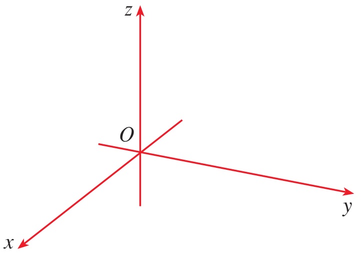


FIGURE 1
Coordinate axes

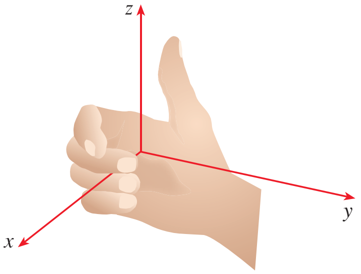


FIGURE 2
Right-hand rule

3D Space

In order to represent points in space, we first choose a fixed point O (the origin) and three directed lines through O that are perpendicular to each other, called the **coordinate axes** and labeled the x-axis, y-axis, and z-axis. Usually we think of the x- and y-axes as being horizontal and the z-axis as being vertical, and we draw the orientation of the axes as in Figure 1. The direction of the z-axis is determined by the **right-hand rule** as illustrated in Figure 2: if you curl the fingers of your right hand around the z-axis in the direction of a 90° counterclockwise rotation from the positive x-axis to the positive y-axis, then your thumb points in the positive direction of the z-axis.

The three coordinate axes determine the three **coordinate planes** illustrated in Figure 3(a). The xy-plane is the plane that contains the x- and y-axes; the yz-plane contains the y- and z-axes; the xz-plane contains the x- and z-axes. These three coordinate planes divide space into eight parts, called **octants**. The **first octant**, in the foreground, is determined by the positive axes.

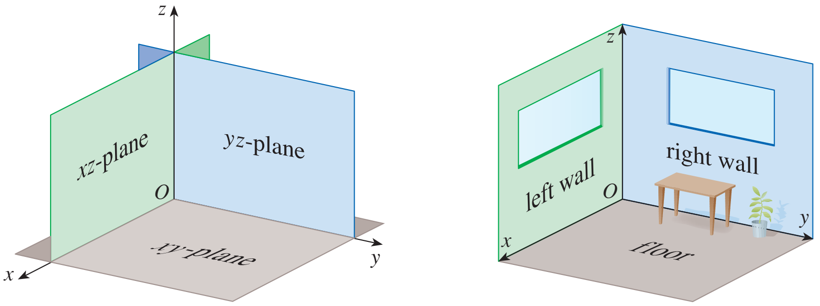


FIGURE 3 (a) Coordinate planes (b)

Because many people have some difficulty visualizing diagrams of three-dimensional figures, you may find it helpful to do the following [see Figure 3(b)]. Look at any bottom corner of a room and call the corner the origin. The wall on your left is in the xz-plane, the wall on your right is in the yz-plane, and the floor is in the xy-plane. The x-axis runs along the intersection of the floor and the left wall. The y-axis runs along the intersection of the floor and the right wall. The z-axis runs up from the floor toward the ceiling along the intersection of the two walls. You are situated in the first octant, and you can now imagine seven other rooms situated in the other seven octants (three on the same floor and four on the floor below), all connected by the common corner point O.

Now if P is any point in space, let a be the (directed) distance from the yz-plane to P, let b be the distance from the xz-plane to P, and let c be the distance from the xy-plane to P. We represent the point P by the ordered triple (a, b, c) of real numbers and we call a, b, and c the **coordinates** of P; a is the x-coordinate, b is the y-coordinate, and c is the z-coordinate. Thus, to locate the point (a, b, c), we can start at the origin O and move a units along the x-axis, then b units parallel to the y-axis, and then c units parallel to the z-axis as in Figure 4.

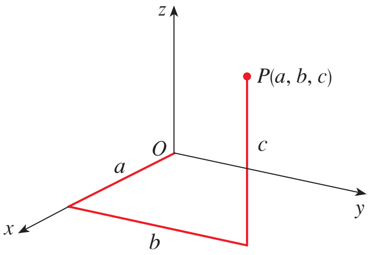


FIGURE 4

The point $P(a, b, c)$ determines a rectangular box as in Figure 5. If we drop a perpendicular from P to the xy-plane, we get a point Q with coordinates $(a, b, 0)$ called the **projection** of P onto the xy-plane. Similarly, $R(0, b, c)$ and $S(a, 0, c)$ are the projections of P onto the yz-plane and xz-plane, respectively.

As numerical illustrations, the points $(-4, 3, -5)$ and $(3, -2, -6)$ are plotted in Figure 6.

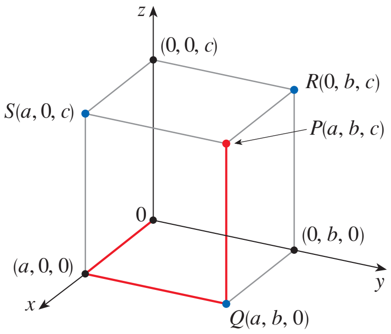


FIGURE 5

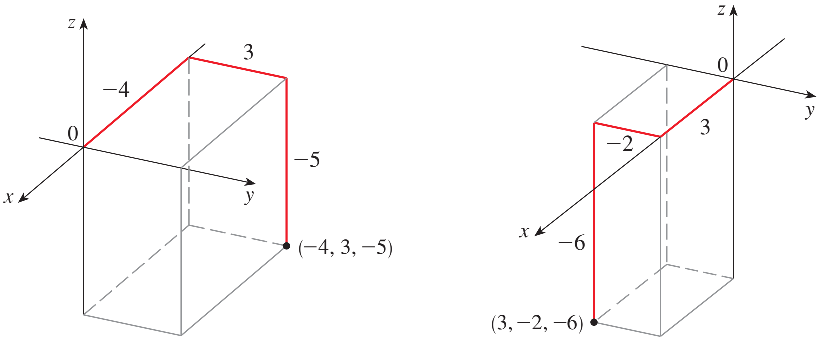


FIGURE 6

The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$. We have given a one-to-one correspondence between points P in space and ordered triples (a, b, c) in $\mathbb{R}^3$. It is called a **three-dimensional rectangular coordinate system**. Notice that, in terms of coordinates, the first octant can be described as the set of points whose coordinates are all positive.

Surfaces and Solids

In two-dimensional analytic geometry, the graph of an equation involving x and y is a curve in $\mathbb{R}^2$. In three-dimensional analytic geometry, an equation in x, y, and z represents a *surface* in $\mathbb{R}^3$.

EXAMPLE 1 What surface in $\mathbb{R}^3$ is represented by each of the following equations?

(a) $z = 3$ (b) $y = 5$

SOLUTION

(a) The equation $z = 3$ represents the set $\{(x, y, z) \mid z = 3\}$, which is the set of all points in $\mathbb{R}^3$ whose z-coordinate is 3 (x and y can each be any value). This is the horizontal plane that is parallel to the xy-plane and three units above it as in Figure 7(a).

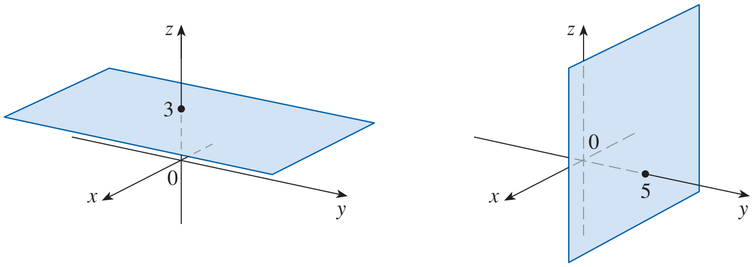


FIGURE 7

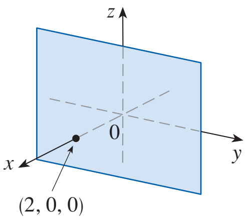
(a) $z = 3$, a plane in $\mathbb{R}^3$

(b) $y = 5$, a plane in $\mathbb{R}^3$

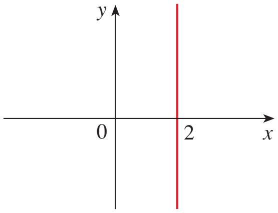
(b) The equation $y = 5$ represents the set of all points in $\mathbb{R}^3$ whose y-coordinate is 5. This is the vertical plane that is parallel to the xz-plane and five units to the right of it as in Figure 7(b). ■

NOTE When an equation is given, we must understand from the context whether it represents a curve in $\mathbb{R}^2$ or a surface in $\mathbb{R}^3$. For example, $x = 2$ represents a plane in $\mathbb{R}^3$, but of course $x = 2$ can also represent a line in $\mathbb{R}^2$ if we are dealing with two-dimensional analytic geometry. See Figure 8.

In general, if k is a constant, then $x = k$ represents a plane parallel to the yz-plane, $y = k$ is a plane parallel to the xz-plane, and $z = k$ is a plane parallel to the xy-plane. In Figure 5, the faces of the rectangular box are formed by the three coordinate planes $x = 0$ (the yz-plane), $y = 0$ (the xz-plane), and $z = 0$ (the xy-plane), and the planes $x = a$, $y = b$, and $z = c$.



(a) In $\mathbb{R}^3$, $x = 2$ is a plane.



(b) In $\mathbb{R}^2$, $x = 2$ is a line.

FIGURE 8

EXAMPLE 2

(a) Which points (x, y, z) satisfy the equations

$$x^2 + y^2 = 1 \qquad \text{and} \qquad z = 3$$

(b) What does the equation $x^2 + y^2 = 1$ represent as a surface in $\mathbb{R}^3$?
(c) What solid region in $\mathbb{R}^3$ is represented by the inequalities $x^2 + y^2 \leqslant 1$, $2 \leqslant z \leqslant 4$?

SOLUTION

(a) Because $z = 3$, the points lie in the horizontal plane $z = 3$ from Example 1(a). Because $x^2 + y^2 = 1$, the points lie on the circle with radius 1 and center on the z-axis. See Figure 9.

(b) Given that $x^2 + y^2 = 1$, with no restriction on z, we see that the point (x, y, z) could lie on a circle in any horizontal plane $z = k$. So the surface $x^2 + y^2 = 1$ in $\mathbb{R}^3$ consists of all possible horizontal circles $x^2 + y^2 = 1$, $z = k$, and is therefore the circular cylinder with radius 1 whose axis is the z-axis. See Figure 10.

(c) Because $x^2 + y^2 \leqslant 1$, any point (x, y, z) in the region must lie on or inside the circle of radius 1, centered on the z-axis, in a horizontal plane $z = k$. We are given that $2 \leqslant z \leqslant 4$, so the given inequalities represent the portion of the solid circular cylinder of radius 1, with axis the z-axis, that lies on or between the planes $z = 2$ and $z = 4$. See Figure 11.

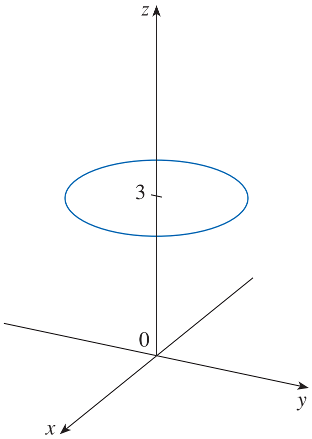


FIGURE 9
The circle $x^2 + y^2 = 1$, $z = 3$

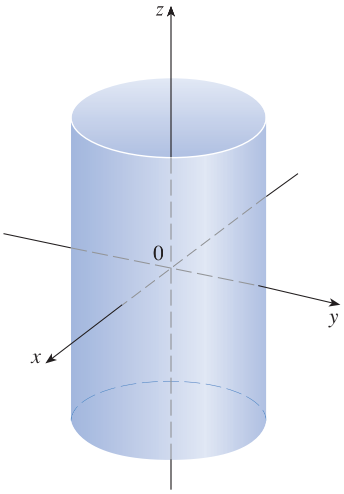


FIGURE 10
The cylinder $x^2 + y^2 = 1$

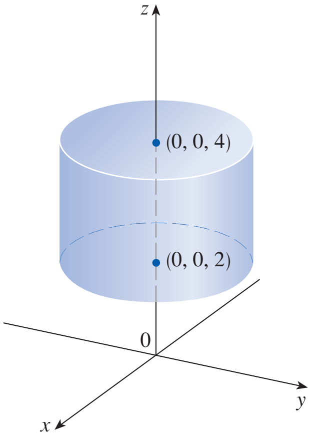


FIGURE 11
The solid region $x^2 + y^2 \leqslant 1$, $2 \leqslant z \leqslant 4$

■

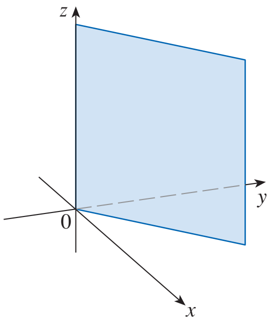


FIGURE 12
Part of the plane $y = x$

EXAMPLE 3 Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $y = x$.

SOLUTION The equation represents the set of all points in $\mathbb{R}^3$ whose x- and y-coordinates are equal, that is, $\{(x, x, z) \mid x \in \mathbb{R}, z \in \mathbb{R}\}$. This is a vertical plane that intersects the xy-plane in the line $y = x$, $z = 0$. The portion of this plane that lies in the first octant is sketched in Figure 12. ■

■ Distance and Spheres

The familiar formula for the distance between two points in a plane is easily extended to the following three-dimensional formula.

Distance Formula in Three Dimensions The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

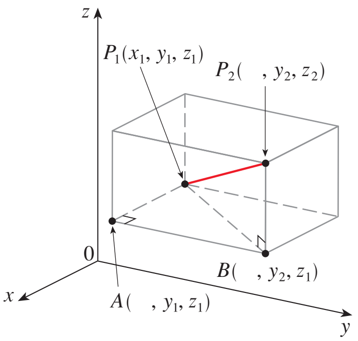


FIGURE 13

To see why this formula is true, we construct a rectangular box as in Figure 13, where P_1 and P_2 are opposite vertices and the faces of the box are parallel to the coordinate planes. If $A(x_2, y_1, z_1)$ and $B(x_2, y_2, z_1)$ are the vertices of the box indicated in the figure, then

$$|P_1A| = |x_2 - x_1| \qquad |AB| = |y_2 - y_1| \qquad |BP_2| = |z_2 - z_1|$$

Because triangles P_1BP_2 and P_1AB are both right-angled, two applications of the Pythagorean Theorem give

$$|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2$$

and

$$|P_1B|^2 = |P_1A|^2 + |AB|^2$$

Combining these equations, we get

$$\begin{aligned} |P_1P_2|^2 &= |P_1A|^2 + |AB|^2 + |BP_2|^2 \\ &= |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \end{aligned}$$

Therefore

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

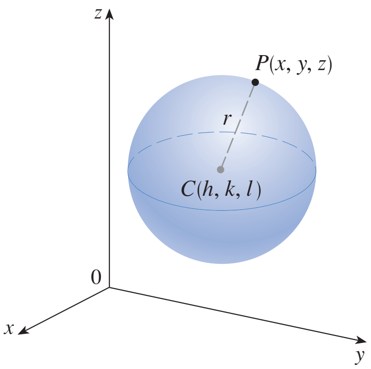


FIGURE 14

EXAMPLE 4 The distance from the point $P(2, -1, 7)$ to the point $Q(1, -3, 5)$ is

$$|PQ| = \sqrt{(1 - 2)^2 + (-3 + 1)^2 + (5 - 7)^2} = \sqrt{1 + 4 + 4} = 3 \qquad ■$$

A sphere with radius r and center $C(h, k, l)$ is defined as the set of all points $P(x, y, z)$ whose distance from C is r. (See Figure 14.) Thus P is on the sphere if and only if $|PC| = r$, that is

$$\sqrt{(x - h)^2 + (y - k)^2 + (z - l)^2} = r$$

Squaring both sides, we have the following result.

Equation of a Sphere An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

In particular, if the center is the origin O, then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$

EXAMPLE 5 Find an equation of the sphere with center $(3, -1, 6)$ that passes through the point $(5, 2, 3)$.

SOLUTION The radius r of the sphere is the distance between the points $(3, -1, 6)$ and $(5, 2, 3)$:

$$r = \sqrt{(5 - 3)^2 + [2 - (-1)]^2 + (3 - 6)^2} = \sqrt{22}$$

Then an equation of the sphere is

$$(x - 3)^2 + [y - (-1)]^2 + (z - 6)^2 = \left(\sqrt{22}\right)^2$$

or

$$(x - 3)^2 + (y + 1)^2 + (z - 6)^2 = 22$$

■

EXAMPLE 6 Show that $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$ is the equation of a sphere, and find its center and radius.

SOLUTION We can rewrite the given equation in the form of an equation of a sphere if we complete squares:

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) + (z^2 + 2z + 1) = -6 + 4 + 9 + 1$$

$$(x + 2)^2 + (y - 3)^2 + (z + 1)^2 = 8$$

Comparing this equation with the standard form, we see that it is the equation of a sphere with center $(-2, 3, -1)$ and radius $\sqrt{8} = 2\sqrt{2}$. ■

EXAMPLE 7 What region in $\mathbb{R}^3$ is represented by the following inequalities?

$$1 \leqslant x^2 + y^2 + z^2 \leqslant 4 \qquad z \leqslant 0$$

SOLUTION The inequalities

$$1 \leqslant x^2 + y^2 + z^2 \leqslant 4$$

can be rewritten as

$$1 \leqslant \sqrt{x^2 + y^2 + z^2} \leqslant 2$$

so they represent the points (x, y, z) whose distance from the origin is at least 1 and at most 2. But we are also given that $z \leqslant 0$, so the points lie on or below the xy-plane. Thus the given inequalities represent the region that lies between (or on) the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ and beneath (or on) the xy-plane. It is sketched in Figure 15. ■

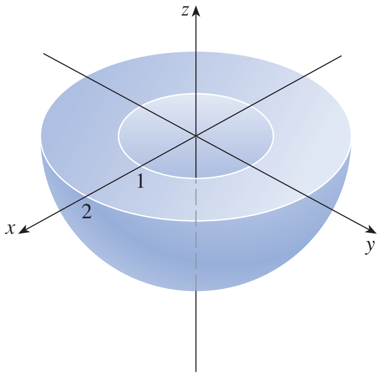


FIGURE 15

12.1 Exercises

1. Suppose you start at the origin, move along the x-axis a distance of 4 units in the positive direction, and then move downward a distance of 3 units. What are the coordinates of your position?

2. Sketch the points $(1, 5, 3)$, $(0, 2, -3)$, $(-3, 0, 2)$, and $(2, -2, -1)$ on a single set of coordinate axes.

3. Which of the points $A(-4, 0, -1)$, $B(3, 1, -5)$, and $C(2, 4, 6)$ is closest to the yz-plane? Which point lies in the xz-plane?

4. What are the projections of the point $(2, 3, 5)$ on the xy-, yz-, and xz-planes? Draw a rectangular box with the origin and $(2, 3, 5)$ as opposite vertices and with its faces parallel to the coordinate planes. Label all vertices of the box. Find the length of the diagonal of the box.

5. What does the equation $x = 4$ represent in $\mathbb{R}^2$? What does it represent in $\mathbb{R}^3$? Illustrate with sketches.

6. What does the equation $y = 3$ represent in $\mathbb{R}^3$? What does $z = 5$ represent? What does the pair of equations $y = 3$, $z = 5$ represent? In other words, describe the set of points (x, y, z) such that $y = 3$ and $z = 5$. Illustrate with a sketch.

7. Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $x + y = 2$.

8. Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $x^2 + z^2 = 9$.

9–10 Find the distance between the given points.

9. $(3, 5, -2)$, $(-1, 1, -4)$ **10.** $(-6, -3, 0)$, $(2, 4, 5)$

11–12 Find the lengths of the sides of the triangle PQR. Is it a right triangle? Is it an isosceles triangle?

11. $P(3, -2, -3)$, $Q(7, 0, 1)$, $R(1, 2, 1)$

12. $P(2, -1, 0)$, $Q(4, 1, 1)$, $R(4, -5, 4)$

---

13. Determine whether the points lie on a straight line.
(a) $A(2, 4, 2)$, $B(3, 7, -2)$, $C(1, 3, 3)$
(b) $D(0, -5, 5)$, $E(1, -2, 4)$, $F(3, 4, 2)$

14. Find the distance from $(4, -2, 6)$ to each of the following.
(a) The xy-plane
(b) The yz-plane
(c) The xz-plane
(d) The x-axis
(e) The y-axis
(f) The z-axis

15. Find an equation of the sphere with center $(-3, 2, 5)$ and radius 4. What is the intersection of this sphere with the yz-plane?

16. Find an equation of the sphere with center $(2, -6, 4)$ and radius 5. Describe its intersection with each of the coordinate planes.

17. Find an equation of the sphere that passes through the point $(4, 3, -1)$ and has center $(3, 8, 1)$.

18. Find an equation of the sphere that passes through the origin and whose center is $(1, 2, 3)$.

19–22 Show that the equation represents a sphere, and find its center and radius.

19. $x^2 + y^2 + z^2 + 8x - 2z = 8$

20. $x^2 + y^2 + z^2 = 6x - 4y - 10z$

21. $2x^2 + 2y^2 + 2z^2 - 2x + 4y + 1 = 0$

22. $4x^2 + 4y^2 + 4z^2 = 16x - 6y - 12$

---

23. Midpoint Formula Prove that the midpoint of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

24. Use the Midpoint Formula in Exercise 23 to find the center of a sphere if one of its diameters has endpoints $(5, 4, 3)$ and $(1, 6, -9)$. Then find an equation of the sphere.

25. Find an equation of the sphere with center $(-1, 4, 5)$ that just touches (at only one point) the (a) xy-plane, (b) yz-plane, and (c) xz-plane.

26. Which coordinate plane is closest to the point $(7, 3, 8)$? Find an equation of the sphere with center $(7, 3, 8)$ that just touches (at one point) that coordinate plane.

27–42 Describe in words the region of $\mathbb{R}^3$ represented by the equation(s) or inequalities.

27. $z = -2$

28. $x = 3$

29. $y \geqslant 1$

30. $x < 4$

31. $-1 \leqslant x \leqslant 2$

32. $z = y$

33. $x^2 + y^2 = 4, \quad z = -1$

34. $x^2 + y^2 = 4$

35. $y^2 + z^2 \leqslant 25$

36. $x^2 + z^2 \leqslant 25, \quad 0 \leqslant y \leqslant 2$

37. $x^2 + y^2 + z^2 = 4$

38. $x^2 + y^2 + z^2 \leqslant 4$

39. $1 \leqslant x^2 + y^2 + z^2 \leqslant 5$

40. $1 \leqslant x^2 + y^2 \leqslant 5$

41. $0 \leqslant x \leqslant 3, \; 0 \leqslant y \leqslant 3, \; 0 \leqslant z \leqslant 3$

42. $x^2 + y^2 + z^2 > 2z$

43–46 Write inequalities to describe the region.

43. The region between the yz-plane and the vertical plane $x = 5$

44. The solid cylinder that lies on or below the plane $z = 8$ and on or above the disk in the xy-plane with center the origin and radius 2

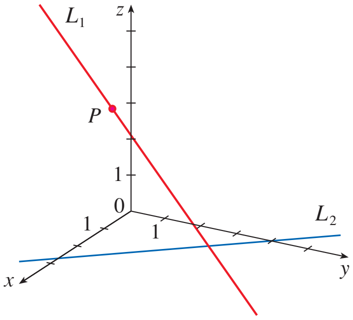
45. The region consisting of all points between (but not on) the spheres of radius r and R centered at the origin, where $r < R$

46. The solid upper hemisphere of the sphere of radius 2 centered at the origin

47. The figure shows a line L_1 in space and a second line L_2, which is the projection of L_1 onto the xy-plane. (In other words, the points on L_2 are directly beneath, or above, the points on L_1.)

(a) Find the coordinates of the point P on the line L_1.

(b) Locate on the diagram the points A, B, and C, where the line L_1 intersects the xy-plane, the yz-plane, and the xz-plane, respectively.



48. Consider the points P such that the distance from P to $A(-1, 5, 3)$ is twice the distance from P to $B(6, 2, -2)$. Show that the set of all such points is a sphere, and find its center and radius.

49. Find an equation of the set of all points equidistant from the points $A(-1, 5, 3)$ and $B(6, 2, -2)$. Describe the set.

50. Find the volume of the solid that lies inside both of the spheres

$$x^2 + y^2 + z^2 + 4x - 2y + 4z + 5 = 0$$

and $$x^2 + y^2 + z^2 = 4$$

51. Find the distance between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 4x + 4y + 4z - 11$.

52. Describe and sketch a solid with the following properties: When illuminated by rays parallel to the z-axis, its shadow is a circular disk. If the rays are parallel to the y-axis, its shadow is a square. If the rays are parallel to the x-axis, its shadow is an isosceles triangle.

12.2 Vectors

The term **vector** is used in mathematics and the sciences to indicate a quantity that has both magnitude and direction. For instance, to describe the velocity of a moving object, we must specify both the speed of the object and the direction of travel. Other examples of vectors include force, displacement, and acceleration.

Geometric Description of Vectors

A vector is often represented by an arrow or a directed line segment. The length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector. We denote a vector by printing a letter in boldface (**v**) or by putting an arrow above the letter ($\vec{v}$).

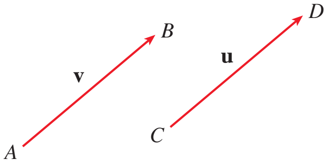


FIGURE 1
Equivalent vectors

For instance, suppose a particle moves along a line segment from point A to point B. The corresponding **displacement vector v**, shown in Figure 1, has **initial point** A (the tail) and **terminal point** B (the tip) and we indicate this by writing $\mathbf{v} = \overrightarrow{AB}$. Notice that the vector $\mathbf{u} = \overrightarrow{CD}$ has the same length and the same direction as **v** even though it is in a different position. We say that **u** and **v** are **equivalent** (or **equal**) and we write $\mathbf{u} = \mathbf{v}$. The **zero vector**, denoted by **0**, has length 0. It is the only vector with no specific direction.

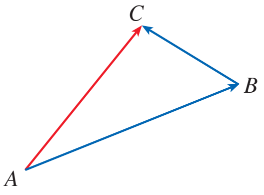


FIGURE 2

We will often find it useful to combine vectors. For example, suppose a particle moves from A to B with displacement vector $\overrightarrow{AB}$, and then the particle changes direction and moves from B to C, with displacement vector $\overrightarrow{BC}$, as shown in Figure 2. The combined effect of these displacements is that the particle has moved from A to C. The resulting displacement vector $\overrightarrow{AC}$ is called the *sum* of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ and we write

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

In general, if we start with vectors **u** and **v**, we first place **v** so that its tail coincides with the tip of **u** and define the sum of **u** and **v** as follows.

Definition of Vector Addition If **u** and **v** are vectors positioned so the initial point of **v** is at the terminal point of **u**, then the **sum u + v** is the vector from the initial point of **u** to the terminal point of **v**.

The definition of vector addition is illustrated in Figure 3. You can see why this definition is sometimes called the **Triangle Law**.

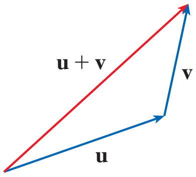


FIGURE 3
Triangle Law

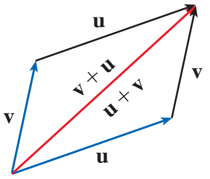


FIGURE 4
Parallelogram Law

In Figure 4 we start with the same vectors **u** and **v** as in Figure 3 and draw another copy of **v** with the same initial point as **u**. Completing the parallelogram, we see that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. This also gives another way to construct the sum: if we place **u** and **v** so they start at the same point, then $\mathbf{u} + \mathbf{v}$ lies along the diagonal of the parallelogram with **u** and **v** as sides. (This is called the **Parallelogram Law**.)

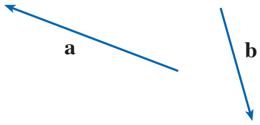


FIGURE 5

EXAMPLE 1 Draw the sum of the vectors **a** and **b** shown in Figure 5.

SOLUTION First we place **b** with its tail at the tip of **a**, being careful to draw a copy of **b** that has the same length and direction. Then we draw the vector $\mathbf{a} + \mathbf{b}$ [see Figure 6(a)] starting at the initial point of **a** and ending at the terminal point of the copy of **b**.

Alternatively, we could place **b** so it starts where **a** starts and construct $\mathbf{a} + \mathbf{b}$ by the Parallelogram Law as shown in Figure 6(b).

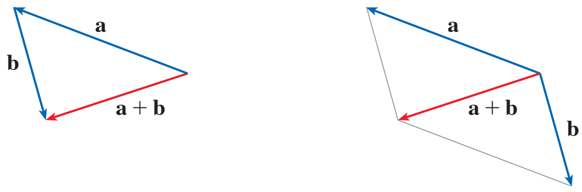


FIGURE 6

(a) (b)

■

We now define multiplication of a vector **v** by a real number c. In this context we call the real number c a **scalar** to distinguish it from a vector. For instance, we want the *scalar multiple* $2\mathbf{v}$ to be the same vector as the sum $\mathbf{v} + \mathbf{v}$, which has the same direction as **v** but is twice as long. In general, we multiply a vector by a scalar as follows.

Definition of Scalar Multiplication If c is a scalar and **v** is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of **v** and whose direction is the same as **v** if $c > 0$ and is opposite to **v** if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

This definition is illustrated in Figure 7. We see that real numbers work like scaling factors here; that's why we call them scalars. Notice that two nonzero vectors are **parallel** if they are scalar multiples of one another. In particular, the vector $-\mathbf{v} = (-1)\mathbf{v}$ has the same length as $\mathbf{v}$ but points in the opposite direction. We call it the **negative** of $\mathbf{v}$.

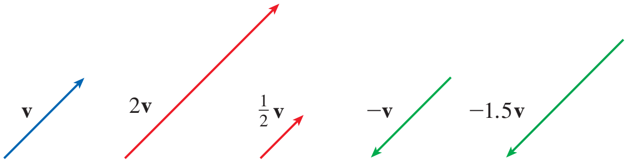


FIGURE 7
Scalar multiples of $\mathbf{v}$

By the **difference** $\mathbf{u} - \mathbf{v}$ of two vectors we mean

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

For the vectors $\mathbf{u}$ and $\mathbf{v}$ shown in Figure 8(a), we can construct the difference $\mathbf{u} - \mathbf{v}$ by first drawing the negative of $\mathbf{v}$, $-\mathbf{v}$, and then adding it to $\mathbf{u}$ by the Parallelogram Law as in Figure 8(b). Alternatively, since $\mathbf{v} + (\mathbf{u} - \mathbf{v}) = \mathbf{u}$, the vector $\mathbf{u} - \mathbf{v}$, when added to $\mathbf{v}$, gives $\mathbf{u}$. So we could construct $\mathbf{u} - \mathbf{v}$ as in Figure 8(c) by means of the Triangle Law. Notice that if $\mathbf{u}$ and $\mathbf{v}$ both start from the same initial point, then $\mathbf{u} - \mathbf{v}$ connects the tip of $\mathbf{v}$ to the tip of $\mathbf{u}$.

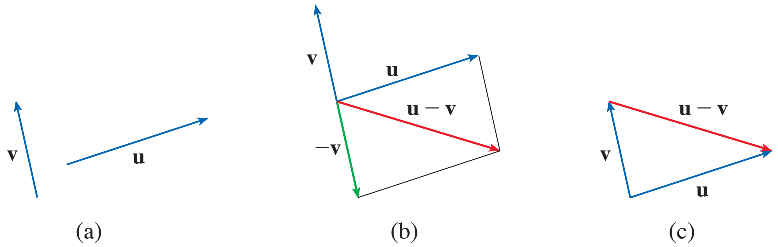


FIGURE 8
Drawing the difference $\mathbf{u} - \mathbf{v}$

EXAMPLE 2 If $\mathbf{a}$ and $\mathbf{b}$ are the vectors shown in Figure 9, draw $\mathbf{a} - 2\mathbf{b}$.

SOLUTION We first draw the vector $-2\mathbf{b}$ pointing in the direction opposite to $\mathbf{b}$ and twice as long. We place it with its tail at the tip of $\mathbf{a}$ and then use the Triangle Law to draw $\mathbf{a} + (-2\mathbf{b})$ as shown in Figure 10.

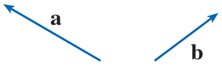


FIGURE 9

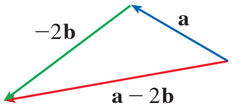


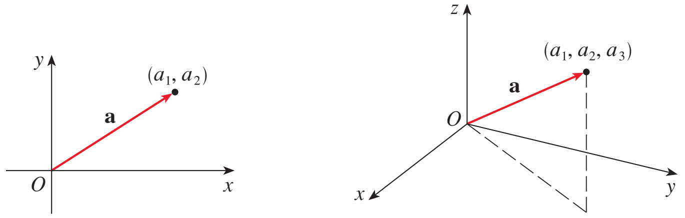
FIGURE 10 ■

■ Components of a Vector

For some purposes it's convenient to introduce a coordinate system that allows us to treat vectors algebraically. If we place the initial point of a vector $\mathbf{a}$ at the origin of a rectangular coordinate system, then the terminal point of $\mathbf{a}$ has coordinates of the form (a_1, a_2) or (a_1, a_2, a_3), depending on whether our coordinate system is two- or three-dimensional (see Figure 11). These coordinates are called the **components** of $\mathbf{a}$ and we write

$$\mathbf{a} = \langle a_1, a_2 \rangle \qquad \text{or} \qquad \mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

We use the notation $\langle a_1, a_2\rangle$ for the ordered pair that refers to a vector so as not to confuse it with the ordered pair (a_1, a_2) that refers to a point in the plane.



$\mathbf{a} = \langle a_1, a_2\rangle$ $\mathbf{a} = \langle a_1, a_2, a_3\rangle$

FIGURE 11

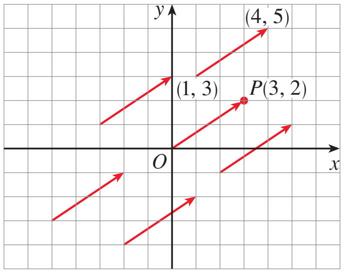


FIGURE 12
Representations of $\mathbf{a} = \langle 3, 2\rangle$

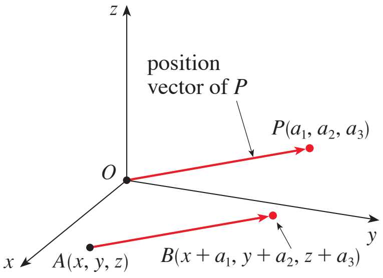


FIGURE 13
Representations of $\mathbf{a} = \langle a_1, a_2, a_3\rangle$

For instance, all of the vectors shown in Figure 12 are equivalent to the vector $\overrightarrow{OP} = \langle 3, 2\rangle$ whose terminal point is $P(3, 2)$. What they have in common is that the terminal point is reached from the initial point by a displacement of three units to the right and two upward. We can think of all these geometric vectors as **representations** of the algebraic vector $\mathbf{a} = \langle 3, 2\rangle$. The particular representation $\overrightarrow{OP}$ from the origin to the point $P(3, 2)$ is called the **position vector** of the point P.

In three dimensions, the vector $\mathbf{a} = \overrightarrow{OP} = \langle a_1, a_2, a_3\rangle$ is the **position vector** of the point $P(a_1, a_2, a_3)$. (See Figure 13.) Let's consider any other representation of $\mathbf{a}$ by a directed line segment $\overrightarrow{AB}$ with initial point $A(x_1, y_1, z_1)$ and terminal point $B(x_2, y_2, z_2)$. Then we must have $x_1 + a_1 = x_2$, $y_1 + a_2 = y_2$, and $z_1 + a_3 = z_2$ and so $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, and $a_3 = z_2 - z_1$. Thus we have the following result.

1 Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$$

EXAMPLE 3 Find the vector represented by the directed line segment with initial point $A(2, -3, 4)$ and terminal point $B(-2, 1, 1)$.

SOLUTION By (1), the vector corresponding to $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle -2 - 2, 1 - (-3), 1 - 4\rangle = \langle -4, 4, -3\rangle$$ ■

The **magnitude** or **length** of the vector $\mathbf{v}$ is the length of any of its representations and is denoted by the symbol $|\mathbf{v}|$ or $\|\mathbf{v}\|$. By using the distance formula to compute the length of a segment OP, we obtain the following formulas.

The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2\rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

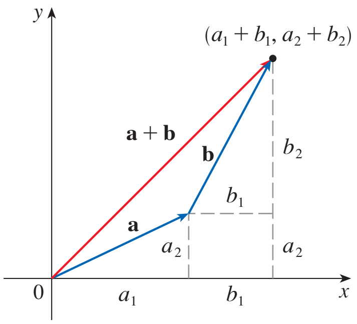


FIGURE 14

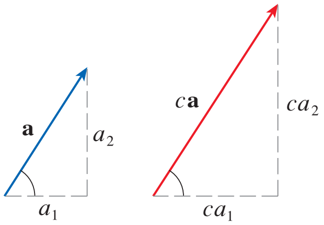


FIGURE 15

How do we add vectors algebraically? Figure 14 shows that if $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then their sum is $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$, at least for the case where the components are positive. In other words, *to add algebraic vectors we add corresponding components*. Similarly, *to subtract vectors we subtract corresponding components*. From the similar triangles in Figure 15 we see that the components of $c\mathbf{a}$ are ca_1 and ca_2. So *to multiply a vector by a scalar we multiply each component by that scalar.*

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \qquad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

EXAMPLE 4 If $\mathbf{a} = \langle 4, 0, 3 \rangle$ and $\mathbf{b} = \langle -2, 1, 5 \rangle$, find $|\mathbf{a}|$ and the vectors $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $3\mathbf{b}$, and $2\mathbf{a} + 5\mathbf{b}$.

SOLUTION

$$|\mathbf{a}| = \sqrt{4^2 + 0^2 + 3^2} = \sqrt{25} = 5$$

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \langle 4, 0, 3 \rangle + \langle -2, 1, 5 \rangle \\ &= \langle 4 + (-2), 0 + 1, 3 + 5 \rangle = \langle 2, 1, 8 \rangle \end{aligned}$$

$$\begin{aligned} \mathbf{a} - \mathbf{b} &= \langle 4, 0, 3 \rangle - \langle -2, 1, 5 \rangle \\ &= \langle 4 - (-2), 0 - 1, 3 - 5 \rangle = \langle 6, -1, -2 \rangle \end{aligned}$$

$$3\mathbf{b} = 3\langle -2, 1, 5 \rangle = \langle 3(-2), 3(1), 3(5) \rangle = \langle -6, 3, 15 \rangle$$

$$\begin{aligned} 2\mathbf{a} + 5\mathbf{b} &= 2\langle 4, 0, 3 \rangle + 5\langle -2, 1, 5 \rangle \\ &= \langle 8, 0, 6 \rangle + \langle -10, 5, 25 \rangle = \langle -2, 5, 31 \rangle \end{aligned}$$ ■

We denote by V_2 the set of all two-dimensional vectors and by V_3 the set of all three-dimensional vectors. More generally, we will later need to consider the set V_n of all n-dimensional vectors. An n-dimensional vector is an ordered n-tuple:

$$\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$$

where $a_1, a_2, \ldots, a_n$ are real numbers that are called the components of $\mathbf{a}$. Addition and scalar multiplication in V_n are defined in terms of components just as for the cases $n = 2$ and $n = 3$.

Vectors in n dimensions are used to list various quantities in an organized way. For instance, the components of a six-dimensional vector

$$\mathbf{p} = \langle p_1, p_2, p_3, p_4, p_5, p_6 \rangle$$

might represent the prices of six different ingredients required to make a particular product. Four-dimensional vectors $\langle x, y, z, t \rangle$ are used in relativity theory, where the first three components specify a position in space and the fourth represents time.

Properties of Vectors If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$

4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$

6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$

7. $(cd)\mathbf{a} = c(d\mathbf{a})$

8. $1\mathbf{a} = \mathbf{a}$

These eight properties of vectors can be readily verified either geometrically or algebraically. For instance, Property 1 can be seen from Figure 4 (it's equivalent to the Parallelogram Law) or as follows for the case $n = 2$:

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle \\ &= \langle b_1 + a_1, b_2 + a_2 \rangle = \langle b_1, b_2 \rangle + \langle a_1, a_2 \rangle \\ &= \mathbf{b} + \mathbf{a}\end{aligned}$$

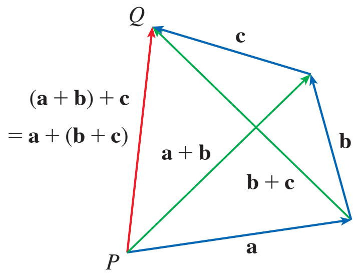


FIGURE 16

We can see why Property 2 (the associative law) is true by looking at Figure 16 and applying the Triangle Law several times: the vector $\overrightarrow{PQ}$ is obtained either by first constructing $\mathbf{a} + \mathbf{b}$ and then adding $\mathbf{c}$ or by adding $\mathbf{a}$ to the vector $\mathbf{b} + \mathbf{c}$.

Three vectors in V_3 play a special role. Let

$$\boxed{\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle}$$

These vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are called the **standard basis vectors**. They have length 1 and point in the directions of the positive x-, y-, and z-axes. Similarly, in two dimensions we define $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$. (See Figure 17.)

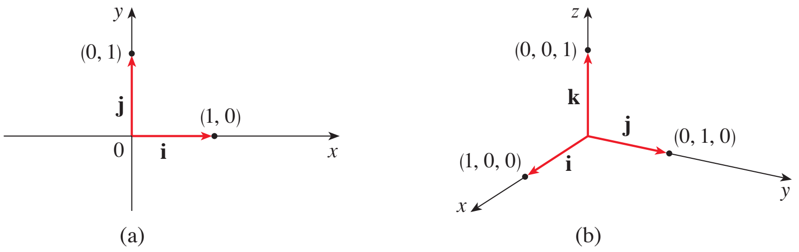


FIGURE 17
Standard basis vectors in V_2 and V_3

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then we can write

$$\begin{aligned}\mathbf{a} &= \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \\ &= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle\end{aligned}$$

$$\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \tag{2}$$

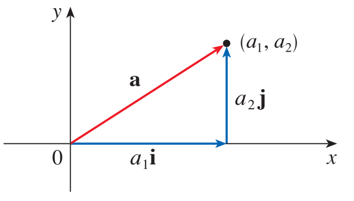
Thus any vector in V_3 can be expressed in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. For instance,

$$\langle 1, -2, 6 \rangle = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

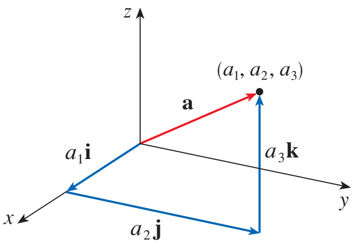
Similarly, in two dimensions, we can write

$$\mathbf{a} = \langle a_1, a_2 \rangle = a_1 \mathbf{i} + a_2 \mathbf{j} \tag{3}$$

See Figure 18 for the geometric interpretation of Equations 3 and 2 and compare with Figure 17.



(a) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$



(b) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

FIGURE 18

EXAMPLE 5 If $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 7\mathbf{k}$, express the vector $2\mathbf{a} + 3\mathbf{b}$ in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

SOLUTION Using Properties 1, 2, 5, 6, and 7 of vectors, we have

$$\begin{aligned}2\mathbf{a} + 3\mathbf{b} &= 2(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + 3(4\mathbf{i} + 7\mathbf{k}) \\ &= 2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k} + 12\mathbf{i} + 21\mathbf{k} = 14\mathbf{i} + 4\mathbf{j} + 15\mathbf{k}\end{aligned}$$ ■

A **unit vector** is a vector whose length is 1. For instance, **i**, **j**, and **k** are all unit vectors. In general, if $\mathbf{a} \neq \mathbf{0}$, then the unit vector that has the same direction as **a** is

$$\boxed{4} \qquad \mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

In order to verify this, we let $c = 1/|\mathbf{a}|$. Then $\mathbf{u} = c\mathbf{a}$ and c is a positive scalar, so **u** has the same direction as **a**. Also

$$|\mathbf{u}| = |c\mathbf{a}| = |c||\mathbf{a}| = \frac{1}{|\mathbf{a}|}|\mathbf{a}| = 1$$

Gibbs

Josiah Willard Gibbs (1839–1903), a professor of mathematical physics at Yale College, published the first book on vectors, *Vector Analysis*, in 1881. More complicated objects, called quaternions, had earlier been invented by Sir William Rowan Hamilton as mathematical tools for describing space, but they weren't easy for scientists to use. Quaternions have a scalar part and a vector part. Gibb's idea was to use the vector part separately. Maxwell and Heaviside had similar ideas, but Gibb's approach has proved to be the most convenient way to study space.

EXAMPLE 6 Find the unit vector in the direction of the vector $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

SOLUTION The given vector has length

$$|2\mathbf{i} - \mathbf{j} - 2\mathbf{k}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$$

so, by Equation 4, the unit vector with the same direction is

$$\tfrac{1}{3}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = \tfrac{2}{3}\mathbf{i} - \tfrac{1}{3}\mathbf{j} - \tfrac{2}{3}\mathbf{k}$$ ■

Applications

Vectors are useful in many aspects of physics and engineering. In Chapter 13 we will see how they describe the velocity and acceleration of objects moving in space. Here we first look at forces.

A force is represented by a vector because it has both magnitude (measured in pounds or newtons) and direction. If several forces are acting on an object, the **resultant force** experienced by the object is the vector sum of these forces.

EXAMPLE 7 A 100 kg weight hangs from two wires as shown in Figure 19. Find the tensions (forces) $\mathbf{T}_1$ and $\mathbf{T}_2$ in the wires and the magnitudes of these tensions.

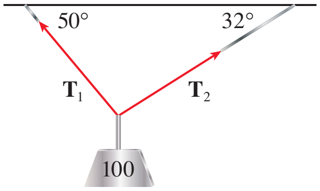


FIGURE 19

SOLUTION We first express $\mathbf{T}_1$ and $\mathbf{T}_2$ in terms of their horizontal and vertical components. From Figure 20 we see that

$$\boxed{5} \qquad \mathbf{T}_1 = -|\mathbf{T}_1|\cos 50°\,\mathbf{i} + |\mathbf{T}_1|\sin 50°\,\mathbf{j}$$

$$\boxed{6} \qquad \mathbf{T}_2 = |\mathbf{T}_2|\cos 32°\,\mathbf{i} + |\mathbf{T}_2|\sin 32°\,\mathbf{j}$$

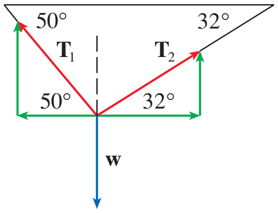


FIGURE 20

The force of gravity acting on the load is $\mathbf{F} = -100(9.8)\,\mathbf{j} = -980\,\mathbf{j}$. The resultant $\mathbf{T}_1 + \mathbf{T}_2$ of the tensions counterbalances **F** and so we must have

$$\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{F} = 980\,\mathbf{j}$$

Thus

$$\bigl(-|\mathbf{T}_1|\cos 50° + |\mathbf{T}_2|\cos 32°\bigr)\,\mathbf{i} + \bigl(|\mathbf{T}_1|\sin 50° + |\mathbf{T}_2|\sin 32°\bigr)\,\mathbf{j} = 980\,\mathbf{j}$$

Equating components, we get

$$-|\mathbf{T}_1|\cos 50° + |\mathbf{T}_2|\cos 32° = 0$$

$$|\mathbf{T}_1|\sin 50° + |\mathbf{T}_2|\sin 32° = 980$$

Solving the first of these equations for $|\mathbf{T}_2|$ and substituting into the second, we get

$$|\mathbf{T}_1|\sin 50° + \frac{|\mathbf{T}_1|\cos 50°}{\cos 32°}\sin 32° = 980$$

$$|\mathbf{T}_1|\left(\sin 50° + \cos 50°\,\frac{\sin 32°}{\cos 32°}\right) = 980$$

So the magnitudes of the tensions are

$$|\mathbf{T}_1| = \frac{980}{\sin 50° + \tan 32° \cos 50°} \approx 839 \text{ N}$$

and $$|\mathbf{T}_2| = \frac{|\mathbf{T}_1|\cos 50°}{\cos 32°} \approx 636 \text{ N}$$

Substituting these values in (5) and (6), we obtain the tension vectors

$$\mathbf{T}_1 \approx -539\,\mathbf{i} + 643\,\mathbf{j}$$

$$\mathbf{T}_2 \approx 539\,\mathbf{i} + 337\,\mathbf{j}$$ ■

If an airplane is flying in wind, then the *true course*, or *track*, of the plane is the direction of the resultant of the velocity vectors of the plane and of the wind. The *ground speed* of the plane is the magnitude of the resultant. Similarly, a boat navigating through flowing water follows a true course in the direction of the resultant of the velocity vectors of the boat and of the water current.

EXAMPLE 8 A woman launches a boat from the south shore of a straight river that flows directly west at 4 km/h. She wants to land at the point directly across on the opposite shore. If the speed of the boat (relative to the water) is 8 km/h, in what direction should she steer the boat in order to arrive at the desired landing point?

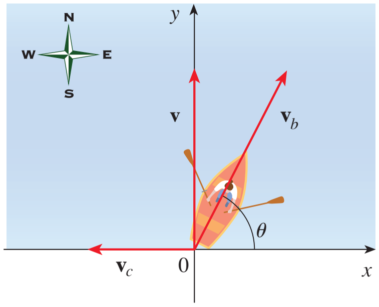


FIGURE 21

When describing directions for navigation, we often use a *bearing*, such as N 20° W, which means from the northerly direction, turn 20° toward west. (Note that a bearing always begins with either north or south.)

SOLUTION Let's choose coordinate axes with the origin at the initial position of the boat, as shown in Figure 21. The velocity of the river current is $\mathbf{v}_c = -4\,\mathbf{i}$ and, since the speed of the boat (in still water) is 8 km/h, the boat's velocity is $\mathbf{v}_b = 8(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$, where θ is as shown in the figure. The resultant velocity is

$$\begin{aligned}\mathbf{v} &= \mathbf{v}_b + \mathbf{v}_c \\ &= 8\cos\theta\,\mathbf{i} + 8\sin\theta\,\mathbf{j} - 4\,\mathbf{i} = (-4 + 8\cos\theta)\mathbf{i} + (8\sin\theta)\mathbf{j}\end{aligned}$$

We want the true course of the boat to be directly north, so the x-component of $\mathbf{v}$ must be zero:

$$-4 + 8\cos\theta = 0 \quad\Longrightarrow\quad \cos\theta = \tfrac{1}{2} \quad\Longrightarrow\quad \theta = 60°$$

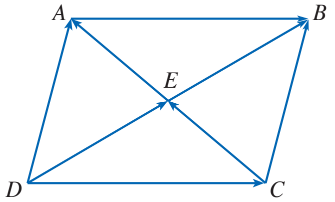
Thus the woman should steer the boat in the direction $\theta = 60°$, or N 30° E. ■

12.2 Exercises

1. Is each of the following quantities a vector or a scalar? Explain.
   (a) The cost of a theater ticket
   (b) The current in a river
   (c) The initial flight path from Houston to Dallas
   (d) The population of the world

2. What is the relationship between the point (4, 7) and the vector $\langle 4, 7\rangle$? Illustrate with a sketch.

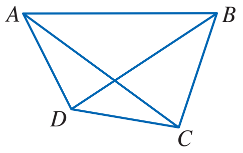
3. Name all the equal vectors in the parallelogram shown.



4. Using the vectors shown in the figure, write each sum or difference as a single vector.

(a) $\overrightarrow{AB} + \overrightarrow{BC}$ (b) $\overrightarrow{CD} + \overrightarrow{DB}$

(c) $\overrightarrow{DB} - \overrightarrow{AB}$ (d) $\overrightarrow{DC} + \overrightarrow{CA} + \overrightarrow{AB}$

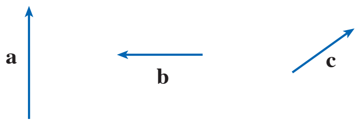


5. Copy the vectors in the figure and use them to draw the following vectors.

(a) $\mathbf{a} + \mathbf{b}$ (b) $\mathbf{b} + \mathbf{c}$

(c) $\mathbf{a} + \mathbf{c}$ (d) $\mathbf{a} - \mathbf{c}$

(e) $\mathbf{b} + \mathbf{a} + \mathbf{c}$ (f) $\mathbf{a} - \mathbf{b} - \mathbf{c}$

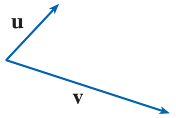


6. Copy the vectors in the figure and use them to draw the following vectors.

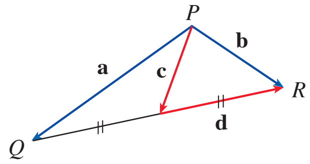
(a) $\mathbf{u} + \mathbf{v}$ (b) $\mathbf{u} - \mathbf{v}$

(c) $2\mathbf{u}$ (d) $-\frac{1}{2}\mathbf{v}$

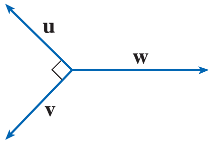
(e) $3\mathbf{u} + \mathbf{v}$ (f) $\mathbf{v} - 2\mathbf{u}$



7. In the figure, the tip of **c** and the tail of **d** are both the midpoint of QR. Express **c** and **d** in terms of **a** and **b**.



8. If the vectors in the figure satisfy $|\mathbf{u}| = |\mathbf{v}| = 1$ and $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$, what is $|\mathbf{w}|$?



9–14 Find a vector **a** with representation given by the directed line segment $\overrightarrow{AB}$. Draw $\overrightarrow{AB}$ and the equivalent representation starting at the origin.

9. $A(-2, 1)$, $B(1, 2)$

10. $A(-5, -1)$, $B(-3, 3)$

11. $A(3, -1)$, $B(2, 3)$

12. $A(3, 2)$, $B(1, 0)$

13. $A(1, -2, 4)$, $B(-2, 3, 0)$

14. $A(3, 0, -2)$, $B(0, 5, 0)$

15–18 Find the sum of the given vectors and illustrate geometrically.

15. $\langle -1, 4\rangle$, $\langle 6, -2\rangle$

16. $\langle 3, -1\rangle$, $\langle -1, 5\rangle$

17. $\langle 3, 0, 1\rangle$, $\langle 0, 8, 0\rangle$

18. $\langle 1, 3, -2\rangle$, $\langle 0, 0, 6\rangle$

19–22 Find $\mathbf{a} + \mathbf{b}$, $4\mathbf{a} + 2\mathbf{b}$, $|\mathbf{a}|$, and $|\mathbf{a} - \mathbf{b}|$.

19. $\mathbf{a} = \langle -3, 4\rangle$, $\mathbf{b} = \langle 9, -1\rangle$

20. $\mathbf{a} = 5\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = -\mathbf{i} - 2\mathbf{j}$

21. $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 4\mathbf{k}$

22. $\mathbf{a} = \langle 8, 1, -4\rangle$, $\mathbf{b} = \langle 5, -2, 1\rangle$

23–25 Find a unit vector that has the same direction as the given vector.

23. $\langle 6, -2\rangle$

24. $-5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

25. $8\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

---

26. Find the vector that has the same direction as $\langle 6, 2, -3\rangle$ but has length 4.

27–28 What is the angle between the given vector and the positive direction of the x-axis?

27. $\mathbf{i} + \sqrt{3}\,\mathbf{j}$

28. $8\mathbf{i} + 6\mathbf{j}$

---

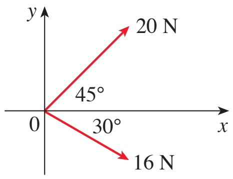
29. The initial point of a vector **v** in V_2 is the origin and the terminal point is in quadrant II. If **v** makes an angle $5\pi/6$ with the positive x-axis and $|\mathbf{v}| = 4$, find **v** in component form.

30. If a child pulls a sled through the snow on a level path with a force of 50 N exerted at an angle of 38° above the horizontal, find the horizontal and vertical components of the force.

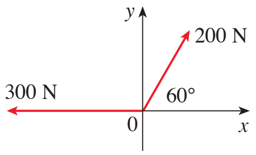
31. A quarterback throws a football with angle of elevation 40° and speed 20 m/s. Find the horizontal and vertical components of the velocity vector.

32–33 Find the magnitude of the resultant force and the angle it makes with the positive x-axis.

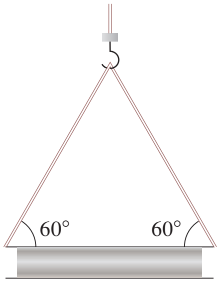
32.



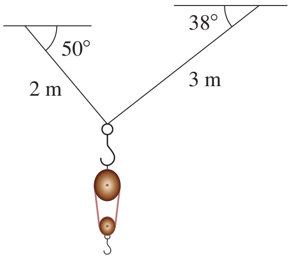
33.



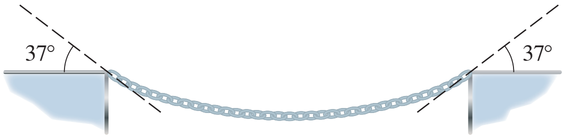
34. A crane suspends a 500 kg steel beam horizontally by support cables (with negligible weight) attached from a hook to each end of the beam. The support cables each make an angle of 60° with the beam. Find the tension vector in each support cable and the magnitude of each tension.



35. A block-and-tackle pulley hoist is suspended in a warehouse by ropes of lengths 2 m and 3 m. The hoist weighs 350 N. The ropes, fastened at different heights, make angles of 50° and 38° with the horizontal. Find the tension in each rope and the magnitude of each tension.



36. The tension vector at each end of a chain has magnitude 25 N (see the figure). What is the weight of the chain?



37. Three forces act on an object. Two of the forces are at an angle of 100° to each other and have magnitudes 25 N and 12 N. The third is perpendicular to the plane of these two forces and has magnitude 4 N. Calculate the magnitude of the force that would exactly counterbalance these three forces.

38. A rower wants to row her kayak across a channel that is 400 m wide and land at a point 250 m upstream from her starting point. She can row (in still water) at 2 m/s and the current in the channel flows at 0.5 m/s.
(a) In what direction should she steer the kayak?
(b) How long will the trip take?

39. A pilot is steering a plane in the direction N 45° W at an airspeed (speed in still air) of 290 km/h. A wind is blowing in the direction S 30° E at a speed of 55 km/h. Find the true course and the ground speed of the plane.

40. A ship is sailing west at a speed of 32 km/h and a dog is running due north on the deck of the ship at 4 km/h. Find the speed and direction of the dog relative to the surface of the water.

41. Find the unit vectors that are parallel to the tangent line to the parabola $y = x^2$ at the point (2, 4).

42. (a) Find the unit vectors that are parallel to the tangent line to the curve $y = 2 \sin x$ at the point $(\pi/6, 1)$.
(b) Find the unit vectors that are perpendicular to the tangent line.
(c) Sketch the curve $y = 2 \sin x$ and the vectors in parts (a) and (b), all starting at $(\pi/6, 1)$.

43. If A, B, and C are the vertices of a triangle, find

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$$

44. Let C be the point on the line segment AB that is twice as far from B as it is from A. If $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, and $\mathbf{c} = \overrightarrow{OC}$, show that $\mathbf{c} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$.

45. (a) Draw the vectors $\mathbf{a} = \langle 3, 2 \rangle$, $\mathbf{b} = \langle 2, -1 \rangle$, and $\mathbf{c} = \langle 7, 1 \rangle$.
(b) Show, by means of a sketch, that there are scalars s and t such that $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$.
(c) Use the sketch to estimate the values of s and t.
(d) Find the exact values of s and t.

46. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors that are not parallel and $\mathbf{c}$ is any vector in the plane determined by $\mathbf{a}$ and $\mathbf{b}$. Give a geometric argument to show that $\mathbf{c}$ can be written as $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$ for suitable scalars s and t. Then give an argument using components.

47. If $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, describe the set of all points (x, y, z) such that $|\mathbf{r} - \mathbf{r}_0| = 1$.

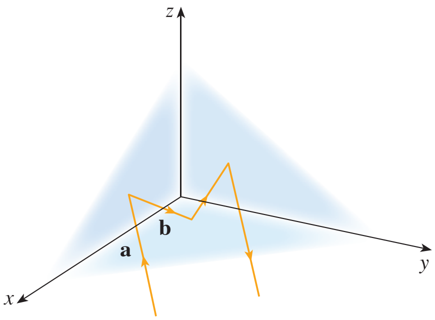
48. If $\mathbf{r} = \langle x, y \rangle$, $\mathbf{r}_1 = \langle x_1, y_1 \rangle$, and $\mathbf{r}_2 = \langle x_2, y_2 \rangle$, describe the set of all points (x, y) such that $|\mathbf{r} - \mathbf{r}_1| + |\mathbf{r} - \mathbf{r}_2| = k$, where $k > |\mathbf{r}_1 - \mathbf{r}_2|$.

49. Figure 16 gives a geometric demonstration of Property 2 of vectors. Use components to give an algebraic proof of this fact for the case $n = 2$.

50. Prove Property 5 of vectors algebraically for the case $n = 3$. Then use similar triangles to give a geometric proof.

51. Use vectors to prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

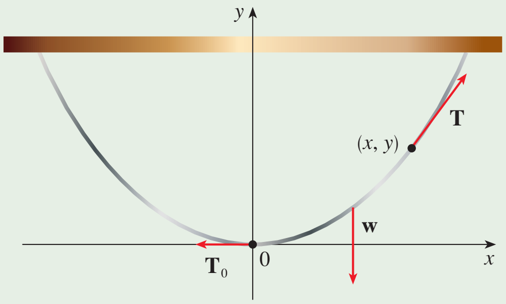
52. **Corner Reflectors** Suppose the three coordinate planes are all mirrored, forming a *corner reflector*, and a light ray given by the vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ first strikes the xz-plane, as shown in the figure. Use the fact that the angle of incidence equals the angle of reflection to show that the direction of the reflected ray is given by $\mathbf{b} = \langle a_1, -a_2, a_3 \rangle$. Deduce that, after being reflected by all three mutually perpendicular mirrors, the resulting ray is parallel to the initial ray. (Scientists have used this principle, together with laser beams and an array of corner reflectors on the moon, to calculate very precisely the distance from Earth to the moon.)

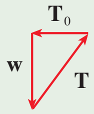


DISCOVERY PROJECT | THE SHAPE OF A HANGING CHAIN

In Section 3.11 we stated that a heavy flexible chain or cable suspended between two points at the same height takes the shape of a curve called a *catenary* (a term reportedly coined by Thomas Jefferson) with equation $y = a \cosh(x/a)$. Here we use the interpretation of the derivative as the slope of a tangent to derive this equation.

Suppose that a chain (or cable) of uniform linear mass density ρ is hanging between two points, as shown in the figure. We place the origin at the vertex of the catenary, and let (x, y) be any point on the curve, $x > 0$. (By symmetry, if $x < 0$ we obtain a similar result.)





Consider the section of the chain from the origin to (x, y). The forces that act on the section are the downward gravitational force $\mathbf{w}$ and the tensions $\mathbf{T}_0$ and $\mathbf{T}$ at each end of the section—each of which is tangent to the curve. Because the section of chain is in equilibrium, we know that

$$\mathbf{T}_0 + \mathbf{T} + \mathbf{w} = \mathbf{0}$$

1. Let $y = f(x)$ be the equation of the curve and let $s(x)$ be the arc length function (Equation 8.1.5) from the origin to the point (x, y). Show that $\mathbf{T} = \langle |\mathbf{T}_0|, g\rho s(x) \rangle$, where g is the acceleration due to gravity.

2. By interpreting dy/dx as the slope of a tangent at (x, y), show that

$$\frac{dy}{dx} = \frac{s(x)}{a}$$

where $a = |\mathbf{T}_0|/(g\rho)$, a constant.

3. Differentiate both sides of the differential equation in Problem 2 and use Equation 8.1.6 to obtain the second-order differential equation

$$\frac{d^2y}{dx^2} = \frac{1}{a}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

with initial conditions $y(0) = 0$ (the curve passes through the origin) and $y'(0) = 0$ (the tangent at the origin is horizontal). Solve this equation by first substituting $z = dy/dx$ and then solving the resulting first-order differential equation. Conclude that the equation of the curve is

$$y = a\cosh\frac{x}{a} - a$$

4. Graph $y = a\cosh(x/a) - a$ for $a = \frac{1}{2}$, $a = 1$, and $a = 3$. How does the value of a affect the shape of the curve?

12.3 The Dot Product

So far we have seen how to add two vectors and how to multiply a vector by a scalar. The question arises: is it possible to multiply two vectors so that their product is a useful quantity? One such product is the dot product, which we now define. Another is the cross product, which is discussed in the next section.

The Dot Product of Two Vectors

To find the dot product of vectors **a** and **b** we multiply corresponding components and add.

1 **Definition of the Dot Product** If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, thenthe **dot product** of **a** and **b** is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

The dot product of two vectors is a real number, not a vector. For this reason, the dot product is sometimes called the **scalar product** (or **inner product**). Although Definition 1 is given for three-dimensional vectors, the dot product of two-dimensional vectors is defined in a similar fashion:

$$\langle a_1, a_2\rangle \cdot \langle b_1, b_2\rangle = a_1b_1 + a_2b_2$$

EXAMPLE 1

$$\langle 2, 4\rangle \cdot \langle 3, -1\rangle = 2(3) + 4(-1) = 2$$

$$\langle -1, 7, 4\rangle \cdot \left\langle 6, 2, -\tfrac{1}{2}\right\rangle = (-1)(6) + 7(2) + 4\left(-\tfrac{1}{2}\right) = 6$$

$$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k}) = 1(0) + 2(2) + (-3)(-1) = 7$$ ■

The dot product obeys many of the laws that hold for ordinary products of real numbers. These are stated in the following theorem.

2 Properties of the Dot Product If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$

5. $\mathbf{0} \cdot \mathbf{a} = 0$

PROOF These properties are easily proved using Definition 1. For instance, here are the proofs of Properties 1 and 3:

1. $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2$

3.
$$\begin{aligned}\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle \\ &= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\ &= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \\ &= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\ &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}\end{aligned}$$

The proofs of the remaining properties are left as exercises. ■

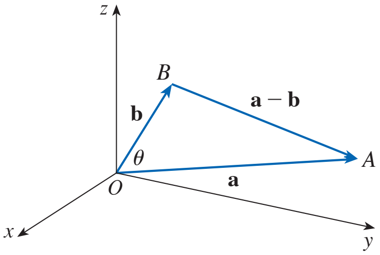


FIGURE 1

The dot product $\mathbf{a} \cdot \mathbf{b}$ can be given a geometric interpretation in terms of the **angle** θ **between a and b**, which is defined to be the angle between the representations of $\mathbf{a}$ and $\mathbf{b}$ that start at the origin, where $0 \leqslant \theta \leqslant \pi$. In other words, θ is the angle between the line segments $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in Figure 1. Note that if $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors, then $\theta = 0$ or $\theta = \pi$.

The formula in the following theorem is used by physicists as the *definition* of the dot product.

3 Theorem If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

The Law of Cosines is reviewed in Appendix D.

PROOF If we apply the Law of Cosines to triangle OAB in Figure 1, we get

$$\boxed{4} \qquad |AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB| \cos \theta$$

(Observe that the Law of Cosines still applies in the limiting cases when $\theta = 0$ or π, or $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.) But $|OA| = |\mathbf{a}|$, $|OB| = |\mathbf{b}|$, and $|AB| = |\mathbf{a} - \mathbf{b}|$, so Equation 4 becomes

$$\boxed{5} \qquad |\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta$$

Using Properties 1, 2, and 3 of the dot product, we can rewrite the left side of this equation as follows:

$$\begin{aligned}|\mathbf{a} - \mathbf{b}|^2 &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\ &= |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2\end{aligned}$$

Therefore Equation 5 gives

$$|\mathbf{a}|^2 - 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$$

Thus

$$-2\mathbf{a}\cdot\mathbf{b} = -2|\mathbf{a}||\mathbf{b}|\cos\theta$$

or

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \qquad \blacksquare$$

EXAMPLE 2 If the vectors **a** and **b** have lengths 4 and 6, and the angle between them is $\pi/3$, find $\mathbf{a}\cdot\mathbf{b}$.

SOLUTION Using Theorem 3, we have

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\pi/3) = 4\cdot 6\cdot \tfrac{1}{2} = 12 \qquad \blacksquare$$

The formula in Theorem 3 also enables us to find the angle between two vectors.

6 Corollary If θ is the angle between the nonzero vectors **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

EXAMPLE 3 Find the angle between the vectors $\mathbf{a} = \langle 2, 2, -1\rangle$ and $\mathbf{b} = \langle 5, -3, 2\rangle$.

SOLUTION Since

$$|\mathbf{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3 \qquad \text{and} \qquad |\mathbf{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

and since

$$\mathbf{a}\cdot\mathbf{b} = 2(5) + 2(-3) + (-1)(2) = 2$$

we have, from Corollary 6,

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{2}{3\sqrt{38}}$$

So the angle between **a** and **b** is

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx 1.46 \quad (\text{or } 84^\circ) \qquad \blacksquare$$

Two nonzero vectors **a** and **b** are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \pi/2$. Then Theorem 3 gives

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\pi/2) = 0$$

and conversely if $\mathbf{a}\cdot\mathbf{b} = 0$, then $\cos\theta = 0$, so $\theta = \pi/2$. The zero vector **0** is considered to be perpendicular to all vectors. Therefore we have the following method for determining whether two vectors are orthogonal.

7 Two vectors **a** and **b** are orthogonal if and only if $\mathbf{a}\cdot\mathbf{b} = 0$.

EXAMPLE 4 Show that $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is perpendicular to $5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

SOLUTION Since

$$(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 2(5) + 2(-4) + (-1)(2) = 0$$

these vectors are perpendicular by (7). ■

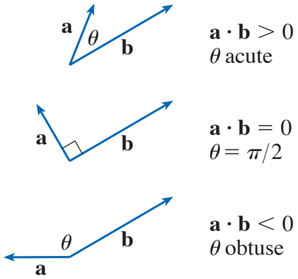

a
θ
b
$\mathbf{a} \cdot \mathbf{b} > 0$
θ acute
a
b
$\mathbf{a} \cdot \mathbf{b} = 0$
$\theta = \pi/2$
θ
b
a
$\mathbf{a} \cdot \mathbf{b} < 0$
θ obtuse

FIGURE 2

Because $\cos\theta > 0$ if $0 \leqslant \theta < \pi/2$ and $\cos\theta < 0$ if $\pi/2 < \theta \leqslant \pi$, we see that $\mathbf{a} \cdot \mathbf{b}$ is positive for $\theta < \pi/2$ and negative for $\theta > \pi/2$. We can think of $\mathbf{a} \cdot \mathbf{b}$ as measuring the extent to which $\mathbf{a}$ and $\mathbf{b}$ point in the same direction. The dot product $\mathbf{a} \cdot \mathbf{b}$ is positive if $\mathbf{a}$ and $\mathbf{b}$ point in the same general direction, 0 if they are perpendicular, and negative if they point in generally opposite directions (see Figure 2). In the extreme case where $\mathbf{a}$ and $\mathbf{b}$ point in exactly the same direction, we have $\theta = 0$, so $\cos\theta = 1$ and

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$$

If $\mathbf{a}$ and $\mathbf{b}$ point in exactly opposite directions, then we have $\theta = \pi$ and so $\cos\theta = -1$ and $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}||\mathbf{b}|$.

Direction Angles and Direction Cosines

The **direction angles** of a nonzero vector $\mathbf{a}$ are the angles α, β, and γ (in the interval $[0, \pi]$) that $\mathbf{a}$ makes with the positive x-, y-, and z-axes, respectively (see Figure 3).

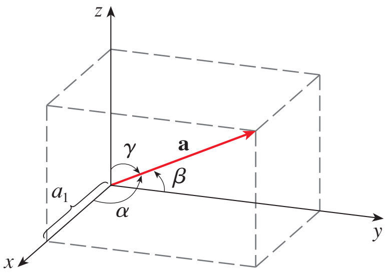


FIGURE 3

The cosines of these direction angles, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, are called the **direction cosines** of the vector $\mathbf{a}$. Using Corollary 6 with $\mathbf{b}$ replaced by $\mathbf{i}$, we obtain

$$\boxed{8} \qquad \cos\alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}$$

(This can also be seen directly from Figure 3.)

Similarly, we also have

$$\boxed{9} \qquad \cos\beta = \frac{a_2}{|\mathbf{a}|} \qquad \cos\gamma = \frac{a_3}{|\mathbf{a}|}$$

By squaring the expressions in Equations 8 and 9 and adding, we see that

$$\boxed{10} \qquad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

We can also use Equations 8 and 9 to write

$$\begin{aligned} \mathbf{a} = \langle a_1, a_2, a_3 \rangle &= \big\langle |\mathbf{a}|\cos\alpha, |\mathbf{a}|\cos\beta, |\mathbf{a}|\cos\gamma \big\rangle \\ &= |\mathbf{a}|\langle \cos\alpha, \cos\beta, \cos\gamma \rangle \end{aligned}$$

Therefore

$$\boxed{11} \qquad \frac{1}{|\mathbf{a}|}\mathbf{a} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

which says that the direction cosines of $\mathbf{a}$ are the components of the unit vector in the direction of $\mathbf{a}$.

EXAMPLE 5 Find the direction angles of the vector $\mathbf{a} = \langle 1, 2, 3 \rangle$.

SOLUTION Since $|\mathbf{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, Equations 8 and 9 give

$$\cos\alpha = \frac{1}{\sqrt{14}} \qquad \cos\beta = \frac{2}{\sqrt{14}} \qquad \cos\gamma = \frac{3}{\sqrt{14}}$$

and so

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) \approx 74^\circ \qquad \beta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) \approx 58^\circ \qquad \gamma = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) \approx 37^\circ$$

■

■ Projections

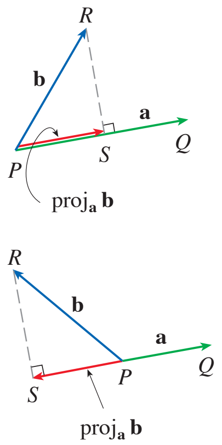

R
b
a
Q
S
P
$\text{proj}_{\mathbf{a}}\,\mathbf{b}$
R
b
a
Q
P
S
$\text{proj}_{\mathbf{a}}\,\mathbf{b}$

FIGURE 4
Vector projections

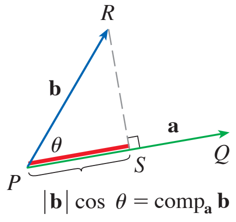

R
b
a
θ
Q
P
S
$|\mathbf{b}|\cos\theta = \text{comp}_{\mathbf{a}}\,\mathbf{b}$

FIGURE 5
Scalar projection

Figure 4 shows representations $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of two vectors **a** and **b** with the same initial point P. If S is the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the **vector projection** of **b** onto **a** and is denoted by $\text{proj}_{\mathbf{a}}\,\mathbf{b}$. (You can think of it as a shadow of **b**.)

The **scalar projection** of **b** onto **a** (also called the **component of b along a**) is defined to be the signed magnitude of the vector projection, which is the number $|\mathbf{b}|\cos\theta$, where θ is the angle between **a** and **b**. (See Figure 5.) This is denoted by $\text{comp}_{\mathbf{a}}\,\mathbf{b}$. Observe that it is negative if $\pi/2 < \theta \leqslant \pi$. The equation

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{a}|(|\mathbf{b}|\cos\theta)$$

shows that the dot product of **a** and **b** can be interpreted as the length of **a** times the scalar projection of **b** onto **a**. Since

$$|\mathbf{b}|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|}\cdot\mathbf{b}$$

the component of **b** along **a** can be computed by taking the dot product of **b** with the unit vector in the direction of **a**. We summarize these ideas as follows.

Scalar projection of **b** onto **a**: $\qquad \text{comp}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$

Vector projection of **b** onto **a**: $\qquad \text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}$

Notice that the vector projection is the scalar projection times the unit vector in the direction of **a**.

EXAMPLE 6 Find the scalar projection and vector projection of $\mathbf{b} = \langle 1, 1, 2\rangle$ onto $\mathbf{a} = \langle -2, 3, 1\rangle$.

SOLUTION Since $|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$, the scalar projection of **b** onto **a** is

$$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

The vector projection is this scalar projection times the unit vector in the direction of **a**:

$$\text{proj}_{\mathbf{a}}\,\mathbf{b} = \frac{3}{\sqrt{14}}\,\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14}\,\mathbf{a} = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\right\rangle$$

■

Application: Work

One use of projections occurs in physics in calculating work. In Section 6.4 we defined the work done by a constant force F in moving an object through a distance d as $W = Fd$, but this applies only when the force is directed along the line of motion of the object. Suppose, however, that the constant force is a vector $\mathbf{F} = \overrightarrow{PR}$ pointing in some other direction, as illustrated in Figure 6. If the force moves the object from P to Q, then the **displacement vector** is $\mathbf{D} = \overrightarrow{PQ}$. The **work** done by this force is defined to be the product of the component of the force along $\mathbf{D}$ and the distance moved:

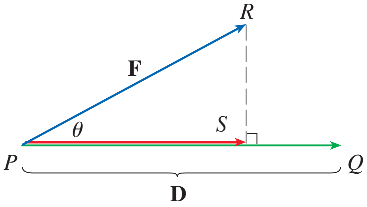


FIGURE 6

$$W = (|\mathbf{F}| \cos \theta)|\mathbf{D}|$$

But then, from Theorem 3, we have

$$\boxed{12} \qquad W = |\mathbf{F}||\mathbf{D}| \cos \theta = \mathbf{F} \cdot \mathbf{D}$$

Thus the work done by a constant force $\mathbf{F}$ is the dot product $\mathbf{F} \cdot \mathbf{D}$, where $\mathbf{D}$ is the displacement vector.

EXAMPLE 7 A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle 35° above the horizontal. Find the work done by the force.

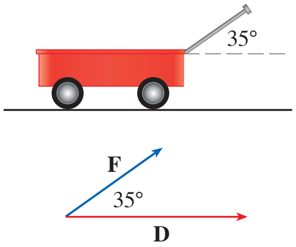


FIGURE 7

SOLUTION If $\mathbf{F}$ and $\mathbf{D}$ are the force and displacement vectors, as pictured in Figure 7, then the work done is

$$W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}| \cos 35°$$

$$= (70)(100) \cos 35° \approx 5734 \text{ N}\cdot\text{m} = 5734 \text{ J}$$

■

EXAMPLE 8 A force is given by a vector $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and moves a particle from the point $P(2, 1, 0)$ to the point $Q(4, 6, 2)$. Find the work done.

SOLUTION The displacement vector is $\mathbf{D} = \overrightarrow{PQ} = \langle 2, 5, 2 \rangle$, so by Equation 12, the work done is

$$W = \mathbf{F} \cdot \mathbf{D} = \langle 3, 4, 5 \rangle \cdot \langle 2, 5, 2 \rangle$$

$$= 6 + 20 + 10 = 36$$

If the unit of length is meters and the magnitude of the force is measured in newtons, then the work done is 36 J. ■

12.3 Exercises

1. Which of the following expressions are meaningful? Which are meaningless? Explain.

(a) $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ (b) $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

(c) $|\mathbf{a}|(\mathbf{b} \cdot \mathbf{c})$ (d) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$

(e) $\mathbf{a} \cdot \mathbf{b} + \mathbf{c}$ (f) $|\mathbf{a}| \cdot (\mathbf{b} + \mathbf{c})$

2–10 Find $\mathbf{a} \cdot \mathbf{b}$.

2. $\mathbf{a} = \langle 5, -2 \rangle$, $\mathbf{b} = \langle 3, 4 \rangle$

3. $\mathbf{a} = \langle 1.5, 0.4 \rangle$, $\mathbf{b} = \langle -4, 6 \rangle$

4. $\mathbf{a} = \langle 6, -2, 3 \rangle$, $\mathbf{b} = \langle 2, 5, -1 \rangle$

5. $\mathbf{a} = \langle 4, 1, \frac{1}{4} \rangle$, $\mathbf{b} = \langle 6, -3, -8 \rangle$

6. $\mathbf{a} = \langle p, -p, 2p \rangle$, $\mathbf{b} = \langle 2q, q, -q \rangle$

7. $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

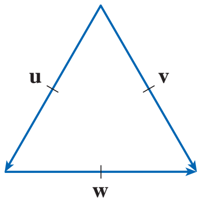
8. $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \mathbf{b} = 4\mathbf{i} + 5\mathbf{k}$

9. $|\mathbf{a}| = 7, \quad |\mathbf{b}| = 4,$ the angle between $\mathbf{a}$ and $\mathbf{b}$ is 30°

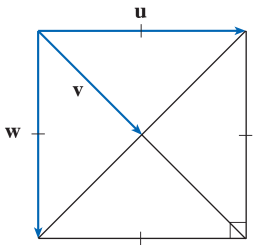
10. $|\mathbf{a}| = 80, \quad |\mathbf{b}| = 50,$ the angle between $\mathbf{a}$ and $\mathbf{b}$ is $3\pi/4$

11–12 If $\mathbf{u}$ is a unit vector, find $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \cdot \mathbf{w}$.

11.



12.



13. (a) Show that $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$.
(b) Show that $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$.

14. A street vendor sells a hamburgers, b hot dogs, and c bottles of water on a given day. He charges \$4 for a hamburger, \$2.50 for a hot dog, and \$1 for a bottle of water. If $\mathbf{A} = \langle a, b, c \rangle$ and $\mathbf{P} = \langle 4, 2.5, 1 \rangle$, what is the meaning of the dot product $\mathbf{A} \cdot \mathbf{P}$?

15–20 Find the angle between the vectors. (First find an exact expression and then approximate to the nearest degree.)

15. $\mathbf{u} = \langle 5, 1 \rangle, \quad \mathbf{v} = \langle 3, 2 \rangle$

16. $\mathbf{a} = \mathbf{i} - 3\mathbf{j}, \quad \mathbf{b} = -3\mathbf{i} + 4\mathbf{j}$

17. $\mathbf{a} = \langle 1, -4, 1 \rangle, \quad \mathbf{b} = \langle 0, 2, -2 \rangle$

18. $\mathbf{a} = \langle -1, 3, 4 \rangle, \quad \mathbf{b} = \langle 5, 2, 1 \rangle$

19. $\mathbf{u} = \mathbf{i} - 4\mathbf{j} + \mathbf{k}, \quad \mathbf{v} = -3\mathbf{i} + \mathbf{j} + 5\mathbf{k}$

20. $\mathbf{a} = 8\mathbf{i} - \mathbf{j} + 4\mathbf{k}, \quad \mathbf{b} = 4\mathbf{j} + 2\mathbf{k}$

21–22 Find, correct to the nearest degree, the three angles of the triangle with the given vertices.

21. $P(2, 0), \quad Q(0, 3), \quad R(3, 4)$

22. $A(1, 0, -1), \quad B(3, -2, 0), \quad C(1, 3, 3)$

23–24 Determine whether the given vectors are orthogonal, parallel, or neither.

23. (a) $\mathbf{a} = \langle 9, 3 \rangle, \quad \mathbf{b} = \langle -2, 6 \rangle$
(b) $\mathbf{a} = \langle 4, 5, -2 \rangle, \quad \mathbf{b} = \langle 3, -1, 5 \rangle$
(c) $\mathbf{a} = -8\mathbf{i} + 12\mathbf{j} + 4\mathbf{k}, \quad \mathbf{b} = 6\mathbf{i} - 9\mathbf{j} - 3\mathbf{k}$
(d) $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 3\mathbf{k}, \quad \mathbf{b} = 5\mathbf{i} + 9\mathbf{j} - 2\mathbf{k}$

24. (a) $\mathbf{u} = \langle -5, 4, -2 \rangle, \quad \mathbf{v} = \langle 3, 4, -1 \rangle$
(b) $\mathbf{u} = 9\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}, \quad \mathbf{v} = -6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$
(c) $\mathbf{u} = \langle c, c, c \rangle, \quad \mathbf{v} = \langle c, 0, -c \rangle$

25. Use vectors to determine whether the triangle with vertices $P(1, -3, -2)$, $Q(2, 0, -4)$, and $R(6, -2, -5)$ is right-angled.

26. Find the values of x such that the angle between the vectors $\langle 2, 1, -1 \rangle$, and $\langle 1, x, 0 \rangle$ is 45°.

27. Find a unit vector that is orthogonal to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{k}$.

28. Find two unit vectors that make an angle of 60° with $\mathbf{v} = \langle 3, 4 \rangle$.

29–30 Find the acute angle between the lines. Use degrees rounded to one decimal place.

29. $y = 4 - 3x, \quad y = 3x + 2$

30. $5x - y = 8, \quad x + 3y = 15$

31–32 Find the acute angles between the curves at their points of intersection. Use degrees rounded to one decimal place. (The angle between two curves is the angle between their tangent lines at the point of intersection.)

31. $y = x^2, \quad y = x^3$

32. $y = \sin x, \quad y = \cos x, \quad 0 \leqslant x \leqslant \pi/2$

33–37 Find the direction cosines and direction angles of the vector. (Give the direction angles correct to the nearest tenth of a degree.)

33. $\langle 4, 1, 8 \rangle$

34. $\langle -6, 2, 9 \rangle$

35. $3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

36. $-0.7\mathbf{i} + 1.2\mathbf{j} - 0.8\mathbf{k}$

37. $\langle c, c, c \rangle, \quad$ where $c > 0$

38. If a vector has direction angles $\alpha = \pi/4$ and $\beta = \pi/3$, find the third direction angle γ.

39–44 Find the scalar and vector projections of $\mathbf{b}$ onto $\mathbf{a}$.

39. $\mathbf{a} = \langle -5, 12 \rangle, \quad \mathbf{b} = \langle 4, 6 \rangle$

40. $\mathbf{a} = \langle 1, 4 \rangle, \quad \mathbf{b} = \langle 2, 3 \rangle$

41. $\mathbf{a} = \langle 4, 7, -4 \rangle, \quad \mathbf{b} = \langle 3, -1, 1 \rangle$

42. $\mathbf{a} = \langle -1, 4, 8 \rangle, \quad \mathbf{b} = \langle 12, 1, 2 \rangle$

43. $\mathbf{a} = 3\mathbf{i} - 3\mathbf{j} + \mathbf{k}, \quad \mathbf{b} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

44. $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \quad \mathbf{b} = 5\mathbf{i} - \mathbf{k}$

45. Show that the vector $\operatorname{orth}_{\mathbf{a}} \mathbf{b} = \mathbf{b} - \operatorname{proj}_{\mathbf{a}} \mathbf{b}$ is orthogonal to $\mathbf{a}$. (It is called an **orthogonal projection** of $\mathbf{b}$.)

46. For the vectors in Exercise 40, find $\operatorname{orth}_{\mathbf{a}} \mathbf{b}$ and illustrate by drawing the vectors $\mathbf{a}$, $\mathbf{b}$, $\operatorname{proj}_{\mathbf{a}} \mathbf{b}$, and $\operatorname{orth}_{\mathbf{a}} \mathbf{b}$.

47. If $\mathbf{a} = \langle 3, 0, -1 \rangle$, find a vector $\mathbf{b}$ such that $\operatorname{comp}_{\mathbf{a}} \mathbf{b} = 2$.

48. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors.

(a) Under what circumstances is $\operatorname{comp}_{\mathbf{a}} \mathbf{b} = \operatorname{comp}_{\mathbf{b}} \mathbf{a}$?

(b) Under what circumstances is $\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \operatorname{proj}_{\mathbf{b}} \mathbf{a}$?

49. Find the work done by a force $\mathbf{F} = 8\,\mathbf{i} - 6\,\mathbf{j} + 9\,\mathbf{k}$ that moves an object from the point $(0, 10, 8)$ along a straight line to the point $(6, 12, 20)$. The distance is measured in meters and the force in newtons.

50. A tow truck drags a stalled car along a road. The chain makes an angle of 30° with the road and the tension in the chain is 1500 N. How much work is done by the truck in pulling the car 1 km?

51. A sled is pulled along a level path through snow by a rope. A 30 N force acting at an angle of 40° above the horizontal moves the sled 80 m. Find the work done by the force.

52. A boat sails south with the help of a wind blowing in the direction S 36° E with magnitude 2000 N. Find the work done by the wind as the boat moves 40 m.

53. Distance from a Point to a Line Use a scalar projection to show that the distance from a point $P_1(x_1, y_1)$ to the line $ax + by + c = 0$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

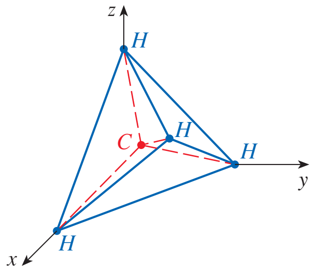
Use this formula to find the distance from the point $(-2, 3)$ to the line $3x - 4y + 5 = 0$.

54. If $\mathbf{r} = \langle x, y, z \rangle$, $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, show that the vector equation $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$ represents a sphere, and find its center and radius.

55. Find the angle, in degrees rounded to one decimal place, between a diagonal of a cube and one of its edges.

56. Find the angle, in degrees rounded to one decimal place, between a diagonal of a cube and a diagonal of one of its faces.

57. A molecule of methane, CH_4, is structured with the four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid. The *bond angle* is the angle formed by the H—C—H combination; it is the angle between the lines that join the carbon atom to two of the hydrogen atoms. Show that the bond angle is about 109.5°.
[*Hint:* Take the vertices of the tetrahedron to be the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 1, 1)$, as shown in the figure. Then the centroid is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.]



58. If $\mathbf{c} = |\mathbf{a}|\mathbf{b} + |\mathbf{b}|\mathbf{a}$, where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are all nonzero vectors, show that $\mathbf{c}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

59. Prove Properties 2, 4, and 5 of the dot product (Theorem 2).

60. Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonals are perpendicular.

61. Cauchy-Schwartz Inequality Use Theorem 3 to prove the Cauchy-Schwarz Inequality:

$$|\mathbf{a} \cdot \mathbf{b}| \leqslant |\mathbf{a}|\,|\mathbf{b}|$$

62. Triangle Inequality The Triangle Inequality for vectors is

$$|\mathbf{a} + \mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|$$

(a) Give a geometric interpretation of the Triangle Inequality.

(b) Use the Cauchy-Schwarz Inequality from Exercise 61 to prove the Triangle Inequality. [*Hint:* Use the fact that $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$ and use Property 3 of the dot product.]

63. Parallelogram Identity The Parallelogram Identity states that

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$$

(a) Give a geometric interpretation of the Parallelogram Identity.

(b) Prove the Parallelogram Identity. (See the hint in Exercise 62.)

64. Show that if $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal, then the vectors $\mathbf{u}$ and $\mathbf{v}$ must have the same length.

65. If θ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$, show that

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b} \cdot \operatorname{proj}_{\mathbf{b}} \mathbf{a} = (\mathbf{a} \cdot \mathbf{b}) \cos^2\theta$$

66. (a) Show that if $\mathbf{u}$ and $\mathbf{v}$ are nonzero orthogonal vectors, then $|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$.

(b) Show that the converse of part (a) is also true: if $|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

12.4 The Cross Product

Given two nonzero vectors, it is very useful to be able to find a nonzero vector that is perpendicular to both of them, as we will see in the next section and in Chapters 13 and 14. We now define an operation, called the cross product, that produces such a vector.

The Cross Product of Two Vectors

Given two nonzero vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, suppose that a nonzero vector $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Then $\mathbf{a} \cdot \mathbf{c} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 0$ and so

$$\boxed{1} \qquad a_1c_1 + a_2c_2 + a_3c_3 = 0$$

$$\boxed{2} \qquad b_1c_1 + b_2c_2 + b_3c_3 = 0$$

To eliminate c_3 we multiply (1) by b_3 and (2) by a_3 and subtract:

$$\boxed{3} \qquad (a_1b_3 - a_3b_1)c_1 + (a_2b_3 - a_3b_2)c_2 = 0$$

Equation 3 has the form $pc_1 + qc_2 = 0$, for which an obvious solution is $c_1 = q$ and $c_2 = -p$. So a solution of (3) is

$$c_1 = a_2b_3 - a_3b_2 \qquad c_2 = a_3b_1 - a_1b_3$$

Substituting these values into (1) and (2), we then get

$$c_3 = a_1b_2 - a_2b_1$$

This means that a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ is

$$\langle c_1, c_2, c_3 \rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

The resulting vector is called the *cross product* of $\mathbf{a}$ and $\mathbf{b}$ and is denoted by $\mathbf{a} \times \mathbf{b}$.

Hamilton

The cross product was invented by the Irish mathematician Sir William Rowan Hamilton (1805–1865), who had created a precursor of vectors, called quaternions. When he was five years old Hamilton could read Latin, Greek, and Hebrew. At age eight he added French and Italian and at ten he could read Arabic and Sanskrit. At the age of 21, while still an undergraduate at Trinity College in Dublin, Hamilton was appointed Professor of Astronomy at the university and Royal Astronomer of Ireland!

$\boxed{4}$ **Definition of the Cross Product** If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$,then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

Notice that the cross product $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ is a vector (whereas the dot product is a scalar). For this reason it is also called the **vector product**. Note that $\mathbf{a} \times \mathbf{b}$ is defined only when $\mathbf{a}$ and $\mathbf{b}$ are *three-dimensional* vectors.

In order to make Definition 4 easier to remember, we use the notation of determinants. A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

(Multiply across the diagonals and subtract.) For example,

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 14$$

A **determinant of order 3** can be defined in terms of second-order determinants:

$$\boxed{5} \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Observe that each term on the right side of Equation 5 involves a number a_i in the first row of the determinant, and a_i is multiplied by the second-order determinant obtained from the left side by deleting the row and column in which a_i appears. Notice also the minus sign in the second term. For example,

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} = 1 \begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix}$$

$$= 1(0 - 4) - 2(6 + 5) + (-1)(12 - 0) = -38$$

If we now rewrite Definition 4 using second-order determinants and the standard basis vectors **i**, **j**, and **k**, we see that the cross product of the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is

$$\boxed{6} \qquad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

In view of the similarity between Equations 5 and 6, we often write

$$\boxed{7} \qquad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Although the first row of the symbolic determinant in Equation 7 consists of vectors, if we expand it as if it were an ordinary determinant using the rule in Equation 5, we obtain Equation 6. The symbolic formula in Equation 7 is probably the easiest way of remembering and computing cross products.

EXAMPLE 1 If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k}$$

$$= (-15 - 28)\mathbf{i} - (-5 - 8)\mathbf{j} + (7 - 6)\mathbf{k} = -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}$$ ■

EXAMPLE 2 Show that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector $\mathbf{a}$ in V_3.

SOLUTION If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then

$$\mathbf{a} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= (a_2a_3 - a_3a_2)\,\mathbf{i} - (a_1a_3 - a_3a_1)\,\mathbf{j} + (a_1a_2 - a_2a_1)\,\mathbf{k}$$

$$= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$ ■

■ Properties of the Cross Product

We constructed the cross product $\mathbf{a} \times \mathbf{b}$ so that it would be perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. This is one of the most important properties of a cross product, so let's emphasize and verify it in the following theorem and give a formal proof.

8 Theorem The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

PROOF In order to show that $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$, we compute their dot product as follows:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3$$

$$= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1)$$

$$= a_1a_2b_3 - a_1b_2a_3 - a_1a_2b_3 + b_1a_2a_3 + a_1b_2a_3 - b_1a_2a_3$$

$$= 0$$

A similar computation shows that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$. Therefore $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. ■

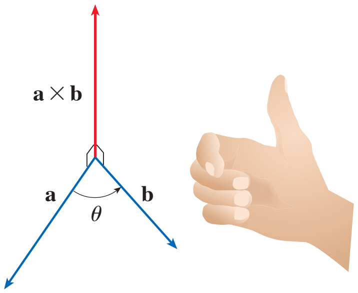


FIGURE 1
The right-hand rule gives the direction of $\mathbf{a} \times \mathbf{b}$.

If $\mathbf{a}$ and $\mathbf{b}$ are represented by directed line segments with the same initial point (as in Figure 1), then Theorem 8 says that the cross product $\mathbf{a} \times \mathbf{b}$ points in a direction perpendicular to the plane through $\mathbf{a}$ and $\mathbf{b}$. It turns out that the direction of $\mathbf{a} \times \mathbf{b}$ is given by the *right-hand rule:* if the fingers of your right hand curl in the direction of a rotation (through an angle less than 180°) from $\mathbf{a}$ to $\mathbf{b}$, then your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

Now that we know the direction of the vector $\mathbf{a} \times \mathbf{b}$, the remaining thing we need to complete its geometric description is its length $|\mathbf{a} \times \mathbf{b}|$. This is given by the following theorem.

9 Theorem If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leqslant \theta \leqslant \pi$), then the length of the cross product $\mathbf{a} \times \mathbf{b}$ is given by

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$$

PROOF From the definitions of the cross product and length of a vector, we have

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\ &= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2 \\ &\qquad + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 \\ &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta \qquad \text{(by Theorem 12.3.3)} \\ &= |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2\theta) \\ &= |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta \end{aligned}$$

Taking square roots and observing that $\sqrt{\sin^2\theta} = \sin\theta$ because $\sin\theta \geqslant 0$ when $0 \leqslant \theta \leqslant \pi$, we have

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$ ■

10 Corollary Two nonzero vectors **a** and **b** are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

PROOF Two nonzero vectors **a** and **b** are parallel if and only if $\theta = 0$ or π. In either case $\sin\theta = 0$, so $|\mathbf{a} \times \mathbf{b}| = 0$ and therefore $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. ■

Geometric characterization of $\mathbf{a} \times \mathbf{b}$

Since a vector is completely determined by its magnitude and direction, we can now say that for nonparallel vectors **a** and **b**, $\mathbf{a} \times \mathbf{b}$ is the vector that is perpendicular to both **a** and **b**, whose orientation is determined by the right-hand rule, and whose length is $|\mathbf{a}||\mathbf{b}|\sin\theta$. In fact, that is exactly how physicists *define* $\mathbf{a} \times \mathbf{b}$.

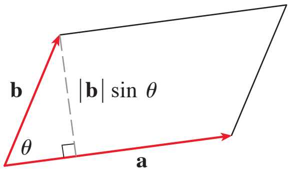


FIGURE 2

The geometric interpretation of Theorem 9 can be seen by looking at Figure 2. If **a** and **b** are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\mathbf{a}|$, altitude $|\mathbf{b}|\sin\theta$, and area

$$A = |\mathbf{a}|\big(|\mathbf{b}|\sin\theta\big) = |\mathbf{a} \times \mathbf{b}|$$

Thus we have the following way of interpreting the magnitude of a cross product.

The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by **a** and **b**.

EXAMPLE 3 Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

SOLUTION The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and is therefore perpendicular to the plane through P, Q, and R. We know from (12.2.1) that

$$\overrightarrow{PQ} = (-2 - 1)\,\mathbf{i} + (5 - 4)\,\mathbf{j} + (-1 - 6)\,\mathbf{k} = -3\mathbf{i} + \mathbf{j} - 7\mathbf{k}$$

$$\overrightarrow{PR} = (1 - 1)\,\mathbf{i} + (-1 - 4)\,\mathbf{j} + (1 - 6)\,\mathbf{k} = -5\mathbf{j} - 5\mathbf{k}$$

We compute the cross product of these vectors:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= (-5 - 35)\,\mathbf{i} - (15 - 0)\,\mathbf{j} + (15 - 0)\,\mathbf{k} = -40\mathbf{i} - 15\mathbf{j} + 15\mathbf{k}$$

So the vector $\langle -40, -15, 15 \rangle$ is perpendicular to the given plane. Any nonzero scalar multiple of this vector, such as $\langle -8, -3, 3 \rangle$, is also perpendicular to the plane. ■

EXAMPLE 4 Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

SOLUTION In Example 3 we computed that $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -40, -15, 15 \rangle$. The area of the parallelogram with adjacent sides PQ and PR is the length of this cross product:

$$\left| \overrightarrow{PQ} \times \overrightarrow{PR} \right| = \sqrt{(-40)^2 + (-15)^2 + 15^2} = 5\sqrt{82}$$

The area A of the triangle PQR is half the area of this parallelogram, that is, $\frac{5}{2}\sqrt{82}$. ■

If we apply Theorems 8 and 9 to the standard basis vectors **i**, **j**, and **k** using $\theta = \pi/2$, we obtain

$$\boxed{\begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}}$$

Observe that $\mathbf{i} \times \mathbf{j} \neq \mathbf{j} \times \mathbf{i}$

⊘ Thus the cross product is not commutative. Also

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

whereas $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$

⊘ So the associative law for multiplication does not usually hold; that is, in general,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

However, some of the usual laws of algebra *do* hold for cross products. The following theorem summarizes the properties of vector products.

11 Properties of the Cross Product If **a**, **b**, and **c** are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

These properties can be proved by writing the vectors in terms of their components and using the definition of a cross product. We give the proof of Property 5 and leave the remaining proofs as exercises.

PROOF OF PROPERTY 5 If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, then

$$\boxed{12} \quad \begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \\ &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \\ &= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3 \\ &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \end{aligned}$$

■

Triple Products

The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ that occurs in Property 5 is called the **scalar triple product** of the vectors **a**, **b**, and **c**. Notice from Equation 12 that we can write the scalar triple product as a determinant:

$$\boxed{13} \qquad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

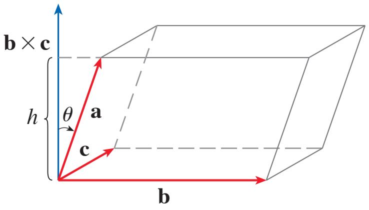


FIGURE 3

The geometric significance of the scalar triple product can be seen by considering the parallelepiped determined by the vectors **a**, **b**, and **c**. (See Figure 3.) The area of the base parallelogram is $A = |\mathbf{b} \times \mathbf{c}|$. If θ is the angle between **a** and $\mathbf{b} \times \mathbf{c}$, then the height h of the parallelepiped is $h = |\mathbf{a}||\cos\theta|$. (We must use $|\cos\theta|$ instead of $\cos\theta$ in case $\theta > \pi/2$.) Therefore the volume of the parallelepiped is

$$V = Ah = |\mathbf{b} \times \mathbf{c}||\mathbf{a}||\cos\theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| \qquad \text{(by Theorem 12.3.3)}$$

Thus we have proved the following formula.

$\boxed{14}$ The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

If we use the formula in (14) and discover that the volume of the parallelepiped determined by **a**, **b**, and **c** is 0, then the vectors must lie in the same plane; that is, they are **coplanar**.

EXAMPLE 5 Use the scalar triple product to show that the vectors $\mathbf{a} = \langle 1, 4, -7 \rangle$, $\mathbf{b} = \langle 2, -1, 4 \rangle$, and $\mathbf{c} = \langle 0, -9, 18 \rangle$ are coplanar.

SOLUTION We use Equation 13 to compute their scalar triple product:

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} \\ &= 1 \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7 \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix} \\ &= 1(18) - 4(36) - 7(-18) = 0 \end{aligned}$$

Therefore, by (14), the volume of the parallelepiped determined by **a**, **b**, and **c** is 0. This means that **a**, **b**, and **c** are coplanar. ■

The product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ that occurs in Property 6 is called the **vector triple product** of **a**, **b**, and **c**. Property 6 will be used to derive Kepler's First Law of planetary motion in Chapter 13. Its proof is left as Exercise 50.

Application: Torque

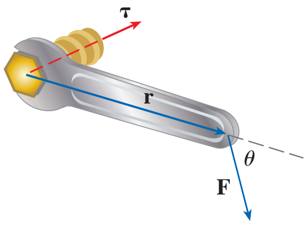


FIGURE 4

The idea of a cross product occurs often in physics. In particular, we consider a force **F** acting on a rigid body at a point given by a position vector **r**. (For instance, if we tighten a bolt by applying a force to a wrench as in Figure 4, we produce a turning effect.) The **torque** $\boldsymbol{\tau}$ (relative to the origin) is defined to be the cross product of the position and force vectors

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

and measures the tendency of the body to rotate about the origin. The direction of the torque vector indicates the axis of rotation. According to Theorem 9, the magnitude of the torque vector is

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin\theta$$

where θ is the angle between the position and force vectors. Observe that the only component of **F** that can cause a rotation is the one perpendicular to **r**, that is, $|\mathbf{F}| \sin\theta$. The magnitude of the torque is equal to the area of the parallelogram determined by **r** and **F**.

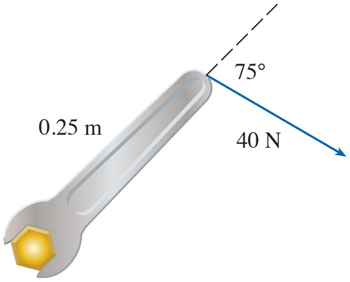


FIGURE 5

EXAMPLE 6 A bolt is tightened by applying a 40-N force to a 0.25-m wrench, as shown in Figure 5. Find the magnitude of the torque about the center of the bolt.

SOLUTION The magnitude of the torque vector is

$$\begin{aligned} |\boldsymbol{\tau}| &= |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin 75° = (0.25)(40) \sin 75° \\ &= 10 \sin 75° \approx 9.66 \text{ N}\cdot\text{m} \end{aligned}$$

If the bolt is right-threaded, then the torque vector itself is

$$\boldsymbol{\tau} = |\boldsymbol{\tau}|\,\mathbf{n} \approx 9.66\,\mathbf{n}$$

where **n** is a unit vector directed down into the page (by the right-hand rule). ■

12.4 Exercises

1–7 Find the cross product $\mathbf{a} \times \mathbf{b}$ and verify that it is orthogonal to both **a** and **b**.

1. $\mathbf{a} = \langle 2, 3, 0 \rangle, \quad \mathbf{b} = \langle 1, 0, 5 \rangle$

2. $\mathbf{a} = \langle 4, 3, -2 \rangle, \quad \mathbf{b} = \langle 2, -1, 1 \rangle$

3. $\mathbf{a} = 2\mathbf{j} - 4\mathbf{k}, \quad \mathbf{b} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

4. $\mathbf{a} = 3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}, \quad \mathbf{b} = 3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$

5. $\mathbf{a} = \frac{1}{2}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{1}{4}\mathbf{k}, \quad \mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

6. $\mathbf{a} = t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin t\,\mathbf{k}, \quad \mathbf{b} = \mathbf{i} - \sin t\,\mathbf{j} + \cos t\,\mathbf{k}$

7. $\mathbf{a} = \langle t^3, t^2, t \rangle, \quad \mathbf{b} = \langle t, 2t, 3t \rangle$

---

8. If $\mathbf{a} = \mathbf{i} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{j} + \mathbf{k}$, find $\mathbf{a} \times \mathbf{b}$. Sketch **a**, **b**, and $\mathbf{a} \times \mathbf{b}$ as vectors starting at the origin.

9–12 Find the vector, not with determinants, but by using properties of cross products.

9. $(\mathbf{i} \times \mathbf{j}) \times \mathbf{k}$

10. $\mathbf{k} \times (\mathbf{i} - 2\mathbf{j})$

11. $(\mathbf{j} - \mathbf{k}) \times (\mathbf{k} - \mathbf{i})$

12. $(\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - \mathbf{j})$

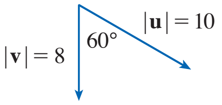
---

13. State whether each expression is meaningful. If not, explain why. If so, state whether it is a vector or a scalar.

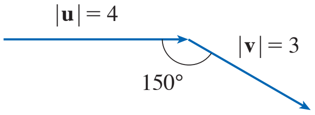
(a) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
(b) $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$
(c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$
(d) $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$
(e) $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$
(f) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$

14–15 Find $|\mathbf{u} \times \mathbf{v}|$ and determine whether $\mathbf{u} \times \mathbf{v}$ is directed into the page or out of the page.

14.



15.



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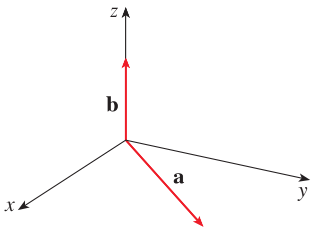
16. The figure shows a vector $\mathbf{a}$ in the xy-plane and a vector $\mathbf{b}$ in the direction of $\mathbf{k}$. Their lengths are $|\mathbf{a}| = 3$ and $|\mathbf{b}| = 2$.

(a) Find $|\mathbf{a} \times \mathbf{b}|$.

(b) Use the right-hand rule to decide whether the components of $\mathbf{a} \times \mathbf{b}$ are positive, negative, or 0.



17. If $\mathbf{a} = \langle 2, -1, 3 \rangle$ and $\mathbf{b} = \langle 4, 2, 1 \rangle$, find $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$.

18. If $\mathbf{a} = \langle 1, 0, 1 \rangle$, $\mathbf{b} = \langle 2, 1, -1 \rangle$, and $\mathbf{c} = \langle 0, 1, 3 \rangle$, show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

19. Find two unit vectors orthogonal to both $\langle 3, 2, 1 \rangle$ and $\langle -1, 1, 0 \rangle$.

20. Find two unit vectors orthogonal to both $\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + \mathbf{j}$.

21. Show that $\mathbf{0} \times \mathbf{a} = \mathbf{0} = \mathbf{a} \times \mathbf{0}$ for any vector $\mathbf{a}$ in V_3.

22. Show that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$ for all vectors $\mathbf{a}$ and $\mathbf{b}$ in V_3.

23–26 Prove the specified property of cross products (Theorem 11).

23. Property 1: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

24. Property 2: $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$

25. Property 3: $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

26. Property 4: $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

---

27. Find the area of the parallelogram with vertices $A(-3, 0)$, $B(-1, 3)$, $C(5, 2)$, and $D(3, -1)$.

28. Find the area of the parallelogram with vertices $P(1, 0, 2)$, $Q(3, 3, 3)$, $R(7, 5, 8)$, and $S(5, 2, 7)$.

29–32 (a) Find a nonzero vector orthogonal to the plane through the points P, Q, and R, and (b) find the area of triangle PQR.

29. $P(3, 1, 1)$, $Q(5, 2, 4)$, $R(8, 5, 3)$

30. $P(-2, 0, 4)$, $Q(1, 3, -2)$, $R(0, 3, 5)$

31. $P(7, -2, 0)$, $Q(3, 1, 3)$, $R(4, -4, 2)$

32. $P(2, -3, 4)$, $Q(-1, -2, 2)$, $R(3, 1, -3)$

33–34 Find the volume of the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

33. $\mathbf{a} = \langle 1, 2, 3 \rangle$, $\mathbf{b} = \langle -1, 1, 2 \rangle$, $\mathbf{c} = \langle 2, 1, 4 \rangle$

34. $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{j} + \mathbf{k}$, $\mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

35–36 Find the volume of the parallelepiped with adjacent edges PQ, PR, and PS.

35. $P(-2, 1, 0)$, $Q(2, 3, 2)$, $R(1, 4, -1)$, $S(3, 6, 1)$

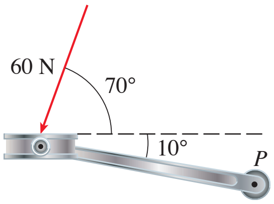
36. $P(3, 0, 1)$, $Q(-1, 2, 5)$, $R(5, 1, -1)$, $S(0, 4, 2)$

---

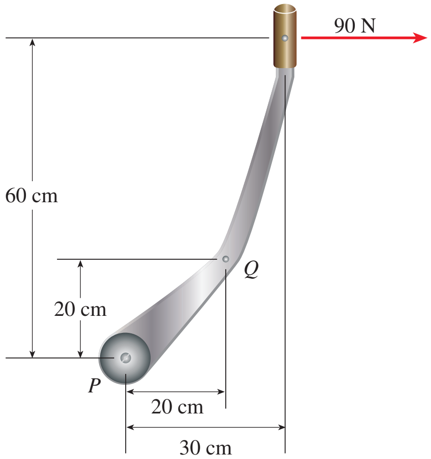
37. Use the scalar triple product to verify that the vectors $\mathbf{u} = \mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$, and $\mathbf{w} = 5\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}$ are coplanar.

38. Use the scalar triple product to determine whether the points $A(1, 3, 2)$, $B(3, -1, 6)$, $C(5, 2, 0)$, and $D(3, 6, -4)$ lie in the same plane.

39. A bicycle pedal is pushed by a foot with a 60-N force as shown in the figure. The shaft of the pedal is 18 cm long. Find the magnitude of the torque about P.



40. (a) A horizontal force of 90 N is applied to the handle of a gearshift lever as shown in the figure. Find the magnitude of the torque about the pivot point P.
(b) Find the magnitude of the torque about P if the same force is applied at the elbow Q of the lever.



41. A wrench 30 cm long lies along the positive y-axis and grips a bolt at the origin. A force is applied in the direction $\langle 0, 3, -4 \rangle$ at the end of the wrench. Find the magnitude of the force needed to supply 100 N·m of torque to the bolt.

42. Let $\mathbf{v} = 5\mathbf{j}$ and let $\mathbf{u}$ be a vector with length 3 that starts at the origin and rotates in the xy-plane. Find the maximum and minimum values of the length of the vector $\mathbf{u} \times \mathbf{v}$. In what direction does $\mathbf{u} \times \mathbf{v}$ point?

43. If $\mathbf{a} \cdot \mathbf{b} = \sqrt{3}$ and $\mathbf{a} \times \mathbf{b} = \langle 1, 2, 2 \rangle$, find the angle between $\mathbf{a}$ and $\mathbf{b}$.

44. (a) Find all vectors $\mathbf{v}$ such that

$$\langle 1, 2, 1 \rangle \times \mathbf{v} = \langle 3, 1, -5 \rangle$$

(b) Explain why there is no vector $\mathbf{v}$ such that

$$\langle 1, 2, 1 \rangle \times \mathbf{v} = \langle 3, 1, 5 \rangle$$

45. **Distance from a Point to a Line** Let P be a point not on the line L that passes through the points Q and R.
(a) Show that the distance d from the point P to the line L is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}$$

where $\mathbf{a} = \overrightarrow{QR}$ and $\mathbf{b} = \overrightarrow{QP}$.
(b) Use the formula in part (a) to find the distance from the point $P(1, 1, 1)$ to the line through $Q(0, 6, 8)$ and $R(-1, 4, 7)$.

46. **Distance from a Point to a Plane** Let P be a point not on the plane that passes through the points Q, R, and S.
(a) Show that the distance d from P to the plane is

$$d = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|}$$

where $\mathbf{a} = \overrightarrow{QR}$, $\mathbf{b} = \overrightarrow{QS}$, and $\mathbf{c} = \overrightarrow{QP}$.
(b) Use the formula in part (a) to find the distance from the point $P(2, 1, 4)$ to the plane through the points $Q(1, 0, 0)$, $R(0, 2, 0)$, and $S(0, 0, 3)$.

47. Show that $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$.

48. If $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, show that

$$\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$$

49. Prove that $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$.

50. Prove Property 6 of cross products, that is,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

51. Use Exercise 50 to prove that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$$

52. Prove that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$$

53. Suppose that $\mathbf{a} \neq \mathbf{0}$.
(a) If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?
(b) If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?
(c) If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

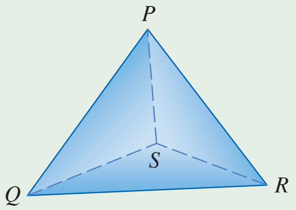
54. If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are noncoplanar vectors, let

$$\mathbf{k}_1 = \frac{\mathbf{v}_2 \times \mathbf{v}_3}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} \qquad \mathbf{k}_2 = \frac{\mathbf{v}_3 \times \mathbf{v}_1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

$$\mathbf{k}_3 = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

(These vectors occur in the study of crystallography. Vectors of the form $n_1\mathbf{v}_1 + n_2\mathbf{v}_2 + n_3\mathbf{v}_3$, where each n_i is an integer, form a *lattice* for a crystal. Vectors written similarly in terms of $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ form the *reciprocal lattice*.)
(a) Show that $\mathbf{k}_i$ is perpendicular to $\mathbf{v}_j$ if $i \neq j$.
(b) Show that $\mathbf{k}_i \cdot \mathbf{v}_i = 1$ for $i = 1, 2, 3$.
(c) Show that $\mathbf{k}_1 \cdot (\mathbf{k}_2 \times \mathbf{k}_3) = \dfrac{1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$.

DISCOVERY PROJECT | THE GEOMETRY OF A TETRAHEDRON



A tetrahedron is a solid with four vertices, P, Q, R, and S, and four triangular faces, as shown in the figure.

1. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ be vectors with lengths equal to the areas of the faces opposite the vertices P, Q, R, and S, respectively, and directions perpendicular to the respective faces and pointing outward. Show that

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$$

2. The volume V of a tetrahedron is one-third the distance from a vertex to the opposite face, times the area of that face.

(a) Find a formula for the volume of a tetrahedron in terms of the coordinates of its vertices P, Q, R, and S.

(b) Find the volume of the tetrahedron whose vertices are $P(1, 1, 1)$, $Q(1, 2, 3)$, $R(1, 1, 2)$, and $S(3, -1, 2)$.

3. Suppose the tetrahedron in the figure has a trirectangular vertex S. (This means that the three angles at S are all right angles.) Let A, B, and C be the areas of the three faces that meet at S, and let D be the area of the opposite face PQR. Using the result of Problem 1, or otherwise, show that

$$D^2 = A^2 + B^2 + C^2$$

(This is a three-dimensional version of the Pythagorean Theorem.)

12.5 Equations of Lines and Planes

Lines

A line in the xy-plane is determined when a point on the line and the direction of the line (its slope or angle of inclination) are given. The equation of the line can then be written using the point-slope form.

Likewise, a line L in three-dimensional space is determined when we know a point $P_0(x_0, y_0, z_0)$ on L and a direction for L, which is conveniently described by a vector $\mathbf{v}$ parallel to the line. Let $P(x, y, z)$ be an arbitrary point on L and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P (that is, they have representations $\overrightarrow{OP_0}$ and $\overrightarrow{OP}$). If $\mathbf{a}$ is the vector with representation $\overrightarrow{P_0P}$, as in Figure 1, then the Triangle Law for vector addition gives $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$.

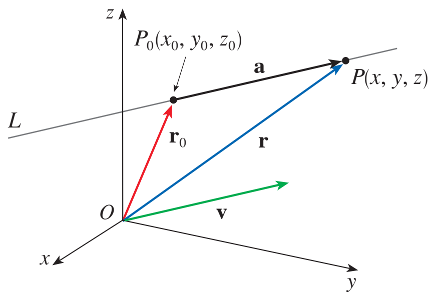


FIGURE 1

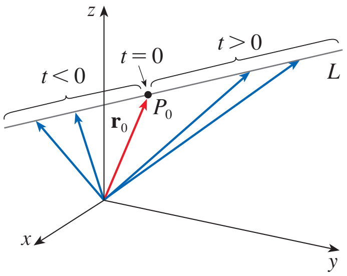


FIGURE 2

Since **a** and **v** are parallel vectors, there is a scalar t such that $\mathbf{a} = t\mathbf{v}$. Thus

1 $$\boxed{\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}}$$

which is a **vector equation** of L. Each value of the **parameter** t gives the position vector **r** of a point on L. In other words, as t varies, the line is traced out by the tip of the vector **r**. As Figure 2 indicates, positive values of t correspond to points on L that lie on one side of P_0, whereas negative values of t correspond to points that lie on the other side of P_0.

If the vector **v** that gives the direction of the line L is written in component form as $\mathbf{v} = \langle a, b, c \rangle$, then we have $t\mathbf{v} = \langle ta, tb, tc \rangle$. We can also write $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, so the vector equation (1) becomes

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Two vectors are equal if and only if corresponding components are equal. Therefore we have the three scalar equations:

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

where $t \in \mathbb{R}$. These equations are called **parametric equations** of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$. Each value of the parameter t gives a point (x, y, z) on L.

2 Parametric equations for a line through the point (x_0, y_0, z_0) and parallel to the direction vector $\langle a, b, c \rangle$ are

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

EXAMPLE 1

(a) Find a vector equation and parametric equations for the line that passes through the point (5, 1, 3) and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.
(b) Find two other points on the line.

Figure 3 shows the line L in Example 1 and its relation to the given point and to the vector that gives its direction.

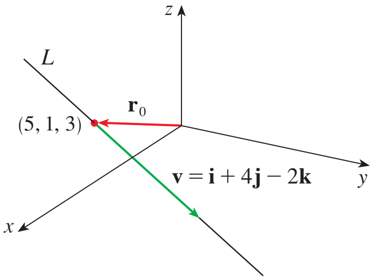


FIGURE 3

SOLUTION

(a) Here $\mathbf{r}_0 = \langle 5, 1, 3 \rangle = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, so the vector equation (1) becomes

$$\mathbf{r} = (5\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

or

$$\mathbf{r} = (5 + t)\,\mathbf{i} + (1 + 4t)\,\mathbf{j} + (3 - 2t)\,\mathbf{k}$$

Parametric equations are

$$x = 5 + t \qquad y = 1 + 4t \qquad z = 3 - 2t$$

(b) Choosing the parameter value $t = 1$ gives $x = 6$, $y = 5$, and $z = 1$, so (6, 5, 1) is a point on the line. Similarly, $t = -1$ gives the point $(4, -3, 5)$. ■

The vector equation and parametric equations of a line are not unique. If we change the point or the parameter or choose a different parallel vector, then the equations change. For instance, if, instead of (5, 1, 3), we choose the point (6, 5, 1) in Example 1, then the parametric equations of the line become

$$x = 6 + t \qquad y = 5 + 4t \qquad z = 1 - 2t$$

Or, if we stay with the point (5, 1, 3) but choose the parallel vector $2\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$, we arrive at the equations

$$x = 5 + 2t \qquad y = 1 + 8t \qquad z = 3 - 4t$$

In general, if a vector $\mathbf{v} = \langle a, b, c \rangle$ is used to describe the direction of a line L, then the numbers a, b, and c are called **direction numbers** of L. Since any vector parallel to $\mathbf{v}$ could also be used, we see that any three numbers proportional to a, b, and c could also be used as a set of direction numbers for L.

Another way of describing a line L is to eliminate the parameter t from Equations 2. If none of a, b, or c is 0, we can solve each of these equations for t:

$$t = \frac{x - x_0}{a} \qquad t = \frac{y - y_0}{b} \qquad t = \frac{z - z_0}{c}$$

Equating the results, we obtain

$$\boxed{3} \qquad \boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

These equations are called **symmetric equations** of L. Notice that the numbers a, b, and c that appear in the denominators of Equations 3 are direction numbers of L, that is, components of a vector parallel to L. If one of a, b, or c is 0, we can still eliminate t. For instance, if $a = 0$, we could write the equations of L as

$$x = x_0 \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

This means that L lies in the vertical plane $x = x_0$.

Figure 4 shows the line L in Example 2 and the point P where it intersects the xy-plane.

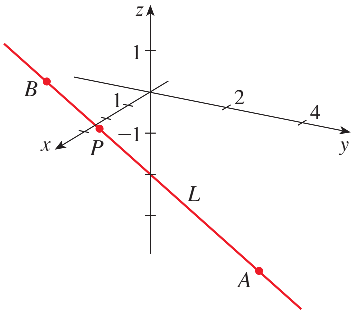


FIGURE 4

EXAMPLE 2

(a) Find parametric equations and symmetric equations of the line that passes through the points $A(2, 4, -3)$ and $B(3, -1, 1)$.
(b) At what point does this line intersect the xy-plane?

SOLUTION

(a) We are not explicitly given a vector parallel to the line, but we observe that the vector $\mathbf{v}$ with representation $\overrightarrow{AB}$ is parallel to the line and

$$\mathbf{v} = \langle 3 - 2, -1 - 4, 1 - (-3) \rangle = \langle 1, -5, 4 \rangle$$

Thus direction numbers are $a = 1$, $b = -5$, and $c = 4$. Taking the point $(2, 4, -3)$ as P_0, we see that parametric equations (2) are

$$x = 2 + t \qquad y = 4 - 5t \qquad z = -3 + 4t$$

and symmetric equations (3) are

$$\frac{x - 2}{1} = \frac{y - 4}{-5} = \frac{z + 3}{4}$$

(b) The line intersects the xy-plane when $z = 0$. From the parametric equations we have $z = -3 + 4t = 0$, which gives $t = \frac{3}{4}$. Using this value of t, we get $x = 2 + \frac{3}{4} = \frac{11}{4}$ and $y = 4 - 5\left(\frac{3}{4}\right) = \frac{1}{4}$. Thus the line intersects the xy-plane at the point $\left(\frac{11}{4}, \frac{1}{4}, 0\right)$.

Alternatively, we can put $z = 0$ in the symmetric equations and obtain

$$\frac{x-2}{1} = \frac{y-4}{-5} = \frac{3}{4}$$

which again gives $x = \frac{11}{4}$ and $y = \frac{1}{4}$. ■

In general, the procedure of Example 2 shows that direction numbers of the line L through the points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ are $x_1 - x_0$, $y_1 - y_0$, and $z_1 - z_0$ and so symmetric equations of L are

$$\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0}$$

Often, we need a description, not of an entire line, but of just a line segment. How, for instance, could we describe the line segment AB in Example 2? If we put $t = 0$ in the parametric equations in Example 2(a), we get the point $(2, 4, -3)$ and if we put $t = 1$ we get $(3, -1, 1)$. So the line segment AB is described by the parametric equations

$$x = 2 + t \qquad y = 4 - 5t \qquad z = -3 + 4t \qquad 0 \leqslant t \leqslant 1$$

or by the corresponding vector equation

$$\mathbf{r}(t) = \langle 2 + t, 4 - 5t, -3 + 4t \rangle \qquad 0 \leqslant t \leqslant 1$$

In general, we know from Equation 1 that the vector equation of a line through the (tip of the) vector $\mathbf{r}_0$ in the direction of a vector $\mathbf{v}$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$. If the line also passes through (the tip of) $\mathbf{r}_1$, then we can take $\mathbf{v} = \mathbf{r}_1 - \mathbf{r}_0$ and so its vector equation is

$$\mathbf{r} = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1$$

The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the parameter interval $0 \leqslant t \leqslant 1$.

4 The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the vector equation

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \leqslant t \leqslant 1$$

The lines L_1 and L_2 in Example 3, shown in Figure 5, are skew lines.

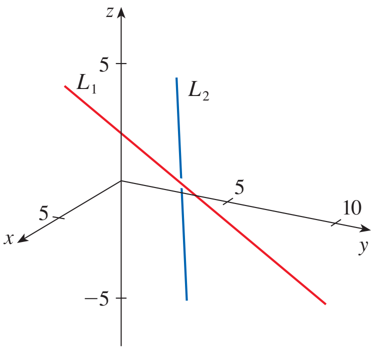


FIGURE 5

EXAMPLE 3 Show that the lines L_1 and L_2 with parametric equations

$$L_1\colon \quad x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$
$$L_2\colon \quad x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

are **skew lines**; that is, they do not intersect and are not parallel (and therefore do not lie in the same plane).

SOLUTION The lines are not parallel because the corresponding direction vectors $\langle 1, 3, -1 \rangle$ and $\langle 2, 1, 4 \rangle$ are not parallel. (Their components are not proportional.) If L_1 and L_2 had a point of intersection, there would be values of t and s such that

$$\begin{aligned} 1 + t &= 2s \\ -2 + 3t &= 3 + s \\ 4 - t &= -3 + 4s \end{aligned}$$

But if we solve the first two equations, we get $t = \frac{11}{5}$ and $s = \frac{8}{5}$, and these values don't satisfy the third equation. Therefore there are no values of t and s that satisfy the three equations, so L_1 and L_2 do not intersect. Thus L_1 and L_2 are skew lines. ■

Planes

Although a line in space is determined by a point and a direction, a plane in space is more difficult to describe. A single vector parallel to a plane is not enough to convey the "direction" of the plane, but a vector perpendicular to the plane does completely specify its direction. Thus a plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n}$ that is orthogonal to the plane. This orthogonal vector $\mathbf{n}$ is called a **normal vector**. Let $P(x, y, z)$ be an arbitrary point in the plane, and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P. Then the vector $\mathbf{r} - \mathbf{r}_0$ is represented by $\overrightarrow{P_0P}$. (See Figure 6.) The normal vector $\mathbf{n}$ is orthogonal to every vector in the given plane. In particular, $\mathbf{n}$ is orthogonal to $\mathbf{r} - \mathbf{r}_0$ and so we have

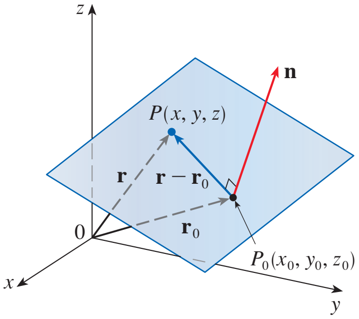


FIGURE 6

$$\boxed{5} \qquad \boxed{\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0}$$

which can be rewritten as

$$\boxed{6} \qquad \boxed{\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0}$$

Either Equation 5 or Equation 6 is called a **vector equation of the plane**.

To obtain a scalar equation for the plane, we write $\mathbf{n} = \langle a, b, c \rangle$, $\mathbf{r} = \langle x, y, z \rangle$, and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$. Then the vector equation (5) becomes

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

Expanding the left side of this equation gives the following.

7 A **scalar equation of the plane** through point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

EXAMPLE 4 Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $\mathbf{n} = \langle 2, 3, 4 \rangle$. Find the intercepts and sketch the plane.

SOLUTION Putting $a = 2$, $b = 3$, $c = 4$, $x_0 = 2$, $y_0 = 4$, and $z_0 = -1$ in Equation 7, we see that an equation of the plane is

$$2(x - 2) + 3(y - 4) + 4(z + 1) = 0$$

or

$$2x + 3y + 4z = 12$$

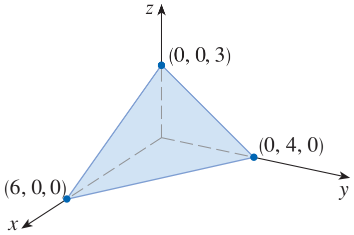


FIGURE 7

To find the x-intercept we set $y = z = 0$ in this equation and obtain $x = 6$. Similarly, the y-intercept is 4 and the z-intercept is 3. This enables us to sketch the portion of the plane that lies in the first octant (see Figure 7). ■

By collecting terms in Equation 7 as we did in Example 4, we can rewrite the equation of a plane as

$$\boxed{8} \qquad \boxed{ax + by + cz + d = 0}$$

where $d = -(ax_0 + by_0 + cz_0)$. Equation 8 is called a **linear equation** in x, y, and z. Conversely, it can be shown that if a, b, and c are not all 0, then the linear equation (8) represents a plane with normal vector $\langle a, b, c \rangle$. (See Exercise 83.)

Figure 8 shows the portion of the plane in Example 5 that is enclosed by triangle *PQR*.

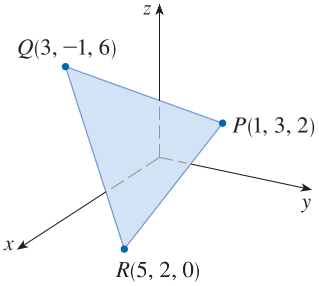


FIGURE 8

EXAMPLE 5 Find an equation of the plane that passes through the points $P(1, 3, 2)$, $Q(3, -1, 6)$, and $R(5, 2, 0)$.

SOLUTION The vectors $\mathbf{a}$ and $\mathbf{b}$ corresponding to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\mathbf{a} = \langle 2, -4, 4 \rangle \qquad \mathbf{b} = \langle 4, -1, -2 \rangle$$

Since both $\mathbf{a}$ and $\mathbf{b}$ lie in the plane, their cross product $\mathbf{a} \times \mathbf{b}$ is orthogonal to the plane and can be taken as the normal vector. Thus

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\,\mathbf{i} + 20\,\mathbf{j} + 14\,\mathbf{k}$$

With the point $P(1, 3, 2)$ and the normal vector $\mathbf{n}$, an equation of the plane is

$$12(x - 1) + 20(y - 3) + 14(z - 2) = 0$$

or

$$6x + 10y + 7z = 50$$

■

EXAMPLE 6 Find the point at which the line with parametric equations $x = 2 + 3t$, $y = -4t$, $z = 5 + t$ intersects the plane $4x + 5y - 2z = 18$.

SOLUTION We substitute the expressions for x, y, and z from the parametric equations into the equation of the plane:

$$4(2 + 3t) + 5(-4t) - 2(5 + t) = 18$$

This simplifies to $-10t = 20$, so $t = -2$. Therefore the point of intersection occurs when the parameter value is $t = -2$. Then $x = 2 + 3(-2) = -4$, $y = -4(-2) = 8$, $z = 5 - 2 = 3$ and so the point of intersection is $(-4, 8, 3)$. ■

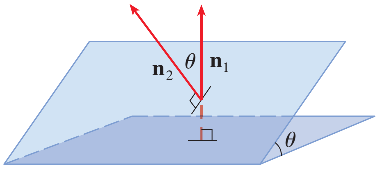


FIGURE 9

Two planes are **parallel** if their normal vectors are parallel. For instance, the planes $x + 2y - 3z = 4$ and $2x + 4y - 6z = 3$ are parallel because their normal vectors are $\mathbf{n}_1 = \langle 1, 2, -3 \rangle$ and $\mathbf{n}_2 = \langle 2, 4, -6 \rangle$ and $\mathbf{n}_2 = 2\mathbf{n}_1$. If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors (see angle θ in Figure 9).

Figure 10 shows the planes in Example 7 and their line of intersection *L*.

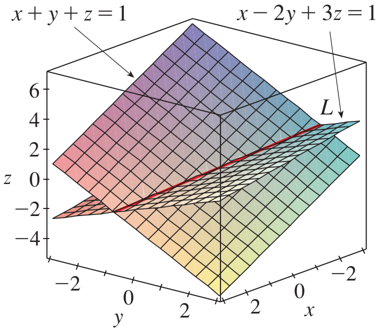


FIGURE 10

EXAMPLE 7

(a) Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.

(b) Find symmetric equations for the line of intersection L of these two planes.

SOLUTION

(a) The normal vectors of these planes are

$$\mathbf{n}_1 = \langle 1, 1, 1 \rangle \qquad \mathbf{n}_2 = \langle 1, -2, 3 \rangle$$

and so, if θ is the angle between the planes, Corollary 12.3.6 gives

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \frac{1(1) + 1(-2) + 1(3)}{\sqrt{1 + 1 + 1}\,\sqrt{1 + 4 + 9}} = \frac{2}{\sqrt{42}}$$

$$\theta = \cos^{-1}\!\left(\frac{2}{\sqrt{42}}\right) \approx 72^\circ$$

(b) We first need to find a point on L. For instance, we can find the point where the line intersects the xy-plane by setting $z = 0$ in the equations of both planes. This gives the equations $x + y = 1$ and $x - 2y = 1$, whose solution is $x = 1$, $y = 0$. So the point $(1, 0, 0)$ lies on L.

Now we observe that, since L lies in both planes, it is perpendicular to both of the normal vectors. Thus a vector $\mathbf{v}$ parallel to L is given by the cross product

Another way to find the line of intersection is to solve the equations of the planes for two of the variables in terms of the third, which can be taken as the parameter.

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\,\mathbf{i} - 2\,\mathbf{j} - 3\,\mathbf{k}$$

and so the symmetric equations of L can be written as

$$\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3} \qquad \blacksquare$$

NOTE Since a linear equation in x, y, and z represents a plane and two nonparallel planes intersect in a line, it follows that two linear equations can represent a line. The points (x, y, z) that satisfy both $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ lie on both of these planes, and so the pair of linear equations represents the line of intersection of the planes (if they are not parallel). For instance, in Example 7 the line L was given as the line of intersection of the planes $x + y + z = 1$ and $x - 2y + 3z = 1$. The symmetric equations that we found for L could be written as

$$\frac{x-1}{5} = \frac{y}{-2} \qquad \text{and} \qquad \frac{y}{-2} = \frac{z}{-3}$$

which is again a pair of linear equations. They exhibit L as the line of intersection of the planes $(x - 1)/5 = y/(-2)$ and $y/(-2) = z/(-3)$. (See Figure 11.)

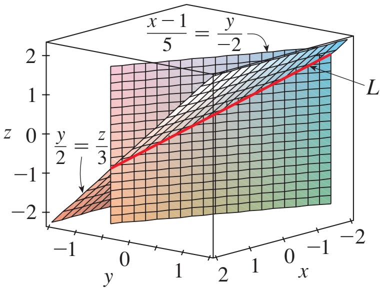


FIGURE 11

Figure 11 shows how the line L in Example 7 can also be regarded as the line of intersection of planes derived from its symmetric equations.

In general, when we write the equations of a line in the symmetric form

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

we can regard the line as the line of intersection of the two planes

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} \qquad \text{and} \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Distances

In order to find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$, we let $P_0(x_0, y_0, z_0)$ be any point in the given plane and $\mathbf{b}$ be the vector corresponding to $\overrightarrow{P_0P_1}$. Then

$$\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

From Figure 12 you can see that the distance D from P_1 to the plane is equal to the absolute value of the scalar projection of $\mathbf{b}$ onto the normal vector $\mathbf{n} = \langle a, b, c \rangle$. (See Section 12.3.) Thus

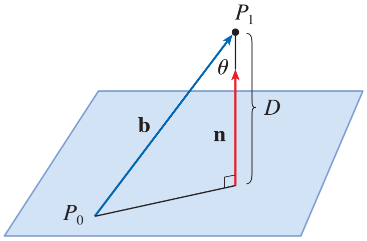


FIGURE 12

$$\begin{aligned} D = |\operatorname{comp}_{\mathbf{n}} \mathbf{b}| &= \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

Since P_0 lies in the plane, its coordinates satisfy the equation of the plane and so we have $ax_0 + by_0 + cz_0 + d = 0$. Thus we have the following formula.

9 The distance D from the point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

EXAMPLE 8 Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

SOLUTION First we note that the planes are parallel because their normal vectors $\langle 10, 2, -2 \rangle$ and $\langle 5, 1, -1 \rangle$ are parallel. To find the distance D between the planes, we choose any point on one plane and calculate its distance to the other plane. In particular, if we put $y = z = 0$ in the equation of the first plane, we get $10x = 5$ and so $\left(\frac{1}{2}, 0, 0\right)$ is a point in this plane. By Formula 9, the distance between $\left(\frac{1}{2}, 0, 0\right)$ and the plane $5x + y - z - 1 = 0$ is

$$D = \frac{\left|5\left(\frac{1}{2}\right) + 1(0) - 1(0) - 1\right|}{\sqrt{5^2 + 1^2 + (-1)^2}} = \frac{\frac{3}{2}}{3\sqrt{3}} = \frac{\sqrt{3}}{6}$$

So the distance between the planes is $\sqrt{3}/6$. ■

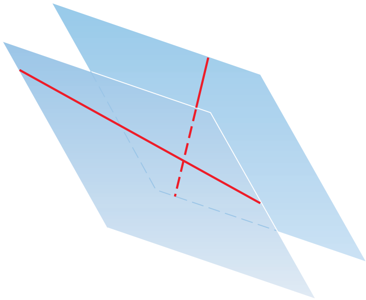


FIGURE 13
Skew lines, like those in Example 9, always lie on (nonidentical) parallel planes.

EXAMPLE 9 In Example 3 we showed that the lines

$$L_1: \quad x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$
$$L_2: \quad x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

are skew. Find the distance between them.

SOLUTION Since the two lines L_1 and L_2 are skew, they can be viewed as lying on two parallel planes P_1 and P_2. The distance between L_1 and L_2 is the same as the distance between P_1 and P_2, which can be computed as in Example 8. The common normal vector to both planes must be orthogonal to both $\mathbf{v}_1 = \langle 1, 3, -1 \rangle$ (the direction of L_1) and $\mathbf{v}_2 = \langle 2, 1, 4 \rangle$ (the direction of L_2). So a normal vector is

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 13\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}$$

If we put $s = 0$ in the equations of L_2, we get the point $(0, 3, -3)$ on L_2 and so an equation for P_2 is

$$13(x - 0) - 6(y - 3) - 5(z + 3) = 0 \qquad \text{or} \qquad 13x - 6y - 5z + 3 = 0$$

If we now set $t = 0$ in the equations for L_1, we get the point $(1, -2, 4)$ on P_1. So the distance between L_1 and L_2 is the same as the distance from $(1, -2, 4)$ to $13x - 6y - 5z + 3 = 0$. By Formula 9, this distance is

$$D = \frac{|13(1) - 6(-2) - 5(4) + 3|}{\sqrt{13^2 + (-6)^2 + (-5)^2}} = \frac{8}{\sqrt{230}} \approx 0.53$$ ■

12.5 Exercises

1. Determine whether each statement is true or false in $\mathbb{R}^3$.
(a) Two lines parallel to a third line are parallel.
(b) Two lines perpendicular to a third line are parallel.
(c) Two planes parallel to a third plane are parallel.
(d) Two planes perpendicular to a third plane are parallel.
(e) Two lines parallel to a plane are parallel.
(f) Two lines perpendicular to a plane are parallel.
(g) Two planes parallel to a line are parallel.
(h) Two planes perpendicular to a line are parallel.
(i) Two planes either intersect or are parallel.
(j) Two lines either intersect or are parallel.
(k) A plane and a line either intersect or are parallel.

2–5 Find a vector equation and parametric equations for the line.

2. The line through the point $(4, 2, -3)$ and parallel to the vector $2\mathbf{i} - \mathbf{j} + 6\mathbf{k}$

3. The line through the point $(-1, 8, 7)$ and parallel to the vector $\langle \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \rangle$

4. The line through the point $(6, 0, -2)$ and parallel to the line

$$x = 4 - 3t \qquad y = -1 + 4t \qquad z = 6 + 5t$$

5. The line through the point $(5, 7, 1)$ and perpendicular to the plane $3x - 2y + 2z = 8$

6–12 Find parametric equations and symmetric equations for the line.

6. The line through the points $(-5, 2, 5)$ and $(1, 6, -2)$

7. The line through the origin and the point $(8, -1, 3)$

8. The line through the points $(0.4, -0.2, 1.1)$ and $(1.3, 0.8, -2.3)$

9. The line through the points $(12, 9, -13)$ and $(-7, 9, 11)$

10. The line through $(2, 1, 0)$ and perpendicular to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$

11. The line through $(-6, 2, 3)$ and parallel to the line $\frac{1}{2}x = \frac{1}{3}y = z + 1$

12. The line of intersection of the planes $x + 2y + 3z = 1$ and $x - y + z = 1$

---

13. Is the line through $(-4, -6, 1)$ and $(-2, 0, -3)$ parallel to the line through $(10, 18, 4)$ and $(5, 3, 14)$?

14. Is the line through $(-2, 4, 0)$ and $(1, 1, 1)$ perpendicular to the line through $(2, 3, 4)$ and $(3, -1, -8)$?

15. (a) Find symmetric equations for the line that passes through the point $(1, -5, 6)$ and is parallel to the vector $\langle -1, 2, -3 \rangle$.
(b) Find the points in which the required line in part (a) intersects the coordinate planes.

16. (a) Find parametric equations for the line through $(2, 4, 6)$ that is perpendicular to the plane $x - y + 3z = 7$.
(b) In what points does this line intersect the coordinate planes?

17. Find a vector equation for the line segment from $(6, -1, 9)$ to $(7, 6, 0)$.

18. Find parametric equations for the line segment from $(-2, 18, 31)$ to $(11, -4, 48)$.

19–22 Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

19. L_1: $x = 3 + 2t, \quad y = 4 - t, \quad z = 1 + 3t$
L_2: $x = 1 + 4s, \quad y = 3 - 2s, \quad z = 4 + 5s$

20. L_1: $x = 5 - 12t, \quad y = 3 + 9t, \quad z = 1 - 3t$
L_2: $x = 3 + 8s, \quad y = -6s, \quad z = 7 + 2s$

21. L_1: $\dfrac{x-2}{1} = \dfrac{y-3}{-2} = \dfrac{z-1}{-3}$
L_2: $\dfrac{x-3}{1} = \dfrac{y+4}{3} = \dfrac{z-2}{-7}$

22. L_1: $\dfrac{x}{1} = \dfrac{y-1}{-1} = \dfrac{z-2}{3}$
L_2: $\dfrac{x-2}{2} = \dfrac{y-3}{-2} = \dfrac{z}{7}$

23–40 Find an equation of the plane.

23. The plane through the point $(3, 2, 1)$ and with normal vector $5\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$

24. The plane through the point $(-3, 4, 2)$ and with normal vector $\langle 6, 1, -1 \rangle$

25. The plane through the point $(5, -2, 4)$ and perpendicular to the vector $-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

26. The plane through the origin and perpendicular to the line

$$x = 1 - 8t \qquad y = -1 - 7t \qquad z = 4 + 2t$$

27. The plane through the point $(1, 3, -1)$ and perpendicular to the line

$$\frac{x+3}{4} = -y = \frac{z-1}{5}$$

28. The plane through the point $(9, -4, -5)$ and parallel to the plane $z = 2x - 3y$

29. The plane through the point $(2.1, 1.7, -0.9)$ and parallel to the plane $2x - y + 3z = 1$

30. The plane that contains the line $x = 1 + t$, $y = 2 - t$, $z = 4 - 3t$ and is parallel to the plane $5x + 2y + z = 1$

31. The plane through the points $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$

32. The plane through the origin and the points $(3, -2, 1)$ and $(1, 1, 1)$

33. The plane through the points $(2, 1, 2)$, $(3, -8, 6)$, and $(-2, -3, 1)$

34. The plane through the points $(3, 0, -1)$, $(-2, -2, 3)$, and $(7, 1, -4)$

35. The plane that passes through the point $(3, 5, -1)$ and contains the line $x = 4 - t$, $y = 2t - 1$, $z = -3t$

36. The plane that passes through the point $(6, -1, 3)$ and contains the line with symmetric equations $x/3 = y + 4 = z/2$

37. The plane that passes through the point $(3, 1, 4)$ and contains the line of intersection of the planes $x + 2y + 3z = 1$ and $2x - y + z = -3$

38. The plane that passes through the points $(0, -2, 5)$ and $(-1, 3, 1)$ and is perpendicular to the plane $2z = 5x + 4y$

39. The plane that passes through the point $(1, 5, 1)$ and is perpendicular to the planes $2x + y - 2z = 2$ and $x + 3z = 4$

40. The plane that passes through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$ and is perpendicular to the plane $x + y - 2z = 1$

41–44 Use intercepts to help sketch the plane.

41. $2x + 5y + z = 10$

42. $3x + y + 2z = 6$

43. $6x - 3y + 4z = 6$

44. $6x + 5y - 3z = 15$

45–47 Find the point at which the line intersects the given plane.

45. $x = 2 - 2t$, $y = 3t$, $z = 1 + t$; $x + 2y - z = 7$

46. $x = t - 1$, $y = 1 + 2t$, $z = 3 - t$; $3x - y + 2z = 5$

47. $5x = y/2 = z + 2$; $10x - 7y + 3z + 24 = 0$

48. Where does the line through $(-3, 1, 0)$ and $(-1, 5, 6)$ intersect the plane $2x + y - z = -2$?

49. Find direction numbers for the line of intersection of the planes $x + y + z = 1$ and $x + z = 0$.

50. Find the cosine of the angle between the planes $x + y + z = 0$ and $x + 2y + 3z = 1$.

51–56 Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them. (Use degrees and round to one decimal place.)

51. $x + 4y - 3z = 1$, $-3x + 6y + 7z = 0$

52. $9x - 3y + 6z = 2$, $2y = 6x + 4z$

53. $x + 2y - z = 2$, $2x - 2y + z = 1$

54. $x - y + 3z = 1$, $3x + y - z = 2$

55. $2x - 3y = z$, $4x = 3 + 6y + 2z$

56. $5x + 2y + 3z = 2$, $y = 4x - 6z$

57–58

(a) Find parametric equations for the line of intersection of the planes.

(b) Find the angle, in degrees rounded to one decimal place, between the planes.

57. $x + y + z = 1$, $x + 2y + 2z = 1$

58. $3x - 2y + z = 1$, $2x + y - 3z = 3$

59–60 Find symmetric equations for the line of intersection of the planes.

59. $5x - 2y - 2z = 1$, $4x + y + z = 6$

60. $z = 2x - y - 5$, $z = 4x + 3y - 5$

61. Find an equation for the plane consisting of all points that are equidistant from the points $(1, 0, -2)$ and $(3, 4, 0)$.

62. Find an equation for the plane consisting of all points that are equidistant from the points $(2, 5, 5)$ and $(-6, 3, 1)$.

63. Find an equation of the plane with x-intercept a, y-intercept b, and z-intercept c.

64. (a) Find the point at which the given lines intersect:

$$\mathbf{r} = \langle 1, 1, 0 \rangle + t\langle 1, -1, 2 \rangle$$

$$\mathbf{r} = \langle 2, 0, 2 \rangle + s\langle -1, 1, 0 \rangle$$

(b) Find an equation of the plane that contains these lines.

65. Find parametric equations for the line through the point $(0, 1, 2)$ that is parallel to the plane $x + y + z = 2$ and perpendicular to the line $x = 1 + t$, $y = 1 - t$, $z = 2t$.

66. Find parametric equations for the line through the point $(0, 1, 2)$ that is perpendicular to the line $x = 1 + t$, $y = 1 - t$, $z = 2t$ and intersects this line.

67. Which of the following four planes are parallel? Are any of them identical?

P_1: $3x + 6y - 3z = 6$

P_2: $4x - 12y + 8z = 5$

P_3: $9y = 1 + 3x + 6z$

P_4: $z = x + 2y - 2$

68. Which of the following four lines are parallel? Are any of them identical?

L_1: $x = 1 + 6t$, $y = 1 - 3t$, $z = 12t + 5$

L_2: $x = 1 + 2t$, $y = t$, $z = 1 + 4t$

L_3: $2x - 2 = 4 - 4y = z + 1$

L_4: $\mathbf{r} = \langle 3, 1, 5 \rangle + t\langle 4, 2, 8 \rangle$

69–70 Use the formula in Exercise 12.4.45 to find the distance from the point to the given line.

69. $(4, 1, -2)$; $x = 1 + t$, $y = 3 - 2t$, $z = 4 - 3t$

70. $(0, 1, 3)$; $x = 2t$, $y = 6 - 2t$, $z = 3 + t$

71–72 Find the distance from the point to the given plane.

71. $(1, -2, 4)$, $3x + 2y + 6z = 5$

72. $(-6, 3, 5)$, $x - 2y - 4z = 8$

73–74 Find the distance between the given parallel planes.

73. $2x - 3y + z = 4$, $4x - 6y + 2z = 3$

74. $6z = 4y - 2x$, $9z = 1 - 3x + 6y$

---

75. Distance between Parallel Planes Show that the distance between the parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

76. Find equations of the planes that are parallel to the plane $x + 2y - 2z = 1$ and two units away from it.

77. Show that the lines with symmetric equations $x = y = z$ and $x + 1 = y/2 = z/3$ are skew, and find the distance between these lines.

78. Find the distance between the skew lines with parametric equations $x = 1 + t$, $y = 1 + 6t$, $z = 2t$, and $x = 1 + 2s$, $y = 5 + 15s$, $z = -2 + 6s$.

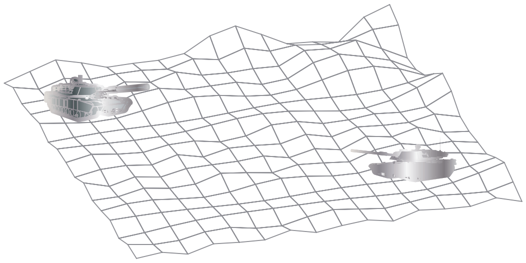
79. Let L_1 be the line through the origin and the point $(2, 0, -1)$. Let L_2 be the line through the points $(1, -1, 1)$ and $(4, 1, 3)$. Find the distance between L_1 and L_2.

80. Let L_1 be the line through the points $(1, 2, 6)$ and $(2, 4, 8)$. Let L_2 be the line of intersection of the planes P_1 and P_2, where P_1 is the plane $x - y + 2z + 1 = 0$ and P_2 is the plane through the points $(3, 2, -1)$, $(0, 0, 1)$, and $(1, 2, 1)$. Calculate the distance between L_1 and L_2.

81. Two tanks are participating in a battle simulation. Tank A is at point $(325, 810, 561)$ and tank B is positioned at point $(765, 675, 599)$.

(a) Find parametric equations for the line of sight between the tanks.

(b) If we divide the line of sight into 5 equal segments, the elevations of the terrain at the four intermediate points from tank A to tank B are 549, 566, 586, and 589. Can the tanks see each other?



82. Give a geometric description of each family of planes.

(a) $x + y + z = c$

(b) $x + y + cz = 1$

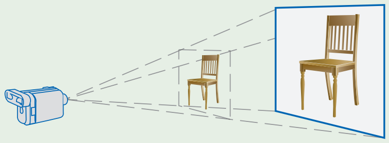
(c) $y \cos\theta + z \sin\theta = 1$

83. If a, b, and c are not all 0, show that the equation $ax + by + cz + d = 0$ represents a plane and $\langle a, b, c \rangle$ is a normal vector to the plane.

Hint: Suppose $a \neq 0$ and rewrite the equation in the form

$$a\left(x + \frac{d}{a}\right) + b(y - 0) + c(z - 0) = 0$$

DISCOVERY PROJECT | PUTTING 3D IN PERSPECTIVE



Computer graphics programmers face the same challenge as the great painters of the past: how to represent a three-dimensional scene as a flat image on a two-dimensional plane (a screen or a canvas). To create the illusion of perspective, in which closer objects appear larger than those farther away, three-dimensional objects in the computer's memory are projected onto a rectangular screen window from a viewpoint where the eye, or camera, is located. The viewing volume—the portion of space that will be visible—is the region contained by the four planes that pass through the viewpoint and an edge of the screen window. If objects in the scene extend beyond these four planes, they must be truncated before pixel data are sent to the screen. These planes are therefore called *clipping planes*.

1. Suppose the screen is represented by a rectangle in the yz-plane with vertices $(0, \pm400, 0)$ and $(0, \pm400, 600)$, and the camera is placed at $(1000, 0, 0)$. A line L in the scene passes through the points $(230, -285, 102)$ and $(860, 105, 264)$. At what points should L be clipped by the clipping planes?

2. If the clipped line segment is projected onto the screen window, identify the resulting line segment.

3. Use parametric equations to plot the edges of the screen window, the clipped line segment, and its projection onto the screen window. Then add sight lines connecting the viewpoint to each end of the clipped segments to verify that the projection is correct.

4. A rectangle with vertices $(621, -147, 206)$, $(563, 31, 242)$, $(657, -111, 86)$, and $(599, 67, 122)$ is added to the scene. The line L intersects this rectangle. To make the rectangle appear opaque, a programmer can use *hidden line rendering,* which removes portions of objects that are behind other objects. Identify the portion of L that should be removed.

12.6 Cylinders and Quadric Surfaces

We have already looked at two special types of surfaces: planes (in Section 12.5) and spheres (in Section 12.1). Here we investigate two other types of surfaces: cylinders and quadric surfaces.

In order to sketch the graph of a surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These curves are called **traces** (or cross-sections) of the surface.

Cylinders

A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.

EXAMPLE 1 Sketch the graph of the surface $z = x^2$.

SOLUTION Notice that the equation of the graph, $z = x^2$, doesn't involve y. This means that any vertical plane with equation $y = k$ (parallel to the xz-plane) intersects the graph in a curve with equation $z = x^2$. So these vertical traces are parabolas. Figure 1 shows how the graph is formed by taking the parabola $z = x^2$ in the xz-plane and moving it in the direction of the y-axis. The graph is a surface, called a **parabolic cylinder**, made up of infinitely many shifted copies of the same parabola. Here the rulings of the cylinder are parallel to the y-axis. ■

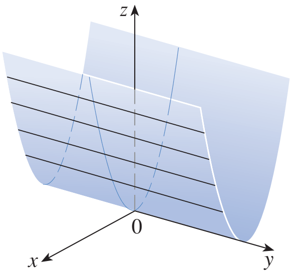


FIGURE 1
The surface $z = x^2$ is a parabolic cylinder.

In Example 1 the variable y is missing from the equation of the cylinder. This is typical of a surface whose rulings are parallel to one of the coordinate axes. If one of the variables x, y, or z is missing from the equation of a surface, then the surface is a cylinder.

EXAMPLE 2 Identify and sketch the surfaces.

(a) $x^2 + y^2 = 1$ (b) $y^2 + z^2 = 1$

SOLUTION

(a) Since z is missing and the equations $x^2 + y^2 = 1$, $z = k$ represent a circle with radius 1 in the plane $z = k$, the surface $x^2 + y^2 = 1$ is a circular cylinder whose axis is the z-axis. (See Figure 2. We first encountered this surface in Example 12.1.2.) Here the rulings are vertical lines.

(b) In this case x is missing and the surface is a circular cylinder whose axis is the x-axis. (See Figure 3.) It is obtained by taking the circle $y^2 + z^2 = 1$, $x = 0$ in the yz-plane and moving it parallel to the x-axis.

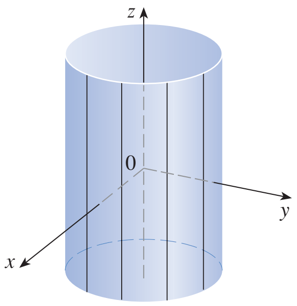


FIGURE 2
$x^2 + y^2 = 1$

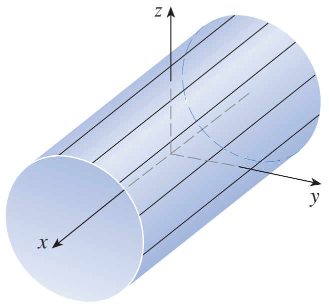


FIGURE 3
$y^2 + z^2 = 1$ ■

⊘ **NOTE** When you are dealing with surfaces, it is important to recognize that an equation like $x^2 + y^2 = 1$ represents a cylinder and not a circle. The trace of the cylinder $x^2 + y^2 = 1$ in the xy-plane is the circle with equations $x^2 + y^2 = 1$, $z = 0$.

■ Quadric Surfaces

A **quadric surface** is the graph of a second-degree equation in three variables x, y, and z. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where $A, B, C, \ldots, J$ are constants, but by translation and rotation it can be brought into one of the two *standard forms*

$$Ax^2 + By^2 + Cz^2 + J = 0 \qquad \text{or} \qquad Ax^2 + By^2 + Iz = 0$$

Quadric surfaces are the counterparts in three dimensions of the conic sections in the plane. (See Section 10.5 for a review of conic sections.)

EXAMPLE 3 Use traces to sketch the quadric surface with equation

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

SOLUTION By substituting $z = 0$, we find that the trace in the xy-plane is $x^2 + y^2/9 = 1$, which we recognize as an equation of an ellipse. In general, the horizontal trace in the plane $z = k$ is

$$x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4} \qquad z = k$$

which is an ellipse, provided that $k^2 < 4$, that is, $-2 < k < 2$. (If $|k| = 2$, the trace consists of a single point, and the trace is empty for $|k| > 2$.)

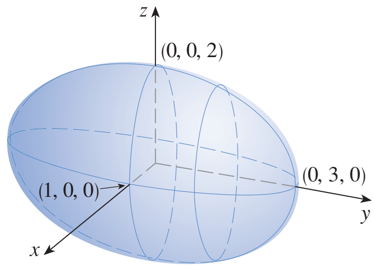


FIGURE 4

The ellipsoid $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

Similarly, vertical traces parallel to the *yz*- and *xz*-planes are also ellipses:

$$\frac{y^2}{9} + \frac{z^2}{4} = 1 - k^2 \qquad x = k \qquad (\text{if } -1 < k < 1)$$

$$x^2 + \frac{z^2}{4} = 1 - \frac{k^2}{9} \qquad y = k \qquad (\text{if } -3 < k < 3)$$

Figure 4 shows how drawing some traces indicates the shape of the surface. It's called an **ellipsoid** because all of its traces are ellipses. Notice that it is symmetric with respect to each coordinate plane; this is because its equation involves only even powers of *x*, *y*, and *z*. ■

EXAMPLE 4 Use traces to sketch the surface $z = 4x^2 + y^2$.

SOLUTION If we put $x = 0$, we get $z = y^2$, so the *yz*-plane intersects the surface in a parabola. If we put $x = k$ (a constant), we get $z = y^2 + 4k^2$. This means that if we slice the graph with any plane parallel to the *yz*-plane, we obtain a parabola that opens upward. Similarly, if $y = k$, the trace is $z = 4x^2 + k^2$, which is again a parabola that opens upward. If we put $z = k$, we get the horizontal traces $4x^2 + y^2 = k$, which we recognize as a family of ellipses ($k > 0$). Knowing the shapes of the traces, we can sketch the graph in Figure 5. Because of the elliptical and parabolic traces, the quadric surface $z = 4x^2 + y^2$ is called an **elliptic paraboloid**. ■

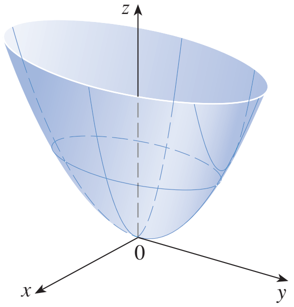
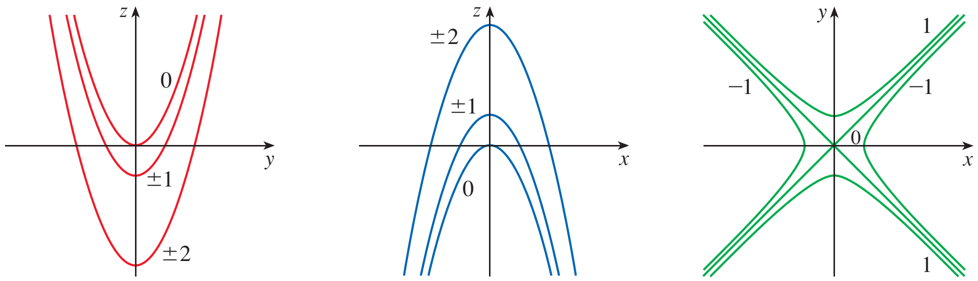


FIGURE 5

The surface $z = 4x^2 + y^2$ is an elliptic paraboloid. Horizontal traces are ellipses; vertical traces are parabolas.

EXAMPLE 5 Sketch the surface $z = y^2 - x^2$.

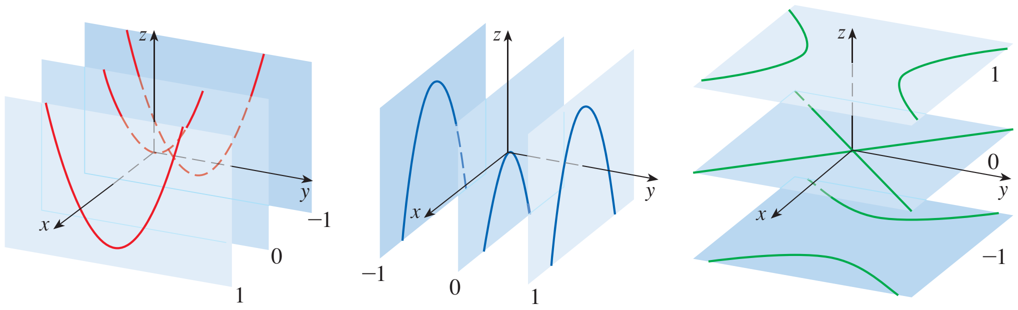
SOLUTION The traces in the vertical planes $x = k$ are the parabolas $z = y^2 - k^2$, which open upward. The traces in $y = k$ are the parabolas $z = -x^2 + k^2$, which open downward. The horizontal traces are $y^2 - x^2 = k$, a family of hyperbolas. We draw the families of traces in Figure 6, and we show how the traces appear when placed in their correct planes in Figure 7.



Traces in $x = k$ are $z = y^2 - k^2$. Traces in $y = k$ are $z = -x^2 + k^2$. Traces in $z = k$ are $y^2 - x^2 = k$.

FIGURE 6

Vertical traces are parabolas; horizontal traces are hyperbolas. All traces are labeled with the value of *k*.



Traces in $x = k$ Traces in $y = k$ Traces in $z = k$

FIGURE 7

Traces moved to their correct planes

In Figure 8 we fit together the traces from Figure 7 to form the surface $z = y^2 - x^2$, a **hyperbolic paraboloid**. Notice that the shape of the surface near the origin resembles that of a saddle. This surface will be investigated further in Section 14.7 when we discuss saddle points.

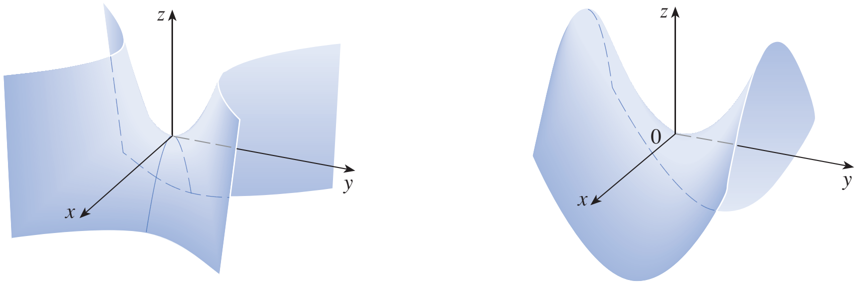


FIGURE 8

Two views of the surface $z = y^2 - x^2$, a hyperbolic paraboloid

■

EXAMPLE 6 Sketch the surface $\dfrac{x^2}{4} + y^2 - \dfrac{z^2}{4} = 1$.

SOLUTION The trace in any horizontal plane $z = k$ is the ellipse

$$\frac{x^2}{4} + y^2 = 1 + \frac{k^2}{4} \qquad z = k$$

but the traces in the xz- and yz-planes are the hyperbolas

$$\frac{x^2}{4} - \frac{z^2}{4} = 1 \quad y = 0 \qquad \text{and} \qquad y^2 - \frac{z^2}{4} = 1 \quad x = 0$$

This surface is called a **hyperboloid of one sheet** and is sketched in Figure 9.

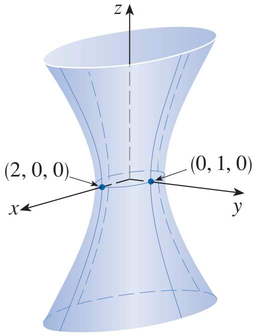


FIGURE 9

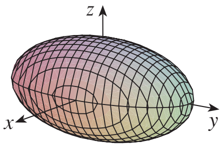
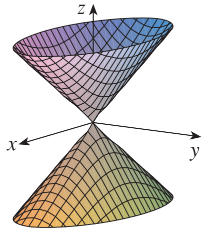
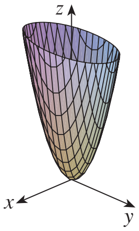
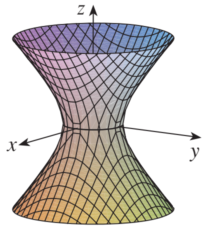
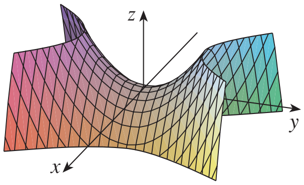
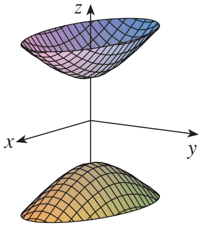
The surface $\dfrac{x^2}{4} + y^2 - \dfrac{z^2}{4} = 1$, a hyperboloid of one sheet

■

The idea of using traces to draw a surface is employed in three-dimensional graphing software. In most such software, traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k.

Table 1 shows computer-drawn graphs of the six basic types of quadric surfaces in standard form. All surfaces are symmetric with respect to the z-axis. If a quadric surface is symmetric about a different axis, its equation changes accordingly.

Table 1 Graphs of Quadric Surfaces

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	**Cone** 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	**Hyperboloid of One Sheet** 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	**Hyperboloid of Two Sheets** 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

EXAMPLE 7 Identify and sketch the surface $4x^2 - y^2 + 2z^2 + 4 = 0$.

SOLUTION Dividing by -4, we first put the equation in standard form:

$$-x^2 + \frac{y^2}{4} - \frac{z^2}{2} = 1$$

Comparing this equation with Table 1, we see that it represents a hyperboloid of two sheets, the only difference being that in this case the axis of the hyperboloid is the y-axis. The traces in the xy- and yz-planes are the hyperbolas

$$-x^2 + \frac{y^2}{4} = 1 \qquad z = 0 \qquad \text{and} \qquad \frac{y^2}{4} - \frac{z^2}{2} = 1 \qquad x = 0$$

The surface has no trace in the xz-plane, but traces in the vertical planes $y = k$ for $|k| > 2$ are the ellipses

$$x^2 + \frac{z^2}{2} = \frac{k^2}{4} - 1 \qquad y = k$$

which can be written as

$$\frac{x^2}{\dfrac{k^2}{4} - 1} + \frac{z^2}{2\left(\dfrac{k^2}{4} - 1\right)} = 1 \qquad y = k$$

These traces are used to make the sketch in Figure 10. ■

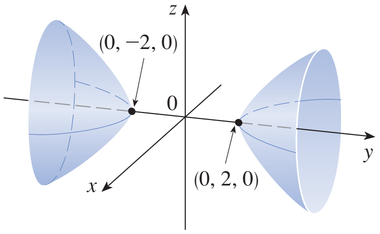


FIGURE 10
The surface $4x^2 - y^2 + 2z^2 + 4 = 0$, a hyperboloid of two sheets

EXAMPLE 8 Classify the quadric surface $x^2 + 2z^2 - 6x - y + 10 = 0$.

SOLUTION By completing the square we rewrite the equation as

$$y - 1 = (x - 3)^2 + 2z^2$$

Comparing this equation with Table 1, we see that it represents an elliptic paraboloid. Here, however, the axis of the paraboloid is parallel to the y-axis, and it has been shifted so that its vertex is the point $(3, 1, 0)$. The traces in the plane $y = k$ $(k > 1)$ are the ellipses

$$(x - 3)^2 + 2z^2 = k - 1 \qquad y = k$$

The trace in the xy-plane is the parabola with equation $y = 1 + (x - 3)^2$, $z = 0$. The paraboloid is sketched in Figure 11. ■

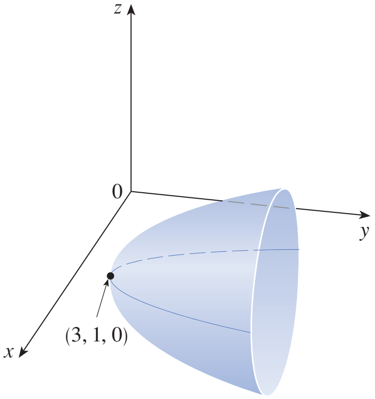


FIGURE 11
$x^2 + 2z^2 - 6x - y + 10 = 0$, a paraboloid

Applications of Quadric Surfaces

Examples of quadric surfaces can be found in the world around us. In fact, the world itself is a good example. Although the earth is commonly modeled as a sphere, a more accurate model is an ellipsoid because the earth's rotation has caused a flattening at the poles. (See Exercise 51.)

Circular paraboloids, obtained by rotating a parabola about its axis, are used to collect and reflect light, sound, and radio and television signals [see Figure 12(a)]. In a radio telescope, for instance, signals from distant stars that strike the bowl are all reflected to the receiver at the focus and are therefore amplified. (The idea is explained in Problem 22 in the Problems Plus following Chapter 3.) The same principle applies to microphones and satellite dishes in the shape of paraboloids.

Cooling towers for nuclear reactors are usually designed in the shape of hyperboloids of one sheet [Figure 12(b)] for reasons of structural stability. Pairs of hyperboloids are used to transmit rotational motion between skew axes. [See Figure 12(c); the cogs of the gears are the generating lines of the hyperboloids. See Exercise 53.]

Petr Student / Shutterstock.com

(a) A satellite dish reflects signals to the focus of a paraboloid.

(b) Nuclear reactors have cooling towers in the shape of hyperboloids.

(c) Gears in the shape of hyperboloids mesh and rotate along skew axes.

FIGURE 12 Applications of quadric surfaces

12.6 Exercises

1. (a) What does the equation $y = x^2$ represent as a curve in $\mathbb{R}^2$?
  (b) What does it represent as a surface in $\mathbb{R}^3$?
  (c) What does the equation $z = y^2$ represent?

2. (a) Sketch the graph of $y = e^x$ as a curve in $\mathbb{R}^2$.
  (b) Sketch the graph of $y = e^x$ as a surface in $\mathbb{R}^3$.
  (c) Describe and sketch the surface $z = e^y$.

3–8 Describe and sketch the surface.

3. $x^2 + z^2 = 4$

4. $y^2 + 9z^2 = 9$

5. $x^2 + y + 1 = 0$

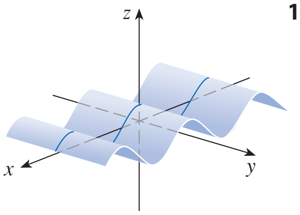
6. $z = -\sqrt{x}$

7. $xy = 1$

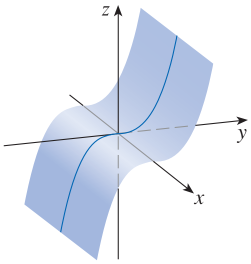
8. $z = \sin y$

9–10 Write an equation whose graph could be the surface shown.

9.



10.



11. (a) Find and identify the traces of the quadric surface $x^2 + y^2 - z^2 = 1$ and explain why the graph looks like the graph of the hyperboloid of one sheet in Table 1.
  (b) If we change the equation in part (a) to $x^2 - y^2 + z^2 = 1$, how is the graph affected?
  (c) What if we change the equation in part (a) to $x^2 + y^2 + 2y - z^2 = 0$?

12. (a) Find and identify the traces of the quadric surface $-x^2 - y^2 + z^2 = 1$ and explain why the graph looks like the graph of the hyperboloid of two sheets in Table 1.
  (b) If the equation in part (a) is changed to $x^2 - y^2 - z^2 = 1$, what happens to the graph? Sketch the new graph.

13–22 Use traces to sketch and identify the surface.

13. $x = y^2 + 4z^2$

14. $4x^2 + 9y^2 + 9z^2 = 36$

15. $x^2 = 4y^2 + z^2$

16. $z^2 - 4x^2 - y^2 = 4$

17. $9y^2 + 4z^2 = x^2 + 36$

18. $3x^2 + y + 3z^2 = 0$

19. $\dfrac{x^2}{9} + \dfrac{y^2}{25} + \dfrac{z^2}{4} = 1$

20. $3x^2 - y^2 + 3z^2 = 0$

21. $y = z^2 - x^2$

22. $x = y^2 - z^2$

23–30 Match the equation with its graph (labeled I–VIII). Give reasons for your choice.

23. $x^2 + 4y^2 + 9z^2 = 1$

24. $9x^2 + 4y^2 + z^2 = 1$

25. $x^2 - y^2 + z^2 = 1$

26. $-x^2 + y^2 - z^2 = 1$

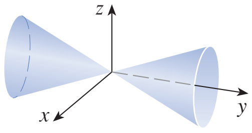
27. $y = 2x^2 + z^2$

28. $y^2 = x^2 + 2z^2$

29. $x^2 + 2z^2 = 1$

30. $y = x^2 - z^2$

I



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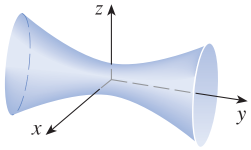
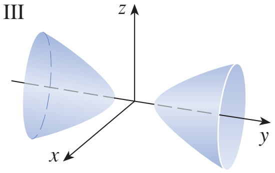
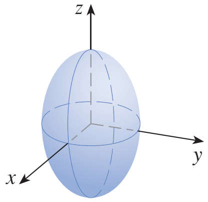


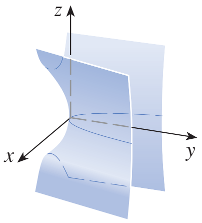
III



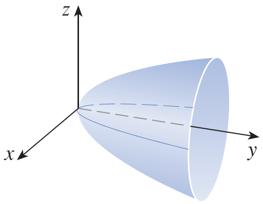
IV



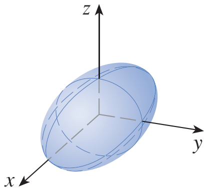
V



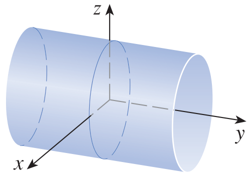
VI



VII

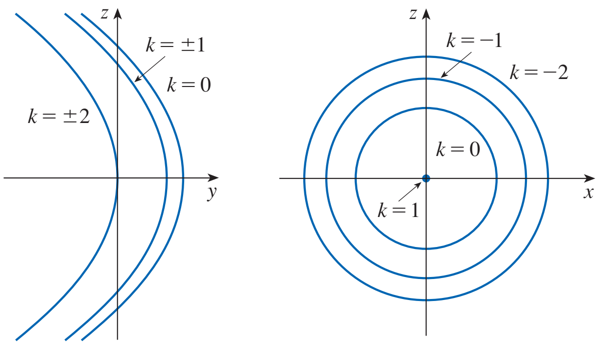


VIII

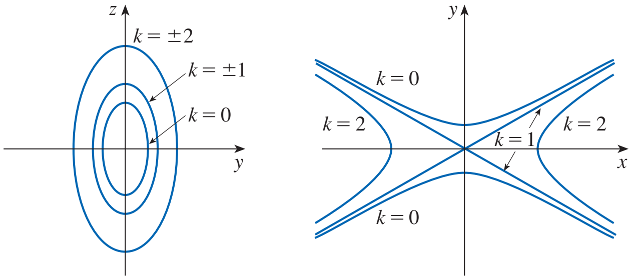


31–32 Sketch and identify a quadric surface that could have the traces shown.

31. Traces in $x = k$ Traces in $y = k$



32. Traces in $x = k$ Traces in $z = k$



33–40 Reduce the equation to one of the standard forms, classify the surface, and sketch it.

33. $y^2 = x^2 + \frac{1}{9}z^2$

34. $4x^2 - y + 2z^2 = 0$

35. $x^2 + 2y - 2z^2 = 0$

36. $y^2 = x^2 + 4z^2 + 4$

37. $x^2 + y^2 - 2x - 6y - z + 10 = 0$

38. $x^2 - y^2 - z^2 - 4x - 2z + 3 = 0$

39. $x^2 - y^2 + z^2 - 4x - 2z = 0$

40. $4x^2 + y^2 + z^2 - 24x - 8y + 4z + 55 = 0$

41–44 Graph the surface. Experiment with viewpoints and with domains for the variables until you get a good view of the surface.

41. $-4x^2 - y^2 + z^2 = 1$

42. $x^2 - y^2 - z = 0$

43. $-4x^2 - y^2 + z^2 = 0$

44. $x^2 - 6x + 4y^2 - z = 0$

45. Sketch the region bounded by the surfaces $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 = 1$ for $1 \leqslant z \leqslant 2$.

46. Sketch the region bounded by the paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$.

47. Find an equation for the surface obtained by rotating the curve $y = \sqrt{x}$ about the x-axis.

48. Find an equation for the surface obtained by rotating the line $z = 2y$ about the z-axis.

49. Find an equation for the surface consisting of all points that are equidistant from the point $(-1, 0, 0)$ and the plane $x = 1$. Identify the surface.

50. Find an equation for the surface consisting of all points P for which the distance from P to the x-axis is twice the distance from P to the yz-plane. Identify the surface.

51. Traditionally, the earth's surface has been modeled as a sphere, but the World Geodetic System of 1984 (WGS-84) uses an ellipsoid as a more accurate model. It places the center of the earth at the origin and the north pole on the positive z-axis. The distance from the center to the poles is 6356.523 km and the distance to a point on the equator is 6378.137 km.

(a) Find an equation of the earth's surface as used by WGS-84.
(b) Curves of equal latitude are traces in the planes $z = k$. What is the shape of these curves?
(c) Meridians (curves of equal longitude) are traces in planes of the form $y = mx$. What is the shape of these meridians?

52. A cooling tower for a nuclear reactor is to be constructed in the shape of a hyperboloid of one sheet [see Figure 12(b)]. The diameter at the base is 280 m and the minimum diameter, 500 m above the base, is 200 m. Find an equation for the tower.

53. Show that if the point (a, b, c) lies on the hyperbolic paraboloid $z = y^2 - x^2$, then the lines with parametric equations $x = a + t$, $y = b + t$, $z = c + 2(b - a)t$ and $x = a + t$, $y = b - t$, $z = c - 2(b + a)t$ both lie entirely on this paraboloid. (This shows that the hyperbolic paraboloid is what is called a **ruled surface**; that is, it can be generated by the motion of a straight line. In fact, this exercise shows that through each point on the hyperbolic paraboloid there are two generating lines. The only other quadric surfaces that are ruled surfaces are cylinders, cones, and hyperboloids of one sheet.)

54. Show that the curve of intersection of the surfaces $x^2 + 2y^2 - z^2 + 3x = 1$ and $2x^2 + 4y^2 - 2z^2 - 5y = 0$ lies in a plane.

55. Graph the surfaces $z = x^2 + y^2$ and $z = 1 - y^2$ on a common screen using the domain $|x| \leqslant 1.2$, $|y| \leqslant 1.2$ and observe the curve of intersection of these surfaces. Show that the projection of this curve onto the xy-plane is an ellipse.

12 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. What is the difference between a vector and a scalar?

2. How do you add two vectors geometrically? How do you add them algebraically?

3. If **a** is a vector and c is a scalar, how is $c\mathbf{a}$ related to **a** geometrically? How do you find $c\mathbf{a}$ algebraically?

4. How do you find the vector from one point to another?

5. How do you find the dot product $\mathbf{a} \cdot \mathbf{b}$ of two vectors if you know their lengths and the angle between them? What if you know their components?

6. How are dot products useful?

7. Write expressions for the scalar and vector projections of **b** onto **a**. Illustrate with diagrams.

8. How do you find the cross product $\mathbf{a} \times \mathbf{b}$ of two vectors if you know their lengths and the angle between them? What if you know their components?

9. How are cross products useful?

10. (a) How do you find the area of the parallelogram determined by **a** and **b**?
(b) How do you find the volume of the parallelepiped determined by **a**, **b**, and **c**?

11. How do you find a vector perpendicular to a plane?

12. How do you find the angle between two intersecting planes?

13. Write a vector equation, parametric equations, and symmetric equations for a line.

14. Write a vector equation and a scalar equation for a plane.

15. (a) How do you tell if two vectors are parallel?
(b) How do you tell if two vectors are perpendicular?
(c) How do you tell if two planes are parallel?

16. (a) Describe a method for determining whether three points P, Q, and R lie on the same line.
(b) Describe a method for determining whether four points P, Q, R, and S lie in the same plane.

17. (a) How do you find the distance from a point to a line?
(b) How do you find the distance from a point to a plane?
(c) How do you find the distance between two lines?

18. What are the traces of a surface? How do you find them?

19. Write equations in standard form of the six types of quadric surfaces.

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$, then $\mathbf{u} \cdot \mathbf{v} = \langle u_1 v_1, u_2 v_2 \rangle$.

2. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3, $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$.

3. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3, $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}||\mathbf{v}|$.

4. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|$.

5. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

6. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3, $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$.

7. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{v} \times \mathbf{u}|$.

8. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3 and any scalar k,

$$k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$$

9. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3 and any scalar k,

$$k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v}$$

10. For any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V_3,

$$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$$

11. For any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V_3,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$$

12. For any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V_3,

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$$

13. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$.

14. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3, $(\mathbf{u} + \mathbf{v}) \times \mathbf{v} = \mathbf{u} \times \mathbf{v}$.

15. The vector $\langle 3, -1, 2 \rangle$ is parallel to the plane

$$6x - 2y + 4z = 1$$

16. A linear equation $Ax + By + Cz + D = 0$ represents a line in space.

17. The set of points $\{(x, y, z) \mid x^2 + y^2 = 1\}$ is a circle.

18. In $\mathbb{R}^3$ the graph of $y = x^2$ is a paraboloid.

19. If $\mathbf{u} \cdot \mathbf{v} = 0$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

20. If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

21. If $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

22. If $\mathbf{u}$ and $\mathbf{v}$ are in V_3, then $|\mathbf{u} \cdot \mathbf{v}| \leqslant |\mathbf{u}||\mathbf{v}|$.

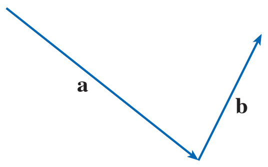
EXERCISES

1. (a) Find an equation of the sphere that passes through the point $(6, -2, 3)$ and has center $(-1, 2, 1)$.
(b) Find the curve in which this sphere intersects the yz-plane.
(c) Find the center and radius of the sphere

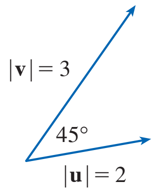
$$x^2 + y^2 + z^2 - 8x + 2y + 6z + 1 = 0$$

2. Copy the vectors in the figure and use them to draw each of the following vectors.

(a) $\mathbf{a} + \mathbf{b}$ (b) $\mathbf{a} - \mathbf{b}$
(c) $-\frac{1}{2}\mathbf{a}$ (d) $2\mathbf{a} + \mathbf{b}$



3. If $\mathbf{u}$ and $\mathbf{v}$ are the vectors shown in the figure, find $\mathbf{u} \cdot \mathbf{v}$ and $|\mathbf{u} \times \mathbf{v}|$. Is $\mathbf{u} \times \mathbf{v}$ directed into the page or out of it?



4. Calculate the given quantity if

$$\mathbf{a} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$$
$$\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$
$$\mathbf{c} = \mathbf{j} - 5\mathbf{k}$$

(a) $2\mathbf{a} + 3\mathbf{b}$ (b) $|\mathbf{b}|$
(c) $\mathbf{a} \cdot \mathbf{b}$ (d) $\mathbf{a} \times \mathbf{b}$

(e) $|\mathbf{b} \times \mathbf{c}|$ (f) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
(g) $\mathbf{c} \times \mathbf{c}$ (h) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$
(i) $\operatorname{comp}_{\mathbf{a}} \mathbf{b}$ (j) $\operatorname{proj}_{\mathbf{a}} \mathbf{b}$
(k) The angle between **a** and **b** (correct to the nearest degree)

5. Find the values of x such that the vectors $\langle 3, 2, x \rangle$ and $\langle 2x, 4, x \rangle$ are orthogonal.

6. Find two unit vectors that are orthogonal to both $\mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

7. Suppose that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 2$. Find the value of each of the following.
(a) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ (b) $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$
(c) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})$ (d) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v}$

8. Show that if **a**, **b**, and **c** are in V_3, then

$$(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})] = [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]^2$$

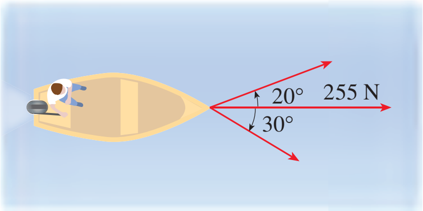
9. Find the acute angle between two diagonals of a cube.

10. Given the points $A(1, 0, 1)$, $B(2, 3, 0)$, $C(-1, 1, 4)$, and $D(0, 3, 2)$, find the volume of the parallelepiped with adjacent edges AB, AC, and AD.

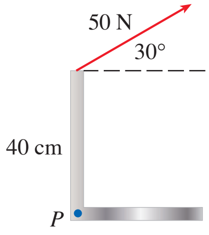
11. (a) Find a vector perpendicular to the plane through the points $A(1, 0, 0)$, $B(2, 0, -1)$, and $C(1, 4, 3)$.
(b) Find the area of triangle ABC.

12. A constant force $\mathbf{F} = 3\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$ moves an object along the line segment from $(1, 0, 2)$ to $(5, 3, 8)$. Find the work done if the distance is measured in meters and the force in newtons.

13. A boat is pulled onto shore using two ropes, as shown in the diagram. If a force of 255 N is needed, find the magnitude of the force in each rope.



14. Find the magnitude of the torque about P if a 50-N force is applied as shown.



15–17 Find parametric equations for the line.

15. The line through $(4, -1, 2)$ and $(1, 1, 5)$

16. The line through $(1, 0, -1)$ and parallel to the line $\frac{1}{3}(x - 4) = \frac{1}{2}y = z + 2$

17. The line through $(-2, 2, 4)$ and perpendicular to the plane $2x - y + 5z = 12$

18–20 Find an equation of the plane.

18. The plane through $(2, 1, 0)$ and parallel to $x + 4y - 3z = 1$

19. The plane through $(3, -1, 1)$, $(4, 0, 2)$, and $(6, 3, 1)$

20. The plane through $(1, 2, -2)$ that contains the line $x = 2t$, $y = 3 - t$, $z = 1 + 3t$

21. Find the point in which the line with parametric equations $x = 2 - t$, $y = 1 + 3t$, $z = 4t$ intersects the plane $2x - y + z = 2$.

22. Find the distance from the origin to the line $x = 1 + t$, $y = 2 - t$, $z = -1 + 2t$.

23. Determine whether the lines given by the symmetric equations

$$\frac{x - 1}{2} = \frac{y - 2}{3} = \frac{z - 3}{4}$$

and

$$\frac{x + 1}{6} = \frac{y - 3}{-1} = \frac{z + 5}{2}$$

are parallel, skew, or intersecting.

24. (a) Show that the planes $x + y - z = 1$ and $2x - 3y + 4z = 5$ are neither parallel nor perpendicular.
(b) Find, correct to the nearest degree, the angle between these planes.

25. Find an equation of the plane through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$ and perpendicular to the plane $x + y - 2z = 1$.

26. (a) Find an equation of the plane that passes through the points $A(2, 1, 1)$, $B(-1, -1, 10)$, and $C(1, 3, -4)$.
(b) Find symmetric equations for the line through B that is perpendicular to the plane in part (a).
(c) A second plane passes through $(2, 0, 4)$ and has normal vector $\langle 2, -4, -3 \rangle$. Show that the acute angle between the planes is approximately 43°.
(d) Find parametric equations for the line of intersection of the two planes.

27. Find the distance between the planes $3x + y - 4z = 2$ and $3x + y - 4z = 24$.

28–36 Identify and sketch the graph of each surface.

28. $x = 3$

29. $x = z$

30. $y = z^2$

31. $x^2 = y^2 + 4z^2$

32. $4x - y + 2z = 4$

33. $-4x^2 + y^2 - 4z^2 = 4$

34. $y^2 + z^2 = 1 + x^2$

35. $4x^2 + 4y^2 - 8y + z^2 = 0$

36. $x = y^2 + z^2 - 2y - 4z + 5$

---

37. An ellipsoid is created by rotating the ellipse $4x^2 + y^2 = 16$ about the x-axis. Find an equation of the ellipsoid.

38. A surface consists of all points P such that the distance from P to the plane $y = 1$ is twice the distance from P to the point $(0, -1, 0)$. Find an equation for this surface and identify it.

Problems Plus

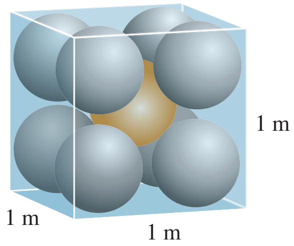


FIGURE FOR PROBLEM 1

1. Each edge of a cubical box has length 1 m. The box contains nine spherical balls with the same radius r. The center of one ball is at the center of the cube and it touches the other eight balls. Each of the other eight balls touches three sides of the box. Thus the balls are tightly packed in the box (see the figure). Find r. (If you have trouble with this problem, read about the problem-solving strategy entitled *Use Analogy* in Principles of Problem Solving following Chapter 1.)

2. Let B be a solid box with length L, width W, and height H. Let S be the set of all points that are a distance at most 1 from some point of B. Express the volume of S in terms of L, W, and H.

3. Let L be the line of intersection of the planes $cx + y + z = c$ and $x - cy + cz = -1$, where c is a real number.

(a) Find symmetric equations for L.

(b) As the number c varies, the line L sweeps out a surface S. Find an equation for the curve of intersection of S with the horizontal plane $z = t$ (the trace of S in the plane $z = t$).

(c) Find the volume of the solid bounded by S and the planes $z = 0$ and $z = 1$.

4. A plane is capable of flying at a speed of 180 km/h in still air. The pilot takes off from an airfield and heads due north according to the plane's compass. After 30 minutes of flight time, the pilot notices that, due to the wind, the plane has actually traveled 80 km in the direction N 5° E.

(a) What is the wind velocity?

(b) In what direction should the pilot have headed to reach the intended destination?

5. Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors with $|\mathbf{v}_1| = 2$, $|\mathbf{v}_2| = 3$, and $\mathbf{v}_1 \cdot \mathbf{v}_2 = 5$. Let $\mathbf{v}_3 = \text{proj}_{\mathbf{v}_1} \mathbf{v}_2$, $\mathbf{v}_4 = \text{proj}_{\mathbf{v}_2} \mathbf{v}_3$, $\mathbf{v}_5 = \text{proj}_{\mathbf{v}_3} \mathbf{v}_4$, and so on. Compute $\sum_{n=1}^{\infty} |\mathbf{v}_n|$.

6. Find an equation of the largest sphere that passes through the point $(-1, 1, 4)$ and is such that each of the points (x, y, z) inside the sphere satisfies the condition

$$x^2 + y^2 + z^2 < 136 + 2(x + 2y + 3z)$$

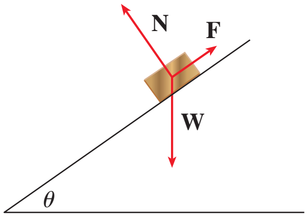


FIGURE FOR PROBLEM 7

7. Suppose a block of mass m is placed on an inclined plane, as shown in the figure. The block's descent down the plane is slowed by friction; if θ is not too large, friction will prevent the block from moving at all. The forces acting on the block are the weight $\mathbf{W}$, where $|\mathbf{W}| = mg$ (g is the acceleration due to gravity); the normal force $\mathbf{N}$ (the normal component of the reactionary force of the plane on the block), where $|\mathbf{N}| = n$; and the force $\mathbf{F}$ due to friction, which acts parallel to the inclined plane, opposing the direction of motion. If the block is at rest and θ is increased, $|\mathbf{F}|$ must also increase until ultimately $|\mathbf{F}|$ reaches its maximum, beyond which the block begins to slide. At this angle θ_s, it has been observed that $|\mathbf{F}|$ is proportional to n. Thus, when $|\mathbf{F}|$ is maximal, we can say that $|\mathbf{F}| = \mu_s n$, where μ_s is called the *coefficient of static friction* and depends on the materials that are in contact.

(a) Observe that $\mathbf{N} + \mathbf{F} + \mathbf{W} = \mathbf{0}$ and deduce that $\mu_s = \tan \theta_s$.

(b) Suppose that, for $\theta > \theta_s$, an additional outside force $\mathbf{H}$ is applied to the block, horizontally from the left, and let $|\mathbf{H}| = h$. If h is small, the block may still slide down the plane; if h is large enough, the block will move up the plane. Let $h_{\min}$ be the smallest value of h that allows the block to remain motionless (so that $|\mathbf{F}|$ is maximal).

By choosing the coordinate axes so that $\mathbf{F}$ lies along the x-axis, resolve each force into components parallel and perpendicular to the inclined plane and show that

$$h_{\min} \sin\theta + mg\cos\theta = n \qquad \text{and} \qquad h_{\min}\cos\theta + \mu_s n = mg\sin\theta$$

(c) Show that

$$h_{\min} = mg\tan(\theta - \theta_s)$$

Does this equation seem reasonable? Does it make sense for $\theta = \theta_s$? Does it make sense as $\theta \to 90°$? Explain.

(d) Let h_{max} be the largest value of h that allows the block to remain motionless. (In which direction is **F** heading?) Show that

$$h_{\text{max}} = mg\tan(\theta + \theta_s)$$

Does this equation seem reasonable? Explain.

8. A solid has the following properties. When illuminated by rays parallel to the z-axis, its shadow is a circular disk. If the rays are parallel to the y-axis, its shadow is a square. If the rays are parallel to the x-axis, its shadow is an isosceles triangle. (In Exercise 12.1.52 you were asked to describe and sketch an example of such a solid, but there are many such solids.) Assume that the projection onto the xz-plane is a square whose sides have length 1.

(a) What is the volume of the largest such solid?

(b) Is there a smallest volume?

The paths of objects moving through space—like the planes pictured here—can be described by vector functions. In Section 13.1 we will see how to use these vector functions to determine whether or not two such objects will collide.

Magdalena Zeglen / EyeEm / Getty Images

13

Vector Functions

THE FUNCTIONS THAT WE HAVE been using so far have been real-valued functions. We now study functions whose values are vectors because such functions are needed to describe curves and surfaces in space. We will also use vector-valued functions to describe the motion of objects through space. In particular, we will use them to derive Kepler's laws of planetary motion.

13.1 Vector Functions and Space Curves

Vector-Valued Functions

In general, a function is a rule that assigns to each element in the domain an element in the range. A **vector-valued function**, or **vector function**, is simply a function whose domain is a set of real numbers and whose range is a set of vectors. We are most interested in vector functions $\mathbf{r}$ whose values are three-dimensional vectors. This means that for every number t in the domain of $\mathbf{r}$ there is a unique vector in V_3 denoted by $\mathbf{r}(t)$. If $f(t)$, $g(t)$, and $h(t)$ are the components of the vector $\mathbf{r}(t)$, then f, g, and h are real-valued functions called the **component functions** of $\mathbf{r}$ and we can write

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$$

We use the letter t to denote the independent variable because it represents time in most applications of vector functions.

EXAMPLE 1 If

$$\mathbf{r}(t) = \left\langle t^3, \ln(3 - t), \sqrt{t}\right\rangle$$

then the component functions are

$$f(t) = t^3 \qquad g(t) = \ln(3 - t) \qquad h(t) = \sqrt{t}$$

By our usual convention, the domain of $\mathbf{r}$ consists of all values of t for which the expression for $\mathbf{r}(t)$ is defined. The expressions t^3, $\ln(3 - t)$, and $\sqrt{t}$ are all defined when $3 - t > 0$ and $t \geqslant 0$. Therefore the domain of $\mathbf{r}$ is the interval $[0, 3)$. ■

Limits and Continuity

The **limit** of a vector function $\mathbf{r}$ is defined by taking the limits of its component functions as follows.

If $\lim_{t\to a}\mathbf{r}(t) = \mathbf{L}$, this definition is equivalent to saying that the length and direction of the vector $\mathbf{r}(t)$ approach the length and direction of the vector $\mathbf{L}$.

1 If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, then

$$\lim_{t\to a}\mathbf{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right\rangle$$

provided the limits of the component functions exist.

Equivalently, we could have used an ε-δ definition (see Exercise 62). Limits of vector functions obey the same rules as limits of real-valued functions (see Exercise 61).

EXAMPLE 2 Find $\lim_{t\to 0}\mathbf{r}(t)$, where $\mathbf{r}(t) = (1 + t^3)\,\mathbf{i} + te^{-t}\,\mathbf{j} + \dfrac{\sin t}{t}\,\mathbf{k}$.

SOLUTION According to Definition 1, the limit of $\mathbf{r}$ is the vector whose components are the limits of the component functions of $\mathbf{r}$:

$$\lim_{t\to 0}\mathbf{r}(t) = \left[\lim_{t\to 0}(1 + t^3)\right]\mathbf{i} + \left[\lim_{t\to 0} te^{-t}\right]\mathbf{j} + \left[\lim_{t\to 0}\frac{\sin t}{t}\right]\mathbf{k}$$

$$= \mathbf{i} + \mathbf{k} \qquad \text{(by Equation 3.3.5)}$$ ■

A vector function $\mathbf{r}$ is **continuous at *a*** if

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$$

In view of Definition 1, we see that $\mathbf{r}$ is continuous at a if and only if its component functions f, g, and h are continuous at a.

Space Curves

There is a close connection between continuous vector functions and space curves. Suppose that f, g, and h are continuous real-valued functions on an interval I. Then the set C of all points (x, y, z) in space, where

$$\boxed{2} \qquad x = f(t) \qquad y = g(t) \qquad z = h(t)$$

and t varies throughout the interval I, is called a **space curve**. The equations in (2) are called **parametric equations of *C*** and t is called a **parameter**. We can think of C as being traced out by a moving particle whose position at time t is $(f(t), g(t), h(t))$. If we now consider the vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then $\mathbf{r}(t)$ is the position vector of the point $P(f(t), g(t), h(t))$ on C. Thus any continuous vector function $\mathbf{r}$ defines a space curve C that is traced out by the tip of the moving vector $\mathbf{r}(t)$, as shown in Figure 1.

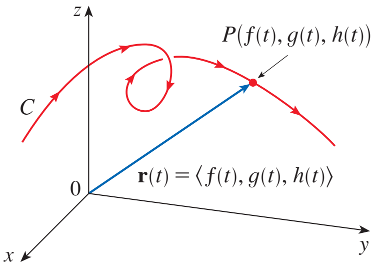


FIGURE 1
C is traced out by the tip of a moving position vector $\mathbf{r}(t)$.

EXAMPLE 3 Describe the curve defined by the vector function

$$\mathbf{r}(t) = \langle 1 + t, 2 + 5t, -1 + 6t \rangle$$

SOLUTION The corresponding parametric equations are

$$x = 1 + t \qquad y = 2 + 5t \qquad z = -1 + 6t$$

which we recognize from Equations 12.5.2 as parametric equations of a line passing through the point $(1, 2, -1)$ and parallel to the vector $\langle 1, 5, 6 \rangle$. Alternatively, we could observe that the function can be written as $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, where $\mathbf{r}_0 = \langle 1, 2, -1 \rangle$ and $\mathbf{v} = \langle 1, 5, 6 \rangle$, and this is the vector equation of a line as given by Equation 12.5.1. ■

Plane curves can also be represented in vector notation. For instance, the curve given by the parametric equations $x = t^2 - 2t$ and $y = t + 1$ (see Example 10.1.1) could also be described by the vector equation

$$\mathbf{r}(t) = \langle t^2 - 2t, t + 1 \rangle = (t^2 - 2t)\,\mathbf{i} + (t + 1)\,\mathbf{j}$$

where $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.

EXAMPLE 4 Sketch the curve whose vector equation is

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

SOLUTION The parametric equations for this curve are

$$x = \cos t \qquad y = \sin t \qquad z = t$$

Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$ for all values of t, the curve must lie on the circular cylinder $x^2 + y^2 = 1$. The point (x, y, z) lies directly above the point $(x, y, 0)$, which moves counterclockwise around the circle $x^2 + y^2 = 1$ in the xy-plane. (The projection of the curve onto the xy-plane has vector equation $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$. See Example 10.1.2.) Since $z = t$, the curve spirals upward around the cylinder as t increases. The curve, shown in Figure 2, is called a **helix**. ■

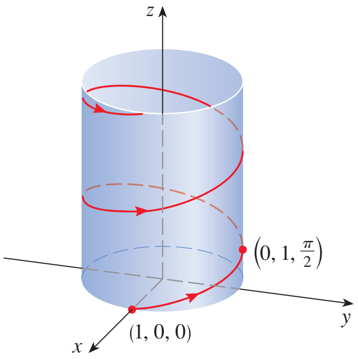


FIGURE 2

FIGURE 3
A double helix

The corkscrew shape of the helix in Example 4 is familiar from its occurrence in coiled springs. It also occurs in the model of DNA (deoxyribonucleic acid, the genetic material of living cells). In 1953 James Watson and Francis Crick showed that the structure of the DNA molecule is that of two linked, parallel helixes that are intertwined as in Figure 3.

In Examples 3 and 4 we were given vector equations of curves and asked for a geometric description or sketch. In the next three examples we are given a geometric description of a curve and are asked to find parametric equations for the curve.

EXAMPLE 5 Find a vector equation and parametric equations for the line segment that joins the point $P(1, 3, -2)$ to the point $Q(2, -1, 3)$.

SOLUTION In Section 12.5 we found a vector equation for the line segment that joins the tip of the vector $\mathbf{r}_0$ to the tip of the vector $\mathbf{r}_1$:

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \leqslant t \leqslant 1$$

(See Equation 12.5.4.) Here we take $\mathbf{r}_0 = \langle 1, 3, -2\rangle$ and $\mathbf{r}_1 = \langle 2, -1, 3\rangle$ to obtain a vector equation of the line segment from P to Q:

$$\mathbf{r}(t) = (1 - t)\,\langle 1, 3, -2\rangle + t\langle 2, -1, 3\rangle \qquad 0 \leqslant t \leqslant 1$$

or $$\mathbf{r}(t) = \langle 1 + t, 3 - 4t, -2 + 5t\rangle \qquad 0 \leqslant t \leqslant 1$$

Figure 4 shows the line segment PQ in Example 5.

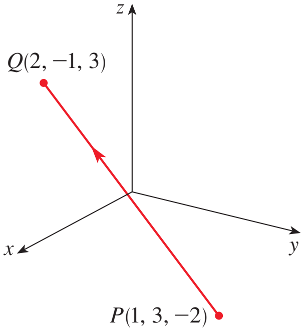


FIGURE 4

The corresponding parametric equations are

$$x = 1 + t \qquad y = 3 - 4t \qquad z = -2 + 5t \qquad 0 \leqslant t \leqslant 1$$ ■

EXAMPLE 6 Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

SOLUTION Figure 5 shows how the plane and the cylinder intersect, and Figure 6 shows the curve of intersection C, which is an ellipse.

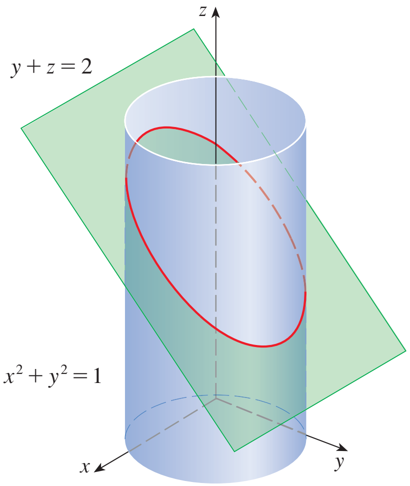


FIGURE 5

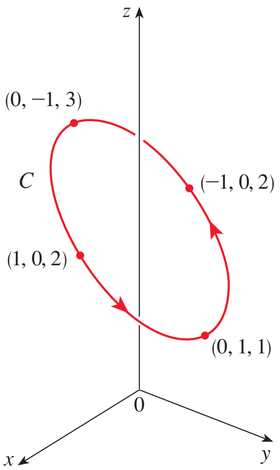


FIGURE 6

The projection of C onto the xy-plane is the circle $x^2 + y^2 = 1$, $z = 0$. So we know from Example 10.1.2 that we can write

$$x = \cos t \qquad y = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

From the equation of the plane, we have

$$z = 2 - y = 2 - \sin t$$

So we can write parametric equations for C as

$$x = \cos t \qquad y = \sin t \qquad z = 2 - \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

The corresponding vector equation is

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + (2 - \sin t)\,\mathbf{k} \qquad 0 \leqslant t \leqslant 2\pi$$

This equation is called a *parametrization* of the curve C. The arrows in Figure 6 indicate the direction in which C is traced as the parameter t increases. ■

Figure 7 shows the surfaces of Example 7 and their curve of intersection.

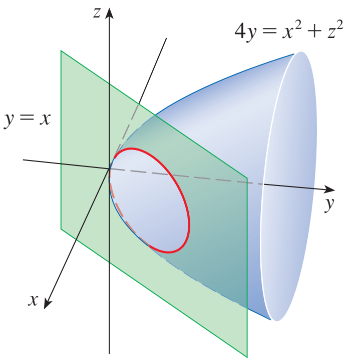


FIGURE 7

EXAMPLE 7 Find parametric equations for the curve of intersection of the paraboloid $4y = x^2 + z^2$ and the plane $y = x$.

SOLUTION Because any point on the curve C of intersection satisfies the equations of both surfaces, we can substitute $y = x$ into the equation of the paraboloid, giving $4x = x^2 + z^2$. Completing the square in x gives $(x - 2)^2 + z^2 = 4$, so C must be contained in the circular cylinder $(x - 2)^2 + z^2 = 4$, and the projection of C onto the xz-plane is the circle $(x - 2)^2 + z^2 = 4$, $y = 0$ [with center $(2, 0, 0)$ and radius 2]. From Example 10.1.4, we can write $x = 2 + 2\cos t$, $z = 2\sin t$, $0 \leqslant t \leqslant 2\pi$, and because $y = x$, parametric equations for C are

$$x = 2 + 2\cos t \qquad y = 2 + 2\cos t \qquad z = 2\sin t \qquad 0 \leqslant t \leqslant 2\pi$$ ■

■ Using Technology to Draw Space Curves

Space curves are inherently more difficult to draw by hand than plane curves; for an accurate representation we need to use technology. For instance, Figure 8 shows a computer-generated graph of the curve with parametric equations

$$x = (4 + \sin 20t)\cos t \qquad y = (4 + \sin 20t)\sin t \qquad z = \cos 20t$$

It's called a **toroidal spiral** because it lies on a torus. Another interesting curve, the **trefoil knot**, with equations

$$x = (2 + \cos 1.5t)\cos t \qquad y = (2 + \cos 1.5t)\sin t \qquad z = \sin 1.5t$$

is graphed in Figure 9. It wouldn't be easy to plot either of these curves by hand.

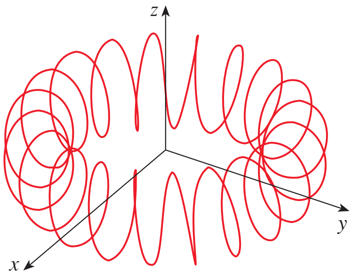


FIGURE 8
A toroidal spiral

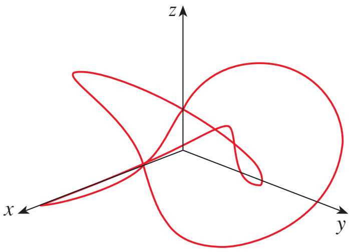
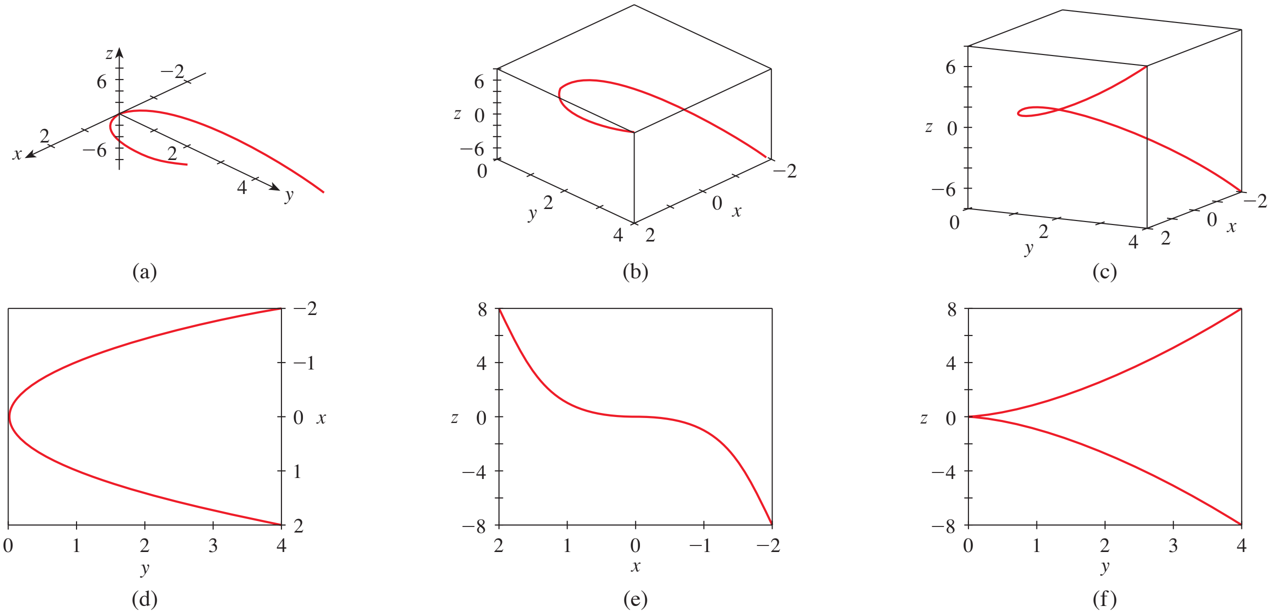


FIGURE 9
A trefoil knot

Even when a computer is used to draw a space curve, optical illusions make it difficult to get a good impression of what the curve really looks like. (This is especially true in Figure 9. See Exercise 60.) The next example shows how to cope with this problem.

EXAMPLE 8 Use a calculator or computer to draw the curve with vector equation $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$. This curve is called a **twisted cubic**.

SOLUTION We start by plotting the curve with parametric equations $x = t$, $y = t^2$, $z = t^3$ for $-2 \leqslant t \leqslant 2$. The result is shown in Figure 10(a), but it's hard to see the true nature of the curve from that graph alone. Some three-dimensional graphing software allows the user to enclose a curve or surface in a box instead of displaying the coordinate axes. When we look at the same curve in a box in Figure 10(b), we have a much clearer picture of the curve. We can see that it climbs from a lower corner of the box to the upper corner nearest us, and it twists as it climbs.



(a) (b) (c)

(d) (e) (f)

FIGURE 10 Views of the twisted cubic

We get an even better idea of the curve when we view it from different vantage points. Part (c) shows the result of rotating the box to give another viewpoint. Parts (d), (e), and (f) show the views we get when we look directly at a face of the box. In particular, part (d) shows the view from directly above the box. It is the projection of the curve onto the xy-plane, namely, the parabola $y = x^2$. Part (e) shows the projection onto the xz-plane, the cubic curve $z = x^3$. It's now obvious why the given curve is called a twisted cubic. ■

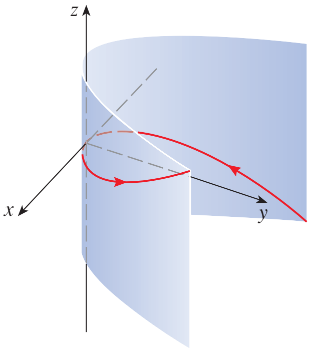


FIGURE 11

Another method of visualizing a space curve is to draw it on a surface. For instance, the twisted cubic in Example 8 lies on the parabolic cylinder $y = x^2$. (Eliminate the parameter from the first two parametric equations, $x = t$ and $y = t^2$.) Figure 11 shows both the cylinder and the twisted cubic, and we see that the curve moves upward through the origin along the surface of the cylinder. We also used this method in Example 4 to visualize the helix lying on the circular cylinder (see Figure 2).

A third method for visualizing the twisted cubic is to realize that it also lies on the cylinder $z = x^3$. So it can be viewed as the curve of intersection of the cylinders $y = x^2$ and $z = x^3$. (See Figure 12.)

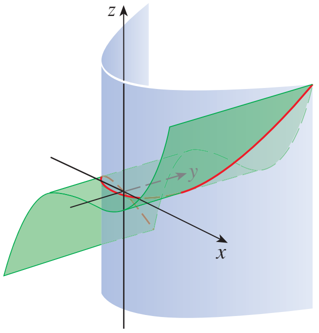
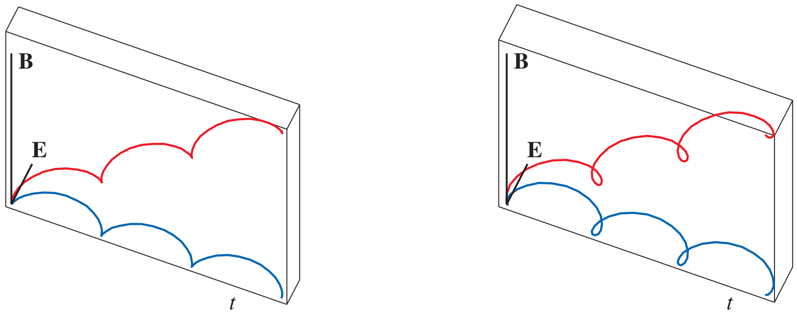


FIGURE 12

Some graphing software provides us with a clearer picture of a space curve by enclosing it in a tube. Such a plot enables us to see whether one part of a curve passes in front of or behind another part of the curve. For example, Figure 14 shows the curve of Figure 13(b) as rendered by the `tubeplot` command in Maple.

We have seen that an interesting space curve, the helix, occurs in the model of DNA. Another notable example of a space curve in science is the trajectory of a positively charged particle in orthogonally oriented electric and magnetic fields **E** and **B**. Depending on the initial velocity given the particle at the origin, the path of the particle is either a space curve whose projection onto the horizontal plane is the cycloid we studied in Section 10.1 [Figure 13(a)] or a curve whose projection is the trochoid investigated in Exercise 10.1.49 [Figure 13(b)].



(a) $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t, t \rangle$

(b) $\mathbf{r}(t) = \left\langle t - \frac{3}{2} \sin t, 1 - \frac{3}{2} \cos t, t \right\rangle$

FIGURE 13
Motion of a charged particle in orthogonally oriented electric and magnetic fields

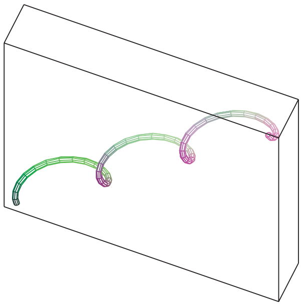


FIGURE 14

For further details concerning the physics involved and animations of the trajectories of the particles, see the following websites:

- www.physics.ucla.edu/plasma-exp/Beam/
- www.phy.ntnu.edu.tw/ntnujava/index.php?topic=36

13.1 Exercises

1–2 Find the domain of the vector function.

1. $\mathbf{r}(t) = \left\langle \ln(t + 1), \dfrac{t}{\sqrt{9 - t^2}}, 2^t \right\rangle$

2. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \ln t\,\mathbf{j} + \dfrac{1}{t - 2}\mathbf{k}$

3–6 Find the limit.

3. $\displaystyle\lim_{t \to 0} \left(e^{-3t}\,\mathbf{i} + \frac{t^2}{\sin^2 t}\,\mathbf{j} + \cos 2t\,\mathbf{k} \right)$

4. $\displaystyle\lim_{t \to 1} \left(\frac{t^2 - t}{t - 1}\,\mathbf{i} + \sqrt{t + 8}\,\mathbf{j} + \frac{\sin \pi t}{\ln t}\,\mathbf{k} \right)$

5. $\lim_{t\to\infty}\left\langle \dfrac{1+t^2}{1-t^2}, \tan^{-1}t, \dfrac{1-e^{-2t}}{t}\right\rangle$

6. $\lim_{t\to\infty}\left\langle te^{-t}, \dfrac{t^3+t}{2t^3-1}, t\sin\dfrac{1}{t}\right\rangle$

7–16 Sketch the curve with the given vector equation. Indicate with an arrow the direction in which t increases.

7. $\mathbf{r}(t) = \langle -\cos t, t\rangle$

8. $\mathbf{r}(t) = \langle t^2 - 1, t\rangle$

9. $\mathbf{r}(t) = \langle 3\sin t, 2\cos t\rangle$

10. $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}$

11. $\mathbf{r}(t) = \langle t, 2-t, 2t\rangle$

12. $\mathbf{r}(t) = \langle \sin \pi t, t, \cos \pi t\rangle$

13. $\mathbf{r}(t) = \langle 3, t, 2-t^2\rangle$

14. $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \mathbf{k}$

15. $\mathbf{r}(t) = t^2\,\mathbf{i} + t^4\,\mathbf{j} + t^6\,\mathbf{k}$

16. $\mathbf{r}(t) = \cos t\,\mathbf{i} - \cos t\,\mathbf{j} + \sin t\,\mathbf{k}$

17–18 Draw the projection of the curve onto the given plane.

17. $\mathbf{r}(t) = \langle t^2, t^3, t^{-3}\rangle$, yz-plane

18. $\mathbf{r}(t) = \langle t+1, 3t+1, \cos(t/2)\rangle$, xy-plane

---

19–20 Draw the projections of the curve onto the three coordinate planes. Use these projections to help sketch the curve.

19. $\mathbf{r}(t) = \langle t, \sin t, 2\cos t\rangle$

20. $\mathbf{r}(t) = \langle t, t, t^2\rangle$

21–24 Find a vector equation and parametric equations for the line segment that joins P to Q.

21. $P(-2, 1, 0)$, $Q(5, 2, -3)$

22. $P(0, 0, 0)$, $Q(-7, 4, 6)$

23. $P(3.5, -1.4, 2.1)$, $Q(1.8, 0.3, 2.1)$

24. $P(a, b, c)$, $Q(u, v, w)$

25–30 Match the parametric equations with the graphs (labeled I–VI). Give reasons for your choices.

25. $x = t\cos t$, $y = t$, $z = t\sin t$, $t \geqslant 0$

26. $x = \cos t$, $y = \sin t$, $z = 1/(1+t^2)$

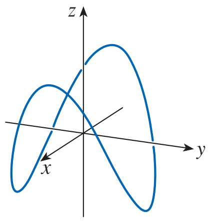
27. $x = t$, $y = 1/(1+t^2)$, $z = t^2$

28. $x = \cos t$, $y = \sin t$, $z = \cos 2t$

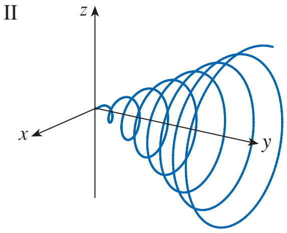
29. $x = \cos 8t$, $y = \sin 8t$, $z = e^{0.8t}$, $t \geqslant 0$

30. $x = \cos^2 t$, $y = \sin^2 t$, $z = t$

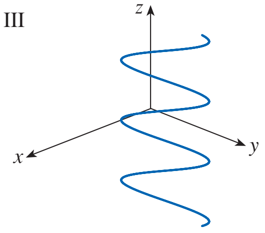
I



II



III



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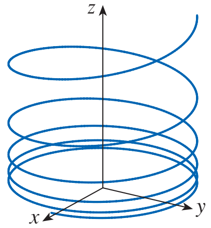
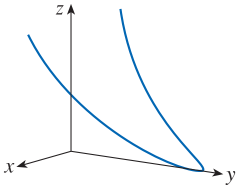
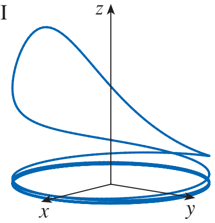


V



VI



31–34 Find an equation of the plane that contains the curve with the given vector equation.

31. $\mathbf{r}(t) = \langle t, 4, t^2\rangle$

32. $\mathbf{r}(t) = \langle t, t^2, t\rangle$

33. $\mathbf{r}(t) = \langle \sin t, \cos t, -\cos t\rangle$

34. $\mathbf{r}(t) = \langle 2t, \sin t, t+1\rangle$

---

35. Show that the curve with parametric equations $x = t\cos t$, $y = t\sin t$, $z = t$ lies on the cone $z^2 = x^2 + y^2$, and use this fact to help sketch the curve.

36. Show that the curve with parametric equations $x = \sin t$, $y = \cos t$, $z = \sin^2 t$ is the curve of intersection of the surfaces $z = x^2$ and $x^2 + y^2 = 1$. Use this fact to help sketch the curve.

37. Find three different surfaces that contain the curve

$$\mathbf{r}(t) = 2t\,\mathbf{i} + e^t\,\mathbf{j} + e^{2t}\,\mathbf{k}$$

38. Find three different surfaces that contain the curve

$$\mathbf{r}(t) = t^2\,\mathbf{i} + \ln t\,\mathbf{j} + (1/t)\,\mathbf{k}$$

39. At what points does the curve $\mathbf{r}(t) = t\,\mathbf{i} + (2t - t^2)\,\mathbf{k}$ intersect the paraboloid $z = x^2 + y^2$?

40. At what points does the helix $\mathbf{r}(t) = \langle \sin t, \cos t, t\rangle$ intersect the sphere $x^2 + y^2 + z^2 = 5$?

41–45 Graph the curve with the given vector equation. Make sure you choose a parameter domain and viewpoints that reveal the true nature of the curve.

41. $\mathbf{r}(t) = \langle \cos t \sin 2t, \sin t \sin 2t, \cos 2t \rangle$

42. $\mathbf{r}(t) = \langle te^t, e^{-t}, t \rangle$

43. $\mathbf{r}(t) = \left\langle \sin 3t \cos t, \frac{1}{4}t, \sin 3t \sin t \right\rangle$

44. $\mathbf{r}(t) = \langle \cos(8 \cos t) \sin t, \sin(8 \cos t) \sin t, \cos t \rangle$

45. $\mathbf{r}(t) = \langle \cos 2t, \cos 3t, \cos 4t \rangle$

---

46. Graph the curve with parametric equations

$$x = \sin t \qquad y = \sin 2t \qquad z = \cos 4t$$

Explain its shape by graphing its projections onto the three coordinate planes.

47. Graph the curve with parametric equations

$$x = (1 + \cos 16t) \cos t$$
$$y = (1 + \cos 16t) \sin t$$
$$z = 1 + \cos 16t$$

Explain the appearance of the graph by showing that it lies on a cone.

48. Graph the curve with parametric equations

$$x = \sqrt{1 - 0.25 \cos^2 10t} \, \cos t$$
$$y = \sqrt{1 - 0.25 \cos^2 10t} \, \sin t$$
$$z = 0.5 \cos 10t$$

Explain the appearance of the graph by showing that it lies on a sphere.

49. Show that the curve with parametric equations $x = t^2$, $y = 1 - 3t$, $z = 1 + t^3$ passes through the points $(1, 4, 0)$ and $(9, -8, 28)$ but not through the point $(4, 7, -6)$.

50–54 Find a vector function that represents the curve of intersection of the two surfaces.

50. The cylinder $x^2 + y^2 = 4$ and the surface $z = xy$

51. The cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1 + y$

52. The paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$

53. The hyperbolic paraboloid $z = x^2 - y^2$ and the cylinder $x^2 + y^2 = 1$

54. The semiellipsoid $x^2 + y^2 + 4z^2 = 4$, $y \geqslant 0$, and the cylinder $x^2 + z^2 = 1$

---

55. Try to sketch by hand the curve of intersection of the circular cylinder $x^2 + y^2 = 4$ and the parabolic cylinder $z = x^2$. Then find parametric equations for this curve and use these equations and a computer to graph the curve.

56. Try to sketch by hand the curve of intersection of the parabolic cylinder $y = x^2$ and the top half of the ellipsoid $x^2 + 4y^2 + 4z^2 = 16$. Then find parametric equations for this curve and use these equations and a computer to graph the curve.

57–58 **Intersection and Collision** If two objects travel through space along two different curves, it's often important to know whether they will collide. (Will a missile hit its moving target? Will two aircraft collide?) Their paths might intersect, but we need to know whether the objects are in the same position *at the same time.* (See Exercises 10.1.55–57.)

57. The trajectories of two particles are given by the vector functions

$$\mathbf{r}_1(t) = \langle t^2, 7t - 12, t^2 \rangle \qquad \mathbf{r}_2(t) = \langle 4t - 3, t^2, 5t - 6 \rangle$$

for $t \geqslant 0$. Do the particles collide?

58. Two particles travel along the space curves

$$\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle \qquad \mathbf{r}_2(t) = \langle 1 + 2t, 1 + 6t, 1 + 14t \rangle$$

Do the particles collide? Do their paths intersect?

---

59. (a) Graph the curve with parametric equations

$$x = \tfrac{27}{26} \sin 8t - \tfrac{8}{39} \sin 18t$$
$$y = -\tfrac{27}{26} \cos 8t + \tfrac{8}{39} \cos 18t$$
$$z = \tfrac{144}{65} \sin 5t$$

(b) Show that the curve lies on the hyperboloid of one sheet $144x^2 + 144y^2 - 25z^2 = 100$.

60. **Trefoil Knot** The view of the trefoil knot shown in Figure 9 is accurate, but it doesn't reveal the whole story. Use the parametric equations

$$x = (2 + \cos 1.5t) \cos t$$
$$y = (2 + \cos 1.5t) \sin t$$
$$z = \sin 1.5t$$

to sketch the curve by hand as viewed from above, with gaps indicating where the curve passes over itself. Start by showing that the projection of the curve onto the xy-plane has polar coordinates $r = 2 + \cos 1.5t$ and $\theta = t$, so r varies between 1 and 3. Then show that z has maximum and minimum values when the projection is halfway between $r = 1$ and $r = 3$.

When you have finished your sketch, use a computer to draw the curve with viewpoint directly above and compare with your sketch. Then plot the curve from several other viewpoints. You can get a better impression of the curve if you plot a tube with radius 0.2 around the curve. (Use the `tubeplot` command in Maple or the `tubecurve` or `Tube` command in Mathematica.)

61. Properties of Limits Suppose **u** and **v** are vector functions that possess limits as $t \to a$ and let c be a constant. Prove the following properties of limits.

(a) $\lim_{t\to a} [\mathbf{u}(t) + \mathbf{v}(t)] = \lim_{t\to a} \mathbf{u}(t) + \lim_{t\to a} \mathbf{v}(t)$

(b) $\lim_{t\to a} c\mathbf{u}(t) = c \lim_{t\to a} \mathbf{u}(t)$

(c) $\lim_{t\to a} [\mathbf{u}(t) \cdot \mathbf{v}(t)] = \lim_{t\to a} \mathbf{u}(t) \cdot \lim_{t\to a} \mathbf{v}(t)$

(d) $\lim_{t\to a} [\mathbf{u}(t) \times \mathbf{v}(t)] = \lim_{t\to a} \mathbf{u}(t) \times \lim_{t\to a} \mathbf{v}(t)$

62. Show that $\lim_{t\to a} \mathbf{r}(t) = \mathbf{b}$ if and only if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } 0 < |t - a| < \delta \quad \text{then} \quad |\mathbf{r}(t) - \mathbf{b}| < \varepsilon$$

13.2 Derivatives and Integrals of Vector Functions

Later in this chapter we are going to use vector functions to describe the motion of planets and other objects through space. Here we prepare the way by developing the calculus of vector functions.

Derivatives

The derivative $\mathbf{r}'$ of a vector function $\mathbf{r}$ is defined in much the same way as for real-valued functions:

$$\boxed{1} \qquad \boxed{\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}}$$

if this limit exists. The geometric significance of this definition is shown in Figure 1. If the points P and Q have position vectors $\mathbf{r}(t)$ and $\mathbf{r}(t+h)$, then $\overrightarrow{PQ}$ represents the vector $\mathbf{r}(t+h) - \mathbf{r}(t)$, which can therefore be regarded as a secant vector. If $h > 0$, the scalar multiple $(1/h)(\mathbf{r}(t+h) - \mathbf{r}(t))$ has the same direction as $\mathbf{r}(t+h) - \mathbf{r}(t)$. As $h \to 0$, it appears that this vector approaches a vector that lies on the tangent line. For this reason, the vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve defined by $\mathbf{r}$ at the point P, provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \mathbf{0}$. The **tangent line** to C at P is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t)$.

Notice that when $0 < h < 1$, multiplying the secant vector by $1/h$ *stretches* the vector, as shown in Figure 1(b).

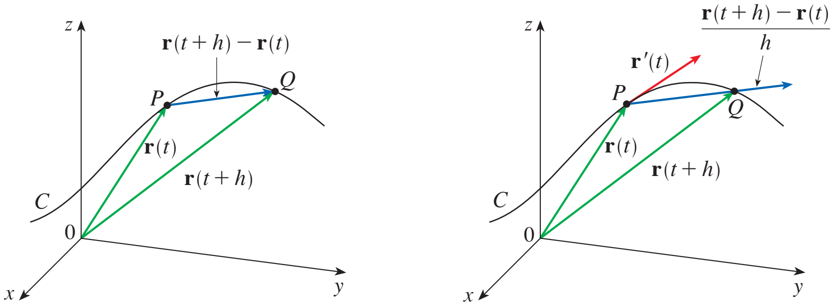


FIGURE 1

(a) The secant vector $\overrightarrow{PQ}$

(b) The tangent vector $\mathbf{r}'(t)$

The following theorem gives us a convenient method for computing the derivative of a vector function $\mathbf{r}$: just differentiate each component of $\mathbf{r}$.

2 Theorem If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$$

PROOF

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\mathbf{r}(t+\Delta t) - \mathbf{r}(t)] \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\langle f(t+\Delta t), g(t+\Delta t), h(t+\Delta t)\rangle - \langle f(t), g(t), h(t)\rangle] \\
&= \lim_{\Delta t \to 0} \left\langle \frac{f(t+\Delta t)-f(t)}{\Delta t}, \frac{g(t+\Delta t)-g(t)}{\Delta t}, \frac{h(t+\Delta t)-h(t)}{\Delta t} \right\rangle \\
&= \left\langle \lim_{\Delta t \to 0} \frac{f(t+\Delta t)-f(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{g(t+\Delta t)-g(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{h(t+\Delta t)-h(t)}{\Delta t} \right\rangle \\
&= \langle f'(t), g'(t), h'(t)\rangle
\end{aligned}$$

■

A unit vector that has the same direction as the tangent vector is called the **unit tangent vector T** and is defined by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

EXAMPLE 1

(a) Find the derivative of $\mathbf{r}(t) = (1 + t^3)\,\mathbf{i} + te^{-t}\,\mathbf{j} + \sin 2t\,\mathbf{k}$.
(b) Find the unit tangent vector at the point where $t = 0$.

SOLUTION

(a) According to Theorem 2, we differentiate each component of **r**:

$$\mathbf{r}'(t) = 3t^2\,\mathbf{i} + (1 - t)e^{-t}\,\mathbf{j} + 2\cos 2t\,\mathbf{k}$$

(b) Since $\mathbf{r}(0) = \mathbf{i}$ and $\mathbf{r}'(0) = \mathbf{j} + 2\mathbf{k}$, the unit tangent vector at the point $(1, 0, 0)$ is

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{\mathbf{j} + 2\mathbf{k}}{\sqrt{1+4}} = \frac{1}{\sqrt{5}}\,\mathbf{j} + \frac{2}{\sqrt{5}}\,\mathbf{k}$$

■

EXAMPLE 2 For the curve $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (2 - t)\,\mathbf{j}$, find $\mathbf{r}'(t)$ and sketch the position vector $\mathbf{r}(1)$ and the tangent vector $\mathbf{r}'(1)$.

SOLUTION We have

$$\mathbf{r}'(t) = \frac{1}{2\sqrt{t}}\,\mathbf{i} - \mathbf{j} \qquad \text{and} \qquad \mathbf{r}'(1) = \frac{1}{2}\,\mathbf{i} - \mathbf{j}$$

The curve is a plane curve and elimination of the parameter from the equations $x = \sqrt{t}$, $y = 2 - t$ gives $y = 2 - x^2$, $x \geqslant 0$. In Figure 2 we draw the position vector $\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$ starting at the origin and the tangent vector $\mathbf{r}'(1)$ starting at the corresponding point $(1, 1)$. ■

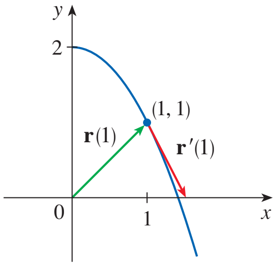


FIGURE 2

Notice from Figure 2 that the tangent vector points in the direction of increasing *t*. (See Exercise 60.)

EXAMPLE 3 Find parametric equations for the tangent line to the helix with parametric equations

$$x = 2\cos t \qquad y = \sin t \qquad z = t$$

at the point $(0, 1, \pi/2)$.

SOLUTION The vector equation of the helix is $\mathbf{r}(t) = \langle 2\cos t, \sin t, t\rangle$, so

$$\mathbf{r}'(t) = \langle -2\sin t, \cos t, 1\rangle$$

The parameter value corresponding to the point $(0, 1, \pi/2)$ is $t = \pi/2$, so the tangent vector there is $\mathbf{r}'(\pi/2) = \langle -2, 0, 1 \rangle$. The tangent line is the line through $(0, 1, \pi/2)$ parallel to the vector $\langle -2, 0, 1 \rangle$, so by Equations 12.5.2 its parametric equations are

$$x = -2t \qquad y = 1 \qquad z = \frac{\pi}{2} + t$$

■

The helix and the tangent line in Example 3 are shown in Figure 3.

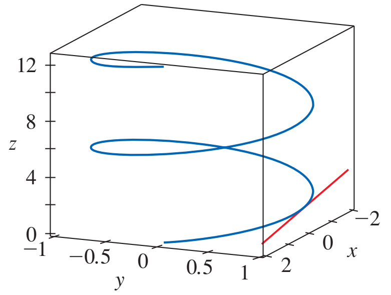


FIGURE 3

In Section 13.4 we will see how $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ can be interpreted as the velocity and acceleration vectors of a particle moving through space with position vector $\mathbf{r}(t)$ at time t.

Just as for real-valued functions, the **second derivative** of a vector function $\mathbf{r}$ is the derivative of $\mathbf{r}'$, that is, $\mathbf{r}'' = (\mathbf{r}')'$. For instance, the second derivative of the function in Example 3 is

$$\mathbf{r}''(t) = \langle -2 \cos t, -\sin t, 0 \rangle$$

Differentiation Rules

The next theorem shows that the differentiation formulas for real-valued functions have their counterparts for vector-valued functions.

$\boxed{3}$ **Theorem** Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\dfrac{d}{dt}[f(t)\,\mathbf{u}(t)] = f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t)$
4. $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
5. $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\,\mathbf{u}'(f(t))$ (Chain Rule)

This theorem can be proved either directly from Definition 1 or by using Theorem 2 and the corresponding differentiation formulas for real-valued functions. The proof of Formula 4 follows; the remaining formulas are left as exercises.

PROOF OF FORMULA 4 Let

$$\mathbf{u}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle \qquad \mathbf{v}(t) = \langle g_1(t), g_2(t), g_3(t) \rangle$$

Then $$\mathbf{u}(t) \cdot \mathbf{v}(t) = f_1(t)\,g_1(t) + f_2(t)\,g_2(t) + f_3(t)\,g_3(t) = \sum_{i=1}^{3} f_i(t)\,g_i(t)$$

so the ordinary Product Rule gives

$$\begin{aligned} \frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] &= \frac{d}{dt} \sum_{i=1}^{3} f_i(t)\,g_i(t) = \sum_{i=1}^{3} \frac{d}{dt}[f_i(t)\,g_i(t)] \\ &= \sum_{i=1}^{3} [f_i'(t)\,g_i(t) + f_i(t)\,g_i'(t)] \\ &= \sum_{i=1}^{3} f_i'(t)\,g_i(t) + \sum_{i=1}^{3} f_i(t)\,g_i'(t) \\ &= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \end{aligned}$$

■

We use Formula 4 to prove the following theorem.

4 Theorem If $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

PROOF Since

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = |\mathbf{r}(t)|^2 = c^2$$

and c^2 is a constant, Formula 4 of Theorem 3 gives

$$0 = \frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2\mathbf{r}'(t) \cdot \mathbf{r}(t)$$

Thus $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$, which says that $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$. ■

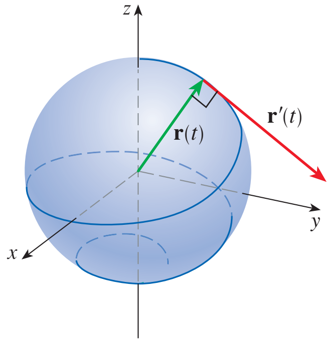


FIGURE 4

Geometrically, Theorem 4 says that if a curve lies on a sphere with center the origin, then the tangent vector $\mathbf{r}'(t)$ is always perpendicular to the position vector $\mathbf{r}(t)$. (See Figure 4.)

Integrals

The **definite integral** of a continuous vector function $\mathbf{r}(t)$ can be defined in much the same way as for real-valued functions except that the integral is a vector. But then we can express the integral of $\mathbf{r}$ in terms of the integrals of its component functions f, g, and h as follows. (We use the notation of Chapter 5.)

$$\begin{aligned} \int_a^b \mathbf{r}(t)\,dt &= \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{r}(t_i^*)\,\Delta t \\ &= \lim_{n\to\infty} \left[\left(\sum_{i=1}^{n} f(t_i^*)\,\Delta t \right) \mathbf{i} + \left(\sum_{i=1}^{n} g(t_i^*)\,\Delta t \right) \mathbf{j} + \left(\sum_{i=1}^{n} h(t_i^*)\,\Delta t \right) \mathbf{k} \right] \end{aligned}$$

and so

$$\boxed{\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt \right) \mathbf{i} + \left(\int_a^b g(t)\,dt \right) \mathbf{j} + \left(\int_a^b h(t)\,dt \right) \mathbf{k}}$$

This means that we can evaluate an integral of a vector function by integrating each component function.

We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where $\mathbf{R}$ is an antiderivative of $\mathbf{r}$, that is, $\mathbf{R}'(t) = \mathbf{r}(t)$. We use the notation $\int \mathbf{r}(t)\,dt$ for indefinite integrals (antiderivatives).

EXAMPLE 4 If $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2t\,\mathbf{k}$, then

$$\begin{aligned}\int \mathbf{r}(t)\,dt &= \left(\int 2\cos t\,dt\right)\mathbf{i} + \left(\int \sin t\,dt\right)\mathbf{j} + \left(\int 2t\,dt\right)\mathbf{k}\\ &= 2\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + t^2\,\mathbf{k} + \mathbf{C}\end{aligned}$$

where $\mathbf{C}$ is a vector constant of integration, and

$$\int_0^{\pi/2} \mathbf{r}(t)\,dt = \Big[2\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + t^2\,\mathbf{k}\Big]_0^{\pi/2} = 2\,\mathbf{i} + \mathbf{j} + \frac{\pi^2}{4}\,\mathbf{k}$$ ■

13.2 Exercises

1. The figure shows a curve C given by a vector function $\mathbf{r}(t)$.

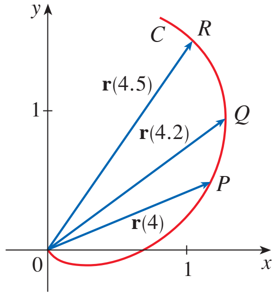
(a) Draw the vectors $\mathbf{r}(4.5) - \mathbf{r}(4)$ and $\mathbf{r}(4.2) - \mathbf{r}(4)$.

(b) Draw the vectors

$$\frac{\mathbf{r}(4.5) - \mathbf{r}(4)}{0.5} \quad \text{and} \quad \frac{\mathbf{r}(4.2) - \mathbf{r}(4)}{0.2}$$

(c) Write expressions for $\mathbf{r}'(4)$ and the unit tangent vector $\mathbf{T}(4)$.

(d) Draw the vector $\mathbf{T}(4)$.



2. (a) Make a large sketch of the curve described by the vector function $\mathbf{r}(t) = \langle t^2, t\rangle$, $0 \leqslant t \leqslant 2$, and draw the vectors $\mathbf{r}(1)$, $\mathbf{r}(1.1)$, and $\mathbf{r}(1.1) - \mathbf{r}(1)$.

(b) Draw the vector $\mathbf{r}'(1)$ starting at (1, 1), and compare it with the vector

$$\frac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1}$$

Explain why these vectors are so close to each other in length and direction.

3–8

(a) Sketch the plane curve with the given vector equation.

(b) Find $\mathbf{r}'(t)$.

(c) Sketch the position vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ for the given value of t.

3. $\mathbf{r}(t) = \langle t - 2, t^2 + 1\rangle$, $t = -1$

4. $\mathbf{r}(t) = \langle t^2, t^3\rangle$, $t = 1$

5. $\mathbf{r}(t) = e^{2t}\,\mathbf{i} + e^t\,\mathbf{j}$, $t = 0$

6. $\mathbf{r}(t) = e^t\,\mathbf{i} + 2t\,\mathbf{j}$, $t = 0$

7. $\mathbf{r}(t) = 4\sin t\,\mathbf{i} - 2\cos t\,\mathbf{j}$, $t = 3\pi/4$

8. $\mathbf{r}(t) = (\cos t + 1)\,\mathbf{i} + (\sin t - 1)\,\mathbf{j}$, $t = -\pi/3$

9–16 Find the derivative of the vector function.

9. $\mathbf{r}(t) = \left\langle \sqrt{t - 2}, 3, 1/t^2\right\rangle$

10. $\mathbf{r}(t) = \langle e^{-t}, t - t^3, \ln t\rangle$

11. $\mathbf{r}(t) = t^2\,\mathbf{i} + \cos(t^2)\,\mathbf{j} + \sin^2 t\,\mathbf{k}$

12. $\mathbf{r}(t) = \dfrac{1}{1+t}\,\mathbf{i} + \dfrac{t}{1+t}\,\mathbf{j} + \dfrac{t^2}{1+t}\,\mathbf{k}$

13. $\mathbf{r}(t) = t\sin t\,\mathbf{i} + e^t\cos t\,\mathbf{j} + \sin t\cos t\,\mathbf{k}$

14. $\mathbf{r}(t) = \sin^2 at\,\mathbf{i} + te^{bt}\,\mathbf{j} + \cos^2 ct\,\mathbf{k}$

15. $\mathbf{r}(t) = \mathbf{a} + t\,\mathbf{b} + t^2\,\mathbf{c}$

16. $\mathbf{r}(t) = t\,\mathbf{a} \times (\mathbf{b} + t\,\mathbf{c})$

17–20 Find the unit tangent vector $\mathbf{T}(t)$ at the point with the given value of the parameter t.

17. $\mathbf{r}(t) = \left\langle t^2 - 2t,\ 1 + 3t,\ \tfrac{1}{3}t^3 + \tfrac{1}{2}t^2 \right\rangle, \quad t = 2$

18. $\mathbf{r}(t) = \langle \tan^{-1} t,\ 2e^{2t},\ 8te^{t} \rangle, \quad t = 0$

19. $\mathbf{r}(t) = \cos t\,\mathbf{i} + 3t\,\mathbf{j} + 2\sin 2t\,\mathbf{k}, \quad t = 0$

20. $\mathbf{r}(t) = \sin^2 t\,\mathbf{i} + \cos^2 t\,\mathbf{j} + \tan^2 t\,\mathbf{k}, \quad t = \pi/4$

21–22 Find the unit tangent vector $\mathbf{T}(t)$ at the given point on the curve.

21. $\mathbf{r}(t) = \langle t^3 + 1,\ 3t - 5,\ 4/t \rangle, \quad (2, -2, 4)$

22. $\mathbf{r}(t) = \sin t\,\mathbf{i} + 5t\,\mathbf{j} + \cos t\,\mathbf{k}, \quad (0, 0, 1)$

23. If $\mathbf{r}(t) = \langle t^4, t, t^2 \rangle$, find $\mathbf{r}'(t)$, $\mathbf{T}(1)$, $\mathbf{r}''(t)$, and $\mathbf{r}'(t) \times \mathbf{r}''(t)$.

24. If $\mathbf{r}(t) = \langle e^{2t}, e^{-3t}, t \rangle$, find $\mathbf{r}'(0)$, $\mathbf{T}(0)$, $\mathbf{r}''(0)$, and $\mathbf{r}'(0) \times \mathbf{r}''(0)$.

25–28 Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point.

25. $x = t^2 + 1, \quad y = 4\sqrt{t}, \quad z = e^{t^2 - t}; \quad (2, 4, 1)$

26. $x = \ln(t + 1), \quad y = t\cos 2t, \quad z = 2^t; \quad (0, 0, 1)$

27. $x = e^{-t}\cos t, \quad y = e^{-t}\sin t, \quad z = e^{-t}; \quad (1, 0, 1)$

28. $x = \sqrt{t^2 + 3}, \quad y = \ln(t^2 + 3), \quad z = t; \quad (2, \ln 4, 1)$

29. Find a vector equation for the tangent line to the curve of intersection of the cylinders $x^2 + y^2 = 25$ and $y^2 + z^2 = 20$ at the point $(3, 4, 2)$.

30. Find the point on the curve $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, e^t \rangle$, $0 \leqslant t \leqslant \pi$, where the tangent line is parallel to the plane $\sqrt{3}x + y = 1$.

31–33 Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point. Illustrate by graphing both the curve and the tangent line on a common screen.

31. $x = t, \ y = e^{-t}, \ z = 2t - t^2; \quad (0, 1, 0)$

32. $x = 2\cos t, \ y = 2\sin t, \ z = 4\cos 2t; \quad \left(\sqrt{3}, 1, 2\right)$

33. $x = t\cos t, \ y = t, \ z = t\sin t; \quad (-\pi, \pi, 0)$

34. (a) Find the point of intersection of the tangent lines to the curve $\mathbf{r}(t) = \langle \sin \pi t, 2\sin \pi t, \cos \pi t \rangle$ at the points where $t = 0$ and $t = 0.5$.

(b) Illustrate by graphing the curve and both tangent lines.

35. The curves $\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{r}_2(t) = \langle \sin t, \sin 2t, t \rangle$ intersect at the origin. Find their angle of intersection correct to the nearest degree.

36. At what point do the curves $\mathbf{r}_1(t) = \langle t, 1 - t, 3 + t^2 \rangle$ and $\mathbf{r}_2(s) = \langle 3 - s, s - 2, s^2 \rangle$ intersect? Find their angle of intersection correct to the nearest degree.

37–42 Evaluate the integral.

37. $\displaystyle\int_0^2 (t\,\mathbf{i} - t^3\,\mathbf{j} + 3t^5\,\mathbf{k})\,dt$

38. $\displaystyle\int_1^4 \left(2t^{3/2}\,\mathbf{i} + (t + 1)\sqrt{t}\,\mathbf{k}\right)dt$

39. $\displaystyle\int_0^1 \left(\frac{1}{t + 1}\,\mathbf{i} + \frac{1}{t^2 + 1}\,\mathbf{j} + \frac{t}{t^2 + 1}\,\mathbf{k}\right)dt$

40. $\displaystyle\int_0^{\pi/4} (\sec t \tan t\,\mathbf{i} + t\cos 2t\,\mathbf{j} + \sin^2 2t\cos 2t\,\mathbf{k})\,dt$

41. $\displaystyle\int \left(\frac{1}{1 + t^2}\,\mathbf{i} + te^{t^2}\,\mathbf{j} + \sqrt{t}\,\mathbf{k}\right)dt$

42. $\displaystyle\int \left(t\cos t^2\,\mathbf{i} + \frac{1}{t}\,\mathbf{j} + \sec^2 t\,\mathbf{k}\right)dt$

43. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} + \sqrt{t}\,\mathbf{k}$ and $\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$.

44. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = t\,\mathbf{i} + e^t\,\mathbf{j} + te^t\,\mathbf{k}$ and $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

45. Prove Formula 1 of Theorem 3.

46. Prove Formula 3 of Theorem 3.

47. Prove Formula 5 of Theorem 3.

48. Prove Formula 6 of Theorem 3.

49. If $\mathbf{u}(t) = \langle \sin t, \cos t, t \rangle$ and $\mathbf{v}(t) = \langle t, \cos t, \sin t \rangle$, use Formula 4 of Theorem 3 to find

$$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$$

50. If $\mathbf{u}$ and $\mathbf{v}$ are the vector functions in Exercise 49, use Formula 5 of Theorem 3 to find

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)]$$

51. Find $f'(2)$, where $f(t) = \mathbf{u}(t) \cdot \mathbf{v}(t)$, $\mathbf{u}(2) = \langle 1, 2, -1 \rangle$, $\mathbf{u}'(2) = \langle 3, 0, 4 \rangle$, and $\mathbf{v}(t) = \langle t, t^2, t^3 \rangle$.

52. If $\mathbf{r}(t) = \mathbf{u}(t) \times \mathbf{v}(t)$, where $\mathbf{u}$ and $\mathbf{v}$ are the vector functions in Exercise 51, find $\mathbf{r}'(2)$.

53. If $\mathbf{r}(t) = \mathbf{a}\cos \omega t + \mathbf{b}\sin \omega t$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, show that $\mathbf{r}(t) \times \mathbf{r}'(t) = \omega\mathbf{a} \times \mathbf{b}$.

54. If $\mathbf{r}$ is the vector function in Exercise 53, show that $\mathbf{r}''(t) + \omega^2\mathbf{r}(t) = \mathbf{0}$.

55. Show that if $\mathbf{r}$ is a vector function such that $\mathbf{r}''$ exists, then

$$\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t)$$

56. Find an expression for $\dfrac{d}{dt}[\mathbf{u}(t) \cdot (\mathbf{v}(t) \times \mathbf{w}(t))]$.

57. If $\mathbf{r}(t) \neq \mathbf{0}$, show that $\dfrac{d}{dt}|\mathbf{r}(t)| = \dfrac{1}{|\mathbf{r}(t)|}\mathbf{r}(t) \cdot \mathbf{r}'(t)$.

[*Hint:* $|\mathbf{r}(t)|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$]

58. Prove the converse of Theorem 4: if a curve has the property that the position vector $\mathbf{r}(t)$ is always orthogonal to the tangent vector $\mathbf{r}'(t)$, then $|\mathbf{r}(t)|$ is constant and thus the curve lies on a sphere with center the origin.

59. If $\mathbf{u}(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}''(t)]$, show that

$$\mathbf{u}'(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}'''(t)]$$

60. Show that the tangent vector to a curve defined by a vector function $\mathbf{r}(t)$ points in the direction of increasing t. [*Hint:* Refer to Figure 1 and consider the cases $h > 0$ and $h < 0$ separately.]

13.3 Arc Length and Curvature

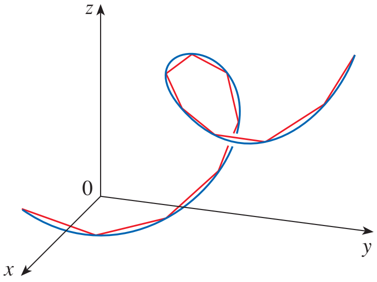


FIGURE 1
The length of a space curve is the limit of lengths of approximating polygonal paths.

Arc Length

In Section 10.2 we defined the length of a plane curve with parametric equations $x = f(t)$, $y = g(t)$, $a \leqslant t \leqslant b$, as the limit of lengths of approximating polygonal paths and, for the case where f' and g' are continuous, we arrived at the formula

$$\boxed{1} \qquad L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The length of a space curve is defined in exactly the same way (see Figure 1). Suppose that the curve has the vector equation $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, $a \leqslant t \leqslant b$, or, equivalently, the parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$, where f', g', and h' are continuous. If the curve is traversed exactly once as t increases from a to b, then it can be shown that its length is

$$\boxed{2} \qquad \boxed{\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \end{aligned}}$$

Notice that both of the arc length formulas (1) and (2) can be put into the more compact form

In Section 13.4 we will see that if $\mathbf{r}(t)$ is the position vector of a moving object at time t, then $\mathbf{r}'(t)$ is the velocity vector and $|\mathbf{r}'(t)|$ is the speed. Thus Equation 3 says that to compute distance traveled, we integrate speed.

$$\boxed{3} \qquad \boxed{L = \int_a^b |\mathbf{r}'(t)|\, dt}$$

because, for plane curves $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$,

$$|\mathbf{r}'(t)| = |f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j}| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

and for space curves $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$,

$$|\mathbf{r}'(t)| = |f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

Figure 2 shows the arc of the helix whose length is computed in Example 1.

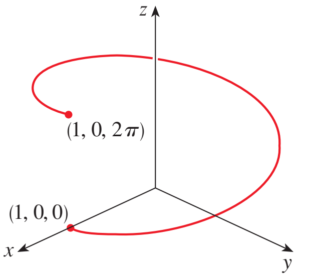


FIGURE 2

EXAMPLE 1 Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

SOLUTION Since $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$, we have

$$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{2}$$

The arc from $(1, 0, 0)$ to $(1, 0, 2\pi)$ is described by the parameter interval $0 \leqslant t \leqslant 2\pi$ and so, from Formula 3, we have

$$L = \int_0^{2\pi} |\mathbf{r}'(t)|\,dt = \int_0^{2\pi} \sqrt{2}\,dt = 2\sqrt{2}\pi$$ ■

A single curve C can be represented by more than one vector function. For instance, the twisted cubic

$$\boxed{4} \qquad \mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle \qquad 1 \leqslant t \leqslant 2$$

could also be represented by the function

$$\boxed{5} \qquad \mathbf{r}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle \qquad 0 \leqslant u \leqslant \ln 2$$

where the connection between the parameters t and u is given by $t = e^u$. We say that Equations 4 and 5 are **parametrizations** of the curve C. If we were to use Equation 3 to compute the length of C using Equations 4 and 5, we would get the same answer. This is because arc length is a geometric property of the curve and hence is independent of the parametrization that is used.

The Arc Length Function

Now we suppose that C is a curve given by a vector function

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k} \qquad a \leqslant t \leqslant b$$

where $\mathbf{r}'$ is continuous and C is traversed exactly once as t increases from a to b. We define its **arc length function** s by

$$\boxed{6} \qquad s(t) = \int_a^t |\mathbf{r}'(u)|\,du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\,du$$

(Compare to Equation 10.2.7.) Thus $s(t)$ is the length of the part of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$. (See Figure 3.) If we differentiate both sides of Equation 6 using Part 1 of the Fundamental Theorem of Calculus, we obtain

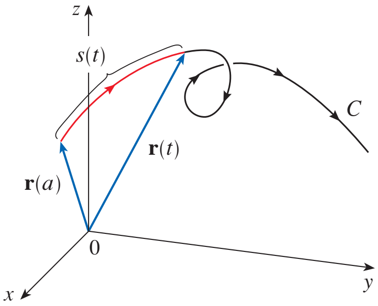


FIGURE 3

$$\boxed{7} \qquad \frac{ds}{dt} = |\mathbf{r}'(t)|$$

It is often useful to **parametrize a curve with respect to arc length** because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system or a particular parametrization. If a curve $\mathbf{r}(t)$ is already given in terms of a parameter t and $s(t)$ is the arc length function given by Equation 6, then we may be able to solve for t as a function of s: $t = t(s)$. Then the curve can be reparametrized in terms of s by substituting for t: $\mathbf{r} = \mathbf{r}(t(s))$. Thus, if $s = 3$ for instance, $\mathbf{r}(t(3))$ is the position vector of the point 3 units of length along the curve from its starting point.

EXAMPLE 2 Reparametrize the helix $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing t.

SOLUTION The initial point $(1, 0, 0)$ corresponds to the parameter value $t = 0$. From Example 1 we have

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{2}$$

and so

$$s = s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{2}\,du = \sqrt{2}\,t$$

Therefore $t = s/\sqrt{2}$ and the required reparametrization is obtained by substituting for t:

$$\mathbf{r}(t(s)) = \cos\left(s/\sqrt{2}\right)\mathbf{i} + \sin\left(s/\sqrt{2}\right)\mathbf{j} + \left(s/\sqrt{2}\right)\mathbf{k}$$

■

Curvature

A parametrization $\mathbf{r}(t)$ is called **smooth** on an interval I if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ on I. A curve is called **smooth** if it has a smooth parametrization. A smooth curve has no sharp corner or cusp; when the tangent vector turns, it does so continuously.

If C is a smooth curve defined by the vector function $\mathbf{r}$, recall that the unit tangent vector $\mathbf{T}(t)$ is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

and indicates the direction of the curve. From Figure 4 you can see that $\mathbf{T}(t)$ changes direction very slowly when C is fairly straight, but it changes direction more quickly when C bends or twists more sharply.

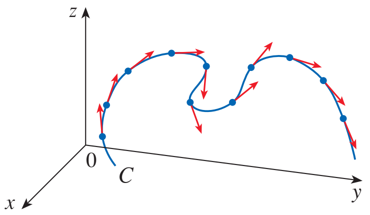


FIGURE 4
Unit tangent vectors at equally spaced points on C

The curvature of C at a given point is a measure of how quickly the curve changes direction at that point. Specifically, we define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length. (We use arc length so that the definition of curvature will be independent of the parametrization.) Because the unit tangent vector has constant length, only changes in direction contribute to the rate of change of $\mathbf{T}$.

8 Definition The **curvature** of a curve is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$$

where $\mathbf{T}$ is the unit tangent vector.

The curvature is easier to compute if it is expressed in terms of the parameter t instead of s, so we use the Chain Rule (Theorem 13.2.3, Formula 6) to write

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} \qquad \Longrightarrow \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right|$$

But $ds/dt = |\mathbf{r}'(t)|$ from Equation 7, so

$$\boxed{9} \qquad \boxed{\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}}$$

EXAMPLE 3 Show that the curvature of a circle of radius a is $1/a$.

SOLUTION We can take the circle to have center the origin, and then a parametrization is

$$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}$$

Therefore $\qquad \mathbf{r}'(t) = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} \qquad$ and $\qquad |\mathbf{r}'(t)| = a$

so
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

and
$$\mathbf{T}'(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$$

This gives $|\mathbf{T}'(t)| = 1$, so using Formula 9, we have

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{a}$$ ■

The result of Example 3 shows that small circles have large curvature and large circles have small curvature, in accordance with our intuition. We can see directly from the definition of curvature that the curvature of a straight line is always 0 because the tangent vector is constant.

Although Formula 9 can be used in all cases to compute the curvature, the formula given by the following theorem is often more convenient to apply.

10 Theorem The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

PROOF Since $\mathbf{T} = \mathbf{r}'/|\mathbf{r}'|$ and $|\mathbf{r}'| = ds/dt$, we have

$$\mathbf{r}' = |\mathbf{r}'|\mathbf{T} = \frac{ds}{dt}\mathbf{T}$$

so the Product Rule (Theorem 13.2.3, Formula 3) gives

$$\mathbf{r}'' = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T}'$$

Using the fact that $\mathbf{T} \times \mathbf{T} = \mathbf{0}$ (see Example 12.4.2), we have

$$\mathbf{r}' \times \mathbf{r}'' = \left(\frac{ds}{dt}\right)^2 (\mathbf{T} \times \mathbf{T}')$$

Now $|\mathbf{T}(t)| = 1$ for all t, so $\mathbf{T}$ and $\mathbf{T}'$ are orthogonal by Theorem 13.2.4. Therefore, by Theorem 12.4.9,

$$|\mathbf{r}' \times \mathbf{r}''| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T} \times \mathbf{T}'| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T}||\mathbf{T}'| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T}'|$$

Thus
$$|\mathbf{T}'| = \frac{|\mathbf{r}' \times \mathbf{r}''|}{(ds/dt)^2} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^2}$$

and
$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$$
■

EXAMPLE 4 Find the curvature of the twisted cubic $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at a general point and at (0, 0, 0).

SOLUTION We first compute the required ingredients:

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle \qquad \mathbf{r}''(t) = \langle 0, 2, 6t \rangle$$

$$|\mathbf{r}'(t)| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2\,\mathbf{i} - 6t\,\mathbf{j} + 2\,\mathbf{k}$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{36t^4 + 36t^2 + 4} = 2\sqrt{9t^4 + 9t^2 + 1}$$

Theorem 10 then gives

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{2\sqrt{1 + 9t^2 + 9t^4}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

At the origin, where $t = 0$, the curvature is $\kappa(0) = 2$. ■

For the special case of a plane curve with equation $y = f(x)$, we choose x as the parameter and write $\mathbf{r}(x) = x\,\mathbf{i} + f(x)\,\mathbf{j}$. Then $\mathbf{r}'(x) = \mathbf{i} + f'(x)\,\mathbf{j}$ and $\mathbf{r}''(x) = f''(x)\,\mathbf{j}$. Since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{j} = \mathbf{0}$, it follows that $\mathbf{r}'(x) \times \mathbf{r}''(x) = f''(x)\,\mathbf{k}$. We also have $|\mathbf{r}'(x)| = \sqrt{1 + [f'(x)]^2}$ and so, by Theorem 10,

11
$$\boxed{\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}}$$

EXAMPLE 5 Find the curvature of the parabola $y = x^2$ at the points (0, 0), (1, 1), and (2, 4).

SOLUTION Since $y' = 2x$ and $y'' = 2$, Formula 11 gives

$$\kappa(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}$$

The curvature at (0, 0) is $\kappa(0) = 2$. At (1, 1) it is $\kappa(1) = 2/5^{3/2} \approx 0.18$. At (2, 4) it is $\kappa(2) = 2/17^{3/2} \approx 0.03$. Observe from the expression for $\kappa(x)$ or the graph of κ in Figure 5 that $\kappa(x) \to 0$ as $x \to \pm\infty$. This corresponds to the fact that the parabola appears to become nearly straight as $x \to \pm\infty$. ■

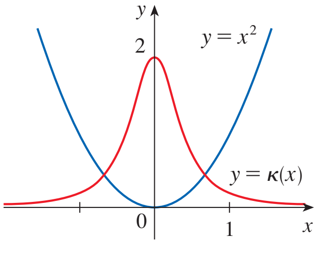


FIGURE 5
The parabola $y = x^2$ and its curvature function

The Normal and Binormal Vectors

At a given point on a smooth space curve $\mathbf{r}(t)$, there are many vectors that are orthogonal to the unit tangent vector $\mathbf{T}(t)$. We single out one by observing that, because $|\mathbf{T}(t)| = 1$ for all t, we have $\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$ by Theorem 13.2.4, so $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$. Note that, typically, $\mathbf{T}'(t)$ is itself not a unit vector. But at any point where $\kappa \neq 0$ we can define the **principal unit normal vector** $\mathbf{N}(t)$ (or simply **unit normal**) as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

We can think of the unit normal vector as indicating the direction in which the curve is turning at each point. The vector

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

is called the **binormal vector**. It is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$ and is also a unit vector. (See Figure 6.)

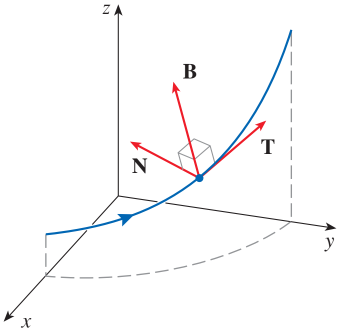


FIGURE 6

Figure 7 illustrates Example 6 by showing the vectors **T**, **N**, and **B** at two locations on the helix. In general, the vectors **T**, **N**, and **B**, starting at the various points on a curve, form a set of orthogonal vectors, called the **TNB** frame, that moves along the curve as t varies. This **TNB** frame plays an important role in the branch of mathematics known as differential geometry and in its applications to the motion of spacecraft.

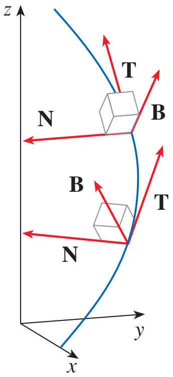


FIGURE 7

EXAMPLE 6 Find the unit normal and binormal vectors for the circular helix

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

SOLUTION We first compute the ingredients needed for the unit normal vector:

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k} \qquad |\mathbf{r}'(t)| = \sqrt{2}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{2}}(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k})$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}}(-\cos t\,\mathbf{i} - \sin t\,\mathbf{j}) \qquad |\mathbf{T}'(t)| = \frac{1}{\sqrt{2}}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j} = \langle -\cos t,\ -\sin t, 0\rangle$$

This shows that the unit normal vector at any point on the helix is horizontal and points toward the z-axis. The binormal vector is

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\langle \sin t,\ -\cos t,\ 1\rangle \qquad \blacksquare$$

EXAMPLE 7 Find the unit tangent, unit normal, and binormal vectors and the curvature for the curve $\mathbf{r}(t) = \langle t, \sqrt{2}\ln t, 1/t\rangle$ at the point $(1, 0, 1)$.

SOLUTION We start by finding $\mathbf{T}$ and $\mathbf{T}'$ as functions of t.

$$\mathbf{r}'(t) = \langle 1, \sqrt{2}/t, -1/t^2\rangle$$

$$|\mathbf{r}'(t)| = \sqrt{1 + \frac{2}{t^2} + \frac{1}{t^4}} = \frac{1}{t^2}\sqrt{t^4 + 2t^2 + 1}$$

$$= \frac{1}{t^2}\sqrt{(t^2+1)^2} = \frac{1}{t^2}(t^2 + 1) \qquad \text{(because } t^2 + 1 > 0\text{)}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{t^2}{(t^2+1)}\left\langle 1, \frac{\sqrt{2}}{t}, -\frac{1}{t^2}\right\rangle = \frac{1}{(t^2+1)}\langle t^2, \sqrt{2}t, -1\rangle$$

We use Formula 3 of Theorem 13.2.3 to differentiate **T**:

$$\mathbf{T}'(t) = \frac{-2t}{(t^2+1)^2}\langle t^2, \sqrt{2}t, -1\rangle + \frac{1}{(t^2+1)}\langle 2t, \sqrt{2}, 0\rangle$$

The point (1, 0, 1) corresponds to $t = 1$, so we have

$$\mathbf{T}(1) = \tfrac{1}{2}\langle 1, \sqrt{2}, -1\rangle$$

$$\mathbf{T}'(1) = -\tfrac{1}{2}\langle 1, \sqrt{2}, -1\rangle + \tfrac{1}{2}\langle 2, \sqrt{2}, 0\rangle = \tfrac{1}{2}\langle 1, 0, 1\rangle$$

$$\mathbf{N}(1) = \frac{\mathbf{T}'(1)}{|\mathbf{T}'(1)|} = \frac{\tfrac{1}{2}\langle 1, 0, 1\rangle}{\tfrac{1}{2}\sqrt{1+0+1}} = \frac{1}{\sqrt{2}}\langle 1, 0, 1\rangle$$

$$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \frac{1}{2\sqrt{2}}\langle \sqrt{2}, -2, -\sqrt{2}\rangle = \tfrac{1}{2}\langle 1, -\sqrt{2}, -1\rangle$$

and, by Formula 9, the curvature is

$$\kappa(1) = \frac{|\mathbf{T}'(1)|}{|\mathbf{r}'(1)|} = \frac{\sqrt{2}/2}{2} = \frac{\sqrt{2}}{4}$$

We could also use Theorem 10 to compute $\kappa(1)$; you can check that we get the same answer. ■

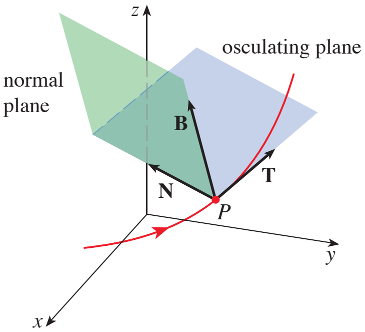


FIGURE 8

The plane determined by the normal and binormal vectors **N** and **B** at a point P on a curve C is called the **normal plane** of C at P. It consists of all lines that are orthogonal to the tangent vector **T**. The plane determined by the vectors **T** and **N** is called the **osculating plane** of C at P. (See Figure 8.) The name comes from the Latin *osculum*, meaning "kiss." It is the plane that comes closest to containing the part of the curve near P. (For a plane curve, the osculating plane is simply the plane that contains the curve.)

The **circle of curvature**, or the **osculating circle**, of C at P is the circle in the osculating plane that passes through P with radius $1/\kappa$ and center a distance $1/\kappa$ from P along the vector **N**. The center of the circle is called the **center of curvature** of C at P. We can think of the circle of curvature as the circle that best describes how C behaves near P—it shares the same tangent, normal, and curvature at P. Figure 9 illustrates two circles of curvature for a plane curve.

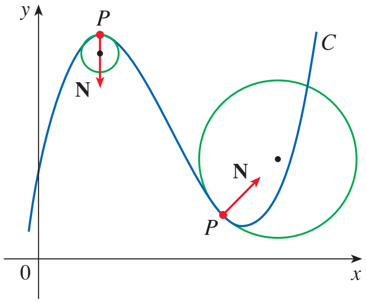


FIGURE 9

EXAMPLE 8 Find equations of the normal plane and osculating plane of the helix in Example 6 at the point $P(0, 1, \pi/2)$.

Figure 10 shows the helix and the osculating plane in Example 8.

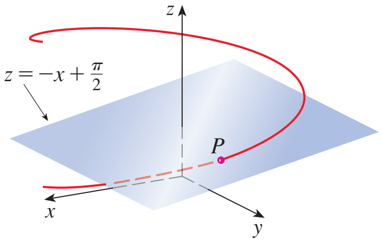


FIGURE 10

SOLUTION The point P corresponds to $t = \pi/2$ and the normal plane there has normal vector $\mathbf{r}'(\pi/2) = \langle -1, 0, 1\rangle$, so an equation of the normal plane is

$$-1(x - 0) + 0(y - 1) + 1\left(z - \frac{\pi}{2}\right) = 0 \qquad \text{or} \qquad z = x + \frac{\pi}{2}$$

The osculating plane at P contains the vectors $\mathbf{T}$ and $\mathbf{N}$, so a vector normal to the osculating plane is $\mathbf{T} \times \mathbf{N} = \mathbf{B}$. From Example 6 we have

$$\mathbf{B}(t) = \frac{1}{\sqrt{2}}\langle \sin t,\ -\cos t,\ 1\rangle \qquad \mathbf{B}\left(\frac{\pi}{2}\right) = \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\rangle$$

The vector $\langle 1, 0, 1\rangle$ is parallel to $\mathbf{B}(\pi/2)$ (so also normal to the osculating plane). Thus an equation of the osculating plane is

$$1(x - 0) + 0(y - 1) + 1\left(z - \frac{\pi}{2}\right) = 0 \qquad \text{or} \qquad z = -x + \frac{\pi}{2}$$ ■

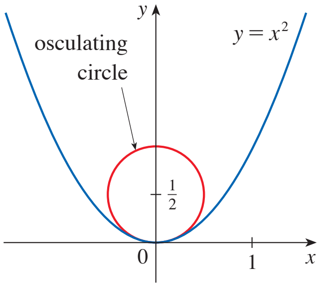


FIGURE 11
Notice that the circle and the parabola appear to bend similarly at the origin.

EXAMPLE 9 Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

SOLUTION From Example 5, the curvature of the parabola at the origin is $\kappa(0) = 2$ so the radius of the osculating circle there is $1/\kappa = \frac{1}{2}$. Moving this distance in the direction of $\mathbf{N} = \langle 0, 1\rangle$ (the tangent vector is horizontal at the origin so the normal vector is vertical) leads us to the center of curvature at $\left(0, \frac{1}{2}\right)$, so an equation of the circle of curvature is

$$x^2 + \left(y - \tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$

This circle is graphed in Figure 11. ■

We summarize here the formulas for unit tangent, unit normal and binormal vectors, and curvature.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

■ Torsion

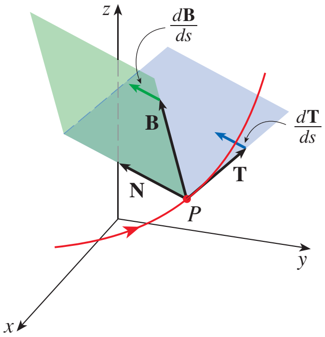


FIGURE 12

Curvature $\kappa = |d\mathbf{T}/ds|$ at a point P on a curve C indicates how tightly the curve "bends." Since $\mathbf{T}$ is a normal vector for the normal plane, $d\mathbf{T}/ds$ tells us how the normal plane changes as P moves along C. [Note that the vector $d\mathbf{T}/ds$ is parallel to $\mathbf{N}$ (Exercise 63), so as P moves along C, the tangent vector at P rotates in the direction of $\mathbf{N}$. A space curve can also lift or "twist" out of the osculating plane at P.] Since $\mathbf{B}$ is normal to the osculating plane, $d\mathbf{B}/ds$ gives us information about how the osculating plane changes as P moves along C. (See Figure 12.)

In Exercise 65 you are asked to show that $d\mathbf{B}/ds$ is parallel to $\mathbf{N}$. Thus there is a scalar τ such that

$$\boxed{12} \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

(It is customary to include the negative sign in Equation 12.) The number τ is called the *torsion* of C at P. If we take the dot product with $\mathbf{N}$ of each side of Equation 12 and note that $\mathbf{N} \cdot \mathbf{N} = 1$, we get the following definition.

13 Definition The **torsion** of a curve is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

Intuitively, the torsion τ at a point P on a curve is a measure of how much the curve "twists" at P. If τ is positive, the curve twists out of the osculating plane at P in the direction of the binormal vector $\mathbf{B}$; if τ is negative, the curve twists in the opposite direction.

Torsion is easier to compute if it is expressed in terms of the parameter t instead of s, so we use the Chain Rule to write

$$\frac{d\mathbf{B}}{dt} = \frac{d\mathbf{B}}{ds}\frac{ds}{dt} \qquad \text{so} \qquad \frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}/dt}{ds/dt} = \frac{\mathbf{B}'(t)}{|\mathbf{r}'(t)|}$$

Now from Definition 13 we have

$$\boxed{\tau(t) = -\frac{\mathbf{B}'(t) \cdot \mathbf{N}(t)}{|\mathbf{r}'(t)|}} \tag{14}$$

EXAMPLE 10 Find the torsion of the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$.

SOLUTION In Example 6 we computed $ds/dt = |\mathbf{r}'(t)| = \sqrt{2}$, $\mathbf{N}(t) = \langle -\cos t, -\sin t, 0 \rangle$, and $\mathbf{B}(t) = \left(1/\sqrt{2}\right)\langle \sin t, -\cos t, 1 \rangle$. Then $\mathbf{B}'(t) = \left(1/\sqrt{2}\right)\langle \cos t, \sin t, 0 \rangle$ and Formula 14 gives

$$\tau(t) = -\frac{\mathbf{B}'(t) \cdot \mathbf{N}(t)}{|\mathbf{r}'(t)|} = -\frac{1}{2}\langle \cos t, \sin t, 0 \rangle \cdot \langle -\cos t, -\sin t, 0 \rangle = \frac{1}{2} \qquad ■$$

Figure 13 shows the unit circle $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$ in the xy-plane and Figure 14 shows the helix of Example 10. Both curves have constant curvature, but the circle has constant torsion 0 whereas the helix has constant torsion $\frac{1}{2}$. We can think of the circle as bending at each point but never twisting, while the helix both bends *and* twists (upward) at each point.

It can be shown that under certain conditions, the shape of a space curve is completely determined by the values of curvature and torsion at each point on the curve.

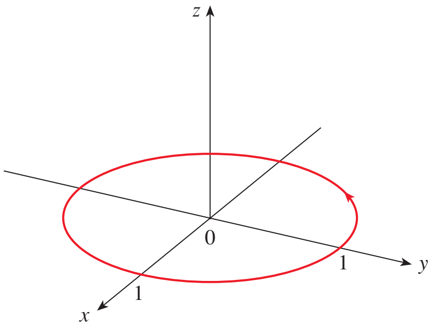


FIGURE 13 $\kappa = 1, \tau = 0$

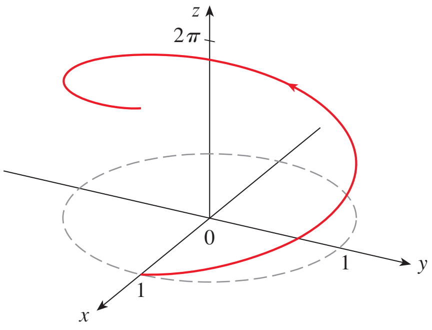


FIGURE 14 $\kappa = \frac{1}{2}, \tau = \frac{1}{2}$

The following theorem gives a formula that is often more convenient for computing torsion; a proof is outlined in Exercise 72.

15 Theorem The torsion of the curve given by the vector function **r** is

$$\tau(t) = \frac{[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2}$$

In Exercises 68–70 you are asked to use Theorem 15 to compute the torsion of a curve.

13.3 Exercises

1–2

(a) Use Equation 2 to compute the length of the given line segment.

(b) Compute the length using the distance formula and compare to your answer from part (a).

1. $\mathbf{r}(t) = \langle 3 - t, 2t, 4t + 1 \rangle, \quad 1 \leqslant t \leqslant 3$

2. $\mathbf{r}(t) = (t + 2)\,\mathbf{i} - t\,\mathbf{j} + (3t - 5)\,\mathbf{k}, \quad -1 \leqslant t \leqslant 2$

3–8 Find the length of the curve.

3. $\mathbf{r}(t) = \langle t, 3\cos t, 3\sin t \rangle, \quad -5 \leqslant t \leqslant 5$

4. $\mathbf{r}(t) = \langle 2t, t^2, \frac{1}{3}t^3 \rangle, \quad 0 \leqslant t \leqslant 1$

5. $\mathbf{r}(t) = \sqrt{2}\,t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}, \quad 0 \leqslant t \leqslant 1$

6. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \ln\cos t\,\mathbf{k}, \quad 0 \leqslant t \leqslant \pi/4$

7. $\mathbf{r}(t) = \mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}, \quad 0 \leqslant t \leqslant 1$

8. $\mathbf{r}(t) = t^2\,\mathbf{i} + 9t\,\mathbf{j} + 4t^{3/2}\,\mathbf{k}, \quad 1 \leqslant t \leqslant 4$

T **9–11** Find the length of the curve correct to four decimal places. (Use a calculator or computer to approximate the integral.)

9. $\mathbf{r}(t) = \langle t^2, t^3, t^4 \rangle, \quad 0 \leqslant t \leqslant 2$

10. $\mathbf{r}(t) = \langle t, e^{-t}, te^{-t} \rangle, \quad 1 \leqslant t \leqslant 3$

11. $\mathbf{r}(t) = \langle \cos \pi t, 2t, \sin 2\pi t \rangle$, from $(1, 0, 0)$ to $(1, 4, 0)$

T **12.** Graph the curve with parametric equations $x = \sin t$, $y = \sin 2t$, $z = \sin 3t$. Find the total length of this curve, correct to four decimal places.

13. Let C be the curve of intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$. Find the exact length of C from the origin to the point $(6, 18, 36)$.

T **14.** Find, correct to four decimal places, the length of the curve of intersection of the cylinder $4x^2 + y^2 = 4$ and the plane $x + y + z = 2$.

15–16

(a) Find the arc length function for the curve measured from the point P in the direction of increasing t and then reparametrize the curve with respect to arc length starting from P.

(b) Find the point 4 units along the curve (in the direction of increasing t) from P.

15. $\mathbf{r}(t) = (5 - t)\,\mathbf{i} + (4t - 3)\,\mathbf{j} + 3t\,\mathbf{k}, \quad P(4, 1, 3)$

16. $\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j} + \sqrt{2}\,e^t\,\mathbf{k}, \quad P(0, 1, \sqrt{2})$

17. Suppose you start at the point $(0, 0, 3)$ and move 5 units along the curve $x = 3\sin t$, $y = 4t$, $z = 3\cos t$ in the positive direction. Where are you now?

18. Reparametrize the curve

$$\mathbf{r}(t) = \left(\frac{2}{t^2 + 1} - 1\right)\mathbf{i} + \frac{2t}{t^2 + 1}\,\mathbf{j}$$

with respect to arc length measured from the point $(1, 0)$ in the direction of increasing t. Express the reparametrization in its simplest form. What can you conclude about the curve?

19–24

(a) Find the unit tangent and unit normal vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

(b) Use Formula 9 to find the curvature.

19. $\mathbf{r}(t) = \langle t^2, \sin t - t\cos t, \cos t + t\sin t \rangle, \quad t > 0$

20. $\mathbf{r}(t) = \langle 5\sin t, t, 5\cos t \rangle$

21. $\mathbf{r}(t) = \langle t, t^2, 4 \rangle$

22. $\mathbf{r}(t) = \langle t, t, \frac{1}{2}t^2 \rangle$

23. $\mathbf{r}(t) = \langle t, \frac{1}{2}t^2, t^2 \rangle$

24. $\mathbf{r}(t) = \langle \sqrt{2}\,t, e^t, e^{-t} \rangle$

25–27 Use Theorem 10 to find the curvature.

25. $\mathbf{r}(t) = t^3\,\mathbf{j} + t^2\,\mathbf{k}$

26. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + e^t\,\mathbf{k}$

27. $\mathbf{r}(t) = \sqrt{6}\,t^2\,\mathbf{i} + 2t\,\mathbf{j} + 2t^3\,\mathbf{k}$

28. Find the curvature of $\mathbf{r}(t) = \langle t^2, \ln t, t \ln t \rangle$ at the point $(1, 0, 0)$.

29. Find the curvature of $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at the point $(1, 1, 1)$.

30. Graph the curve with parametric equations $x = \cos t$, $y = \sin t$, $z = \sin 5t$ and find the curvature at the point $(1, 0, 0)$.

31–33 Use Formula 11 to find the curvature.

31. $y = x^4$

32. $y = \tan x$

33. $y = xe^x$

34–35 At what point does the curve have maximum curvature? What happens to the curvature as $x \to \infty$?

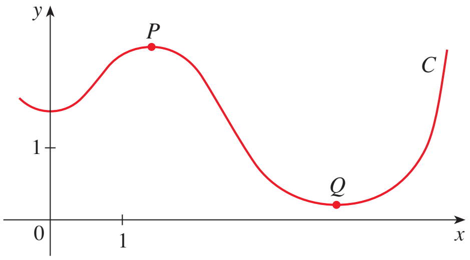
34. $y = \ln x$

35. $y = e^x$

---

36. Find an equation of a parabola that has curvature 4 at the origin.

37. (a) Is the curvature of the curve C shown in the figure greater at P or at Q? Explain.
(b) Estimate the curvature at P and at Q by sketching the osculating circles at those points.



38–39 Use a graphing calculator or computer to graph both the curve and its curvature function $\kappa(x)$ on the same screen. Is the graph of κ what you would expect?

38. $y = x^4 - 2x^2$

39. $y = x^{-2}$

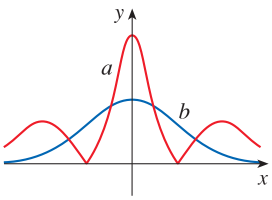
T **40–41** Use a computer algebra system to compute the curvature function $\kappa(t)$. Then graph the space curve and its curvature function. Comment on how the curvature reflects the shape of the curve.

40. $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t, 4\cos(t/2) \rangle, \quad 0 \leqslant t \leqslant 8\pi$

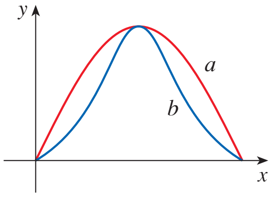
41. $\mathbf{r}(t) = \left\langle te^t, e^{-t}, \sqrt{2}\,t \right\rangle, \quad -5 \leqslant t \leqslant 5$

42–43 Two graphs, a and b, are shown. One is a curve $y = f(x)$ and the other is the graph of its curvature function $y = \kappa(x)$. Identify each curve and explain your choices.

42.



43.



T **44.** (a) Graph the curve $\mathbf{r}(t) = \langle \sin 3t, \sin 2t, \sin 3t \rangle$. At how many points on the curve does it appear that the curvature has a local or absolute maximum?
(b) Use a computer algebra system to find and graph the curvature function. Does this graph confirm your conclusion from part (a)?

T **45.** The graph of $\mathbf{r}(t) = \left\langle t - \frac{3}{2}\sin t, 1 - \frac{3}{2}\cos t, t \right\rangle$ is shown in Figure 13.1.13(b). Where do you think the curvature is largest? Use a computer algebra system to find and graph the curvature function. For which values of t is the curvature largest?

46–49 Curvature of Plane Parametric Curves The curvature of a plane parametric curve $x = f(t)$, $y = g(t)$ is given by

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$$

where the dots indicate derivatives with respect to t.

46. Use Theorem 10 to prove the given formula for curvature.

47. Find the curvature of the curve $x = t^2$, $y = t^3$.

48. Find the curvature of the curve $x = a \cos \omega t$, $y = b \sin \omega t$.

49. Find the curvature of the curve $x = e^t \cos t$, $y = e^t \sin t$.

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50. Consider the curvature at $x = 0$ for each member of the family of functions $f(x) = e^{cx}$. For which members is $\kappa(0)$ largest?

51–52 Find the vectors **T**, **N**, and **B** at the given point.

51. $\mathbf{r}(t) = \left\langle t^2, \frac{2}{3}t^3, t \right\rangle, \quad \left(1, \frac{2}{3}, 1\right)$

52. $\mathbf{r}(t) = \langle \cos t, \sin t, \ln \cos t \rangle, \quad (1, 0, 0)$

53–54 Find equations of the normal plane and osculating plane of the curve at the given point.

53. $x = \sin 2t, \; y = -\cos 2t, \; z = 4t; \quad (0, 1, 2\pi)$

54. $x = \ln t, \; y = 2t, \; z = t^2; \quad (0, 2, 1)$

55. Find equations of the osculating circles of the ellipse $9x^2 + 4y^2 = 36$ at the points $(2, 0)$ and $(0, 3)$. Use a graphing calculator or computer to graph the ellipse and both osculating circles on the same screen.

56. Find equations of the osculating circles of the parabola $y = \frac{1}{2}x^2$ at the points $(0, 0)$ and $\left(1, \frac{1}{2}\right)$. Graph both osculating circles and the parabola on the same screen.

57. At what point on the curve $x = t^3$, $y = 3t$, $z = t^4$ is the normal plane parallel to the plane $6x + 6y - 8z = 1$?

T **58.** Is there a point on the curve in Exercise 57 where the osculating plane is parallel to the plane $x + y + z = 1$? [*Note:* You will need a computer algebra system for differentiating, for simplifying, and for computing a cross product.]

59. Find equations of the normal and osculating planes of the curve of intersection of the parabolic cylinders $x = y^2$ and $z = x^2$ at the point $(1, 1, 1)$.

60. Show that the osculating plane at every point on the curve $\mathbf{r}(t) = \left\langle t + 2, 1 - t, \frac{1}{2}t^2 \right\rangle$ is the same plane. What can you conclude about the curve?

61. Show that at every point on the curve

$$\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle$$

the angle between the unit tangent vector and the z-axis is the same. Then show that the same result holds true for the unit normal and binormal vectors.

62. The Rectifying Plane The *rectifying plane* of a curve at a point is the plane that contains the vectors $\mathbf{T}$ and $\mathbf{B}$ at that point. Find the rectifying plane of the curve $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \tan t\,\mathbf{k}$ at the point $\left(\sqrt{2}/2, \sqrt{2}/2, 1\right)$.

63. Show that the curvature κ is related to the tangent and normal vectors by the equation

$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}$$

64. Show that the curvature of a plane curve is $\kappa = \left| d\phi/ds \right|$, where ϕ is the angle between $\mathbf{T}$ and $\mathbf{i}$; that is, ϕ is the angle of inclination of the tangent line. (This shows that the definition of curvature is consistent with the definition for plane curves given in Exercises 10.2.79–83.)

65. (a) Show that $d\mathbf{B}/ds$ is perpendicular to $\mathbf{B}$.
(b) Show that $d\mathbf{B}/ds$ is perpendicular to $\mathbf{T}$.
(c) Deduce from parts (a) and (b) that $d\mathbf{B}/ds$ is parallel to $\mathbf{N}$.

66–67 Use Formula 14 to find the torsion at the given value of t.

66. $\mathbf{r}(t) = \langle \sin t, 3t, \cos t \rangle, \quad t = \pi/2$

67. $\mathbf{r}(t) = \left\langle \frac{1}{2}t^2, 2t, t \right\rangle, \quad t = 1$

68–70 Use Theorem 15 to find the torsion of the given curve at a general point and at the point corresponding to $t = 0$.

68. $\mathbf{r}(t) = \left\langle t, \frac{1}{2}t^2, \frac{1}{3}t^3 \right\rangle$

69. $\mathbf{r}(t) = \langle e^t, e^{-t}, t \rangle$

70. $\mathbf{r}(t) = \langle \cos t, \sin t, \sin t \rangle$

71–72 Frenet-Serret Formulas The following formulas, called the *Frenet-Serret formulas*, are of fundamental importance in differential geometry:

1. $d\mathbf{T}/ds = \kappa \mathbf{N}$
2. $d\mathbf{N}/ds = -\kappa \mathbf{T} + \tau \mathbf{B}$
3. $d\mathbf{B}/ds = -\tau \mathbf{N}$

(Formula 1 comes from Exercise 63 and Formula 3 is Equation 12.)

71. Use the fact that $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ to deduce Formula 2 from Formulas 1 and 3.

72. Use the Frenet-Serret formulas to prove each of the following. (Primes denote derivatives with respect to t. Start as in the proof of Theorem 10.)
(a) $\mathbf{r}'' = s''\mathbf{T} + \kappa (s')^2 \mathbf{N}$
(b) $\mathbf{r}' \times \mathbf{r}'' = \kappa (s')^3 \mathbf{B}$
(c) $\mathbf{r}''' = [s''' - \kappa^2 (s')^3]\mathbf{T} + [3\kappa s' s'' + \kappa' (s')^2]\mathbf{N} + \kappa \tau (s')^3 \mathbf{B}$
(d) $\tau = \dfrac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}$

73. Show that the circular helix $\mathbf{r}(t) = \langle a \cos t, a \sin t, bt \rangle$, where a and b are positive constants, has constant curvature and constant torsion. (Use Theorem 15.)

74. Find the curvature and torsion of the curve $x = \sinh t$, $y = \cosh t$, $z = t$ at the point $(0, 1, 0)$.

75. Evolute of a Curve The *evolute* of a smooth curve C is the curve generated by the centers of curvature of C.
(a) Explain why the evolute of a curve given by $\mathbf{r}$ is

$$\mathbf{r}_e(t) = \mathbf{r}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t) \qquad \kappa(t) \neq 0$$

(b) Find the evolute of the helix in Example 6.
(c) Find the evolute of the parabola in Example 5.

76. Planar Curves A space curve C given by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ is called *planar* if it lies in a plane.
(a) Show that C is planar if and only if there exist scalars a, b, c, and d, not all zero, such that $ax(t) + by(t) + cz(t) = d$ for all t.
(b) Show that if C is planar, then the binormal vector $\mathbf{B}$ is normal to the plane containing C.
(c) Show that if C is a planar curve then the torsion of C is zero for all t.
(d) Show that the curve $\mathbf{r}(t) = \langle t, 2t, t^2 \rangle$ is planar and find an equation of the plane that contains the curve. Use this equation to find the binormal vector $\mathbf{B}$.

77. The DNA molecule has the shape of a double helix (see Figure 13.1.3). The radius of each helix is about 10 angstroms (1 Å $= 10^{-8}$ cm). Each helix rises about 34 Å during each complete turn, and there are about 2.9×10^8 complete turns. Estimate the length of each helix.

78. Let's consider the problem of designing a railroad track to make a smooth transition between sections of straight track. Existing track along the negative x-axis is to be joined smoothly to a track along the line $y = 1$ for $x \geqslant 1$.

(a) Find a polynomial $P = P(x)$ of degree 5 such that the function F defined by

$$F(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ P(x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geqslant 1 \end{cases}$$

is continuous and has continuous slope and continuous curvature.

(b) Graph F.

13.4 Motion in Space: Velocity and Acceleration

In this section we show how the ideas of tangent and normal vectors and curvature can be used in physics to study the motion of an object—including its velocity and acceleration—along a space curve. In particular, we follow in the footsteps of Newton by using these methods to derive Kepler's First Law of planetary motion.

Velocity, Speed, and Acceleration

Suppose a particle moves through space so that its position vector at time t is $\mathbf{r}(t)$. Notice from Figure 1 that, for small values of h, the vector

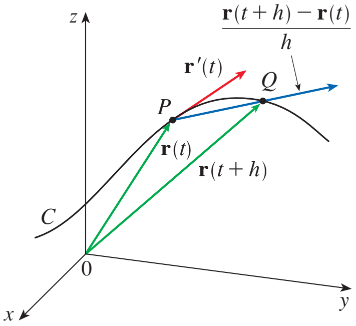


FIGURE 1

$$\boxed{1} \qquad \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

approximates the direction of the particle moving along the curve $\mathbf{r}(t)$. Its magnitude measures the size of the displacement vector per unit time. The vector (1) gives the average velocity over a time interval of length h and its limit is the **velocity vector** $\mathbf{v}(t)$ at time t:

$$\boxed{2} \qquad \boxed{\mathbf{v}(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t)}$$

Thus the velocity vector is also the tangent vector and points in the direction of the tangent line.

The **speed** of the particle at time t is the magnitude of the velocity vector, that is, $|\mathbf{v}(t)|$. This is appropriate because, from (2) and from Equation 13.3.7, we have

Compare to Equation 10.2.8, where we defined speed for plane parametric curves.

$$|\mathbf{v}(t)| = |\mathbf{r}'(t)| = \frac{ds}{dt} = \text{rate of change of distance with respect to time}$$

As in the case of one-dimensional motion, the **acceleration** of the particle is defined as the derivative of the velocity:

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$$

EXAMPLE 1 The position vector of an object moving in a plane is given by $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j}$. Find its velocity, speed, and acceleration when $t = 1$ and illustrate geometrically.

SOLUTION The velocity and acceleration at time t are

$$\mathbf{v}(t) = \mathbf{r}'(t) = 3t^2\,\mathbf{i} + 2t\,\mathbf{j} \qquad \mathbf{a}(t) = \mathbf{r}''(t) = 6t\,\mathbf{i} + 2\,\mathbf{j}$$

and the speed is

$$|\mathbf{v}(t)| = \sqrt{(3t^2)^2 + (2t)^2} = \sqrt{9t^4 + 4t^2}$$

When $t = 1$, we have

$$\mathbf{v}(1) = 3\,\mathbf{i} + 2\,\mathbf{j} \qquad \mathbf{a}(1) = 6\,\mathbf{i} + 2\,\mathbf{j} \qquad |\mathbf{v}(1)| = \sqrt{13}$$

These velocity and acceleration vectors are shown in Figure 2. ■

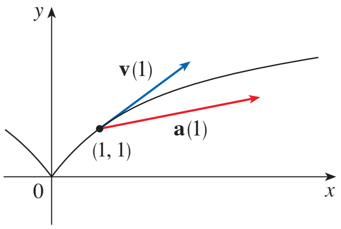


FIGURE 2

EXAMPLE 2 Find the velocity, acceleration, and speed of a particle with position vector $\mathbf{r}(t) = \langle t^2, e^t, te^t \rangle$.

SOLUTION

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, e^t, (1 + t)e^t \rangle$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, e^t, (2 + t)e^t \rangle$$

$$|\mathbf{v}(t)| = \sqrt{4t^2 + e^{2t} + (1 + t)^2 e^{2t}}$$ ■

Figure 3 shows the path of the particle in Example 2 with the velocity and acceleration vectors when $t = 1$.

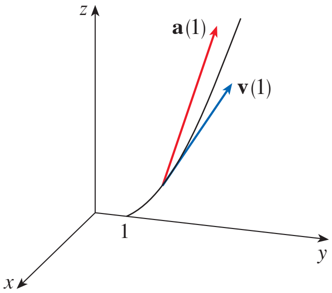


FIGURE 3

NOTE Earlier in the chapter we saw that a curve can be parametrized in different ways but the geometric properties of a curve—arc length, curvature, and torsion—are independent of the choice of parametrization. On the other hand, velocity, speed, and acceleration *do* depend on the parametrizations used. You can think of the curve as a road and a parametrization as describing how you travel along that road. The length and curvature of the road do not depend on how you travel on it, but your velocity and acceleration do.

The vector integrals that were introduced in Section 13.2 can be used to find position vectors when velocity or acceleration vectors are known, as in the next example.

EXAMPLE 3 A moving particle starts at an initial position $\mathbf{r}(0) = \langle 1, 0, 0 \rangle$ with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$. Its acceleration is $\mathbf{a}(t) = 4t\,\mathbf{i} + 6t\,\mathbf{j} + \mathbf{k}$. Find its velocity and position at time t.

SOLUTION Since $\mathbf{a}(t) = \mathbf{v}'(t)$, we have

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int (4t\,\mathbf{i} + 6t\,\mathbf{j} + \mathbf{k})\,dt$$

$$= 2t^2\,\mathbf{i} + 3t^2\,\mathbf{j} + t\,\mathbf{k} + \mathbf{C}$$

To determine the value of the constant vector $\mathbf{C}$, we use the fact that $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$. The preceding equation gives $\mathbf{v}(0) = \mathbf{C}$, so $\mathbf{C} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and

$$\mathbf{v}(t) = 2t^2\,\mathbf{i} + 3t^2\,\mathbf{j} + t\,\mathbf{k} + \mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$= (2t^2 + 1)\,\mathbf{i} + (3t^2 - 1)\,\mathbf{j} + (t + 1)\,\mathbf{k}$$

Since $\mathbf{v}(t) = \mathbf{r}'(t)$, we have

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt$$

$$= \int [(2t^2 + 1)\,\mathbf{i} + (3t^2 - 1)\,\mathbf{j} + (t + 1)\,\mathbf{k}]\,dt$$

$$= \left(\tfrac{2}{3}t^3 + t\right)\mathbf{i} + (t^3 - t)\,\mathbf{j} + \left(\tfrac{1}{2}t^2 + t\right)\mathbf{k} + \mathbf{D}$$

Putting $t = 0$, we find that $\mathbf{D} = \mathbf{r}(0) = \mathbf{i}$, so the position at time t is given by

$$\mathbf{r}(t) = \left(\tfrac{2}{3}t^3 + t + 1\right)\mathbf{i} + (t^3 - t)\,\mathbf{j} + \left(\tfrac{1}{2}t^2 + t\right)\mathbf{k}$$ ■

The expression for $\mathbf{r}(t)$ that we obtained in Example 3 was used to plot the path of the particle in Figure 4 for $0 \leqslant t \leqslant 3$.

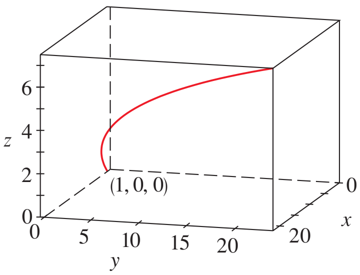


FIGURE 4

In general, vector integrals allow us to recover velocity when acceleration is known and position when velocity is known:

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}(u)\,du \qquad \mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(u)\,du$$

If the force that acts on a particle is known, then the acceleration can be found from **Newton's Second Law of Motion**. The vector version of this law states that if, at any time t, a force $\mathbf{F}(t)$ acts on an object of mass m producing an acceleration $\mathbf{a}(t)$, then

$$\mathbf{F}(t) = m\mathbf{a}(t)$$

EXAMPLE 4 An object with mass m that moves in a circular path with constant angular speed ω has position vector $\mathbf{r}(t) = a\cos\omega t\,\mathbf{i} + a\sin\omega t\,\mathbf{j}$. Find the force acting on the object and show that it is directed toward the origin.

The object moving with position P has angular speed $\omega = d\theta/dt$, where θ is the angle shown in Figure 5.

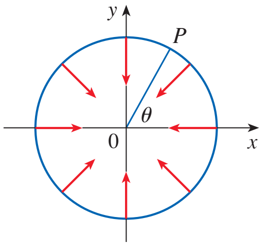


FIGURE 5

SOLUTION To find the force, we first need to know the acceleration:

$$\mathbf{v}(t) = \mathbf{r}'(t) = -a\omega\sin\omega t\,\mathbf{i} + a\omega\cos\omega t\,\mathbf{j}$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = -a\omega^2\cos\omega t\,\mathbf{i} - a\omega^2\sin\omega t\,\mathbf{j}$$

Therefore Newton's Second Law gives the force as

$$\mathbf{F}(t) = m\mathbf{a}(t) = -m\omega^2(a\cos\omega t\,\mathbf{i} + a\sin\omega t\,\mathbf{j})$$

Notice that $\mathbf{F}(t) = -m\omega^2\mathbf{r}(t)$. This shows that the force acts in the direction opposite to the radius vector $\mathbf{r}(t)$ and therefore points toward the origin (see Figure 5). Such a force is called a *centripetal* (center-seeking) force. ■

■ Projectile Motion

EXAMPLE 5 A projectile is fired with angle of elevation α and initial velocity $\mathbf{v}_0$. (See Figure 6.) Assuming that air resistance is negligible and the only external force is due to gravity, find the position function $\mathbf{r}(t)$ of the projectile. What value of α maximizes the range (the horizontal distance traveled)?

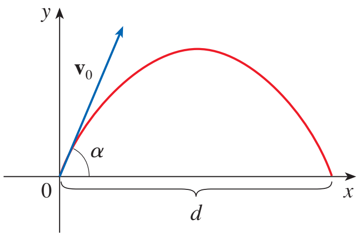


FIGURE 6

SOLUTION We set up the axes so that the projectile starts at the origin. Since the force due to gravity acts downward, we have

$$\mathbf{F} = m\mathbf{a} = -mg\,\mathbf{j}$$

where $g = |\mathbf{a}| \approx 9.8\ \text{m/s}^2$. Thus

$$\mathbf{a} = -g\,\mathbf{j}$$

Since $\mathbf{v}'(t) = \mathbf{a}$, we have $\qquad \mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{C}$

where $\mathbf{C} = \mathbf{v}(0) = \mathbf{v}_0$. Therefore

$$\mathbf{r}'(t) = \mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{v}_0$$

Integrating again, we obtain

$$\mathbf{r}(t) = -\tfrac{1}{2}gt^2\,\mathbf{j} + t\,\mathbf{v}_0 + \mathbf{D}$$

But $\mathbf{D} = \mathbf{r}(0) = \mathbf{0}$, so the position vector of the projectile is given by

$$\boxed{3} \qquad \mathbf{r}(t) = -\tfrac{1}{2}gt^2\,\mathbf{j} + t\,\mathbf{v}_0$$

If we write $|\mathbf{v}_0| = v_0$ (the initial speed of the projectile), then

$$\mathbf{v}_0 = v_0 \cos\alpha\,\mathbf{i} + v_0 \sin\alpha\,\mathbf{j}$$

and Equation 3 becomes

$$\mathbf{r}(t) = (v_0\cos\alpha)t\,\mathbf{i} + \left[(v_0\sin\alpha)t - \tfrac{1}{2}gt^2\right]\mathbf{j}$$

The parametric equations of the trajectory are therefore

If you eliminate t from Equations 4, you will see that y is a quadratic function of x. So the path of the projectile is part of a parabola.

$$\boxed{4} \qquad \boxed{x = (v_0\cos\alpha)t \qquad y = (v_0\sin\alpha)t - \tfrac{1}{2}gt^2}$$

The horizontal distance d is the value of x when $y = 0$. Setting $y = 0$, we obtain $t = 0$ or $t = (2v_0\sin\alpha)/g$. This second value of t then gives

$$d = x = (v_0\cos\alpha)\frac{2v_0\sin\alpha}{g} = \frac{v_0^2(2\sin\alpha\cos\alpha)}{g} = \frac{v_0^2\sin 2\alpha}{g}$$

Clearly, d has its maximum value when $\sin 2\alpha = 1$, that is, $\alpha = 45°$. ■

EXAMPLE 6 A projectile is fired with initial speed 150 m/s and angle of elevation 30° from a position 10 m above ground level. Where does the projectile hit the ground, and with what speed?

SOLUTION If we place the origin at ground level, then the initial position of the projectile is (0, 10) and so we need to adjust Equations 4 by adding 10 to the expression for y. With $v_0 = 150$ m/s, $\alpha = 30°$, and $g = 9.8\ \text{m/s}^2$, we have

$$x = 150\cos(30°)t = 75\sqrt{3}\,t$$

$$y = 10 + 150\sin(30°)t - \tfrac{1}{2}(9.8)t^2 = 10 + 75t - 4.9t^2$$

Impact occurs when $y = 0$, that is, $4.9t^2 - 75t - 10 = 0$. Using the quadratic formula to solve this equation (and taking only the positive value of t), we get

$$t = \frac{75 + \sqrt{5625 + 196}}{9.8} \approx 15.44$$

Then $x \approx 75\sqrt{3}(15.44) \approx 2006$, so the projectile hits the ground about 2006 m away.

The velocity of the projectile is

$$\mathbf{v}(t) = \mathbf{r}'(t) = 75\sqrt{3}\,\mathbf{i} + (75 - 9.8t)\,\mathbf{j}$$

So its speed at impact is

$$|\mathbf{v}(15.44)| = \sqrt{(75\sqrt{3})^2 + (75 - 9.8 \cdot 15.44)^2} \approx 151 \text{ m/s}$$ ■

Tangential and Normal Components of Acceleration

When we study the motion of a particle, it is often useful to resolve the acceleration into two components, one in the direction of the tangent and the other in the direction of the normal. If we write $v = |\mathbf{v}|$ for the speed of the particle, then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{v}}{v}$$

and so

$$\mathbf{v} = v\mathbf{T}$$

If we differentiate both sides of this equation with respect to t, we get

$$\mathbf{a} = \mathbf{v}' = v'\mathbf{T} + v\mathbf{T}' \tag{5}$$

If we use the expression for the curvature given by Equation 13.3.9, then we have

$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{T}'|}{v} \qquad \text{so} \qquad |\mathbf{T}'| = \kappa v \tag{6}$$

The unit normal vector was defined in Section 13.3 as $\mathbf{N} = \mathbf{T}'/|\mathbf{T}'|$, so (6) gives

$$\mathbf{T}' = |\mathbf{T}'|\mathbf{N} = \kappa v\mathbf{N}$$

and Equation 5 becomes

$$\boxed{\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}} \tag{7}$$

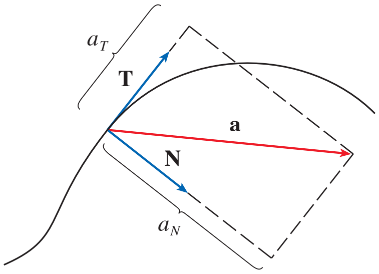


FIGURE 7

Writing a_T and a_N for the tangential and normal components of acceleration, we have

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$$

where

$$a_T = v' \qquad \text{and} \qquad a_N = \kappa v^2 \tag{8}$$

This resolution is illustrated in Figure 7.

Let's look at what Formula 7 says. The first thing to notice is that the binormal vector **B** is absent. No matter how an object moves through space, its acceleration always lies in the plane of **T** and **N** (the osculating plane). (Recall that **T** gives the direction of motion and **N** points in the direction the curve is turning.) Next we notice that the tangential component of acceleration is v', the rate of change of speed, and the normal component of acceleration is κv^2, the curvature times the square of the speed. This makes sense if we think of a passenger in a car—a sharp turn in a road means a large value of the curvature κ, so the component of the acceleration perpendicular to the motion is large and the passenger is thrown against the car door. High speed around the turn has the same effect; in fact, if you double your speed, a_N is increased by a factor of 4.

Although we have expressions for the tangential and normal components of acceleration in Equations 8, it's desirable to have expressions that depend only on $\mathbf{r}$, $\mathbf{r}'$, and $\mathbf{r}''$. To this end we take the dot product of $\mathbf{v} = v\mathbf{T}$ with $\mathbf{a}$ as given by Equation 7:

$$\begin{aligned}\mathbf{v}\cdot\mathbf{a} &= v\mathbf{T}\cdot(v'\mathbf{T} + \kappa v^2\mathbf{N})\\ &= vv'\mathbf{T}\cdot\mathbf{T} + \kappa v^3\mathbf{T}\cdot\mathbf{N}\\ &= vv' \qquad (\text{since } \mathbf{T}\cdot\mathbf{T} = 1 \text{ and } \mathbf{T}\cdot\mathbf{N} = 0)\end{aligned}$$

Therefore

$$a_T = v' = \frac{\mathbf{v}\cdot\mathbf{a}}{v} = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} \tag{9}$$

Using the formula for curvature given by Theorem 13.3.10, we have

$$a_N = \kappa v^2 = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}|\mathbf{r}'(t)|^2 = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} \tag{10}$$

EXAMPLE 7 A particle moves with position function $\mathbf{r}(t) = \langle t^2, t^2, t^3 \rangle$. Find the tangential and normal components of acceleration.

SOLUTION

$$\begin{aligned} \mathbf{r}(t) &= t^2\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k} \\ \mathbf{r}'(t) &= 2t\,\mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k} \\ \mathbf{r}''(t) &= 2\,\mathbf{i} + 2\,\mathbf{j} + 6t\,\mathbf{k} \\ |\mathbf{r}'(t)| &= \sqrt{8t^2 + 9t^4} \end{aligned}$$

Therefore Equation 9 gives the tangential component as

$$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}}$$

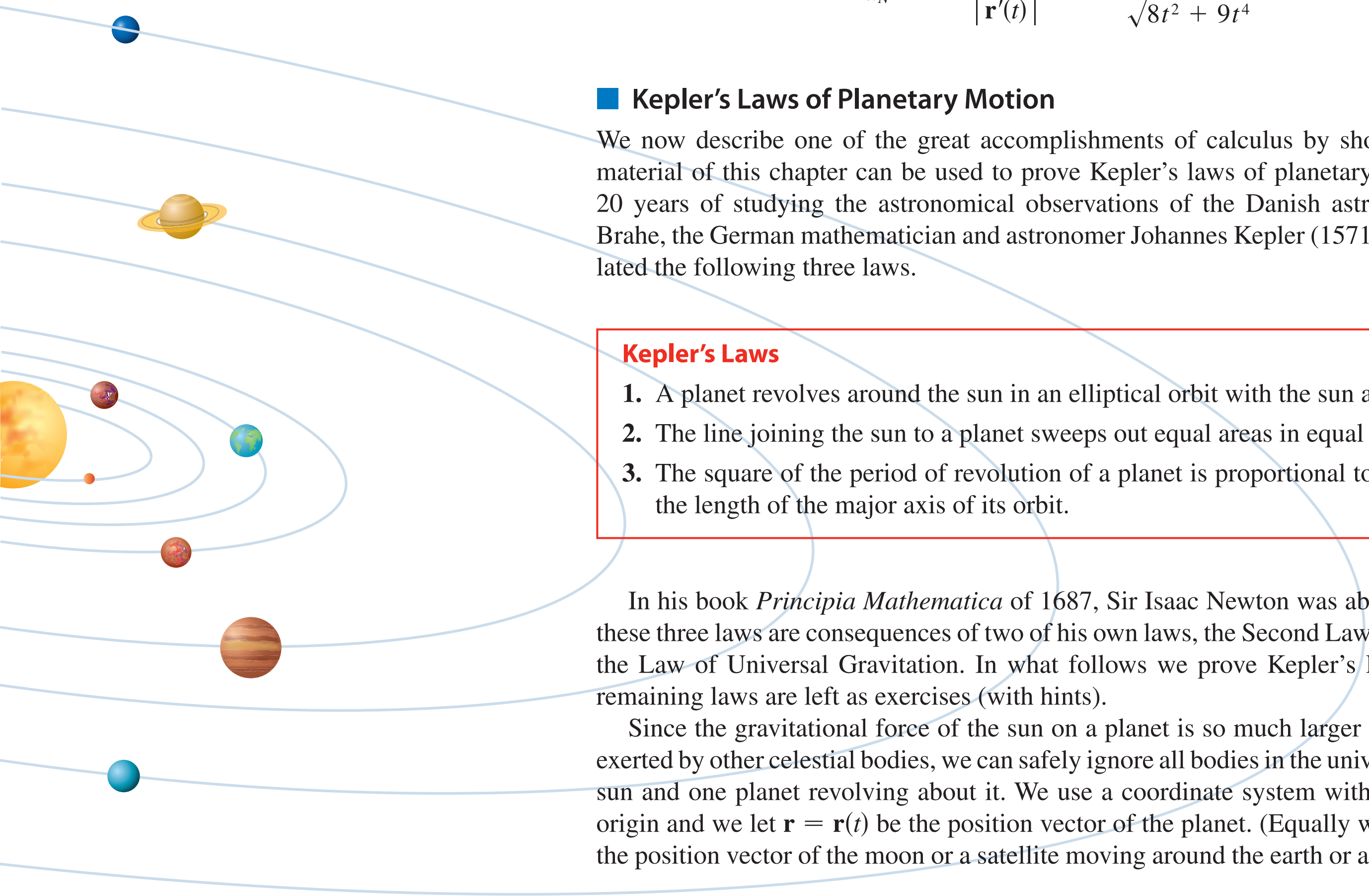
Since

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t & 2t & 3t^2 \\ 2 & 2 & 6t \end{vmatrix} = 6t^2\,\mathbf{i} - 6t^2\,\mathbf{j}$$

Equation 10 gives the normal component as

$$a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{6\sqrt{2}\,t^2}{\sqrt{8t^2 + 9t^4}}$$

■



Kepler's Laws of Planetary Motion

We now describe one of the great accomplishments of calculus by showing how the material of this chapter can be used to prove Kepler's laws of planetary motion. After 20 years of studying the astronomical observations of the Danish astronomer Tycho Brahe, the German mathematician and astronomer Johannes Kepler (1571–1630) formulated the following three laws.

Kepler's Laws

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.
2. The line joining the sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

In his book *Principia Mathematica* of 1687, Sir Isaac Newton was able to show that these three laws are consequences of two of his own laws, the Second Law of Motion and the Law of Universal Gravitation. In what follows we prove Kepler's First Law. The remaining laws are left as exercises (with hints).

Since the gravitational force of the sun on a planet is so much larger than the forces exerted by other celestial bodies, we can safely ignore all bodies in the universe except the sun and one planet revolving about it. We use a coordinate system with the sun at the origin and we let $\mathbf{r} = \mathbf{r}(t)$ be the position vector of the planet. (Equally well, $\mathbf{r}$ could be the position vector of the moon or a satellite moving around the earth or a comet moving around a star.) The velocity vector is $\mathbf{v} = \mathbf{r}'$ and the acceleration vector is $\mathbf{a} = \mathbf{r}''$. We use the following laws of Newton:

$$\text{Second Law of Motion:} \qquad \mathbf{F} = m\mathbf{a}$$

$$\text{Law of Gravitation:} \qquad \mathbf{F} = -\frac{GMm}{r^3}\mathbf{r} = -\frac{GMm}{r^2}\mathbf{u}$$

where $\mathbf{F}$ is the gravitational force on the planet, m and M are the masses of the planet and the sun, G is the gravitational constant, $r = |\mathbf{r}|$, and $\mathbf{u} = (1/r)\mathbf{r}$ is the unit vector in the direction of $\mathbf{r}$.

We first show that the planet moves in one plane. By equating the expressions for $\mathbf{F}$ in Newton's two laws, we find that

$$\mathbf{a} = -\frac{GM}{r^3}\mathbf{r}$$

and so $\mathbf{a}$ is parallel to $\mathbf{r}$. It follows that $\mathbf{r} \times \mathbf{a} = \mathbf{0}$. We use Formula 5 in Theorem 13.2.3 to write

$$\begin{aligned} \frac{d}{dt}(\mathbf{r} \times \mathbf{v}) &= \mathbf{r}' \times \mathbf{v} + \mathbf{r} \times \mathbf{v}' \\ &= \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a} = \mathbf{0} + \mathbf{0} = \mathbf{0} \end{aligned}$$

Therefore

$$\mathbf{r} \times \mathbf{v} = \mathbf{h}$$

where $\mathbf{h}$ is a constant vector. (We may assume that $\mathbf{h} \neq \mathbf{0}$; that is, $\mathbf{r}$ and $\mathbf{v}$ are not parallel.) This means that the vector $\mathbf{r} = \mathbf{r}(t)$ is perpendicular to $\mathbf{h}$ for all values of t, so the planet always lies in the plane through the origin perpendicular to $\mathbf{h}$. Thus the orbit of the planet is a plane curve.

To prove Kepler's First Law we rewrite the vector $\mathbf{h}$ as follows:

$$\begin{aligned} \mathbf{h} &= \mathbf{r} \times \mathbf{v} = \mathbf{r} \times \mathbf{r}' = r\mathbf{u} \times (r\mathbf{u})' \\ &= r\mathbf{u} \times (r\mathbf{u}' + r'\mathbf{u}) = r^2(\mathbf{u} \times \mathbf{u}') + rr'(\mathbf{u} \times \mathbf{u}) \\ &= r^2(\mathbf{u} \times \mathbf{u}') \end{aligned}$$

Then

$$\begin{aligned} \mathbf{a} \times \mathbf{h} &= \frac{-GM}{r^2}\mathbf{u} \times (r^2\mathbf{u} \times \mathbf{u}') = -GM\,\mathbf{u} \times (\mathbf{u} \times \mathbf{u}') \\ &= -GM[(\mathbf{u} \cdot \mathbf{u}')\mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{u}'] \qquad \text{(by Theorem 12.4.11, Property 6)} \end{aligned}$$

But $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 = 1$ and, since $|\mathbf{u}(t)| = 1$, it follows from Theorem 13.2.4 that

$$\mathbf{u} \cdot \mathbf{u}' = 0$$

Therefore

$$\mathbf{a} \times \mathbf{h} = GM\,\mathbf{u}'$$

and so

$$(\mathbf{v} \times \mathbf{h})' = \mathbf{v}' \times \mathbf{h} + \mathbf{v} \times \mathbf{h}' = \mathbf{v}' \times \mathbf{h} = \mathbf{a} \times \mathbf{h} = GM\,\mathbf{u}'$$

Integrating both sides of this equation, we get

$$\boxed{11} \qquad \mathbf{v} \times \mathbf{h} = GM\,\mathbf{u} + \mathbf{c}$$

where $\mathbf{c}$ is a constant vector.

At this point it is convenient to choose the coordinate axes so that the standard basis vector $\mathbf{k}$ points in the direction of the vector $\mathbf{h}$. Then the planet moves in the

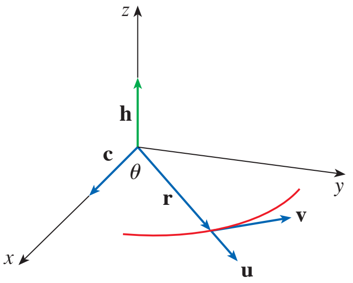


FIGURE 8

xy-plane. Since both $\mathbf{v} \times \mathbf{h}$ and $\mathbf{u}$ are perpendicular to $\mathbf{h}$, Equation 11 shows that $\mathbf{c}$ lies in the xy-plane. This means that we can choose the x- and y-axes so that the vector $\mathbf{i}$ lies in the direction of $\mathbf{c}$, as shown in Figure 8.

If θ is the angle between $\mathbf{c}$ and $\mathbf{r}$, then (r, θ) are polar coordinates of the planet. From Equation 11 we have

$$\begin{aligned} \mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) &= \mathbf{r} \cdot (GM\,\mathbf{u} + \mathbf{c}) = GM\,\mathbf{r} \cdot \mathbf{u} + \mathbf{r} \cdot \mathbf{c} \\ &= GMr\,\mathbf{u} \cdot \mathbf{u} + |\mathbf{r}||\mathbf{c}|\cos\theta = GMr + rc\cos\theta \end{aligned}$$

where $c = |\mathbf{c}|$. Then

$$r = \frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{GM + c\cos\theta} = \frac{1}{GM}\,\frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{1 + e\cos\theta}$$

where $e = c/(GM)$. But

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{h} = \mathbf{h} \cdot \mathbf{h} = |\mathbf{h}|^2 = h^2$$

where $h = |\mathbf{h}|$. So

$$r = \frac{h^2/(GM)}{1 + e\cos\theta} = \frac{eh^2/c}{1 + e\cos\theta}$$

Writing $d = h^2/c$, we obtain the equation

$$r = \frac{ed}{1 + e\cos\theta} \tag{12}$$

Comparing with Theorem 10.6.6, we see that Equation 12 is the polar equation of a conic section with focus at the origin and eccentricity e. We know that the orbit of a planet is a closed curve and so the conic must be an ellipse.

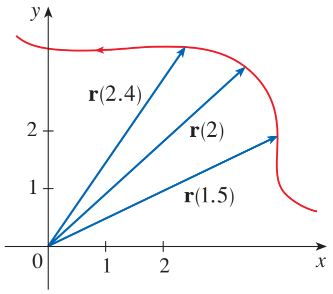
This completes the derivation of Kepler's First Law. We will guide you through the derivation of the Second and Third Laws in the Applied Project following this section. The proofs of these three laws show that the methods of this chapter provide a powerful tool for describing some of the laws of nature.

13.4 Exercises

1. The table gives coordinates of a particle moving through space along a smooth curve.
   (a) Find the average velocities over the time intervals $[0, 1]$, $[0.5, 1]$, $[1, 2]$, and $[1, 1.5]$.
   (b) Estimate the velocity and speed of the particle at $t = 1$.

t	x	y	z
0	2.7	9.8	3.7
0.5	3.5	7.2	3.3
1.0	4.5	6.0	3.0
1.5	5.9	6.4	2.8
2.0	7.3	7.8	2.7

2. The figure shows the path of a particle that moves with position vector $\mathbf{r}(t)$ at time t.
   (a) Draw a vector that represents the average velocity of the particle over the time interval $2 \leqslant t \leqslant 2.4$.
   (b) Draw a vector that represents the average velocity over the time interval $1.5 \leqslant t \leqslant 2$.
   (c) Write an expression for the velocity vector $\mathbf{v}(2)$.
   (d) Draw an approximation to the vector $\mathbf{v}(2)$ and estimate the speed of the particle at $t = 2$.



3–8 Find the velocity, acceleration, and speed of a particle with the given position function. Sketch the path of the particle and draw the velocity and acceleration vectors for the specified value of t.

3. $\mathbf{r}(t) = \left\langle -\frac{1}{2}t^2, t \right\rangle, \quad t = 2$

4. $\mathbf{r}(t) = \langle t^2, 1/t^2 \rangle, \quad t = 1$

5. $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}, \quad t = \pi/3$

6. $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{2t}\,\mathbf{j}, \quad t = 0$

7. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\mathbf{j} + 2\,\mathbf{k}, \quad t = 1$

8. $\mathbf{r}(t) = t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \sin t\,\mathbf{k}, \quad t = 0$

9–14 Find the velocity, acceleration, and speed of a particle with the given position function.

9. $\mathbf{r}(t) = \langle t^2 + t, t^2 - t, t^3 \rangle$

10. $\mathbf{r}(t) = \langle 2\cos t, 3t, 2\sin t \rangle$

11. $\mathbf{r}(t) = \sqrt{2}\,t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}$

12. $\mathbf{r}(t) = t^2\,\mathbf{i} + 2t\,\mathbf{j} + \ln t\,\mathbf{k}$

13. $\mathbf{r}(t) = e^t(\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k})$

14. $\mathbf{r}(t) = \langle t^2, \sin t - t\cos t, \cos t + t\sin t \rangle, \quad t \geqslant 0$

15–16 Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position.

15. $\mathbf{a}(t) = 2\,\mathbf{i} + 2t\,\mathbf{k}, \quad \mathbf{v}(0) = 3\,\mathbf{i} - \mathbf{j}, \quad \mathbf{r}(0) = \mathbf{j} + \mathbf{k}$

16. $\mathbf{a}(t) = \sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + 6t\,\mathbf{k}, \quad \mathbf{v}(0) = -\mathbf{k},$
$\mathbf{r}(0) = \mathbf{j} - 4\,\mathbf{k}$

17–18

(a) Find the position vector of a particle that has the given acceleration and the specified initial velocity and position.

(b) Graph the path of the particle.

17. $\mathbf{a}(t) = 2t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos 2t\,\mathbf{k}, \quad \mathbf{v}(0) = \mathbf{i}, \quad \mathbf{r}(0) = \mathbf{j}$

18. $\mathbf{a}(t) = t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}, \quad \mathbf{v}(0) = \mathbf{k}, \quad \mathbf{r}(0) = \mathbf{j} + \mathbf{k}$

19. The position function of a particle is given by $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t \rangle$. When is the speed a minimum?

20. What force is required so that a particle of mass m has the position function $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$?

21. A force with magnitude 20 N acts directly upward from the xy-plane on an object with mass 4 kg. The object starts at the origin with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$. Find its position function and its speed at time t.

22. Show that if a particle moves with constant speed, then the velocity and acceleration vectors are orthogonal.

23. A projectile is fired with an initial speed of 200 m/s and angle of elevation 60°. Find (a) the range of the projectile, (b) the maximum height reached, and (c) the speed at impact.

24. Rework Exercise 23 if the projectile is fired from a position 100 m above the ground.

25. A ball is thrown upward at an angle of 45° to the ground. If the ball lands 90 m away, what was the initial speed of the ball?

26. A projectile is fired from a tank with initial speed 400 m/s. Find two angles of elevation that can be used to hit a target 3000 m away.

27. A rifle is fired with angle of elevation 36°. What is the initial speed if the maximum height of the bullet is 500 m?

28. A batter hits a baseball 1 m above the ground toward the center field fence, which is 4 m high and 120 m from home plate. The ball leaves the bat with speed 35 m/s at an angle 50° above the horizontal. Is it a home run? (In other words, does the ball clear the fence?)

29. A medieval city has the shape of a square and is protected by walls with length 500 m and height 15 m. You are the commander of an attacking army and the closest you can get to the wall is 100 m. Your plan is to set fire to the city by catapulting heated rocks over the wall (with an initial speed of 80 m/s). At what range of angles should you tell your men to set the catapult? (Assume the path of the rocks is perpendicular to the wall.)

30. Show that a projectile reaches three-quarters of its maximum height in half the time needed to reach its maximum height.

31. A ball is thrown eastward into the air from the origin (in the direction of the positive x-axis). The initial velocity is $50\,\mathbf{i} + 80\,\mathbf{k}$, with speed measured in meters per second. The spin of the ball results in a southward acceleration of 4 m/s^2, so the acceleration vector is $\mathbf{a} = -4\,\mathbf{j} - 32\,\mathbf{k}$. Where does the ball land and with what speed?

32. A ball with mass 0.8 kg is thrown southward into the air with a speed of 30 m/s at an angle of 30° to the ground. A west wind applies a steady force of 4 N to the ball in an easterly direction. Where does the ball land and with what speed?

33. Water traveling along a straight portion of a river normally flows fastest in the middle, and the speed slows to almost zero at the banks. Consider a long straight stretch of river flowing north, with parallel banks 40 m apart. If the maximum water speed is 3 m/s, we can use a quadratic function as a basic model for the rate of water flow x units from the west bank: $f(x) = \frac{3}{400}x(40 - x)$.

(a) A boat proceeds at a constant speed of 5 m/s from a point A on the west bank while maintaining a heading perpendicular to the bank. How far down the river on the opposite bank will the boat touch shore? Graph the path of the boat.

(b) Suppose we would like to pilot the boat to land at the point B on the east bank directly opposite A. If we maintain a constant speed of 5 m/s and a constant heading, find the angle at which the boat should head. Then graph the actual path the boat follows. Does the path seem realistic?

34. Another reasonable model for the water speed of the river in Exercise 33 is a sine function: $f(x) = 3\sin(\pi x/40)$. If a boater would like to cross the river from A to B with constant heading and a constant speed of 5 m/s, determine the angle at which the boat should head.

35. A particle has position function $\mathbf{r}(t)$. If $\mathbf{r}'(t) = \mathbf{c} \times \mathbf{r}(t)$, where $\mathbf{c}$ is a constant vector, describe the path of the particle.

36. (a) If a particle moves along a straight line, what can you say about its acceleration vector?

(b) If a particle moves with constant speed along a curve, what can you say about its acceleration vector?

37–40 Find the tangential and normal components of the acceleration vector.

37. $\mathbf{r}(t) = (t^2 + 1)\,\mathbf{i} + t^3\,\mathbf{j}, \quad t \geqslant 0$

38. $\mathbf{r}(t) = 2t^2\,\mathbf{i} + \left(\frac{2}{3}t^3 - 2t\right)\mathbf{j}$

39. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$

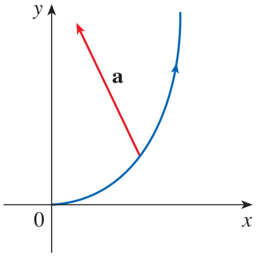
40. $\mathbf{r}(t) = t\,\mathbf{i} + 2e^t\,\mathbf{j} + e^{2t}\,\mathbf{k}$

41–42 Find the tangential and normal components of the acceleration vector at the given point.

41. $\mathbf{r}(t) = \ln t\,\mathbf{i} + (t^2 + 3t)\,\mathbf{j} + 4\sqrt{t}\,\mathbf{k}, \quad (0, 4, 4)$

42. $\mathbf{r}(t) = \dfrac{1}{t}\,\mathbf{i} + \dfrac{1}{t^2}\,\mathbf{j} + \dfrac{1}{t^3}\,\mathbf{k}, \quad (1, 1, 1)$

43. The magnitude of the acceleration vector $\mathbf{a}$ is 10 cm/s². Use the figure to estimate the tangential and normal components of $\mathbf{a}$.



44. Angular Momentum and Torque If a particle with mass m moves with position vector $\mathbf{r}(t)$, then its *angular momentum* is defined as $\mathbf{L}(t) = m\mathbf{r}(t) \times \mathbf{v}(t)$ and its *torque* as $\boldsymbol{\tau}(t) = m\mathbf{r}(t) \times \mathbf{a}(t)$. Show that $\mathbf{L}'(t) = \boldsymbol{\tau}(t)$. Deduce that if $\boldsymbol{\tau}(t) = \mathbf{0}$ for all t, then $\mathbf{L}(t)$ is constant. (This is the *law of conservation of angular momentum.*)

45. The position function of a spacecraft is

$$\mathbf{r}(t) = (3 + t)\,\mathbf{i} + (2 + \ln t)\,\mathbf{j} + \left(7 - \frac{4}{t^2 + 1}\right)\mathbf{k}$$

and the coordinates of a space station are (6, 4, 9). The captain wants the craft to coast into the space station. When should the engines be turned off?

46. A rocket burning its onboard fuel while moving through space has velocity $\mathbf{v}(t)$ and mass $m(t)$ at time t. If the exhaust gases escape with velocity $\mathbf{v}_e$ relative to the rocket, it can be deduced from Newton's Second Law of Motion that

$$m\frac{d\mathbf{v}}{dt} = \frac{dm}{dt}\mathbf{v}_e$$

(a) Show that $\mathbf{v}(t) = \mathbf{v}(0) - \ln\dfrac{m(0)}{m(t)}\mathbf{v}_e$.

(b) For the rocket to accelerate in a straight line from rest to twice the speed of its own exhaust gases, what fraction of its initial mass would the rocket have to burn as fuel?

APPLIED PROJECT | KEPLER'S LAWS

Johannes Kepler stated the following three laws of planetary motion on the basis of massive amounts of data on the positions of the planets at various times.

Kepler's Laws

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.
2. The line joining the sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

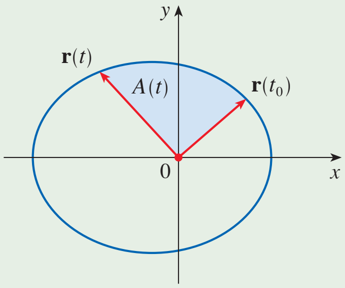
(continued)

Kepler formulated these laws because they fitted the astronomical data. He wasn't able to see why they were true or how they related to each other. But Sir Isaac Newton, in his *Principia Mathematica* of 1687, showed how to deduce Kepler's three laws from two of Newton's own laws, the Second Law of Motion and the Law of Universal Gravitation. In Section 13.4 we proved Kepler's First Law using the calculus of vector functions. In this project we guide you through the proofs of Kepler's Second and Third Laws and explore some of their consequences.

1. Use the following steps to prove Kepler's Second Law. The notation is the same as in the proof of the First Law in Section 13.4. In particular, use polar coordinates so that $\mathbf{r} = (r\cos\theta)\,\mathbf{i} + (r\sin\theta)\,\mathbf{j}$.

(a) Show that $\mathbf{h} = r^2 \dfrac{d\theta}{dt}\mathbf{k}$.

(b) Deduce that $r^2 \dfrac{d\theta}{dt} = h$.



(c) If $A = A(t)$ is the area swept out by the radius vector $\mathbf{r} = \mathbf{r}(t)$ in the time interval $[t_0, t]$ as in the figure, show that

$$\frac{dA}{dt} = \tfrac{1}{2}r^2\frac{d\theta}{dt}$$

(d) Deduce that

$$\frac{dA}{dt} = \tfrac{1}{2}h = \text{constant}$$

This says that the rate at which A is swept out is constant and proves Kepler's Second Law.

2. Let T be the period of a planet about the sun; that is, T is the time required for it to travel once around its elliptical orbit. Suppose that the lengths of the major and minor axes of the ellipse are $2a$ and $2b$.

(a) Use part (d) of Problem 1 to show that $T = 2\pi ab/h$.

(b) Show that $\dfrac{h^2}{GM} = ed = \dfrac{b^2}{a}$.

(c) Use parts (a) and (b) to show that $T^2 = \dfrac{4\pi^2}{GM}a^3$.

This proves Kepler's Third Law. [Notice that the proportionality constant $4\pi^2/(GM)$ is independent of the planet.]

3. The period of the earth's orbit is approximately 365.25 days. Use this fact and Kepler's Third Law to find the length of the major axis of the earth's orbit. You will need the mass of the sun, $M = 1.99 \times 10^{30}$ kg, and the gravitational constant, $G = 6{,}67 \times 10^{-11}\ \mathrm{N{\cdot}m^2/kg^2}$.

4. It's possible to place a satellite into orbit about the earth so that it remains fixed above a given location on the equator. Compute the altitude that is needed for such a satellite. The earth's mass is 5.98×10^{24} kg; its radius is 6.37×10^{6} m. (This orbit is called the Clarke Geosynchronous Orbit after Arthur C. Clarke, who first proposed the idea in 1945. The first such satellite, *Syncom II,* was launched in July 1963.)

13 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. What is a vector function? How do you find its derivative and its integral?

2. What is the connection between vector functions and space curves?

3. How do you find the tangent vector to a smooth curve at a point? How do you find the tangent line? The unit tangent vector?

4. If $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function, write the rules for differentiating the following vector functions.

(a) $\mathbf{u}(t) + \mathbf{v}(t)$ (b) $c\mathbf{u}(t)$ (c) $f(t)\,\mathbf{u}(t)$

(d) $\mathbf{u}(t) \cdot \mathbf{v}(t)$ (e) $\mathbf{u}(t) \times \mathbf{v}(t)$ (f) $\mathbf{u}(f(t))$

5. How do you find the length of a space curve given by a vector function $\mathbf{r}(t)$?

6. (a) What is the definition of curvature?

(b) Write a formula for curvature in terms of $\mathbf{r}'(t)$ and $\mathbf{T}'(t)$.

(c) Write a formula for curvature in terms of $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$.

(d) Write a formula for the curvature of a plane curve with equation $y = f(x)$.

7. (a) Write formulas for the unit normal and binormal vectors of a smooth space curve $\mathbf{r}(t)$.

(b) What is the normal plane of a curve at a point? What is the osculating plane? What is the osculating circle?

8. (a) How do you find the velocity, speed, and acceleration of a particle that moves along a space curve?

(b) Write the acceleration in terms of its tangential and normal components.

9. State Kepler's Laws.

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. The curve with vector equation $\mathbf{r}(t) = t^3\,\mathbf{i} + 2t^3\,\mathbf{j} + 3t^3\,\mathbf{k}$ is a line.

2. The curve $\mathbf{r}(t) = \langle 0, t^2, 4t \rangle$ is a parabola.

3. The curve $\mathbf{r}(t) = \langle 2t, 3 - t, 0 \rangle$ is a line that passes through the origin.

4. The derivative of a vector function is obtained by differentiating each component function.

5. If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are differentiable vector functions, then

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}'(t)$$

6. If $\mathbf{r}(t)$ is a differentiable vector function, then

$$\frac{d}{dt}|\mathbf{r}(t)| = |\mathbf{r}'(t)|$$

7. If $\mathbf{T}(t)$ is the unit tangent vector of a smooth curve, then the curvature is $\kappa = |d\mathbf{T}/dt|$.

8. The binormal vector is $\mathbf{B}(t) = \mathbf{N}(t) \times \mathbf{T}(t)$.

9. Suppose f is twice continuously differentiable. At an inflection point of the curve $y = f(x)$, the curvature is 0.

10. If $\kappa(t) = 0$ for all t, the curve is a straight line.

11. If $|\mathbf{r}(t)| = 1$ for all t, then $|\mathbf{r}'(t)|$ is a constant.

12. If $|\mathbf{r}(t)| = 1$ for all t, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

13. The osculating circle of a curve C at a point has the same tangent vector, normal vector, and curvature as C at that point.

14. Different parametrizations of the same curve result in identical tangent vectors at a given point on the curve.

15. The projection of the curve $\mathbf{r}(t) = \langle \cos 2t, t, \sin 2t \rangle$ onto the xz-plane is a circle.

16. The vector equations $\mathbf{r}(t) = \langle t, 2t, t + 1 \rangle$ and $\mathbf{r}(t) = \langle t - 1, 2t - 2, t \rangle$ are parametrizations of the same line.

EXERCISES

1. (a) Sketch the curve with vector function

$$\mathbf{r}(t) = t\,\mathbf{i} + \cos \pi t\,\mathbf{j} + \sin \pi t\,\mathbf{k} \qquad t \geqslant 0$$

(b) Find $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$.

2. Let $\mathbf{r}(t) = \left\langle \sqrt{2 - t}, (e^t - 1)/t, \ln(t + 1) \right\rangle$.

(a) Find the domain of $\mathbf{r}$.

(b) Find $\lim_{t \to 0} \mathbf{r}(t)$.

(c) Find $\mathbf{r}'(t)$.

3. Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 16$ and the plane $x + z = 5$.

4. Find parametric equations for the tangent line to the curve $x = 2 \sin t$, $y = 2 \sin 2t$, $z = 2 \sin 3t$ at the point $(1, \sqrt{3}, 2)$. Graph the curve and the tangent line on a common screen.

5. If $\mathbf{r}(t) = t^2\,\mathbf{i} + t \cos \pi t\,\mathbf{j} + \sin \pi t\,\mathbf{k}$, evaluate $\int_0^1 \mathbf{r}(t)\, dt$.

6. Let C be the curve with equations $x = 2 - t^3$, $y = 2t - 1$, $z = \ln t$. Find (a) the point where C intersects the xz-plane, (b) parametric equations of the tangent line at $(1, 1, 0)$, and (c) an equation of the normal plane to C at $(1, 1, 0)$.

7. Use Simpson's Rule with $n = 6$ to estimate the length of the arc of the curve with equations $x = t^2$, $y = t^3$, $z = t^4$, $0 \leqslant t \leqslant 3$.

8. Find the length of the curve $\mathbf{r}(t) = \langle 2t^{3/2}, \cos 2t, \sin 2t \rangle$, $0 \leqslant t \leqslant 1$.

9. The helix $\mathbf{r}_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ intersects the curve $\mathbf{r}_2(t) = (1 + t)\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$ at the point $(1, 0, 0)$. Find the angle of intersection of these curves.

10. Reparametrize the curve $\mathbf{r}(t) = e^t\,\mathbf{i} + e^t \sin t\,\mathbf{j} + e^t \cos t\,\mathbf{k}$ with respect to arc length measured from the point $(1, 0, 1)$ in the direction of increasing t.

11. For the curve given by $\mathbf{r}(t) = \langle \sin^3 t, \cos^3 t, \sin^2 t \rangle$, $0 \leqslant t \leqslant \pi/2$, find

(a) the unit tangent vector.

(b) the unit normal vector.

(c) the unit binormal vector.

(d) the curvature.

(e) the torsion.

12. Find the curvature of the ellipse $x = 3 \cos t$, $y = 4 \sin t$ at the points $(3, 0)$ and $(0, 4)$.

13. Find the curvature of the curve $y = x^4$ at the point $(1, 1)$.

14. Find an equation of the osculating circle of the curve $y = x^4 - x^2$ at the origin. Graph both the curve and its osculating circle.

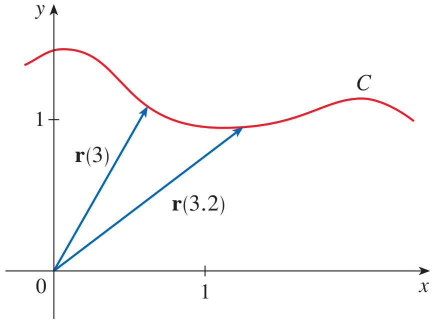
15. Find an equation of the osculating plane of the curve $x = \sin 2t$, $y = t$, $z = \cos 2t$ at the point $(0, \pi, 1)$.

16. The figure shows the curve C traced by a particle with position vector $\mathbf{r}(t)$ at time t.

(a) Draw a vector that represents the average velocity of the particle over the time interval $3 \leqslant t \leqslant 3.2$.

(b) Write an expression for the velocity $\mathbf{v}(3)$.

(c) Write an expression for the unit tangent vector $\mathbf{T}(3)$ and draw it.



17. A particle moves with position function $\mathbf{r}(t) = t \ln t\,\mathbf{i} + t\,\mathbf{j} + e^{-t}\,\mathbf{k}$. Find the velocity, speed, and acceleration of the particle.

18. Find the velocity, speed, and acceleration of a particle moving with position function $\mathbf{r}(t) = (2t^2 - 3)\,\mathbf{i} + 2t\,\mathbf{j}$. Sketch the path of the particle and draw the position, velocity, and acceleration vectors for $t = 1$.

19. A particle starts at the origin with initial velocity $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Its acceleration is $\mathbf{a}(t) = 6t\,\mathbf{i} + 12t^2\,\mathbf{j} - 6t\,\mathbf{k}$. Find its position function.

20. An athlete throws a shot at an angle of 45° to the horizontal at an initial speed of 13 m/s. It leaves the athlete's hand 2 m above the ground.

(a) Where is the shot 2 seconds later?

(b) How high does the shot go?

(c) Where does the shot land?

21. A projectile is launched with an initial speed of 40 m/s from the floor of a tunnel whose height is 30 m. What angle of elevation should be used to achieve the maximum possible horizontal range of the projectile? What is the maximum range?

22. Find the tangential and normal components of the acceleration vector of a particle with position function

$$\mathbf{r}(t) = t\,\mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k}$$

23. A disk of radius 1 is rotating in the counterclockwise direction at a constant angular speed ω. A particle starts at the center of the disk and moves toward the edge along a fixed radius so that its position at time t, $t \geqslant 0$, is given by $\mathbf{r}(t) = t\mathbf{R}(t)$, where

$$\mathbf{R}(t) = \cos \omega t\, \mathbf{i} + \sin \omega t\, \mathbf{j}$$

(a) Show that the velocity $\mathbf{v}$ of the particle is

$$\mathbf{v} = \cos \omega t\, \mathbf{i} + \sin \omega t\, \mathbf{j} + t\mathbf{v}_d$$

where $\mathbf{v}_d = \mathbf{R}'(t)$ is the velocity of a point on the edge of the disk.

(b) Show that the acceleration $\mathbf{a}$ of the particle is

$$\mathbf{a} = 2\mathbf{v}_d + t\mathbf{a}_d$$

where $\mathbf{a}_d = \mathbf{R}''(t)$ is the acceleration of a point on the edge of the disk. The extra term $2\mathbf{v}_d$ is called the *Coriolis acceleration*; it is the result of the interaction of the rotation of the disk and the motion of the particle. One can obtain a physical demonstration of this acceleration by walking toward the edge of a moving merry-go-round.

(c) Determine the Coriolis acceleration of a particle that moves on a rotating disk according to the equation

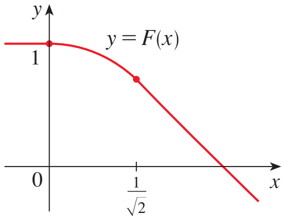
$$\mathbf{r}(t) = e^{-t}\cos \omega t\, \mathbf{i} + e^{-t}\sin \omega t\, \mathbf{j}$$

24. In designing *transfer curves* to connect sections of straight railroad tracks, it's important to realize that the acceleration of the train should be continuous so that the reactive force exerted by the train on the track is also continuous. Because of the formulas for the components of acceleration in Section 13.4, this will be the case if the curvature varies continuously.

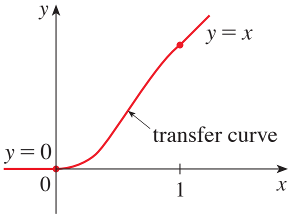
(a) A logical candidate for a transfer curve to join existing tracks given by $y = 1$ for $x \leqslant 0$ and $y = \sqrt{2} - x$ for $x \geqslant 1/\sqrt{2}$ might be the function $f(x) = \sqrt{1 - x^2}$, $0 < x < 1/\sqrt{2}$, whose graph is the arc of the circle shown in the figure. It looks reasonable at first glance. Show that the function

$$F(x) = \begin{cases} 1 & \text{if } x \leqslant 0 \\ \sqrt{1 - x^2} & \text{if } 0 < x < 1/\sqrt{2} \\ \sqrt{2} - x & \text{if } x \geqslant 1/\sqrt{2} \end{cases}$$

is continuous and has continuous slope, but does not have continuous curvature. Therefore f is not an appropriate transfer curve.



(b) Find a fifth-degree polynomial to serve as a transfer curve between the following straight line segments: $y = 0$ for $x \leqslant 0$ and $y = x$ for $x \geqslant 1$. Could this be done with a fourth-degree polynomial? Use a graphing calculator or computer to sketch the graph of the "connected" function and check to see that it looks like the one in the figure.



Problems Plus

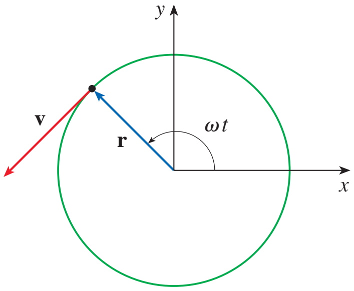


FIGURE FOR PROBLEM 1

1. A particle P moves with constant angular speed ω around a circle whose center is at the origin and whose radius is R. The particle is said to be in *uniform circular motion.* Assume that the motion is counterclockwise and that the particle is at the point $(R, 0)$ when $t = 0$. The position vector at time $t \geqslant 0$ is $\mathbf{r}(t) = R\cos\omega t\,\mathbf{i} + R\sin\omega t\,\mathbf{j}$.

(a) Find the velocity vector $\mathbf{v}$ and show that $\mathbf{v}\cdot\mathbf{r} = 0$. Conclude that $\mathbf{v}$ is tangent to the circle and points in the direction of the motion.

(b) Show that the speed $|\mathbf{v}|$ of the particle is the constant ωR. The *period* T of the particle is the time required for one complete revolution. Conclude that

$$T = \frac{2\pi R}{|\mathbf{v}|} = \frac{2\pi}{\omega}$$

(c) Find the acceleration vector $\mathbf{a}$. Show that it is proportional to $\mathbf{r}$ and that it points toward the origin. An acceleration with this property is called a *centripetal acceleration.* Show that the magnitude of the acceleration vector is $|\mathbf{a}| = R\omega^2$.

(d) Suppose that the particle has mass m. Show that the magnitude of the force $\mathbf{F}$ that is required to produce this motion, called a *centripetal force,* is

$$|\mathbf{F}| = \frac{m|\mathbf{v}|^2}{R}$$

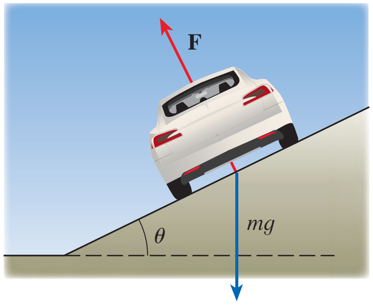


FIGURE FOR PROBLEM 2

2. A circular curve of radius R on a highway is banked at an angle θ so that a car can safely traverse the curve without skidding when there is no friction between the road and the tires. The loss of friction could occur, for example, if the road is covered with a film of water or ice. The rated speed v_R of the curve is the maximum speed that a car can attain without skidding. Suppose a car of mass m is traversing the curve at the rated speed v_R. Two forces are acting on the car: the vertical force, mg, due to the weight of the car, and a force $\mathbf{F}$ exerted by, and normal to, the road (see the figure).

The vertical component of $\mathbf{F}$ balances the weight of the car, so that $|\mathbf{F}|\cos\theta = mg$. The horizontal component of $\mathbf{F}$ produces a centripetal force on the car so that, by Newton's Second Law and part (d) of Problem 1,

$$|\mathbf{F}|\sin\theta = \frac{mv_R^2}{R}$$

(a) Show that $v_R^2 = Rg\tan\theta$.

(b) Find the rated speed of a circular curve with radius 120 m that is banked at an angle of 12°.

(c) Suppose the design engineers want to keep the banking at 12°, but wish to increase the rated speed by 50%. What should the radius of the curve be?

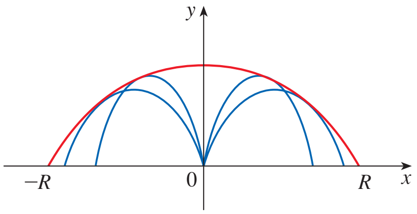


FIGURE FOR PROBLEM 3

3. A projectile is fired from the origin with angle of elevation α and initial speed v_0. Assuming that air resistance is negligible and that the only force acting on the projectile is gravity, g, we showed in Example 13.4.5 that the position vector of the projectile is

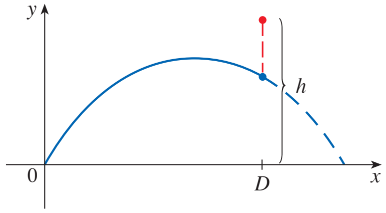
$$\mathbf{r}(t) = (v_0\cos\alpha)t\,\mathbf{i} + \left[(v_0\sin\alpha)t - \tfrac{1}{2}gt^2\right]\mathbf{j}$$

We also showed that the maximum horizontal distance of the projectile is achieved when $\alpha = 45°$ and in this case the range is $R = v_0^2/g$.

(a) At what angle should the projectile be fired to achieve maximum height and what is the maximum height?

(b) Fix the initial speed v_0 and consider the parabola $x^2 + 2Ry - R^2 = 0$, whose graph is shown in the figure at the left. Show that the projectile can hit any target inside or on the boundary of the region bounded by the parabola and the x-axis, and it can't hit any target outside this region.

(c) Suppose that the gun is elevated to an angle of inclination α in order to aim at a target that is suspended at a height h directly over a point D units downrange (see the following figure). The target is released at the instant the gun is fired. Show that the projectile always hits the target, regardless of the value v_0, provided the projectile does not hit the ground "before" D.



4. (a) A projectile is fired from the origin down an inclined plane that makes an angle θ with the horizontal. The angle of elevation of the gun and the initial speed of the projectile are α and v_0, respectively. Find the position vector of the projectile and the parametric equations of the path of the projectile as functions of the time t. (Ignore air resistance.)

(b) Show that the angle of elevation α that will maximize the downhill range is the angle halfway between the plane and the vertical.

(c) Suppose the projectile is fired up an inclined plane whose angle of inclination is θ. Show that, in order to maximize the (uphill) range, the projectile should be fired in the direction halfway between the plane and the vertical.

(d) In a paper presented in 1686, Edmond Halley summarized the laws of gravity and projectile motion and applied them to gunnery. One problem he posed involved firing a projectile to hit a target a distance R up an inclined plane. Show that the angle at which the projectile should be fired to hit the target but use the least amount of energy is the same as the angle in part (c). (Use the fact that the energy needed to fire the projectile is proportional to the square of the initial speed, so minimizing the energy is equivalent to minimizing the initial speed.)

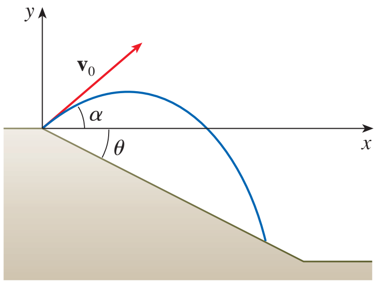


FIGURE FOR PROBLEM 4

5. A ball rolls off a table with a speed of 0.5 m/s. The table is 1.2 m high.

(a) Determine the point at which the ball hits the floor and find its speed at the instant of impact.

(b) Find the angle θ between the path of the ball and the vertical line drawn through the point of impact (see the figure).

(c) Suppose the ball rebounds from the floor at the same angle with which it hits the floor, but loses 20% of its speed due to energy absorbed by the ball on impact. Where does the ball strike the floor on the second bounce?

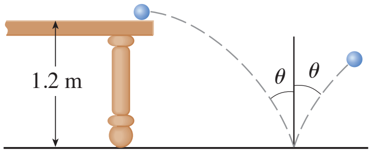


FIGURE FOR PROBLEM 5

6. Find the curvature of the curve with parametric equations

$$x = \int_0^t \sin\left(\tfrac{1}{2}\pi\theta^2\right) d\theta \qquad y = \int_0^t \cos\left(\tfrac{1}{2}\pi\theta^2\right) d\theta$$

T **7.** If a projectile is fired with angle of elevation α and initial speed v, then parametric equations for its trajectory are

$$x = (v \cos \alpha)t \qquad y = (v \sin \alpha)t - \tfrac{1}{2}gt^2$$

(See Example 13.4.5.) We know that the range (horizontal distance traveled) is maximized when $\alpha = 45°$. What value of α maximizes the total distance traveled by the projectile? (State your answer correct to the nearest degree.)

8. A cable has radius r and length L and is wound around a spool with radius R without overlapping. What is the shortest length along the spool that is covered by the cable?

9. Show that the curve with vector equation

$$\mathbf{r}(t) = \langle a_1t^2 + b_1t + c_1, a_2t^2 + b_2t + c_2, a_3t^2 + b_3t + c_3 \rangle$$

lies in a plane and find an equation of the plane.

A function of two variables can describe the shape of a surface like the one formed by these sand dunes. In Exercise 14.6.40 you are asked to use partial derivatives to compute the rate of change of elevation as a hiker walks in different directions.

SeppFriedhuber / E+ / Getty Images

14

Partial Derivatives

SO FAR WE HAVE DEALT with the calculus of functions of a single variable. But, in the real world, physical quantities often depend on two or more variables, so in this chapter we turn our attention to functions of several variables and extend the basic ideas of differential calculus to such functions.

14.1 Functions of Several Variables

In this section we study functions of two or more variables from four points of view:

• verbally	(by a description in words)
• numerically	(by a table of values)
• algebraically	(by an explicit formula)
• visually	(by a graph or level curves)

Functions of Two Variables

The temperature T at a point on the surface of the earth at any given time depends on the longitude x and latitude y of the point. We can think of T as being a function of the two variables x and y, or as a function of the pair (x, y). We indicate this functional dependence by writing $T = f(x, y)$.

The volume V of a circular cylinder depends on its radius r and its height h. In fact, we know that $V = \pi r^2 h$. We say that V is a function of r and h, and we can write $V(r, h) = \pi r^2 h$.

> **Definition** A **function f of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

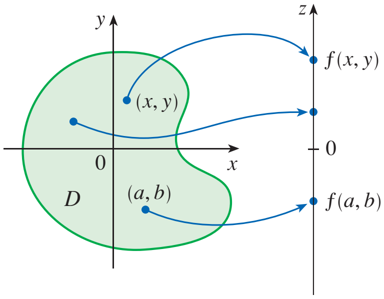


FIGURE 1

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y). The variables x and y are **independent variables** and z is the **dependent variable**. [Compare this with the notation $y = f(x)$ for functions of a single variable.]

A function of two variables is just a function whose domain is a subset of $\mathbb{R}^2$ and whose range is a subset of $\mathbb{R}$. One way of visualizing such a function is by means of an arrow diagram (see Figure 1), where the domain D is represented as a subset of the xy-plane and the range is a set of numbers on a real line, shown as a z-axis. For instance, if $f(x, y)$ represents the temperature at a point (x, y) in a flat metal plate with the shape of D, we can think of the z-axis as a thermometer displaying the recorded temperatures.

If a function f is given by a formula and no domain is specified, then the domain of f is understood to be the set of all pairs (x, y) for which the given expression defines a real number.

EXAMPLE 1 For each of the following functions, evaluate $f(3, 2)$ and find and sketch the domain.

(a) $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$ (b) $f(x, y) = x \ln(y^2 - x)$

SOLUTION

(a) $f(3, 2) = \dfrac{\sqrt{3 + 2 + 1}}{3 - 1} = \dfrac{\sqrt{6}}{2}$

The expression for f makes sense if the denominator is not 0 and the quantity under the square root sign is nonnegative. So the domain of f is

$$D = \{(x, y) \mid x + y + 1 \geqslant 0,\ x \neq 1\}$$

The inequality $x + y + 1 \geqslant 0$, or $y \geqslant -x - 1$, describes the points that lie on or above the line $y = -x - 1$, while $x \neq 1$ means that the points on the line $x = 1$ must be excluded from the domain (see Figure 2).

(b) $f(3, 2) = 3 \ln(2^2 - 3) = 3 \ln 1 = 0$

Since $\ln(y^2 - x)$ is defined only when $y^2 - x > 0$, that is, $x < y^2$, the domain of f is $D = \{(x, y) \mid x < y^2\}$. This is the set of points to the left of the parabola $x = y^2$. (See Figure 3.)

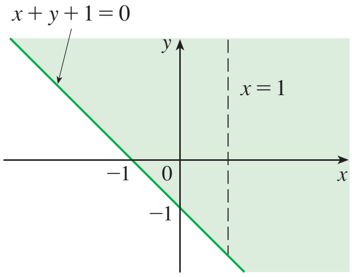


FIGURE 2

Domain of $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$

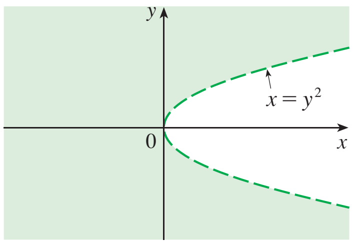


FIGURE 3

Domain of $f(x, y) = x \ln(y^2 - x)$

■

EXAMPLE 2 Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

SOLUTION The domain of g is

$$D = \{(x, y) \mid 9 - x^2 - y^2 \geqslant 0\} = \{(x, y) \mid x^2 + y^2 \leqslant 9\}$$

which is the disk with center $(0, 0)$ and radius 3. (See Figure 4.) The range of g is

$$\left\{z \mid z = \sqrt{9 - x^2 - y^2},\ (x, y) \in D\right\}$$

Since z is a positive square root, $z \geqslant 0$. Also, because $9 - x^2 - y^2 \leqslant 9$, we have

$$\sqrt{9 - x^2 - y^2} \leqslant 3$$

So the range is

$$\{z \mid 0 \leqslant z \leqslant 3\} = [0, 3]$$

■

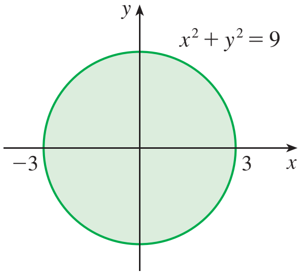


FIGURE 4

Domain of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Not all functions can be represented by explicit formulas. The function in the next example is described verbally and by numerical estimates of its values.

EXAMPLE 3 In regions with severe winter weather, the *wind-chill index* is often used to describe the apparent severity of the cold. This index W is a subjective temperature that depends on the actual temperature T and the wind speed v. So W is a function of

T and v, and we can write $W = f(T, v)$. Table 1 records values of W compiled by the US National Weather Service and the Meteorological Service of Canada.

Table 1 Wind-chill index as a function of air temperature and wind speed

Wind speed (km/h)

Actual temperature (°C)

T \ v	5	10	15	20	25	30	40	50	60	70	80
5	4	3	2	1	1	0	−1	−1	−2	−2	−3
0	−2	−3	−4	−5	−6	−6	−7	−8	−9	−9	−10
−5	−7	−9	−11	−12	−12	−13	−14	−15	−16	−16	−17
−10	−13	−15	−17	−18	−19	−20	−21	−22	−23	−23	−24
−15	−19	−21	−23	−24	−25	−26	−27	−29	−30	−30	−31
−20	−24	−27	−29	−30	−32	−33	−34	−35	−36	−37	−38
−25	−30	−33	−35	−37	−38	−39	−41	−42	−43	−44	−45
−30	−36	−39	−41	−43	−44	−46	−48	−49	−50	−51	−52
−35	−41	−45	−48	−49	−51	−52	−54	−56	−57	−58	−60
−40	−47	−51	−54	−56	−57	−59	−61	−63	−64	−65	−67

The Wind-Chill Index
The wind-chill index measures how cold it feels when it's windy. It is based on a model of how fast a human face loses heat. It was developed through clinical trials in which volunteers were exposed to a variety of temperatures and wind speeds in a refrigerated wind tunnel.

For instance, the table shows that if the actual temperature is −5°C and the wind speed is 50 km/h, then subjectively it would feel as cold as a temperature of about −15°C with no wind. So

$$f(-5, 50) = -15$$

■

Table 2

Year	P	L	K
1899	100	100	100
1900	101	105	107
1901	112	110	114
1902	122	117	122
1903	124	122	131
1904	122	121	138
1905	143	125	149
1906	152	134	163
1907	151	140	176
1908	126	123	185
1909	155	143	198
1910	159	147	208
1911	153	148	216
1912	177	155	226
1913	184	156	236
1914	169	152	244
1915	189	156	266
1916	225	183	298
1917	227	198	335
1918	223	201	366
1919	218	196	387
1920	231	194	407
1921	179	146	417
1922	240	161	431

EXAMPLE 4 In 1928 Charles Cobb and Paul Douglas published a study in which they modeled the growth of the American economy during the period 1899–1922. They considered a simplified view of the economy in which production output is determined by the amount of labor involved and the amount of capital invested. While many other factors affect economic performance, this model proved to be remarkably accurate. The function Cobb and Douglas used to model production was of the form

$$\boxed{1} \qquad P(L, K) = bL^{\alpha}K^{1-\alpha}$$

where P is the total production (the monetary value of all goods produced in a year), L is the amount of labor (the total number of person-hours worked in a year), and K is the amount of capital invested (the monetary worth of all machinery, equipment, and buildings). In the Discovery Project following Section 14.3 we will show how the form of Equation 1 follows from certain economic assumptions.

Cobb and Douglas used economic data published by the government to obtain Table 2. They took the year 1899 as a baseline and P, L, and K for 1899 were each assigned the value 100. The values for other years were expressed as percentages of the 1899 values.

Cobb and Douglas used the method of least squares to fit the data of Table 2 to the function

$$\boxed{2} \qquad P(L, K) = 1.01L^{0.75}K^{0.25}$$

(See Exercise 81 for the details.)

If we use the model given by the function in Equation 2 to compute the production in the years 1910 and 1920, we get the values

$$P(147, 208) = 1.01(147)^{0.75}(208)^{0.25} \approx 161.9$$

$$P(194, 407) = 1.01(194)^{0.75}(407)^{0.25} \approx 235.8$$

which are quite close to the actual values, 159 and 231.

The production function (1) has subsequently been used in many settings, ranging from individual firms to global economics. It has become known as the **Cobb-Douglas production function**. Its domain is $\{(L, K) \mid L \geqslant 0, K \geqslant 0\}$ because L and K represent labor and capital and are therefore never negative. ■

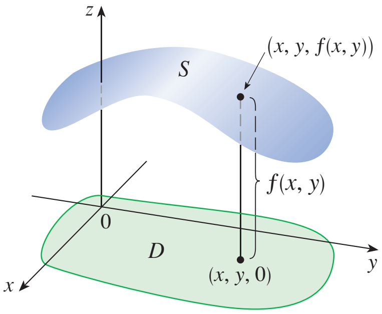


FIGURE 5

■ Graphs

Another way of visualizing the behavior of a function of two variables is to consider its graph.

> **Definition** If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

The graph of a function f of two variables is a surface S with equation $z = f(x, y)$. We can visualize the graph S of f as lying directly above or below its domain D in the xy-plane (see Figure 5).

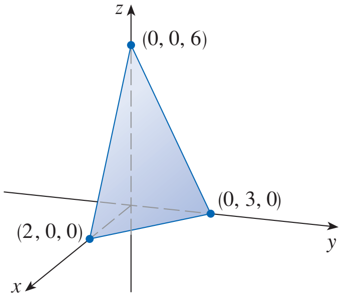


FIGURE 6

EXAMPLE 5 Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$.

SOLUTION The graph of f has the equation $z = 6 - 3x - 2y$, or $3x + 2y + z = 6$, which represents a plane. To graph the plane we first find the intercepts. Putting $y = z = 0$ in the equation, we get $x = 2$ as the x-intercept. Similarly, the y-intercept is 3 and the z-intercept is 6. This helps us sketch the portion of the graph that lies in the first octant in Figure 6. ■

The function in Example 5 is a special case of the function

$$f(x, y) = ax + by + c$$

which is called a **linear function**. The graph of such a function has the equation

$$z = ax + by + c \qquad \text{or} \qquad ax + by - z + c = 0$$

so it is a plane (see Section 12.5). In much the same way that linear functions of one variable are important in single-variable calculus, we will see that linear functions of two variables play a central role in multivariable calculus.

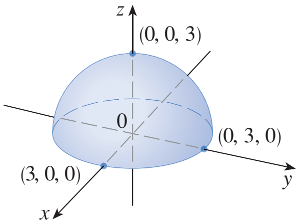


FIGURE 7
Graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$

EXAMPLE 6 Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

SOLUTION In Example 2 we found that the domain of g is the disk with center $(0, 0)$ and radius 3. The graph of g has equation $z = \sqrt{9 - x^2 - y^2}$. We square both sides of this equation to obtain $z^2 = 9 - x^2 - y^2$, or $x^2 + y^2 + z^2 = 9$, which we recognize as an equation of the sphere with center the origin and radius 3. But, since $z \geqslant 0$, the graph of g is just the top half of this sphere (see Figure 7). ■

NOTE An entire sphere can't be represented by a single function of x and y. As we saw in Example 6, the upper hemisphere of the sphere $x^2 + y^2 + z^2 = 9$ is represented by the function $g(x, y) = \sqrt{9 - x^2 - y^2}$. The lower hemisphere is represented by the function $h(x, y) = -\sqrt{9 - x^2 - y^2}$.

EXAMPLE 7 Use a computer to draw the graph of the Cobb-Douglas production function $P(L, K) = 1.01L^{0.75}K^{0.25}$.

SOLUTION Figure 8 shows the graph of P for values of the labor L and capital K that lie between 0 and 300. The computer has drawn the surface by plotting vertical traces. We see from these traces that the value of the production P increases as either L or K increases, as expected.

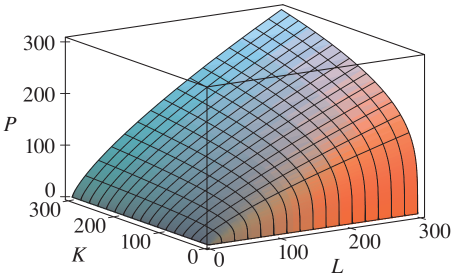


FIGURE 8

■

EXAMPLE 8 Find the domain and range and sketch the graph of $h(x, y) = 4x^2 + y^2$.

SOLUTION Notice that $h(x, y)$ is defined for all possible ordered pairs of real numbers (x, y), so the domain is $\mathbb{R}^2$, the entire xy-plane. The range of h is the set $[0, \infty)$ of all nonnegative real numbers. [Notice that $x^2 \geqslant 0$ and $y^2 \geqslant 0$, so $h(x, y) \geqslant 0$ for all x and y.] The graph of h has the equation $z = 4x^2 + y^2$, which is the elliptic paraboloid that we sketched in Example 12.6.4. Horizontal traces are ellipses and vertical traces are parabolas (see Figure 9).

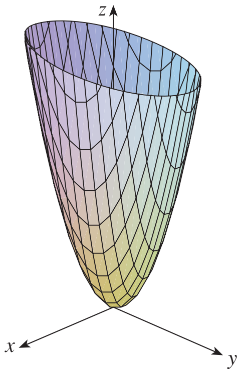
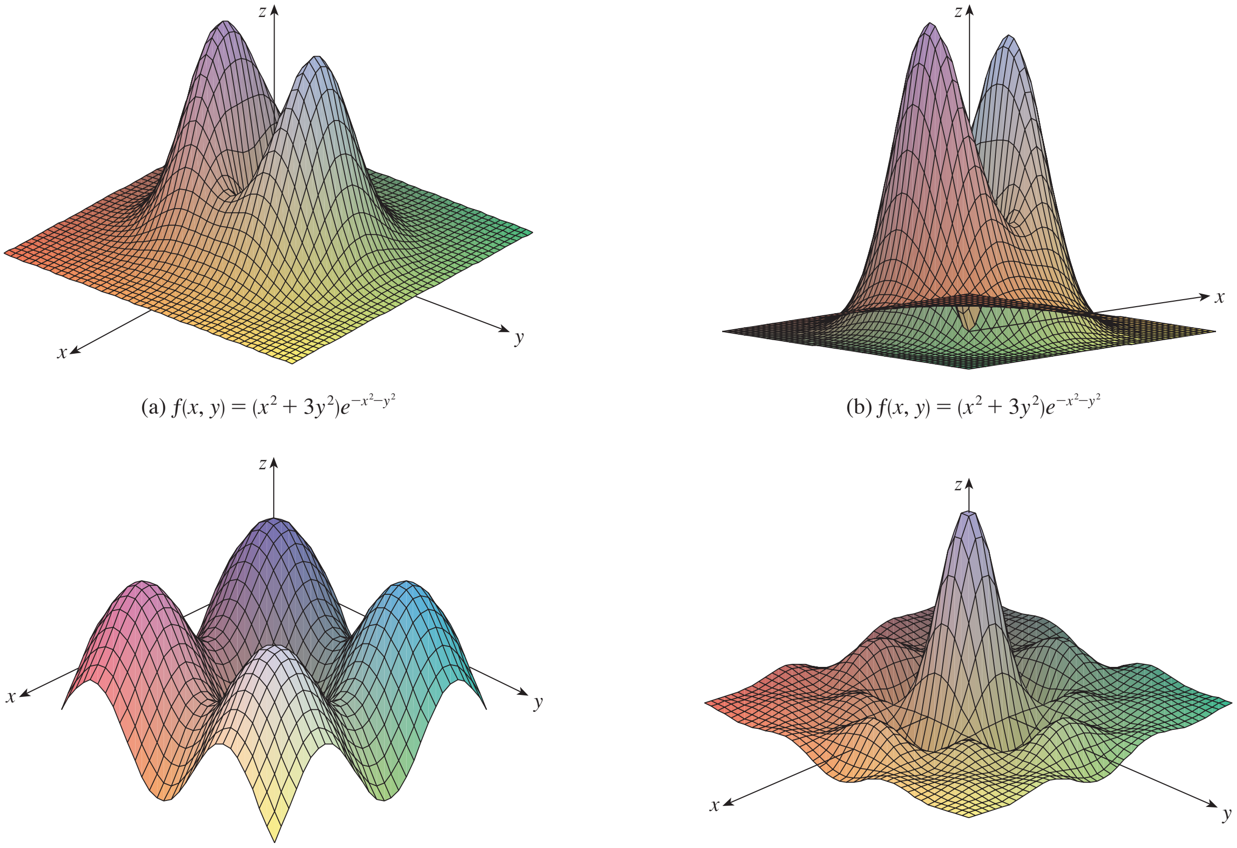


FIGURE 9
Graph of $h(x, y) = 4x^2 + y^2$

■

Many software applications are available for graphing functions of two variables. In some programs, traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k.

Figure 10 shows computer-generated graphs of several functions. Notice that we get an especially good picture of a function when rotation is used to give views from different vantage points. In parts (a) and (b) the graph of f is very flat and close to the xy-plane except near the origin; this is because $e^{-x^2-y^2}$ is very small when x or y is large.



(a) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$

(b) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$

FIGURE 10

(c) $f(x, y) = \sin x + \sin y$

(d) $f(x, y) = \dfrac{\sin x \sin y}{xy}$

Level Curves and Contour Maps

So far we have two methods for visualizing functions: arrow diagrams and graphs. A third method, borrowed from mapmakers, is a *contour map* on which points of constant elevation are joined to form *contour curves*, or *level curves*.

Definition The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k. In other words, it is a curve in the xy-plane that shows where the graph of f has height k (above or below the xy-plane). A collection of level curves is called a **contour map**. Contour maps are most descriptive when the level curves

$f(x, y) = k$ are drawn for equally spaced values of k, and we assume that this is the case unless indicated otherwise.

You can see from Figure 11 the relation between level curves and horizontal traces. The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy-plane. So if you draw a contour map of a function and visualize the level curves being lifted up to the surface at the indicated height, then you can mentally piece together a picture of the graph. The surface is steeper where the level curves are close together and somewhat flatter where they are farther apart.

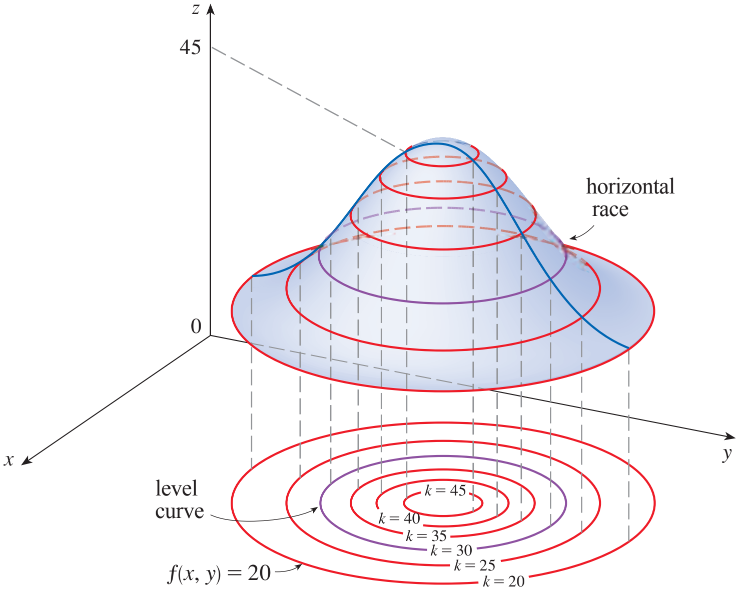


FIGURE 11

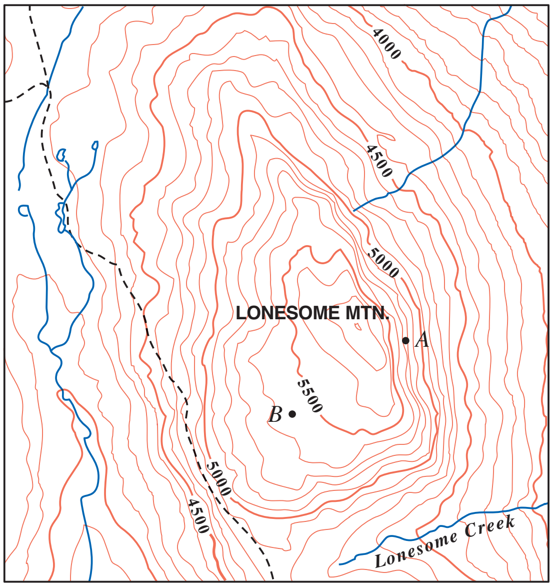


FIGURE 12

One common example of level curves occurs in topographic maps of mountainous regions, such as the map in Figure 12. The level curves are curves of constant elevation above sea level. If you walk along one of these contour lines, you neither ascend nor descend. Another common example is the temperature function introduced in the opening paragraph of this section. Here the level curves are called **isothermals**; they join locations with the same temperature. Figure 13 shows a weather map of the world indicating the average July temperatures. The isothermals are the curves that separate the colored bands.

In weather maps of atmospheric pressure at a given time as a function of longitude and latitude, the level curves are called **isobars**; they join locations with the same pressure (see Exercise 34). Surface winds tend to flow from areas of high pressure across the isobars toward areas of low pressure and are strongest where the isobars are tightly packed.

A contour map of worldwide precipitation is shown in Figure 14. Here the level curves are not labeled but they separate the colored regions and the amount of precipitation in each region is indicated in the color key.

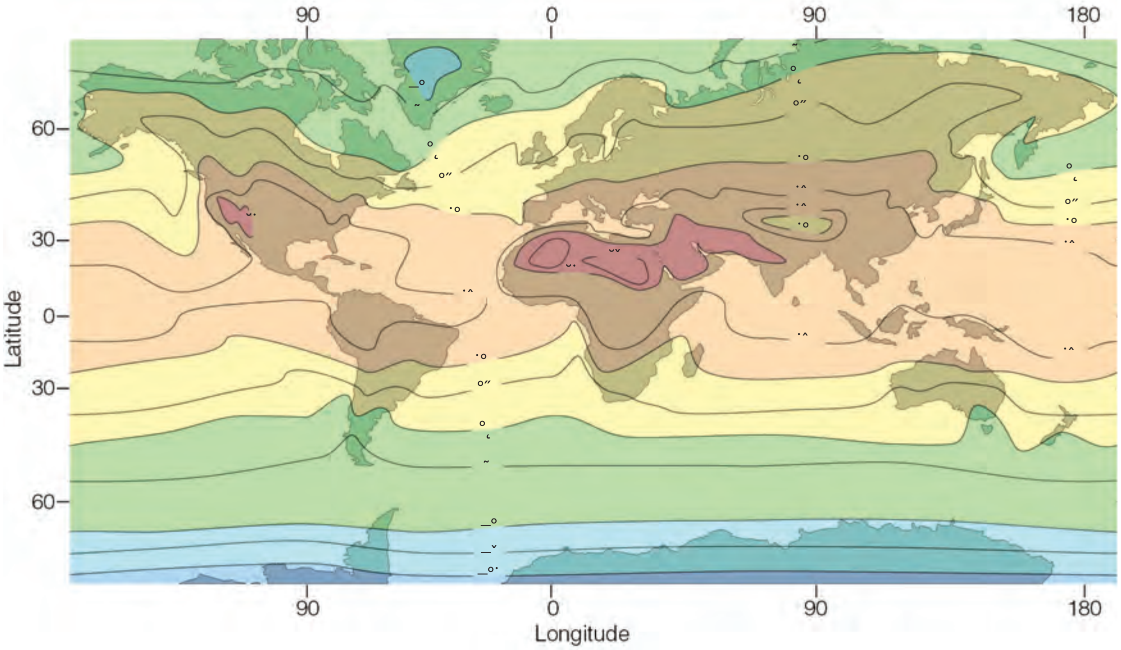
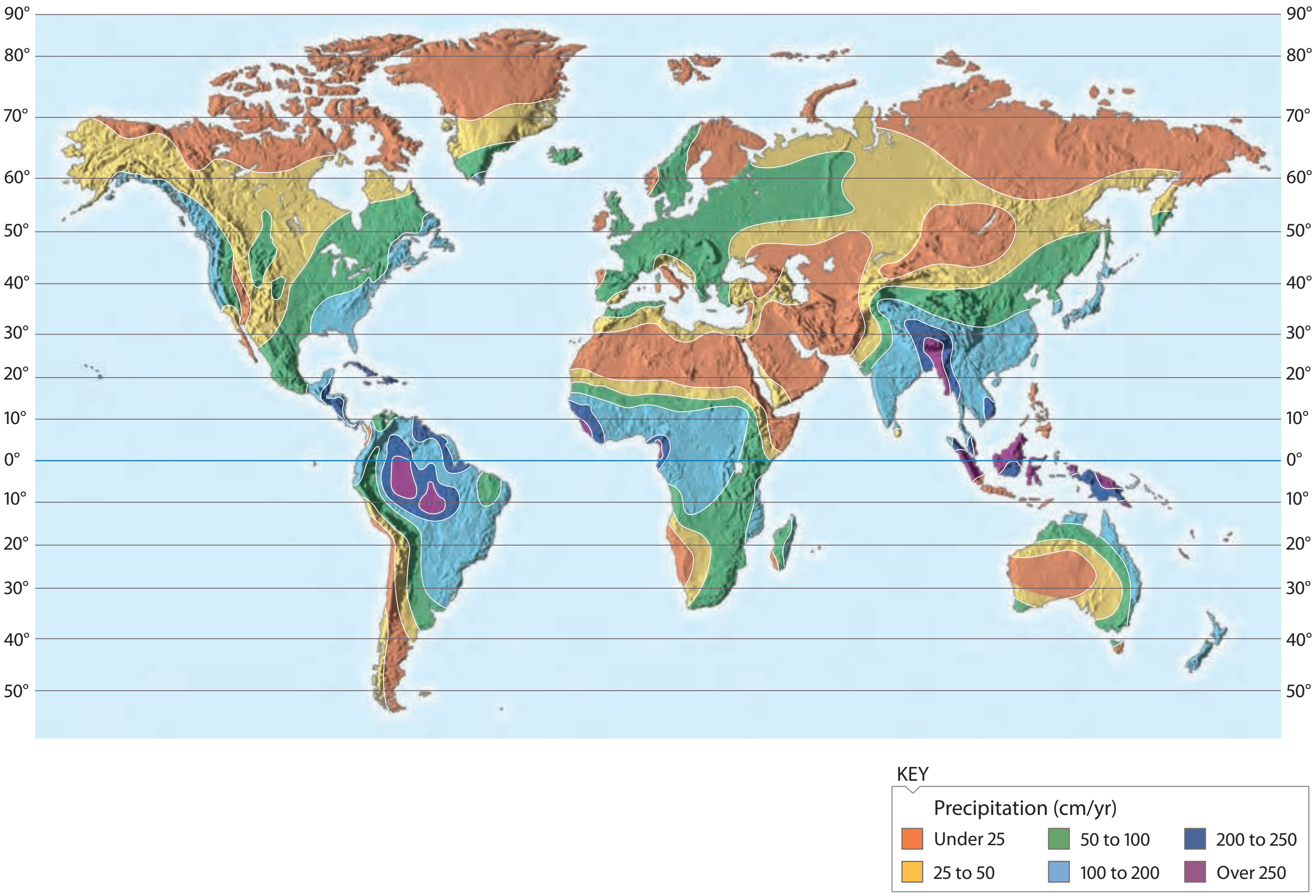

90
0
90
180
60
30
0
30
60
Latitude
90
0
90
180
Longitude

FIGURE 13 Average air temperature near sea level in July (degrees Celsius)


90°
80°
70°
60°
50°
40°
30°
20°
10°
0°
10°
20°
30°
40°
50°
KEY
Precipitation (cm/yr)
Under 25
25 to 50
50 to 100
100 to 200
200 to 250
Over 250

EXAMPLE 9 A contour map for a function f is shown in Figure 15. Use it to estimate the values of $f(1, 3)$ and $f(4, 5)$.

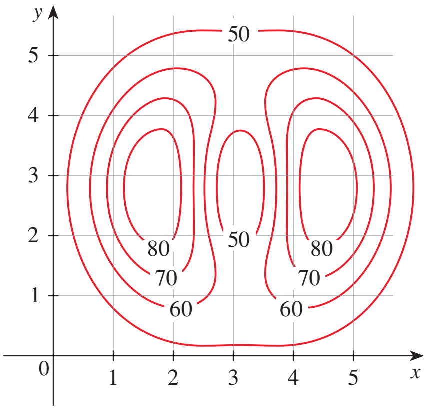


FIGURE 15

SOLUTION The point $(1, 3)$ lies partway between the level curves with z-values 70 and 80. We estimate that

$$f(1, 3) \approx 73$$

Similarly, we estimate that

$$f(4, 5) \approx 56$$

■

EXAMPLE 10 Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12$.

SOLUTION The level curves are

$$6 - 3x - 2y = k \qquad \text{or} \qquad 3x + 2y + (k - 6) = 0$$

This is a family of lines with slope $-\frac{3}{2}$. The four particular level curves with $k = -6, 0, 6$, and 12 are $3x + 2y - 12 = 0$, $3x + 2y - 6 = 0$, $3x + 2y = 0$, and $3x + 2y + 6 = 0$. They are sketched in Figure 16. For equally spaced values of k the level curves are equally spaced parallel lines because the graph of f is a plane (see Figure 6).

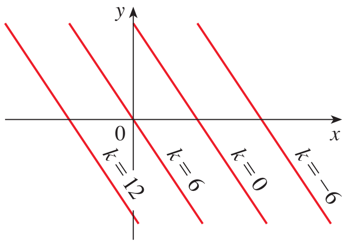


FIGURE 16
Contour map of
$f(x, y) = 6 - 3x - 2y$

■

EXAMPLE 11 Sketch the level curves of the function

$$g(x, y) = \sqrt{9 - x^2 - y^2} \qquad \text{for} \quad k = 0, 1, 2, 3$$

SOLUTION The level curves are

$$\sqrt{9 - x^2 - y^2} = k \qquad \text{or} \qquad x^2 + y^2 = 9 - k^2$$

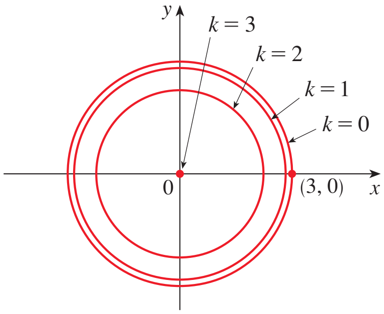


FIGURE 17
Contour map of
$g(x, y) = \sqrt{9 - x^2 - y^2}$

This is a family of concentric circles with center $(0, 0)$ and radius $\sqrt{9 - k^2}$. The cases $k = 0, 1, 2, 3$ are shown in Figure 17. Try to visualize these level curves lifted up to form a surface and compare with the graph of g (a hemisphere) in Figure 7. ■

EXAMPLE 12 Sketch some level curves of the function $h(x, y) = 4x^2 + y^2 + 1$.

SOLUTION The level curves are

$$4x^2 + y^2 + 1 = k \qquad \text{or} \qquad \frac{x^2}{\frac{1}{4}(k - 1)} + \frac{y^2}{k - 1} = 1$$

which, for $k > 1$, describes a family of ellipses with semiaxes $\frac{1}{2}\sqrt{k - 1}$ and $\sqrt{k - 1}$. Figure 18(a) shows a contour map of h drawn by a computer. Figure 18(b) shows these level curves lifted up to the graph of h (an elliptic paraboloid) where they become horizontal traces. We see from Figure 18 how the graph of h is put together from the level curves.

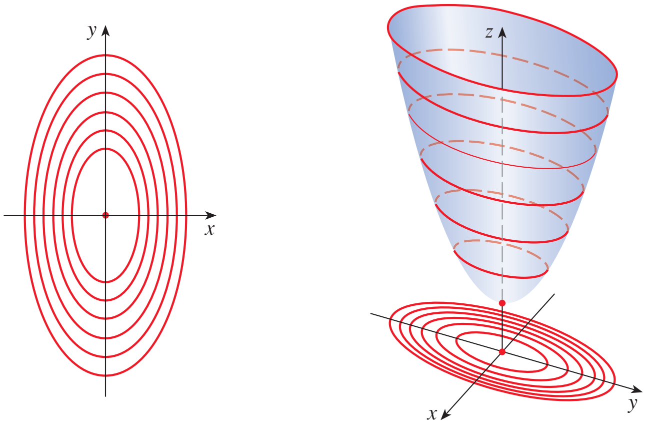


FIGURE 18
The graph of $h(x, y) = 4x^2 + y^2 + 1$ is formed by lifting the level curves.

(a) Contour map

(b) Horizontal traces are raised level curves. ■

EXAMPLE 13 Plot level curves for the Cobb-Douglas production function of Example 4.

SOLUTION In Figure 19 we use a computer to draw a contour plot for the Cobb-Douglas production function

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

Level curves are labeled with the value of the production P. For instance, the level curve labeled 140 shows all values of the labor L and capital investment K that result in a production of $P = 140$. We see that, for a fixed value of P, as L increases K decreases, and vice versa. ■

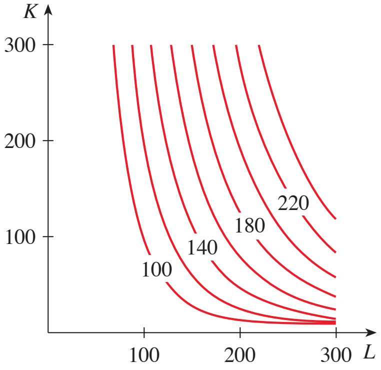
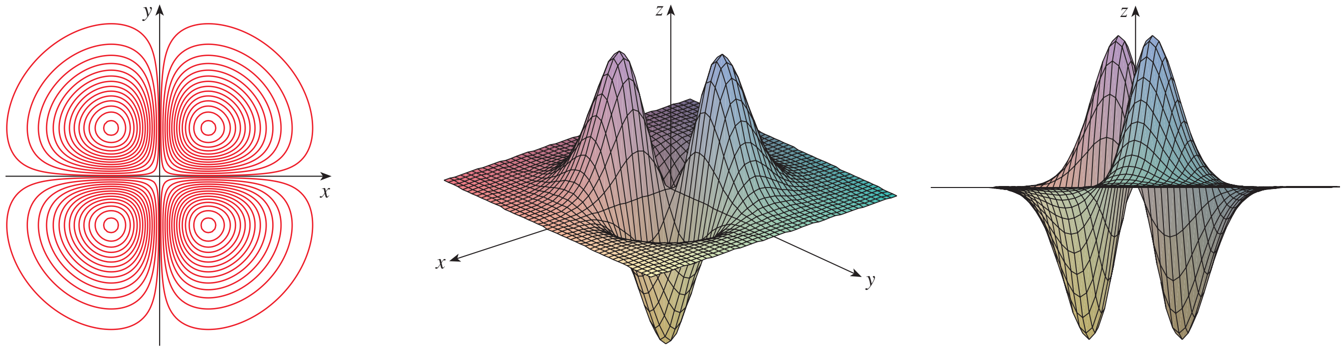


FIGURE 19

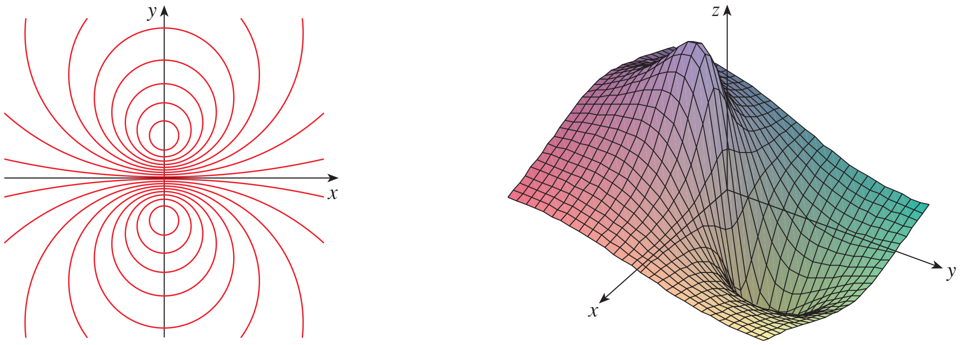
For some purposes, a contour map is more useful than a graph. That is certainly true in Example 13. (Compare Figure 19 with Figure 8.) It is also true in estimating function values, as in Example 9.

Figure 20 shows some computer-generated level curves together with the corresponding computer-generated graphs. Notice that the level curves in part (c) crowd together near the origin. That corresponds to the fact that the graph in part (d) is very steep near the origin.



(a) Level curves of $f(x, y) = -xye^{-x^2-y^2}$

(b) Two views of $f(x, y) = -xye^{-x^2-y^2}$



(c) Level curves of $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$

(d) $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$

FIGURE 20

Functions of Three or More Variables

A **function of three variables**, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$. For instance, the temperature T at a point on the surface of the earth depends on the longitude x and latitude y of the point and on the time t, so we could write $T = f(x, y, t)$.

EXAMPLE 14 Find the domain of f if

$$f(x, y, z) = \ln(z - y) + xy \sin z$$

SOLUTION The expression for $f(x, y, z)$ is defined as long as $z - y > 0$, so the domain of f is

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid z > y\}$$

This is a **half-space** consisting of all points that lie above the plane $z = y$. ■

It's very difficult to visualize a function f of three variables by its graph, since that would lie in a four-dimensional space. However, we do gain some insight into f by examining its **level surfaces**, which are the surfaces with equations $f(x, y, z) = k$, where k is a constant. If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

EXAMPLE 15 Find the level surfaces of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

SOLUTION The level surfaces are $x^2 + y^2 + z^2 = k$, where $k \geqslant 0$. These form a family of concentric spheres with radius $\sqrt{k}$. (See Figure 21.) Thus, as (x, y, z) varies over any sphere with center O, the value of $f(x, y, z)$ remains fixed.

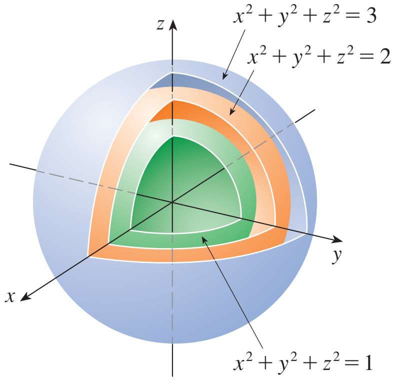


FIGURE 21 ■

EXAMPLE 16 Describe the level surfaces of the function

$$f(x, y, z) = x^2 - y - z^2$$

SOLUTION The level surfaces are $x^2 - y - z^2 = k$, or $y = x^2 - z^2 - k$, a family of hyperbolic paraboloids. Figure 22 shows the level surfaces for $k = 0$ and $k = \pm 5$.

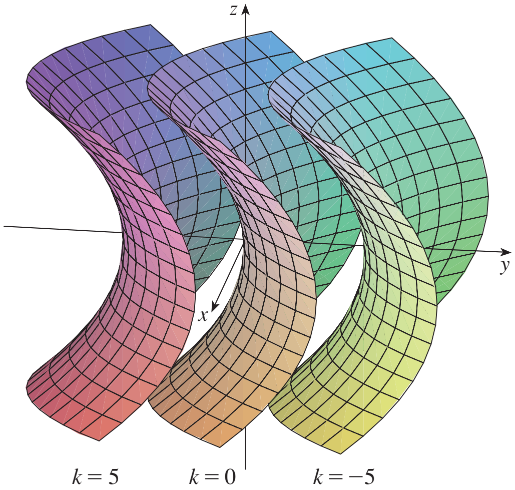


FIGURE 22 ■

Functions of any number of variables can be considered. A **function of *n* variables** is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an n-tuple $(x_1, x_2, \ldots, x_n)$

of real numbers. We denote by $\mathbb{R}^n$ the set of all such n-tuples. For example, if a company uses n different ingredients in making a food product, c_i is the cost per unit of the ith ingredient, and x_i units of the ith ingredient are used, then the total cost C of the ingredients is a function of the n variables $x_1, x_2, \ldots, x_n$:

$$\boxed{3} \qquad C = f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

The function f is a real-valued function whose domain is a subset of $\mathbb{R}^n$. Sometimes we use vector notation to write such functions more compactly: If $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$, we often write $f(\mathbf{x})$ in place of $f(x_1, x_2, \ldots, x_n)$. With this notation we can rewrite the function defined in Equation 3 as

$$f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$$

where $\mathbf{c} = \langle c_1, c_2, \ldots, c_n \rangle$ and $\mathbf{c} \cdot \mathbf{x}$ denotes the dot product of the vectors $\mathbf{c}$ and $\mathbf{x}$ in V_n.

In view of the one-to-one correspondence between points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ and their position vectors $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$ in V_n, we have three ways of looking at a function f defined on a subset of $\mathbb{R}^n$:

1. As a function of n real variables $x_1, x_2, \ldots, x_n$
2. As a function of a single point variable $(x_1, x_2, \ldots, x_n)$
3. As a function of a single vector variable $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$

We will see that all three points of view are useful.

14.1 Exercises

1. If $f(x, y) = x^2y/(2x - y^2)$, find
   (a) $f(1, 3)$ (b) $f(-2, -1)$
   (c) $f(x + h, y)$ (d) $f(x, x)$

2. If $g(x, y) = x \sin y + y \sin x$, find
   (a) $g(\pi, 0)$ (b) $g(\pi/2, \pi/4)$
   (c) $g(0, y)$ (d) $g(x, y + h)$

3. Let $g(x, y) = x^2 \ln(x + y)$.
   (a) Evaluate $g(3, 1)$.
   (b) Find and sketch the domain of g.
   (c) Find the range of g.

4. Let $h(x, y) = e^{\sqrt{y - x^2}}$.
   (a) Evaluate $h(-2, 5)$.
   (b) Find and sketch the domain of h.
   (c) Find the range of h.

5. Let $F(x, y, z) = \sqrt{y} - \sqrt{x - 2z}$.
   (a) Evaluate $F(3, 4, 1)$.
   (b) Find and describe the domain of F.

6. Let $f(x, y, z) = \ln\left(z - \sqrt{x^2 + y^2}\right)$.
   (a) Evaluate $f(4, -3, 6)$.
   (b) Find and describe the domain of f.

7–16 Find and sketch the domain of the function.

7. $f(x, y) = \sqrt{x - 2} + \sqrt{y - 1}$

8. $f(x, y) = \sqrt[4]{x - 3y}$

9. $q(x, y) = \sqrt{x} + \sqrt{4 - 4x^2 - y^2}$

10. $g(x, y) = \ln(x^2 + y^2 - 9)$

11. $g(x, y) = \dfrac{x - y}{x + y}$

12. $g(x, y) = \dfrac{\ln(2 - x)}{1 - x^2 - y^2}$

13. $p(x, y) = \dfrac{\sqrt{xy}}{x + 1}$

14. $f(x, y) = \sin^{-1}(x + y)$

15. $f(x, y, z) = \sqrt{4 - x^2} + \sqrt{9 - y^2} + \sqrt{1 - z^2}$

16. $f(x, y, z) = \ln(16 - 4x^2 - 4y^2 - z^2)$

---

17. A model for the surface area of a human body is given by the function

$$S = f(w, h) = 0.0072w^{0.425}h^{0.725}$$

where w is the weight (in kilograms), h is the height (in centimeters), and S is measured in square meters.
   (a) Find $f(73, 178)$ and interpret it.
   (b) What is your own surface area?

18. A manufacturer has modeled its yearly production function P (the monetary value of its entire production in millions of dollars) as a Cobb-Douglas function

$$P(L, K) = 1.47L^{0.65}K^{0.35}$$

where L is the number of labor hours (in thousands) and K is the invested capital (in millions of dollars). Find $P(120, 20)$ and interpret it.

19. In Example 3 we considered the function $W = f(T, v)$, where W is the wind-chill index, T is the actual temperature, and v is the wind speed. A numerical representation is given in Table 1.

(a) What is the value of $f(-15, 40)$? What is its meaning?
(b) Describe in words the meaning of the question "For what value of v is $f(-20, v) = -30$?" Then answer the question.
(c) Describe in words the meaning of the question "For what value of T is $f(T, 20) = -49$?" Then answer the question.
(d) What is the meaning of the function $W = f(-5, v)$? Describe the behavior of this function.
(e) What is the meaning of the function $W = f(T, 50)$? Describe the behavior of this function.

20. The *temperature-humidity index* I (or humidex, for short) is the perceived air temperature when the actual temperature is T and the relative humidity is h, so we can write $I = f(T, h)$. The following table of values of I is an excerpt from a table compiled by the National Oceanic & Atmospheric Administration.

Table 3 **Apparent temperature as a function of temperature and humidity**

Relative humidity (%)

Actual temperature (°C)

h T	20	30	40	50	60	70
20	20	20	20	21	22	23
25	25	25	26	28	30	32
30	30	31	34	36	38	41
35	36	39	42	45	48	51
40	43	47	51	55	59	63

(a) What is the value of $f(95, 70)$? What is its meaning?
(b) For what value of h is $f(90, h) = 100$?
(c) For what value of T is $f(T, 50) = 88$?
(d) What are the meanings of the functions $I = f(80, h)$ and $I = f(100, h)$? Compare the behavior of these two functions of h.

21. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in Table 4.

(a) What is the value of $f(40, 15)$? What is its meaning?
(b) What is the meaning of the function $h = f(30, t)$? Describe the behavior of this function.
(c) What is the meaning of the function $h = f(v, 30)$? Describe the behavior of this function.

Table 4 **Wave height as a function of wind speed and duration**

Duration (hours)

Wind speed (km/h)

t v	5	10	15	20	30	40	50
20	0.6	0.6	0.6	0.6	0.6	0.6	0.6
30	1.2	1.3	1.5	1.5	1.5	1.6	1.6
40	1.5	2.2	2.4	2.5	2.7	2.8	2.8
60	2.8	4.0	4.9	5.2	5.5	5.8	5.9
80	4.3	6.4	7.7	8.6	9.5	10.1	10.2
100	5.8	8.9	11.0	12.2	13.8	14.7	15.3
120	7.4	11.3	14.4	16.6	19.0	20.5	21.1

22. A company makes three sizes of cardboard boxes: small, medium, and large. It costs \$2.50 to make a small box, \$4.00 for a medium box, and \$4.50 for a large box. Fixed costs are \$8000.

(a) Express the cost of making x small boxes, y medium boxes, and z large boxes as a function of three variables: $C = f(x, y, z)$.
(b) Find $f(3000, 5000, 4000)$ and interpret it.
(c) What is the domain of f?

23–31 Sketch the graph of the function.

23. $f(x, y) = y$

24. $f(x, y) = x^2$

25. $f(x, y) = 10 - 4x - 5y$

26. $f(x, y) = \cos y$

27. $f(x, y) = \sin x$

28. $f(x, y) = 2 - x^2 - y^2$

29. $f(x, y) = x^2 + 4y^2 + 1$

30. $f(x, y) = \sqrt{4x^2 + y^2}$

31. $f(x, y) = \sqrt{4 - 4x^2 - y^2}$

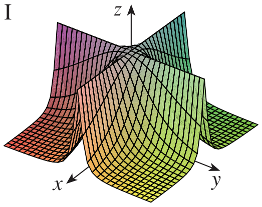
32. Match the function with its graph (labeled I–VI). Give reasons for your choices.

(a) $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$ (b) $f(x, y) = \dfrac{1}{1 + x^2y^2}$

(c) $f(x, y) = \ln(x^2 + y^2)$ (d) $f(x, y) = \cos\sqrt{x^2 + y^2}$

(e) $f(x, y) = |xy|$ (f) $f(x, y) = \cos(xy)$

I



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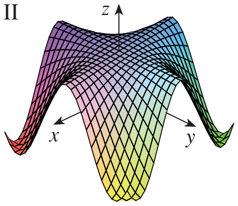

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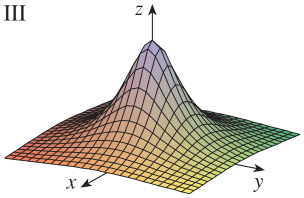

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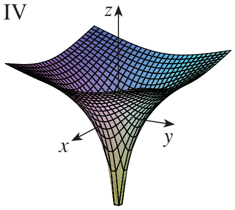

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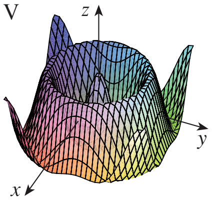

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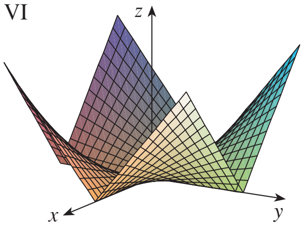
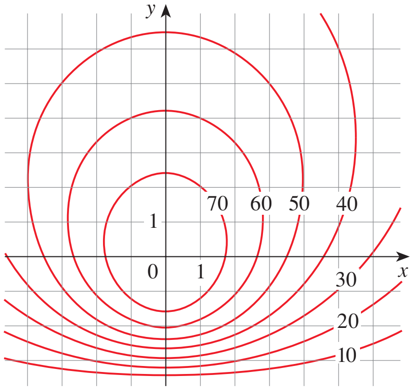


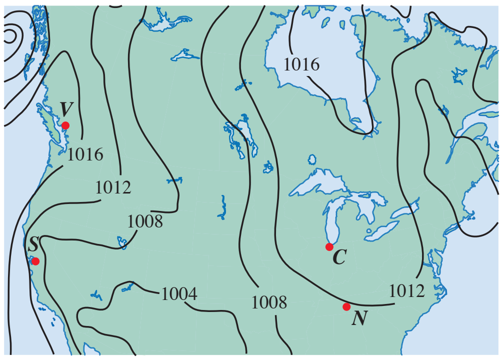
33. A contour map for a function f is shown. Use it to estimate the values of $f(-3, 3)$ and $f(3, -2)$. What can you say about the shape of the graph?



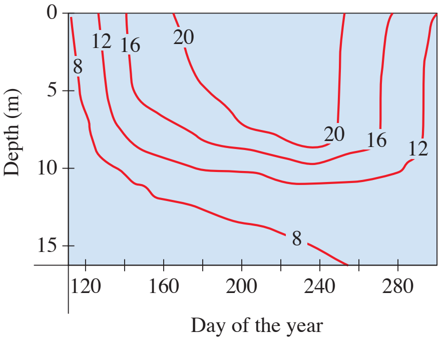
34. Shown is a contour map of atmospheric pressure in North America on a particular day. On the level curves (isobars) the pressure is indicated in millibars (mb).

(a) Estimate the pressure at C (Chicago), N (Nashville), S (San Francisco), and V (Vancouver).

(b) At which of these locations were the winds strongest? (See the discussion preceding Example 9.)

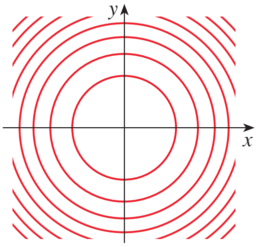


35. Level curves (isothermals) are shown for the typical water temperature (in °C) in Long Lake (Minnesota) as a function of depth and time of year. Estimate the temperature in the lake on June 9 (day 160) at a depth of 10 m and on June 29 (day 180) at a depth of 5 m.



36. Two contour maps are shown. One is for a function f whose graph is a cone. The other is for a function g whose graph is a paraboloid. Which is which, and why?

I



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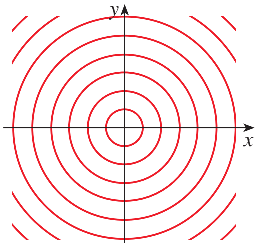
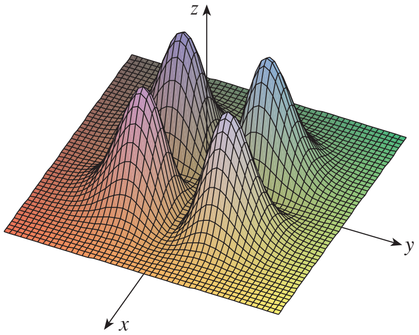


37. Locate the points A and B on the map of Lonesome Mountain (Figure 12). How would you describe the terrain near A? Near B?

38. Make a rough sketch of a contour map for the function whose graph is shown.



39. The *body mass index* (BMI) of a person is defined by

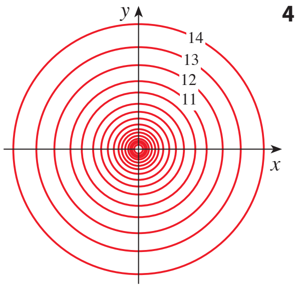
$$B(m, h) = \frac{m}{h^2}$$

where m is the person's mass (in kilograms) and h is the person's height (in meters). Draw the level curves $B(m, h) = 18.5$, $B(m, h) = 25$, $B(m, h) = 30$, and $B(m, h) = 40$. A rough guideline is that a person is underweight if the BMI is less than 18.5; optimal if the BMI lies between 18.5 and 25; overweight if the BMI lies between 25 and 30; and obese if the BMI exceeds 30. Shade the region corresponding to optimal BMI. Does someone who weighs 62 kg and is 152 cm tall fall into the optimal category?

40. The body mass index is defined in Exercise 39. Draw the level curve of this function corresponding to someone who is 200 cm tall and weighs 80 kg. Find the weights and heights of two other people with that same level curve.

41–44 A contour map of a function is shown. Use it to make a rough sketch of the graph of f.

41.



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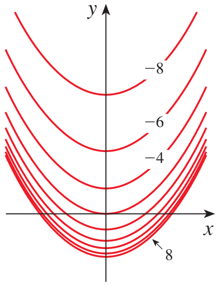

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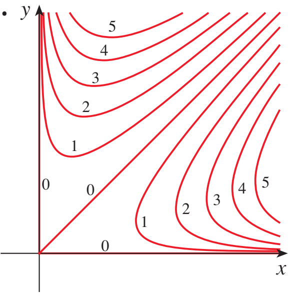

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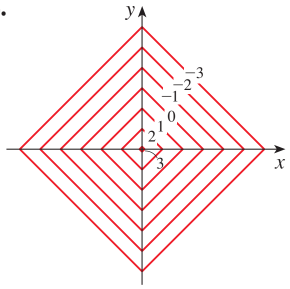


45–52 Draw a contour map of the function showing several level curves.

45. $f(x, y) = x^2 - y^2$

46. $f(x, y) = xy$

47. $f(x, y) = \sqrt{x} + y$

48. $f(x, y) = \ln(x^2 + 4y^2)$

49. $f(x, y) = ye^x$

50. $f(x, y) = y - \arctan x$

51. $f(x, y) = \sqrt[3]{x^2 + y^2}$

52. $f(x, y) = y/(x^2 + y^2)$

53–54 Sketch both a contour map and a graph of the given function and compare them.

53. $f(x, y) = x^2 + 9y^2$

54. $f(x, y) = \sqrt{36 - 9x^2 - 4y^2}$

55. A thin metal plate, located in the xy-plane, has temperature $T(x, y)$ at the point (x, y). Sketch some level curves (isothermals) if the temperature function is given by

$$T(x, y) = \frac{100}{1 + x^2 + 2y^2}$$

56. If $V(x, y)$ is the electric potential at a point (x, y) in the xy-plane, then the level curves of V are called *equipotential curves* because at all points on such a curve the electric potential is the same. Sketch some equipotential curves if $V(x, y) = c/\sqrt{r^2 - x^2 - y^2}$, where c is a positive constant.

57–60 Graph the function using various domains and viewpoints. If your software also produces level curves, then plot some contour lines of the same function and compare with the graph.

57. $f(x, y) = xy^2 - x^3$ (monkey saddle)

58. $f(x, y) = xy^3 - yx^3$ (dog saddle)

59. $f(x, y) = e^{-(x^2+y^2)/3}(\sin(x^2) + \cos(y^2))$

60. $f(x, y) = \cos x \cos y$

61–66 Match the function (a) with its graph (labeled A–F below) and (b) with its contour map (labeled I–VI). Give reasons for your choices.

61. $z = \sin(xy)$

62. $z = e^x \cos y$

63. $z = \sin(x - y)$

64. $z = \sin x - \sin y$

65. $z = (1 - x^2)(1 - y^2)$

66. $z = \dfrac{x - y}{1 + x^2 + y^2}$

67–70 Describe the level surfaces of the function.

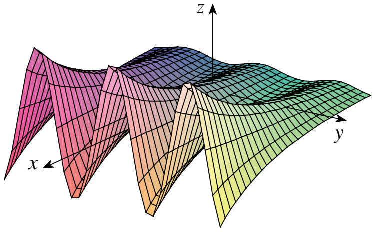
67. $f(x, y, z) = 2y - z + 1$

68. $g(x, y, z) = x + y^2 - z^2$

69. $g(x, y, z) = x^2 + y^2 - z^2$

70. $f(x, y, z) = x^2 + 2y^2 + 3z^2$

A



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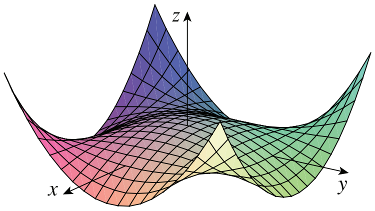

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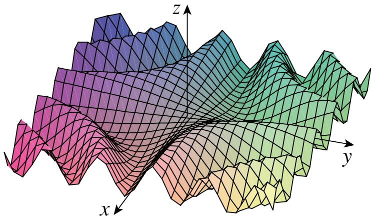

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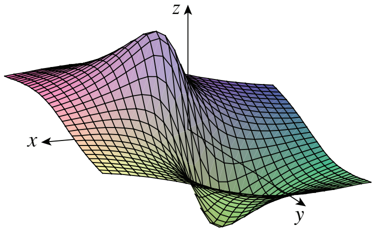

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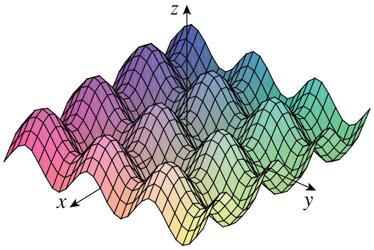

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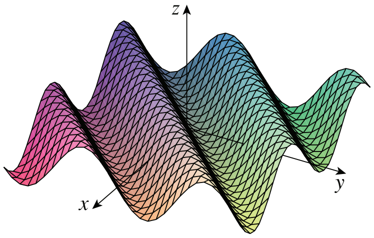

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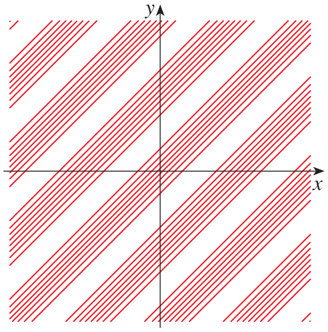

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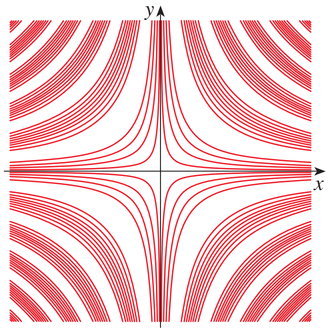

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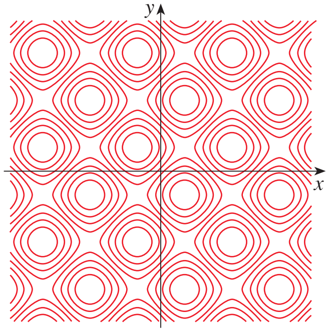

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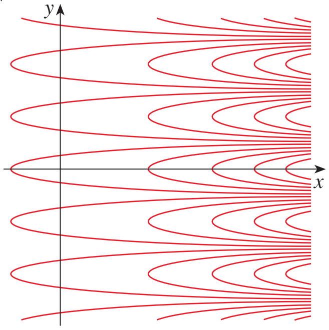

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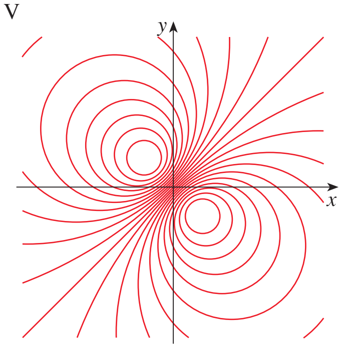

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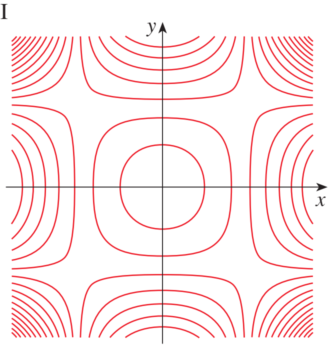


71–72 Describe how the graph of g is obtained from the graph of f.

71. (a) $g(x, y) = f(x, y) + 2$
(b) $g(x, y) = 2f(x, y)$
(c) $g(x, y) = -f(x, y)$
(d) $g(x, y) = 2 - f(x, y)$

72. (a) $g(x, y) = f(x - 2, y)$
(b) $g(x, y) = f(x, y + 2)$
(c) $g(x, y) = f(x + 3, y - 4)$

73–74 Graph the function using various domains and viewpoints that give good views of the "peaks and valleys." Would you say the function has a maximum value? Can you identify any points on the graph that you might consider to be "local maximum points"? What about "local minimum points"?

73. $f(x, y) = 3x - x^4 - 4y^2 - 10xy$

74. $f(x, y) = xye^{-x^2-y^2}$

75–76 Graph the function using various domains and viewpoints. Comment on the limiting behavior of the function. What happens as both x and y become large? What happens as (x, y) approaches the origin?

75. $f(x, y) = \dfrac{x + y}{x^2 + y^2}$ **76.** $f(x, y) = \dfrac{xy}{x^2 + y^2}$

---

77. Investigate the family of functions $f(x, y) = e^{cx^2+y^2}$. How does the shape of the graph depend on c?

78. Investigate the family of surfaces

$$z = (ax^2 + by^2)e^{-x^2-y^2}$$

How does the shape of the graph depend on the numbers a and b?

79. Investigate the family of surfaces $z = x^2 + y^2 + cxy$. In particular, you should determine the transitional values of c for which the surface changes from one type of quadric surface to another.

80. Graph the functions

$$f(x, y) = \sqrt{x^2 + y^2}$$

$$f(x, y) = e^{\sqrt{x^2+y^2}}$$

$$f(x, y) = \ln\sqrt{x^2 + y^2}$$

$$f(x, y) = \sin\left(\sqrt{x^2 + y^2}\right)$$

and $$f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$$

In general, if g is a function of one variable, how is the graph of

$$f(x, y) = g\left(\sqrt{x^2 + y^2}\right)$$

obtained from the graph of g?

81. (a) Show that, by taking logarithms, the general Cobb-Douglas function $P = bL^{\alpha}K^{1-\alpha}$ can be expressed as

$$\ln\frac{P}{K} = \ln b + \alpha \ln\frac{L}{K}$$

(b) If we let $x = \ln(L/K)$ and $y = \ln(P/K)$, the equation in part (a) becomes the linear equation $y = \alpha x + \ln b$. Use Table 2 (in Example 4) to make a table of values of $\ln(L/K)$ and $\ln(P/K)$ for the years 1899–1922. Then find the least squares regression line through the points $(\ln(L/K), \ln(P/K))$.

(c) Deduce that the Cobb-Douglas production function is $P = 1.01L^{0.75}K^{0.25}$.

14.2 Limits and Continuity

Limits of Functions of Two Variables

Let's compare the behavior of the functions

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2} \qquad \text{and} \qquad g(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

as x and y both approach 0 [and therefore the point (x, y) approaches the origin].

Tables 1 and 2 show values of $f(x, y)$ and $g(x, y)$, correct to three decimal places, for points (x, y) near the origin. (Notice that neither function is defined at the origin.)

Table 1 **Values of** $f(x, y)$

x \ y	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
−0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
−0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

Table 2 **Values of** $g(x, y)$

x \ y	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
−0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
−0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0	−1.000	−1.000	−1.000		−1.000	−1.000	−1.000
0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

It appears that as (x, y) approaches $(0, 0)$, the values of $f(x, y)$ are approaching 1 whereas the values of $g(x, y)$ aren't approaching any particular number. It turns out that these guesses based on numerical evidence are correct, and we write

$$\lim_{(x, y) \to (0, 0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1 \qquad \text{and} \qquad \lim_{(x, y) \to (0, 0)} \frac{x^2 - y^2}{x^2 + y^2} \quad \text{does not exist}$$

In general, we use the notation

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

to indicate that the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) (staying within the domain of f). In other words, we can make the values of $f(x, y)$ as close to L as we like by taking the point (x, y) sufficiently close to the point (a, b), but not equal to (a, b). A more precise definition follows.

1 **Definition** Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y) \in D \quad \text{and} \quad 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \varepsilon$$

Other notations for the limit in Definition 1 are

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = L \qquad \text{and} \qquad f(x, y) \to L \;\text{ as }\; (x, y) \to (a, b)$$

Notice that $|f(x, y) - L|$ is the distance between the numbers $f(x, y)$ and L, and $\sqrt{(x - a)^2 + (y - b)^2}$ is the distance between the point (x, y) and the point (a, b). Thus Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small, but not 0. (Compare to the definition of a limit for a function of a single variable, Definition 2.4.2.) Figure 1 illustrates Definition 1 by means of an arrow diagram. If any small interval $(L - \varepsilon, L + \varepsilon)$ is given around L, then we can find a disk D_δ with center (a, b) and radius $\delta > 0$ such that f maps all the points in D_δ [except possibly (a, b)] into the interval $(L - \varepsilon, L + \varepsilon)$.

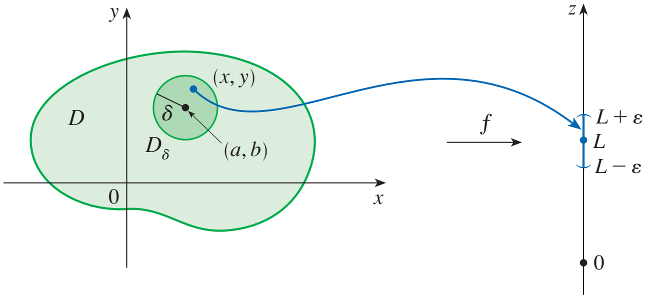


FIGURE 1

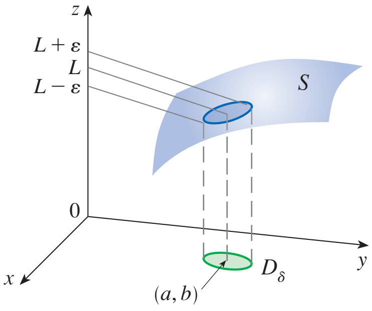


FIGURE 2

Another illustration of Definition 1 is given in Figure 2 where the surface S is the graph of f. If $\varepsilon > 0$ is given, we can find $\delta > 0$ such that if (x, y) is restricted to lie in the disk D_δ and $(x, y) \neq (a, b)$, then the corresponding part of S lies between the horizontal planes $z = L - \varepsilon$ and $z = L + \varepsilon$.

Showing That a Limit Does Not Exist

For functions of a single variable, when we let x approach a, there are only two possible directions of approach, from the left or from the right. We recall from Chapter 2 that if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.

For functions of two variables, the situation is not as simple because we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever (see Figure 3) as long as (x, y) stays within the domain of f.

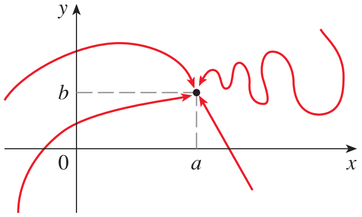


FIGURE 3
Different paths approaching (a, b)

Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0). The definition refers only to the *distance* between (x, y) and (a, b). It does not refer to the direction of approach. Therefore, if the limit exists, then $f(x, y)$ must approach the same limit *no matter how* (x, y) approaches (a, b). Thus one way to show that $\lim_{(x, y) \to (a, b)} f(x, y)$ does not exist is to find different paths of approach along which the function has different limits.

> If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x, y) \to (a, b)} f(x, y)$ does not exist.

EXAMPLE 1 Show that $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

SOLUTION Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$. First let's approach $(0, 0)$ along the x-axis. On this path $y = 0$ for every point (x, y), so the function becomes $f(x, 0) = x^2/x^2 = 1$ for all $x \neq 0$ and thus

$$f(x, y) \to 1 \qquad \text{as} \qquad (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

We now approach along the y-axis by putting $x = 0$. Then $f(0, y) = \dfrac{-y^2}{y^2} = -1$ for all $y \neq 0$, so

$$f(x, y) \to -1 \qquad \text{as} \qquad (x, y) \to (0, 0) \text{ along the } y\text{-axis}$$

(See Figure 4.) Since f has two different limits as (x, y) approaches $(0, 0)$ along two different lines, the given limit does not exist. (This confirms the conjecture we made on the basis of numerical evidence at the beginning of this section.) ■

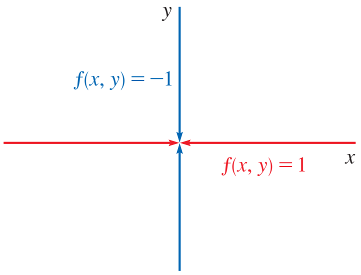


FIGURE 4

EXAMPLE 2 If $f(x, y) = \dfrac{xy}{x^2 + y^2}$, does $\displaystyle\lim_{(x, y)\to(0, 0)} f(x, y)$ exist?

SOLUTION If $y = 0$, then $f(x, 0) = 0/x^2 = 0$. Therefore

$$f(x, y) \to 0 \qquad \text{as} \qquad (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

If $x = 0$, then $f(0, y) = 0/y^2 = 0$, so

$$f(x, y) \to 0 \qquad \text{as} \qquad (x, y) \to (0, 0) \text{ along the } y\text{-axis}$$

Although we have obtained identical limits along the two axes, that does *not* show that the given limit is 0. Let's now approach $(0, 0)$ along another line, say $y = x$. For all $x \neq 0$,

$$f(x, x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

Therefore $\qquad f(x, y) \to \frac{1}{2} \qquad \text{as} \qquad (x, y) \to (0, 0) \text{ along } y = x$

(See Figure 5.) Since we have obtained different limits along different paths, the given limit does not exist. ■

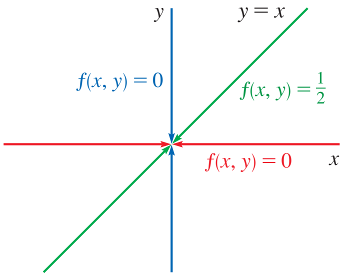


FIGURE 5

Figure 6 sheds some light on Example 2. The ridge that occurs above the line $y = x$ corresponds to the fact that $f(x, y) = \frac{1}{2}$ for all points (x, y) on that line except the origin.

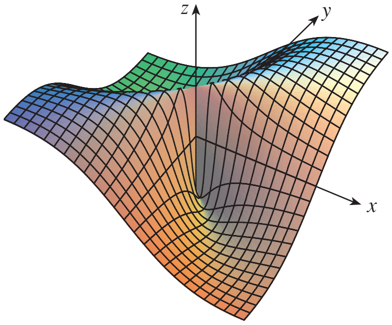


FIGURE 6

$f(x, y) = \dfrac{xy}{x^2 + y^2}$

EXAMPLE 3 If $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$, does $\displaystyle\lim_{(x, y)\to(0, 0)} f(x, y)$ exist?

SOLUTION With the solution of Example 2 in mind, let's try to save time by letting $(x, y) \to (0, 0)$ along any line through the origin. If the line is not the y-axis, then $y = mx$, where m is the slope, and

$$f(x, y) = f(x, mx) = \frac{x(mx)^2}{x^2 + (mx)^4} = \frac{m^2x^3}{x^2 + m^4x^4} = \frac{m^2x}{1 + m^4x^2}$$

Figure 7 shows the graph of the function in Example 3. Notice the ridge above the parabola $x = y^2$.

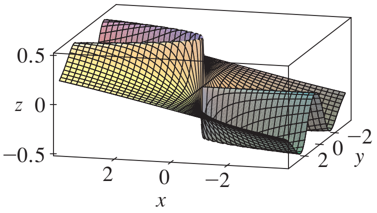


FIGURE 7

So $\qquad f(x, y) \to 0 \qquad \text{as} \qquad (x, y) \to (0, 0)$ along $y = mx$

We get the same result as $(x, y) \to (0, 0)$ along the line $x = 0$. Thus f has the same limiting value along every line through the origin. But that does not show that the given limit is 0, for if we now let $(x, y) \to (0, 0)$ along the parabola $x = y^2$, we have

$$f(x, y) = f(y^2, y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$$

so $\qquad f(x, y) \to \frac{1}{2} \qquad \text{as} \qquad (x, y) \to (0, 0)$ along $x = y^2$

Since different paths lead to different limiting values, the given limit does not exist. ■

Properties of Limits

Just as for functions of one variable, the calculation of limits for functions of two variables can be greatly simplified by the use of properties of limits. The Limit Laws listed in Section 2.3 can be extended to functions of two variables. Assuming that the indicated limits exist, we can state these laws verbally as follows:

Sum Law **1.** The limit of a sum is the sum of the limits.

Difference Law **2.** The limit of a difference is the difference of the limits.

Constant Multiple Law **3.** The limit of a constant times a function is the constant times the limit of the function.

Product Law **4.** The limit of a product is the product of the limits.

Quotient Law **5.** The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

In Exercise 54, you are asked to prove the following special limits:

$$\boxed{2} \qquad \lim_{(x, y) \to (a, b)} x = a \qquad \lim_{(x, y) \to (a, b)} y = b \qquad \lim_{(x, y) \to (a, b)} c = c$$

A **polynomial function** of two variables (or polynomial, for short) is a sum of terms of the form $cx^m y^n$, where c is a constant and m and n are nonnegative integers. A **rational function** is a ratio of two polynomials. For instance,

$$p(x, y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$$

is a polynomial, whereas

$$q(x, y) = \frac{2xy + 1}{x^2 + y^2}$$

is a rational function.

The special limits in (2) along with the limit laws allow us to evaluate the limit of any polynomial function p by direct substitution:

$$\boxed{3} \qquad \lim_{(x, y) \to (a, b)} p(x, y) = p(a, b)$$

Similarly, for any rational function $q(x, y) = p(x, y)/r(x, y)$ we have

$$\boxed{4} \qquad \lim_{(x, y) \to (a, b)} q(x, y) = \lim_{(x, y) \to (a, b)} \frac{p(x, y)}{r(x, y)} = \frac{p(a, b)}{r(a, b)} = q(a, b)$$

provided that (a, b) is in the domain of q.

EXAMPLE 4 Evaluate $\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$.

SOLUTION Since $f(x, y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial, we can find the limit by direct substitution:

$$\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2 = 11 \quad \blacksquare$$

EXAMPLE 5 Evaluate $\lim_{(x,y)\to(-2,3)} \dfrac{x^2y + 1}{x^3y^2 - 2x}$ if it exists.

SOLUTION The function $f(x, y) = (x^2y + 1)/(x^3y^2 - 2x)$ is a rational function and the point $(-2, 3)$ is in its domain (the denominator is not 0 there), so we can evaluate the limit by direct substitution:

$$\lim_{(x,y)\to(-2,3)} \frac{x^2y + 1}{x^3y^2 - 2x} = \frac{(-2)^2(3) + 1}{(-2)^3(3)^2 - 2(-2)} = -\frac{13}{68} \quad \blacksquare$$

The Squeeze Theorem also holds for functions of two or more variables. In the next example we find a limit in two different ways: by using the definition of limit and by using the Squeeze Theorem.

EXAMPLE 6 Find $\lim_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2 + y^2}$ if it exists.

SOLUTION 1 As in Example 3, we could show that the limit along any line through the origin is 0. This doesn't prove that the given limit is 0, but the limits along the parabolas $y = x^2$ and $x = y^2$ also turn out to be 0, so we begin to suspect that the limit does exist and is equal to 0.

Let $\varepsilon > 0$. We want to find $\delta > 0$ such that

$$\text{if} \quad 0 < \sqrt{x^2 + y^2} < \delta \quad \text{then} \quad \left| \frac{3x^2y}{x^2 + y^2} - 0 \right| < \varepsilon$$

that is,

$$\text{if} \quad 0 < \sqrt{x^2 + y^2} < \delta \quad \text{then} \quad \frac{3x^2|y|}{x^2 + y^2} < \varepsilon$$

But $x^2 \leqslant x^2 + y^2$ since $y^2 \geqslant 0$, so $x^2/(x^2 + y^2) \leqslant 1$ and therefore

$$\boxed{5} \qquad \frac{3x^2|y|}{x^2 + y^2} \leqslant 3|y| = 3\sqrt{y^2} \leqslant 3\sqrt{x^2 + y^2}$$

Thus if we choose $\delta = \varepsilon/3$ and let $0 < \sqrt{x^2 + y^2} < \delta$, then by (5) we have

$$\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| \leqslant 3\sqrt{x^2 + y^2} < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

Hence, by Definition 1,

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2} = 0$$

SOLUTION 2 As in Solution 1,

$$\left|\frac{3x^2y}{x^2+y^2}\right| = \frac{3x^2|y|}{x^2+y^2} \leqslant 3|y|$$

so

$$-3|y| \leqslant \frac{3x^2y}{x^2+y^2} \leqslant 3|y|$$

Now $|y| \to 0$ as $y \to 0$ so $\lim_{(x,y)\to(0,0)} (-3|y|) = 0$ and $\lim_{(x,y)\to(0,0)} (3|y|) = 0$ (using Limit Law 3). Thus, by the Squeeze Theorem,

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

■

■ Continuity

Recall that evaluating limits of *continuous* functions of a single variable is easy. It can be accomplished by direct substitution because the defining property of a continuous function is $\lim_{x\to a} f(x) = f(a)$. Continuous functions of two variables are also defined by the direct substitution property.

6 Definition A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

We say that f is **continuous on** D if f is continuous at every point (a, b) in D.

The intuitive meaning of continuity is that if the point (x, y) changes by a small amount, then the value of $f(x, y)$ changes by a small amount. This means that a surface that is the graph of a continuous function has no hole or break.

We have already seen that limits of polynomial functions can be evaluated by direct substitution (Equation 3). It follows by the definition of continuity that *all polynomials are continuous on* $\mathbb{R}^2$. Likewise, Equation 4 shows that *any rational function is continuous on its domain*. In general, using properties of limits, you can see that sums, differences, products, and quotients of continuous functions are continuous on their domains.

EXAMPLE 7 Where is the function $f(x, y) = \dfrac{x^2-y^2}{x^2+y^2}$ continuous?

SOLUTION The function f is discontinuous at $(0, 0)$ because it is not defined there. Since f is a rational function, it is continuous on its domain, which is the set $D = \{(x, y) \mid (x, y) \neq (0, 0)\}$. ■

EXAMPLE 8 Let

$$g(x, y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Here g is defined at $(0, 0)$ but g is still discontinuous there because $\lim_{(x,y)\to(0,0)} g(x, y)$ does not exist (see Example 1). ■

Figure 8 shows the graph of the continuous function in Example 9.

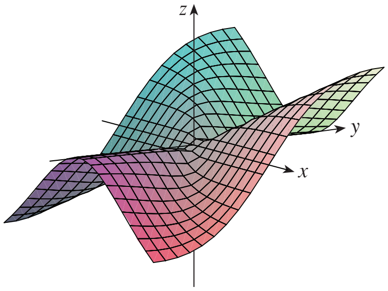


FIGURE 8

EXAMPLE 9 Let

$$f(x, y) = \begin{cases} \dfrac{3x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

We know f is continuous for $(x, y) \neq (0, 0)$ since it is equal to a rational function there. Also, from Example 6, we have

$$\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{(x, y) \to (0, 0)} \frac{3x^2y}{x^2 + y^2} = 0 = f(0, 0)$$

Therefore f is continuous at $(0, 0)$, and so it is continuous on $\mathbb{R}^2$. ■

Just as for functions of one variable, composition is another way of combining two continuous functions to get a third. In fact, it can be shown that if f is a continuous function of two variables and g is a continuous function of a single variable that is defined on the range of f, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also a continuous function.

EXAMPLE 10 Where is the function $h(x, y) = e^{-(x^2+y^2)}$ continuous?

SOLUTION The function $f(x, y) = x^2 + y^2$ is a polynomial and thus is continuous on $\mathbb{R}^2$. Because the function $g(t) = e^{-t}$ is continuous for all values of t, the composite function

$$h(x, y) = g(f(x, y)) = e^{-(x^2+y^2)}$$

is continuous on $\mathbb{R}^2$. The function h is graphed in Figure 9.

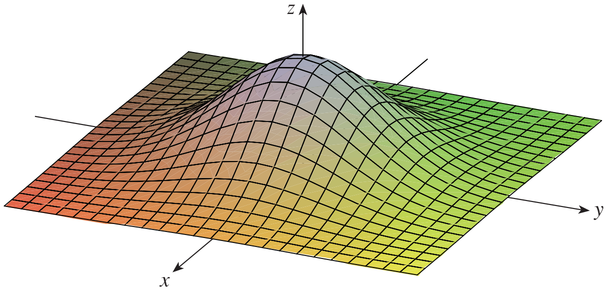


FIGURE 9
The function $h(x, y) = e^{-(x^2+y^2)}$ is continuous everywhere.

■

EXAMPLE 11 Where is the function $h(x, y) = \arctan(y/x)$ continuous?

SOLUTION The function $f(x, y) = y/x$ is a rational function and therefore continuous except on the line $x = 0$. The function $g(t) = \arctan t$ is continuous everywhere. So the composite function

$$g(f(x, y)) = \arctan(y/x) = h(x, y)$$

is continuous except where $x = 0$. The graph in Figure 10 shows the break in the graph of h above the y-axis.

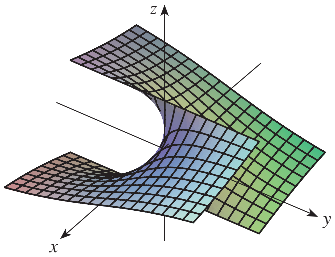


FIGURE 10
The function $h(x, y) = \arctan(y/x)$ is discontinuous where $x = 0$.

■

Functions of Three or More Variables

Everything that we have done in this section can be extended to functions of three or more variables. The notation

$$\lim_{(x, y, z) \to (a, b, c)} f(x, y, z) = L$$

means that the values of $f(x, y, z)$ approach the number L as the point (x, y, z) approaches the point (a, b, c) (staying within the domain of f). Because the distance between two points (x, y, z) and (a, b, c) in $\mathbb{R}^3$ is given by $\sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}$, we can write the precise definition as follows: for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } (x, y, z) \text{ is in the domain of } f \text{ and } 0 < \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2} < \delta$$
$$\text{then } |f(x, y, z) - L| < \varepsilon$$

The function f is **continuous** at (a, b, c) if

$$\lim_{(x, y, z) \to (a, b, c)} f(x, y, z) = f(a, b, c)$$

For instance, the function

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 - 1}$$

is a rational function of three variables and so is continuous at every point in $\mathbb{R}^3$ except where $x^2 + y^2 + z^2 = 1$. In other words, it is discontinuous on the sphere with center the origin and radius 1.

If we use the vector notation introduced at the end of Section 14.1, then we can write the definitions of a limit for functions of two or three variables in a single compact form as follows.

7 If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } \quad \mathbf{x} \in D \quad \text{and} \quad 0 < |\mathbf{x} - \mathbf{a}| < \delta \quad \text{then} \quad |f(\mathbf{x}) - L| < \varepsilon$$

Notice that if $n = 1$, then $\mathbf{x} = x$ and $\mathbf{a} = a$, and (7) is just the definition of a limit for functions of a single variable (Definition 2.4.2). For the case $n = 2$, we have $\mathbf{x} = \langle x, y \rangle$, $\mathbf{a} = \langle a, b \rangle$, and $|\mathbf{x} - \mathbf{a}| = \sqrt{(x - a)^2 + (y - b)^2}$, so (7) becomes Definition 1. If $n = 3$, then $\mathbf{x} = \langle x, y, z \rangle$, $\mathbf{a} = \langle a, b, c \rangle$, and (7) becomes the definition of a limit of a function of three variables. In each case the definition of continuity can be written as

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$

14.2 Exercises

1. Suppose that $\lim_{(x,y)\to(3,1)} f(x,y) = 6$. What can you say about the value of $f(3,1)$? What if f is continuous?

2. Explain why each function is continuous or discontinuous.
- (a) The outdoor temperature as a function of longitude, latitude, and time
- (b) Elevation (height above sea level) as a function of longitude, latitude, and time
- (c) The cost of a taxi ride as a function of distance traveled and time

3–4 Use a table of numerical values of $f(x,y)$ for (x,y) near the origin to make a conjecture about the value of the limit of $f(x,y)$ as $(x,y) \to (0,0)$. Then explain why your guess is correct.

3. $f(x,y) = \dfrac{x^2y^3 + x^3y^2 - 5}{2 - xy}$ **4.** $f(x,y) = \dfrac{2xy}{x^2 + 2y^2}$

5–12 Find the limit.

5. $\displaystyle\lim_{(x,y)\to(3,2)} (x^2y^3 - 4y^2)$

6. $\displaystyle\lim_{(x,y)\to(5,-2)} (x^2y + 3xy^2 + 4)$

7. $\displaystyle\lim_{(x,y)\to(-3,1)} \frac{x^2y - xy^3}{x - y + 2}$ **8.** $\displaystyle\lim_{(x,y)\to(2,-1)} \frac{x^2y + xy^2}{x^2 - y^2}$

9. $\displaystyle\lim_{(x,y)\to(\pi,\pi/2)} y\sin(x - y)$ **10.** $\displaystyle\lim_{(x,y)\to(3,2)} e^{\sqrt{2x-y}}$

11. $\displaystyle\lim_{(x,y)\to(1,1)} \left(\frac{x^2y^3 - x^3y^2}{x^2 - y^2}\right)$ **12.** $\displaystyle\lim_{(x,y)\to(\pi,\pi/2)} \frac{\cos y - \sin 2y}{\cos x \cos y}$

13–18 Show that the limit does not exist.

13. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2 + y^2}$ **14.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + 3y^2}$

15. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x + y)^2}{x^2 + y^2}$ **16.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + xy^2}{x^4 + y^2}$

17. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2 x}{x^4 + y^4}$ **18.** $\displaystyle\lim_{(x,y)\to(1,1)} \frac{y - x}{1 - y + \ln x}$

19–30 Find the limit, if it exists, or show that the limit does not exist.

19. $\displaystyle\lim_{(x,y)\to(-1,-2)} (x^2y - xy^2 + 3)^3$

20. $\displaystyle\lim_{(x,y)\to(\pi,1/2)} e^{xy}\sin xy$

21. $\displaystyle\lim_{(x,y)\to(2,3)} \frac{3x - 2y}{4x^2 - y^2}$ **22.** $\displaystyle\lim_{(x,y)\to(1,2)} \frac{2x - y}{4x^2 - y^2}$

23. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2\cos y}{x^2 + y^4}$ **24.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2 + xy + y^2}$

25. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$

26. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2 + y^8}$

27. $\displaystyle\lim_{(x,y,z)\to(6,1,-2)} \sqrt{x + z}\cos(\pi y)$

28. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz}{x^2 + y^2 + z^2}$

29. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4}$

30. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^4 + y^2 + z^3}{x^4 + 2y^2 + z}$

31–34 Use the Squeeze Theorem to find the limit.

31. $\displaystyle\lim_{(x,y)\to(0,0)} xy\sin\frac{1}{x^2 + y^2}$ **32.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$

33. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4 + y^4}$

34. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2z^2}{x^2 + y^2 + z^2}$

35–36 Use a graph of the function to explain why the limit does not exist.

35. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2 + 3xy + 4y^2}{3x^2 + 5y^2}$ **36.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2 + y^6}$

37–38 Find $h(x,y) = g(f(x,y))$ and the set of points at which h is continuous.

37. $g(t) = t^2 + \sqrt{t}, \quad f(x,y) = 2x + 3y - 6$

38. $g(t) = t + \ln t, \quad f(x,y) = \dfrac{1 - xy}{1 + x^2y^2}$

39–40 Graph the function and observe where it is discontinuous. Then use the formula to explain what you have observed.

39. $f(x,y) = e^{1/(x-y)}$ **40.** $f(x,y) = \dfrac{1}{1 - x^2 - y^2}$

41–50 Determine the set of points at which the function is continuous.

41. $F(x,y) = \dfrac{xy}{1 + e^{x-y}}$ **42.** $F(x,y) = \cos\sqrt{1 + x - y}$

43. $F(x,y) = \dfrac{1 + x^2 + y^2}{1 - x^2 - y^2}$ **44.** $H(x,y) = \dfrac{e^x + e^y}{e^{xy} - 1}$

45. $G(x, y) = \sqrt{x} + \sqrt{1 - x^2 - y^2}$

46. $G(x, y) = \ln(1 + x - y)$

47. $f(x, y, z) = \arcsin(x^2 + y^2 + z^2)$

48. $f(x, y, z) = \sqrt{y - x^2}\,\ln z$

49. $f(x, y) = \begin{cases} \dfrac{x^2y^3}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$

50. $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + xy + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

51–53 Use polar coordinates to find the limit. [If (r, θ) are polar coordinates of the point (x, y) with $r \geqslant 0$, note that $r \to 0^+$ as $(x, y) \to (0, 0)$.]

51. $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{x^3 + y^3}{x^2 + y^2}$

52. $\displaystyle\lim_{(x, y)\to(0, 0)} (x^2 + y^2)\ln(x^2 + y^2)$

53. $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2}$

54. Prove the three special limits in (2).

55. At the beginning of this section we considered the function

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

and guessed on the basis of numerical evidence that $f(x, y) \to 1$ as $(x, y) \to (0, 0)$. Use polar coordinates to confirm the value of the limit. Then graph the function.

56. Graph and discuss the continuity of the function

$$f(x, y) = \begin{cases} \dfrac{\sin xy}{xy} & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$$

57. Let

$$f(x, y) = \begin{cases} 0 & \text{if } y \leqslant 0 \quad \text{or} \quad y \geqslant x^4 \\ 1 & \text{if } 0 < y < x^4 \end{cases}$$

(a) Show that $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along any path through $(0, 0)$ of the form $y = mx^a$ with $0 < a < 4$.
(b) Despite part (a), show that f is discontinuous at $(0, 0)$.
(c) Show that f is discontinuous on two entire curves.

58. Show that the function f given by $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on $\mathbb{R}^n$. [*Hint:* Consider $|\mathbf{x} - \mathbf{a}|^2 = (\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$.]

59. If $\mathbf{c} \in V_n$, show that the function f given by $f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$ is continuous on $\mathbb{R}^n$.

14.3 Partial Derivatives

Partial Derivatives of Functions of Two Variables

On a hot day, extreme humidity makes us think the temperature is higher than it really is, whereas in very dry air we perceive the temperature to be lower than the thermometer indicates. The National Weather Service has devised the *heat index* (also called the temperature-humidity index, or humidex, in some countries) to describe the combined effects of temperature and humidity. The heat index I is the perceived air temperature when the actual temperature is T and the relative humidity is H. So I is a function of T and H and we can write $I = f(T, H)$. The following table of values of I is an excerpt from a table compiled by the National Weather Service.

Table 1 Heat index I as a function of temperature and humidity

		Relative humidity (%)								
	T \ H	40	45	50	55	60	65	70	75	80
Actual temperature (°C)	26	28	28	29	31	31	32	33	34	35
	28	31	32	33	34	35	36	37	38	39
	30	34	35	36	37	38	40	41	42	43
	32	37	38	39	41	42	43	45	46	47
	34	41	42	43	45	47	48	49	51	52
	36	43	45	47	48	50	51	53	54	56

If we concentrate on the highlighted column of the table, which corresponds to a relative humidity of $H = 60\%$, we are considering the heat index as a function of the single variable T for a fixed value of H. Let's write $g(T) = f(T, 60)$. Then $g(T)$ describes how the heat index I increases as the actual temperature T increases when the relative humidity is 60%. The derivative of g when $T = 30°\text{C}$ is the rate of change of I with respect to T when $T = 30°\text{C}$:

$$g'(30) = \lim_{h\to 0} \frac{g(30 + h) - g(30)}{h} = \lim_{h\to 0} \frac{f(30 + h, 60) - f(30, 60)}{h}$$

We can approximate $g'(30)$ using the values in Table 1 by taking $h = 2$ and -2:

$$g'(30) \approx \frac{g(32) - g(30)}{2} = \frac{f(32, 60) - f(30, 60)}{2} = \frac{42 - 38}{2} = 2$$

$$g'(30) \approx \frac{g(28) - g(30)}{-2} = \frac{f(28, 60) - f(30, 60)}{-2} = \frac{35 - 38}{-2} = 1.5$$

Averaging these values, we can say that the derivative $g'(30)$ is approximately 1.75. This means that, when the actual temperature is 30°C and the relative humidity is 60%, the apparent temperature (heat index) rises by about 1.75°C for every degree that the actual temperature rises.

Now let's look at the highlighted row in Table 1, which corresponds to a fixed temperature of $T = 30°\text{C}$. The numbers in this row are values of the function $G(H) = f(30, H)$, which describes how the heat index increases as the relative humidity H increases when the actual temperature is $T = 30°\text{C}$. The derivative of this function when $H = 60\%$ is the rate of change of I with respect to H when $H = 60\%$:

$$G'(60) = \lim_{h\to 0} \frac{G(60 + h) - G(60)}{h} = \lim_{h\to 0} \frac{f(30, 60 + h) - f(30, 60)}{h}$$

By taking $h = 5$ and -5, we approximate $G'(60)$ using the tabular values:

$$G'(60) \approx \frac{G(65) - G(60)}{5} = \frac{f(30, 65) - f(30, 60)}{5} = \frac{42 - 38}{5} = 0.4$$

$$G'(60) \approx \frac{G(55) - G(60)}{-5} = \frac{f(30, 55) - f(30, 60)}{-5} = \frac{37 - 38}{-5} = 0.2$$

By averaging these values we get the estimate $G'(60) \approx 0.3$. This says that, when the temperature is 30°C and the relative humidity is 60%, the heat index rises about 0.3°C for every percent that the relative humidity rises.

In general, if f is a function of two variables x and y, suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant. Then we are really considering a function of a single variable x, namely, $g(x) = f(x, b)$. If g has a derivative at a, then we call it the **partial derivative of f with respect to x at (a, b)** and denote it by $f_x(a, b)$. Thus

$$\boxed{f_x(a, b) = g'(a) \qquad \text{where} \qquad g(x) = f(x, b)} \tag{1}$$

By the definition of a derivative, we have

$$g'(a) = \lim_{h\to 0} \frac{g(a + h) - g(a)}{h}$$

and so Equation 1 becomes

2

$$\boxed{f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}}$$

Similarly, the **partial derivative of f with respect to y at (a, b)**, denoted by $f_y(a, b)$, is obtained by keeping x fixed ($x = a$) and finding the ordinary derivative at b of the function $G(y) = f(a, y)$:

3

$$\boxed{f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}}$$

With this notation for partial derivatives, we can write the rates of change of the heat index I with respect to the actual temperature T and relative humidity H when $T = 30°\text{C}$ and $H = 60\%$ as follows:

$$f_T(30, 60) \approx 1.75 \qquad f_H(30, 60) \approx 0.3$$

If we now let the point (a, b) vary in Equations 2 and 3, f_x and f_y become functions of two variables.

4 Definition If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

There are many alternative notations for partial derivatives. For instance, instead of f_x we can write f_1 or $D_1 f$ (to indicate differentiation with respect to the *first* variable) or $\partial f / \partial x$. But here $\partial f / \partial x$ can't be interpreted as a ratio of differentials.

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

To compute partial derivatives, all we have to do is remember from Equation 1 that the partial derivative with respect to x is just the *ordinary* derivative of the function g of a single variable that we get by keeping y fixed. Thus we have the following rule.

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

EXAMPLE 1 If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

SOLUTION Holding y constant and differentiating with respect to x, we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

and so

$$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

Holding x constant and differentiating with respect to y, we get

$$f_y(x, y) = 3x^2y^2 - 4y$$

$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

■

EXAMPLE 2 If $f(x, y) = \sin\left(\dfrac{x}{1 + y}\right)$, calculate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

SOLUTION Using the Chain Rule for functions of one variable, we have

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1 + y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1 + y}\right) = \cos\left(\frac{x}{1 + y}\right) \cdot \frac{1}{1 + y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1 + y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1 + y}\right) = -\cos\left(\frac{x}{1 + y}\right) \cdot \frac{x}{(1 + y)^2}$$

■

Interpretations of Partial Derivatives

To give a geometric interpretation of partial derivatives, we recall that the equation $z = f(x, y)$ represents a surface S (the graph of f). If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S. By fixing $y = b$, we are restricting our attention to the curve C_1 in which the vertical plane $y = b$ intersects S. (In other words, C_1 is the trace of S in the plane $y = b$.) Likewise, the vertical plane $x = a$ intersects S in a curve C_2. Both of the curves C_1 and C_2 pass through the point P. (See Figure 1.)

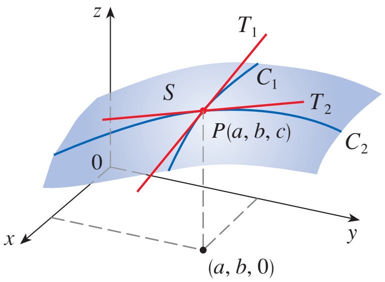


FIGURE 1
The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

Note that the curve C_1 is the graph of the function $g(x) = f(x, b)$, so the slope of its tangent T_1 at P is $g'(a) = f_x(a, b)$. The curve C_2 is the graph of the function $G(y) = f(a, y)$, so the slope of its tangent T_2 at P is $G'(b) = f_y(a, b)$.

Thus the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.

EXAMPLE 3 If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

SOLUTION We have

$$f_x(x, y) = -2x \qquad f_y(x, y) = -4y$$

$$f_x(1, 1) = -2 \qquad f_y(1, 1) = -4$$

The graph of f is the paraboloid $z = 4 - x^2 - 2y^2$ and the vertical plane $y = 1$ intersects it in the parabola $z = 2 - x^2$, $y = 1$. (As in the preceding discussion, we label it C_1 in Figure 2.) The slope of the tangent line to this parabola at the point $(1, 1, 1)$ is $f_x(1, 1) = -2$. (Notice that the tangent line slopes downward in the positive x-direction.) Similarly, the curve C_2 in which the plane $x = 1$ intersects the paraboloid is the parabola $z = 3 - 2y^2$, $x = 1$, and the slope of the tangent line at $(1, 1, 1)$ is $f_y(1, 1) = -4$. (See Figure 3.)

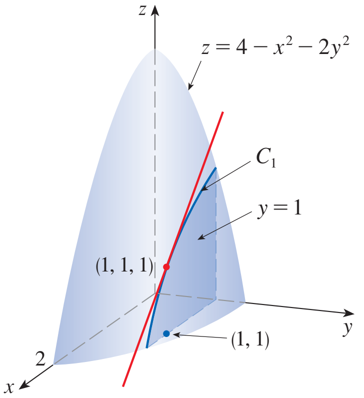


FIGURE 2

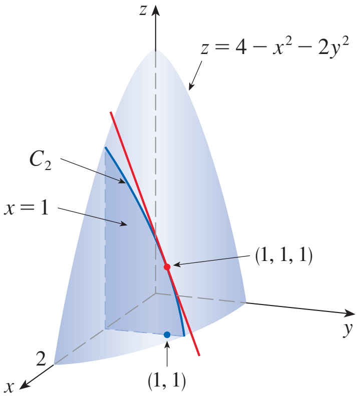


FIGURE 3

As we have seen in the case of the heat index function at the beginning of this section, partial derivatives can also be interpreted as *rates of change*. If $z = f(x, y)$, then $\partial z/\partial x$ represents the rate of change of z with respect to x when y is fixed. Similarly, $\partial z/\partial y$ represents the rate of change of z with respect to y when x is fixed.

EXAMPLE 4 In Exercise 14.1.39 we defined the body mass index (BMI) of a person as

$$B(m, h) = \frac{m}{h^2}$$

Calculate the partial derivatives of B for a young man with $m = 64$ kg and $h = 1.68$ m and interpret them.

SOLUTION Regarding h as a constant, we see that the partial derivative with respect to m is

$$\frac{\partial B}{\partial m}(m, h) = \frac{\partial}{\partial m}\left(\frac{m}{h^2}\right) = \frac{1}{h^2}$$

so $$\frac{\partial B}{\partial m}(64, 1.68) = \frac{1}{(1.68)^2} \approx 0.35\ (\text{kg/m}^2)/\text{kg}$$

This is the rate at which the man's BMI increases with respect to his weight when he weighs 64 kg and his height is 1.68 m. So if his weight increases by a small amount, one kilogram for instance, and his height remains unchanged, then his BMI will increase from $B(64, 1.68) \approx 22.68$ by about 0.35.

Now we regard m as a constant. The partial derivative with respect to h is

$$\frac{\partial B}{\partial h}(m, h) = \frac{\partial}{\partial h}\left(\frac{m}{h^2}\right) = m\left(-\frac{2}{h^3}\right) = -\frac{2m}{h^3}$$

so $$\frac{\partial B}{\partial h}(64, 1.68) = -\frac{2 \cdot 64}{(1.68)^3} \approx -27\ (\text{kg/m}^2)/\text{m}$$

This is the rate at which the man's BMI increases with respect to his height when he weighs 64 kg and his height is 1.68 m. So if the man is still growing and his weight stays unchanged while his height increases by a small amount, say 1 cm, then his BMI will *decrease* by about $27(0.01) = 0.27$. ■

EXAMPLE 5 Find $\partial z/\partial x$ and $\partial z/\partial y$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz + 4 = 0$$

Then evaluate these partial derivatives at the point $(-1, 1, 2)$.

Some software can plot surfaces defined by implicit equations in three variables. Figure 4 shows such a plot of the surface defined by the equation in Example 5.

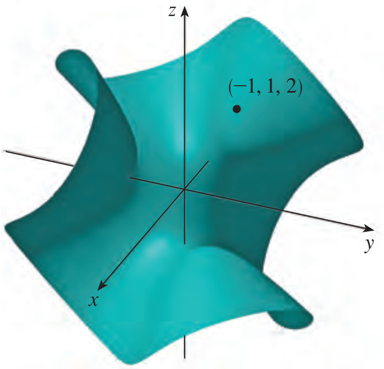


FIGURE 4

SOLUTION To find $\partial z/\partial x$, we differentiate implicitly with respect to x, being careful to treat y as a constant and z as a function (of x):

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0$$

Solving this equation for $\partial z/\partial x$, we obtain

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

Similarly, implicit differentiation with respect to y gives

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

Notice that the point $(-1, 1, 2)$ satisfies the equation $x^3 + y^3 + z^3 + 6xyz + 4 = 0$ so it lies on the surface. At this point

$$\frac{\partial z}{\partial x} = -\frac{(-1)^2 + 2 \cdot 1 \cdot 2}{2^2 + 2(-1) \cdot 1} = -\frac{5}{2} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -\frac{1^2 + 2(-1) \cdot 2}{2^2 + 2(-1) \cdot 1} = \frac{3}{2}$$ ■

Functions of Three or More Variables

Partial derivatives can also be defined for functions of three or more variables. For example, if f is a function of three variables x, y, and z, then its partial derivative with respect to x is defined as

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$$

and it is found by regarding y and z as constants and differentiating $f(x, y, z)$ with respect to x. If $w = f(x, y, z)$, then $f_x = \partial w/\partial x$ can be interpreted as the rate of change of w with respect to x when y and z are held fixed. But we can't interpret it geometrically because the graph of f lies in four-dimensional space.

In general, if u is a function of n variables, $u = f(x_1, x_2, \ldots, x_n)$, its partial derivative with respect to the ith variable x_i is

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

and we also write

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f$$

EXAMPLE 6 Find f_x, f_y, and f_z if $f(x, y, z) = e^{xy} \ln z$.

SOLUTION Holding y and z constant and differentiating with respect to x, we have

$$f_x = ye^{xy} \ln z$$

Similarly,

$$f_y = xe^{xy} \ln z \qquad \text{and} \qquad f_z = \frac{e^{xy}}{z}$$ ■

Higher Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f. If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus the notation f_{xy} (or $\partial^2 f/\partial y\,\partial x$) means that we first differentiate with respect to x and then with respect to y, whereas in computing f_{yx} the order is reversed.

EXAMPLE 7 Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

SOLUTION In Example 1 we found that

$$f_x(x, y) = 3x^2 + 2xy^3 \qquad f_y(x, y) = 3x^2y^2 - 4y$$

Therefore

$$f_{xx} = \frac{\partial}{\partial x}(3x^2 + 2xy^3) = 6x + 2y^3 \qquad f_{xy} = \frac{\partial}{\partial y}(3x^2 + 2xy^3) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}(3x^2y^2 - 4y) = 6xy^2 \qquad f_{yy} = \frac{\partial}{\partial y}(3x^2y^2 - 4y) = 6x^2y - 4 \quad ■$$

Notice that $f_{xy} = f_{yx}$ in Example 7. This is not just a coincidence. It turns out that the mixed partial derivatives f_{xy} and f_{yx} are equal for most functions that one meets in practice. The following theorem, which was discovered by the French mathematician Alexis Clairaut (1713–1765), gives conditions under which we can assert that $f_{xy} = f_{yx}$. The proof is given in Appendix F.

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Clairaut

Alexis Clairaut was a child prodigy in mathematics: he read l'Hospital's textbook on calculus when he was 10 and presented a paper on geometry to the French Academy of Sciences when he was 13. At the age of 18, Clairaut published *Recherches sur les courbes à double courbure,* which was the first systematic treatise on three-dimensional analytic geometry and included the calculus of space curves.

Partial derivatives of order 3 or higher can also be defined. For instance,

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y\,\partial x}\right) = \frac{\partial^3 f}{\partial y^2\,\partial x}$$

and using Clairaut's Theorem it can be shown that $f_{xyy} = f_{yxy} = f_{yyx}$ if these functions are continuous.

EXAMPLE 8 Calculate f_{xxyz} if $f(x, y, z) = \sin(3x + yz)$.

SOLUTION

$$\begin{aligned} f_x &= 3\cos(3x + yz) \\ f_{xx} &= -9\sin(3x + yz) \\ f_{xxy} &= -9z\cos(3x + yz) \\ f_{xxyz} &= -9\cos(3x + yz) + 9yz\sin(3x + yz) \end{aligned}$$

■

■ Partial Differential Equations

Partial derivatives occur in *partial differential equations* that express certain physical laws. For instance, the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called **Laplace's equation** after Pierre Laplace (1749–1827). Solutions of this equation are called **harmonic functions**; they play a role in problems of heat conduction, fluid flow, and electric potential.

EXAMPLE 9 Show that the function $u(x, y) = e^x \sin y$ is a solution of Laplace's equation.

SOLUTION We first compute the needed second-order partial derivatives:

$$u_x = e^x \sin y \qquad u_y = e^x \cos y$$

$$u_{xx} = e^x \sin y \qquad u_{yy} = -e^x \sin y$$

So $$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$$

Therefore u satisfies Laplace's equation. ■

The **wave equation**

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

describes the motion of a waveform, which could be an ocean wave, a sound wave, a light wave, or a wave traveling along a vibrating string. For instance, if $u(x, t)$ represents the displacement of a vibrating violin string at time t and at a distance x from one end of the string (as in Figure 5), then $u(x, t)$ satisfies the wave equation. Here the constant a depends on the density of the string and on the tension in the string.

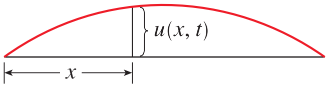


FIGURE 5

EXAMPLE 10 Verify that the function $u(x, t) = \sin(x - at)$ satisfies the wave equation.

SOLUTION

$$u_x = \cos(x - at) \qquad u_t = -a\cos(x - at)$$

$$u_{xx} = -\sin(x - at) \qquad u_{tt} = -a^2 \sin(x - at) = a^2 u_{xx}$$

So u satisfies the wave equation. ■

Partial differential equations involving functions of three variables are also very important in science and engineering. The three-dimensional Laplace equation is

$$\boxed{5} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

and one application is in geophysics. If $u(x, y, z)$ represents magnetic field strength at position (x, y, z), then it satisfies Equation 5. The strength of the magnetic field indicates the distribution of iron-rich minerals and reflects different rock types and the location of faults.

14.3 Exercises

1. At the beginning of this section we discussed the function $I = f(T, H)$, where I is the heat index, T is the actual temperature, and H is the relative humidity. Use Table 1 to estimate $f_T(34, 75)$ and $f_H(34, 75)$. What are the practical interpretations of these values?

2. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in the following table.

Duration (hours)

Wind speed (km/h)

v \ t	5	10	15	20	30	40	50
20	0.6	0.6	0.6	0.6	0.6	0.6	0.6
30	1.2	1.3	1.5	1.5	1.5	1.6	1.6
40	1.5	2.2	2.4	2.5	2.7	2.8	2.8
60	2.8	4.0	4.9	5.2	5.5	5.8	5.9
80	4.3	6.4	7.7	8.6	9.5	10.1	10.2
100	5.8	8.9	11.0	12.2	13.8	14.7	15.3
120	7.4	11.3	14.4	16.6	19.0	20.5	21.1

(a) What are the meanings of the partial derivatives $\partial h/\partial v$ and $\partial h/\partial t$?

(b) Estimate the values of $f_v(40, 15)$ and $f_t(40, 15)$. What are the practical interpretations of these values?

(c) What appears to be the value of the following limit?

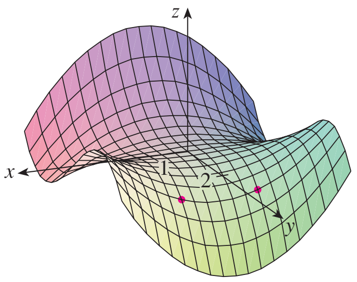
$$\lim_{t\to\infty} \frac{\partial h}{\partial t}$$

3. The temperature T (in °C) at a location in the Northern Hemisphere depends on the longitude x, latitude y, and time t, so we can write $T = f(x, y, t)$. Let's measure time in hours from the beginning of January.

(a) What are the meanings of the partial derivatives $\partial T/\partial x$, $\partial T/\partial y$, and $\partial T/\partial t$?

(b) Honolulu has longitude 158°W and latitude 21°N. Suppose that at 9:00 AM on January 1 the wind is blowing hot air to the northeast, so the air to the west and south is warm and the air to the north and east is cooler. Would you expect $f_x(158, 21, 9)$, $f_y(158, 21, 9)$, and $f_t(158, 21, 9)$ to be positive or negative? Explain.

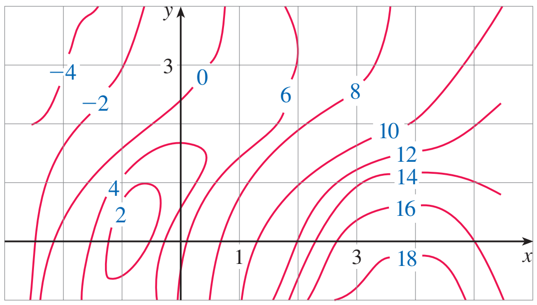
4–5 Determine the signs of the partial derivatives for the function f whose graph is shown.



4. (a) $f_x(1, 2)$ (b) $f_y(1, 2)$

5. (a) $f_x(-1, 2)$ (b) $f_y(-1, 2)$

6. A contour map is given for a function f. Use it to estimate $f_x(2, 1)$ and $f_y(2, 1)$.



7. If $f(x, y) = 16 - 4x^2 - y^2$, find $f_x(1, 2)$ and $f_y(1, 2)$ and interpret these numbers as slopes. Illustrate with either hand-drawn sketches or computer plots.

8. If $f(x, y) = \sqrt{4 - x^2 - 4y^2}$, find $f_x(1, 0)$ and $f_y(1, 0)$ and interpret these numbers as slopes. Illustrate with either hand-drawn sketches or computer plots.

9–36 Find the first partial derivatives of the function.

9. $f(x, y) = x^4 + 5xy^3$

10. $f(x, y) = x^2y - 3y^4$

11. $g(x, y) = x^3 \sin y$

12. $g(x, t) = e^{xt}$

13. $z = \ln(x + t^2)$

14. $w = \dfrac{u}{v^2}$

15. $f(x, y) = ye^{xy}$

16. $g(x, y) = (x^2 + xy)^3$

17. $g(x, y) = y(x + x^2y)^5$

18. $f(x, y) = \dfrac{x}{(x + y)^2}$

19. $f(x, y) = \dfrac{ax + by}{cx + dy}$

20. $w = \dfrac{e^v}{u + v^2}$

21. $g(u, v) = (u^2v - v^3)^5$

22. $u(r, \theta) = \sin(r \cos \theta)$

23. $R(p, q) = \tan^{-1}(pq^2)$

24. $f(x, y) = x^y$

25. $F(x, y) = \displaystyle\int_y^x \cos(e^t)\, dt$

26. $F(\alpha, \beta) = \displaystyle\int_\alpha^\beta \sqrt{t^3 + 1}\, dt$

27. $f(x, y, z) = x^3yz^2 + 2yz$

28. $f(x, y, z) = xy^2e^{-xz}$

29. $w = \ln(x + 2y + 3z)$

30. $w = y \tan(x + 2z)$

31. $p = \sqrt{t^4 + u^2 \cos v}$

32. $u = x^{y/z}$

33. $h(x, y, z, t) = x^2y \cos(z/t)$

34. $\phi(x, y, z, t) = \dfrac{\alpha x + \beta y^2}{\gamma z + \delta t^2}$

35. $u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

36. $u = \sin(x_1 + 2x_2 + \cdots + nx_n)$

37–40 Find the indicated partial derivative.

37. $R(s, t) = te^{s/t}$; $R_t(0, 1)$

38. $f(x, y) = y \sin^{-1}(xy)$; $f_y\left(1, \tfrac{1}{2}\right)$

39. $f(x, y, z) = \ln \dfrac{1 - \sqrt{x^2 + y^2 + z^2}}{1 + \sqrt{x^2 + y^2 + z^2}}$; $f_y(1, 2, 2)$

40. $f(x, y, z) = x^{yz}$; $f_z(e, 1, 0)$

41–44 Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$.

41. $x^2 + 2y^2 + 3z^2 = 1$

42. $x^2 - y^2 + z^2 - 2z = 4$

43. $e^z = xyz$

44. $yz + x \ln y = z^2$

45–46 Find $\partial z/\partial x$ and $\partial z/\partial y$.

45. (a) $z = f(x) + g(y)$ (b) $z = f(x + y)$

46. (a) $z = f(x)g(y)$ (b) $z = f(xy)$

(c) $z = f(x/y)$

47–52 Find all the second partial derivatives.

47. $f(x, y) = x^4y - 2x^3y^2$

48. $f(x, y) = \ln(ax + by)$

49. $z = \dfrac{y}{2x + 3y}$

50. $T = e^{-2r} \cos \theta$

51. $v = \sin(s^2 - t^2)$

52. $z = \arctan \dfrac{x + y}{1 - xy}$

53–56 Verify that the conclusion of Clairaut's Theorem holds, that is, $u_{xy} = u_{yx}$.

53. $u = x^4y^3 - y^4$

54. $u = e^{xy} \sin y$

55. $u = \cos(x^2y)$

56. $u = \ln(x + 2y)$

57–64 Find the indicated partial derivative(s).

57. $f(x, y) = x^4y^2 - x^3y$; f_{xxx}, f_{xyx}

58. $f(x, y) = \sin(2x + 5y)$; f_{yxy}

59. $f(x, y, z) = e^{xyz^2}$; f_{xyz}

60. $g(r, s, t) = e^r \sin(st)$; g_{rst}

61. $W = \sqrt{u + v^2}$; $\dfrac{\partial^3 W}{\partial u^2\, \partial v}$

62. $V = \ln(r + s^2 + t^3)$; $\dfrac{\partial^3 V}{\partial r\, \partial s\, \partial t}$

63. $w = \dfrac{x}{y + 2z}$; $\dfrac{\partial^3 w}{\partial z\, \partial y\, \partial x}$, $\dfrac{\partial^3 w}{\partial x^2\, \partial y}$

64. $u = x^a y^b z^c$; $\dfrac{\partial^6 u}{\partial x\, \partial y^2\, \partial z^3}$

65–66 Use Definition 4 to find $f_x(x, y)$ and $f_y(x, y)$.

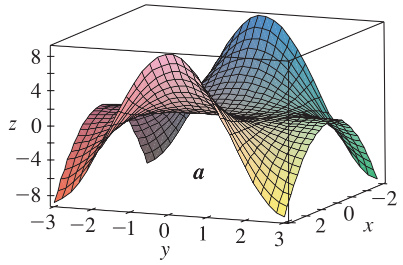
65. $f(x, y) = xy^2 - x^3y$

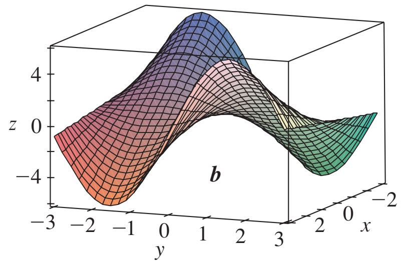
66. $f(x, y) = \dfrac{x}{x + y^2}$

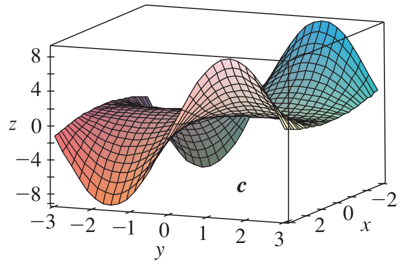
67. If $f(x, y, z) = xy^2z^3 + \arcsin\left(x\sqrt{z}\right)$, find f_{xzy}. [*Hint:* Which order of differentiation is easiest?]

68. If $g(x, y, z) = \sqrt{1 + xz} + \sqrt{1 - xy}$, find g_{xyz}. [*Hint:* Use a different order of differentiation for each term.]

69. The following surfaces, labeled *a*, *b*, and *c*, are graphs of a function f and its partial derivatives f_x and f_y. Identify each surface and give reasons for your choices.







70–71 Find f_x and f_y and graph f, f_x, and f_y with domains and viewpoints that enable you to see the relationships between them.

70. $f(x, y) = \dfrac{y}{1 + x^2y^2}$ **71.** $f(x, y) = x^2y^3$

---

72. Determine the signs of the partial derivatives for the function f whose graph is shown in Exercises 4–5.

(a) $f_{xx}(-1, 2)$ (b) $f_{yy}(-1, 2)$

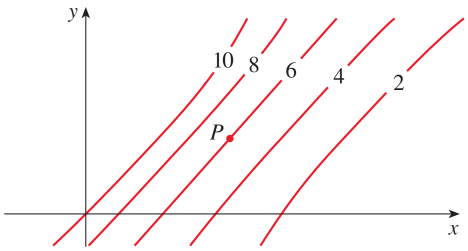
(c) $f_{xy}(1, 2)$ (d) $f_{xy}(-1, 2)$

73. Use the table of values of $f(x, y)$ to estimate the values of $f_x(3, 2)$, $f_x(3, 2.2)$, and $f_{xy}(3, 2)$.

x \ y	1.8	2.0	2.2
2.5	12.5	10.2	9.3
3.0	18.1	17.5	15.9
3.5	20.0	22.4	26.1

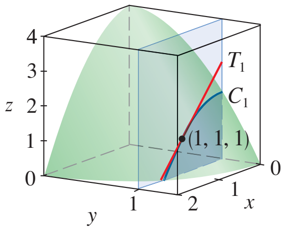
74. Level curves are shown for a function f. Determine whether the following partial derivatives are positive or negative at the point P.

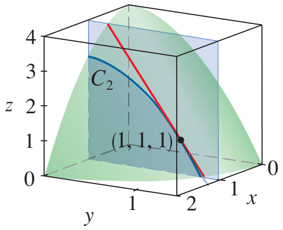
(a) f_x (b) f_y (c) f_{xx} (d) f_{xy} (e) f_{yy}



75. (a) In Example 3 we found that $f_x(1, 1) = -2$ for the function $f(x, y) = 4 - x^2 - 2y^2$. We interpreted this result geometrically as the slope of the tangent line to the curve C_1 at the point $P(1, 1, 1)$, where C_1 is the trace of the graph of f in the plane $y = 1$. (See the figure.) Verify this interpretation by finding a vector equation for C_1, computing the tangent vector to C_1 at P, and then finding the slope of the tangent line to C_1 at P in the plane $y = 1$.

(b) Use a similar method to verify that $f_y(1, 1) = -4$.





76. If $u = e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n}$, where $a_1^2 + a_2^2 + \cdots + a_n^2 = 1$, show that

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = u$$

77. Show that the function $u = u(x, t)$ is a solution of the wave equation $u_{tt} = a^2u_{xx}$.

(a) $u = \sin(kx)\sin(akt)$

(b) $u = t/(a^2t^2 - x^2)$

(c) $u = (x - at)^6 + (x + at)^6$

(d) $u = \sin(x - at) + \ln(x + at)$

78. Determine whether each of the following functions is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$.

(a) $u = x^2 + y^2$ (b) $u = x^2 - y^2$

(c) $u = x^3 + 3xy^2$ (d) $u = \ln\sqrt{x^2 + y^2}$

(e) $u = \sin x \cosh y + \cos x \sinh y$

(f) $u = e^{-x}\cos y - e^{-y}\cos x$

79. Verify that the function $u = 1/\sqrt{x^2 + y^2 + z^2}$ is a solution of the three-dimensional Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$.

80. The Heat Equation Verify that the function $u = e^{-\alpha^2 k^2 t} \sin kx$ is a solution of the *heat conduction equation* $u_t = \alpha^2 u_{xx}$.

81. The Diffusion Equation The *diffusion equation*

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}$$

where D is a positive constant, describes the diffusion of heat through a solid, or the concentration of a pollutant at time t at a distance x from the source of the pollution, or the invasion of alien species into a new habitat. Verify that the function

$$c(x, t) = \frac{1}{\sqrt{4\pi D t}} e^{-x^2/(4Dt)}$$

is a solution of the diffusion equation.

82. The temperature at a point (x, y) on a flat metal plate is given by $T(x, y) = 60/(1 + x^2 + y^2)$, where T is measured in °C and x, y in meters. Find the rate of change of temperature with respect to distance at the point $(2, 1)$ in (a) the x-direction and (b) the y-direction.

83. The total resistance R produced by three conductors with resistances R_1, R_2, R_3 connected in a parallel electrical circuit is given by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Find $\partial R/\partial R_1$.

84. Ideal Gas Law The gas law for a fixed mass m of an ideal gas at absolute temperature T, pressure P, and volume V is $PV = mRT$, where R is the gas constant.

(a) Show that $\dfrac{\partial P}{\partial V}\dfrac{\partial V}{\partial T}\dfrac{\partial T}{\partial P} = -1$.

(b) Show that $T\dfrac{\partial P}{\partial T}\dfrac{\partial V}{\partial T} = mR$.

85. Van der Waals Equation The *Van der Waals equation* for n moles of a gas is

$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$$

where P is the pressure, V is the volume, and T is the temperature of the gas. The constant R is the universal gas constant and a and b are positive constants that are characteristic of a particular gas. Calculate $\partial T/\partial P$ and $\partial P/\partial V$.

86. The wind-chill index is modeled by the function

$$W = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$$

where T is the temperature (°C) and v is the wind speed (in km/h). When $T = -15$°C and $v = 30$ km/h, by how much would you expect the apparent temperature W to drop if the actual temperature decreases by 1°C? What if the wind speed increases by 1 km/h?

87. A model for the surface area of a human body is given by the function

$$S = f(w, h) = 0.0072w^{0.425}h^{0.725}$$

where w is the weight (in kilograms), h is the height (in centimeters), and S is measured in square meters. Calculate and interpret the partial derivatives.

(a) $\dfrac{\partial S}{\partial w}(73, 178)$ (b) $\dfrac{\partial S}{\partial h}(73, 178)$

88. One of Poiseuille's laws states that the resistance of blood flowing through an artery is

$$R = C\frac{L}{r^4}$$

where L and r are the length and radius of the artery and C is a positive constant determined by the viscosity of the blood. Calculate $\partial R/\partial L$ and $\partial R/\partial r$ and interpret them.

89. In the project following Section 4.7 we expressed the power needed by a bird during its flapping mode as

$$P(v, x, m) = Av^3 + \frac{B(mg/x)^2}{v}$$

where A and B are constants specific to a species of bird, v is the velocity of the bird, m is the mass of the bird, and x is the fraction of the flying time spent in flapping mode. Calculate $\partial P/\partial v$, $\partial P/\partial x$, and $\partial P/\partial m$ and interpret them.

90. In a study of frost penetration it was found that the temperature T at time t (measured in days) at a depth x (measured in meters) can be modeled by the function

$$T(x, t) = T_0 + T_1 e^{-\lambda x}\sin(\omega t - \lambda x)$$

where $\omega = 2\pi/365$ and λ is a positive constant.

(a) Find $\partial T/\partial x$. What is its physical significance?

(b) Find $\partial T/\partial t$. What is its physical significance?

(c) Show that T satisfies the heat equation $T_t = kT_{xx}$ for a certain constant k.

(d) Graph $T(x, t)$ for $\lambda = 0.2$, $T_0 = 0$, and $T_1 = 10$.

(e) What is the physical significance of the term $-\lambda x$ in the expression $\sin(\omega t - \lambda x)$?

91. The kinetic energy of a body with mass m and velocity v is $K = \frac{1}{2}mv^2$. Show that

$$\frac{\partial K}{\partial m}\frac{\partial^2 K}{\partial v^2} = K$$

92. The average energy E (in kcal) needed for a lizard to walk or run a distance of 1 km has been modeled by the equation

$$E(m, v) = 2.65m^{0.66} + \frac{3.5m^{0.75}}{v}$$

where m is the body mass of the lizard (in grams) and v is its speed (in km/h). Calculate $E_m(400, 8)$ and $E_v(400, 8)$ and interpret your answers.

Source: C. Robbins, *Wildlife Feeding and Nutrition*, 2d ed. (San Diego: Academic Press, 1993).

93. The ellipsoid $4x^2 + 2y^2 + z^2 = 16$ intersects the plane $y = 2$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 2)$.

94. The paraboloid $z = 6 - x - x^2 - 2y^2$ intersects the plane $x = 1$ in a parabola. Find parametric equations for the tangent line to this parabola at the point $(1, 2, -4)$. Use a computer to graph the paraboloid, the parabola, and the tangent line on the same screen.

95. You are told that there is a function f whose partial derivatives are $f_x(x, y) = x + 4y$ and $f_y(x, y) = 3x - y$. Should you believe it?

96. If a, b, c are the sides of a triangle and A, B, C are the opposite angles, find $\partial A/\partial a$, $\partial A/\partial b$, $\partial A/\partial c$ by implicit differentiation of the Law of Cosines.

97. Use Clairaut's Theorem to show that if the third-order partial derivatives of f are continuous, then

$$f_{xyy} = f_{yxy} = f_{yyx}$$

98. (a) How many nth-order partial derivatives does a function of two variables have?

(b) If these partial derivatives are all continuous, how many of them can be distinct?

(c) Answer the question in part (a) for a function of three variables.

99. If

$$f(x, y) = x(x^2 + y^2)^{-3/2} e^{\sin(x^2 y)}$$

find $f_x(1, 0)$. [*Hint:* Instead of finding $f_x(x, y)$ first, note that it's easier to use Equation 1 or Equation 2.]

100. If $f(x, y) = \sqrt[3]{x^3 + y^3}$, find $f_x(0, 0)$.

101. Let

$$f(x, y) = \begin{cases} \dfrac{x^3 y - xy^3}{x^2 + y^2} & \text{if} \quad (x, y) \neq (0, 0) \\ 0 & \text{if} \quad (x, y) = (0, 0) \end{cases}$$

(a) Graph f.

(b) Find $f_x(x, y)$ and $f_y(x, y)$ when $(x, y) \neq (0, 0)$.

(c) Find $f_x(0, 0)$ and $f_y(0, 0)$ using Equations 2 and 3.

(d) Show that $f_{xy}(0, 0) = -1$ and $f_{yx}(0, 0) = 1$.

(e) Does the result of part (d) contradict Clairaut's Theorem? Use graphs of f_{xy} and f_{yx} to illustrate your answer.

DISCOVERY PROJECT | DERIVING THE COBB-DOUGLAS PRODUCTION FUNCTION

In Example 14.1.4 we described the work of Cobb and Douglas in modeling the total production P of an economic system as a function of the amount of labor L and the capital investment K. If the production function is denoted by $P = P(L, K)$, then $\partial P/\partial L$, the rate at which production changes with respect to the amount of labor, is called the **marginal productivity of labor**. Similarly, $\partial P/\partial K$ is the **marginal productivity of capital**.

Here we use these partial derivatives to show how the particular form of the model used by Cobb and Douglas follows from the following assumptions they made about the economy.

(i) If either labor or capital vanishes, then so will production.

(ii) The marginal productivity of labor is proportional to the amount of production per unit of labor (P/L).

(iii) The marginal productivity of capital is proportional to the amount of production per unit of capital (P/K).

1. Assumption (ii) says that

$$\frac{\partial P}{\partial L} = \alpha \frac{P}{L}$$

for some constant α. If K is held constant $(K = K_0)$, then this partial differential equation becomes the ordinary differential equation

$$\frac{dP}{dL} = \alpha \frac{P}{L}$$

Solve this separable differential equation by the methods of Section 9.3 to get $P(L, K_0) = C_1(K_0)\, L^{\alpha}$, where the constant C_1 is written as $C_1(K_0)$ because it could depend on the value of K_0.

(continued)

2. Similarly, show that assumption (iii) implies that if L is held constant ($L = L_0$), then $P(L_0, K) = C_2(L_0)K^\beta$.

3. Comparing the results of Problems 1 and 2, conclude that

$$P(L, K) = bL^\alpha K^\beta$$

where b is a constant that is independent of both L and K. Cobb and Douglas assumed that $\alpha + \beta = 1$, so that

$$P(L, K) = bL^\alpha K^{1-\alpha}$$

In this case, if labor and capital are both increased by a factor m, then by what factor is production increased?

4. Show that $P(L, K) = bL^\alpha K^{1-\alpha}$ satisfies the partial differential equation

$$L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = P$$

5. Cobb and Douglas used the function $P(L, K) = 1.01L^{0.75}K^{0.25}$ to model the American economy from 1899 to 1922. Find the marginal productivity of labor and the marginal productivity of capital in the year 1920, when $L = 194$ and $K = 407$, and interpret the results. In that year, which would have benefited production more, an increase in capital investment or an increase in spending on labor?

14.4 Tangent Planes and Linear Approximations

One of the most important ideas in single-variable calculus is that as we zoom in toward a point on the graph of a differentiable function, the graph becomes indistinguishable from its tangent line and we can approximate the function by a linear function. (See Section 3.10.) Here we develop similar ideas in three dimensions. As we zoom in toward a point on a surface that is the graph of a differentiable function of two variables, the surface looks more and more like a plane (its tangent plane) and we can approximate the function by a linear function of two variables. We also extend the idea of a differential to functions of two or more variables.

Tangent Planes

Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S. As in Section 14.3, let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. Then the point P lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P. Then the **tangent plane** to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2. (See Figure 1.)

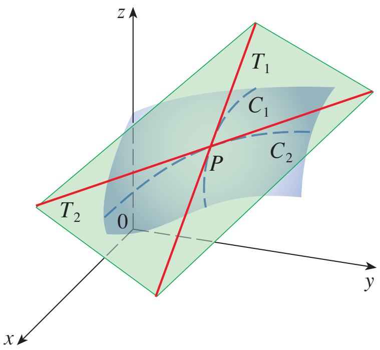


FIGURE 1
The tangent plane contains the tangent lines T_1 and T_2.

We will see in Section 14.6 that if C is any other curve that lies on the surface S and passes through P, then its tangent line at P also lies in the tangent plane. Therefore you can think of the tangent plane to S at P as consisting of all possible tangent lines at P to curves that lie on S and pass through P. The tangent plane at P is the plane that most closely approximates the surface S near the point P.

We know from Equation 12.5.7 that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

By dividing this equation by C and letting $a = -A/C$ and $b = -B/C$, we can write it in the form

$$\boxed{1} \qquad z - z_0 = a(x - x_0) + b(y - y_0)$$

If Equation 1 represents the tangent plane at P, then its intersection with the plane $y = y_0$ must be the tangent line T_1. Setting $y = y_0$ in Equation 1 gives

$$z - z_0 = a(x - x_0) \qquad \text{where } \; y = y_0$$

and we recognize this as the equation (in point-slope form) of a line with slope a. But from Section 14.3 we know that the slope of the tangent T_1 is $f_x(x_0, y_0)$. Therefore $a = f_x(x_0, y_0)$.

Similarly, putting $x = x_0$ in Equation 1, we get $z - z_0 = b(y - y_0)$, which must represent the tangent line T_2, so $b = f_y(x_0, y_0)$.

Note the similarity between the equation of a tangent plane and the equation of a tangent line:

$$y - y_0 = f'(x_0)(x - x_0)$$

2 Equation of a Tangent Plane Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

EXAMPLE 1 Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

SOLUTION Let $f(x, y) = 2x^2 + y^2$. Then

$$\begin{aligned} f_x(x, y) &= 4x & \qquad f_y(x, y) &= 2y \\ f_x(1, 1) &= 4 & \qquad f_y(1, 1) &= 2 \end{aligned}$$

Then (2) gives the equation of the tangent plane at $(1, 1, 3)$ as

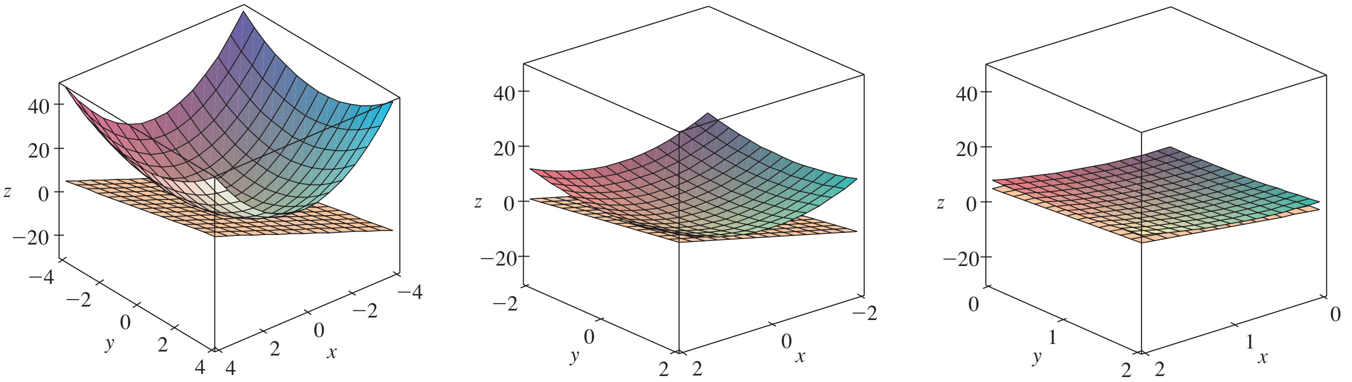
$$z - 3 = 4(x - 1) + 2(y - 1)$$

or

$$z = 4x + 2y - 3$$

■

Figure 2(a) shows the elliptic paraboloid and its tangent plane at (1, 1, 3) that we found in Example 1. In parts (b) and (c) we zoom in toward the point (1, 1, 3). Notice that the more we zoom in, the flatter the graph appears and the more it resembles its tangent plane.



(a) (b) (c)

FIGURE 2 The elliptic paraboloid $z = 2x^2 + y^2$ appears to coincide with its tangent plane as we zoom in toward (1, 1, 3).

In Figure 3 we corroborate this impression by zooming in toward the point (1, 1) on a contour map of the function $f(x, y) = 2x^2 + y^2$. Notice that the more we zoom in, the more the level curves look like equally spaced parallel lines, which is characteristic of a plane.

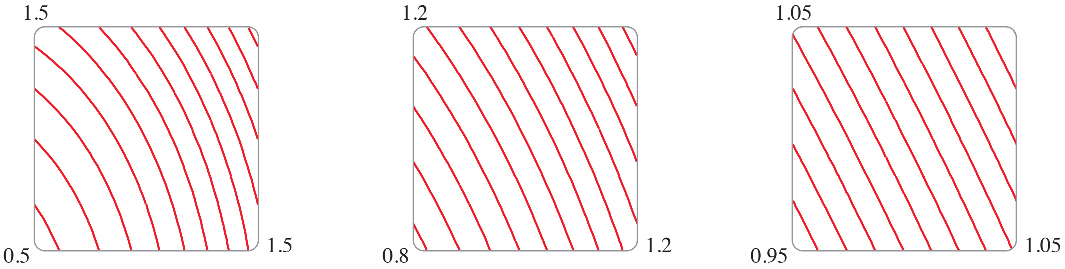


FIGURE 3
Zooming in toward (1, 1) on a contour map of $f(x, y) = 2x^2 + y^2$

Linear Approximations

In Example 1 we found that an equation of the tangent plane to the graph of the function $f(x, y) = 2x^2 + y^2$ at the point (1, 1, 3) is $z = 4x + 2y - 3$. Therefore, in view of the visual evidence in Figures 2 and 3, the linear function of two variables

$$L(x, y) = 4x + 2y - 3$$

is a good approximation to $f(x, y)$ when (x, y) is near (1, 1). The function L is called the *linearization* of f at (1, 1) and the approximation

$$f(x, y) \approx 4x + 2y - 3$$

is called the *linear approximation* or *tangent plane approximation* of f at (1, 1).

For instance, at the point (1.1, 0.95) the linear approximation gives

$$f(1.1, 0.95) \approx 4(1.1) + 2(0.95) - 3 = 3.3$$

which is quite close to the true value of $f(1.1, 0.95) = 2(1.1)^2 + (0.95)^2 = 3.3225$. But if we take a point farther away from (1, 1), such as (2, 3), we no longer get a good approximation. In fact, $L(2, 3) = 11$ whereas $f(2, 3) = 17$.

In general, we know from (2) that an equation of the tangent plane to the graph of a function f of two variables at the point $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The linear function whose graph is this tangent plane, namely

$$\boxed{3} \qquad \boxed{L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)}$$

is called the **linearization** of f at (a, b) and the approximation

$$\boxed{4} \qquad \boxed{f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)}$$

is called the **linear approximation** or the **tangent plane approximation** of f at (a, b).

We have defined tangent planes for surfaces $z = f(x, y)$, where f has continuous first partial derivatives. What happens if f_x and f_y are not continuous? Figure 4 pictures such a function; its equation is

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

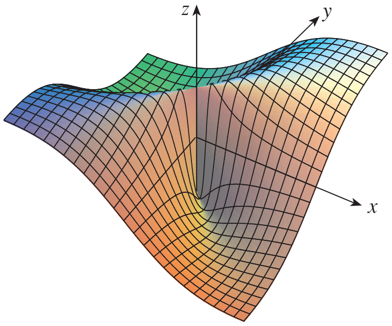


FIGURE 4
$f(x, y) = \dfrac{xy}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$,
$f(0, 0) = 0$

You can verify (see Exercise 54) that its partial derivatives exist at the origin and, in fact, $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$, but f_x and f_y are not continuous. The linear approximation would be $f(x, y) \approx 0$, but $f(x, y) = \frac{1}{2}$ at all points on the line $y = x$. So a function of two variables can behave badly even though both of its partial derivatives exist. To rule out such behavior, we formulate the idea of a differentiable function of two variables.

Recall that for a function of one variable, $y = f(x)$, if x changes from a to $a + \Delta x$, we defined the increment of y as

$$\Delta y = f(a + \Delta x) - f(a)$$

In Chapter 3 we showed that if f is differentiable at a, then

This is Equation 3.4.7.

$$\boxed{5} \qquad \Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x \qquad \text{where } \varepsilon \to 0 \text{ as } \Delta x \to 0$$

Now consider a function of two variables, $z = f(x, y)$, and suppose x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$. Then the corresponding **increment** of z is

$$\boxed{6} \qquad \Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

Thus the increment Δz represents the change in the value of f when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$. By analogy with (5) we define the differentiability of a function of two variables as follows.

7 Definition If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and ε_2 are functions of Δx and Δy such that ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \ \Delta y) \to (0, 0)$.

Definition 7 says that a differentiable function is one for which the linear approximation (4) is a good approximation when (x, y) is near (a, b). In other words, the tangent plane approximates the graph of f well near the point of tangency.

It's sometimes hard to use Definition 7 directly to check the differentiability of a function, but the next theorem provides a convenient sufficient condition for differentiability.

Theorem 8 is proved in Appendix F.

8 Theorem If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

EXAMPLE 2 Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

SOLUTION The partial derivatives are

$$f_x(x, y) = e^{xy} + xye^{xy} \qquad f_y(x, y) = x^2e^{xy}$$
$$f_x(1, 0) = 1 \qquad f_y(1, 0) = 1$$

Both f_x and f_y are continuous functions, so f is differentiable by Theorem 8. The linearization is

$$\begin{aligned} L(x, y) &= f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) \\ &= 1 + 1(x - 1) + 1 \cdot y = x + y \end{aligned}$$

The corresponding linear approximation is

$$xe^{xy} \approx x + y$$

so $$f(1.1, -0.1) \approx 1.1 - 0.1 = 1$$

Compare this with the actual value of $f(1.1, -0.1) = 1.1e^{-0.11} \approx 0.98542$. ■

Figure 5 shows the graphs of the function f and its linearization L in Example 2.

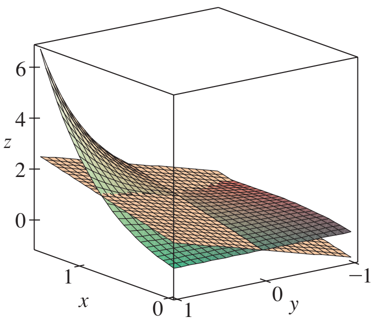


FIGURE 5

EXAMPLE 3 At the beginning of Section 14.3 we discussed the heat index (perceived temperature) I as a function of the actual temperature T and the relative humidity H and gave the following table of values from the National Weather Service.

Relative humidity (%)

Actual temperature (°C)

T \ H	40	45	50	55	60	65	70	75	80
26	28	28	29	31	31	32	33	34	35
28	31	32	33	34	35	36	37	38	39
30	34	35	36	37	38	40	41	42	43
32	37	38	39	41	42	43	45	46	47
34	41	42	43	45	47	48	49	51	52
36	43	45	47	48	50	51	53	54	56

Find a linear approximation for the heat index $I = f(T, H)$ when T is near 30°C and H is near 60%. Use it to estimate the heat index when the actual temperature is 31°C and the relative humidity is 62%.

SOLUTION We read from the table that $f(30, 60) = 38$. At the beginning of Section 14.3 we used the tabular values to estimate that $f_T(30, 60) \approx 1.75$ and $f_H(30, 60) \approx 0.3$. So the linear approximation is

$$\begin{aligned} f(T, H) &\approx f(30, 60) + f_T(30, 60)(T - 30) + f_H(30, 60)(H - 60) \\ &\approx 38 + 1.75(T - 30) + 0.3(H - 60) \end{aligned}$$

In particular,

$$f(31, 62) \approx 38 + 1.75(1) + 0.3(2) = 40.35$$

Therefore, when $T = 31$°C and $H = 62$%, the heat index is

$$I \approx 40.4°\text{C}$$ ■

Differentials

For a differentiable function of one variable, $y = f(x)$, we define the differential dx to be an independent variable; that is, dx can be given the value of any real number. The differential of y is then defined as

$$\boxed{9} \qquad dy = f'(x)\,dx$$

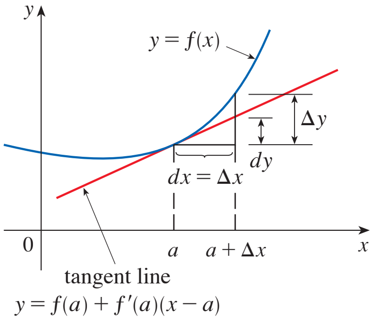


FIGURE 6

(See Section 3.10.) Figure 6 shows the relationship between the increment Δy and the differential dy: Δy represents the change in height of the curve $y = f(x)$ and dy represents the change in height of the tangent line when x changes by an amount $dx = \Delta x$.

For a differentiable function of two variables, $z = f(x, y)$, we define the **differentials** dx and dy to be independent variables; that is, they can be given any values. Then the **differential** dz, also called the **total differential**, is defined by

$$\boxed{10} \qquad \boxed{dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy}$$

(Compare with Equation 9.) Sometimes the notation df is used in place of dz.

If we take $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$ in Equation 10, then the differential of z is

$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

So, in the notation of differentials, the linear approximation (4) can be written as

$$f(x, y) \approx f(a, b) + dz$$

Figure 7 is the three-dimensional counterpart of Figure 6 and shows the geometric interpretation of the differential dz and the increment Δz: dz represents the change in height of the tangent plane, whereas Δz represents the change in height of the surface $z = f(x, y)$ when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

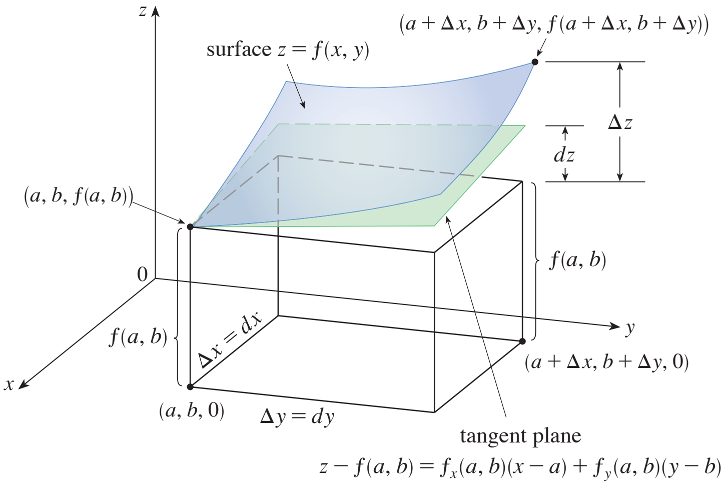


FIGURE 7

EXAMPLE 4

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.
(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

In Example 4, dz is close to Δz because the tangent plane is a good approximation to the surface $z = x^2 + 3xy - y^2$ near $(2, 3, 13)$. (See Figure 8.)

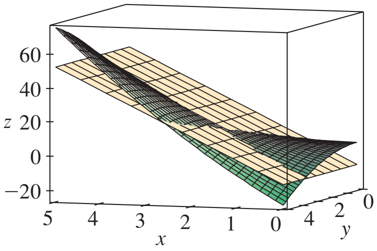


FIGURE 8

SOLUTION

(a) Definition 10 gives

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = (2x + 3y)\,dx + (3x - 2y)\,dy$$

(b) Putting $x = 2$, $dx = \Delta x = 0.05$, $y = 3$, and $dy = \Delta y = -0.04$, we get

$$dz = [2(2) + 3(3)]0.05 + [3(2) - 2(3)](-0.04) = 0.65$$

The increment of z is

$$\begin{aligned}\Delta z &= f(2.05, 2.96) - f(2, 3)\\ &= [(2.05)^2 + 3(2.05)(2.96) - (2.96)^2] - [2^2 + 3(2)(3) - 3^2]\\ &= 0.6449\end{aligned}$$

Notice that $\Delta z \approx dz$ but dz is easier to compute. ■

EXAMPLE 5 The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as ε cm in each.

(a) Use differentials to estimate the maximum error in the calculated volume of the cone.

(b) What is the estimated maximum error in volume if the radius and height are measured with errors up to 0.1 cm?

SOLUTION

(a) The volume V of a cone with base radius r and height h is $V = \pi r^2 h/3$. So the differential of V is

$$dV = \frac{\partial V}{\partial r}\,dr + \frac{\partial V}{\partial h}\,dh = \frac{2\pi rh}{3}\,dr + \frac{\pi r^2}{3}\,dh$$

Since each error is at most ε cm, we have $|\Delta r| \leqslant \varepsilon$, $|\Delta h| \leqslant \varepsilon$. To estimate the largest error in the volume, we take the largest error in the measurement of r and of h. Therefore we take $dr = \varepsilon$ and $dh = \varepsilon$ along with $r = 10$, $h = 25$. This gives

$$\Delta V \approx dV = \frac{500\pi}{3}\,\varepsilon + \frac{100\pi}{3}\,\varepsilon = 200\pi\varepsilon$$

Thus the maximum error in the calculated volume is about $200\pi\varepsilon$ cm^3.

(b) If the largest error in each measurement is $\varepsilon = 0.1$ cm, then $dV = 200\pi(0.1) \approx 63$, so the estimated maximum error in volume is about 63 cm^3. (Note that since the measured volume of the cone is $V = \pi(10)^2(25)/3 \approx 2618$, this is a relative error of $63/2618 \approx 0.024$ or 2.4%.) ■

Functions of Three or More Variables

Linear approximations, differentiability, and differentials can be defined in a similar manner for functions of more than two variables. A differentiable function is defined by an expression similar to the one in Definition 7. For such functions the **linear approximation** is

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

and the linearization $L(x, y, z)$ is the right side of this expression.

If $w = f(x, y, z)$, then the **increment** of w is

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

The **differential** dw is defined in terms of the differentials dx, dy, and dz of the independent variables by

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz$$

EXAMPLE 6 The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within ε cm.
(a) Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.
(b) What is the estimated maximum error in the calculated volume if the measured dimensions are correct to within 0.2 cm?

SOLUTION
(a) If the dimensions of the box are x, y, and z, then its volume is $V = xyz$ and so

$$dV = \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz = yz\,dx + xz\,dy + xy\,dz$$

We are given that $|\Delta x| \leqslant \varepsilon$, $|\Delta y| \leqslant \varepsilon$, and $|\Delta z| \leqslant \varepsilon$. To estimate the largest error in the volume, we therefore use $dx = \varepsilon$, $dy = \varepsilon$, and $dz = \varepsilon$ together with $x = 75$, $y = 60$, and $z = 40$:

$$\Delta V \approx dV = (60)(40)\varepsilon + (75)(40)\varepsilon + (75)(60)\varepsilon = 9900\varepsilon$$

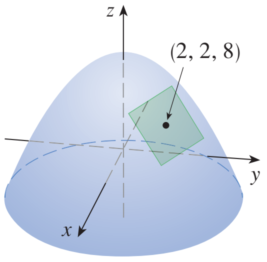
Thus the maximum error in the calculated volume is about 9900 times larger than the error in each measurement taken.

(b) If the largest error in each measurement is $\varepsilon = 0.2$ cm, then $dV = 9900(0.2) = 1980$, so an error of only 0.2 cm in measuring each dimension could lead to an error of approximately 1980 cm^3 in the calculated volume. (This may seem like a large error, but you can verify that it's only about 1% of the volume of the box.) ■

14.4 Exercises

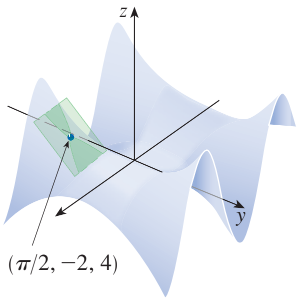
1–2 The graph of a function f is shown. Find an equation of the tangent plane to the surface $z = f(x, y)$ at the specified point.

1. $f(x, y) = 16 - x^2 - y^2$



$z = 16 - x^2 - y^2$

2. $f(x, y) = y^2 \sin x$



$z = y^2 \sin x$

3–10 Find an equation of the tangent plane to the given surface at the specified point.

3. $z = 2x^2 + y^2 - 5y$, $(1, 2, -4)$

4. $z = (x + 2)^2 - 2(y - 1)^2 - 5$, $(2, 3, 3)$

5. $z = e^{x-y}$, $(2, 2, 1)$

6. $z = y^2 e^x$, $(0, 3, 9)$

7. $z = 2\sqrt{y}/x$, $(-1, 1, -2)$

8. $z = x/y^2$, $(-4, 2, -1)$

9. $z = x \sin(x + y)$, $(-1, 1, 0)$

10. $z = \ln(x - 2y)$, $(3, 1, 0)$

11–12 Graph the surface and the tangent plane at the given point. (Choose the domain and viewpoint so that you get a good view of both the surface and the tangent plane.) Then zoom in until the surface and the tangent plane become indistinguishable.

11. $z = x^2 + xy + 3y^2$, $(1, 1, 5)$

12. $z = \sqrt{9 + x^2y^2}$, $(2, 2, 5)$

13–14 Draw the graph of f and its tangent plane at the given point. (Use a computer to compute the partial derivatives.) Then zoom in until the surface and the tangent plane become indistinguishable.

13. $f(x, y) = \dfrac{1 + \cos^2(x - y)}{1 + \cos^2(x + y)}$, $\left(\dfrac{\pi}{3}, \dfrac{\pi}{6}, \dfrac{7}{4}\right)$

14. $f(x, y) = e^{-xy/10}\left(\sqrt{x} + \sqrt{y} + \sqrt{xy}\right)$, $(1, 1, 3e^{-0.1})$

15–22 Explain why the function is differentiable at the given point. Then find the linearization $L(x, y)$ of the function at that point.

15. $f(x, y) = x^3y^2$, $(-2, 1)$

16. $f(x, y) = y \tan x$, $(\pi/4, 2)$

17. $f(x, y) = 1 + x \ln(xy - 5)$, $(2, 3)$

18. $f(x, y) = \sqrt{xy}$, $(1, 4)$

19. $f(x, y) = x^2e^y$, $(1, 0)$

20. $f(x, y) = \dfrac{1 + y}{1 + x}$, $(1, 3)$

21. $f(x, y) = 4 \arctan(xy)$, $(1, 1)$

22. $f(x, y) = y + \sin(x/y)$, $(0, 3)$

23–24 Verify the linear approximation at $(0, 0)$.

23. $e^x \cos(xy) \approx x + 1$

24. $\dfrac{y - 1}{x + 1} \approx x + y - 1$

25. Given that f is a differentiable function with $f(2, 5) = 6$, $f_x(2, 5) = 1$, and $f_y(2, 5) = -1$, use a linear approximation to estimate $f(2.2, 4.9)$.

26. Find the linear approximation of the function $f(x, y) = 1 - xy \cos \pi y$ at $(1, 1)$ and use it to approximate $f(1.02, 0.97)$. Illustrate by graphing f and the tangent plane.

27. Find the linear approximation of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at $(3, 2, 6)$ and use it to approximate the number $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$.

28. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in meters in the following table. Use the table to find a linear approximation to the wave height function when v is near 40 km/h and t is near 20 hours. Then estimate the wave heights when the wind has been blowing for 24 hours at 43 km/h.

Duration (hours)

Wind speed (km/h)

v \ t	5	10	15	20	30	40	50
40	1.5	2.2	2.4	2.5	2.7	2.8	2.8
60	2.8	4.0	4.9	5.2	5.5	5.8	5.9
80	4.3	6.4	7.7	8.6	9.5	10.1	10.2
100	5.8	8.9	11.0	12.2	13.8	14.7	15.3
120	7.4	11.3	14.4	16.6	19.0	20.5	21.1

29. Use the table in Example 3 to find a linear approximation to the heat index function when the actual temperature is near 32°C and the relative humidity is near 65%. Then estimate the heat index when the actual temperature is 33°C and the relative humidity is 63%.

30. The wind-chill index W is the perceived temperature when the actual temperature is T and the wind speed is v, so we can write $W = f(T, v)$. The following table of values is an excerpt from Table 1 in Section 14.1. Use the table to find a linear approximation to the wind-chill index function when T is near -15°C and v is near 50 km/h. Then estimate the wind-chill index when the temperature is -17°C and the wind speed is 55 km/h.

Wind speed (km/h)

Actual temperature (°C)

T \ v	20	30	40	50	60	70
−10	−18	−20	−21	−22	−23	−23
−15	−24	−26	−27	−29	−30	−30
−20	−30	−33	−34	−35	−36	−37
−25	−37	−39	−41	−42	−43	−44

31–38 Find the differential of the function.

31. $m = p^5q^3$

32. $z = x \ln(y^2 + 1)$

33. $z = e^{-2x} \cos 2\pi t$

34. $u = \sqrt{x^2 + 3y^2}$

35. $H = x^2y^4 + y^3z^5$

36. $w = xze^{-y^2 - z^2}$

37. $R = \alpha\beta^2 \cos \gamma$

38. $T = \dfrac{v}{1 + uvw}$

39. If $z = 5x^2 + y^2$ and (x, y) changes from $(1, 2)$ to $(1.05, 2.1)$, compare the values of Δz and dz.

40. If $z = x^2 - xy + 3y^2$ and (x, y) changes from $(3, -1)$ to $(2.96, -0.95)$, compare the values of Δz and dz.

41. The length and width of a rectangle are measured as 30 cm and 24 cm, respectively, with an error in measurement of at most 0.1 cm in each. Use differentials to estimate the maximum error in the calculated area of the rectangle.

42. Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick.

43. Use differentials to estimate the amount of tin in a closed tin can with diameter 8 cm and height 12 cm if the tin is 0.04 cm thick.

44. The base and height of a triangle are measured as 70 cm and 40 cm, respectively. Suppose that each measurement has a possible error of at most ε inches.

(a) Use differentials to estimate the maximum error in the calculated area of the triangle.

(b) What is the estimated maximum error in the area of the triangle if the base and height are measured with errors at most 0.64 cm?

45. The radius of a right circular cylinder is measured as 1 m, and the height is measured as 4 m. Suppose that each measurement has a possible error of at most ε feet.

(a) Use differentials to estimate the maximum error in the calculated volume of the cylinder.

(b) If the computed volume must be accurate to within one cubic foot, determine the largest allowable value of ε.

46. The wind-chill index is modeled by the function

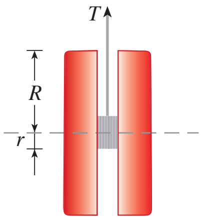
$$W = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$$

where T is the actual temperature (in °C) and v is the wind speed (in km/h). The wind speed is measured as 26 km/h, with a possible error of ± 2 km/h, and the actual temperature is measured as -11°C, with a possible error of ± 1°C. Use differentials to estimate the maximum error in the calculated value of W due to the measurement errors in T and v.

47. The tension T in the string of the yo-yo in the figure is

$$T = \frac{mgR}{2r^2 + R^2}$$

where m is the mass of the yo-yo and g is acceleration due to gravity. Use differentials to estimate the change in the tension if R is increased from 3 cm to 3.1 cm and r is increased from 0.7 cm to 0.8 cm. Does the tension increase or decrease?



48. The pressure, volume, and temperature of a mole of an ideal gas are related by the equation $PV = 8.31T$, where P is measured in kilopascals, V in liters, and T in kelvins. Use differentials to find the approximate change in the pressure if the volume increases from 12 L to 12.3 L and the temperature decreases from 310 K to 305 K.

49. If R is the total resistance of three resistors, connected in parallel, with resistances R_1, R_2, R_3, then

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

If the resistances are measured in ohms as $R_1 = 25\ \Omega$, $R_2 = 40\ \Omega$, and $R_3 = 50\ \Omega$, with a possible error of 0.5% in each case, estimate the maximum error in the calculated value of R.

50. A model for the surface area of a human body is given by $S = 0.0072w^{0.425}h^{0.725}$, where w is the weight (in kilograms), h is the height (in centimeters), and S is measured in square meters. If the errors in measurement of w and h are at most 2%, use differentials to estimate the maximum percentage error in the calculated surface area.

51. In Exercise 14.1.39 and Example 14.3.4, the body mass index of a person was defined as $B(m, h) = m/h^2$, where m is the mass in kilograms and h is the height in meters.

(a) What is the linear approximation of $B(m, h)$ for a child with mass 23 kg and height 1.10 m?

(b) If the child's mass increases by 1 kg and height by 3 cm, use the linear approximation to estimate the new BMI. Compare with the actual new BMI.

52. Suppose you need to know an equation of the tangent plane to a surface S at the point $P(2, 1, 3)$. You don't have an equation for S but you know that the curves

$$\mathbf{r}_1(t) = \langle 2 + 3t, 1 - t^2, 3 - 4t + t^2 \rangle$$

$$\mathbf{r}_2(u) = \langle 1 + u^2, 2u^3 - 1, 2u + 1 \rangle$$

both lie on S. Find an equation of the tangent plane at P.

53. Prove that if f is a function of two variables that is differentiable at (a, b), then f is continuous at (a, b).

Hint: Show that

$$\lim_{(\Delta x, \Delta y) \to (0,0)} f(a + \Delta x, b + \Delta y) = f(a, b)$$

54. (a) The function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

was graphed in Figure 4. Show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist but f is not differentiable at $(0, 0)$. [*Hint:* Use the result of Exercise 53.]

(b) Explain why f_x and f_y are not continuous at $(0, 0)$.

APPLIED PROJECT THE SPEEDO LZR RACER

Many technological advances have occurred in sports that have contributed to increased athletic performance. One of the best known is the introduction, in 2008, of the Speedo LZR racer. It was claimed that this full-body swimsuit reduced a swimmer's drag in the water. Figure 1 shows the number of world records broken in men's and women's long-course freestyle swimming events from 1990 to 2011.[1] The dramatic increase in 2008 when the suit was introduced led people to claim that such suits were a form of technological doping. As a result, all full-body suits were banned from competition starting in 2010.

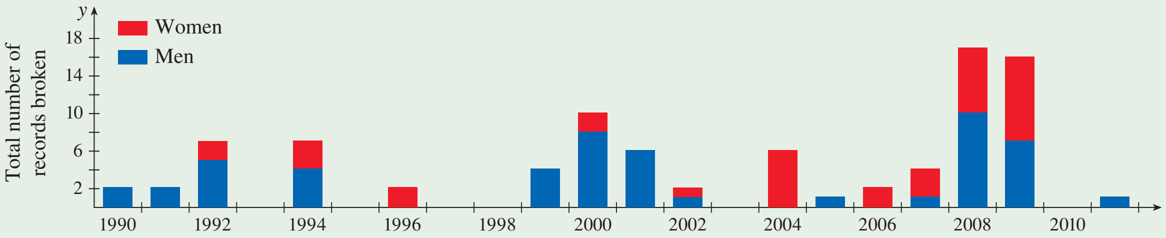


FIGURE 1 Number of world records set in long-course men's and women's freestyle swimming event 1990–2011

It might be surprising that a simple reduction in drag could have such a big effect on performance. We can gain some insight into this using a simple mathematical model.[2]

The speed v of an object being propelled through water is given by

$$v(P, C) = \left(\frac{2P}{kC}\right)^{1/3}$$

where P is the power being used to propel the object, C is the drag coefficient, and k is a positive constant. Athletes can therefore increase their swimming speeds by increasing their power or reducing their drag coefficients. But how effective is each of these?

To compare the effect of increasing power versus reducing drag, we need to somehow compare the two in common units. A frequently used approach is to determine the percentage change in speed that results from a given percentage change in power and in drag.

If we work with percentages as fractions, then when power is changed by a fraction x (with x corresponding to $100x$ percent), P changes from P to $P + xP$. Likewise, if the drag coefficient is changed by a fraction y, this means that it has changed from C to $C + yC$. Finally, the fractional change in speed resulting from both effects is

$$\boxed{1} \qquad \frac{v(P + xP, C + yC) - v(P, C)}{v(P, C)}$$

1. Expression 1 gives the fractional change in speed that results from a change x in power and a change y in drag. Show that this reduces to the function

$$f(x, y) = \left(\frac{1 + x}{1 + y}\right)^{1/3} - 1$$

Given the context, what is the domain of f?

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[1] L. Foster et al., "Influence of Full Body Swimsuits on Competitive Performance," *Procedia Engineering* 34 (2012): 712–17.

[2] Adapted from http://plus.maths.org/content/swimming.

2. Suppose that the possible changes in power x and drag y are small. Find the linear approximation to the function $f(x, y)$. What does this approximation tell you about the effect of a small increase in power versus a small decrease in drag?

3. Calculate $f_{xx}(x, y)$ and $f_{yy}(x, y)$. Based on the signs of these derivatives, does the linear approximation in Problem 2 result in an overestimate or an underestimate for an increase in power? What about for a decrease in drag? Use your answer to explain why, for changes in power or drag that are not very small, a decrease in drag is more effective.

4. Graph the level curves of $f(x, y)$. Explain how the shapes of these curves relate to your answers to Problems 2 and 3.

14.5 The Chain Rule

Recall that the Chain Rule for functions of a single variable gives the rule for differentiating a composite function: if $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

In this section we extend the Chain Rule to functions of more than one variable.

The Chain Rule: Case 1

For functions of more than one variable, the Chain Rule has several versions, each of them giving a rule for differentiating a composite function. The first version (Theorem 1) deals with the case where $z = f(x, y)$ and each of the variables x and y is, in turn, a function of a variable t. This means that z is indirectly a function of t, $z = f(g(t), h(t))$, and the Chain Rule gives a formula for differentiating z as a function of t. We assume that f is differentiable (Definition 14.4.7). Recall that this is the case when f_x and f_y are continuous (Theorem 14.4.8).

1 The Chain Rule (Case 1) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

PROOF A change of Δt in t produces changes of Δx in x and Δy in y. These, in turn, produce a change of Δz in z, and from Definition 14.4.7 we have

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where $\varepsilon_1 \longrightarrow 0$ and $\varepsilon_2 \longrightarrow 0$ as $(\Delta x,\ \Delta y) \longrightarrow (0, 0)$. [If the functions ε_1 and ε_2 are not defined at $(0, 0)$, we can define them to be 0 there.] Dividing both sides of this equation by Δt, we have

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1\frac{\Delta x}{\Delta t} + \varepsilon_2\frac{\Delta y}{\Delta t}$$

If we now let $\Delta t \to 0$, then $\Delta x = g(t + \Delta t) - g(t) \to 0$ because g is differentiable and therefore continuous. Similarly, $\Delta y \to 0$. This, in turn, means that $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, so

$$\begin{aligned}\frac{dz}{dt} &= \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} \\ &= \frac{\partial f}{\partial x} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y} \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} + \left(\lim_{\Delta t \to 0} \varepsilon_1\right) \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \left(\lim_{\Delta t \to 0} \varepsilon_2\right) \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + 0 \cdot \frac{dx}{dt} + 0 \cdot \frac{dy}{dt} \\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \qquad ■\end{aligned}$$

Since we often write $\partial z/\partial x$ in place of $\partial f/\partial x$, we can rewrite the Chain Rule in the form

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

Notice the similarity to the definition of the differential:

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

EXAMPLE 1 If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

SOLUTION The Chain Rule gives

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)\end{aligned}$$

It's not necessary to substitute the expressions for x and y in terms of t. We simply observe that when $t = 0$, we have $x = \sin 0 = 0$ and $y = \cos 0 = 1$. Therefore

$$\left.\frac{dz}{dt}\right|_{t=0} = (0 + 3)(2\cos 0) + (0 + 0)(-\sin 0) = 6 \qquad ■$$

The derivative in Example 1 can be interpreted as the rate of change of z with respect to t as the point (x, y) moves along the curve C with parametric equations $x = \sin 2t$, $y = \cos t$. (See Figure 1.) In particular, when $t = 0$, the point (x, y) is $(0, 1)$ and $dz/dt = 6$ is the rate of increase as we move along the curve C through $(0, 1)$. If, for instance, $z = T(x, y) = x^2y + 3xy^4$ represents the temperature at the point (x, y), then the composite function $z = T(\sin 2t, \cos t)$ represents the temperature at points on C and the derivative dz/dt represents the rate at which the temperature changes along C.

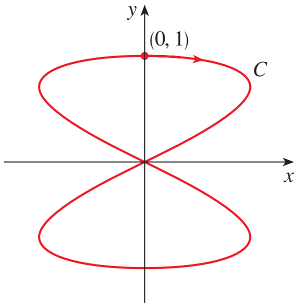


FIGURE 1
The curve $x = \sin 2t$, $y = \cos t$

EXAMPLE 2 The pressure P (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

SOLUTION If t represents the time elapsed in seconds, then at the given instant we have $T = 300$, $dT/dt = 0.1$, $V = 100$, $dV/dt = 0.2$. Since

$$P = 8.31\frac{T}{V}$$

the Chain Rule gives

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} = \frac{8.31}{V}\frac{dT}{dt} - \frac{8.31T}{V^2}\frac{dV}{dt}$$

$$= \frac{8.31}{100}(0.1) - \frac{8.31(300)}{100^2}(0.2) = -0.04155$$

The pressure is decreasing at a rate of about 0.042 kPa/s. ■

■ The Chain Rule: Case 2

We now consider the situation where $z = f(x, y)$ but each of x and y is a function of two variables s and t: $x = g(s, t)$, $y = h(s, t)$. Then z is indirectly a function of s and t and we wish to find $\partial z/\partial s$ and $\partial z/\partial t$. Recall that in computing $\partial z/\partial t$ we hold s fixed and compute the ordinary derivative of z with respect to t. Therefore we can apply Theorem 1 to obtain

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

A similar argument holds for $\partial z/\partial s$ and so we have proved the following version of the Chain Rule.

2 The Chain Rule (Case 2) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

EXAMPLE 3 If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

SOLUTION Applying Case 2 of the Chain Rule, we get

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2)$$

If we wish, we can now express $\partial z/\partial s$ and $\partial z/\partial t$ solely in terms of s and t by substituting $x = st^2$, $y = s^2t$, to get

$$\frac{\partial z}{\partial s} = t^2 e^{st^2} \sin(s^2t) + 2ste^{st^2} \cos(s^2t)$$

$$\frac{\partial z}{\partial t} = 2ste^{st^2} \sin(s^2t) + s^2e^{st^2} \cos(s^2t)$$ ■

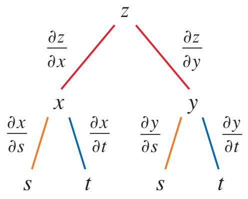

z
$\frac{\partial z}{\partial x}$
$\frac{\partial z}{\partial y}$
x
y
$\frac{\partial x}{\partial s}$
$\frac{\partial x}{\partial t}$
$\frac{\partial y}{\partial s}$
$\frac{\partial y}{\partial t}$
s
t
s
t

FIGURE 2

Case 2 of the Chain Rule contains three types of variables: s and t are **independent** variables, x and y are called **intermediate** variables, and z is the **dependent** variable. Notice that Theorem 2 has one term for each intermediate variable and each of these terms resembles the one-dimensional Chain Rule (see Equation 3.4.2).

To remember the Chain Rule, it's helpful to draw the **tree diagram** in Figure 2. We draw branches from the dependent variable z to the intermediate variables x and y to indicate that z is a function of x and y. Then we draw branches from x and y to the independent variables s and t. On each branch we write the corresponding partial derivative. To find $\partial z/\partial s$, we find the product of the partial derivatives along each path from z to s and then add these products:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Similarly, we find $\partial z/\partial t$ by using the paths from z to t.

The Chain Rule: General Version

Now we consider the general situation in which a dependent variable u is a function of n intermediate variables $x_1, \ldots, x_n$, each of which is, in turn, a function of m independent variables $t_1, \ldots, t_m$. Notice that there are n terms, one for each intermediate variable. The proof is similar to that of Case 1.

3 The Chain Rule (General Version) Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

EXAMPLE 4 Write out the Chain Rule for the case where $w = f(x, y, z, t)$ and $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, and $t = t(u, v)$.

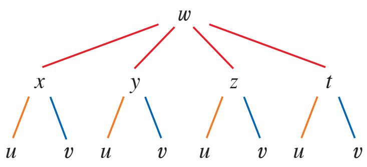


FIGURE 3

SOLUTION We apply Theorem 3 with $n = 4$ and $m = 2$. Figure 3 shows the tree diagram. Although we haven't written the derivatives on the branches, it's understood that if a branch leads from y to u, then the partial derivative for that branch is $\partial y/\partial u$. With the aid of the tree diagram, we can now write the required expressions:

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$ ■

EXAMPLE 5 If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find the value of $\partial u/\partial s$ when $r = 2$, $s = 1$, $t = 0$.

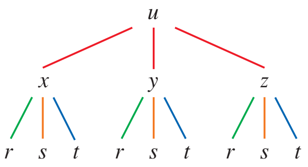


FIGURE 4

SOLUTION With the help of the tree diagram in Figure 4, we have

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2\sin t)$$

When $r = 2$, $s = 1$, and $t = 0$, we have $x = 2$, $y = 2$, and $z = 0$, so

$$\frac{\partial u}{\partial s} = (64)(2) + (16)(4) + (0)(0) = 192$$ ■

EXAMPLE 6 If $g(s, t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

SOLUTION Let $x = s^2 - t^2$ and $y = t^2 - s^2$. Then $g(s, t) = f(x, y)$ and the Chain Rule gives

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$$

Therefore

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = \left(2st\frac{\partial f}{\partial x} - 2st\frac{\partial f}{\partial y}\right) + \left(-2st\frac{\partial f}{\partial x} + 2st\frac{\partial f}{\partial y}\right) = 0 \qquad \blacksquare$$

EXAMPLE 7 If $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$, find expressions for (a) $\partial z/\partial r$ and (b) $\partial^2 z/\partial r^2$.

SOLUTION

(a) The Chain Rule gives

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}(2r) + \frac{\partial z}{\partial y}(2s)$$

(b) Applying the Product Rule to the expression in part (a), we get

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r}\left(2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}\right)\\ &= 2\frac{\partial z}{\partial x} + 2r\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + 2s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)\end{aligned} \tag{4}$$

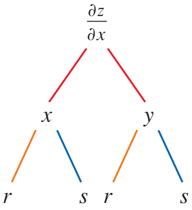


FIGURE 5

But, using the Chain Rule again (see Figure 5), we have

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x^2}(2r) + \frac{\partial^2 z}{\partial y\,\partial x}(2s)$$

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x\,\partial y}(2r) + \frac{\partial^2 z}{\partial y^2}(2s)$$

Putting these expressions into Equation 4 and using the equality of the mixed second-order derivatives, we obtain

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= 2\frac{\partial z}{\partial x} + 2r\left(2r\frac{\partial^2 z}{\partial x^2} + 2s\frac{\partial^2 z}{\partial y\,\partial x}\right) + 2s\left(2r\frac{\partial^2 z}{\partial x\,\partial y} + 2s\frac{\partial^2 z}{\partial y^2}\right)\\ &= 2\frac{\partial z}{\partial x} + 4r^2\frac{\partial^2 z}{\partial x^2} + 8rs\frac{\partial^2 z}{\partial x\,\partial y} + 4s^2\frac{\partial^2 z}{\partial y^2}\end{aligned} \qquad \blacksquare$$

Implicit Differentiation

The Chain Rule can be used to give a more complete description of the process of implicit differentiation that was introduced in Sections 3.5 and 14.3. We suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x, that is, $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f. If F is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x. Since both x and y are functions of x, we obtain

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

But $dx/dx = 1$, so if $\partial F/\partial y \neq 0$ we solve for dy/dx and obtain

5

$$\boxed{\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}}$$

To derive this equation we assumed that $F(x, y) = 0$ defines y implicitly as a function of x. The **Implicit Function Theorem**, proved in advanced calculus, gives conditions under which this assumption is valid: it states that if F is defined on a disk containing (a, b), where $F(a, b) = 0$, $F_y(a, b) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x, y) = 0$ defines y as a function of x near the point (a, b) and the derivative of this function is given by Equation 5.

EXAMPLE 8 Find y' if $x^3 + y^3 = 6xy$.

SOLUTION The given equation can be written as

$$F(x, y) = x^3 + y^3 - 6xy = 0$$

so Equation 5 gives

The solution to Example 8 should be compared to the one in Example 3.5.2.

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x} \qquad \blacksquare$$

Now we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f. If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ as follows:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

But $$\frac{\partial}{\partial x}(x) = 1 \qquad \text{and} \qquad \frac{\partial}{\partial x}(y) = 0$$

so this equation becomes

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

If $\partial F/\partial z \neq 0$, we solve for $\partial z/\partial x$ and obtain the first formula in Equations 6. The formula for $\partial z/\partial y$ is obtained in a similar manner.

$$\boxed{\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}} = -\frac{F_y}{F_z}} \tag{6}$$

Again, a version of the **Implicit Function Theorem** stipulates conditions under which our assumption is valid: if F is defined within a sphere containing (a, b, c), where $F(a, b, c) = 0$, $F_z(a, b, c) \neq 0$, and F_x, F_y, and F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (a, b, c) and this function is differentiable, with partial derivatives given by (6).

EXAMPLE 9 Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz + 4 = 0$.

SOLUTION Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz + 4$. Then, from Equations 6, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$ ■

The solution to Example 9 should be compared to the one in Example 14.3.5.

14.5 Exercises

1–2 Find dz/dt in two ways: by using the Chain Rule, and by first substituting the expressions for x and y to write z as a function of t. Do your answers agree?

1. $z = x^2y + xy^2, \quad x = 3t, \quad y = t^2$

2. $z = xye^y, \quad x = t^2, \quad y = 5t$

3–8 Use the Chain Rule to find dz/dt or dw/dt.

3. $z = xy^3 - x^2y, \quad x = t^2 + 1, \quad y = t^2 - 1$

4. $z = \dfrac{x - y}{x + 2y}, \quad x = e^{\pi t}, \quad y = e^{-\pi t}$

5. $z = \sin x \cos y, \quad x = \sqrt{t}, \quad y = 1/t$

6. $z = \sqrt{1 + xy}, \quad x = \tan t, \quad y = \arctan t$

7. $w = xe^{y/z}, \quad x = t^2, \quad y = 1 - t, \quad z = 1 + 2t$

8. $w = \ln\sqrt{x^2 + y^2 + z^2}, \quad x = \sin t, \quad y = \cos t, \quad z = \tan t$

9–10 Find $\partial z/\partial s$ and $\partial z/\partial t$ in two ways: by using the Chain Rule, and by first substituting the expressions for x and y to write z as a function of s and t. Do your answers agree?

9. $z = x^2 + y^2, \quad x = 2s + 3t, \quad y = s + t$

10. $z = x^2 \sin y, \quad x = s^2t, \quad y = st$

11–16 Use the Chain Rule to find $\partial z/\partial s$ and $\partial z/\partial t$.

11. $z = (x - y)^5, \quad x = s^2t, \quad y = st^2$

12. $z = \tan^{-1}(x^2 + y^2), \quad x = s \ln t, \quad y = te^s$

13. $z = \ln(3x + 2y), \quad x = s \sin t, \quad y = t \cos s$

14. $z = \sqrt{x}\, e^{xy}, \quad x = 1 + st, \quad y = s^2 - t^2$

15. $z = (\sin \theta)/r, \quad r = st, \quad \theta = s^2 + t^2$

16. $z = \tan(u/v), \quad u = 2s + 3t, \quad v = 3s - 2t$

17. Suppose f is a differentiable function of x and y, and $p(t) = f(g(t), h(t))$, $g(2) = 4$, $g'(2) = -3$, $h(2) = 5$, $h'(2) = 6$, $f_x(4, 5) = 2$, $f_y(4, 5) = 8$. Find $p'(2)$.

18. Let $R(s, t) = G(u(s, t), v(s, t))$, where G, u, and v are differentiable, $u(1, 2) = 5$, $u_s(1, 2) = 4$, $u_t(1, 2) = -3$, $v(1, 2) = 7$, $v_s(1, 2) = 2$, $v_t(1, 2) = 6$, $G_u(5, 7) = 9$, $G_v(5, 7) = -2$. Find $R_s(1, 2)$ and $R_t(1, 2)$.

19. Suppose f is a differentiable function of x and y, and $g(u, v) = f(e^u + \sin v, e^u + \cos v)$. Use the table of values to calculate $g_u(0, 0)$ and $g_v(0, 0)$.

	f	g	f_x	f_y
(0, 0)	3	6	4	8
(1, 2)	6	3	2	5

20. Suppose f is a differentiable function of x and y, and $g(r, s) = f(2r - s, s^2 - 4r)$. Use the table of values in Exercise 19 to calculate $g_r(1, 2)$ and $g_s(1, 2)$.

21–24 Use a tree diagram to write out the Chain Rule for the given case. Assume all functions are differentiable.

21. $u = f(x, y)$, where $x = x(r, s, t)$, $y = y(r, s, t)$

22. $w = f(x, y, z)$, where $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$

23. $T = F(p, q, r)$, where $p = p(x, y, z)$, $q = q(x, y, z)$, $r = r(x, y, z)$

24. $R = F(t, u)$ where $t = t(w, x, y, z)$, $u = u(w, x, y, z)$

25–30 Use the Chain Rule to find the indicated partial derivatives.

25. $z = x^4 + x^2y$, $x = s + 2t - u$, $y = stu^2$;

$$\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}, \frac{\partial z}{\partial u} \quad \text{when } s = 4, t = 2, u = 1$$

26. $T = \dfrac{v}{2u + v}$, $u = pq\sqrt{r}$, $v = p\sqrt{q}\,r$;

$$\frac{\partial T}{\partial p}, \frac{\partial T}{\partial q}, \frac{\partial T}{\partial r} \quad \text{when } p = 2, q = 1, r = 4$$

27. $w = xy + yz + zx$, $x = r\cos\theta$, $y = r\sin\theta$, $z = r\theta$;

$$\frac{\partial w}{\partial r}, \frac{\partial w}{\partial \theta} \quad \text{when } r = 2, \theta = \pi/2$$

28. $P = \sqrt{u^2 + v^2 + w^2}$, $u = xe^y$, $v = ye^x$, $w = e^{xy}$;

$$\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y} \quad \text{when } x = 0, y = 2$$

29. $N = \dfrac{p + q}{p + r}$, $p = u + vw$, $q = v + uw$, $r = w + uv$;

$$\frac{\partial N}{\partial u}, \frac{\partial N}{\partial v}, \frac{\partial N}{\partial w} \quad \text{when } u = 2, v = 3, w = 4$$

30. $u = xe^{ty}$, $x = \alpha^2\beta$, $y = \beta^2\gamma$, $t = \gamma^2\alpha$;

$$\frac{\partial u}{\partial \alpha}, \frac{\partial u}{\partial \beta}, \frac{\partial u}{\partial \gamma} \quad \text{when } \alpha = -1, \beta = 2, \gamma = 1$$

31–34 Use Equation 5 to find dy/dx.

31. $y\cos x = x^2 + y^2$

32. $\cos(xy) = 1 + \sin y$

33. $\tan^{-1}(x^2y) = x + xy^2$

34. $e^y \sin x = x + xy$

35–38 Use Equations 6 to find $\partial z/\partial x$ and $\partial z/\partial y$.

35. $x^2 + 2y^2 + 3z^2 = 1$

36. $x^2 - y^2 + z^2 - 2z = 4$

37. $e^z = xyz$

38. $yz + x\ln y = z^2$

39. The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1 + t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds?

40. Wheat production W in a given year depends on the average temperature T and the annual rainfall R. Scientists estimate that the average temperature is rising at a rate of 0.15°C/year and rainfall is decreasing at a rate of 0.1 cm/year. They also estimate that at current production levels, $\partial W/\partial T = -2$ and $\partial W/\partial R = 8$.

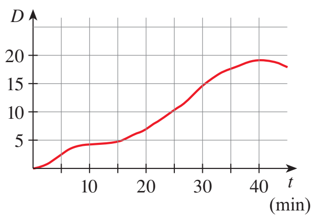
(a) What is the significance of the signs of these partial derivatives?

(b) Estimate the current rate of change of wheat production, dW/dt.

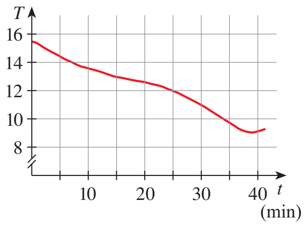
41. The speed of sound traveling through ocean water with salinity 35 parts per thousand has been modeled by the equation

$$C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$$

where C is the speed of sound (in meters per second), T is the temperature (in degrees Celsius), and D is the depth below the ocean surface (in meters). A scuba diver began a leisurely dive into the ocean water; the diver's depth and the surrounding water temperature over time are recorded in the following graphs. Estimate the rate of change (with respect to time) of the speed of sound through the ocean water experienced by the diver 20 minutes into the dive. What are the units?



Depth



Water temperature

42. The radius of a right circular cone is increasing at a rate of 4.6 cm/s while its height is decreasing at a rate of 6.5 cm/s. At what rate is the volume of the cone changing when the radius is 300 cm and the height is 350 cm?

43. The length ℓ, width w, and height h of a box change with time. At a certain instant the dimensions are $\ell = 1$ m and $w = h = 2$ m, and ℓ and w are increasing at a rate of 2 m/s while h is decreasing at a rate of 3 m/s. At that instant find the rates at which the following quantities are changing.

(a) The volume
(b) The surface area
(c) The length of a diagonal

44. The voltage V in a simple electrical circuit is slowly decreasing as the battery wears out. The resistance R is slowly increasing as the resistor heats up. Use Ohm's Law, $V = IR$, to find how the current I is changing at the moment when $R = 400\ \Omega$, $I = 0.08$ A, $dV/dt = -0.01$ V/s, and $dR/dt = 0.03\ \Omega$/s.

45. The pressure of 1 mole of an ideal gas is increasing at a rate of 0.05 kPa/s and the temperature is increasing at a rate of 0.15 K/s. Use the equation $PV = 8.31T$ in Example 2 to find the rate of change of the volume when the pressure is 20 kPa and the temperature is 320 K.

46. A manufacturer has modeled its yearly production function P (the value of its entire production, in millions of dollars) as a Cobb-Douglas function

$$P(L, K) = 1.47L^{0.65}K^{0.35}$$

where L is the number of labor hours (in thousands) and K is the invested capital (in millions of dollars). Suppose that when $L = 30$ and $K = 8$, the labor force is decreasing at a rate of 2000 labor hours per year and capital is increasing at a rate of \$500,000 per year. Find the rate of change of production.

47. One side of a triangle is increasing at a rate of 3 cm/s and a second side is decreasing at a rate of 2 cm/s. If the area of the triangle remains constant, at what rate does the angle between the sides change when the first side is 20 cm long, the second side is 30 cm, and the angle is $\pi/6$?

48. Doppler Effect A sound with frequency f_s is produced by a source traveling along a line with speed v_s. If an observer is traveling with speed v_o along the same line from the opposite direction toward the source, then the frequency of the sound heard by the observer is

$$f_o = \left(\frac{c + v_o}{c - v_s}\right) f_s$$

where c is the speed of sound, about 332 m/s. (This is the *Doppler effect*.) Suppose that, at a particular moment, you are in a train traveling at 34 m/s and accelerating at 1.2 m/s^2. A train is approaching you from the opposite direction on the other track at 40 m/s, accelerating at 1.4 m/s^2, and sounds its whistle, which has a frequency of 460 Hz. At that instant, what is the perceived frequency that you hear and how fast is it changing?

49–50 Assume that all the given functions are differentiable.

49. If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, (a) find $\partial z/\partial r$ and $\partial z/\partial\theta$ and (b) show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

50. If $u = f(x, y)$, where $x = e^s\cos t$ and $y = e^s\sin t$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{-2s}\left[\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]$$

51–55 Assume that all the given functions have continuous second-order partial derivatives.

51. Show that any function of the form

$$z = f(x + at) + g(x - at)$$

is a solution of the wave equation

$$\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2}$$

[*Hint:* Let $u = x + at$, $v = x - at$.]

52. If $u = f(x, y)$, where $x = e^s\cos t$ and $y = e^s\sin t$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2s}\left[\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}\right]$$

53. If $z = f(x, y)$, where $x = r^2 + s^2$ and $y = 2rs$, find $\partial^2 z/\partial r\,\partial s$. (Compare with Example 7.)

54. If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, find (a) $\partial z/\partial r$, (b) $\partial z/\partial\theta$, and (c) $\partial^2 z/\partial r\,\partial\theta$.

55. If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}$$

56–58 Homogeneous Functions A function f is called *homogeneous of degree n* if it satisfies the equation

$$f(tx, ty) = t^n f(x, y)$$

for all t, where n is a positive integer and f has continuous second-order partial derivatives.

56. Verify that $f(x, y) = x^2y + 2xy^2 + 5y^3$ is homogeneous of degree 3.

57. Show that if f is homogeneous of degree n, then

(a) $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf(x, y)$

[*Hint:* Use the Chain Rule to differentiate $f(tx, ty)$ with respect to t.]

(b) $x^2\dfrac{\partial^2 f}{\partial x^2} + 2xy\dfrac{\partial^2 f}{\partial x\,\partial y} + y^2\dfrac{\partial^2 f}{\partial y^2} = n(n-1)f(x, y)$

58. If f is homogeneous of degree n, show that

$$f_x(tx, ty) = t^{n-1}f_x(x, y)$$

59. Suppose that the equation $F(x, y, z) = 0$ implicitly defines each of the three variables x, y, and z as functions of the other two: $z = f(x, y)$, $y = g(x, z)$, $x = h(y, z)$. If F is differentiable and F_x, F_y, and F_z are all nonzero, show that

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = -1$$

60. Equation 5 is a formula for the derivative dy/dx of a function defined implicitly by an equation $F(x, y) = 0$, provided that F is differentiable and $F_y \neq 0$. Prove that if F has continuous second derivatives, then a formula for the second derivative of y is

$$\frac{d^2y}{dx^2} = -\frac{F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2}{F_y^3}$$

14.6 Directional Derivatives and the Gradient Vector

The weather map in Figure 1 shows a contour map of the temperature function $T(x, y)$ for the states of California and Nevada at 3:00 PM on a day in October. The level curves, or isothermals, join locations with the same temperature. The partial derivative T_x at a location such as Reno is the rate of change of temperature with respect to distance if we travel east from Reno; T_y is the rate of change of temperature if we travel north. But what if we want to know the rate of change of temperature when we travel southeast (toward Las Vegas), or in some other direction? In this section we introduce a type of derivative, called a *directional derivative,* that enables us to find the rate of change of a function of two or more variables in any direction.

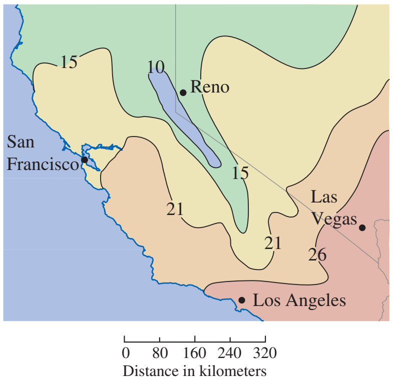


FIGURE 1

Directional Derivatives

Recall that if $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as

$$\boxed{1}\qquad \begin{aligned} f_x(x_0, y_0) &= \lim_{h \to 0}\frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \\[2ex] f_y(x_0, y_0) &= \lim_{h \to 0}\frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} \end{aligned}$$

and represent the rates of change of z in the x- and y-directions, that is, in the directions of the unit vectors **i** and **j**.

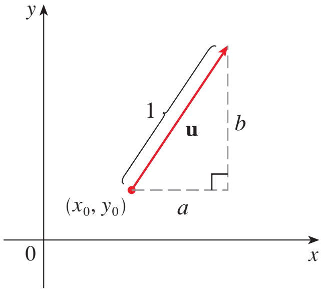


FIGURE 2
A unit vector $\mathbf{u} = \langle a, b \rangle$

Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$. (See Figure 2.) To do this we consider the surface S with the equation $z = f(x, y)$ (the graph of f) and we let $z_0 = f(x_0, y_0)$. Then the point $P(x_0, y_0, z_0)$ lies on S. The vertical plane that passes through P in the direction of $\mathbf{u}$ intersects S in a curve C. (See Figure 3.) The slope of the tangent line T to C at the point P is the rate of change of z in the direction of $\mathbf{u}$.

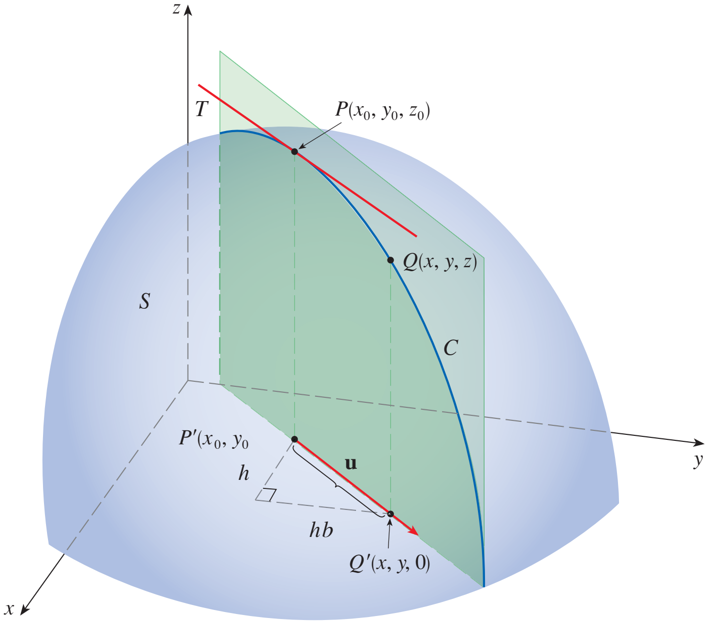


FIGURE 3

If $Q(x, y, z)$ is another point on C and P', Q' are the projections of P, Q onto the xy-plane, then the vector $\overrightarrow{P'Q'}$ is parallel to $\mathbf{u}$ and so

$$\overrightarrow{P'Q'} = h\mathbf{u} = \langle ha, hb \rangle$$

for some scalar h. Therefore $x - x_0 = ha$, $y - y_0 = hb$, so $x = x_0 + ha$, $y = y_0 + hb$, and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

If we take the limit as $h \to 0$, we obtain the rate of change of z (with respect to distance) in the direction of $\mathbf{u}$, which is called the directional derivative of f in the direction of $\mathbf{u}$.

2 **Definition** The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

By comparing Definition 2 with Equations 1, we see that if $\mathbf{u} = \mathbf{i} = \langle 1, 0 \rangle$, then $D_{\mathbf{i}} f = f_x$ and if $\mathbf{u} = \mathbf{j} = \langle 0, 1 \rangle$, then $D_{\mathbf{j}} f = f_y$. In other words, the partial derivatives of f with respect to x and y are just special cases of the directional derivative.

EXAMPLE 1 Use the weather map in Figure 1 to estimate the value of the directional derivative of the temperature function at Reno in the southeasterly direction.

SOLUTION We start by drawing a line through Reno toward the southeast [in the direction of $\mathbf{u} = (\mathbf{i} - \mathbf{j})/\sqrt{2}$; see Figure 4].

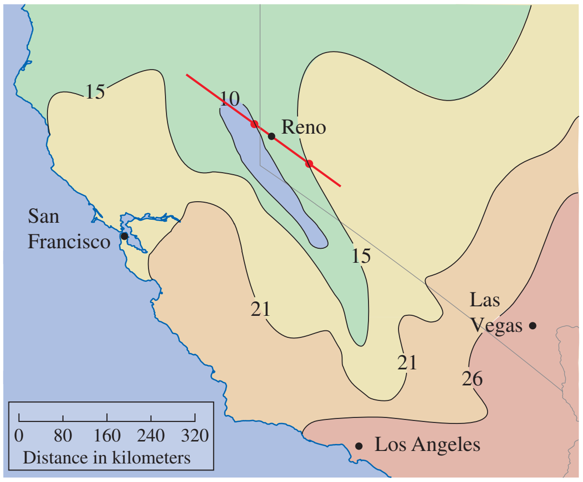


FIGURE 4

We approximate the directional derivative $D_{\mathbf{u}} T$ by the average rate of change of the temperature between the points where this line intersects the isothermals $T = 10$ and $T = 15$. The temperature at the point southeast of Reno is $T = 15°\text{C}$ and the temperature at the point northwest of Reno is $T = 10°\text{C}$. The distance between these points looks to be about 120 kilometers. So the rate of change of the temperature in the southeasterly direction is

$$D_{\mathbf{u}} T \approx \frac{15 - 10}{120} = \frac{1}{24} \approx 0.04°\text{C/km} \qquad ■$$

When we compute the directional derivative of a function defined by a formula, we generally use the following theorem.

3 Theorem If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

PROOF If we define a function g of the single variable h by

$$g(h) = f(x_0 + ha, y_0 + hb)$$

then, by the definition of a derivative, we have

$$\boxed{4} \qquad g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

$$= D_{\mathbf{u}} f(x_0, y_0)$$

On the other hand, we can write $g(h) = f(x, y)$, where $x = x_0 + ha$, $y = y_0 + hb$, so Case 1 of the Chain Rule (Theorem 14.5.1) gives

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x, y)\,a + f_y(x, y)\,b$$

If we now put $h = 0$, then $x = x_0$, $y = y_0$, and

$$g'(0) = f_x(x_0, y_0)\,a + f_y(x_0, y_0)\,b \tag{5}$$

Comparing Equations 4 and 5, we see that

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)\,a + f_y(x_0, y_0)\,b$$ ■

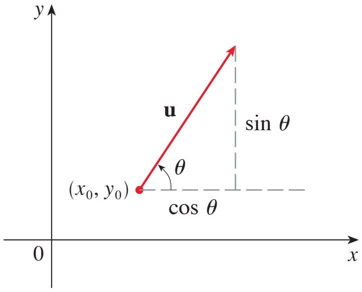


FIGURE 5 A unit vector $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$

If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis (as in Figure 5), then we can write $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ and the formula in Theorem 3 becomes

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta \tag{6}$$

The directional derivative $D_{\mathbf{u}} f(1, 2)$ in Example 2 represents the rate of change of z in the direction of $\mathbf{u}$. This is the slope of the tangent line to the curve of intersection of the surface $z = x^3 - 3xy + 4y^2$ and the vertical plane through $(1, 2, 0)$ in the direction of $\mathbf{u}$ shown in Figure 6.

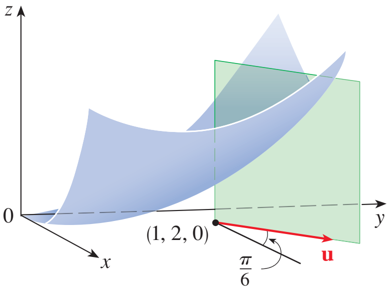


FIGURE 6

EXAMPLE 2 Find the directional derivative $D_{\mathbf{u}} f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and $\mathbf{u}$ is the unit vector in the direction given by angle $\theta = \pi/6$, measured from the positive x-axis. What is $D_{\mathbf{u}} f(1, 2)$?

SOLUTION Formula 6 gives

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= f_x(x, y)\cos\frac{\pi}{6} + f_y(x, y)\sin\frac{\pi}{6} \\ &= (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\frac{1}{2} \\ &= \tfrac{1}{2}\left[3\sqrt{3}x^2 - 3x + \left(8 - 3\sqrt{3}\right)y\right] \end{aligned}$$

Therefore

$$D_{\mathbf{u}} f(1, 2) = \tfrac{1}{2}\left[3\sqrt{3}(1)^2 - 3(1) + \left(8 - 3\sqrt{3}\right)(2)\right] = \frac{13 - 3\sqrt{3}}{2}$$ ■

■ The Gradient Vector

Notice from Theorem 3 that the directional derivative of a differentiable function can be written as the dot product of two vectors:

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= f_x(x, y)\,a + f_y(x, y)\,b \\ &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle a, b \rangle \\ &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \mathbf{u} \end{aligned} \tag{7}$$

The first vector in this dot product occurs not only in computing directional derivatives but in many other contexts as well. So we give it a special name (the *gradient* of f) and a special notation (**grad** f or ∇f, which is read “del f”).

8 Definition If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

EXAMPLE 3 If $f(x, y) = \sin x + e^{xy}$, then

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle \cos x + ye^{xy}, xe^{xy} \rangle$$

and $$\nabla f(0, 1) = \langle 2, 0 \rangle$$ ■

With this notation for the gradient vector, we can rewrite Equation 7 for the directional derivative of a differentiable function as

9 $$\boxed{D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}}$$

This expresses the directional derivative in the direction of a unit vector **u** as the scalar projection of the gradient vector onto **u**.

The gradient vector $\nabla f(2, -1)$ in Example 4 is shown in Figure 7 with initial point $(2, -1)$. Also shown is the vector **v** that gives the direction of the directional derivative. Both of these vectors are superimposed on a contour plot of the graph of f.

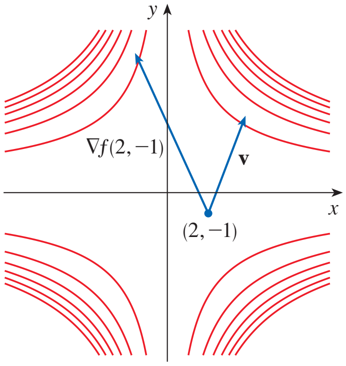


FIGURE 7

EXAMPLE 4 Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

SOLUTION We first compute the gradient vector at $(2, -1)$:

$$\nabla f(x, y) = 2xy^3\mathbf{i} + (3x^2y^2 - 4)\mathbf{j}$$

$$\nabla f(2, -1) = -4\mathbf{i} + 8\mathbf{j}$$

Note that **v** is not a unit vector, but since $|\mathbf{v}| = \sqrt{29}$, the unit vector in the direction of **v** is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j}$$

Therefore, by Equation 9, we have

$$D_{\mathbf{u}} f(2, -1) = \nabla f(2, -1) \cdot \mathbf{u} = (-4\mathbf{i} + 8\mathbf{j}) \cdot \left(\frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j}\right)$$

$$= \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}}$$ ■

■ Functions of Three Variables

For functions of three variables we can define directional derivatives in a similar manner. Again $D_{\mathbf{u}} f(x, y, z)$ can be interpreted as the rate of change of the function in the direction of a unit vector **u**.

10 Definition The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

If we use vector notation, then we can write both definitions (2 and 10) of the directional derivative in the compact form

$$\boxed{D_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}} \tag{11}$$

where $\mathbf{x}_0 = \langle x_0, y_0 \rangle$ if $n = 2$ and $\mathbf{x}_0 = \langle x_0, y_0, z_0 \rangle$ if $n = 3$. This is reasonable because the vector equation of the line through $\mathbf{x}_0$ in the direction of the vector $\mathbf{u}$ is given by $\mathbf{x} = \mathbf{x}_0 + t\mathbf{u}$ (Equation 12.5.1) and so $f(\mathbf{x}_0 + h\mathbf{u})$ represents the value of f at a point on this line.

If $f(x, y, z)$ is differentiable and $\mathbf{u} = \langle a, b, c \rangle$, then the same method that was used to prove Theorem 3 can be used to show that

$$D_{\mathbf{u}} f(x, y, z) = f_x(x, y, z)\, a + f_y(x, y, z)\, b + f_z(x, y, z)\, c \tag{12}$$

For a function f of three variables, the **gradient vector**, denoted by ∇f or **grad** f, is

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

or, for short,

$$\boxed{\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}} \tag{13}$$

Then, just as with functions of two variables, Formula 12 for the directional derivative can be rewritten as

$$\boxed{D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}} \tag{14}$$

EXAMPLE 5 If $f(x, y, z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

SOLUTION

(a) The gradient of f is

$$\begin{aligned} \nabla f(x, y, z) &= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \\ &= \langle \sin yz, xz \cos yz, xy \cos yz \rangle \end{aligned}$$

(b) At $(1, 3, 0)$ we have $\nabla f(1, 3, 0) = \langle 0, 0, 3 \rangle$. The unit vector in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is

$$\mathbf{u} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$$

Therefore Equation 14 gives

$$\begin{aligned} D_{\mathbf{u}} f(1, 3, 0) &= \nabla f(1, 3, 0) \cdot \mathbf{u} \\ &= 3\mathbf{k} \cdot \left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k} \right) \\ &= 3\left(-\frac{1}{\sqrt{6}} \right) = -\sqrt{\frac{3}{2}} \end{aligned}$$

■

■ Maximizing the Directional Derivative

Suppose we have a function f of two or three variables and we consider all possible directional derivatives of f at a given point. These give the rates of change of f in all possible directions. We can then ask the questions: in which of these directions does f change fastest and what is the maximum rate of change? The answers are provided by the following theorem.

15 **Theorem** Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

PROOF From Equation 9 or 14 and using Theorem 12.3.3, we have

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos\theta = |\nabla f| \cos\theta$$

where θ is the angle between ∇f and $\mathbf{u}$. The maximum value of $\cos\theta$ is 1 and this occurs when $\theta = 0$. Therefore the maximum value of $D_{\mathbf{u}} f$ is $|\nabla f|$ and it occurs when $\theta = 0$, that is, when $\mathbf{u}$ has the same direction as ∇f. ■

EXAMPLE 6

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

SOLUTION

(a) We first compute the gradient vector:

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle$$

$$\nabla f(2, 0) = \langle 1, 2 \rangle$$

The unit vector in the direction of $\overrightarrow{PQ} = \left\langle -\frac{3}{2}, 2 \right\rangle$ is $\mathbf{u} = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$, so the rate of change of f in the direction from P to Q is

$$D_{\mathbf{u}} f(2, 0) = \nabla f(2, 0) \cdot \mathbf{u} = \langle 1, 2 \rangle \cdot \left\langle -\tfrac{3}{5}, \tfrac{4}{5} \right\rangle = 1$$

(b) According to Theorem 15, f increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2 \rangle$. The maximum rate of change is

$$|\nabla f(2, 0)| = |\langle 1, 2 \rangle| = \sqrt{5}$$ ■

At (2, 0) the function in Example 6 increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2 \rangle$. Notice from Figure 8 that this vector appears to be perpendicular to the level curve through (2, 0). Figure 9 shows the graph of f and the gradient vector.

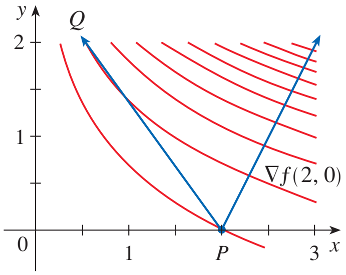


FIGURE 8

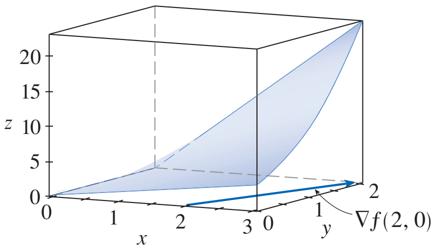


FIGURE 9

EXAMPLE 7 Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

SOLUTION The gradient of T is

$$\begin{aligned} \nabla T &= \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k} \\ &= -\frac{160x}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{i} - \frac{320y}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{j} - \frac{480z}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{k} \\ &= \frac{160}{(1 + x^2 + 2y^2 + 3z^2)^2}(-x\,\mathbf{i} - 2y\,\mathbf{j} - 3z\,\mathbf{k}) \end{aligned}$$

At the point $(1, 1, -2)$ the gradient vector is

$$\nabla T(1, 1, -2) = \tfrac{160}{256}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}) = \tfrac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$$

By Theorem 15 the temperature increases fastest in the direction of the gradient vector $\nabla T(1, 1, -2) = \frac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$ or, equivalently, in the direction of $-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ or the unit vector $(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})/\sqrt{41}$. The maximum rate of increase is the length of the gradient vector:

$$|\nabla T(1, 1, -2)| = \tfrac{5}{8}|-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}| = \tfrac{5}{8}\sqrt{41}$$

Therefore the maximum rate of increase of temperature is $\frac{5}{8}\sqrt{41} \approx 4°\mathrm{C/m}$. ■

Tangent Planes to Level Surfaces

Suppose S is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of a function F of three variables, and let $P(x_0, y_0, z_0)$ be a point on S. Let C be any curve that lies on the surface S and passes through the point P. Recall from Section 13.1 that the curve C is described by a continuous vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$. Let t_0 be the parameter value corresponding to P; that is, $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$. Since C lies on S, any point $(x(t), y(t), z(t))$ must satisfy the equation of S, that is,

$$\boxed{16} \qquad F(x(t), y(t), z(t)) = k$$

If x, y, and z are differentiable functions of t and F is also differentiable, then we can use the Chain Rule to differentiate both sides of Equation 16 as follows:

$$\boxed{17} \qquad \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

But, since $\nabla F = \langle F_x, F_y, F_z \rangle$ and $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$, Equation 17 can be written in terms of a dot product as

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

In particular, when $t = t_0$ we have $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$, so

$$\boxed{18} \qquad \nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0$$

Equation 18 says that *the gradient vector at P, $\nabla F(x_0, y_0, z_0)$, is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ to any curve C on S that passes through P.* (See Figure 10.) If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, it is therefore natural to define the **tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$** as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$. Using the standard equation of a plane (Equation 12.5.7), we can write the equation of this tangent plane as

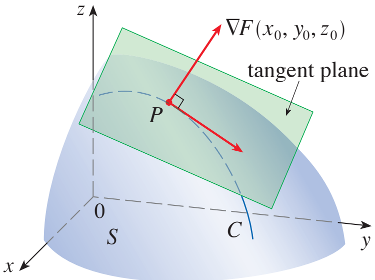


FIGURE 10

$$\boxed{\boxed{19} \qquad F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0}$$

The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane. The direction of the normal line is therefore given by the gradient vector $\nabla F(x_0, y_0, z_0)$ and so, by Equation 12.5.3, its symmetric equations are

$$\boxed{20} \qquad \frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

EXAMPLE 8 Find the equations of the tangent plane and normal line to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

at the point $(-2, 1, -3)$.

SOLUTION The ellipsoid is the level surface (with $k = 3$) of the function

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$$

Figure 11 shows the ellipsoid, tangent plane, and normal line in Example 8.

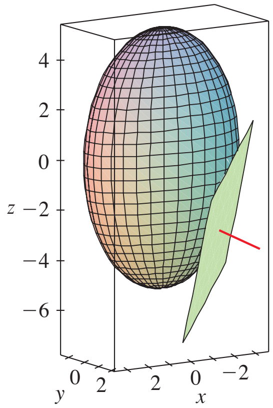


FIGURE 11

Therefore we have

$$F_x(x, y, z) = \frac{x}{2} \qquad F_y(x, y, z) = 2y \qquad F_z(x, y, z) = \frac{2z}{9}$$

$$F_x(-2, 1, -3) = -1 \qquad F_y(-2, 1, -3) = 2 \qquad F_z(-2, 1, -3) = -\tfrac{2}{3}$$

Then Equation 19 gives the equation of the tangent plane at $(-2, 1, -3)$ as

$$-1(x + 2) + 2(y - 1) - \tfrac{2}{3}(z + 3) = 0$$

which simplifies to $3x - 6y + 2z + 18 = 0$.

By Equation 20, symmetric equations of the normal line are

$$\frac{x + 2}{-1} = \frac{y - 1}{2} = \frac{z + 3}{-\frac{2}{3}}$$ ■

In the special case in which the equation of a surface S is of the form $z = f(x, y)$ (that is, S is the graph of a function f of two variables), we can rewrite the equation as

$$F(x, y, z) = f(x, y) - z = 0$$

and regard S as a level surface (with $k = 0$) of F. Then

$$F_x(x_0, y_0, z_0) = f_x(x_0, y_0)$$

$$F_y(x_0, y_0, z_0) = f_y(x_0, y_0)$$

$$F_z(x_0, y_0, z_0) = -1$$

so Equation 19 becomes

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

which is equivalent to Equation 14.4.2. Thus our new, more general, definition of a tangent plane is consistent with the definition that was given for the special case of Section 14.4.

EXAMPLE 9 Find the tangent plane to the surface $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

SOLUTION The surface $z = 2x^2 + y^2$ or, equivalently, $2x^2 + y^2 - z = 0$ is a level surface (with $k = 0$) of the function

$$F(x, y, z) = 2x^2 + y^2 - z$$

Then

$$F_x(x, y, z) = 4x \qquad F_y(x, y, z) = 2y \qquad F_z(x, y, z) = -1$$

$$F_x(1, 1, 3) = 4 \qquad F_y(1, 1, 3) = 2 \qquad F_z(1, 1, 3) = -1$$

By Equation 19 the equation of the tangent plane at $(1, 1, 3)$ is

$$4(x - 1) + 2(y - 1) - (z - 3) = 0$$

which simplifies to $z = 4x + 2y - 3$. ■

Compare the solution to Example 9 to the one in Example 14.4.1.

Significance of the Gradient Vector

We first consider a function f of three variables and a point $P(x_0, y_0, z_0)$ in its domain. We know from Theorem 15 that the gradient vector $\nabla f(x_0, y_0, z_0)$ gives the direction of fastest increase of f. We also know that $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface S of f through P. (Refer to Figure 10.) These two properties are quite compatible intuitively because as we move away from P on the level surface S, the value of f does not change at all. So it seems reasonable that if we move in the perpendicular direction, we get the maximum increase.

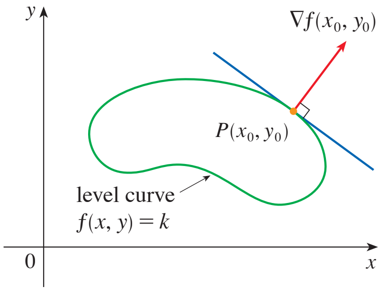


FIGURE 12

In like manner we consider a function f of two variables and a point $P(x_0, y_0)$ in its domain. Again the gradient vector $\nabla f(x_0, y_0)$ gives the direction of fastest increase of f. Also, by considerations similar to our discussion of tangent planes, it can be shown that $\nabla f(x_0, y_0)$ is perpendicular to the level curve $f(x, y) = k$ that passes through P. Again this is intuitively plausible because the values of f remain constant as we move along the curve (see Figure 12).

We now summarize the ways in which the gradient vector is significant.

Properties of the Gradient Vector Let f be a differentiable function of two or three variables and suppose that $\nabla f(\mathbf{x}) \neq \mathbf{0}$.

- The directional derivative of f at $\mathbf{x}$ in the direction of a unit vector $\mathbf{u}$ is given by $D_{\mathbf{u}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$.
- $\nabla f(\mathbf{x})$ points in the direction of maximum rate of increase of f at $\mathbf{x}$, and that maximum rate of change is $|\nabla f(\mathbf{x})|$.
- $\nabla f(\mathbf{x})$ is perpendicular to the level curve or level surface of f through $\mathbf{x}$.

If we consider a topographical map of a hill and let $f(x, y)$ represent the height above sea level at a point with coordinates (x, y), then a curve of steepest ascent can be drawn as in Figure 13 by making it perpendicular to all of the contour lines. This phenomenon can also be noticed in Figure 14.1.12, where Lonesome Creek follows a curve of steepest descent.

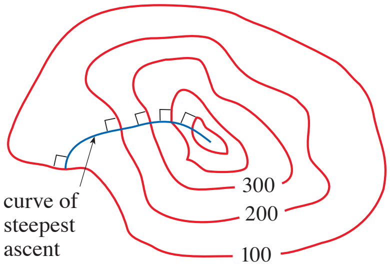


FIGURE 13

Mathematical software can plot sample gradient vectors, where each gradient vector $\nabla f(a, b)$ is plotted starting at the point (a, b). Figure 14 shows such a plot (called a *gradient vector field*) for the function $f(x, y) = x^2 - y^2$ superimposed on a contour map of f. As expected, the gradient vectors point "uphill" and are perpendicular to the level curves.

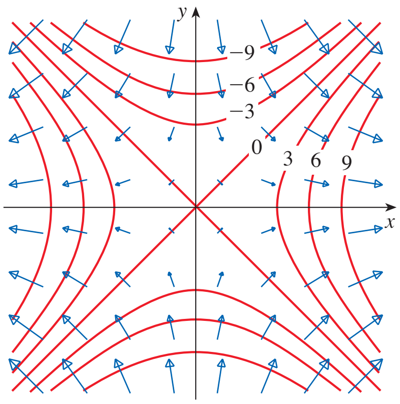
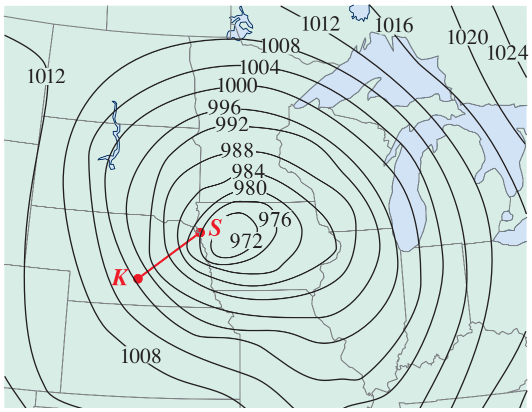


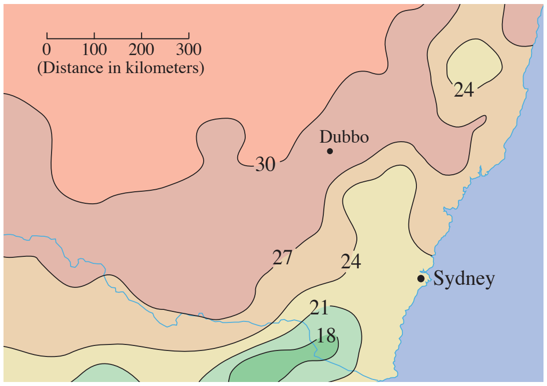
FIGURE 14

14.6 Exercises

1. Level curves for barometric pressure (in millibars) are shown for 6:00 AM on a day in November. A deep low with pressure 972 mb is moving over northeast Iowa. The distance along the red line from K (Kearney, Nebraska) to S (Sioux City, Iowa) is 300 km. Estimate the value of the directional derivative of the pressure function at Kearney in the direction of Sioux City. What are the units of the directional derivative?



2. The contour map shows the average maximum temperature for November 2004 (in °C). Estimate the value of the directional derivative of this temperature function at Dubbo, New South Wales, in the direction of Sydney. What are the units?



3. The wind-chill index W is the perceived temperature when the actual temperature is T and the wind speed is v, so we can write $W = f(T, v)$. The following table of values is an excerpt from Table 1 in Section 14.1. Use the table to estimate the value of $D_{\mathbf{u}} f(-20, 30)$, where $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$.

Wind speed (km/h)

Actual temperature (°C)

T \ v	20	30	40	50	60	70
−10	−18	−20	−21	−22	−23	−23
−15	−24	−26	−27	−29	−30	−30
−20	−30	−33	−34	−35	−36	−37
−25	−37	−39	−41	−42	−43	−44

4–7 Find the directional derivative of f at the given point in the direction indicated by the angle θ.

4. $f(x, y) = xy^3 - x^2, \quad (1, 2), \quad \theta = \pi/3$

5. $f(x, y) = y\cos(xy), \quad (0, 1), \quad \theta = \pi/4$

6. $f(x, y) = \sqrt{2x + 3y}, \quad (3, 1), \quad \theta = -\pi/6$

7. $f(x, y) = \arctan(xy), \quad (2, -3), \quad \theta = 3\pi/4$

8–12

(a) Find the gradient of f.

(b) Evaluate the gradient at the point P.

(c) Find the rate of change of f at P in the direction of the vector $\mathbf{u}$.

8. $f(x, y) = x^2 e^y, \quad P(3, 0), \quad \mathbf{u} = \frac{1}{5}(3\mathbf{i} - 4\mathbf{j})$

9. $f(x, y) = x/y, \quad P(2, 1), \quad \mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$

10. $f(x, y) = x^2 \ln y, \quad P(3, 1), \quad \mathbf{u} = -\frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}$

11. $f(x, y, z) = x^2yz - xyz^3, \quad P(2, -1, 1), \quad \mathbf{u} = \left\langle 0, \frac{4}{5}, -\frac{3}{5} \right\rangle$

12. $f(x, y, z) = y^2 e^{xyz}, \quad P(0, 1, -1), \quad \mathbf{u} = \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \right\rangle$

13–19 Find the directional derivative of the function at the given point in the direction of the vector $\mathbf{v}$.

13. $f(x, y) = e^x \sin y, \quad (0, \pi/3), \quad \mathbf{v} = \langle -6, 8 \rangle$

14. $f(x, y) = \dfrac{x}{x^2 + y^2}, \quad (1, 2), \quad \mathbf{v} = \langle 3, 5 \rangle$

15. $g(s, t) = s\sqrt{t}, \quad (2, 4), \quad \mathbf{v} = 2\mathbf{i} - \mathbf{j}$

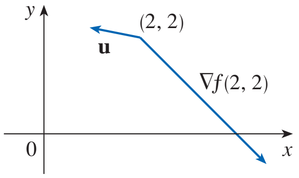
16. $g(u, v) = u^2 e^{-v}, \quad (3, 0), \quad \mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$

17. $f(x, y, z) = x^2y + y^2z, \quad (1, 2, 3), \quad \mathbf{v} = \langle 2, -1, 2 \rangle$

18. $f(x, y, z) = xy^2 \tan^{-1} z, \quad (2, 1, 1), \quad \mathbf{v} = \langle 1, 1, 1 \rangle$

19. $h(r, s, t) = \ln(3r + 6s + 9t), \quad (1, 1, 1),$
$\mathbf{v} = 4\mathbf{i} + 12\mathbf{j} + 6\mathbf{k}$

20. Use the figure to estimate $D_{\mathbf{u}} f(2, 2)$.



21–25 Find the directional derivative of the function at the point P in the direction of the point Q.

21. $f(x, y) = x^2y^2 - y^3, \quad P(1, 2), \quad Q(-3, 5)$

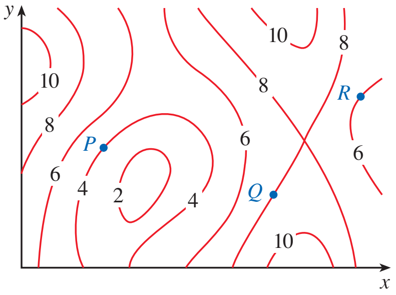
22. $f(x, y) = \dfrac{x}{y^2}, \quad P(3, -1), \quad Q(-2, 11)$

23. $f(x, y) = \sqrt{xy}, \quad P(2, 8), \quad Q(5, 4)$

24. $f(x, y, z) = xy^2z^3, \quad P(2, 1, 1), \quad Q(0, -3, 5)$

25. $f(x, y, z) = xy - xy^2z^2, \quad P(2, -1, 1), \quad Q(5, 1, 7)$

26. The contour map of a function f is shown. At points P, Q, and R, draw an arrow to indicate the direction of the gradient vector.



27–32 Find the maximum rate of change of f at the given point and the direction in which it occurs.

27. $f(x, y) = 5xy^2, \quad (3, -2)$

28. $f(s, t) = \dfrac{s}{s^2 + t^2}, \quad (-1, 1)$

29. $f(x, y) = \sin(xy), \quad (1, 0)$

30. $f(x, y, z) = x\ln(yz), \quad \left(1, 2, \frac{1}{2}\right)$

31. $f(x, y, z) = x/(y + z), \quad (8, 1, 3)$

32. $f(p, q, r) = \arctan(pqr), \quad (1, 2, 1)$

33. Direction of Most Rapid Decrease

(a) Show that a differentiable function f decreases most rapidly at $\mathbf{x}$ in the direction opposite the gradient vector, that is, in the direction of $-\nabla f(\mathbf{x})$, and that the maximum rate of decrease is $-|\nabla f(\mathbf{x})|$.

(b) Use the result of part (a) to find the direction in which the function $f(x, y) = x^4y - x^2y^3$ decreases fastest at the point $(2, -3)$. What is the rate of decrease?

34. Find the directions in which the directional derivative of $f(x, y) = x^2 + xy^3$ at the point $(2, 1)$ has the value 2.

35. Find all points at which the direction of greatest rate of change of the function $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\mathbf{i} + \mathbf{j}$.

36. Near a buoy, the depth of a lake at the point with coordinates (x, y) is $z = 200 + 0.02x^2 - 0.001y^3$, where x, y, and z are measured in meters. A fisherman in a small boat starts at the point $(80, 60)$ and moves toward the buoy, which is located at $(0, 0)$. Is the water under the boat getting deeper or shallower when he departs? Explain.

37. The temperature T in a metal ball is inversely proportional to the distance from the center of the ball, which we take to be the origin. The temperature at the point $(1, 2, 2)$ is 120°.

(a) Find the rate of change of T at $(1, 2, 2)$ in the direction toward the point $(2, 1, 3)$.

(b) Show that at any point in the ball the direction of greatest increase in temperature is given by a vector that points toward the origin.

38. The temperature at a point (x, y, z) is given by

$$T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$$

where T is measured in °C and x, y, z in meters.

(a) Find the rate of change of temperature at the point $P(2, -1, 2)$ in the direction toward the point $(3, -3, 3)$.

(b) In which direction does the temperature increase fastest at P?

(c) Find the maximum rate of increase at P.

39. Suppose that over a certain region of space the electrical potential V is given by $V(x, y, z) = 5x^2 - 3xy + xyz$.

(a) Find the rate of change of the potential at $P(3, 4, 5)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.

(b) In which direction does V change most rapidly at P?

(c) What is the maximum rate of change at P?

40. Suppose you are climbing a hill whose shape is given by the equation $z = 1000 - 0.005x^2 - 0.01y^2$, where x, y, and z are measured in meters, and you are standing at a point with coordinates $(60, 40, 966)$. The positive x-axis points east and the positive y-axis points north.

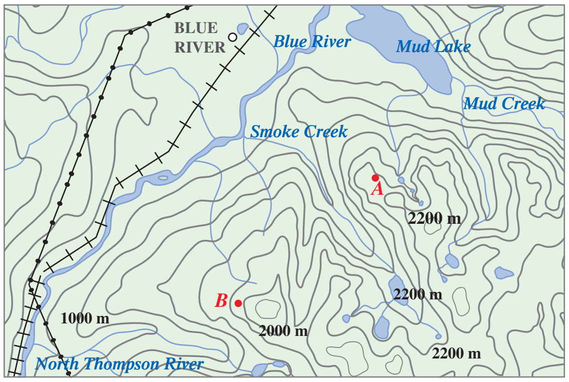
(a) If you walk due south, will you start to ascend or descend? At what rate?

(b) If you walk northwest, will you start to ascend or descend? At what rate?

(c) In which direction is the slope largest? What is the rate of ascent in that direction? At what angle above the horizontal does the path in that direction begin?

41. Let f be a function of two variables that has continuous partial derivatives and consider the points $A(1, 3)$, $B(3, 3)$, $C(1, 7)$, and $D(6, 15)$. The directional derivative of f at A in the direction of the vector $\overrightarrow{AB}$ is 3, and the directional derivative at A in the direction of $\overrightarrow{AC}$ is 26. Find the directional derivative of f at A in the direction of the vector $\overrightarrow{AD}$.

42. Shown is a topographic map of Blue River Pine Provincial Park in British Columbia. Draw curves of steepest descent from point A (descending to Mud Lake) and from point B.



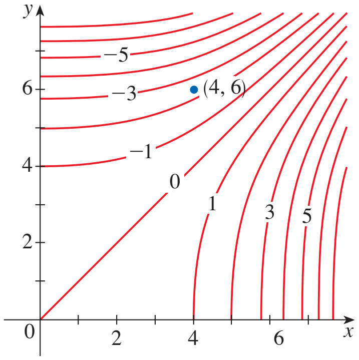
43. Show that the operation of taking the gradient of a function has the given property. Assume that u and v are differentiable functions of x and y and that a, b are constants.

(a) $\nabla(au + bv) = a\,\nabla u + b\,\nabla v$

(b) $\nabla(uv) = u\,\nabla v + v\,\nabla u$

(c) $\nabla\left(\dfrac{u}{v}\right) = \dfrac{v\,\nabla u - u\,\nabla v}{v^2}$ (d) $\nabla u^n = nu^{n-1}\,\nabla u$

44. Sketch the gradient vector $\nabla f(4, 6)$ for the function f whose level curves are shown. Explain how you chose the direction and length of this vector.



45–46 Second Directional Derivatives The *second directional derivative* of $f(x, y)$ is

$$D_{\mathbf{u}}^2 f(x, y) = D_{\mathbf{u}}[D_{\mathbf{u}} f(x, y)]$$

45. If $f(x, y) = x^3 + 5x^2y + y^3$ and $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$, calculate $D_{\mathbf{u}}^2 f(2, 1)$.

46. (a) If $\mathbf{u} = \langle a, b \rangle$ is a unit vector and f has continuous second partial derivatives, show that

$$D_{\mathbf{u}}^2 f = f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2$$

(b) Find the second directional derivative of $f(x, y) = xe^{2y}$ in the direction of $\mathbf{v} = \langle 4, 6 \rangle$.

47–52 Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

47. $2(x - 2)^2 + (y - 1)^2 + (z - 3)^2 = 10$, $(3, 3, 5)$

48. $x = y^2 + z^2 + 1$, $(3, 1, -1)$

49. $xy^2z^3 = 8$, $(2, 2, 1)$

50. $xy + yz + zx = 5$, $(1, 2, 1)$

51. $x + y + z = e^{xyz}$, $(0, 0, 1)$

52. $x^4 + y^4 + z^4 = 3x^2y^2z^2$, $(1, 1, 1)$

53–54 Graph the surface, the tangent plane, and the normal line at the given point on the same screen. Choose a viewpoint so that you get a good view of all three objects.

53. $xy + yz + zx = 3$, $(1, 1, 1)$ **54.** $xyz = 6$, $(1, 2, 3)$

55. If $f(x, y) = xy$, find the gradient vector $\nabla f(3, 2)$ and use it to find the tangent line to the level curve $f(x, y) = 6$ at the point $(3, 2)$. Sketch the level curve, the tangent line, and the gradient vector.

56. If $g(x, y) = x^2 + y^2 - 4x$, find the gradient vector $\nabla g(1, 2)$ and use it to find the tangent line to the level curve $g(x, y) = 1$ at the point $(1, 2)$. Sketch the level curve, the tangent line, and the gradient vector.

57. Show that the equation of the tangent plane to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at the point (x_0, y_0, z_0) can be written as

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1$$

58. Find the equation of the tangent plane to the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ at (x_0, y_0, z_0) and express it in a form similar to the one in Exercise 57.

59. Show that the equation of the tangent plane to the elliptic paraboloid $z/c = x^2/a^2 + y^2/b^2$ at the point (x_0, y_0, z_0) can be written as

$$\frac{2xx_0}{a^2} + \frac{2yy_0}{b^2} = \frac{z + z_0}{c}$$

60. At what point on the ellipsoid $x^2 + y^2 + 2z^2 = 1$ is the tangent plane parallel to the plane $x + 2y + z = 1$?

61. Are there any points on the hyperboloid $x^2 - y^2 - z^2 = 1$ where the tangent plane is parallel to the plane $z = x + y$?

62. Show that the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at the point $(1, 1, 2)$. (This means that they have a common tangent plane at the point.)

63. Show that every plane that is tangent to the cone $x^2 + y^2 = z^2$ passes through the origin.

64. Show that every normal line to the sphere $x^2 + y^2 + z^2 = r^2$ passes through the center of the sphere.

65. Where does the normal line to the paraboloid $z = x^2 + y^2$ at the point $(1, 1, 2)$ intersect the paraboloid a second time?

66. At what points does the normal line through the point $(1, 2, 1)$ on the ellipsoid $4x^2 + y^2 + 4z^2 = 12$ intersect the sphere $x^2 + y^2 + z^2 = 102$?

67. Show that the sum of the x-, y-, and z-intercepts of any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$ is a constant.

68. Show that the pyramids cut off from the first octant by any tangent planes to the surface $xyz = 1$ at points in the first octant must all have the same volume.

69. Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1, 1, 2)$.

70. (a) The plane $y + z = 3$ intersects the cylinder $x^2 + y^2 = 5$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 1)$.

(b) Graph the cylinder, the plane, and the tangent line on the same screen.

71. Where does the helix $\mathbf{r}(t) = \langle \cos \pi t, \sin \pi t, t \rangle$ intersect the paraboloid $z = x^2 + y^2$? What is the angle of intersection between the helix and the paraboloid? (This is the angle between the tangent vector to the curve and the tangent plane to the paraboloid.)

72. The helix $\mathbf{r}(t) = \langle \cos(\pi t/2), \sin(\pi t/2), t \rangle$ intersects the sphere $x^2 + y^2 + z^2 = 2$ in two points. Find the angle of intersection at each point.

73–74 Orthogonal Surfaces Two surfaces are called *orthogonal* at a point of intersection if their normal lines are perpendicular at that point.

73. Show that surfaces with equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$ are orthogonal at a point P where $\nabla F \neq \mathbf{0}$ and $\nabla G \neq \mathbf{0}$ if and only if

$$F_x G_x + F_y G_y + F_z G_z = 0 \quad \text{at } P$$

74. Use Exercise 73 to show that the surfaces $z^2 = x^2 + y^2$ and $x^2 + y^2 + z^2 = r^2$ are orthogonal at every point of intersection. Can you see why this is true without using calculus?

---

75. Suppose that the directional derivatives of $f(x, y)$ are known at a given point in two nonparallel directions given by unit vectors $\mathbf{u}$ and $\mathbf{v}$. Is it possible to find ∇f at this point? If so, how would you do it?

76. (a) Show that the function $f(x, y) = \sqrt[3]{xy}$ is continuous and the partial derivatives f_x and f_y exist at the origin, but the directional derivatives in all other directions do not exist.

(b) Graph f near the origin and comment on how the graph confirms part (a).

77. Show that if $z = f(x, y)$ is differentiable at $\mathbf{x}_0 = \langle x_0, y_0 \rangle$, then

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{f(\mathbf{x}) - [f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)]}{|\mathbf{x} - \mathbf{x}_0|} = 0$$

[*Hint:* Use Definition 14.4.7 directly.]

14.7 Maximum and Minimum Values

Local Maximum and Minimum Values

As we saw in Chapter 4, one of the main uses of ordinary derivatives is in finding maximum and minimum values (extreme values). In this section we see how to use partial derivatives to locate maxima and minima of functions of two variables. In particular, in Example 6 we will see how to maximize the volume of a box without a lid if we have a fixed amount of cardboard to work with.

Look at the hills and valleys in the graph of f shown in Figure 1. There are two points (a, b) where f has a *local maximum*, that is, where $f(a, b)$ is larger than nearby values of $f(x, y)$. Likewise, f has two *local minima*, where $f(a, b)$ is smaller than nearby values. The largest value of $f(x, y)$ on the domain of f is the *absolute maximum*, and the smallest value is the *absolute minimum*.

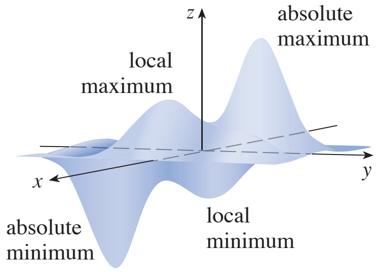


FIGURE 1

1 Definition A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leqslant f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geqslant f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

Fermat's Theorem (Section 4.1) states that, for single-variable functions, if f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$. The following theorem states a similar result for functions of two variables.

Notice that the conclusion of Theorem 2 can be stated in the notation of gradient vectors as $\nabla f(a, b) = \mathbf{0}$.

2 Theorem If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

PROOF Let $g(x) = f(x, b)$. If f has a local maximum (or minimum) at (a, b), then g has a local maximum (or minimum) at a, so $g'(a) = 0$ by Fermat's Theorem (see Theorem 4.1.4). But $g'(a) = f_x(a, b)$ (see Equation 14.3.1) and so $f_x(a, b) = 0$. Similarly, by applying Fermat's Theorem to the function $G(y) = f(a, y)$, we obtain $f_y(a, b) = 0$. ■

If we put $f_x(a, b) = 0$ and $f_y(a, b) = 0$ in the equation of a tangent plane (Equation 14.4.2), we get $z = z_0$. Thus the geometric interpretation of Theorem 2 is that if the graph of f has a tangent plane at a local maximum or minimum, then the tangent plane must be horizontal.

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist. Theorem 2 says that if f has a local maximum or minimum at (a, b), then (a, b) is a critical point of f. However, as in single-variable calculus, not all critical points give rise to maxima or minima.

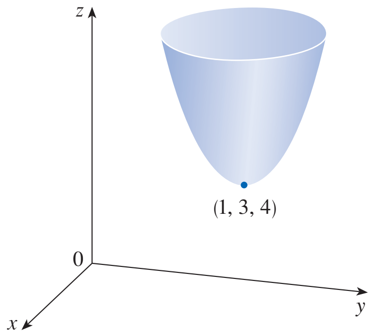


FIGURE 2
$z = x^2 + y^2 - 2x - 6y + 14$

EXAMPLE 1 Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Then

$$f_x(x, y) = 2x - 2 \qquad f_y(x, y) = 2y - 6$$

These partial derivatives are equal to 0 when $x = 1$ and $y = 3$, so the only critical point is $(1, 3)$. By completing the square, we find that

$$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$$

Since $(x - 1)^2 \geqslant 0$ and $(y - 3)^2 \geqslant 0$, we have $f(x, y) \geqslant 4$ for all values of x and y. Therefore $f(1, 3) = 4$ is a local minimum, and in fact it is the absolute minimum of f. This can be confirmed geometrically from the graph of f, which is the elliptic paraboloid with vertex $(1, 3, 4)$ shown in Figure 2. ■

EXAMPLE 2 Find the extreme values of $f(x, y) = y^2 - x^2$.

SOLUTION Since $f_x = -2x$ and $f_y = 2y$, the only critical point is $(0, 0)$. Notice that for points on the x-axis we have $y = 0$, so $f(x, y) = -x^2 < 0$ (if $x \neq 0$). However, for points on the y-axis we have $x = 0$, so $f(x, y) = y^2 > 0$ (if $y \neq 0$). Thus every disk with center $(0, 0)$ contains points where f takes on positive values as well as points where f takes on negative values. Therefore $f(0, 0) = 0$ can't be an extreme value for f, so f has no extreme value. ■

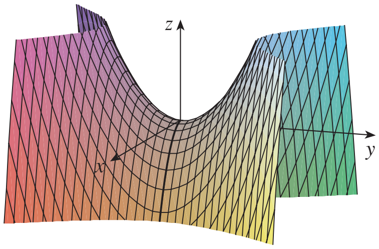


FIGURE 3
$z = y^2 - x^2$

Example 2 illustrates the fact that a function need not have a maximum or minimum value at a critical point. Figure 3 shows one way in which this can happen. The graph of f is the hyperbolic paraboloid $z = y^2 - x^2$, which has a horizontal tangent plane ($z = 0$) at the origin. You can see that $f(0, 0) = 0$ is a maximum in the direction of the x-axis but a minimum in the direction of the y-axis.

Photo by Stan Wagon, Macalester College

A mountain pass also has the shape of a saddle; for people hiking in one direction the saddle point is the lowest point on their route, whereas for those traveling in a different direction the saddle point is the highest point.

Recall that for functions of a single variable, a critical number c where $f'(c) = 0$ may correspond to a local maximum, a local minimum, or neither. An analogous situation occurs for functions of two variables. If (a, b) is a critical point of a function f, where $f_x(a, b) = 0$ and $f_y(a, b) = 0$, then $f(a, b)$ may be a local maximum, a local minimum, or neither. In the last case, we say that (a, b) is a **saddle point** of f. The name is suggested by the shape of the surface in Figure 3 near the origin. In general, the graph of a function at a saddle point need not resemble an actual saddle, but the graph crosses the tangent plane at that point.

We need to be able to determine whether or not a function has an extreme value at a critical point. The following test, which is proved at the end of this section, is analogous to the Second Derivative Test for functions of one variable.

3 Second Derivatives Test Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [so (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)\, f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then (a, b) is a saddle point of f.

NOTE 1 If $D = 0$, the test gives no information: f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point of f.

NOTE 2 To remember the formula for D, it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

EXAMPLE 3 Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

SOLUTION We first find the partial derivatives:

$$f_x = 4x^3 - 4y \qquad f_y = 4y^3 - 4x$$

Since these partial derivatives exist everywhere, the critical points occur where both partial derivatives are zero:

$$x^3 - y = 0 \qquad \text{and} \qquad y^3 - x = 0$$

To solve these equations we substitute $y = x^3$ from the first equation into the second one. This gives

$$0 = x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1)$$

so there are three real solutions: $x = 0, 1, -1$. The three critical points are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

Next we calculate the second partial derivatives and $D(x, y)$:

$$f_{xx} = 12x^2 \qquad f_{xy} = -4 \qquad f_{yy} = 12y^2$$

$$D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

Since $D(0, 0) = -16 < 0$, it follows from case (c) of the Second Derivatives Test that the origin is a saddle point. Since $D(1, 1) = 128 > 0$ and $f_{xx}(1, 1) = 12 > 0$, we see from case (a) of the test that $f(1, 1) = -1$ is a local minimum. This means that -1 is a local minimum value, and it occurs at the point $(1, 1)$. Similarly, we have $D(-1, -1) = 128 > 0$ and $f_{xx}(-1, -1) = 12 > 0$, so $f(-1, -1) = -1$ is also a local minimum.

The graph of f is shown in Figure 4. ■

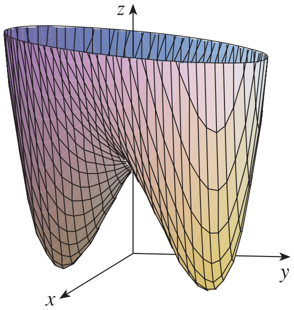


FIGURE 4

$z = x^4 + y^4 - 4xy + 1$

A contour map of the function f in Example 3 is shown in Figure 5. The level curves near $(1, 1)$ and $(-1, -1)$ are oval in shape and indicate that as we move away from $(1, 1)$ or $(-1, -1)$ in any direction the values of f are increasing. The level curves near $(0, 0)$, on the other hand, resemble hyperbolas. They reveal that as we move away from the origin (where the value of f is 1), the values of f decrease in some directions but increase in other directions. Thus the contour map suggests the presence of the minima and saddle point that we found in Example 3.

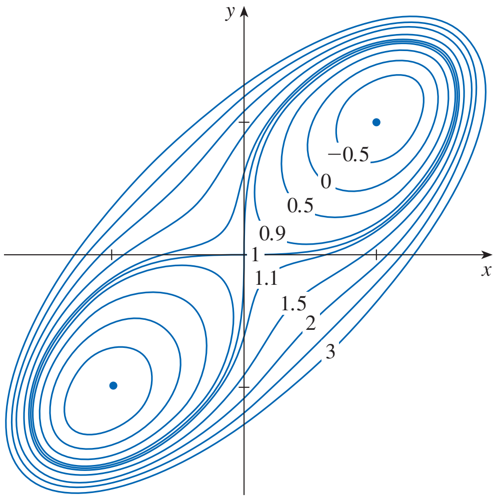


FIGURE 5

EXAMPLE 4 Find and classify the critical points of the function

$$f(x, y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$$

Also find the highest point on the graph of f.

SOLUTION The first-order partial derivatives are

$$f_x = 20xy - 10x - 4x^3 \qquad f_y = 10x^2 - 8y - 8y^3$$

So to find the critical points we need to solve the equations

$$2x(10y - 5 - 2x^2) = 0 \tag{4}$$

$$5x^2 - 4y - 4y^3 = 0 \tag{5}$$

From Equation 4 we see that either

$$x = 0 \qquad \text{or} \qquad 10y - 5 - 2x^2 = 0$$

In the first case ($x = 0$), Equation 5 becomes $-4y(1 + y^2) = 0$, so $y = 0$ and we have the critical point $(0, 0)$.

In the second case ($10y - 5 - 2x^2 = 0$), we get

$$x^2 = 5y - 2.5 \tag{6}$$

and, putting this in Equation 5, we have $25y - 12.5 - 4y - 4y^3 = 0$ or, equivalently,

$$4y^3 - 21y + 12.5 = 0$$

Using a graphing calculator or computer to solve this equation numerically, we obtain

$$y \approx -2.5452 \qquad y \approx 0.6468 \qquad y \approx 1.8984$$

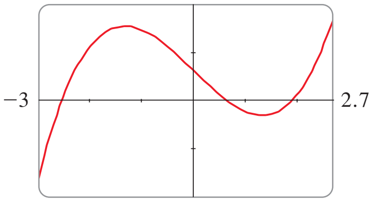


FIGURE 6

(Alternatively, we could graph the function $g(y) = 4y^3 - 21y + 12.5$, as in Figure 6, and find the intercepts.) From Equation 6, the corresponding x-values are given by

$$x = \pm\sqrt{5y - 2.5}$$

If $y \approx -2.5452$, then x has no corresponding real values. If $y \approx 0.6468$, then $x \approx \pm 0.8567$. If $y \approx 1.8984$, then $x \approx \pm 2.6442$. So we have a total of five critical points, which are analyzed in the following chart. All quantities are rounded to two decimal places.

Critical point	Value of f	f_{xx}	D	Conclusion
$(0, 0)$	0.00	−10.00	80.00	local maximum
$(\pm 2.64, 1.90)$	8.50	−55.93	2488.72	local maximum
$(\pm 0.86, 0.65)$	−1.48	−5.87	−187.64	saddle point

Figures 7 and 8 give two views of the graph of f and we see that the surface opens downward. [This can also be seen from the expression for $f(x, y)$: the dominant terms are $-x^4 - 2y^4$ when $|x|$ and $|y|$ are large.] Comparing the values of f at its local maximum points, we see that the absolute maximum value of f is $f(\pm 2.64, 1.90) \approx 8.50$. In other words, the highest points on the graph of f are $(\pm 2.64, 1.90, 8.50)$.

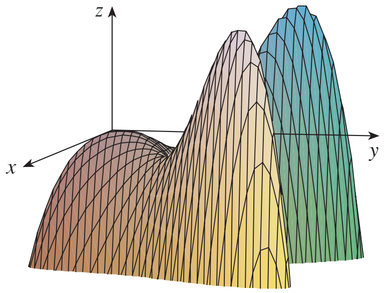


FIGURE 7

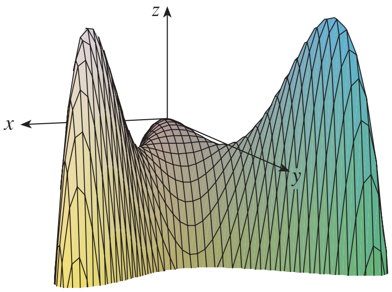


FIGURE 8

■

The five critical points of the function f in Example 4 are shown in red in the contour map of f in Figure 9.

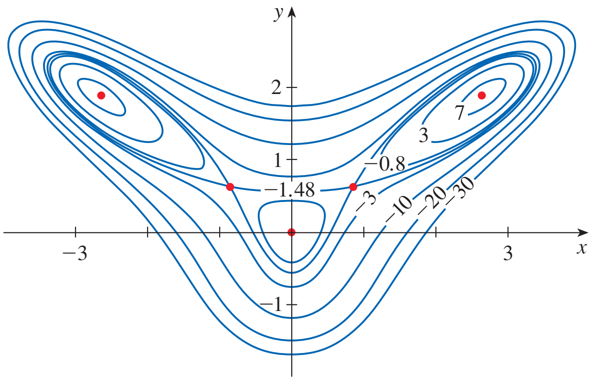


FIGURE 9

EXAMPLE 5 Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

SOLUTION The distance from any point (x, y, z) to the point $(1, 0, -2)$ is

$$d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$$

but if (x, y, z) lies on the plane $x + 2y + z = 4$, then $z = 4 - x - 2y$ and so we have $d = \sqrt{(x-1)^2 + y^2 + (6 - x - 2y)^2}$. We can minimize d by minimizing the simpler expression

$$d^2 = f(x, y) = (x-1)^2 + y^2 + (6 - x - 2y)^2$$

By solving the equations

$$f_x = 2(x-1) - 2(6 - x - 2y) = 4x + 4y - 14 = 0$$

$$f_y = 2y - 4(6 - x - 2y) = 4x + 10y - 24 = 0$$

we find that the only critical point is $\left(\frac{11}{6}, \frac{5}{3}\right)$. Since $f_{xx} = 4$, $f_{xy} = 4$, and $f_{yy} = 10$, we have $D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 24 > 0$ and $f_{xx} > 0$, so by the Second Derivatives Test f has a local minimum at $\left(\frac{11}{6}, \frac{5}{3}\right)$. Intuitively, we can see that this local minimum is actually an absolute minimum because there must be a point on the given plane that is closest to $(1, 0, -2)$. If $x = \frac{11}{6}$ and $y = \frac{5}{3}$, then

Example 5 could also be solved using vectors. Compare with the methods of Section 12.5.

$$d = \sqrt{(x-1)^2 + y^2 + (6 - x - 2y)^2} = \sqrt{\left(\tfrac{5}{6}\right)^2 + \left(\tfrac{5}{3}\right)^2 + \left(\tfrac{5}{6}\right)^2} = \tfrac{5}{6}\sqrt{6}$$

The shortest distance from $(1, 0, -2)$ to the plane $x + 2y + z = 4$ is $\frac{5}{6}\sqrt{6}$. ■

EXAMPLE 6 A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

SOLUTION Let the length, width, and height of the box (in meters) be x, y, and z, as shown in Figure 10. Then the volume of the box is

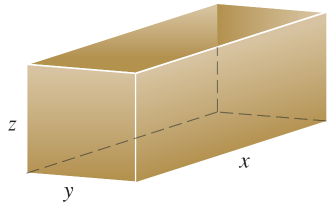


FIGURE 10

$$V = xyz$$

We can express V as a function of just two variables x and y by using the fact that the area of the four sides and the bottom of the box is

$$2xz + 2yz + xy = 12$$

Solving this equation for z, we get $z = (12 - xy)/[2(x + y)]$, so the expression for V becomes

$$V = xy\,\frac{12 - xy}{2(x+y)} = \frac{12xy - x^2y^2}{2(x+y)}$$

We compute the partial derivatives:

$$\frac{\partial V}{\partial x} = \frac{y^2(12 - 2xy - x^2)}{2(x+y)^2} \qquad \frac{\partial V}{\partial y} = \frac{x^2(12 - 2xy - y^2)}{2(x+y)^2}$$

If V is a maximum, then $\partial V/\partial x = \partial V/\partial y = 0$, but $x = 0$ or $y = 0$ gives $V = 0$. It remains to solve the equations

$$12 - 2xy - x^2 = 0 \qquad 12 - 2xy - y^2 = 0$$

These imply that $x^2 = y^2$ and so $x = y$. (Note that x and y must both be nonnegative in this problem.) If we put $x = y$ in either equation we get $12 - 3x^2 = 0$, which gives $x = 2$, $y = 2$, and $z = (12 - 2 \cdot 2)/[2(2 + 2)] = 1$.

We could use the Second Derivatives Test to show that this gives a local maximum of V, or we could simply argue from the physical nature of this problem that there must be an absolute maximum volume, which has to occur at a critical point of V, so it must occur when $x = 2$, $y = 2$, $z = 1$. Then $V = 2 \cdot 2 \cdot 1 = 4$, so the maximum volume of the box is 4 m^3. ■

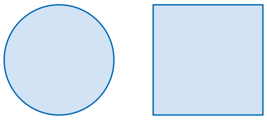
Absolute Maximum and Minimum Values

Just as for single-variable functions, the absolute maximum and minimum values of a function f of two variables are the largest and smallest values that f achieves on its domain.

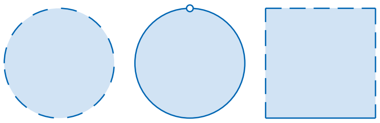
7 Definition Let (a, b) be a point in the domain D of a function f of two variables. Then $f(a, b)$ is the

- **absolute maximum** value of f on D if $f(a, b) \geqslant f(x, y)$ for all (x, y) in D.
- **absolute minimum** value of f on D if $f(a, b) \leqslant f(x, y)$ for all (x, y) in D.

For a function f of one variable, the Extreme Value Theorem says that if f is continuous on a closed interval $[a, b]$, then f has an absolute minimum value and an absolute maximum value. According to the Closed Interval Method in Section 4.1, we found these by evaluating f not only at the critical numbers but also at the endpoints a and b.



(a) Closed sets



(b) Sets that are not closed

FIGURE 11

There is a similar situation for functions of two variables. Just as a closed interval contains its endpoints, a **closed set** in $\mathbb{R}^2$ is one that contains all its boundary points. [A boundary point of D is a point (a, b) such that every disk with center (a, b) contains points in D and also points not in D.] For instance, the disk

$$D = \{(x, y) \mid x^2 + y^2 \leqslant 1\}$$

which consists of all points on or inside the circle $x^2 + y^2 = 1$, is a closed set because it contains all of its boundary points (which are the points on the circle $x^2 + y^2 = 1$). But if even one point on the boundary curve were omitted, the set would not be closed. (See Figure 11.)

A **bounded set** in $\mathbb{R}^2$ is one that is contained within some disk. In other words, it is finite in extent. Then, in terms of closed and bounded sets, we can state the following counterpart of the Extreme Value Theorem in two dimensions.

8 Extreme Value Theorem for Functions of Two Variables If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To find the extreme values guaranteed by Theorem 8, we note that, by Theorem 2, if f has an extreme value at (x_1, y_1), then (x_1, y_1) is either a critical point of f or a boundary point of D. Thus we have the following extension of the Closed Interval Method.

9 To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

EXAMPLE 7 Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 2\}$.

SOLUTION Since f is a polynomial, it is continuous on the closed, bounded rectangle D, so Theorem 8 tells us there is both an absolute maximum and an absolute minimum. According to step 1 in (9), we first find the critical points. These occur when

$$f_x = 2x - 2y = 0$$

$$f_y = -2x + 2 = 0$$

so the only critical point is $(1, 1)$. This point is in D and the value of f there is $f(1, 1) = 1$.

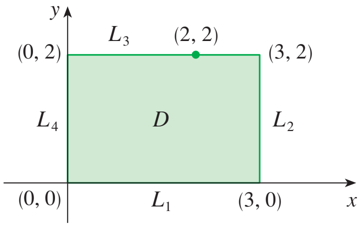


FIGURE 12

In step 2 we look at the values of f on the boundary of D, which consists of the four line segments L_1, L_2, L_3, L_4 shown in Figure 12. On L_1 we have $y = 0$ and

$$f(x, 0) = x^2 \qquad 0 \leqslant x \leqslant 3$$

This is an increasing function of x, so its minimum value is $f(0, 0) = 0$ and its maximum value is $f(3, 0) = 9$. On L_2 we have $x = 3$ and

$$f(3, y) = 9 - 4y \qquad 0 \leqslant y \leqslant 2$$

This is a decreasing function of y, so its maximum value is $f(3, 0) = 9$ and its minimum value is $f(3, 2) = 1$. On L_3 we have $y = 2$ and

$$f(x, 2) = x^2 - 4x + 4 \qquad 0 \leqslant x \leqslant 3$$

By the methods of Chapter 4, or simply by observing that $f(x, 2) = (x - 2)^2$, we see that the minimum value of this function is $f(2, 2) = 0$ and the maximum value is $f(0, 2) = 4$. Finally, on L_4 we have $x = 0$ and

$$f(0, y) = 2y \qquad 0 \leqslant y \leqslant 2$$

with maximum value $f(0, 2) = 4$ and minimum value $f(0, 0) = 0$. Thus, on the boundary, the minimum value of f is 0 and the maximum is 9.

In step 3 we compare these values with the value $f(1, 1) = 1$ at the critical point and conclude that the absolute maximum value of f on D is $f(3, 0) = 9$ and the absolute minimum value is $f(0, 0) = f(2, 2) = 0$. Figure 13 shows the graph of f. ■

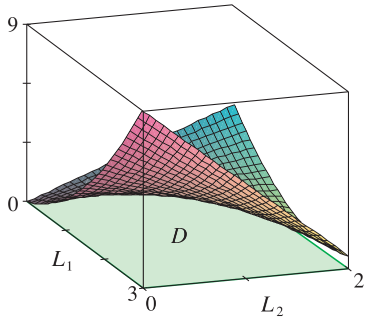


FIGURE 13
$f(x, y) = x^2 - 2xy + 2y$

Proof of the Second Derivatives Test

We close this section by giving a proof of the first part of the Second Derivatives Test. Part (b) has a similar proof.

PROOF OF THEOREM 3, PART (a) We compute the second-order directional derivative of f in the direction of $\mathbf{u} = \langle h, k \rangle$. The first-order derivative is given by Theorem 14.6.3:

$$D_{\mathbf{u}} f = f_x h + f_y k$$

Applying this theorem a second time, we have

$$\begin{aligned} D_{\mathbf{u}}^2 f = D_{\mathbf{u}}(D_{\mathbf{u}} f) &= \frac{\partial}{\partial x}(D_{\mathbf{u}} f)h + \frac{\partial}{\partial y}(D_{\mathbf{u}} f)k \\ &= (f_{xx}h + f_{yx}k)h + (f_{xy}h + f_{yy}k)k \\ &= f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 \qquad \text{(by Clairaut's Theorem)} \end{aligned}$$

If we complete the square in this expression, we obtain

$$\boxed{10} \qquad D_{\mathbf{u}}^2 f = f_{xx}\left(h + \frac{f_{xy}}{f_{xx}}k\right)^2 + \frac{k^2}{f_{xx}}(f_{xx}f_{yy} - f_{xy}^2)$$

We are given that $f_{xx}(a, b) > 0$ and $D(a, b) > 0$. But f_{xx} and $D = f_{xx}f_{yy} - f_{xy}^2$ are continuous functions, so there is a disk B with center (a, b) and radius $\delta > 0$ such that $f_{xx}(x, y) > 0$ and $D(x, y) > 0$ whenever (x, y) is in B. Therefore, by looking at Equation 10, we see that $D_{\mathbf{u}}^2 f(x, y) > 0$ whenever (x, y) is in B. This means that if C is the curve obtained by intersecting the graph of f with the vertical plane through $P(a, b, f(a, b))$ in the direction of $\mathbf{u}$, then C is concave upward on an interval of length 2δ. This is true in the direction of every vector $\mathbf{u}$, so if we restrict (x, y) to lie in B, the graph of f lies above its horizontal tangent plane at P. Thus $f(x, y) \geqslant f(a, b)$ whenever (x, y) is in B. This shows that $f(a, b)$ is a local minimum. ■

14.7 Exercises

1. Suppose $(1, 1)$ is a critical point of a function f with continuous second derivatives. In each case, what can you say about f?

(a) $f_{xx}(1, 1) = 4, \quad f_{xy}(1, 1) = 1, \quad f_{yy}(1, 1) = 2$

(b) $f_{xx}(1, 1) = 4, \quad f_{xy}(1, 1) = 3, \quad f_{yy}(1, 1) = 2$

2. Suppose $(0, 2)$ is a critical point of a function g with continuous second derivatives. In each case, what can you say about g?

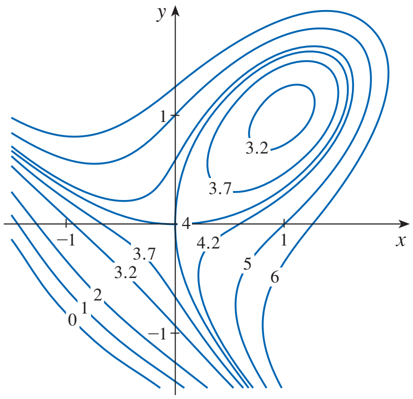
(a) $g_{xx}(0, 2) = -1, \quad g_{xy}(0, 2) = 6, \quad g_{yy}(0, 2) = 1$

(b) $g_{xx}(0, 2) = -1, \quad g_{xy}(0, 2) = 2, \quad g_{yy}(0, 2) = -8$

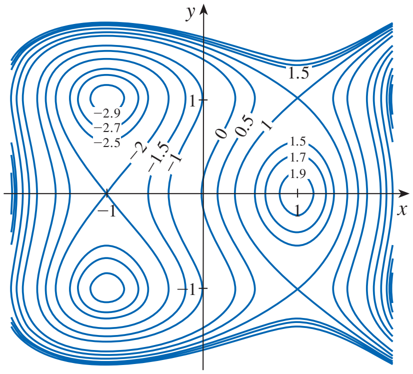
(c) $g_{xx}(0, 2) = 4, \quad g_{xy}(0, 2) = 6, \quad g_{yy}(0, 2) = 9$

3–4 Use the level curves in the figure to predict the location of the critical points of f and whether f has a saddle point or a local maximum or minimum at each critical point. Explain your reasoning. Then use the Second Derivatives Test to confirm your predictions.

3. $f(x, y) = 4 + x^3 + y^3 - 3xy$



4. $f(x, y) = 3x - x^3 - 2y^2 + y^4$



5–22 Find the local maximum and minimum values and saddle point(s) of the function. You are encouraged to use a calculator or computer to graph the function with a domain and viewpoint that reveals all the important aspects of the function.

5. $f(x, y) = x^2 + xy + y^2 + y$

6. $f(x, y) = xy - 2x - 2y - x^2 - y^2$

7. $f(x, y) = 2x^2 - 8xy + y^4 - 4y^3$

8. $f(x, y) = x^3 + y^3 + 3xy$

9. $f(x, y) = (x - y)(1 - xy)$

10. $f(x, y) = y(e^x - 1)$

11. $f(x, y) = y\sqrt{x} - y^2 - 2x + 7y$

12. $f(x, y) = 2 - x^4 + 2x^2 - y^2$

13. $f(x, y) = x^3 - 3x + 3xy^2$

14. $f(x, y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$

15. $f(x, y) = x^4 - 2x^2 + y^3 - 3y$

16. $f(x, y) = x^2 + y^4 + 2xy$

17. $f(x, y) = xy - x^2y - xy^2$

18. $f(x, y) = (6x - x^2)(4y - y^2)$

19. $f(x, y) = e^x \cos y$

20. $f(x, y) = (x^2 + y^2)e^{-x}$

21. $f(x, y) = y^2 - 2y\cos x, \quad -1 \leqslant x \leqslant 7$

22. $f(x, y) = \sin x \sin y, \quad -\pi < x < \pi, \quad -\pi < y < \pi$

---

23. Show that $f(x, y) = x^2 + 4y^2 - 4xy + 2$ has an infinite number of critical points and that $D = 0$ at each one. Then show that f has a local (and absolute) minimum at each critical point.

24. Show that $f(x, y) = x^2ye^{-x^2-y^2}$ has maximum values at $(\pm 1, 1/\sqrt{2})$ and minimum values at $(\pm 1, -1/\sqrt{2})$. Show also that f has infinitely many other critical points and $D = 0$ at each of them. Which of them give rise to maximum values? Minimum values? Saddle points?

25–28 Use a graph or level curves or both to estimate the local maximum and minimum values and saddle point(s) of the function. Then use calculus to find these values precisely.

25. $f(x, y) = x^2 + y^2 + x^{-2}y^{-2}$

26. $f(x, y) = (x - y)e^{-x^2-y^2}$

27. $f(x, y) = \sin x + \sin y + \sin(x + y)$, $0 \leqslant x \leqslant 2\pi,\ 0 \leqslant y \leqslant 2\pi$

28. $f(x, y) = \sin x + \sin y + \cos(x + y)$, $0 \leqslant x \leqslant \pi/4,\ 0 \leqslant y \leqslant \pi/4$

29–32 Find the critical points of f correct to three decimal places (as in Example 4). Then classify the critical points and find the highest or lowest points on the graph, if any.

29. $f(x, y) = x^4 + y^4 - 4x^2y + 2y$

30. $f(x, y) = y^6 - 2y^4 + x^2 - y^2 + y$

31. $f(x, y) = x^4 + y^3 - 3x^2 + y^2 + x - 2y + 1$

32. $f(x, y) = 20e^{-x^2-y^2}\sin 3x \cos 3y, \quad |x| \leqslant 1, \quad |y| \leqslant 1$

33–40 Find the absolute maximum and minimum values of f on the set D.

33. $f(x, y) = x^2 + y^2 - 2x$, D is the closed triangular region with vertices $(2, 0)$, $(0, 2)$, and $(0, -2)$

34. $f(x, y) = x + y - xy$, D is the closed triangular region with vertices $(0, 0)$, $(0, 2)$, and $(4, 0)$

35. $f(x, y) = x^2 + y^2 + x^2y + 4$, $D = \{(x, y) \mid |x| \leqslant 1, |y| \leqslant 1\}$

36. $f(x, y) = x^2 + xy + y^2 - 6y$, $D = \{(x, y) \mid -3 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 5\}$

37. $f(x, y) = x^2 + 2y^2 - 2x - 4y + 1$, $D = \{(x, y) \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 3\}$

38. $f(x, y) = xy^2, \quad D = \{(x, y) \mid x \geqslant 0,\ y \geqslant 0,\ x^2 + y^2 \leqslant 3\}$

39. $f(x, y) = 2x^3 + y^4, \quad D = \{(x, y) \mid x^2 + y^2 \leqslant 1\}$

40. $f(x, y) = x^3 - 3x - y^3 + 12y$, D is the quadrilateral whose vertices are $(-2, 3)$, $(2, 3)$, $(2, 2)$, and $(-2, -2)$

---

41. For functions of one variable it is impossible for a continuous function to have two local maxima and no local minimum. But for functions of two variables such functions exist. Show that the function

$$f(x, y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2$$

has only two critical points, but has local maxima at both of them. Then produce a graph with a carefully chosen domain and viewpoint to see how this is possible.

42. If a function of one variable is continuous on an interval and has only one critical number, then a local maximum has to be an absolute maximum. But this is not true for functions of two variables. Show that the function

$$f(x, y) = 3xe^y - x^3 - e^{3y}$$

has exactly one critical point and that f has a local maximum there that is not an absolute maximum. Produce a graph with a carefully chosen domain and viewpoint to see how this is possible.

43. Find the shortest distance from the point $(2, 0, -3)$ to the plane $x + y + z = 1$.

44. Find the point on the plane $x - 2y + 3z = 6$ that is closest to the point $(0, 1, 1)$.

45. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.

46. Find the points on the surface $y^2 = 9 + xz$ that are closest to the origin.

47. Find three positive numbers whose sum is 100 and whose product is a maximum.

48. Find three positive numbers whose sum is 12 and the sum of whose squares is as small as possible.

49. Find the maximum volume of a rectangular box that is inscribed in a sphere of radius r.

50. Find the dimensions of the box with volume 1000 cm^3 that has minimal surface area.

51. Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $x + 2y + 3z = 6$.

52. Find the dimensions of the rectangular box with largest volume if the total surface area is given as 64 cm^2.

53. Find the dimensions of a rectangular box of maximum volume such that the sum of the lengths of its 12 edges is a constant c.

54. The base of an aquarium with given volume V is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, find the dimensions of the aquarium that minimize the cost of the materials.

55. A cardboard box without a lid is to have a volume of 32,000 cm^3. Find the dimensions that minimize the amount of cardboard used.

56. A rectangular building is being designed to minimize heat loss. The east and west walls lose heat at a rate of 10 units/m^2 per day, the north and south walls at a rate of 8 units/m^2 per day, the floor at a rate of 1 unit/m^2 per day, and the roof at a rate of 5 units/m^2 per day. Each wall must be at least 30 m long, the height must be at least 4 m, and the volume must be exactly 4000 m^3.

(a) Find and sketch the domain of the heat loss as a function of the lengths of the sides.

(b) Find the dimensions that minimize heat loss. (Check both the critical points and the points on the boundary of the domain.)

(c) Could you design a building with even less heat loss if the restrictions on the lengths of the walls were removed?

57. If the length of the diagonal of a rectangular box must be L, what is the largest possible volume?

58. A model for the yield Y of an agricultural crop as a function of the nitrogen level N and phosphorus level P in the soil (measured in appropriate units) is

$$Y(N, P) = kNPe^{-N-P}$$

where k is a positive constant. What levels of nitrogen and phosphorus result in the best yield?

59. The Shannon index (sometimes called the Shannon-Wiener index or Shannon-Weaver index) is a measure of diversity in an ecosystem. For the case of three species, it is defined as

$$H = -p_1 \ln p_1 - p_2 \ln p_2 - p_3 \ln p_3$$

where p_i is the proportion of species i in the ecosystem.

(a) Express H as a function of two variables using the fact that $p_1 + p_2 + p_3 = 1$.

(b) What is the domain of H?

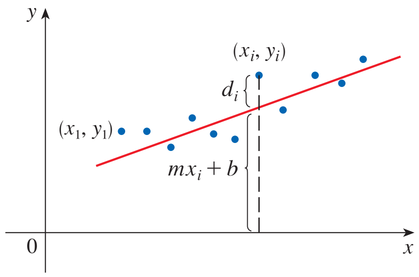
(c) Find the maximum value of H. For what values of p_1, p_2, p_3 does it occur?

60. Three alleles (alternative versions of a gene) A, B, and O determine the four blood types A (AA or AO), B (BB or BO), O (OO), and AB. The Hardy-Weinberg Law states that the proportion of individuals in a population who carry two different alleles is

$$P = 2pq + 2pr + 2rq$$

where p, q, and r are the proportions of A, B, and O in the population. Use the fact that $p + q + r = 1$ to show that P is at most $\frac{2}{3}$.

61. Method of Least Squares Suppose that a scientist has reason to believe that two quantities x and y are related linearly, that is, $y = mx + b$, at least approximately, for some values of m and b. The scientist performs an experiment and collects data in the form of points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, and then plots these points. The points don't lie exactly on a straight line, so the scientist wants to find constants m and b so that the line $y = mx + b$ "fits" the points as well as possible (see the figure).



Let $d_i = y_i - (mx_i + b)$ be the vertical deviation of the point (x_i, y_i) from the line. The *method of least squares* determines m and b so as to minimize $\sum_{i=1}^{n} d_i^2$, the sum of the squares of these deviations. Show that, according to this method, the line of best fit is obtained when

$$m \sum_{i=1}^{n} x_i + bn = \sum_{i=1}^{n} y_i$$

and

$$m \sum_{i=1}^{n} x_i^2 + b \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i$$

Thus the line is found by solving these two equations in the two unknowns m and b. (See Section 1.2 for a further discussion and applications of the method of least squares.)

62. Find an equation of the plane that passes through the point (1, 2, 3) and cuts off the smallest volume in the first octant.

DISCOVERY PROJECT | QUADRATIC APPROXIMATIONS AND CRITICAL POINTS

The Taylor polynomial approximation to functions of one variable that we discussed in Chapter 11 can be extended to functions of two or more variables. Here we investigate quadratic approximations to functions of two variables and use them to give insight into the Second Derivatives Test for classifying critical points.

In Section 14.4 we discussed the linearization of a function f of two variables at a point (a, b):

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Recall that the graph of L is the tangent plane to the surface $z = f(x, y)$ at $(a, b, f(a, b))$ and the corresponding linear approximation is $f(x, y) \approx L(x, y)$. The linearization L is also called the **first-degree Taylor polynomial** of f at (a, b).

1. If f has continuous second-order partial derivatives at (a, b), then the **second-degree Taylor polynomial** of f at (a, b) is

$$\begin{aligned} Q(x, y) = {} & f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ & + \tfrac{1}{2} f_{xx}(a, b)(x - a)^2 + f_{xy}(a, b)(x - a)(y - b) + \tfrac{1}{2} f_{yy}(a, b)(y - b)^2 \end{aligned}$$

and the approximation $f(x, y) \approx Q(x, y)$ is called the **quadratic approximation** to f at (a, b). Verify that Q has the same first- and second-order partial derivatives as f at (a, b).

2. (a) Find the first- and second-degree Taylor polynomials L and Q of $f(x, y) = e^{-x^2-y^2}$ at (0, 0).

(b) Graph f, L, and Q. Comment on how well L and Q approximate f.

3. (a) Find the first- and second-degree Taylor polynomials L and Q for $f(x, y) = xe^y$ at (1, 0).

(b) Compare the values of L, Q, and f at (0.9, 0.1).

(c) Graph f, L, and Q. Comment on how well L and Q approximate f.

4. In this problem we analyze the behavior of the polynomial $f(x, y) = ax^2 + bxy + cy^2$ (without using the Second Derivatives Test) by identifying the graph as a paraboloid.

(a) By completing the square, show that if $a \neq 0$, then

$$f(x, y) = ax^2 + bxy + cy^2 = a\left[\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right)y^2\right]$$

(b) Let $D = 4ac - b^2$. Show that if $D > 0$ and $a > 0$, then f has a local minimum at (0, 0).

(c) Show that if $D > 0$ and $a < 0$, then f has a local maximum at (0, 0).

(d) Show that if $D < 0$, then (0, 0) is a saddle point.

(*continued*)

5. (a) Suppose f is any function with continuous second-order partial derivatives such that $f(0, 0) = 0$ and $(0, 0)$ is a critical point of f. Write an expression for the second-degree Taylor polynomial, Q, of f at $(0, 0)$.

(b) What can you conclude about Q from Problem 4?

(c) In view of the quadratic approximation $f(x, y) \approx Q(x, y)$, what does part (b) suggest about f?

14.8 Lagrange Multipliers

In Example 14.7.6 we maximized a volume function $V = xyz$ subject to the constraint $2xz + 2yz + xy = 12$, which expressed the side condition that the surface area was 12 m^2. In this section we present Lagrange's method for maximizing or minimizing a general function $f(x, y, z)$ subject to a constraint (or side condition) of the form $g(x, y, z) = k$.

Lagrange Multipliers: One Constraint

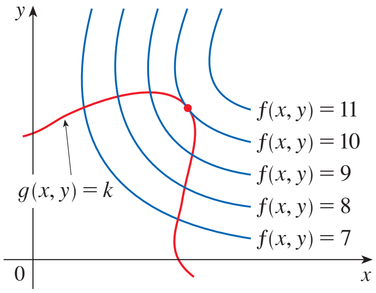


FIGURE 1

First we explain the geometric basis of Lagrange's method for functions of two variables. We start by trying to find the extreme values of $f(x, y)$ subject to a constraint of the form $g(x, y) = k$. In other words, we seek the extreme values of $f(x, y)$ when the point (x, y) is restricted to lie on the level curve $g(x, y) = k$. Figure 1 shows this curve together with several level curves of f. These have the equations $f(x, y) = c$, where $c = 7, 8, 9, 10, 11$. To maximize $f(x, y)$ subject to $g(x, y) = k$ is to find the largest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = k$. It appears from Figure 1 that this happens when these curves just touch each other, that is, when they have a common tangent line. (Otherwise, the value of c could be increased further.) This means that the normal lines at the point (x_0, y_0) where they touch are identical. So the gradient vectors are parallel; that is, $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some scalar λ.

This kind of argument also applies to the problem of finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$. Thus the point (x, y, z) is restricted to lie on the level surface S with equation $g(x, y, z) = k$. Instead of the level curves in Figure 1, we consider the level surfaces $f(x, y, z) = c$ and argue that if the maximum value of f is $f(x_0, y_0, z_0) = c$, then the level surface $f(x, y, z) = c$ is tangent to the level surface $g(x, y, z) = k$ and so the corresponding gradient vectors are parallel.

This intuitive argument can be made precise as follows. Suppose that a function f has an extreme value at a point $P(x_0, y_0, z_0)$ on the surface S and let C be a curve with vector equation $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ that lies on S and passes through P. If t_0 is the parameter value corresponding to the point P, then $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$. The composite function $h(t) = f(x(t), y(t), z(t))$ represents the values that f takes on the curve C. Since f has an extreme value at (x_0, y_0, z_0), it follows that h has an extreme value at t_0, so $h'(t_0) = 0$. But if f is differentiable, we can use the Chain Rule to write

$$\begin{aligned} 0 &= h'(t_0) \\ &= f_x(x_0, y_0, z_0)x'(t_0) + f_y(x_0, y_0, z_0)y'(t_0) + f_z(x_0, y_0, z_0)z'(t_0) \\ &= \nabla f(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) \end{aligned}$$

This shows that the gradient vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the tangent vector $\mathbf{r}'(t_0)$ to every such curve C. But we already know from Section 14.6 that the gradient vector of g, $\nabla g(x_0, y_0, z_0)$, is also orthogonal to $\mathbf{r}'(t_0)$ for every such curve (see Equation 14.6.18). This means that the gradient vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ must be parallel. Therefore, if $\nabla g(x_0, y_0, z_0) \neq \mathbf{0}$, there is a number λ such that

$$\boxed{\nabla f(x_0, y_0, z_0) = \lambda \, \nabla g(x_0, y_0, z_0)} \qquad \boxed{1}$$

Lagrange multipliers are named after the French-Italian mathematician Joseph-Louis Lagrange (1736–1813). See Section 4.2 for a biographical sketch of Lagrange.

The number λ in Equation 1 is called a **Lagrange multiplier**. The procedure based on Equation 1 is as follows.

Method of Lagrange Multipliers To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

1. Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \, \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

2. Evaluate f at all the points (x, y, z) that result from step 1. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

In deriving Lagrange's method we assumed that $\nabla g \neq \mathbf{0}$. In each of our examples you can check that $\nabla g \neq \mathbf{0}$ at all points where $g(x, y, z) = k$. See Exercise 35 for what can go wrong if $\nabla g = \mathbf{0}$. Exercise 34 shows what can happen if ∇g is undefined.

If we write the vector equation $\nabla f = \lambda \, \nabla g$ in terms of components, then the equations in step 1 become

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z \qquad g(x, y, z) = k$$

This is a system of four equations in the four unknowns x, y, z, and λ, and we must find *all* possible solutions (although the explicit values of λ are not needed for the conclusion of the method). If $x = x_0$, $y = y_0$, $z = z_0$ is a solution to this system of equations and the corresponding value of λ is not 0, then $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ are parallel (as we argued geometrically at the beginning of the section). If the value of λ is 0, then $\nabla f(x_0, y_0, z_0) = \mathbf{0}$ and so (x_0, y_0, z_0) is a critical point of f. It follows that $f(x_0, y_0, z_0)$ is a possible local extreme value of f on its domain, and hence also a possible extreme value of f subject to the given constraint (see Exercise 61).

For functions of two variables the method of Lagrange multipliers is similar to the method just described. To find the extreme values of $f(x, y)$ subject to the constraint $g(x, y) = k$, we look for values of x, y, and λ such that

$$\nabla f(x, y) = \lambda \, \nabla g(x, y) \qquad \text{and} \qquad g(x, y) = k$$

This amounts to solving three equations in three unknowns:

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = k$$

EXAMPLE 1 Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

SOLUTION We are asked for the extreme values of f subject to the constraint $g(x, y) = x^2 + y^2 = 1$. Using Lagrange multipliers, we solve the equations $\nabla f = \lambda \, \nabla g$

and $g(x, y) = 1$, which can be written as

$$f_x = \lambda\, g_x \qquad f_y = \lambda\, g_y \qquad g(x, y) = 1$$

or as

$$\boxed{2} \qquad 2x = 2x\lambda$$

$$\boxed{3} \qquad 4y = 2y\lambda$$

$$\boxed{4} \qquad x^2 + y^2 = 1$$

From (2) we have $2x(1 - \lambda) = 0$, so $x = 0$ or $\lambda = 1$. If $x = 0$, then (4) gives $y = \pm 1$. If $\lambda = 1$, then $y = 0$ from (3), so then (4) gives $x = \pm 1$. Therefore f has possible extreme values at the points $(0, 1)$, $(0, -1)$, $(1, 0)$, and $(-1, 0)$. Evaluating f at these four points, we find that

$$f(0, 1) = 2 \qquad f(0, -1) = 2 \qquad f(1, 0) = 1 \qquad f(-1, 0) = 1$$

Therefore the maximum value of f on the circle $x^2 + y^2 = 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(\pm 1, 0) = 1$. In geometric terms, these correspond to the highest and lowest points on the curve C in Figure 2, where C consists of those points on the paraboloid $z = x^2 + 2y^2$ that are directly above the constraint circle $x^2 + y^2 = 1$.

Figure 3 shows a contour map of f. The extreme values of $f(x, y) = x^2 + 2y^2$ correspond to the level curves of f that just touch the circle $x^2 + y^2 = 1$.

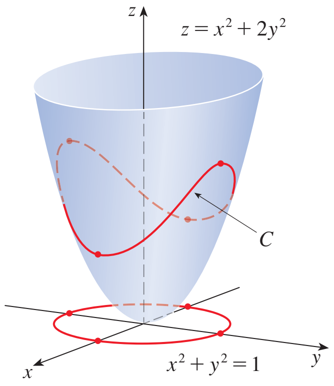


FIGURE 2

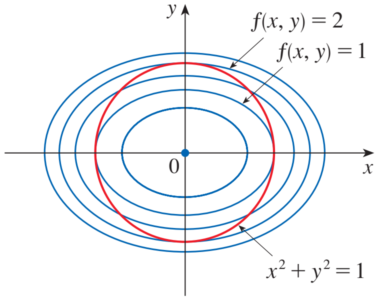


FIGURE 3 ■

Our next illustration of Lagrange's method is to reconsider the problem given in Example 14.7.6.

Many of the optimization problems that we encountered in Section 4.7 can be viewed as optimizing a function of two variables subject to a constraint. In Exercises 17–22 you are asked to revisit several problems from Section 4.7 and solve them using the method of Lagrange multipliers.

EXAMPLE 2 A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

SOLUTION As in Example 14.7.6, we let x, y, and z be the length, width, and height, respectively, of the box in meters. Then we wish to maximize

$$V = xyz$$

subject to the constraint

$$g(x, y, z) = 2xz + 2yz + xy = 12$$

Using the method of Lagrange multipliers, we look for values of x, y, z, and λ such that $\nabla V = \lambda \nabla g$ and $g(x, y, z) = 12$. This gives the equations

$$V_x = \lambda g_x$$

$$V_y = \lambda g_y$$

$$V_z = \lambda g_z$$

$$2xz + 2yz + xy = 12$$

which become

$$\boxed{5} \qquad yz = \lambda(2z + y)$$

$$\boxed{6} \qquad xz = \lambda(2z + x)$$

$$\boxed{7} \qquad xy = \lambda(2x + 2y)$$

$$\boxed{8} \qquad 2xz + 2yz + xy = 12$$

There are no general rules for solving systems of equations. Sometimes some ingenuity is required. In the present example you might notice that if we multiply (5) by x, (6) by y, and (7) by z, then the left sides of these equations will be identical. Doing this, we have

Another method for solving the system of equations (5–8) is to solve each of Equations 5, 6, and 7 for λ and then to equate the resulting expressions.

$$\boxed{9} \qquad xyz = \lambda(2xz + xy)$$

$$\boxed{10} \qquad xyz = \lambda(2yz + xy)$$

$$\boxed{11} \qquad xyz = \lambda(2xz + 2yz)$$

In general λ can be 0, but here we observe that $\lambda \neq 0$ because $\lambda = 0$ would imply $yz = xz = xy = 0$ from (5), (6), and (7) and this would contradict (8). Therefore, from (9) and (10), we have

$$2xz + xy = 2yz + xy$$

which gives $xz = yz$. But $z \neq 0$ (since $z = 0$ would give $V = 0$), so $x = y$. From (10) and (11) we have

$$2yz + xy = 2xz + 2yz$$

which gives $2xz = xy$ and so (since $x \neq 0$) $y = 2z$. If we now put $x = y = 2z$ in (8), we get

$$4z^2 + 4z^2 + 4z^2 = 12$$

Since x, y, and z are all positive, we therefore have $z = 1$ and so $x = 2$ and $y = 2$. Thus we have only one point where f may have an extreme value; how do we know if this point corresponds to a maximum or minimum? As in Example 14.7.6, we argue that there must be a maximum volume, which must occur at the point we found. ■

EXAMPLE 3 Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

SOLUTION The distance from a point (x, y, z) to the point $(3, 1, -1)$ is

$$d = \sqrt{(x - 3)^2 + (y - 1)^2 + (z + 1)^2}$$

but the algebra is simpler if we instead maximize and minimize the square of the distance:

$$d^2 = f(x, y, z) = (x - 3)^2 + (y - 1)^2 + (z + 1)^2$$

The constraint is that the point (x, y, z) lies on the sphere, that is,

$$g(x, y, z) = x^2 + y^2 + z^2 = 4$$

According to the method of Lagrange multipliers, we solve $\nabla f = \lambda \nabla g$, $g = 4$. This gives

$$\boxed{12} \qquad 2(x - 3) = 2x\lambda$$

$$\boxed{13} \qquad 2(y - 1) = 2y\lambda$$

$$\boxed{14} \qquad 2(z + 1) = 2z\lambda$$

$$\boxed{15} \qquad x^2 + y^2 + z^2 = 4$$

The simplest way to solve these equations is to solve for x, y, and z in terms of λ from (12), (13), and (14), and then substitute these values into (15). From (12) we have

$$x - 3 = x\lambda \quad \Longrightarrow \quad x(1 - \lambda) = 3 \quad \Longrightarrow \quad x = \frac{3}{1 - \lambda}$$

[Note that $1 - \lambda \neq 0$ because $\lambda = 1$ is impossible from (12).] Similarly, (13) and (14) give

$$y = \frac{1}{1 - \lambda} \qquad z = -\frac{1}{1 - \lambda}$$

Figure 4 shows the sphere and the nearest point P in Example 3. Can you see how to find the coordinates of P without using calculus?

Therefore, from (15), we have

$$\frac{3^2}{(1 - \lambda)^2} + \frac{1^2}{(1 - \lambda)^2} + \frac{(-1)^2}{(1 - \lambda)^2} = 4$$

which gives $(1 - \lambda)^2 = \frac{11}{4}$, $1 - \lambda = \pm\sqrt{11}/2$, so

$$\lambda = 1 \pm \frac{\sqrt{11}}{2}$$

These values of λ then give the corresponding points (x, y, z):

$$\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right) \quad \text{and} \quad \left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$$

It's easy to see that f has a smaller value at the first of these points, so the closest point is $\left(6/\sqrt{11}, 2/\sqrt{11}, -2/\sqrt{11}\right)$ and the farthest is $\left(-6/\sqrt{11}, -2/\sqrt{11}, 2/\sqrt{11}\right)$. ■

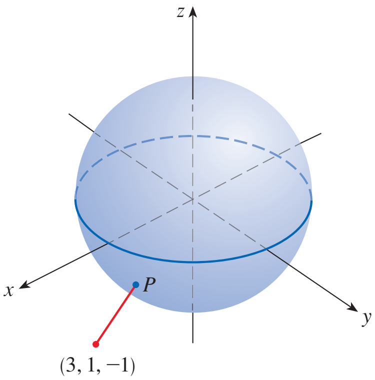


FIGURE 4

EXAMPLE 4 Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the disk $D = \{(x, y) \mid x^2 + y^2 \leqslant 1\}$.

SOLUTION According to the procedure in (14.7.9), we compare the values of f at the critical points in D with the extreme values of f on the boundary of D. Since $f_x = 2x$ and $f_y = 4y$, the only critical point is $(0, 0)$. We compare the value of f at that point with the extreme values on the boundary that we found in Example 1 using Lagrange multipliers:

$$f(0, 0) = 0 \qquad f(\pm 1, 0) = 1 \qquad f(0, \pm 1) = 2$$

Therefore the maximum value of f on D is $f(0, \pm 1) = 2$ and the minimum value is $f(0, 0) = 0$. Figure 5 shows the portion of the graph of f above the disk D. You can see that the highest point on the surface occurs at $(0, \pm 1)$ and the lowest point is at the origin. Figure 6 shows a contour map of f superimposed on the disk D.

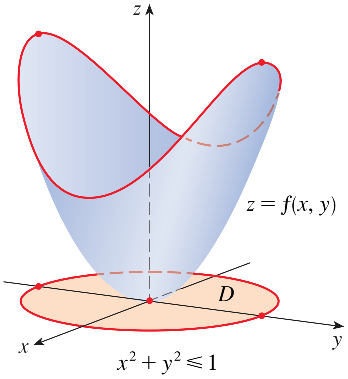


FIGURE 5

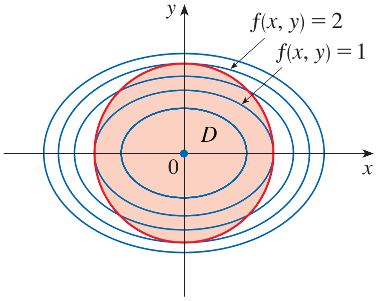


FIGURE 6 ■

■ Lagrange Multipliers: Two Constraints

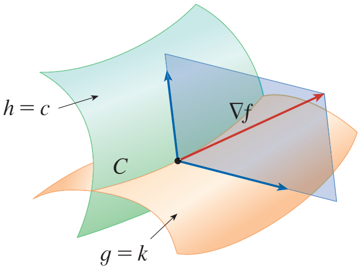


FIGURE 7

Suppose now that we want to find the maximum and minimum values of a function $f(x, y, z)$ subject to two constraints (side conditions) of the form $g(x, y, z) = k$ and $h(x, y, z) = c$. Geometrically, this means that we are looking for the extreme values of f when (x, y, z) is restricted to lie on the curve of intersection C of the level surfaces $g(x, y, z) = k$ and $h(x, y, z) = c$. (See Figure 7.) Suppose f has such an extreme value at a point $P(x_0, y_0, z_0)$. We know from the beginning of this section that ∇f is orthogonal to C at P. But we also know that ∇g is orthogonal to $g(x, y, z) = k$ and ∇h is orthogonal to $h(x, y, z) = c$, so ∇g and ∇h are both orthogonal to C. This means that the gradient vector $\nabla f(x_0, y_0, z_0)$ is in the plane determined by $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$. (We assume that these gradient vectors are not zero and not parallel.) So there are numbers λ and μ (both called Lagrange multipliers) such that

$$\boxed{16} \qquad \boxed{\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)}$$

In this case Lagrange's method is to look for extreme values by solving five equations in the five unknowns x, y, z, λ, and μ. These equations are obtained by writing Equation 16 in terms of its components and using the constraint equations:

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

$$g(x, y, z) = k$$

$$h(x, y, z) = c$$

The cylinder $x^2 + y^2 = 1$ intersects the plane $x - y + z = 1$ in an ellipse (Figure 8). Example 5 asks for the maximum value of f when (x, y, z) is restricted to lie on the ellipse.

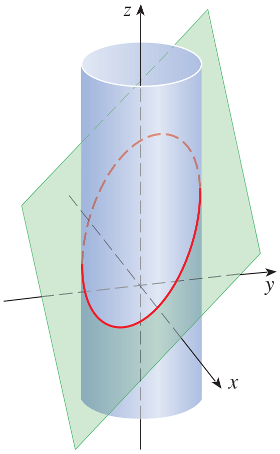


FIGURE 8

EXAMPLE 5 Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

SOLUTION We maximize the function $f(x, y, z) = x + 2y + 3z$ subject to the constraints $g(x, y, z) = x - y + z = 1$ and $h(x, y, z) = x^2 + y^2 = 1$. The Lagrange condition is $\nabla f = \lambda \nabla g + \mu \nabla h$, so we solve the equations

$$\boxed{17} \qquad 1 = \lambda + 2x\mu$$

$$\boxed{18} \qquad 2 = -\lambda + 2y\mu$$

$$\boxed{19} \qquad 3 = \lambda$$

$$\boxed{20} \qquad x - y + z = 1$$

$$\boxed{21} \qquad x^2 + y^2 = 1$$

Putting $\lambda = 3$ [from (19)] in (17), we get $2x\mu = -2$, so $x = -1/\mu$. Similarly, (18) gives $y = 5/(2\mu)$. Substitution in (21) then gives

$$\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1$$

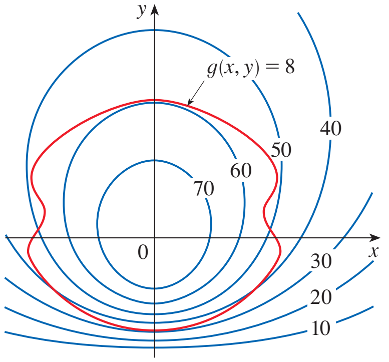
and so $\mu^2 = \frac{29}{4}$, $\mu = \pm\sqrt{29}/2$. Then $x = \mp 2/\sqrt{29}$, $y = \pm 5/\sqrt{29}$, and, from (20), $z = 1 - x + y = 1 \pm 7/\sqrt{29}$. The corresponding values of f are

$$\mp\frac{2}{\sqrt{29}} + 2\left(\pm\frac{5}{\sqrt{29}}\right) + 3\left(1 \pm \frac{7}{\sqrt{29}}\right) = 3 \pm \sqrt{29}$$

Therefore the maximum value of f on the given curve is $3 + \sqrt{29}$. ■

14.8 Exercises

1. Pictured are a contour map of f and a curve with equation $g(x, y) = 8$. Estimate the maximum and minimum values of f subject to the constraint that $g(x, y) = 8$. Explain your reasoning.



2. (a) Use a graphing calculator or computer to graph the circle $x^2 + y^2 = 1$. On the same screen, graph several curves of the form $x^2 + y = c$ until you find two that just touch the circle. What is the significance of the values of c for these two curves?

(b) Use Lagrange multipliers to find the extreme values of $f(x, y) = x^2 + y$ subject to the constraint $x^2 + y^2 = 1$. Compare your answers with those in part (a).

3–16 Each of these extreme value problems has a solution with both a maximum value and a minimum value. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint.

3. $f(x, y) = x^2 - y^2, \quad x^2 + y^2 = 1$

4. $f(x, y) = x^2y, \quad x^2 + y^4 = 5$

5. $f(x, y) = xy, \quad 4x^2 + y^2 = 8$

6. $f(x, y) = xe^y, \quad x^2 + y^2 = 2$

7. $f(x, y) = 2x^2 + 6y^2, \quad x^4 + 3y^4 = 1$

8. $f(x, y) = xye^{-x^2-y^2}, \quad 2x - y = 0$

9. $f(x, y, z) = 2x + 2y + z, \quad x^2 + y^2 + z^2 = 9$

10. $f(x, y, z) = e^{xyz}, \quad 2x^2 + y^2 + z^2 = 24$

11. $f(x, y, z) = xy^2z, \quad x^2 + y^2 + z^2 = 4$

12. $f(x, y, z) = x^2 + y^2 + z^2, \quad x^2 + y^2 + z^2 + xy = 12$

13. $f(x, y, z) = x^2 + y^2 + z^2, \quad x^4 + y^4 + z^4 = 1$

14. $f(x, y, z) = x^4 + y^4 + z^4, \quad x^2 + y^2 + z^2 = 1$

15. $f(x, y, z, t) = x + y + z + t, \quad x^2 + y^2 + z^2 + t^2 = 1$

16. $f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \cdots + x_n,$
$x_1^2 + x_2^2 + \cdots + x_n^2 = 1$

17–22 Use Lagrange multipliers to give an alternate solution to the indicated exercise in Section 4.7.

17. Exercise 3

18. Exercise 8

19. Exercise 7

20. Exercise 18

21. Exercise 25

22. Exercise 24

23–24 The method of Lagrange multipliers assumes that the extreme values exist, but that is not always the case. Show that the problem of finding the minimum value of f subject to the given constraint can be solved using Lagrange multipliers, but f does not have a maximum value with that constraint.

23. $f(x, y) = x^2 + y^2, \quad xy = 1$

24. $f(x, y, z) = x^2 + 2y^2 + 3z^2, \quad x + 2y + 3z = 10$

25–26 Use Lagrange multipliers to find the maximum value of f subject to the given constraint. Then show that f has no minimum value with that constraint.

25. $f(x, y) = e^{xy}, \quad x^3 + y^3 = 16$

26. $f(x, y, z) = 4x + 2y + z, \quad x^2 + y + z^2 = 1$

27–29 Find the extreme values of f on the region described by the inequality.

27. $f(x, y) = x^2 + y^2 + 4x - 4y, \quad x^2 + y^2 \leqslant 9$

28. $f(x, y) = 2x^2 + 3y^2 - 4x - 5, \quad x^2 + y^2 \leqslant 16$

29. $f(x, y) = e^{-xy}, \quad x^2 + 4y^2 \leqslant 1$

30–33 Find the extreme values of f subject to both constraints.

30. $f(x, y, z) = z; \quad x^2 + y^2 = z^2, \quad x + y + z = 24$

31. $f(x, y, z) = x + y + z; \quad x^2 + z^2 = 2, \quad x + y = 1$

32. $f(x, y, z) = x^2 + y^2 + z^2; \quad x - y = 1, \quad y^2 - z^2 = 1$

33. $f(x, y, z) = yz + xy; \quad xy = 1, \quad y^2 + z^2 = 1$

34. Consider the problem of maximizing the function $f(x, y) = 2x + 3y$ subject to the constraint $\sqrt{x} + \sqrt{y} = 5$.
(a) Try using Lagrange multipliers to solve the problem.
(b) Does $f(25, 0)$ give a larger value than the one in part (a)?
(c) Solve the problem by graphing the constraint equation and several level curves of f.
(d) Explain why the method of Lagrange multipliers fails to solve the problem.
(e) What is the significance of $f(9, 4)$?

35. Consider the problem of minimizing the function $f(x, y) = x$ on the curve $y^2 + x^4 - x^3 = 0$ (a piriform).
(a) Try using Lagrange multipliers to solve the problem.
(b) Show that the minimum value is $f(0, 0) = 0$ but the Lagrange condition $\nabla f(0, 0) = \lambda \nabla g(0, 0)$ is not satisfied for any value of λ.
(c) Explain why Lagrange multipliers fail to find the minimum value in this case.

36. (a) Use software that plots implicitly defined curves to estimate the minimum and maximum values of $f(x, y) = x^3 + y^3 + 3xy$ subject to the constraint $(x - 3)^2 + (y - 3)^2 = 9$ by graphical methods.
(b) Solve the problem in part (a) with the aid of Lagrange multipliers. You will need to solve the equations numerically. Compare your answers with those in part (a).

37. The total production P of a certain product depends on the amount L of labor used and the amount K of capital investment. In Section 14.1 and the project following Section 14.3 we discussed how the Cobb-Douglas model $P = bL^{\alpha}K^{1-\alpha}$ follows from certain economic assumptions, where b and α are positive constants and $\alpha < 1$. If the cost of a unit of labor is m and the cost of a unit of capital is n, and the company can spend only p dollars as its total budget, then maximizing the production P is subject to the constraint $mL + nK = p$. Show that the maximum production occurs when

$$L = \frac{\alpha p}{m} \qquad \text{and} \qquad K = \frac{(1 - \alpha)p}{n}$$

38. Referring to Exercise 37, we now suppose that the production is fixed at $bL^{\alpha}K^{1-\alpha} = Q$, where Q is a constant. What values of L and K minimize the cost function $C(L, K) = mL + nK$?

39. Use Lagrange multipliers to prove that the rectangle with maximum area that has a given perimeter p is a square.

40. Use Lagrange multipliers to prove that the triangle with maximum area that has a given perimeter p is equilateral.
Hint: Use Heron's formula for the area:

$$A = \sqrt{s(s - x)(s - y)(s - z)}$$

where $s = p/2$ and x, y, z are the lengths of the sides.

41–53 Use Lagrange multipliers to give an alternate solution to the indicated exercise in Section 14.7.

41. Exercise 43
42. Exercise 44
43. Exercise 45
44. Exercise 46
45. Exercise 47
46. Exercise 48
47. Exercise 49
48. Exercise 50
49. Exercise 51
50. Exercise 52
51. Exercise 53
52. Exercise 54
53. Exercise 57

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54. A package in the shape of a rectangular box can be mailed by the US Postal Service if the sum of its length and girth (the perimeter of a cross-section perpendicular to the length; see Exercise 4.7.23) is at most 108 inches. Use Lagrange multipliers to find the dimensions of the package with largest volume that can be mailed.

55. A grain silo is to be built by attaching a hemispherical roof and a flat floor onto a circular cylinder. Use Lagrange multipliers to show that for a total surface area S, the volume of the silo is maximized when the radius and height of the cylinder are equal.

56. Find the maximum and minimum volumes of a rectangular box whose surface area is 1500 cm^2 and whose total edge length is 200 cm.

57. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.

58. The plane $4x - 3y + 8z = 5$ intersects the cone $z^2 = x^2 + y^2$ in an ellipse.

(a) Graph the cone and the plane, and observe the elliptical intersection.

(b) Use Lagrange multipliers to find the highest and lowest points on the ellipse.

59–60 Find the maximum and minimum values of f subject to the given constraints. Use a computer algebra system to solve the system of equations that arises in using Lagrange multipliers. (If your CAS finds only one solution, you may need to use additional commands.)

59. $f(x, y, z) = ye^{x-z}$; $\quad 9x^2 + 4y^2 + 36z^2 = 36,\; xy + yz = 1$

60. $f(x, y, z) = x + y + z$; $\quad x^2 - y^2 = z,\; x^2 + z^2 = 4$

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61. Use Lagrange multipliers to find the extreme values of $f(x, y) = 3x^2 + y^2$ subject to the constraint $x^2 + y^2 = 4y$. Show that the minimum value corresponds to $\lambda = 0$.

62. (a) Maximize $\sum_{i=1}^{n} x_i y_i$ subject to the constraints $\sum_{i=1}^{n} x_i^2 = 1$ and $\sum_{i=1}^{n} y_i^2 = 1$.

(b) Put

$$x_i = \frac{a_i}{\sqrt{\sum a_j^2}} \quad \text{and} \quad y_i = \frac{b_i}{\sqrt{\sum b_j^2}}$$

to show that

$$\sum a_i b_i \leqslant \sqrt{\sum a_j^2}\,\sqrt{\sum b_j^2}$$

for any numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$. This inequality is known as the *Cauchy-Schwarz Inequality*.

63. (a) Find the maximum value of

$$f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$$

given that $x_1, x_2, \ldots, x_n$ are positive numbers and $x_1 + x_2 + \cdots + x_n = c$, where c is a constant.

(b) Deduce from part (a) that if $x_1, x_2, \ldots, x_n$ are positive numbers, then

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leqslant \frac{x_1 + x_2 + \cdots + x_n}{n}$$

This inequality says that the geometric mean of n numbers is no larger than the arithmetic mean of the numbers. Under what circumstances are these two means equal?

APPLIED PROJECT | ROCKET SCIENCE

NASA

Many rockets—such as the *Saturn V* that first put men on the moon—are designed to use three stages in their ascent into space. A large first stage initially propels the rocket until its fuel is consumed, at which point the stage is jettisoned to reduce the mass of the rocket. The smaller second and third stages function similarly in order to place the rocket's payload into orbit about the earth. (With this design, at least two stages are required in order to reach the necessary velocities, and using three stages has proven to be a good compromise between cost and performance.) Our goal here is to determine the individual masses of the three stages, which are to be designed to minimize the total mass of the rocket while enabling it to reach a desired velocity.

NASA/Lori Losey

For a single-stage rocket consuming fuel at a constant rate, the change in velocity resulting from the acceleration of the rocket vehicle has been modeled by

$$\Delta V = -c \ln\left(1 - \frac{(1 - S)M_r}{P + M_r}\right)$$

where M_r is the mass of the rocket engine including initial fuel, P is the mass of the payload, S is a *structural factor* determined by the design of the rocket (specifically, it is the ratio of the mass of the rocket vehicle without fuel to the total mass of the rocket with fuel), and c is the (constant) speed of exhaust relative to the rocket.

Now consider a rocket with three stages and a payload of mass A. Assume that outside forces are negligible and that c and S remain constant for each stage. If M_i is the mass of the ith stage, we can initially consider the rocket engine to have mass M_1 and its payload to have mass $M_2 + M_3 + A$; the second and third stages can be handled similarly.

1. Show that the velocity attained by the rocket after all three stages have been jettisoned is given by

$$v_f = c\left[\ln\left(\frac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}\right) + \ln\left(\frac{M_2 + M_3 + A}{SM_2 + M_3 + A}\right) + \ln\left(\frac{M_3 + A}{SM_3 + A}\right)\right]$$

2. We wish to minimize the total mass $M = M_1 + M_2 + M_3$ of the rocket engine subject to the constraint that the desired velocity v_f from Problem 1 is attained. The method of Lagrange multipliers is appropriate here, but difficult to implement using the current expressions. To simplify, we define variables N_i so that the constraint equation may be expressed as $v_f = c(\ln N_1 + \ln N_2 + \ln N_3)$. Since M is now difficult to express in terms of the N_i's, we wish to use a simpler function that will be minimized at the same place as M. Show that

$$\frac{M_1 + M_2 + M_3 + A}{M_2 + M_3 + A} = \frac{(1 - S)N_1}{1 - SN_1}$$

$$\frac{M_2 + M_3 + A}{M_3 + A} = \frac{(1 - S)N_2}{1 - SN_2}$$

$$\frac{M_3 + A}{A} = \frac{(1 - S)N_3}{1 - SN_3}$$

and conclude that

$$\frac{M + A}{A} = \frac{(1 - S)^3 N_1 N_2 N_3}{(1 - SN_1)(1 - SN_2)(1 - SN_3)}$$

3. Verify that $\ln((M + A)/A)$ is minimized at the same location as M; use Lagrange multipliers and the results of Problem 2 to find expressions for the values of N_i where the minimum occurs subject to the constraint $v_f = c(\ln N_1 + \ln N_2 + \ln N_3)$. [*Hint:* Use properties of logarithms to help simplify the expressions.]

4. Find an expression for the minimum value of M as a function of v_f.

5. If we want to put a three-stage rocket into orbit 160 km above the earth's surface, a final velocity of approximately 28,000 km/h is required. Suppose that each stage is built with a structural factor $S = 0.2$ and an exhaust speed of $c = 9600$ km/h.
   (a) Find the minimum total mass M of the rocket engines as a function of A.
   (b) Find the mass of each individual stage as a function of A. (They are not equally sized.)

6. The same rocket would require a final velocity of approximately 39,700 km/h in order to escape earth's gravity. Find the mass of each individual stage that would minimize the total mass of the rocket engines and allow the rocket to propel a 200-kg probe into deep space.

APPLIED PROJECT HYDRO-TURBINE OPTIMIZATION

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At a hydroelectric generating station, water is piped from a dam to the power station. The rate at which the water flows through the pipe varies, depending on external conditions.

The power station has three different hydroelectric turbines, each with a known (and unique) power function that gives the amount of electric power generated as a function of the water flow arriving at the turbine. The incoming water can be apportioned in different volumes to each turbine, so the goal of this project is to determine how to distribute water among the turbines to give the maximum total energy production for any rate of flow.

Using experimental evidence and *Bernoulli's equation*, the following quadratic models were determined for the power output of each turbine, along with the allowable flows of operation:

$$KW_1 = (-18.89 + 0.1277Q_1 - 4.08 \cdot 10^{-5}Q_1^2)(170 - 1.6 \cdot 10^{-6}Q_T^2)$$
$$KW_2 = (-24.51 + 0.1358Q_2 - 4.69 \cdot 10^{-5}Q_2^2)(170 - 1.6 \cdot 10^{-6}Q_T^2)$$
$$KW_3 = (-27.02 + 0.1380Q_3 - 3.84 \cdot 10^{-5}Q_3^2)(170 - 1.6 \cdot 10^{-6}Q_T^2)$$

$$250 \leqslant Q_1 \leqslant 1110, \quad 250 \leqslant Q_2 \leqslant 1110, \quad 250 \leqslant Q_3 \leqslant 1225$$

where

Q_i = flow through turbine i in cubic meters per second

KW_i = power generated by turbine i in kilowatts

Q_T = total flow through the station in cubic meters per second

1. If all three turbines are being used, we wish to determine the flow Q_i to each turbine that will give the maximum total energy production. Our limitations are that the flows must sum to the total incoming flow and the given domain restrictions must be observed. Consequently, use Lagrange multipliers to find the values for the individual flows (as functions of Q_T) that maximize the total energy production

   $$KW_1 + KW_2 + KW_3$$

   subject to the constraints

   $$Q_1 + Q_2 + Q_3 = Q_T$$

   and the domain restrictions on each Q_i.

2. For which values of Q_T is your result valid?

3. For an incoming flow of 70 m^3/s, determine the distribution to the turbines and verify (by trying some nearby distributions) that your result is indeed a maximum.

4. Until now we have assumed that all three turbines are operating; is it possible in some situations that more power could be produced by using only one turbine? Make a graph of the three power functions and use it to help decide if an incoming flow of 30 m^3/s should be distributed to all three turbines or routed to just one. (If you determine that only one turbine should be used, which one would it be?) What if the flow is only 17 m^3/s?

5. Perhaps for some flow levels it would be advantageous to use two turbines. If the incoming flow is 40 m^3/s, which two turbines would you recommend using? Use Lagrange multi-pliers to determine how the flow should be distributed between the two turbines to maximize the energy produced. For this flow, is using two turbines more efficient than using all three?

6. If the incoming flow is 96 m^3/s, what distribution would you recommend to the station management?

14 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. (a) What is a function of two variables?
(b) Describe three methods for visualizing a function of two variables.

2. What is a function of three variables? How can you visualize such a function?

3. What does

$$\lim_{(x, y)\to(a, b)} f(x, y) = L$$

mean? How can you show that such a limit does not exist?

4. (a) What does it mean to say that f is continuous at (a, b)?
(b) If f is continuous on $\mathbb{R}^2$, what can you say about its graph?

5. (a) Write expressions for the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ as limits.
(b) How do you interpret $f_x(a, b)$ and $f_y(a, b)$ geometrically? How do you interpret them as rates of change?
(c) If $f(x, y)$ is given by a formula, how do you calculate f_x and f_y?

6. What does Clairaut's Theorem say?

7. How do you find a tangent plane to each of the following types of surfaces?
(a) A graph of a function of two variables, $z = f(x, y)$
(b) A level surface of a function of three variables, $F(x, y, z) = k$

8. Define the linearization of f at (a, b). What is the corresponding linear approximation? What is the geometric interpretation of the linear approximation?

9. (a) What does it mean to say that f is differentiable at (a, b)?
(b) How do you usually verify that f is differentiable?

10. If $z = f(x, y)$, what are the differentials dx, dy, and dz?

11. State the Chain Rule for the case where $z = f(x, y)$ and x and y are functions of one variable. What if x and y are functions of two variables?

12. If z is defined implicitly as a function of x and y by an equation of the form $F(x, y, z) = 0$, how do you find $\partial z/\partial x$ and $\partial z/\partial y$?

13. (a) Write an expression as a limit for the directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$. How do you interpret it as a rate? How do you interpret it geometrically?
(b) If f is differentiable, write an expression for $D_{\mathbf{u}} f(x_0, y_0)$ in terms of f_x and f_y.

14. (a) Define the gradient vector ∇f for a function f of two or three variables.
(b) Express $D_{\mathbf{u}} f$ in terms of ∇f.
(c) Explain the geometric significance of the gradient.

15. What do the following statements mean?
(a) f has a local maximum at (a, b).
(b) f has an absolute maximum at (a, b).
(c) f has a local minimum at (a, b).
(d) f has an absolute minimum at (a, b).
(e) f has a saddle point at (a, b).

16. (a) If f has a local maximum at (a, b), what can you say about its partial derivatives at (a, b)?
(b) What is a critical point of f?

17. State the Second Derivatives Test.

18. (a) What is a closed set in $\mathbb{R}^2$? What is a bounded set?
(b) State the Extreme Value Theorem for functions of two variables.
(c) How do you find the values that the Extreme Value Theorem guarantees?

19. Explain how the method of Lagrange multipliers works in finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$. What if there is a second constraint $h(x, y, z) = c$?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. $f_y(a, b) = \lim_{y\to b} \dfrac{f(a, y) - f(a, b)}{y - b}$

2. There exists a function f with continuous second-order partial derivatives such that $f_x(x, y) = x + y^2$ and $f_y(x, y) = x - y^2$.

3. $f_{xy} = \dfrac{\partial^2 f}{\partial x\, \partial y}$

4. $D_{\mathbf{k}} f(x, y, z) = f_z(x, y, z)$

5. If $f(x, y) \to L$ as $(x, y) \to (a, b)$ along every straight line through (a, b), then $\lim_{(x, y)\to(a, b)} f(x, y) = L$.

6. If $f_x(a, b)$ and $f_y(a, b)$ both exist, then f is differentiable at (a, b).

7. If f has a local minimum at (a, b) and f is differentiable at (a, b), then $\nabla f(a, b) = \mathbf{0}$.

8. If f is a function, then

$$\lim_{(x, y)\to(2, 5)} f(x, y) = f(2, 5)$$

9. If $f(x, y) = \ln y$, then $\nabla f(x, y) = 1/y$.

10. If $(2, 1)$ is a critical point of f and

$$f_{xx}(2, 1)\, f_{yy}(2, 1) < [f_{xy}(2, 1)]^2$$

then f has a saddle point at $(2, 1)$.

11. If $f(x, y) = \sin x + \sin y$, then $-\sqrt{2} \leqslant D_{\mathbf{u}} f(x, y) \leqslant \sqrt{2}$.

12. If $f(x, y)$ has two local maxima, then f must have a local minimum.

EXERCISES

1–2 Find and sketch the domain of the function.

1. $f(x, y) = \ln(x + y + 1)$

2. $f(x, y) = \sqrt{4 - x^2 - y^2} + \sqrt{1 - x^2}$

3–4 Sketch the graph of the function.

3. $f(x, y) = 1 - y^2$

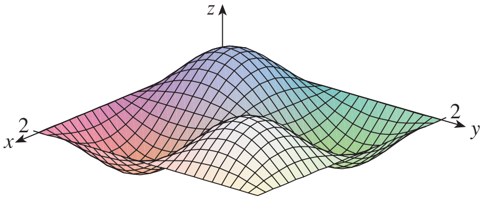
4. $f(x, y) = x^2 + (y - 2)^2$

5–6 Sketch several level curves of the function.

5. $f(x, y) = \sqrt{4x^2 + y^2}$

6. $f(x, y) = e^x + y$

7. Make a rough sketch of a contour map for the function whose graph is shown.

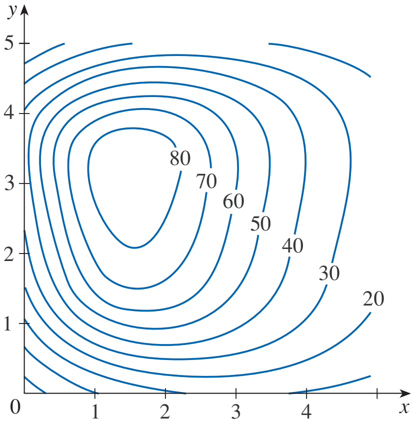


8. The contour map of a function f is shown.

(a) Estimate the value of $f(3, 2)$.

(b) Is $f_x(3, 2)$ positive or negative? Explain.

(c) Which is greater, $f_y(2, 1)$ or $f_y(2, 2)$? Explain.



9–10 Evaluate the limit or show that it does not exist.

9. $\displaystyle\lim_{(x, y)\to(1, 1)} \frac{2xy}{x^2 + 2y^2}$

10. $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{2xy}{x^2 + 2y^2}$

11. A metal plate is situated in the xy-plane and occupies the rectangle $0 \leqslant x \leqslant 10$, $0 \leqslant y \leqslant 8$, where x and y are measured in meters. The temperature at the point (x, y) in the plate is $T(x, y)$, where T is measured in degrees Celsius. Temperatures at equally spaced points were measured and recorded in the table.

(a) Estimate the values of the partial derivatives $T_x(6, 4)$ and $T_y(6, 4)$. What are the units?

(b) Estimate the value of $D_{\mathbf{u}} T(6, 4)$, where $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$. Interpret your result.

(c) Estimate the value of $T_{xy}(6, 4)$.

y x	0	2	4	6	8
0	30	38	45	51	55
2	52	56	60	62	61
4	78	74	72	68	66
6	98	87	80	75	71
8	96	90	86	80	75
10	92	92	91	87	78

12. Find a linear approximation to the temperature function $T(x, y)$ in Exercise 11 near the point $(6, 4)$. Then use it to estimate the temperature at the point $(5, 3.8)$.

13–17 Find the first partial derivatives.

13. $f(x, y) = (5y^3 + 2x^2y)^8$

14. $g(u, v) = \dfrac{u + 2v}{u^2 + v^2}$

15. $F(\alpha, \beta) = \alpha^2 \ln(\alpha^2 + \beta^2)$

16. $G(x, y, z) = e^{xz} \sin(y/z)$

17. $S(u, v, w) = u \arctan(v\sqrt{w})$

18. The speed of sound traveling through ocean water is a function of temperature, salinity, and pressure. It has been modeled by the function

$$C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S - 35) + 0.016D$$

where C is the speed of sound (in meters per second), T is the temperature (in degrees Celsius), S is the salinity (the concentration of salts in parts per thousand, which means the number of grams of dissolved solids per 1000 g of water), and D is the depth below the ocean surface (in meters). Compute $\partial C/\partial T$, $\partial C/\partial S$, and $\partial C/\partial D$ when $T = 10°\text{C}$, $S = 35$ parts per thousand, and $D = 100$ m. Explain the physical significance of these partial derivatives.

19–22 Find all second partial derivatives of f.

19. $f(x, y) = 4x^3 - xy^2$

20. $z = xe^{-2y}$

21. $f(x, y, z) = x^k y^l z^m$

22. $v = r\cos(s + 2t)$

23. If $z = xy + xe^{y/x}$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = xy + z$.

24. If $z = \sin(x + \sin t)$, show that

$$\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x\,\partial t} = \frac{\partial z}{\partial t}\frac{\partial^2 z}{\partial x^2}$$

25–29 Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

25. $z = 3x^2 - y^2 + 2x$, $(1, -2, 1)$

26. $z = e^x \cos y$, $(0, 0, 1)$

27. $x^2 + 2y^2 - 3z^2 = 3$, $(2, -1, 1)$

28. $xy + yz + zx = 3$, $(1, 1, 1)$

29. $\sin(xyz) = x + 2y + 3z$, $(2, -1, 0)$

30. Use a computer to graph the surface $z = x^2 + y^4$ and its tangent plane and normal line at $(1, 1, 2)$ on the same screen. Choose the domain and viewpoint so that you get a good view of all three objects.

31. Find the points on the hyperboloid

$$x^2 + 4y^2 - z^2 = 4$$

where the tangent plane is parallel to the plane

$$2x + 2y + z = 5$$

32. Find du if $u = \ln(1 + se^{2t})$.

33. Find the linear approximation of the function $f(x, y, z) = x^3\sqrt{y^2 + z^2}$ at the point $(2, 3, 4)$ and use it to estimate the number $(1.98)^3\sqrt{(3.01)^2 + (3.97)^2}$.

34. The two legs of a right triangle are measured as 5 m and 12 m with a possible error in measurement of at most 0.2 cm in each. Use differentials to estimate the maximum error in the calculated value of (a) the area of the triangle and (b) the length of the hypotenuse.

35. If $u = x^2y^3 + z^4$, where $x = p + 3p^2$, $y = pe^p$, and $z = p\sin p$, use the Chain Rule to find du/dp.

36. If $v = x^2\sin y + ye^{xy}$, where $x = s + 2t$ and $y = st$, use the Chain Rule to find $\partial v/\partial s$ and $\partial v/\partial t$ when $s = 0$ and $t = 1$.

37. Suppose $z = f(x, y)$, where $x = g(s, t)$, $y = h(s, t)$, $g(1, 2) = 3$, $g_s(1, 2) = -1$, $g_t(1, 2) = 4$, $h(1, 2) = 6$, $h_s(1, 2) = -5$, $h_t(1, 2) = 10$, $f_x(3, 6) = 7$, and $f_y(3, 6) = 8$. Find $\partial z/\partial s$ and $\partial z/\partial t$ when $s = 1$ and $t = 2$.

38. Use a tree diagram to write out the Chain Rule for the case where $w = f(t, u, v)$, $t = t(p, q, r, s)$, $u = u(p, q, r, s)$, and $v = v(p, q, r, s)$ are all differentiable functions.

39. If $z = y + f(x^2 - y^2)$, where f is differentiable, show that

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x$$

40. The length x of a side of a triangle is increasing at a rate of 3 in/s, the length y of another side is decreasing at a rate of 2 in/s, and the contained angle θ is increasing at a rate of 0.05 radian/s. How fast is the area of the triangle changing when $x = 40$ inches, $y = 50$ inches, and $\theta = \pi/6$?

41. If $z = f(u, v)$, where $u = xy$, $v = y/x$, and f has continuous second partial derivatives, show that

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = -4uv\frac{\partial^2 z}{\partial u\,\partial v} + 2v\frac{\partial z}{\partial v}$$

42. If $\cos(xyz) = 1 + x^2y^2 + z^2$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

43. Find the gradient of the function $f(x, y, z) = x^2e^{yz^2}$.

44. (a) When is the directional derivative of f a maximum?
(b) When is it a minimum?
(c) When is it 0?
(d) When is it half of its maximum value?

45–46 Find the directional derivative of f at the given point in the indicated direction.

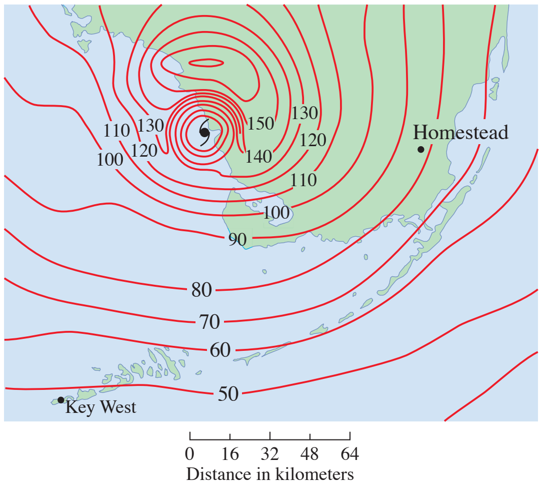
45. $f(x, y) = x^2e^{-y}$, $(-2, 0)$, in the direction toward the point $(2, -3)$

46. $f(x, y, z) = x^2y + x\sqrt{1 + z}$, $(1, 2, 3)$, in the direction of $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

47. Find the maximum rate of change of $f(x, y) = x^2y + \sqrt{y}$ at the point $(2, 1)$. In which direction does it occur?

48. Find the direction in which $f(x, y, z) = ze^{xy}$ increases most rapidly at the point $(0, 1, 2)$. What is the maximum rate of increase?

49. The contour map shows wind speed in km/h during Hurricane Andrew on August 24, 1992. Use it to estimate the value of the directional derivative of the wind speed at Homestead, Florida, in the direction of the eye of the hurricane.



50. Find parametric equations of the tangent line at the point $(-2, 2, 4)$ to the curve of intersection of the surface $z = 2x^2 - y^2$ and the plane $z = 4$.

51–54 Find the local maximum and minimum values and saddle points of the function. You are encouraged to graph the function with a domain and viewpoint that reveals all the important aspects of the function.

51. $f(x, y) = x^2 - xy + y^2 + 9x - 6y + 10$

52. $f(x, y) = x^3 - 6xy + 8y^3$

53. $f(x, y) = 3xy - x^2y - xy^2$

54. $f(x, y) = (x^2 + y)e^{y/2}$

55–56 Find the absolute maximum and minimum values of f on the set D.

55. $f(x, y) = 4xy^2 - x^2y^2 - xy^3$; D is the closed triangular region in the xy-plane with vertices $(0, 0)$, $(0, 6)$, and $(6, 0)$

56. $f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2)$; D is the disk $x^2 + y^2 \leqslant 4$

---

57. Use a graph or level curves or both to estimate the local maximum and minimum values and saddle points of $f(x, y) = x^3 - 3x + y^4 - 2y^2$. Then use calculus to find these values precisely.

58. Use a graphing calculator or computer (or Newton's method) to find the critical points of

$$f(x, y) = 12 + 10y - 2x^2 - 8xy - y^4$$

correct to three decimal places. Then classify the critical points and find the highest point on the graph.

59–62 Use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraint(s).

59. $f(x, y) = x^2y, \quad x^2 + y^2 = 1$

60. $f(x, y) = \dfrac{1}{x} + \dfrac{1}{y}, \quad \dfrac{1}{x^2} + \dfrac{1}{y^2} = 1$

61. $f(x, y, z) = xyz, \quad x^2 + y^2 + z^2 = 3$

62. $f(x, y, z) = x^2 + 2y^2 + 3z^2$;
$x + y + z = 1, \quad x - y + 2z = 2$

---

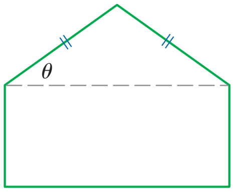
63. Find the points on the surface $xy^2z^3 = 2$ that are closest to the origin.

64. In this problem we identify a point (a, b) on the line $16x + 15y = 100$ such that the sum of the distances from $(-3, 0)$ to (a, b) and from (a, b) to $(3, 0)$ is a minimum.

(a) Write a function f that gives the sum of the distances from $(-3, 0)$ to a point (x, y) and from (x, y) to $(3, 0)$. Let $g(x, y) = 16x + 15y$. Following the method of Lagrange multipliers, we wish to find the minimum value of f subject to the constraint $g(x, y) = 100$. Graph the constraint curve along with several level curves of f, and then use the graph to estimate the minimum value of f. What point (a, b) on the line minimizes f?

(b) Verify that the gradient vectors $\nabla f(a, b)$ and $\nabla g(a, b)$ are parallel.

65. A pentagon is formed by placing an isosceles triangle on a rectangle, as shown in the figure. If the pentagon has fixed perimeter P, find the lengths of the sides of the pentagon that maximize the area of the pentagon.



Problems Plus

1. A rectangle with length L and width W is cut into four smaller rectangles by two lines parallel to the sides. Find the maximum and minimum values of the sum of the squares of the areas of the smaller rectangles.

2. Marine biologists have determined that when a shark detects the presence of blood in the water, it will swim in the direction in which the concentration of the blood increases most rapidly. Based on certain tests, the concentration of blood (in parts per million) at a point $P(x, y)$ on the surface of seawater is approximated by

$$C(x, y) = e^{-(x^2+2y^2)/10^4}$$

where x and y are measured in meters in a rectangular coordinate system with the blood source at the origin.

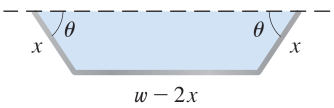
(a) Identify the level curves of the concentration function and sketch several members of this family together with a path that a shark will follow to the source.

(b) Suppose a shark is at the point (x_0, y_0) when it first detects the presence of blood in the water. Find an equation of the shark's path by setting up and solving a differential equation.

3. A long piece of galvanized sheet metal with width w is to be bent into a symmetric form with three straight sides to make a rain gutter. A cross-section is shown in the figure.

(a) Determine the dimensions that allow the maximum possible flow; that is, find the dimensions that give the maximum possible cross-sectional area.

(b) Would it be better to bend the metal into a gutter with a semicircular cross-section?



4. For what values of the number r is the function

$$f(x, y, z) = \begin{cases} \dfrac{(x + y + z)^r}{x^2 + y^2 + z^2} & \text{if } (x, y, z) \neq (0, 0, 0) \\ 0 & \text{if } (x, y, z) = (0, 0, 0) \end{cases}$$

continuous on $\mathbb{R}^3$?

5. Suppose f is a differentiable function of one variable. Show that all tangent planes to the surface $z = xf(y/x)$ intersect in a common point.

6. (a) Newton's method for approximating a solution of an equation $f(x) = 0$ (see Section 4.8) can be adapted to approximating a solution of a system of equations $f(x, y) = 0$ and $g(x, y) = 0$. The surfaces $z = f(x, y)$ and $z = g(x, y)$ intersect in a curve that intersects the xy-plane at the point (r, s), which is the solution of the system. If an initial approximation (x_1, y_1) is close to this point, then the tangent planes to the surfaces at (x_1, y_1) intersect in a straight line that intersects the xy-plane in a point (x_2, y_2), which should be closer to (r, s). (Compare with Figure 4.8.2.) Show that

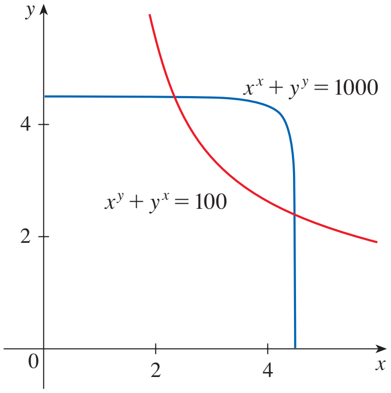
$$x_2 = x_1 - \frac{fg_y - f_y g}{f_x g_y - f_y g_x} \qquad \text{and} \qquad y_2 = y_1 - \frac{f_x g - fg_x}{f_x g_y - f_y g_x}$$

where f, g, and their partial derivatives are evaluated at (x_1, y_1). If we continue this procedure, we obtain successive approximations (x_n, y_n).

(b) It was Thomas Simpson (1710–1761) who formulated Newton's method as we know it today and who extended it to functions of two variables as in part (a). (See the biography of Simpson in Section 7.7.) The example that he gave to illustrate the method was to solve the system of equations

$$x^x + y^y = 1000 \qquad x^y + y^x = 100$$

In other words, he found the points of intersection of the curves in the figure. Use the method of part (a) to find the coordinates of the points of intersection correct to six decimal places.



7. If the ellipse $x^2/a^2 + y^2/b^2 = 1$ is to enclose the circle $x^2 + y^2 = 2y$, what values of a and b minimize the area of the ellipse?

8. Show that the maximum value of the function

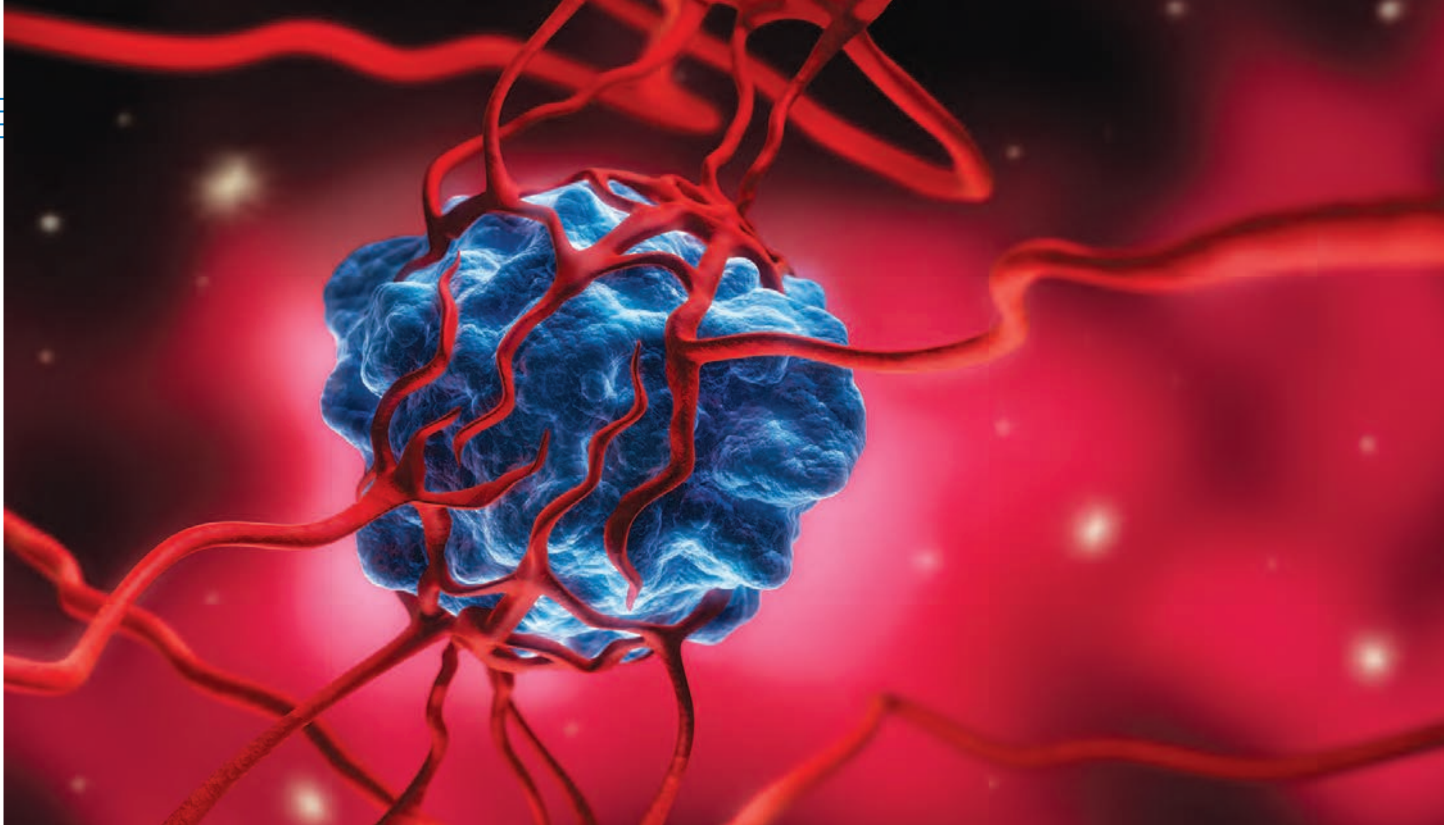
$$f(x, y) = \frac{(ax + by + c)^2}{x^2 + y^2 + 1}$$

is $a^2 + b^2 + c^2$.

Hint: One method for attacking this problem is to use the Cauchy-Schwarz Inequality:

$$|\mathbf{a} \cdot \mathbf{b}| \leqslant |\mathbf{a}||\mathbf{b}|$$

(See Exercise 12.3.61.)



Tumors, such as the one illustrated here, have been modeled as "bumpy spheres." In Exercise 15.8.49 you are asked to compute the volume enclosed by such a surface.

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15

Multiple Integrals

IN THIS CHAPTER WE EXTEND the idea of a definite integral to double and triple integrals of functions of two or three variables. These ideas are then used to compute volumes, masses, and centroids of more general regions than we were able to consider in Chapters 6 and 8. We also use double integrals to calculate probabilities when two random variables are involved.

We will see that polar coordinates are useful in computing double integrals over some types of regions. In a similar way, we will introduce two new coordinate systems in three-dimensional space—cylindrical coordinates and spherical coordinates—that greatly simplify the computation of triple integrals over certain commonly occurring solid regions.

15.1 Double Integrals over Rectangles

In much the same way that our attempt to solve the area problem led to the definition of a definite integral, we now seek to find the volume of a solid and in the process we arrive at the definition of a double integral.

Review of the Definite Integral

First let's recall the basic facts concerning definite integrals of functions of a single variable. If $f(x)$ is defined for $a \leqslant x \leqslant b$, we start by dividing the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/n$ and we choose sample points x_i^* in these subintervals. Then we form the Riemann sum

$$\boxed{1} \qquad \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

and take the limit of such sums as $n \to \infty$ to obtain the definite integral of f from a to b:

$$\boxed{2} \qquad \int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

In the special case where $f(x) \geqslant 0$, the Riemann sum can be interpreted as the sum of the areas of the approximating rectangles in Figure 1, and $\int_a^b f(x)\, dx$ represents the area under the curve $y = f(x)$ from a to b.

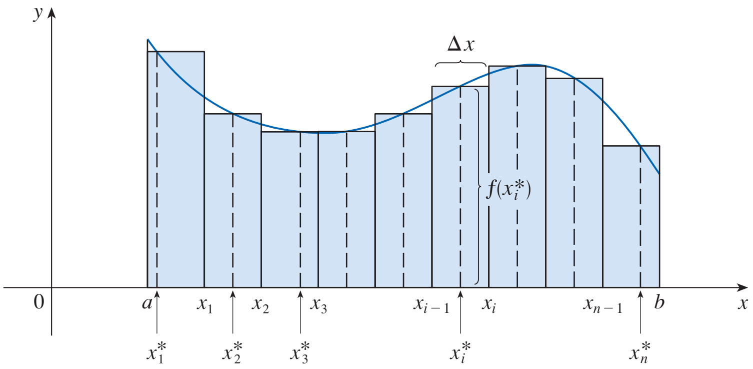


FIGURE 1

Volumes and Double Integrals

In a similar manner we consider a function f of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d] = \left\{(x, y) \in \mathbb{R}^2 \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\right\}$$

and we first suppose that $f(x, y) \geqslant 0$. The graph of f is a surface with equation $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f, that is,

$$S = \left\{(x, y, z) \in \mathbb{R}^3 \mid 0 \leqslant z \leqslant f(x, y),\ (x, y) \in R\right\}$$

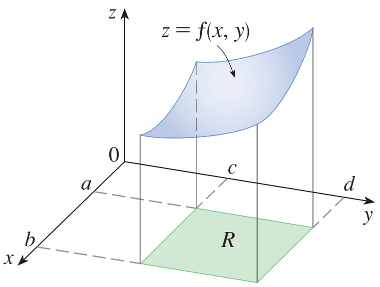


FIGURE 2

(See Figure 2.) Our goal is to find the volume of S.

The first step is to divide the rectangle R into subrectangles. We accomplish this by dividing the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/m$ and dividing $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = (d - c)/n$. By drawing lines parallel to the coordinate axes through the endpoints of these subintervals, as in Figure 3, we form the subrectangles

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leqslant x \leqslant x_i,\ y_{j-1} \leqslant y \leqslant y_j\}$$

each with area $\Delta A = \Delta x\, \Delta y$.

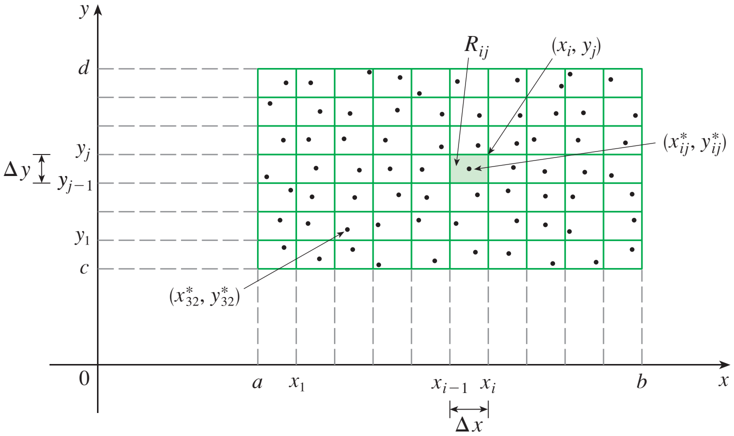


FIGURE 3
Dividing R into subrectangles

If we choose a **sample point** (x_{ij}^*, y_{ij}^*) in each R_{ij}, then we can approximate the part of S that lies above each R_{ij} by a thin rectangular box (or "column") with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$ as shown in Figure 4. (Compare with Figure 1.) The volume of this box is the height of the box times the area of the base rectangle:

$$f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

If we follow this procedure for all the rectangles and add the volumes of the corresponding boxes, we get an approximation to the total volume of S:

$$\boxed{3} \qquad V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

(See Figure 5.) This double sum means that for each subrectangle we evaluate f at the chosen point and multiply by the area of the subrectangle, and then we add the results.

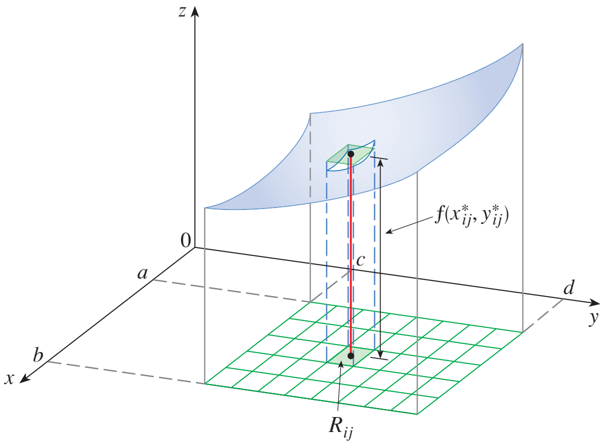


FIGURE 4

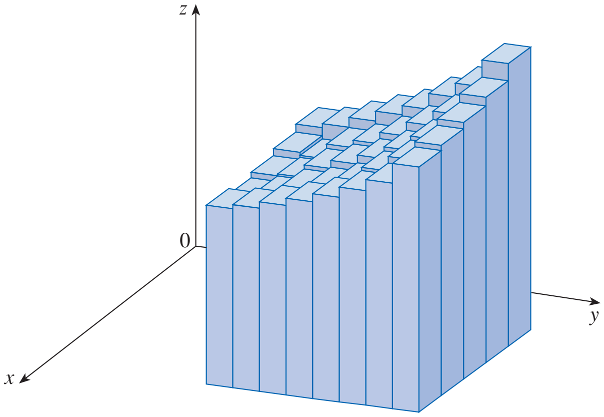


FIGURE 5

Our intuition tells us that the approximation given in (3) becomes better as m and n become larger and so we would expect that

$$\boxed{4} \qquad V = \lim_{m,\,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

The meaning of the double limit in Equation 4 is that we can make the double sum as close as we like to the number V [for any choice of (x_{ij}^*, y_{ij}^*) in R_{ij}] by taking m and n sufficiently large.

We use the expression in Equation 4 to define the **volume** of the solid S that lies under the graph of f and above the rectangle R. (It can be shown that this definition is consistent with our formula for volume in Section 6.2.)

Limits of the type that appear in Equation 4 occur frequently, not just in finding volumes but in a variety of other situations as well—as we will see in Section 15.4—even when f is not a positive function. So we make the following definition.

5 Definition The **double integral** of f over the rectangle R is

$$\iint_R f(x, y)\, dA = \lim_{m,\,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

if this limit exists.

Notice the similarity between Definition 5 and the definition of a single integral in Equation 2.

The precise meaning of the limit in Definition 5 is that for every number $\varepsilon > 0$ there is an integer N such that

$$\left| \iint_R f(x, y)\, dA - \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A \right| < \varepsilon$$

for all integers m and n greater than N and for any choice of sample points (x_{ij}^*, y_{ij}^*) in R_{ij}.

Although we have defined the double integral by dividing R into equal-sized subrectangles, we could have used subrectangles R_{ij} of unequal size. But then we would have to ensure that all of their dimensions approach 0 in the limiting process.

A function f is called **integrable** if the limit in Definition 5 exists. It is shown in courses on advanced calculus that all continuous functions are integrable. In fact, the double integral of f exists provided that f is "not too discontinuous." In particular, if f is bounded on R [that is, there is a constant M such that $|f(x, y)| \leqslant M$ for all (x, y) in R], and f is continuous there, except possibly on a finite number of smooth curves, then f is integrable over R.

The sample point (x_{ij}^*, y_{ij}^*) can be chosen to be any point in the subrectangle R_{ij}, but if we choose it to be the upper right-hand corner of R_{ij} [namely (x_i, y_j), see Figure 3], then the expression for the double integral looks simpler:

$$\boxed{6} \qquad \iint_R f(x, y)\, dA = \lim_{m,\,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\,\Delta A$$

By comparing Definitions 4 and 5, we see that a volume can be written as a double integral:

If $f(x, y) \geqslant 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y)\, dA$$

The sum in Definition 5,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

is called a **double Riemann sum** and is used as an approximation to the value of the double integral. [Notice how similar it is to the Riemann sum in (1) for a function of a single variable.] If f happens to be a *positive* function, then the double Riemann sum represents the sum of volumes of columns, as in Figure 5, and is an approximation to the volume under the graph of f.

EXAMPLE 1 Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.

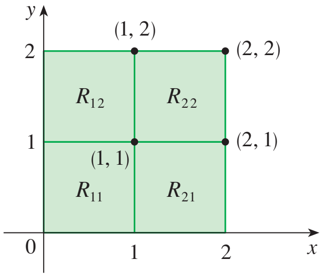


FIGURE 6

SOLUTION The squares are shown in Figure 6. The paraboloid is the graph of $f(x, y) = 16 - x^2 - 2y^2$ and the area of each square is $\Delta A = 1$. Approximating the volume by the Riemann sum with $m = n = 2$, we have

$$\begin{aligned} V &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j)\, \Delta A \\ &= f(1, 1)\, \Delta A + f(1, 2)\, \Delta A + f(2, 1)\, \Delta A + f(2, 2)\, \Delta A \\ &= 13(1) + 7(1) + 10(1) + 4(1) = 34 \end{aligned}$$

This is the volume of the approximating rectangular boxes shown in Figure 7. ■

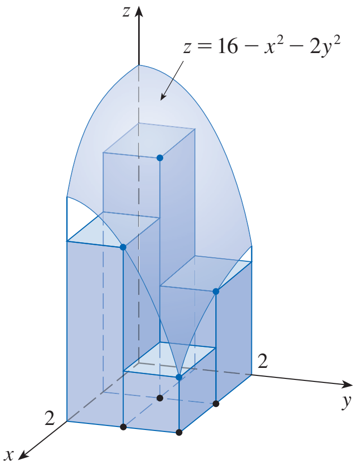
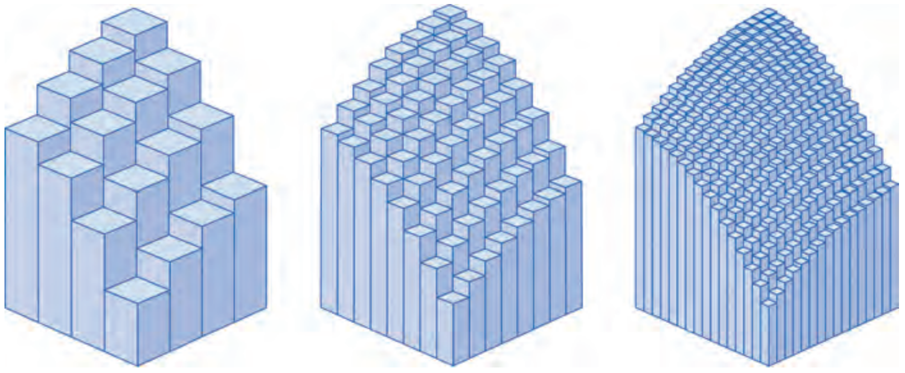


FIGURE 7

We get better approximations to the volume in Example 1 if we increase the number of squares. Figure 8 shows how the columns start to look more like the actual solid and the corresponding approximations become more accurate when we use 16, 64, and 256 squares. In Example 7 we will be able to show that the exact volume is 48.



(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

FIGURE 8

The Riemann sum approximations to the volume under $z = 16 - x^2 - 2y^2$ become more accurate as m and n increase.

EXAMPLE 2 If $R = \{(x, y) \mid -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2\}$, evaluate the integral

$$\iint_R \sqrt{1 - x^2}\, dA$$

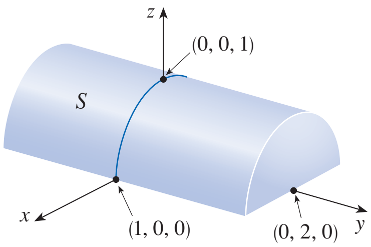


FIGURE 9

SOLUTION It would be very difficult to evaluate this integral directly from Definition 5 but, because $\sqrt{1 - x^2} \geqslant 0$, we can compute the integral by interpreting it as a volume. If $z = \sqrt{1 - x^2}$, then $x^2 + z^2 = 1$ and $z \geqslant 0$, so the given double integral represents the volume of the solid S that lies below the circular cylinder $x^2 + z^2 = 1$ and above the rectangle R. (See Figure 9.) The volume of S is the area of a semicircle with radius 1 times the length of the cylinder. Thus

$$\iint_R \sqrt{1 - x^2}\, dA = \tfrac{1}{2}\pi(1)^2 \times 4 = 2\pi$$ ■

The Midpoint Rule

The methods that we used for approximating single integrals (the Midpoint Rule, the Trapezoidal Rule, Simpson's Rule) all have counterparts for double integrals. Here we consider only the Midpoint Rule for double integrals. This means that we use a double Riemann sum to approximate the double integral, where the sample point (x_{ij}^*, y_{ij}^*) in R_{ij} is chosen to be the center $(\bar{x}_i, \bar{y}_j)$ of R_{ij}. In other words, $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Midpoint Rule for Double Integrals

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\, \Delta A$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

EXAMPLE 3 Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral $\iint_R (x - 3y^2)\, dA$, where $R = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 1 \leqslant y \leqslant 2\}$.

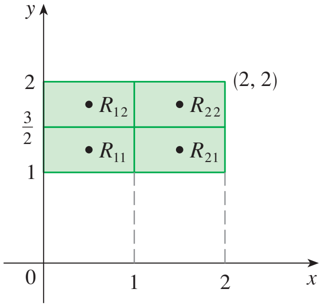


FIGURE 10

SOLUTION In using the Midpoint Rule with $m = n = 2$, we evaluate $f(x, y) = x - 3y^2$ at the centers of the four subrectangles shown in Figure 10. So $\bar{x}_1 = \frac{1}{2}$, $\bar{x}_2 = \frac{3}{2}$, $\bar{y}_1 = \frac{5}{4}$, and $\bar{y}_2 = \frac{7}{4}$. The area of each subrectangle is $\Delta A = \frac{1}{2}$. Thus

$$\begin{aligned}
\iint_R (x - 3y^2)\, dA &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_j)\, \Delta A \\
&= f(\bar{x}_1, \bar{y}_1)\, \Delta A + f(\bar{x}_1, \bar{y}_2)\, \Delta A + f(\bar{x}_2, \bar{y}_1)\, \Delta A + f(\bar{x}_2, \bar{y}_2)\, \Delta A \\
&= f\left(\tfrac{1}{2}, \tfrac{5}{4}\right) \Delta A + f\left(\tfrac{1}{2}, \tfrac{7}{4}\right) \Delta A + f\left(\tfrac{3}{2}, \tfrac{5}{4}\right) \Delta A + f\left(\tfrac{3}{2}, \tfrac{7}{4}\right) \Delta A \\
&= \left(-\tfrac{67}{16}\right)\tfrac{1}{2} + \left(-\tfrac{139}{16}\right)\tfrac{1}{2} + \left(-\tfrac{51}{16}\right)\tfrac{1}{2} + \left(-\tfrac{123}{16}\right)\tfrac{1}{2} \\
&= -\tfrac{95}{8} = -11.875
\end{aligned}$$

Thus we have

$$\iint_R (x - 3y^2)\, dA \approx -11.875$$ ■

NOTE In Example 5 we will see that the exact value of the double integral given in Example 3 is -12. (Remember that the interpretation of a double integral as a volume is valid only when the integrand f is a *positive* function. The integrand in Example 3 is not a positive function, so its integral is not a volume. In Examples 5 and 6 we will discuss how to interpret integrals of functions that are not always positive in terms of volumes.) If we keep dividing each subrectangle in Figure 10 into four smaller ones with similar

Number of subrectangles	Midpoint Rule approximation
1	-11.5000
4	-11.8750
16	-11.9687
64	-11.9922
256	-11.9980
1024	-11.9995

shape, we get the Midpoint Rule approximations displayed in the table in the margin. Notice how these approximations approach the exact value of the double integral, -12.

Iterated Integrals

Recall that it is usually difficult to evaluate single integrals directly from the definition of an integral, but the Fundamental Theorem of Calculus provides a much easier method. The evaluation of double integrals from first principles is even more difficult, but here we see how to express a double integral as an iterated integral, which can then be evaluated by calculating two single integrals.

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$. We use the notation $\int_c^d f(x, y)\, dy$ to mean that x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$. This procedure is called *partial integration with respect to y*. (Notice its similarity to partial differentiation.) Now $\int_c^d f(x, y)\, dy$ is a number that depends on the value of x, so it defines a function of x:

$$A(x) = \int_c^d f(x, y)\, dy$$

If we now integrate the function A with respect to x from $x = a$ to $x = b$, we get

$$\boxed{7} \qquad \int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

The integral on the right side of Equation 7 is called an **iterated integral**. Usually the brackets are omitted. Thus

$$\boxed{8} \qquad \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

means that we first integrate with respect to y (holding x fixed) from $y = c$ to $y = d$, and then we integrate the resulting function of x with respect to x from $x = a$ to $x = b$.

Similarly, the iterated integral

$$\boxed{9} \qquad \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$

means that we first integrate with respect to x (holding y fixed) from $x = a$ to $x = b$ and then we integrate the resulting function of y with respect to y from $y = c$ to $y = d$. Notice that in both Equations 8 and 9 we work *from the inside out.*

EXAMPLE 4 Evaluate the iterated integrals.

(a) $\displaystyle\int_0^3 \int_1^2 x^2 y\, dy\, dx$ (b) $\displaystyle\int_1^2 \int_0^3 x^2 y\, dx\, dy$

SOLUTION

(a) Regarding x as a constant, we obtain

$$\int_1^2 x^2 y\, dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} = x^2\left(\frac{2^2}{2}\right) - x^2\left(\frac{1^2}{2}\right) = \tfrac{3}{2}x^2$$

Thus the function A in the preceding discussion is given by $A(x) = \frac{3}{2}x^2$ in this example. We now integrate this function of x from 0 to 3:

$$\int_0^3 \int_1^2 x^2 y\, dy\, dx = \int_0^3 \left[\int_1^2 x^2 y\, dy \right] dx = \int_0^3 \tfrac{3}{2}x^2\, dx = \frac{x^3}{2}\bigg]_0^3 = \frac{27}{2}$$

(b) Here we first integrate with respect to x, regarding y as a constant:

$$\int_1^2 \int_0^3 x^2 y \, dx \, dy = \int_1^2 \left[\int_0^3 x^2 y \, dx \right] dy = \int_1^2 \left[\frac{x^3}{3} y \right]_{x=0}^{x=3} dy$$

$$= \int_1^2 9y \, dy = 9 \frac{y^2}{2} \Bigg]_1^2 = \frac{27}{2}$$

■

Notice that in Example 4 we obtained the same answer whether we integrated with respect to y or x first. In general, it turns out (see Theorem 10) that the two iterated integrals in Equations 8 and 9 are always equal; that is, the order of integration does not matter. (This is similar to Clairaut's Theorem on the equality of the mixed partial derivatives.)

The following theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral (in either order).

Theorem 10 is named after the Italian mathematician Guido Fubini (1879–1943), who proved a very general version of this theorem in 1907. But the version for continuous functions was known to the French mathematician Augustin-Louis Cauchy almost a century earlier.

10 **Fubini's Theorem** If f is continuous on the rectangle

$$R = \{(x, y) \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$$

then

$$\iint\limits_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

The proof of Fubini's Theorem is too difficult to include in this book, but we can at least give an intuitive indication of why it is true for the case where $f(x, y) \geqslant 0$. Recall that if f is positive, then we can interpret the double integral $\iint_R f(x, y) \, dA$ as the volume V of the solid S that lies above R and under the surface $z = f(x, y)$. But we have another formula that we used for volume in Section 6.2, namely,

$$V = \int_a^b A(x) \, dx$$

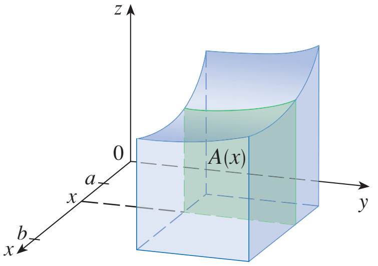


FIGURE 11

where $A(x)$ is the area of a cross-section of S in the plane through x perpendicular to the x-axis. From Figure 11 you can see that $A(x)$ is the area under the curve C whose equation is $z = f(x, y)$, where x is held constant and $c \leqslant y \leqslant d$. Therefore

$$A(x) = \int_c^d f(x, y) \, dy$$

and we have

$$\iint\limits_R f(x, y) \, dA = V = \int_a^b A(x) \, dx = \int_a^b \int_c^d f(x, y) \, dy \, dx$$

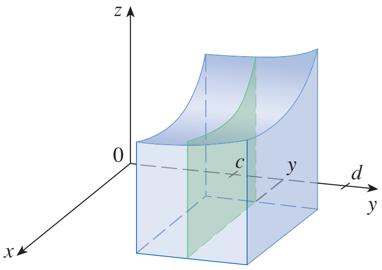


FIGURE 12

A similar argument, using cross-sections perpendicular to the y-axis as in Figure 12, shows that

$$\iint\limits_R f(x, y) \, dA = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

EXAMPLE 5 Evaluate the double integral $\iint_R (x - 3y^2)\,dA$, where $R = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 1 \leqslant y \leqslant 2\}$. (Compare with Example 3.)

SOLUTION 1 Fubini's Theorem gives

$$\iint_R (x - 3y^2)\,dA = \int_0^2 \int_1^2 (x - 3y^2)\,dy\,dx = \int_0^2 \Big[xy - y^3\Big]_{y=1}^{y=2} dx$$

$$= \int_0^2 (x - 7)\,dx = \frac{x^2}{2} - 7x \bigg]_0^2 = -12$$

SOLUTION 2 Again applying Fubini's Theorem, but this time integrating with respect to x first, we have

$$\iint_R (x - 3y^2)\,dA = \int_1^2 \int_0^2 (x - 3y^2)\,dx\,dy = \int_1^2 \left[\frac{x^2}{2} - 3xy^2\right]_{x=0}^{x=2} dy$$

$$= \int_1^2 (2 - 6y^2)\,dy = 2y - 2y^3\Big]_1^2 = -12 \qquad \blacksquare$$

Notice the negative answer in Example 5; nothing is wrong with that. The function f is not a positive function, so its integral doesn't represent a volume. From Figure 13 we see that f is always negative on R, so the value of the integral is the *negative* of the volume that lies *above* the graph of f and *below* R.

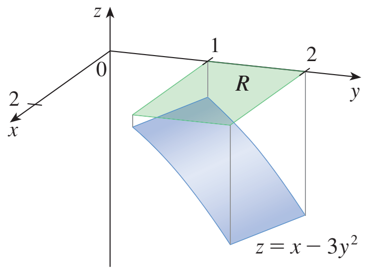


FIGURE 13

EXAMPLE 6 Evaluate $\iint_R y \sin(xy)\,dA$, where $R = [1, 2] \times [0, \pi]$.

SOLUTION If we first integrate with respect to x, we get

$$\iint_R y \sin(xy)\,dA = \int_0^\pi \int_1^2 y \sin(xy)\,dx\,dy$$

$$= \int_0^\pi y \left[-\frac{1}{y}\cos(xy)\right]_{x=1}^{x=2} dy$$

$$= \int_0^\pi (-\cos 2y + \cos y)\,dy$$

$$= -\tfrac{1}{2}\sin 2y + \sin y\Big]_0^\pi = 0 \qquad \blacksquare$$

For a function f that takes on both positive and negative values, $\iint_R f(x, y)\,dA$ is a difference of volumes: $V_1 - V_2$, where V_1 is the volume above R and below the graph of f, and V_2 is the volume below R and above the graph. The fact that the integral in Example 6 is 0 means that these two volumes V_1 and V_2 are equal. (See Figure 14.)

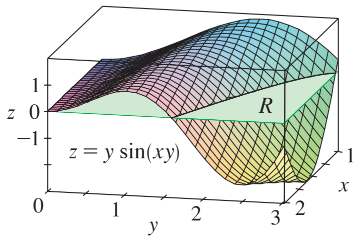


FIGURE 14

NOTE In Example 6, if we reverse the order of integration and first integrate with respect to y, we get

$$\iint_R y \sin(xy)\,dA = \int_1^2 \int_0^\pi y \sin(xy)\,dy\,dx$$

but this order of integration is much more difficult than the method given in the example because it involves integration by parts twice. Therefore, when we evaluate double integrals it is wise to choose the order of integration that gives simpler integrals.

EXAMPLE 7 Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

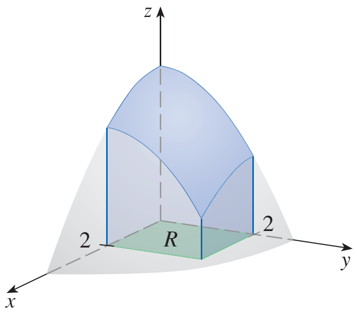


FIGURE 15

SOLUTION We first observe that S is the solid that lies under the surface $z = 16 - x^2 - 2y^2$ and above the square $R = [0, 2] \times [0, 2]$. (See Figure 15.) This solid was considered in Example 1, but we are now in a position to evaluate the double integral using Fubini's Theorem. Therefore

$$V = \iint_R (16 - x^2 - 2y^2)\,dA = \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\,dx\,dy$$
$$= \int_0^2 \Big[16x - \tfrac{1}{3}x^3 - 2y^2x\Big]_{x=0}^{x=2}\,dy$$
$$= \int_0^2 \left(\tfrac{88}{3} - 4y^2\right)dy = \Big[\tfrac{88}{3}y - \tfrac{4}{3}y^3\Big]_0^2 = 48 \qquad ■$$

In the special case where $f(x, y)$ can be factored as the product of a function of x only and a function of y only, the double integral of f can be written in a particularly simple form. To be specific, suppose that $f(x, y) = g(x)h(y)$ and $R = [a, b] \times [c, d]$. Then Fubini's Theorem gives

$$\iint_R f(x, y)\,dA = \int_c^d \int_a^b g(x)h(y)\,dx\,dy = \int_c^d \left[\int_a^b g(x)h(y)\,dx\right] dy$$

In the inner integral, y is a constant, so $h(y)$ is a constant and we can write

$$\int_c^d \left[\int_a^b g(x)h(y)\,dx\right] dy = \int_c^d \left[h(y)\left(\int_a^b g(x)\,dx\right)\right] dy = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

since $\int_a^b g(x)\,dx$ is a constant. Therefore, in this case the double integral of f can be written as the product of two single integrals:

$$\boxed{11} \quad \iint_R g(x)\,h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy \quad \text{where } R = [a, b] \times [c, d]$$

EXAMPLE 8 If $R = [0, \pi/2] \times [0, \pi/2]$, then, by Equation 11,

$$\iint_R \sin x \cos y\,dA = \int_0^{\pi/2} \sin x\,dx \int_0^{\pi/2} \cos y\,dy$$
$$= \Big[-\cos x\Big]_0^{\pi/2} \Big[\sin y\Big]_0^{\pi/2} = 1 \cdot 1 = 1 \qquad ■$$

The function $f(x, y) = \sin x \cos y$ in Example 8 is positive on R, so the integral represents the volume of the solid that lies above R and below the graph of f shown in Figure 16.

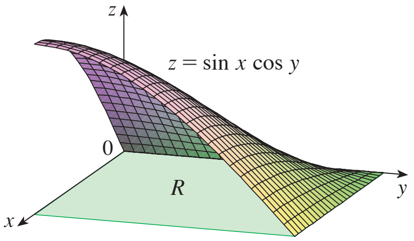


FIGURE 16

Average Value

Recall from Section 6.5 that the average value of a function f of one variable defined on an interval $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b - a} \int_a^b f(x)\, dx$$

In a similar fashion we define the **average value** of a function f of two variables defined on a rectangle R to be

$$f_{\text{avg}} = \frac{1}{A(R)} \iint\limits_R f(x, y)\, dA$$

where $A(R)$ is the area of R.

If $f(x, y) \geqslant 0$, the equation

$$A(R) \times f_{\text{avg}} = \iint\limits_R f(x, y)\, dA$$

says that the box with base R and height f_{avg} has the same volume as the solid that lies under the graph of f. [If $z = f(x, y)$ describes a mountainous region and you chop off the tops of the mountains at height f_{avg}, then you can use them to fill in the valleys so that the region becomes completely flat. See Figure 17.]

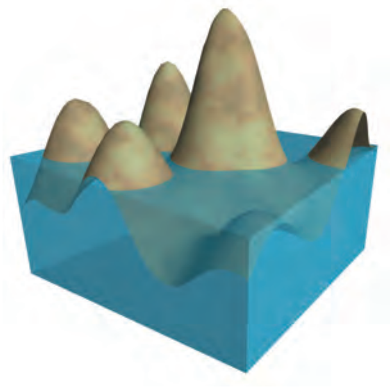


FIGURE 17

EXAMPLE 9 The contour map in Figure 18 shows the snowfall, in centimeters, that fell on the state of Colorado on December 20 and 21, 2006. (The state is in the shape of a rectangle that measures 624 km west to east and 444 km south to north.) Use the contour map to estimate the average snowfall for the entire state of Colorado on those days.

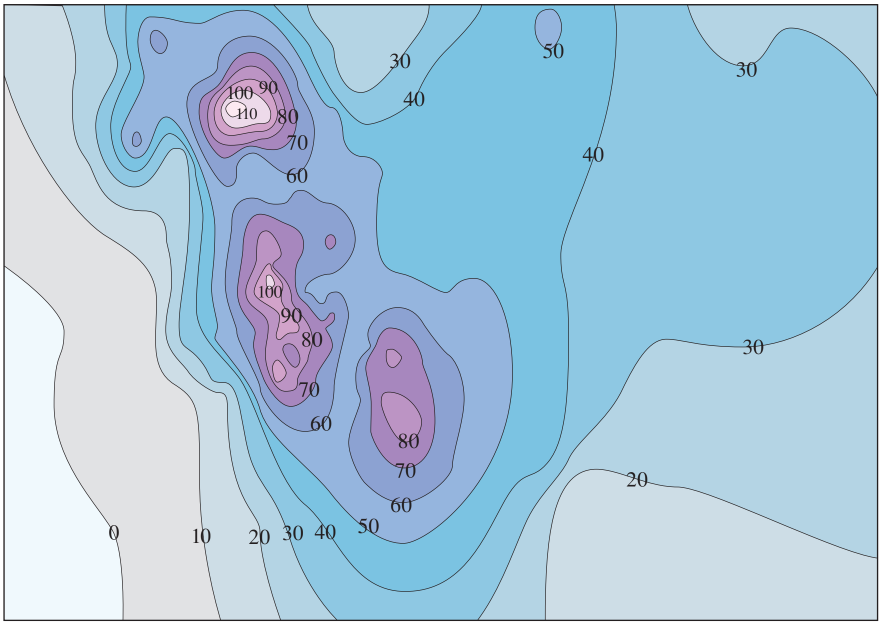


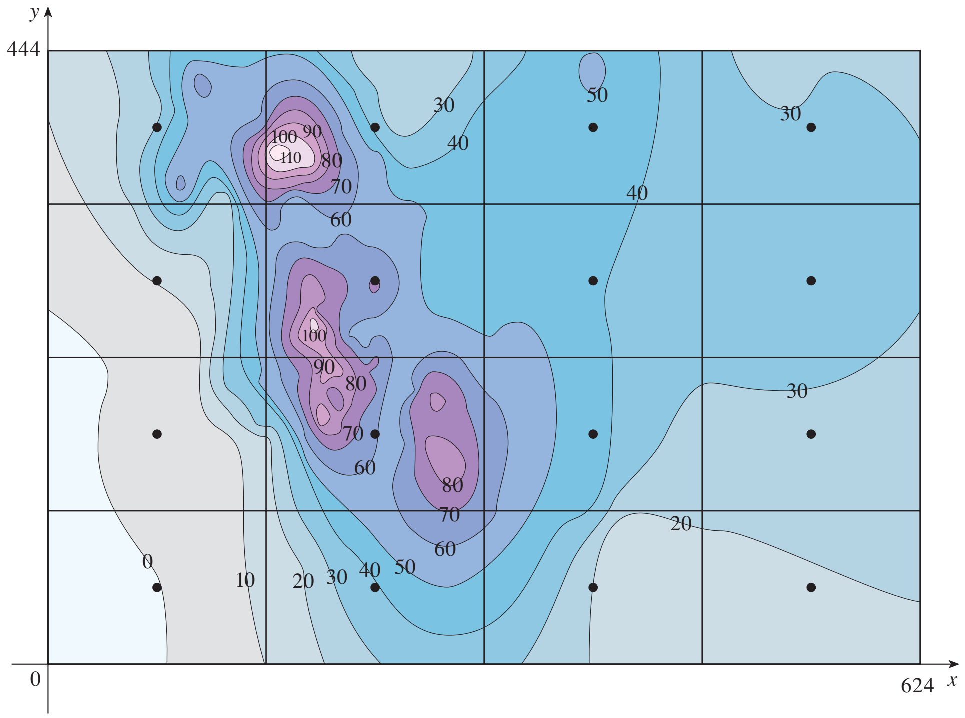
FIGURE 18

SOLUTION Let's place the origin at the southwest corner of the state. Then $0 \leqslant x \leqslant 624$, $0 \leqslant y \leqslant 444$, and $f(x, y)$ is the snowfall, in centimeters, at a location x kilometers to the east and y kilometers to the north of the origin. If R is the rectangle that represents Colorado, then the average snowfall for the state on December 20–21 was

$$f_{\text{avg}} = \frac{1}{A(R)} \iint\limits_R f(x, y)\, dA$$

where $A(R) = 624 \cdot 444$. To estimate the value of this double integral, let's use the Midpoint Rule with $m = n = 4$. In other words, we divide R into 16 subrectangles of equal size, as in Figure 19. The area of each subrectangle is

$$\Delta A = \tfrac{1}{16}(624)(444) = 17{,}316 \text{ km}^2$$



y
444
0
624
x
100
90
110
80
70
60
30
40
50
40
30
100
90
80
70
60
80
70
60
30
20
0
10
20
30
40
50

FIGURE 19

Using the contour map to estimate the value of f at the center of each subrectangle, we get

$$\begin{aligned}\iint_R f(x, y)\, dA &\approx \sum_{i=1}^{4} \sum_{j=1}^{4} f(\bar{x}_i, \bar{y}_j)\, \Delta A \\ &\approx \Delta A[0 + 38 + 20 + 18 + 5 + 64 + 47 + 28 \\ &\qquad + 11 + 70 + 43 + 34 + 30 + 38 + 44 + 33] \\ &= (17{,}316)(523)\end{aligned}$$

Therefore $$f_{\text{avg}} \approx \frac{(17{,}316)(523)}{(624)(444)} \approx 32.7$$

On December 20–21, 2006, Colorado received an average of approximately 32.7 centimeters of snow. ■

15.1 Exercises

1. (a) Estimate the volume of the solid that lies below the surface $z = xy$ and above the rectangle

$$R = \{(x, y) \mid 0 \leqslant x \leqslant 6, 0 \leqslant y \leqslant 4\}$$

Use a Riemann sum with $m = 3$, $n = 2$, and take the sample point to be the upper right corner of each square.

(b) Use the Midpoint Rule to estimate the volume of the solid in part (a).

2. If $R = [0, 4] \times [-1, 2]$, use a Riemann sum with $m = 2$, $n = 3$ to estimate the value of $\iint_R (1 - xy^2)\, dA$. Take the sample points to be (a) the lower right corners and (b) the upper left corners of the rectangles.

3. (a) Use a Riemann sum with $m = n = 2$ to estimate the value of $\iint_R xe^{-xy}\, dA$, where $R = [0, 2] \times [0, 1]$. Take the sample points to be upper right corners.

(b) Use the Midpoint Rule to estimate the integral in part (a).

4. (a) Estimate the volume of the solid that lies below the surface $z = 1 + x^2 + 3y$ and above the rectangle $R = [1, 2] \times [0, 3]$. Use a Riemann sum with $m = n = 2$ and choose the sample points to be lower left corners.

(b) Use the Midpoint Rule to estimate the volume in part (a).

5. Let V be the volume of the solid that lies under the graph of $f(x, y) = \sqrt{52 - x^2 - y^2}$ and above the rectangle given by $2 \leqslant x \leqslant 4$, $2 \leqslant y \leqslant 6$. Use the lines $x = 3$ and $y = 4$ to divide R into subrectangles. Let L and U be the Riemann sums computed using lower left corners and upper right corners, respectively. Without calculating the numbers V, L, and U, arrange them in increasing order and explain your reasoning.

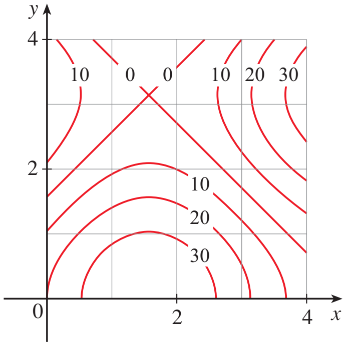
6. A 8-meter by 12-meter swimming pool is filled with water. The depth is measured at 2-m intervals, starting at one corner of the pool, and the values are recorded in the table. Estimate the volume of water in the pool.

	0	2	4	6	8	10	12
0	1	1.5	2	2.4	2.8	3	3
2	1	1.5	2	2.8	3	3.6	3
4	1	1.8	2.7	3	3.6	4	3.2
6	1	1.5	2	2.3	2.7	3	2.5
8	1	1	1	1	1.5	2	2

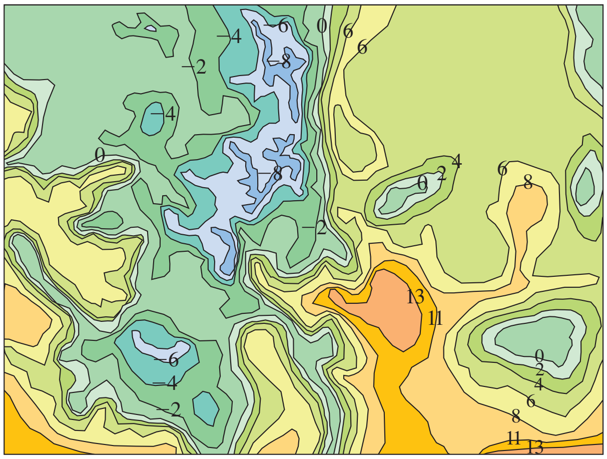
7. A contour map is shown for a function f on the square $R = [0, 4] \times [0, 4]$.

(a) Use the Midpoint Rule with $m = n = 2$ to estimate the value of $\iint_R f(x, y)\, dA$.

(b) Estimate the average value of f.



8. The contour map shows the temperature, in degrees Celsius, at 4:00 PM on a day in February in Colorado. (The state measures 624 km west to east and 444 km south to north.) Use the Midpoint Rule with $m = n = 4$ to estimate the average temperature in Colorado at that time.



9–11 Evaluate the double integral by first identifying it as the volume of a solid.

9. $\iint_R \sqrt{2}\, dA$, $\quad R = \{(x, y) \mid 2 \leqslant x \leqslant 6, -1 \leqslant y \leqslant 5\}$

10. $\iint_R (2x + 1)\, dA$, $\quad R = \{(x, y) \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 4\}$

11. $\iint_R (4 - 2y)\, dA$, $\quad R = [0, 1] \times [0, 1]$

---

12. The integral $\iint_R \sqrt{9 - y^2}\, dA$, where $R = [0, 4] \times [0, 2]$, represents the volume of a solid. Sketch the solid.

13–14 Find $\int_0^2 f(x, y)\, dx$ and $\int_0^3 f(x, y)\, dy$

13. $f(x, y) = x + 3x^2y^2$ **14.** $f(x, y) = y\sqrt{x + 2}$

15–26 Calculate the iterated integral.

15. $\int_1^4 \int_0^2 (6x^2y - 2x)\,dy\,dx$ **16.** $\int_0^1 \int_0^1 (x + y)^2\,dx\,dy$

17. $\int_0^1 \int_1^2 (x + e^{-y})\,dx\,dy$

18. $\int_{-3}^1 \int_1^2 (x^2 + y^{-2})\,dy\,dx$

19. $\int_{-3}^3 \int_0^{\pi/2} (y + y^2 \cos x)\,dx\,dy$

20. $\int_1^3 \int_1^5 \frac{\ln y}{xy}\,dy\,dx$

21. $\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dy\,dx$ **22.** $\int_0^1 \int_0^2 ye^{x-y}\,dx\,dy$

23. $\int_0^3 \int_0^{\pi/2} t^2 \sin^3\phi\,d\phi\,dt$ **24.** $\int_0^1 \int_0^1 xy\sqrt{x^2 + y^2}\,dy\,dx$

25. $\int_0^1 \int_0^1 v(u + v^2)^4\,du\,dv$

26. $\int_0^1 \int_0^1 \sqrt{s + t}\,ds\,dt$

27–34 Calculate the double integral.

27. $\iint_R x \sec^2 y\,dA, \quad R = \{(x, y) \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant \pi/4\}$

28. $\iint_R (y + xy^{-2})\,dA, \quad R = \{(x, y) \mid 0 \leqslant x \leqslant 2, 1 \leqslant y \leqslant 2\}$

29. $\iint_R \frac{xy^2}{x^2 + 1}\,dA, \quad R = \{(x, y) \mid 0 \leqslant x \leqslant 1, -3 \leqslant y \leqslant 3\}$

30. $\iint_R \frac{\tan\theta}{\sqrt{1 - t^2}}\,dA, \quad R = \{(\theta, t) \mid 0 \leqslant \theta \leqslant \pi/3, 0 \leqslant t \leqslant \frac{1}{2}\}$

31. $\iint_R x \sin(x + y)\,dA, \quad R = [0, \pi/6] \times [0, \pi/3]$

32. $\iint_R \frac{x}{1 + xy}\,dA, \quad R = [0, 1] \times [0, 1]$

33. $\iint_R ye^{-xy}\,dA, \quad R = [0, 2] \times [0, 3]$

34. $\iint_R \frac{1}{1 + x + y}\,dA, \quad R = [1, 3] \times [1, 2]$

35–37 Sketch the solid whose volume is given by the iterated integral.

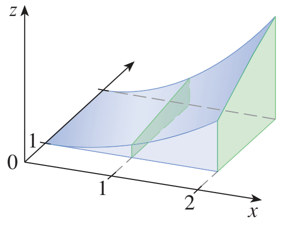
35. $\int_0^1 \int_0^1 (4 - x - 2y)\,dx\,dy$

36. $\int_0^1 \int_0^1 (2 - x^2 - y^2)\,dy\,dx$

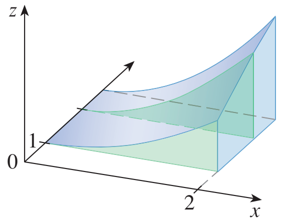
37. $\int_{-2}^2 \int_{-1}^3 (4 - x^2)\,dy\,dx$

38. Consider the solid region S that lies under the surface $z = x^2\sqrt{y}$ and above the rectangle $R = [0, 2] \times [1, 4]$.

(a) Find a formula for the area of a cross-section of S in the plane perpendicular to the x-axis at x for $0 \leqslant x \leqslant 2$. Then use the formula to compute the areas of the cross-sections illustrated.



(b) Find a formula for the area of a cross-section of S in the plane perpendicular to the y-axis at y for $1 \leqslant y \leqslant 4$. Then use the formula to compute the areas of the cross-sections illustrated.



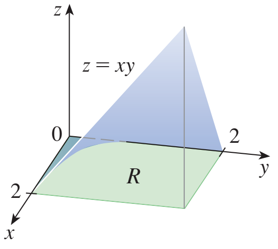
(c) Find the volume of S.

39–42 The figure shows a surface and a rectangle R in the xy-plane.

(a) Set up an iterated integral for the volume of the solid that lies under the surface and above R.

(b) Evaluate the iterated integral to find the volume of the solid.

39.



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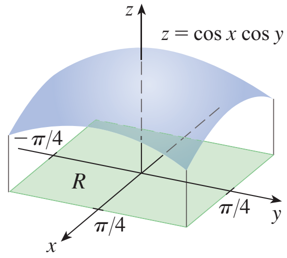

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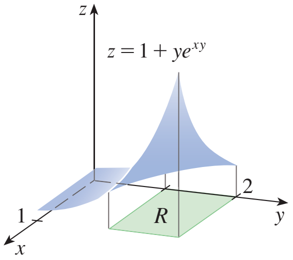
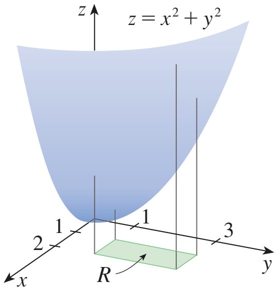


42.



43. Find the volume of the solid that lies under the plane $4x + 6y - 2z + 15 = 0$ and above the rectangle $R = \{(x, y) \mid -1 \leqslant x \leqslant 2, -1 \leqslant y \leqslant 1\}$.

44. Find the volume of the solid that lies under the hyperbolic paraboloid $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1, 1] \times [1, 2]$.

45. Find the volume of the solid lying under the elliptic paraboloid $x^2/4 + y^2/9 + z = 1$ and above the rectangle $R = [-1, 1] \times [-2, 2]$.

46. Find the volume of the solid enclosed by the surface $z = x^2 + xy^2$ and the planes $z = 0$, $x = 0$, $x = 5$, and $y = \pm 2$.

47. Find the volume of the solid enclosed by the surface $z = 1 + x^2ye^y$ and the planes $z = 0$, $x = \pm 1$, $y = 0$, and $y = 1$.

48. Find the volume of the solid in the first octant bounded by the cylinder $z = 16 - x^2$ and the plane $y = 5$.

49. Find the volume of the solid enclosed by the paraboloid $z = 2 + x^2 + (y - 2)^2$ and the planes $z = 1$, $x = 1$, $x = -1$, $y = 0$, and $y = 4$.

50. Graph the solid that lies between the surface $z = 2xy/(x^2 + 1)$ and the plane $z = x + 2y$ and is bounded by the planes $x = 0$, $x = 2$, $y = 0$, and $y = 4$. Then find its volume.

51. Use a computer algebra system to find the exact value of the integral $\iint_R x^5y^3e^{xy}\,dA$, where $R = [0, 1] \times [0, 1]$. Then use the CAS to draw the solid whose volume is given by the integral.

52. Graph the solid that lies between the surfaces $z = e^{-x^2}\cos(x^2 + y^2)$ and $z = 2 - x^2 - y^2$ for $|x| \leqslant 1$, $|y| \leqslant 1$. Use a computer algebra system to approximate the volume of this solid correct to four decimal places.

53–54 Find the average value of f over the given rectangle.

53. $f(x, y) = x^2y$,
R has vertices $(-1, 0)$, $(-1, 5)$, $(1, 5)$, $(1, 0)$

54. $f(x, y) = e^y\sqrt{x + e^y}$, $\quad R = [0, 4] \times [0, 1]$

55–56 Use symmetry to evaluate the double integral.

55. $\displaystyle\iint_R \frac{xy}{1 + x^4}\,dA, \quad R = \{(x, y) \mid -1 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$

56. $\displaystyle\iint_R (1 + x^2\sin y + y^2\sin x)\,dA, \quad R = [-\pi, \pi] \times [-\pi, \pi]$

57. Use a computer algebra system to compute the iterated integrals

$$\int_0^1\int_0^1 \frac{x - y}{(x + y)^3}\,dy\,dx \qquad \text{and} \qquad \int_0^1\int_0^1 \frac{x - y}{(x + y)^3}\,dx\,dy$$

Do the answers contradict Fubini's Theorem? Explain what is happening.

58. (a) In what way are the theorems of Fubini and Clairaut similar?

(b) If $f(x, y)$ is continuous on $[a, b] \times [c, d]$ and

$$g(x, y) = \int_a^x\int_c^y f(s, t)\,dt\,ds$$

for $a < x < b$, $c < y < d$, show that

$$g_{xy} = g_{yx} = f(x, y)$$

15.2 Double Integrals over General Regions

For single integrals, the region over which we integrate is always an interval. But for double integrals, we want to be able to integrate a function not just over rectangles but also over regions of more general shape.

General Regions

Consider a general region D like the one illustrated in Figure 1. We suppose that D is a bounded region, which means that D can be enclosed in a rectangular region R as in Figure 2. In order to integrate a function f over D we define a new function F with domain R by

$$\boxed{1} \qquad F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

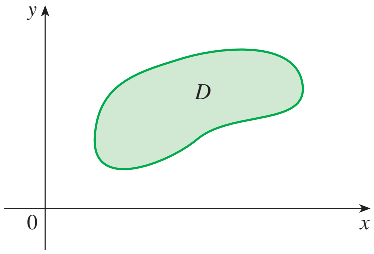


FIGURE 1

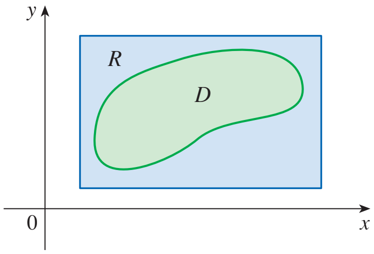


FIGURE 2

If F is integrable over R, then we define the **double integral of f over D** by

$$\boxed{2} \qquad \iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA \qquad \text{where } F \text{ is given by Equation 1}$$

Definition 2 makes sense because R is a rectangle and so $\iint_R F(x, y)\, dA$ has been previously defined in Section 15.1. The procedure that we have used is reasonable because the values of $F(x, y)$ are 0 when (x, y) lies outside D and so they contribute nothing to the integral. This means that it doesn't matter what rectangle R we use as long as it contains D.

In the case where $f(x, y) \geqslant 0$, we can still interpret $\iint_D f(x, y)\, dA$ as the volume of the solid that lies above D and under the surface $z = f(x, y)$ (the graph of f). You can see that this is reasonable by comparing the graphs of f and F in Figures 3 and 4 and remembering that $\iint_R F(x, y)\, dA$ is the volume under the graph of F.

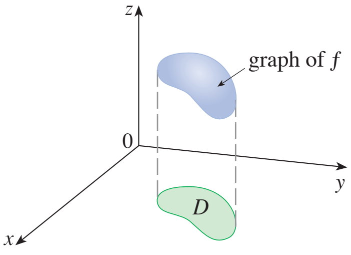


FIGURE 3

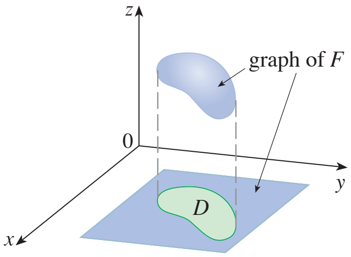


FIGURE 4

Figure 4 also shows that F is likely to have discontinuities at the boundary points of D. Nonetheless, if f is continuous on D and the boundary curve of D is "well behaved" (in a sense outside the scope of this book), then it can be shown that $\iint_R F(x, y)\, dA$ exists and therefore $\iint_D f(x, y)\, dA$ exists. In particular, this is the case for the following two types of regions.

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in Figure 5.

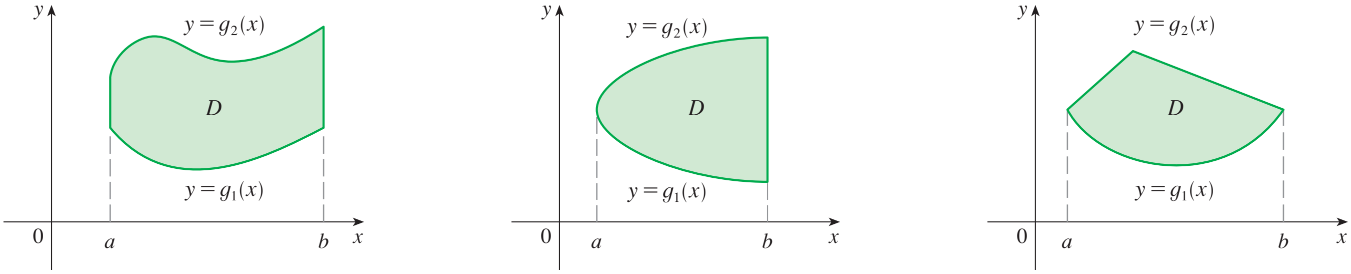


FIGURE 5
Some type I regions

NOTE For a type I region, the functions g_1 and g_2 must be continuous but they do not need to be defined by a single formula. For instance, in the third region of Figure 5, g_2 is a continuous piecewise defined function.

In order to evaluate $\iint_D f(x, y)\, dA$ when D is a region of type I, we choose a rectangle $R = [a, b] \times [c, d]$ that contains D, as in Figure 6, and we let F be the function given by Equation 1; that is, F agrees with f on D and F is 0 outside D. Then, by Fubini's Theorem,

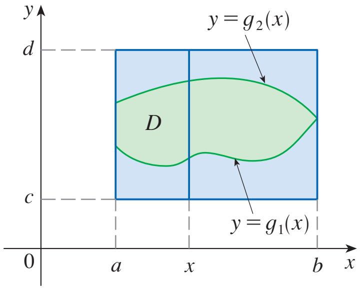


FIGURE 6

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA = \int_a^b \int_c^d F(x, y)\, dy\, dx$$

Observe that $F(x, y) = 0$ if $y < g_1(x)$ or $y > g_2(x)$ because (x, y) then lies outside D. Therefore

$$\int_c^d F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

because $F(x, y) = f(x, y)$ when $g_1(x) \leqslant y \leqslant g_2(x)$. Thus we have the following formula that enables us to evaluate the double integral as an iterated integral.

3 If f is continuous on a type I region D described by

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then $$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

The integral on the right side of (3) is an iterated integral that is similar to the ones we considered in Section 15.1, except that in the inner integral we regard x as being constant not only in $f(x, y)$ but also in the limits of integration, $g_1(x)$ and $g_2(x)$.

We also consider plane regions of **type II**, which can be expressed as

$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$

where h_1 and h_2 are continuous. Three such regions are illustrated in Figure 7.

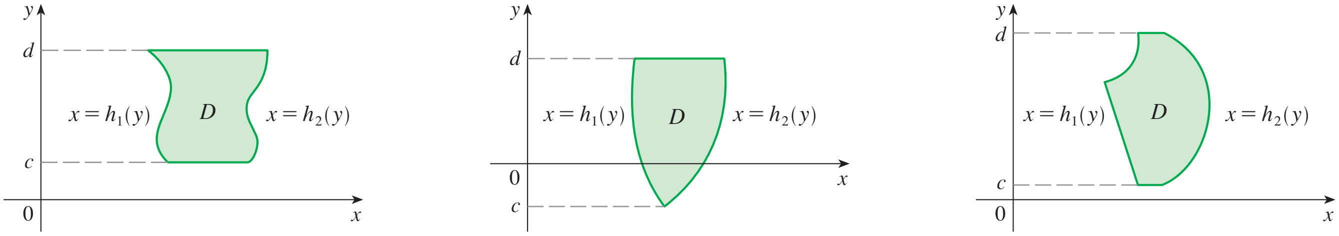


FIGURE 7
Some type II regions

Using the same methods that were used in establishing (3), we can show that the following result holds.

4 If f is continuous on a type II region D described by

$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$

then
$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

EXAMPLE 1 Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

SOLUTION The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = \pm 1$. We note that the region D, sketched in Figure 8, is a type I region but not a type II region and we can write

$$D = \{(x, y) \mid -1 \leqslant x \leqslant 1,\ 2x^2 \leqslant y \leqslant 1 + x^2\}$$

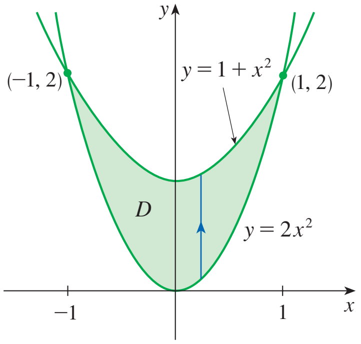


FIGURE 8

Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, Equation 3 gives

$$\begin{aligned}
\iint_D (x + 2y)\, dA &= \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx \\
&= \int_{-1}^{1} \Big[xy + y^2\Big]_{y=2x^2}^{y=1+x^2} dx \\
&= \int_{-1}^{1} [x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2]\, dx \\
&= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx \\
&= -3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\Big]_{-1}^{1} = \frac{32}{15}
\end{aligned}$$

■

NOTE When we set up a double integral as in Example 1, it is essential to draw a diagram. Often it is helpful to draw a vertical arrow as in Figure 8. Then the limits of integration for the *inner* integral can be read from the diagram as follows: The arrow starts at the lower boundary $y = g_1(x)$, which gives the lower limit in the integral, and the arrow ends at the upper boundary $y = g_2(x)$, which gives the upper limit of integration. For a type II region the arrow is drawn horizontally from the left boundary to the right boundary.

EXAMPLE 2 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

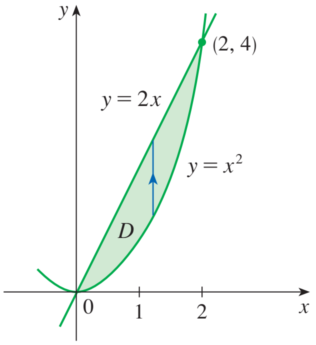


FIGURE 9
D as a type I region

SOLUTION 1 From Figure 9 we see that D is a type I region and

$$D = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ x^2 \leqslant y \leqslant 2x\}$$

Therefore the volume under $z = x^2 + y^2$ and above D is

$$\begin{aligned}
V &= \iint_D (x^2 + y^2)\,dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\,dy\,dx \\
&= \int_0^2 \left[x^2 y + \frac{y^3}{3}\right]_{y=x^2}^{y=2x} dx \\
&= \int_0^2 \left[x^2(2x) + \frac{(2x)^3}{3} - x^2x^2 - \frac{(x^2)^3}{3}\right] dx \\
&= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3}\right) dx \\
&= -\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6}\Bigg]_0^2 = \frac{216}{35}
\end{aligned}$$

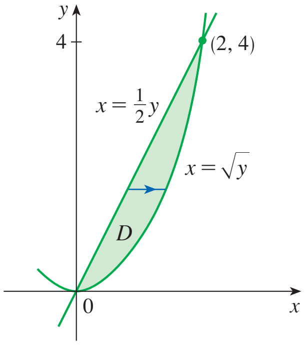


FIGURE 10
D as a type II region

SOLUTION 2 From Figure 10 we see that D can also be written as a type II region:

$$D = \{(x, y) \mid 0 \leqslant y \leqslant 4,\ \tfrac{1}{2}y \leqslant x \leqslant \sqrt{y}\}$$

Therefore another expression for V is

$$\begin{aligned}
V &= \iint_D (x^2 + y^2)\,dA = \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (x^2 + y^2)\,dx\,dy \\
&= \int_0^4 \left[\frac{x^3}{3} + y^2x\right]_{x=\frac{1}{2}y}^{x=\sqrt{y}} dy = \int_0^4 \left(\frac{y^{3/2}}{3} + y^{5/2} - \frac{y^3}{24} - \frac{y^3}{2}\right) dy \\
&= \tfrac{2}{15}y^{5/2} + \tfrac{2}{7}y^{7/2} - \tfrac{13}{96}y^4\Big]_0^4 = \tfrac{216}{35}
\end{aligned}$$

■

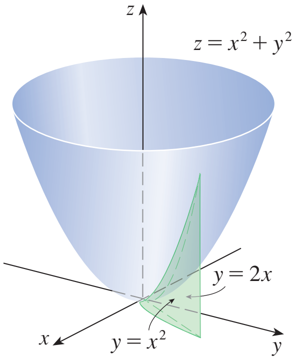


FIGURE 11

Figure 11 shows the solid whose volume is calculated in Example 2. It lies above the xy-plane, below the paraboloid $z = x^2 + y^2$, and between the plane $y = 2x$ and the parabolic cylinder $y = x^2$.

EXAMPLE 3 Evaluate $\iint_D xy\,dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

SOLUTION The region D is shown in Figure 12. Again D is both type I and type II, but the description of D as a type I region is more complicated because the lower boundary consists of two parts. Therefore we prefer to express D as a type II region:

$$D = \left\{(x, y) \mid -2 \leqslant y \leqslant 4,\ \tfrac{1}{2}y^2 - 3 \leqslant x \leqslant y + 1\right\}$$

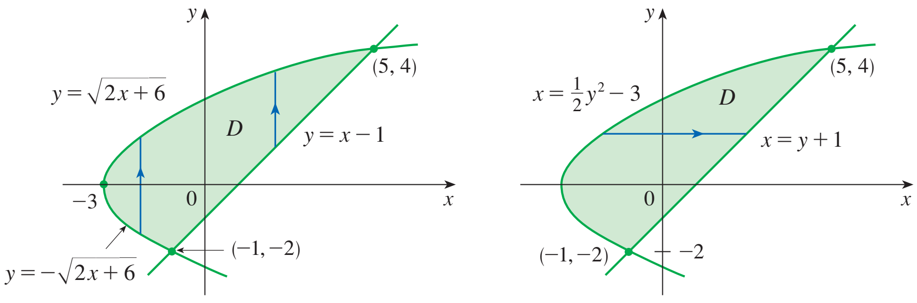


FIGURE 12

(a) D as a type I region

(b) D as a type II region

Then (4) gives

$$\begin{aligned}
\iint\limits_D xy\,dA &= \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx\,dy = \int_{-2}^{4}\left[\frac{x^2}{2}\,y\right]_{x=\frac{1}{2}y^2-3}^{x=y+1} dy \\
&= \tfrac{1}{2}\int_{-2}^{4} y\left[(y+1)^2 - \left(\tfrac{1}{2}y^2 - 3\right)^2\right] dy \\
&= \tfrac{1}{2}\int_{-2}^{4}\left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right) dy \\
&= \frac{1}{2}\left[-\frac{y^6}{24} + y^4 + 2\,\frac{y^3}{3} - 4y^2\right]_{-2}^{4} = 36
\end{aligned}$$

■

In Example 3, if we had expressed D as a type I region using Figure 12(a), then the lower boundary curve would be

$$g_1(x) = \begin{cases} -\sqrt{2x+6} & \text{if } -3 \leqslant x \leqslant -1 \\ x - 1 & \text{if } -1 < x \leqslant 5 \end{cases}$$

and we would have obtained

$$\iint\limits_D xy\,dA = \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\,dy\,dx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} xy\,dy\,dx$$

which would have involved more work than the other method.

EXAMPLE 4 Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

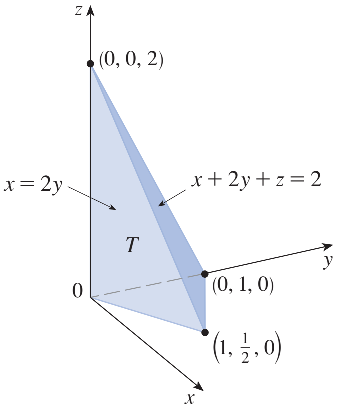


FIGURE 13

SOLUTION In a question such as this, it's wise to draw two diagrams: one of the three-dimensional solid and another of the plane region D over which it lies. Figure 13 shows the tetrahedron T bounded by the coordinate planes $x = 0$, $z = 0$, the vertical plane $x = 2y$, and the plane $x + 2y + z = 2$. Since the plane $x + 2y + z = 2$ intersects the xy-plane (whose equation is $z = 0$) in the line $x + 2y = 2$, we see that T lies above the triangular region D in the xy-plane bounded by the lines $x = 2y$, $x + 2y = 2$, and $x = 0$. (See Figure 14.)

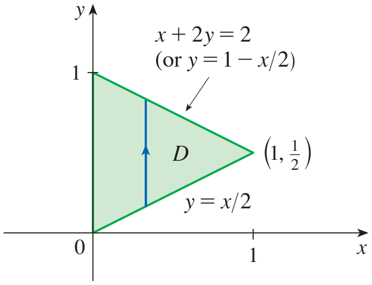


FIGURE 14

The plane $x + 2y + z = 2$ can be written as $z = 2 - x - 2y$, so the required volume lies under the graph of the function $z = 2 - x - 2y$ and above

$$D = \{(x, y) \mid 0 \leqslant x \leqslant 1, x/2 \leqslant y \leqslant 1 - x/2\}$$

Therefore

$$\begin{aligned}
V &= \iint_D (2 - x - 2y)\, dA \\
&= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx \\
&= \int_0^1 \Big[2y - xy - y^2\Big]_{y=x/2}^{y=1-x/2}\, dx \\
&= \int_0^1 \left[2 - x - x\left(1 - \frac{x}{2}\right) - \left(1 - \frac{x}{2}\right)^2 - x + \frac{x^2}{2} + \frac{x^2}{4}\right] dx \\
&= \int_0^1 (x^2 - 2x + 1)\, dx = \frac{x^3}{3} - x^2 + x\Big]_0^1 = \frac{1}{3}
\end{aligned}$$ ■

■ Changing the Order of Integration

Fubini's Theorem tells us that we can express a double integral as an iterated integral in two different orders. Sometimes one order is much more difficult to evaluate than the other—or even impossible. The next example shows how we can change the order of integration when presented with an iterated integral that is difficult to evaluate.

EXAMPLE 5 Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.

SOLUTION If we try to evaluate the integral as it stands, we are faced with the task of first evaluating $\int \sin(y^2)\, dy$. But it's impossible to do so in finite terms since $\int \sin(y^2)\, dy$ is not an elementary function. (See the end of Section 7.5.) So we must change the order of integration. This is accomplished by first expressing the given iterated integral as a double integral. Using (3) backward, we have

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx = \iint_D \sin(y^2)\, dA$$

where $$D = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ x \leqslant y \leqslant 1\}$$

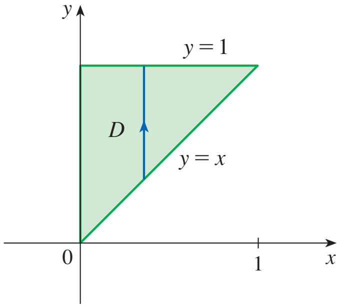


FIGURE 15

D as a type I region

We sketch this region D in Figure 15. Then from Figure 16 we see that an alternative description of D is

$$D = \{(x, y) \mid 0 \leqslant y \leqslant 1,\ 0 \leqslant x \leqslant y\}$$

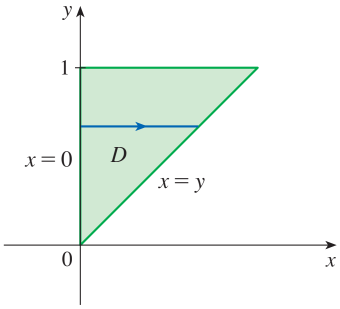


FIGURE 16

D as a type II region

This enables us to use (4) to express the double integral as an iterated integral in the reverse order:

$$\begin{aligned}
\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx &= \iint_D \sin(y^2)\, dA \\
&= \int_0^1 \int_0^y \sin(y^2)\, dx\, dy = \int_0^1 \Big[x \sin(y^2)\Big]_{x=0}^{x=y}\, dy \\
&= \int_0^1 y \sin(y^2)\, dy = -\tfrac{1}{2}\cos(y^2)\Big]_0^1 = \tfrac{1}{2}(1 - \cos 1)
\end{aligned}$$ ■

Properties of Double Integrals

We assume that all of the following integrals exist. For rectangular regions D the first three properties can be proved in the same manner as in Section 5.2. And then for general regions the properties follow from Definition 2.

5 $$\iint_D [f(x, y) + g(x, y)]\, dA = \iint_D f(x, y)\, dA + \iint_D g(x, y)\, dA$$

6 $$\iint_D cf(x, y)\, dA = c \iint_D f(x, y)\, dA \qquad \text{where } c \text{ is a constant}$$

If $f(x, y) \geqslant g(x, y)$ for all (x, y) in D, then

7 $$\iint_D f(x, y)\, dA \geqslant \iint_D g(x, y)\, dA$$

The next property of double integrals is similar to the property of single integrals given by the equation $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$ (Property 5 in Section 5.2).

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 17), then

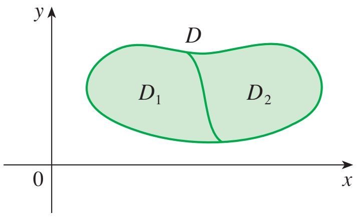


FIGURE 17

8 $$\boxed{\iint_D f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA}$$

Property 8 can be used to evaluate double integrals over regions D that are neither type I nor type II but can be expressed as a union of regions of type I or type II. Figure 18 illustrates this procedure. (See Exercises 67 and 68.)

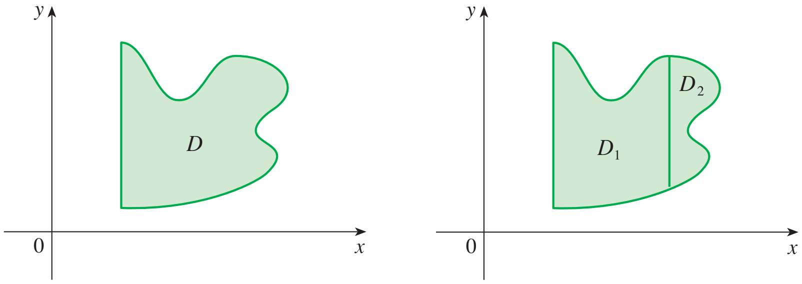


FIGURE 18

(a) D is neither type I nor type II.

(b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

The next property of integrals says that if we integrate the constant function $f(x, y) = 1$ over a region D, we get the area of D:

9 $$\boxed{\iint_D 1\, dA = A(D)}$$

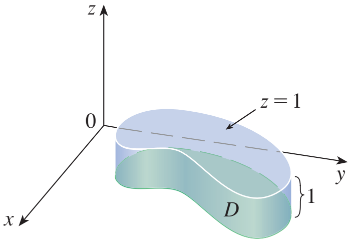


FIGURE 19
Cylinder with base D and height 1

Figure 19 illustrates why Equation 9 is true: A solid cylinder whose base is D and whose height is 1 has volume $A(D) \cdot 1 = A(D)$, but we know that we can also write its volume as $\iint_D 1\, dA$.

Finally, we can combine Properties 6, 7, and 9 to prove the following property. (See Exercise 73.)

10 If $m \leqslant f(x, y) \leqslant M$ for all (x, y) in D, then

$$m \cdot A(D) \leqslant \iint_D f(x, y)\, dA \leqslant M \cdot A(D)$$

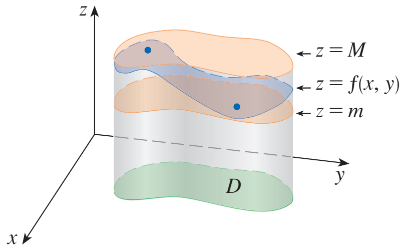


FIGURE 20

Figure 20 illustrates Property 10 for the case $m > 0$. The volume of the solid below the graph of $z = f(x, y)$ and above D is between the volumes of the cylinders with base D and heights m and M. (Compare to Figure 5.2.17, which illustrates the analogous property for single integrals.)

EXAMPLE 6 Use Property 10 to estimate the integral $\iint_D e^{\sin x \cos y}\, dA$, where D is the disk with center the origin and radius 2.

SOLUTION Since $-1 \leqslant \sin x \leqslant 1$ and $-1 \leqslant \cos y \leqslant 1$, we have $-1 \leqslant \sin x \cos y \leqslant 1$ and, because the natural exponential function is increasing, we have

$$e^{-1} \leqslant e^{\sin x \cos y} \leqslant e^1 = e$$

Thus, using $m = e^{-1} = 1/e$, $M = e$, and $A(D) = \pi(2)^2$ in Property 10, we obtain

$$\frac{4\pi}{e} \leqslant \iint_D e^{\sin x \cos y}\, dA \leqslant 4\pi e$$ ■

15.2 Exercises

1–6 Evaluate the iterated integral.

1. $\int_1^5 \int_0^x (8x - 2y)\, dy\, dx$

2. $\int_0^2 \int_0^{y^2} x^2 y\, dx\, dy$

3. $\int_0^1 \int_0^y x e^{y^3}\, dx\, dy$

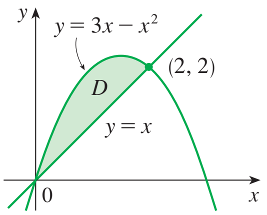
4. $\int_0^{\pi/2} \int_0^x x \sin y\, dy\, dx$

5. $\int_0^1 \int_0^{s^2} \cos(s^3)\, dt\, ds$

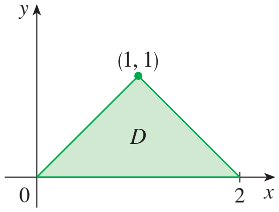
6. $\int_0^1 \int_0^{e^v} \sqrt{1 + e^v}\, dw\, dv$

7–10
(a) Express the double integral $\iint_D f(x, y)\, dA$ as an iterated integral for the given function f and region D.
(b) Evaluate the iterated integral.

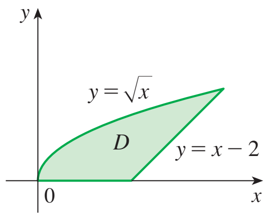
7. $f(x, y) = 2y$



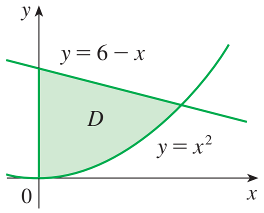
8. $f(x, y) = x + y$



9. $f(x, y) = xy$



10. $f(x, y) = x$



11–14 Evaluate the double integral.

11. $\iint_D \frac{y}{x^2 + 1}\, dA$, $\quad D = \{(x, y) \mid 0 \leqslant x \leqslant 4,\ 0 \leqslant y \leqslant \sqrt{x}\}$

12. $\iint_D (2x + y)\, dA$, $\quad D = \{(x, y) \mid 1 \leqslant y \leqslant 2,\ y - 1 \leqslant x \leqslant 1\}$

13. $\iint_D e^{-y^2}\, dA$, $\quad D = \{(x, y) \mid 0 \leqslant y \leqslant 3,\ 0 \leqslant x \leqslant y\}$

14. $\iint_D y\sqrt{x^2 - y^2}\, dA$, $\quad D = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant x\}$

15. Draw an example of a region that is
(a) type I but not type II
(b) type II but not type I

16. Draw an example of a region that is

(a) both type I and type II

(b) neither type I nor type II

17–18 Express D as a region of type I and also as a region of type II. Then evaluate the double integral in two ways.

17. $\displaystyle\iint_D x\,dA$, D is enclosed by the lines $y = x$, $y = 0$, $x = 1$

18. $\displaystyle\iint_D xy\,dA$, D is enclosed by the curves $y = x^2$, $y = 3x$

19–22 Set up iterated integrals for both orders of integration. Then evaluate the double integral using the easier order and explain why it's easier.

19. $\displaystyle\iint_D y\,dA$, D is bounded by $y = x - 2$, $x = y^2$

20. $\displaystyle\iint_D y^2 e^{xy}\,dA$, D is bounded by $y = x$, $y = 4$, $x = 0$

21. $\displaystyle\iint_D \sin^2 x\,dA$,

D is bounded by $y = \cos x$, $0 \leqslant x \leqslant \pi/2$, $y = 0$, $x = 0$

22. $\displaystyle\iint_D 6x^2\,dA$, D is bounded by $y = x^3$, $y = 2x + 4$, $x = 0$

23–28 Evaluate the double integral.

23. $\displaystyle\iint_D x\cos y\,dA$, D is bounded by $y = 0$, $y = x^2$, $x = 1$

24. $\displaystyle\iint_D (x^2 + 2y)\,dA$, D is bounded by $y = x$, $y = x^3$, $x \geqslant 0$

25. $\displaystyle\iint_D y^2\,dA$,

D is the triangular region with vertices $(0, 1)$, $(1, 2)$, $(4, 1)$

26. $\displaystyle\iint_D xy\,dA$, D is enclosed by the quarter-circle

$y = \sqrt{1 - x^2}$, $x \geqslant 0$, and the axes

27. $\displaystyle\iint_D (2x - y)\,dA$,

D is bounded by the circle with center the origin and radius 2

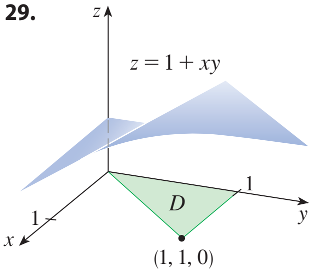
28. $\displaystyle\iint_D y\,dA$, D is the triangular region with vertices $(0, 0)$, $(1, 1)$, and $(4, 0)$

29–30 The figure shows a surface and a region D in the xy-plane.

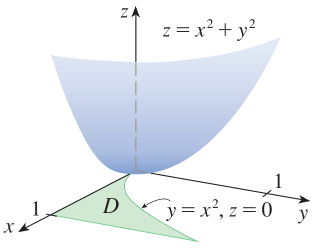
(a) Set up an iterated double integral for the volume of the solid that lies under the surface and above D.

(b) Evaluate the iterated integral to find the volume of the solid.

29.



30.



31–40 Find the volume of the given solid.

31. Under the plane $3x + 2y - z = 0$ and above the region enclosed by the parabolas $y = x^2$ and $x = y^2$

32. Under the surface $z = 1 + x^2y^2$ and above the region enclosed by $x = y^2$ and $x = 4$

33. Under the surface $z = xy$ and above the triangle with vertices $(1, 1)$, $(4, 1)$, and $(1, 2)$

34. Enclosed by the paraboloid $z = x^2 + y^2 + 1$ and the planes $x = 0$, $y = 0$, $z = 0$, and $x + y = 2$

35. The tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$

36. Bounded by the planes $z = x$, $y = x$, $x + y = 2$, and $z = 0$

37. Enclosed by the cylinders $z = x^2$, $y = x^2$ and the planes $z = 0$, $y = 4$

38. Bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0$, $z = 0$ in the first octant

39. Bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y = z$, $x = 0$, $z = 0$ in the first octant

40. Bounded by the cylinders $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$

---

41. Use a graph to estimate the x-coordinates of the points of intersection of the curves $y = x^4$ and $y = 3x - x^2$. If D is the region bounded by these curves, estimate $\iint_D x\,dA$.

42. Find the approximate volume of the solid in the first octant that is bounded by the planes $y = x$, $z = 0$, and $z = x$ and the cylinder $y = \cos x$. (Use a graph to estimate the points of intersection.)

43–46 Find the volume of the solid by subtracting two volumes.

43. The solid enclosed by the parabolic cylinders $y = 1 - x^2$, $y = x^2 - 1$ and the planes $x + y + z = 2$, $2x + 2y - z + 10 = 0$

44. The solid enclosed by the parabolic cylinder $y = x^2$ and the planes $z = 3y$, $z = 2 + y$

45. The solid under the plane $z = 3$, above the plane $z = y$, and between the parabolic cylinders $y = x^2$ and $y = 1 - x^2$

46. The solid in the first octant under the plane $z = x + y$, above the surface $z = xy$, and enclosed by the surfaces $x = 0$, $y = 0$, and $x^2 + y^2 = 4$

47–50 Sketch the solid whose volume is given by the iterated integral.

47. $\int_0^1 \int_0^{1-x} (1 - x - y)\, dy\, dx$ **48.** $\int_0^1 \int_0^{1-x^2} (1 - x)\, dy\, dx$

49. $\int_0^3 \int_0^y \sqrt{9 - x^2}\, dx\, dy$ **50.** $\int_{-2}^2 \int_{-1}^{3-x^2} e^{-y}\, dy\, dx$

T **51–54** Use a computer algebra system to find the exact volume of the solid.

51. Under the surface $z = x^3y^4 + xy^2$ and above the region bounded by the curves $y = x^3 - x$ and $y = x^2 + x$ for $x \geqslant 0$

52. Between the paraboloids $z = 2x^2 + y^2$ and $z = 8 - x^2 - 2y^2$ and inside the cylinder $x^2 + y^2 = 1$

53. Enclosed by $z = 1 - x^2 - y^2$ and $z = 0$

54. Enclosed by $z = x^2 + y^2$ and $z = 2y$

55–60 Sketch the region of integration and change the order of integration.

55. $\int_0^1 \int_0^y f(x, y)\, dx\, dy$ **56.** $\int_0^2 \int_{x^2}^4 f(x, y)\, dy\, dx$

57. $\int_0^{\pi/2} \int_{\sin x}^1 f(x, y)\, dy\, dx$ **58.** $\int_{-2}^2 \int_0^{\sqrt{4-y^2}} f(x, y)\, dx\, dy$

59. $\int_1^2 \int_0^{\ln x} f(x, y)\, dy\, dx$ **60.** $\int_0^1 \int_{\arctan x}^{\pi/4} f(x, y)\, dy\, dx$

61–66 Evaluate the integral by reversing the order of integration.

61. $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$ **62.** $\int_0^1 \int_{x^2}^1 \sqrt{y}\, \sin y\, dy\, dx$

63. $\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3 + 1}\, dy\, dx$

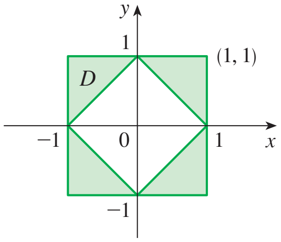
64. $\int_0^2 \int_{y/2}^1 y \cos(x^3 - 1)\, dx\, dy$

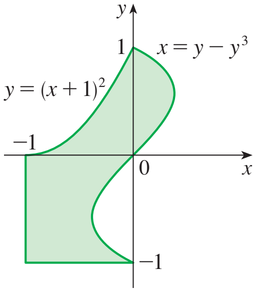
65. $\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1 + \cos^2 x}\, dx\, dy$

66. $\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx\, dy$

67–68 Express D as a union of regions of type I or type II and evaluate the integral.

67. $\iint_D x^2\, dA$ **68.** $\iint_D y\, dA$





69–70 Use Property 10 to estimate the value of the integral.

69. $\iint_S \sqrt{4 - x^2y^2}\, dA$,

$S = \{(x, y) \mid x^2 + y^2 \leqslant 1, x \geqslant 0\}$

70. $\iint_T \sin^4(x + y)\, dA$, T is the triangle enclosed by the lines $y = 0$, $y = 2x$, and $x = 1$

71–72 Find the average value of f over the region D.

71. $f(x, y) = xy$,
D is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$

72. $f(x, y) = x \sin y$,
D is enclosed by the curves $y = 0$, $y = x^2$, and $x = 1$

73. Prove Property 10.

74. In evaluating a double integral over a region D, a sum of iterated integrals was obtained as follows:

$$\iint_D f(x, y)\, dA = \int_0^1 \int_0^{2y} f(x, y)\, dx\, dy + \int_1^3 \int_0^{3-y} f(x, y)\, dx\, dy$$

Sketch the region D and express the double integral as an iterated integral with reversed order of integration.

75–79 Use geometry or symmetry, or both, to evaluate the double integral.

75. $\iint_D (x + 2)\, dA$,

$D = \left\{(x, y) \mid 0 \leqslant y \leqslant \sqrt{9 - x^2}\right\}$

76. $\iint_D \sqrt{R^2 - x^2 - y^2}\, dA$,

D is the disk with center the origin and radius R

77. $\displaystyle\iint_D (2x + 3y)\,dA,$

D is the rectangle $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$

78. $\displaystyle\iint_D (2 + x^2y^3 - y^2\sin x)\,dA,$

$D = \{(x, y) \mid |x| + |y| \leqslant 1\}$

79. $\displaystyle\iint_D \left(ax^3 + by^3 + \sqrt{a^2 - x^2}\right) dA,$

$D = [-a, a] \times [-b, b]$

80–81 Mean Value Theorem for Double Integrals The *Mean Value Theorem for double integrals* says that if f is a continuous function on a plane region D that is of type I or type II, then there exists a point (x_0, y_0) in D such that

$$\iint_D f(x, y)\,dA = f(x_0, y_0)\,A(D)$$

80. Use the Extreme Value Theorem (14.7.8) and Property 15.2.10 of integrals to prove the Mean Value Theorem for double integrals. (Use the proof of the single-variable version in Section 6.5 as a guide.)

81. Suppose that f is continuous on a disk that contains the point (a, b). Let D_r be the closed disk with center (a, b) and radius r. Use the Mean Value Theorem for double integrals to show that

$$\lim_{r\to 0} \frac{1}{\pi r^2} \iint_{D_r} f(x, y)\,dA = f(a, b)$$

T **82.** Graph the solid bounded by the plane $x + y + z = 1$ and the paraboloid $z = 4 - x^2 - y^2$ and find its exact volume. (Use a computer algebra system to find the equations of the boundary curves of the region of integration and to evaluate the double integral.)

15.3 Double Integrals in Polar Coordinates

Suppose that we want to evaluate a double integral $\iint_R f(x, y)\,dA$, where the region R is a circular disk centered at the origin. In this case the description of R in terms of rectangular coordinates is rather complicated, but R is readily described using polar coordinates. In general, if R is a region that is more easily described using polar coordinates, it is often advantageous to evaluate the double integral by first converting it to polar coordinates.

Review of Polar Coordinates

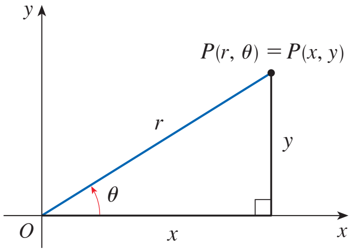


FIGURE 1

Polar coordinates were introduced in Section 10.3. Recall from Figure 1 that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) of that point by the equations

$$\boxed{r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta}$$

Equations of circles centered at the origin are particularly simple in polar coordinates. The unit circle has equation $r = 1$; the region enclosed by this circle is shown in Figure 2(a). Figure 2(b) illustrates another region that is conveniently described in polar coordinates.

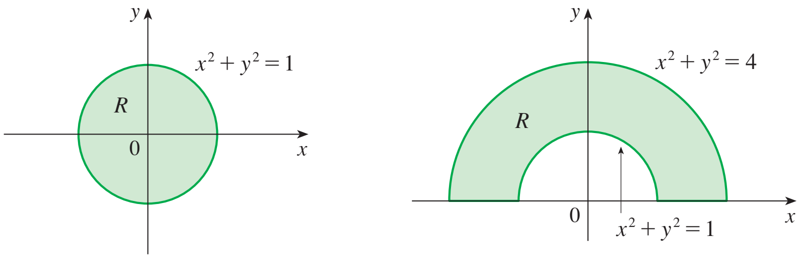


FIGURE 2

(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$

(b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

You may wish to review Table 10.3.1 for other common curves suitably described in polar coordinates.

Double Integrals in Polar Coordinates

The regions in Figure 2 are special cases of a **polar rectangle**

$$R = \{(r, \theta) \mid a \leqslant r \leqslant b,\ \alpha \leqslant \theta \leqslant \beta\}$$

which is shown in Figure 3. In order to compute the double integral $\iint_R f(x, y)\, dA$, where R is a polar rectangle, we divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\Delta r = (b - a)/m$ and we divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ of equal width $\Delta\theta = (\beta - \alpha)/n$. Then the circles $r = r_i$ and the rays $\theta = \theta_j$ divide the polar rectangle R into the small polar rectangles R_{ij} shown in Figure 4.

Compare Figure 4 with Figure 15.1.3.

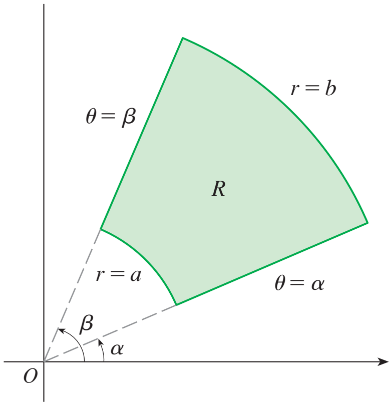


FIGURE 3 Polar rectangle

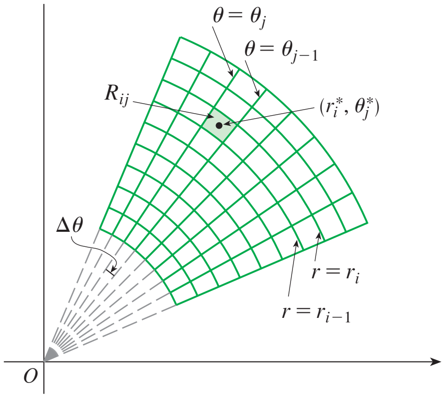


FIGURE 4 Dividing R into polar subrectangles

The "center" of the polar subrectangle

$$R_{ij} = \{(r, \theta) \mid r_{i-1} \leqslant r \leqslant r_i,\ \theta_{j-1} \leqslant \theta \leqslant \theta_j\}$$

has polar coordinates

$$r_i^* = \tfrac{1}{2}(r_{i-1} + r_i) \qquad \theta_j^* = \tfrac{1}{2}(\theta_{j-1} + \theta_j)$$

We compute the area of R_{ij} using the fact that the area of a sector of a circle with radius r and central angle θ is $\frac{1}{2}r^2\theta$. Subtracting the areas of two such sectors, each of which has central angle $\Delta\theta = \theta_j - \theta_{j-1}$, we find that the area of R_{ij} is

$$\begin{aligned}\Delta A_i &= \tfrac{1}{2}r_i^2\,\Delta\theta - \tfrac{1}{2}r_{i-1}^2\,\Delta\theta = \tfrac{1}{2}\,(r_i^2 - r_{i-1}^2)\,\Delta\theta \\ &= \tfrac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\,\Delta\theta = r_i^*\,\Delta r\,\Delta\theta\end{aligned}$$

Although we have defined the double integral $\iint_R f(x, y)\, dA$ in terms of ordinary rectangles, it can be shown that, for continuous functions f, we always obtain the same answer using polar rectangles. The rectangular coordinates of the center of R_{ij} are $(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)$, so a typical Riemann sum is

$$\boxed{1} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\,\Delta A_i = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\, r_i^*\,\Delta r\,\Delta\theta$$

If we write $g(r, \theta) = rf(r\cos\theta, r\sin\theta)$, then the Riemann sum in Equation 1 can be written as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)\,\Delta r\,\Delta\theta$$

which is a Riemann sum for the double integral

$$\int_{\alpha}^{\beta}\int_{a}^{b} g(r,\theta)\,dr\,d\theta$$

Therefore we have

$$\begin{aligned}\iint_R f(x,y)\,dA &= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*,\ r_i^*\sin\theta_j^*)\,\Delta A_i\\ &= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*,\theta_j^*)\,\Delta r\,\Delta\theta = \int_{\alpha}^{\beta}\int_{a}^{b} g(r,\theta)\,dr\,d\theta\\ &= \int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta,\ r\sin\theta)\,r\,dr\,d\theta\end{aligned}$$

2 Change to Polar Coodinates in a Double Integral If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

$$\iint_R f(x,y)\,dA = \int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta,\ r\sin\theta)\,r\,dr\,d\theta$$

The formula in (2) says that we convert from rectangular to polar coordinates in a double integral by writing $x = r\cos\theta$ and $y = r\sin\theta$, using the appropriate limits of integration for r and θ, and replacing dA by $r\,dr\,d\theta$. Be careful not to forget the additional factor r on the right side of Formula 2. A classical method for remembering this is shown in Figure 5, where the "infinitesimal" polar rectangle can be thought of as an ordinary rectangle with dimensions $r\,d\theta$ and dr and therefore has "area" $dA = r\,dr\,d\theta$.

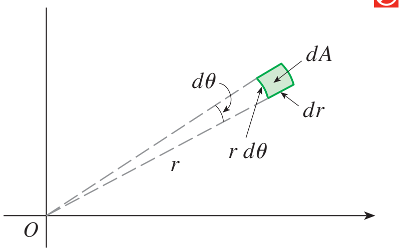


FIGURE 5

EXAMPLE 1 Evaluate $\iint_R (3x + 4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

SOLUTION The region R can be described as

$$R = \{(x,y) \mid y \geqslant 0,\ 1 \leqslant x^2 + y^2 \leqslant 4\}$$

It is the half-ring shown in Figure 2(b), and in polar coordinates it is given by $1 \leqslant r \leqslant 2$, $0 \leqslant \theta \leqslant \pi$. Therefore, by Formula 2,

$$\begin{aligned}\iint_R (3x + 4y^2)\,dA &= \int_0^{\pi}\int_1^2 [3(r\cos\theta) + 4(r\sin\theta)^2]\,r\,dr\,d\theta\\ &= \int_0^{\pi}\int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\,dr\,d\theta\\ &= \int_0^{\pi}\Big[r^3\cos\theta + r^4\sin^2\theta\Big]_{r=1}^{r=2} d\theta = \int_0^{\pi}(7\cos\theta + 15\sin^2\theta)\,d\theta\\ &= \int_0^{\pi}\Big[7\cos\theta + \tfrac{15}{2}(1 - \cos 2\theta)\Big]\,d\theta\\ &= 7\sin\theta + \frac{15\theta}{2} - \frac{15}{4}\sin 2\theta\Big]_0^{\pi} = \frac{15\pi}{2}\end{aligned}$$

■

Here we use the trigonometric identity

$$\sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)$$

See Section 7.2 for advice on integrating trigonometric functions.

EXAMPLE 2 Evaluate the double integral

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}(x^2+y^2)\,dy\,dx$$

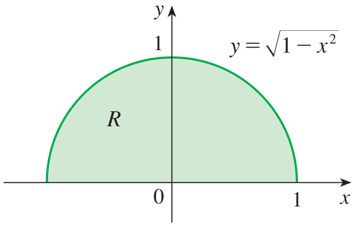


FIGURE 6

SOLUTION This iterated integral is a double integral over the region R shown in Figure 6 and described by

$$R=\left\{(x,y)\mid -1\leqslant x\leqslant 1,\ 0\leqslant y\leqslant\sqrt{1-x^2}\right\}$$

The region is a half-disk, so it is more simply described in polar coordinates:

$$R=\left\{(r,\theta)\mid 0\leqslant\theta\leqslant\pi,\ 0\leqslant r\leqslant 1\right\}$$

Therefore we have

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}(x^2+y^2)\,dy\,dx&=\int_{0}^{\pi}\int_{0}^{1}(r^2)\,r\,dr\,d\theta\\&=\int_{0}^{\pi}\left[\frac{r^4}{4}\right]_{r=0}^{r=1}d\theta=\frac{1}{4}\int_{0}^{\pi}d\theta=\frac{\pi}{4}\end{aligned}$$ ■

EXAMPLE 3 Find the volume of the solid bounded by the plane $z=0$ and the paraboloid $z=1-x^2-y^2$.

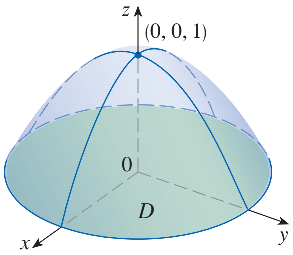


FIGURE 7

SOLUTION If we put $z=0$ in the equation of the paraboloid, we get $x^2+y^2=1$. This means that the plane intersects the paraboloid in the circle $x^2+y^2=1$, so the solid lies under the paraboloid and above the circular disk D given by $x^2+y^2\leqslant 1$ [see Figures 7 and 2(a)]. In polar coordinates D is given by $0\leqslant r\leqslant 1$, $0\leqslant\theta\leqslant 2\pi$. Since $1-x^2-y^2=1-r^2$, the volume is

$$\begin{aligned}V&=\iint_D(1-x^2-y^2)\,dA=\int_{0}^{2\pi}\int_{0}^{1}(1-r^2)\,r\,dr\,d\theta\\&=\int_{0}^{2\pi}d\theta\int_{0}^{1}(r-r^3)\,dr=2\pi\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_{0}^{1}=\frac{\pi}{2}\end{aligned}$$ ■

In Example 3, if we had used rectangular coordinates instead of polar coordinates, we would have obtained

$$V=\iint_D(1-x^2-y^2)\,dA=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(1-x^2-y^2)\,dy\,dx$$

which is not easy to evaluate because it involves finding $\int(1-x^2)^{3/2}\,dx$.

What we have done so far can be extended to the more complicated type of region shown in Figure 8. It's similar to the type II rectangular regions we considered in Section 15.2. In fact, by combining Formula 2 in this section with Formula 15.2.4, we obtain the following formula.

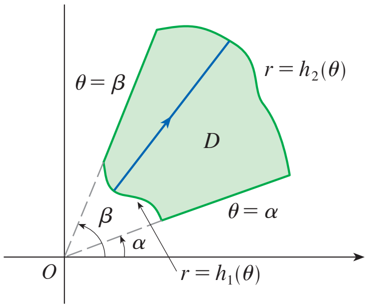


FIGURE 8

$D=\{(r,\theta)\mid\alpha\leqslant\theta\leqslant\beta,\,h_1(\theta)\leqslant r\leqslant h_2(\theta)\}$

3 If f is continuous on a polar region of the form

$$D=\left\{(r,\theta)\mid\alpha\leqslant\theta\leqslant\beta,\ h_1(\theta)\leqslant r\leqslant h_2(\theta)\right\}$$

then

$$\iint_D f(x,y)\,dA=\int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)}f(r\cos\theta,r\sin\theta)\,r\,dr\,d\theta$$

In particular, taking $f(x, y) = 1$, $h_1(\theta) = 0$, and $h_2(\theta) = h(\theta)$ in this formula, we see that the area of the region D bounded by $\theta = \alpha$, $\theta = \beta$, and $r = h(\theta)$ is

$$
\begin{aligned}
A(D) = \iint_D 1\,dA &= \int_\alpha^\beta \int_0^{h(\theta)} r\,dr\,d\theta \\
&= \int_\alpha^\beta \left[\frac{r^2}{2}\right]_0^{h(\theta)} d\theta = \int_\alpha^\beta \tfrac{1}{2}[h(\theta)]^2\,d\theta
\end{aligned}
$$

and this agrees with Formula 10.4.3.

EXAMPLE 4 Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

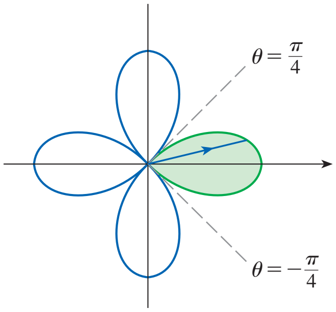


FIGURE 9

SOLUTION From the sketch of the curve in Figure 9, we see that a loop is given by the region

$$D = \{(r, \theta) \mid -\pi/4 \le \theta \le \pi/4,\ 0 \le r \le \cos 2\theta\}$$

So the area is

$$
\begin{aligned}
A(D) = \iint_D dA &= \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r\,dr\,d\theta \\
&= \int_{-\pi/4}^{\pi/4} \left[\tfrac{1}{2}r^2\right]_0^{\cos 2\theta} d\theta = \tfrac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\,d\theta \\
&= \tfrac{1}{4}\int_{-\pi/4}^{\pi/4} (1 + \cos 4\theta)\,d\theta = \tfrac{1}{4}\Big[\theta + \tfrac{1}{4}\sin 4\theta\Big]_{-\pi/4}^{\pi/4} = \frac{\pi}{8} \qquad \blacksquare
\end{aligned}
$$

EXAMPLE 5 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

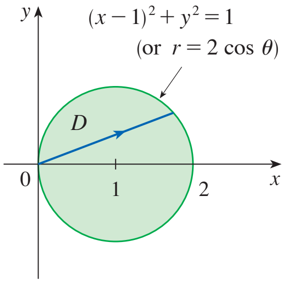


FIGURE 10

SOLUTION The solid lies above the disk D whose boundary circle has equation $x^2 + y^2 = 2x$ or, after completing the square,

$$(x - 1)^2 + y^2 = 1$$

(See Figures 10 and 11.)

In polar coordinates we have $x^2 + y^2 = r^2$ and $x = r\cos\theta$, so the boundary circle $x^2 + y^2 = 2x$ becomes $r^2 = 2r\cos\theta$, or $r = 2\cos\theta$. Thus the disk D is given by

$$D = \{(r, \theta) \mid -\pi/2 \le \theta \le \pi/2,\ 0 \le r \le 2\cos\theta\}$$

and, by Formula 3, we have

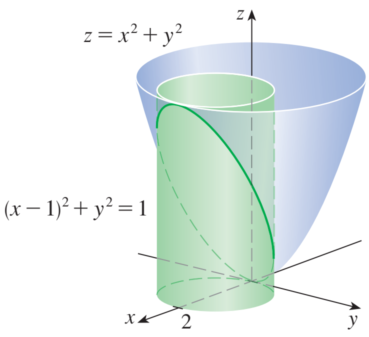


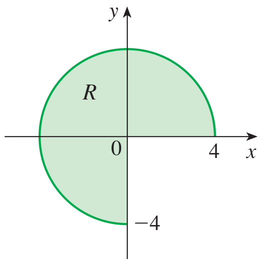
FIGURE 11

$$
\begin{aligned}
V = \iint_D (x^2 + y^2)\,dA &= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2\,r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta \\
&= 4\int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta = 8\int_0^{\pi/2} \cos^4\theta\,d\theta = 8\int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2}\right)^2 d\theta \\
&= 2\int_0^{\pi/2} \Big[1 + 2\cos 2\theta + \tfrac{1}{2}(1 + \cos 4\theta)\Big]\,d\theta \\
&= 2\Big[\tfrac{3}{2}\theta + \sin 2\theta + \tfrac{1}{8}\sin 4\theta\Big]_0^{\pi/2} = 2\left(\frac{3}{2}\right)\left(\frac{\pi}{2}\right) = \frac{3\pi}{2} \qquad \blacksquare
\end{aligned}
$$

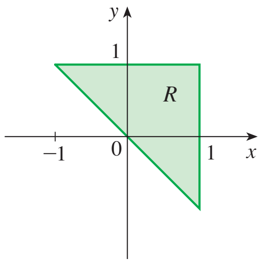
15.3 Exercises

1–6 A region R is shown. Decide whether to use polar coordinates or rectangular coordinates and write $\iint_R f(x, y)\, dA$ as an iterated integral, where f is an arbitrary continuous function on R.

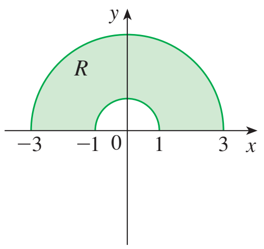
1.



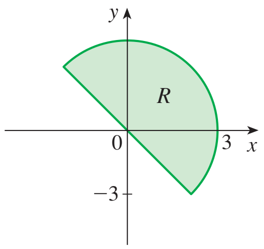
2.



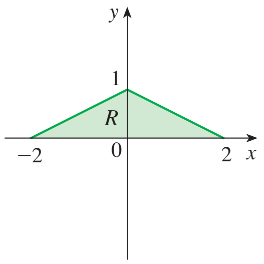
3.



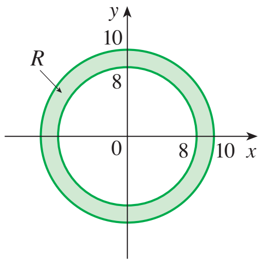
4.



5.



6.



7–8 Sketch the region whose area is given by the integral and evaluate the integral.

7. $\displaystyle\int_{\pi/4}^{3\pi/4}\int_1^2 r\, dr\, d\theta$

8. $\displaystyle\int_{\pi/2}^{\pi}\int_0^{2\sin\theta} r\, dr\, d\theta$

9–16 Evaluate the given integral by changing to polar coordinates.

9. $\iint_D x^2 y\, dA$, where D is the top half of the disk with center the origin and radius 5

10. $\iint_R (2x - y)\, dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$

11. $\iint_R \sin(x^2 + y^2)\, dA$, where R is the region in the first quadrant between the circles with center the origin and radii 1 and 3

12. $\displaystyle\iint_R \frac{y^2}{x^2 + y^2}\, dA$, where R is the region that lies between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ with $0 < a < b$

13. $\iint_D e^{-x^2-y^2}\, dA$, where D is the region bounded by the semicircle $x = \sqrt{4 - y^2}$ and the y-axis

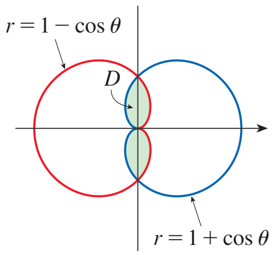
14. $\iint_D \cos\sqrt{x^2 + y^2}\, dA$, where D is the disk with center the origin and radius 2

15. $\iint_R \arctan(y/x)\, dA$,
where $R = \{(x, y) \mid 1 \leqslant x^2 + y^2 \leqslant 4,\ 0 \leqslant y \leqslant x\}$

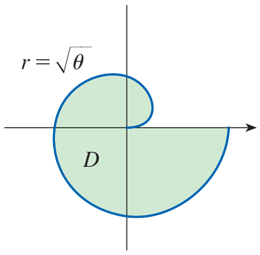
16. $\iint_D x\, dA$, where D is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$

17–22 Use a double integral to find the area of the region D.

17.



18.



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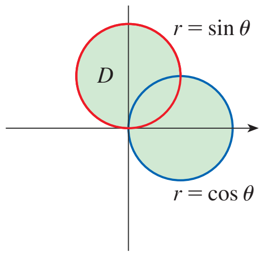
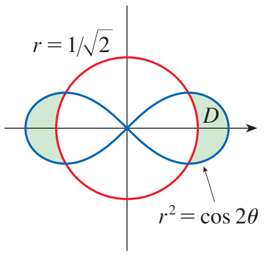


20.

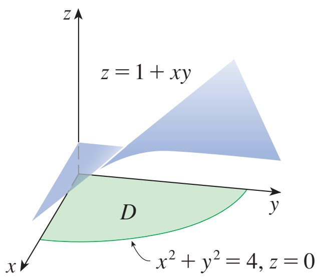


21. D is the loop of the rose $r = \sin 3\theta$ in the first quadrant.

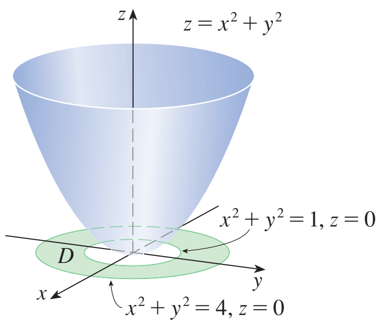
22. D is the region inside the circle $(x - 1)^2 + y^2 = 1$ and outside the circle $x^2 + y^2 = 1$.

23–24
(a) Set up an iterated integral in polar coordinates for the volume of the solid under the surface and above the region D.
(b) Evaluate the iterated integral to find the volume of the solid.

23.



24.

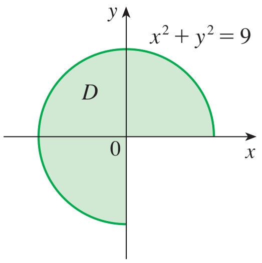


25–28

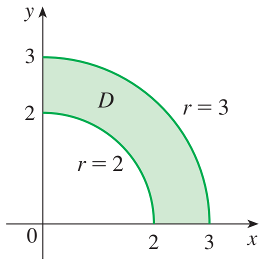
(a) Set up an iterated integral in polar coordinates for the volume of the solid under the graph of the given function and above the region D.

(b) Evaluate the iterated integral to find the volume of the solid.

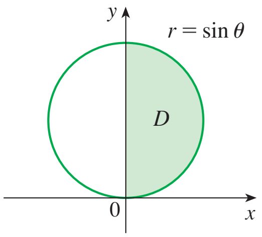
25. $f(x, y) = y$



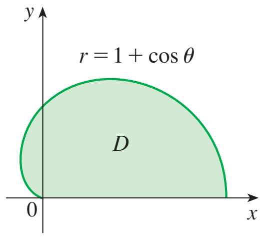
26. $f(x, y) = xy^2$



27. $f(x, y) = x$



28. $f(x, y) = 1$



29–37 Use polar coordinates to find the volume of the given solid.

29. Under the paraboloid $z = x^2 + y^2$ and above the disk $x^2 + y^2 \leqslant 25$

30. Below the cone $z = \sqrt{x^2 + y^2}$ and above the ring $1 \leqslant x^2 + y^2 \leqslant 4$

31. Below the plane $2x + y + z = 4$ and above the disk $x^2 + y^2 \leqslant 1$

32. Inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$

33. A sphere of radius a

34. Bounded by the paraboloid $z = 1 + 2x^2 + 2y^2$ and the plane $z = 7$ in the first octant

35. Above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$

36. Bounded by the paraboloids $z = 6 - x^2 - y^2$ and $z = 2x^2 + 2y^2$

37. Inside both the cylinder $x^2 + y^2 = 4$ and the ellipsoid $4x^2 + 4y^2 + z^2 = 64$

38. (a) A cylindrical drill with radius r_1 is used to bore a hole through the center of a sphere of radius r_2. Find the volume of the ring-shaped solid that remains.

(b) Express the volume in part (a) in terms of the height h of the ring. Notice that the volume depends only on h, not on r_1 or r_2.

39–42 Evaluate the iterated integral by converting to polar coordinates.

39. $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\,dy\,dx$

40. $\displaystyle\int_0^a\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (2x + y)\,dx\,dy$

41. $\displaystyle\int_0^{1/2}\int_{\sqrt{3}\,y}^{\sqrt{1-y^2}} xy^2\,dx\,dy$

42. $\displaystyle\int_0^2\int_0^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2}\,dy\,dx$

T **43–44** Express the double integral in terms of a single integral with respect to r. Then use a calculator (or computer) to evaluate the integral correct to four decimal places.

43. $\iint_D e^{(x^2+y^2)^2}\,dA$, where D is the disk with center the origin and radius 1

44. $\iint_D xy\sqrt{1 + x^2 + y^2}\,dA$, where D is the portion of the disk $x^2 + y^2 \leqslant 1$ that lies in the first quadrant

45. A swimming pool is circular with a 10-meter diameter. The depth is constant along east-west lines and increases linearly from 1 m at the south end to 2 m at the north end. Find the volume of water in the pool.

46. An agricultural sprinkler distributes water in a circular pattern of radius 50 m. It supplies water to a depth of e^{-r} meters per hour at a distance of r meters from the sprinkler.

(a) If $0 < R \leqslant 50$, what is the total amount of water supplied per hour to the region inside the circle of radius R centered at the sprinkler?

(b) Determine an expression for the average amount of water per hour per square meter supplied to the region inside the circle of radius R.

47. Find the average value of the function $f(x, y) = 1/\sqrt{x^2 + y^2}$ on the annular region $a^2 \leqslant x^2 + y^2 \leqslant b^2$, where $0 < a < b$.

48. Let D be the disk with center the origin and radius a. What is the average distance from points in D to the origin?

49. Use polar coordinates to combine the sum

$$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx + \int_{1}^{\sqrt{2}}\int_{0}^{x} xy\,dy\,dx + \int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xy\,dy\,dx$$

into one double integral. Then evaluate the double integral.

50. (a) We define the improper integral (over the entire plane $\mathbb{R}^2$)

$$\begin{aligned} I &= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dy\,dx \\ &= \lim_{a\to\infty}\iint_{D_a} e^{-(x^2+y^2)}\,dA \end{aligned}$$

where D_a is the disk with radius a and center the origin. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dA = \pi$$

(b) An equivalent definition of the improper integral in part (a) is

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \lim_{a\to\infty}\iint_{S_a} e^{-(x^2+y^2)}\,dA$$

where S_a is the square with vertices $(\pm a, \pm a)$. Use this to show that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy = \pi$$

(c) Deduce that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

(d) By making the change of variable $t = \sqrt{2}\,x$, show that

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

(This is a fundamental result for probability and statistics.)

51. Use the result of Exercise 50(c) to evaluate the following integrals.

(a) $\displaystyle\int_0^{\infty} x^2 e^{-x^2}\,dx$ (b) $\displaystyle\int_0^{\infty} \sqrt{x}\,e^{-x}\,dx$

15.4 Applications of Double Integrals

We have already seen one application of double integrals: computing volumes. Another geometric application is finding areas of surfaces and this will be done in the next section. In this section we explore physical applications such as computing mass, electric charge, center of mass, and moment of inertia. We will see that these physical ideas are also important when applied to probability density functions of two random variables.

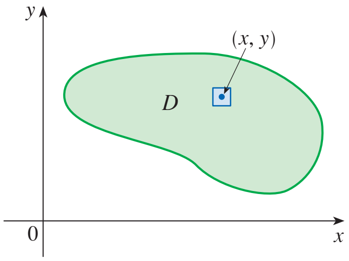


FIGURE 1

Density and Mass

In Section 8.3 we were able to use single integrals to compute moments and the center of mass of a thin plate or lamina with constant density. But now, equipped with the double integral, we can consider a lamina with variable density. Suppose the lamina occupies a region D of the xy-plane and its **density** (in units of mass per unit area) at a point (x, y) in D is given by $\rho(x, y)$, where ρ is a continuous function on D. This means that

$$\rho(x, y) = \lim \frac{\Delta m}{\Delta A}$$

where Δm and ΔA are the mass and area of a small rectangle that contains (x, y) and the limit is taken as the dimensions of the rectangle approach 0. (See Figure 1.)

To find the total mass m of the lamina we divide a rectangle R containing D into subrectangles R_{ij} of the same size (as in Figure 2) and consider $\rho(x, y)$ to be 0 outside D. If we choose a point (x_{ij}^*, y_{ij}^*) in R_{ij}, then the mass of the part of the lamina that occupies R_{ij} is approximately $\rho(x_{ij}^*, y_{ij}^*)\,\Delta A$, where ΔA is the area of R_{ij}. If we add all such masses, we get an approximation to the total mass:

$$m \approx \sum_{i=1}^{k}\sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\,\Delta A$$

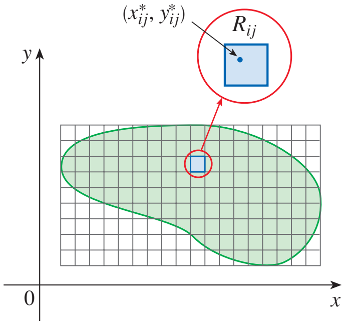


FIGURE 2 The mass of each subrectangle R_{ij} is approximated by $\rho(x_{ij}^*, y_{ij}^*)\,\Delta A$.

If we now increase the number of subrectangles, we obtain the total mass m of the lamina as the limiting value of the approximations:

$$\boxed{m = \lim_{k,\, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D \rho(x, y)\, dA} \tag{1}$$

Physicists also consider other types of density that can be treated in the same manner. For example, if an electric charge is distributed over a region D and the charge density (in units of charge per unit area) is given by $\sigma(x, y)$ at a point (x, y) in D, then the total **electric charge** Q is given by

$$Q = \iint_D \sigma(x, y)\, dA \tag{2}$$

EXAMPLE 1 Charge is distributed over the triangular region D in Figure 3 so that the charge density at (x, y) is $\sigma(x, y) = xy$, measured in coulombs per square meter (C/m^2). Find the total charge.

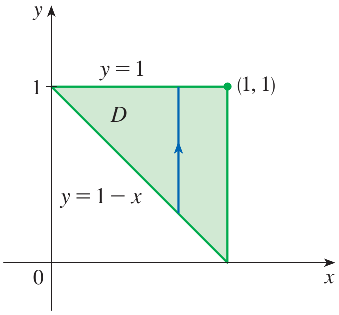


FIGURE 3

SOLUTION From Equation 2 and Figure 3 we have

$$Q = \iint_D \sigma(x, y)\, dA = \int_0^1 \int_{1-x}^1 xy\, dy\, dx = \int_0^1 \left[x\frac{y^2}{2} \right]_{y=1-x}^{y=1} dx = \int_0^1 \frac{x}{2}[1^2 - (1 - x)^2]\, dx$$

$$= \tfrac{1}{2} \int_0^1 (2x^2 - x^3)\, dx = \frac{1}{2}\left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{5}{24}$$

Thus the total charge is $\frac{5}{24}$ C. ■

Moments and Centers of Mass

In Section 8.3 we found the center of mass of a lamina with constant density; here we consider a lamina with variable density. Suppose the lamina occupies a region D and has density function $\rho(x, y)$. Recall from Chapter 8 that we defined the moment of a particle about an axis as the product of its mass and its directed distance from the axis. We divide D into small rectangles as in Figure 2. Then the mass of R_{ij} is approximately $\rho(x_{ij}^*, y_{ij}^*)\, \Delta A$, so we can approximate the moment of R_{ij} with respect to the x-axis by

$$[\rho(x_{ij}^*, y_{ij}^*)\, \Delta A]\, y_{ij}^*$$

If we now add these quantities and take the limit as the number of subrectangles becomes large, we obtain the **moment** of the entire lamina **about the *x*-axis**:

$$\boxed{M_x = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^*\, \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D y\, \rho(x, y)\, dA} \tag{3}$$

Similarly, the **moment about the *y*-axis** is

$$\boxed{M_y = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^*\, \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D x\, \rho(x, y)\, dA} \tag{4}$$

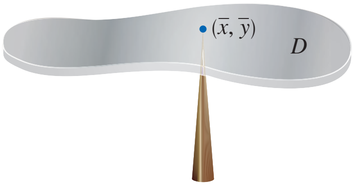


FIGURE 4

As before, we define the center of mass $(\bar{x}, \bar{y})$ so that $m\bar{x} = M_y$ and $m\bar{y} = M_x$. The physical significance is that the lamina behaves as if its entire mass is concentrated at its center of mass. Thus the lamina balances horizontally when supported at its center of mass (see Figure 4).

5 The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x, y)\, dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x, y)\, dA$$

where the mass m is given by

$$m = \iint_D \rho(x, y)\, dA$$

EXAMPLE 2 Find the mass and center of mass of a triangular lamina with vertices $(0, 0)$, $(1, 0)$, and $(0, 2)$ if the density function is $\rho(x, y) = 1 + 3x + y$.

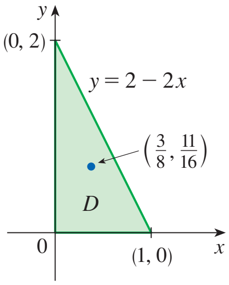


FIGURE 5

SOLUTION The triangle is shown in Figure 5. (Note that the equation of the upper boundary is $y = 2 - 2x$.) The mass of the lamina is

$$\begin{aligned} m &= \iint_D \rho(x, y)\, dA = \int_0^1 \int_0^{2-2x} (1 + 3x + y)\, dy\, dx \\ &= \int_0^1 \left[y + 3xy + \frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx \\ &= 4\int_0^1 (1 - x^2)\, dx = 4\left[x - \frac{x^3}{3} \right]_0^1 = \frac{8}{3} \end{aligned}$$

Then the formulas in (5) give

$$\begin{aligned} \bar{x} &= \frac{1}{m}\iint_D x\rho(x, y)\, dA = \tfrac{3}{8}\int_0^1 \int_0^{2-2x} (x + 3x^2 + xy)\, dy\, dx \\ &= \frac{3}{8}\int_0^1 \left[xy + 3x^2y + x\frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx \\ &= \frac{3}{2}\int_0^1 (x - x^3)\, dx = \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{3}{8} \end{aligned}$$

$$\begin{aligned} \bar{y} &= \frac{1}{m}\iint_D y\rho(x, y)\, dA = \tfrac{3}{8}\int_0^1 \int_0^{2-2x} (y + 3xy + y^2)\, dy\, dx \\ &= \frac{3}{8}\int_0^1 \left[\frac{y^2}{2} + 3x\frac{y^2}{2} + \frac{y^3}{3} \right]_{y=0}^{y=2-2x} dx = \tfrac{1}{4}\int_0^1 (7 - 9x - 3x^2 + 5x^3)\, dx \\ &= \frac{1}{4}\left[7x - 9\frac{x^2}{2} - x^3 + 5\frac{x^4}{4} \right]_0^1 = \frac{11}{16} \end{aligned}$$

The center of mass is at the point $\left(\frac{3}{8}, \frac{11}{16}\right)$. ■

EXAMPLE 3 The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass of the lamina.

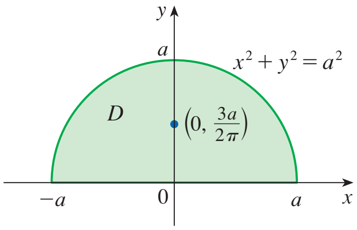


FIGURE 6

SOLUTION Let's place the lamina as the upper half of the circle $x^2 + y^2 = a^2$. (See Figure 6.) Then the distance from a point (x, y) to the center of the circle (the origin) is $\sqrt{x^2 + y^2}$. Therefore the density function is

$$\rho(x, y) = K\sqrt{x^2 + y^2}$$

where K is some constant. Both the density function and the shape of the lamina suggest that we convert to polar coordinates. Then $\sqrt{x^2 + y^2} = r$ and the region D is given by $0 \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant \pi$. Thus the mass of the lamina is

$$m = \iint_D \rho(x, y)\, dA = \iint_D K\sqrt{x^2 + y^2}\, dA$$

$$= \int_0^\pi \int_0^a (Kr)\, r\, dr\, d\theta = K \int_0^\pi d\theta \int_0^a r^2\, dr = K\pi \left.\frac{r^3}{3}\right]_0^a = \frac{K\pi a^3}{3}$$

Both the lamina and the density function are symmetric with respect to the y-axis, so the center of mass must lie on the y-axis, that is, $\bar{x} = 0$. The y-coordinate is given by

$$\bar{y} = \frac{1}{m} \iint_D y\rho(x, y)\, dA = \frac{3}{K\pi a^3} \int_0^\pi \int_0^a r \sin\theta\, (Kr)\, r\, dr\, d\theta$$

$$= \frac{3}{\pi a^3} \int_0^\pi \sin\theta\, d\theta \int_0^a r^3\, dr = \frac{3}{\pi a^3} \Big[-\cos\theta\Big]_0^\pi \left[\frac{r^4}{4}\right]_0^a$$

$$= \frac{3}{\pi a^3} \frac{2a^4}{4} = \frac{3a}{2\pi}$$

Therefore the center of mass is located at the point $(0, 3a/(2\pi))$. ■

Compare the location of the center of mass in Example 3 with Example 8.3.4, where we found that the center of mass of a lamina with the same shape but uniform density is located at the point $(0, 4a/(3\pi))$.

Moment of Inertia

The **moment of inertia** (also called the **second moment**) of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis. We extend this concept to a lamina with density function $\rho(x, y)$ and occupying a region D by proceeding as we did for ordinary moments. We divide D into small rectangles, approximate the moment of inertia of each subrectangle about the x-axis, and take the limit of the sum as the number of subrectangles becomes large. The result is the **moment of inertia** of the lamina **about the x-axis**:

$$\boxed{6} \qquad \boxed{I_x = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D y^2 \rho(x, y)\, dA}$$

Similarly, the **moment of inertia about the y-axis** is

$$\boxed{7} \qquad \boxed{I_y = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D x^2 \rho(x, y)\, dA}$$

We also consider the **moment of inertia about the origin**, also called the **polar moment of inertia**:

$$\boxed{\mathbf{8}\quad I_0 = \lim_{m,\,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[(x_{ij}^*)^2 + (y_{ij}^*)^2\right] \rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_D (x^2 + y^2)\,\rho(x, y)\,dA}$$

Note that $I_0 = I_x + I_y$.

EXAMPLE 4 Find the moments of inertia I_x, I_y, and I_0 of a homogeneous disk D with density $\rho(x, y) = \rho$, center the origin, and radius a.

SOLUTION The boundary of D is the circle $x^2 + y^2 = a^2$ and in polar coordinates D is described by $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant r \leqslant a$. By Formula 6,

$$\begin{aligned} I_x &= \iint_D y^2 \rho\, dA = \rho \int_0^{2\pi} \int_0^a (r \sin\theta)^2\, r\, dr\, d\theta \\ &= \rho \int_0^{2\pi} \sin^2\theta\, d\theta \int_0^a r^3\, dr = \rho \int_0^{2\pi} \tfrac{1}{2}(1 - \cos 2\theta)\, d\theta \int_0^a r^3\, dr \\ &= \frac{\rho}{2}\Big[\theta - \tfrac{1}{2}\sin 2\theta\Big]_0^{2\pi} \left[\frac{r^4}{4}\right]_0^a = \frac{\pi\rho a^4}{4} \end{aligned}$$

Similarly, Formula 7 gives

$$\begin{aligned} I_y &= \iint_D x^2 \rho\, dA = \rho \int_0^{2\pi} \int_0^a (r \cos\theta)^2\, r\, dr\, d\theta \\ &= \rho \int_0^{2\pi} \tfrac{1}{2}(1 + \cos 2\theta)\, d\theta \int_0^a r^3\, dr = \frac{\pi\rho a^4}{4} \end{aligned}$$

(From the symmetry of the problem, it is expected that $I_x = I_y$.) We could use Formula 8 to compute I_0 directly, or use

$$I_0 = I_x + I_y = \frac{\pi\rho a^4}{4} + \frac{\pi\rho a^4}{4} = \frac{\pi\rho a^4}{2}$$ ■

In Example 4 notice that the mass of the disk is

$$m = \text{density} \times \text{area} = \rho(\pi a^2)$$

so the moment of inertia of the disk about the origin (like a wheel about its axle) can be written as

$$I_0 = \frac{\pi\rho a^4}{2} = \tfrac{1}{2}(\rho\pi a^2)a^2 = \tfrac{1}{2}ma^2$$

Thus if we increase the mass or the radius of the disk, we thereby increase the moment of inertia. In general, the moment of inertia plays much the same role in rotational motion that mass plays in linear motion. The moment of inertia of a wheel is what makes it difficult to start or stop the rotation of the wheel, just as the mass of a car is what makes it difficult to start or stop the motion of the car.

The **radius of gyration of a lamina about an axis** is the number R such that

$$\boxed{9} \qquad mR^2 = I$$

where m is the mass of the lamina and I is the moment of inertia about the given axis. Equation 9 says that if the mass of the lamina were concentrated at a distance R from the axis, then the moment of inertia of this "point mass" would be the same as the moment of inertia of the lamina.

In particular, the radius of gyration $\bar{\bar{y}}$ with respect to the x-axis and the radius of gyration $\bar{\bar{x}}$ with respect to the y-axis are given by the equations

$$\boxed{10} \qquad m\bar{\bar{y}}^2 = I_x \qquad m\bar{\bar{x}}^2 = I_y$$

Thus $(\bar{\bar{x}}, \bar{\bar{y}})$ is the point at which the mass of the lamina can be concentrated without changing the moments of inertia with respect to the coordinate axes. (Note the analogy with the center of mass.)

EXAMPLE 5 Find the radius of gyration about the x-axis of the disk in Example 4.

SOLUTION As noted, the mass of the disk is $m = \rho\pi a^2$, so from Equations 10 we have

$$\bar{\bar{y}}^2 = \frac{I_x}{m} = \frac{\frac{1}{4}\pi\rho a^4}{\rho\pi a^2} = \frac{a^2}{4}$$

Therefore the radius of gyration about the x-axis is $\bar{\bar{y}} = \frac{1}{2}a$, which is half the radius of the disk. ■

■ Probability

In Section 8.5 we considered the *probability density function* f of a continuous random variable X. This means that $f(x) \geqslant 0$ for all x, $\int_{-\infty}^{\infty} f(x)\,dx = 1$, and the probability that X lies between a and b is found by integrating f from a to b:

$$P(a \leqslant X \leqslant b) = \int_a^b f(x)\,dx$$

Now we consider a pair of continuous random variables X and Y, such as the lifetimes of two components of a machine or the height and weight of an adult female chosen at random. The **joint density function** of X and Y is a function f of two variables such that the probability that (X, Y) lies in a region D is

$$P((X, Y) \in D) = \iint_D f(x, y)\,dA$$

In particular, if the region is a rectangle, then the probability that X lies between a and b and Y lies between c and d is

$$P(a \leqslant X \leqslant b,\ c \leqslant Y \leqslant d) = \int_a^b \int_c^d f(x, y)\,dy\,dx$$

(See Figure 7.)

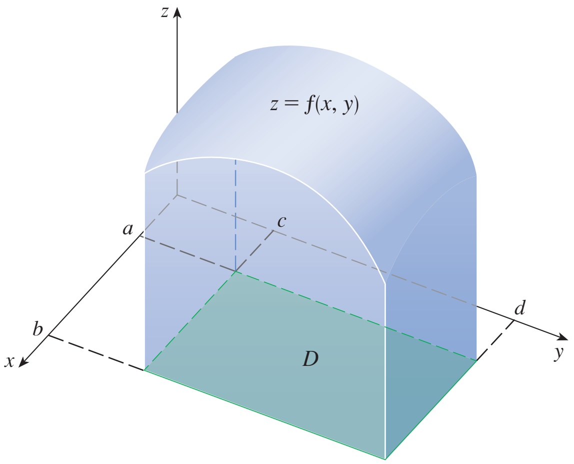


FIGURE 7
The probability that X lies between a and b and Y lies between c and d is the volume that lies above the rectangle $D = [a, b] \times [c, d]$ and below the graph of the joint density function.

Because probabilities aren't negative and are measured on a scale from 0 to 1, the joint density function has the following properties:

$$f(x, y) \geqslant 0 \qquad \iint_{\mathbb{R}^2} f(x, y)\, dA = 1$$

As in Exercise 15.3.50, the double integral over $\mathbb{R}^2$ is an improper integral defined as the limit of double integrals over expanding circles or squares, and we can write

$$\iint_{\mathbb{R}^2} f(x, y)\, dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$$

EXAMPLE 6 If the joint density function for X and Y is given by

$$f(x, y) = \begin{cases} C(x + 2y) & \text{if } 0 \leqslant x \leqslant 10,\ 0 \leqslant y \leqslant 10 \\ 0 & \text{otherwise} \end{cases}$$

find the value of the constant C. Then find $P(X \leqslant 7, Y \geqslant 2)$.

SOLUTION We find the value of C by ensuring that the double integral of f over $\mathbb{R}^2$ is equal to 1. Because $f(x, y) = 0$ outside the rectangle $[0, 10] \times [0, 10]$, we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dy\, dx = \int_0^{10} \int_0^{10} C(x + 2y)\, dy\, dx = C \int_0^{10} \Big[xy + y^2\Big]_{y=0}^{y=10} dx$$

$$= C \int_0^{10} (10x + 100)\, dx = 1500C$$

Therefore $1500C = 1$ and so $C = \frac{1}{1500}$.

Now we can compute the probability that X is at most 7 and Y is at least 2:

$$P(X \leqslant 7, Y \geqslant 2) = \int_{-\infty}^{7} \int_2^{\infty} f(x, y)\, dy\, dx = \int_0^7 \int_2^{10} \tfrac{1}{1500}(x + 2y)\, dy\, dx$$

$$= \tfrac{1}{1500} \int_0^7 \Big[xy + y^2\Big]_{y=2}^{y=10} dx = \tfrac{1}{1500} \int_0^7 (8x + 96)\, dx$$

$$= \tfrac{868}{1500} \approx 0.5787$$

■

Suppose X is a random variable with probability density function $f_1(x)$ and Y is a random variable with density function $f_2(y)$. Then X and Y are called **independent random variables** if their joint density function is the product of their individual density functions:

$$f(x, y) = f_1(x)f_2(y)$$

In Section 8.5 we modeled waiting times by using exponential density functions

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \mu^{-1}e^{-t/\mu} & \text{if } t \geqslant 0 \end{cases}$$

where μ is the mean waiting time. In the next example we consider a situation with two independent waiting times.

EXAMPLE 7 The manager of a movie theater determines that the average time moviegoers wait in line to buy a ticket for a film is 10 minutes and the average time they wait to buy popcorn is 5 minutes. Assuming that the waiting times are independent, find the probability that a moviegoer waits a total of less than 20 minutes before taking his or her seat.

SOLUTION Assuming that both the waiting time X for the ticket purchase and the waiting time Y in the refreshment line are modeled by exponential probability density functions, we can write the individual density functions as

$$f_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{10}e^{-x/10} & \text{if } x \geqslant 0 \end{cases} \qquad f_2(y) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{1}{5}e^{-y/5} & \text{if } y \geqslant 0 \end{cases}$$

Since X and Y are independent, the joint density function is the product:

$$f(x, y) = f_1(x)f_2(y) = \begin{cases} \frac{1}{50}e^{-x/10}e^{-y/5} & \text{if } x \geqslant 0, y \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

We are asked for the probability that $X + Y < 20$:

$$P(X + Y < 20) = P((X, Y) \in D)$$

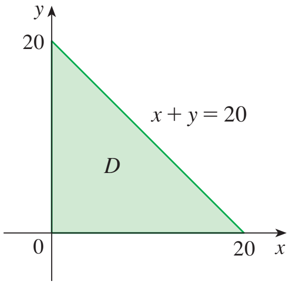


FIGURE 8

where D is the triangular region shown in Figure 8. Thus

$$\begin{aligned} P(X + Y < 20) &= \iint_D f(x, y)\, dA = \int_0^{20}\int_0^{20-x} \frac{1}{50}e^{-x/10}e^{-y/5}\, dy\, dx \\ &= \frac{1}{50}\int_0^{20} \left[e^{-x/10}(-5)e^{-y/5}\right]_{y=0}^{y=20-x} dx = \frac{1}{10}\int_0^{20} e^{-x/10}(1 - e^{(x-20)/5})\, dx \\ &= \frac{1}{10}\int_0^{20} (e^{-x/10} - e^{-4}e^{x/10})\, dx = 1 + e^{-4} - 2e^{-2} \approx 0.7476 \end{aligned}$$

This means that about 75% of the moviegoers wait less than 20 minutes before taking their seats. ■

■ Expected Values

Recall from Section 8.5 that if X is a random variable with probability density function f, then its *mean* is

$$\mu = \int_{-\infty}^{\infty} xf(x)\, dx$$

Now if X and Y are random variables with joint density function f, we define the ***X*-mean** and ***Y*-mean**, also called the **expected values** of X and Y, to be

$$\boxed{11} \qquad \mu_1 = \iint\limits_{\mathbb{R}^2} x f(x, y)\, dA \qquad\qquad \mu_2 = \iint\limits_{\mathbb{R}^2} y f(x, y)\, dA$$

Notice how closely the expressions for μ_1 and μ_2 in (11) resemble the moments M_x and M_y of a lamina with density function ρ in Equations 3 and 4. In fact, we can think of probability as being like continuously distributed mass. We calculate probability the way we calculate mass—by integrating a density function. And because the total "probability mass" is 1, the expressions for $\bar{x}$ and $\bar{y}$ in (5) show that we can think of the expected values of X and Y, μ_1 and μ_2, as the coordinates of the "center of mass" of the probability distribution.

In the next example we deal with normal distributions. As in Section 8.5, a single random variable is *normally distributed* if its probability density function is of the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}$$

where μ is the mean and σ is the standard deviation.

EXAMPLE 8 A factory produces (cylindrically shaped) roller bearings that are sold as having diameter 4.0 cm and length 6.0 cm. In fact, the diameters X are normally distributed with mean 4.0 cm and standard deviation 0.01 cm while the lengths Y are normally distributed with mean 6.0 cm and standard deviation 0.01 cm. Assuming that X and Y are independent, write the joint density function and graph it. Find the probability that a bearing randomly chosen from the production line has either length or diameter that differs from the mean by more than 0.02 cm.

SOLUTION We are given that X and Y are normally distributed with $\mu_1 = 4.0$, $\mu_2 = 6.0$, and $\sigma_1 = \sigma_2 = 0.01$. So the individual density functions for X and Y are

$$f_1(x) = \frac{1}{0.01\sqrt{2\pi}}\, e^{-(x-4)^2/0.0002} \qquad\qquad f_2(y) = \frac{1}{0.01\sqrt{2\pi}}\, e^{-(y-6)^2/0.0002}$$

Since X and Y are independent, the joint density function is the product:

$$\begin{aligned} f(x, y) = f_1(x) f_2(y) &= \frac{1}{0.0002\pi}\, e^{-(x-4)^2/0.0002} e^{-(y-6)^2/0.0002} \\ &= \frac{5000}{\pi}\, e^{-5000[(x-4)^2+(y-6)^2]} \end{aligned}$$

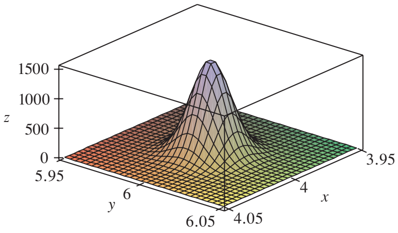


FIGURE 9
Graph of the bivariate normal joint density function in Example 8

A graph of this function is shown in Figure 9.

Let's first calculate the probability that both X and Y differ from their means by less than 0.02 cm. Using a calculator or computer to estimate the integral, we have

$$\begin{aligned} P(3.98 < X < 4.02,\ 5.98 < Y < 6.02) &= \int_{3.98}^{4.02} \int_{5.98}^{6.02} f(x, y)\, dy\, dx \\ &= \frac{5000}{\pi} \int_{3.98}^{4.02} \int_{5.98}^{6.02} e^{-5000[(x-4)^2+(y-6)^2]}\, dy\, dx \\ &\approx 0.91 \end{aligned}$$

Then the probability that either X or Y differs from its mean by more than 0.02 cm is approximately

$$1 - 0.91 = 0.09 \qquad \blacksquare$$

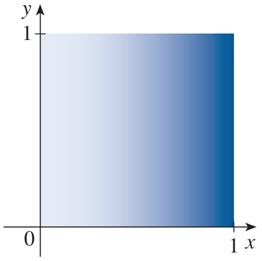
15.4 Exercises

1. Electric charge is distributed over the rectangle $0 \leqslant x \leqslant 5$, $2 \leqslant y \leqslant 5$ so that the charge density at (x, y) is $\sigma(x, y) = 2x + 4y$ (measured in coulombs per square meter). Find the total charge on the rectangle.

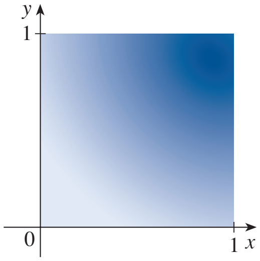
2. Electric charge is distributed over the disk $x^2 + y^2 \leqslant 1$ so that the charge density at (x, y) is $\sigma(x, y) = \sqrt{x^2 + y^2}$ (measured in coulombs per square meter). Find the total charge on the disk.

3–4 The figure shows a lamina that is shaded according to the given density function: darker shading indicates higher density. Estimate the location of the center of mass of the lamina, and then calculate its exact location.

3. $\rho(x, y) = x^2$



4. $\rho(x, y) = xy$



5–12 Find the mass and center of mass of the lamina that occupies the region D and has the given density function ρ.

5. $D = \{(x, y) \mid 1 \leqslant x \leqslant 3,\ 1 \leqslant y \leqslant 4\}$; $\rho(x, y) = ky^2$

6. $D = \{(x, y) \mid 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b\}$; $\rho(x, y) = 1 + x^2 + y^2$

7. D is the triangular region with vertices $(0, 0)$, $(2, 1)$, $(0, 3)$; $\rho(x, y) = x + y$

8. D is the triangular region enclosed by the lines $y = 0$, $y = 2x$, and $x + 2y = 1$; $\rho(x, y) = x$

9. D is bounded by $y = 1 - x^2$ and $y = 0$; $\rho(x, y) = ky$

10. D is bounded by $y = x + 2$ and $y = x^2$; $\rho(x, y) = kx^2$

11. D is bounded by the curves $y = e^{-x}$, $y = 0$, $x = 0$, $x = 1$; $\rho(x, y) = xy$

12. D is enclosed by the curves $y = 0$ and $y = \cos x$, $-\pi/2 \leqslant x \leqslant \pi/2$; $\rho(x, y) = y$

13. A lamina occupies the part of the disk $x^2 + y^2 \leqslant 1$ in the first quadrant. Find its center of mass if the density at any point is proportional to its distance from the x-axis.

14. Find the center of mass of the lamina in Exercise 13 if the density at any point is proportional to the square of its distance from the origin.

15. The boundary of a lamina consists of the semicircles $y = \sqrt{1 - x^2}$ and $y = \sqrt{4 - x^2}$ together with the portions of the x-axis that join them. Find the center of mass of the lamina if the density at any point is proportional to its distance from the origin.

16. Find the center of mass of the lamina in Exercise 15 if the density at any point is inversely proportional to its distance from the origin.

17. Find the center of mass of a lamina in the shape of an isosceles right triangle with equal sides of length a if the density at any point is proportional to the square of the distance from the vertex opposite the hypotenuse.

18. A lamina occupies the region inside the circle $x^2 + y^2 = 2y$ but outside the circle $x^2 + y^2 = 1$. Find the center of mass if the density at any point is inversely proportional to its distance from the origin.

19. Find the moments of inertia I_x, I_y, I_0 for the lamina of Exercise 5.

20. Find the moments of inertia I_x, I_y, I_0 for the lamina of Exercise 8.

21. Find the moments of inertia I_x, I_y, I_0 for the lamina of Exercise 17.

22. Consider a square fan blade with sides of length 2 and the lower left corner placed at the origin. If the density of the blade is $\rho(x, y) = 1 + 0.1x$, is it more difficult to rotate the blade about the x-axis or the y-axis?

23–26 A lamina with constant density $\rho(x, y) = \rho$ occupies the given region. Find the moments of inertia I_x and I_y and the radii of gyration $\bar{\bar{x}}$ and $\bar{\bar{y}}$.

23. The rectangle $0 \leqslant x \leqslant b$, $0 \leqslant y \leqslant h$

24. The triangle with vertices $(0, 0)$, $(b, 0)$, and $(0, h)$

25. The part of the disk $x^2 + y^2 \leqslant a^2$ in the first quadrant

26. The region under the curve $y = \sin x$ from $x = 0$ to $x = \pi$

T **27–28** Use a computer algebra system to find the mass, center of mass, and moments of inertia of the lamina that occupies the region D and has the given density function.

27. D is enclosed by the right loop of the four-leaved rose $r = \cos 2\theta$; $\rho(x, y) = x^2 + y^2$

28. $D = \{(x, y) \mid 0 \leqslant y \leqslant xe^{-x},\ 0 \leqslant x \leqslant 2\}$; $\rho(x, y) = x^2y^2$

29. The joint density function for a pair of random variables X and Y is

$$f(x, y) = \begin{cases} Cx(1 + y) & \text{if } 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant C.
(b) Find $P(X \leqslant 1, Y \leqslant 1)$.
(c) Find $P(X + Y \leqslant 1)$.

30. (a) Verify that

$$f(x, y) = \begin{cases} 4xy & \text{if } 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

is a joint density function.
(b) If X and Y are random variables whose joint density function is the function f in part (a), find
(i) $P\left(X \geqslant \frac{1}{2}\right)$ (ii) $P\left(X \geqslant \frac{1}{2}, Y \leqslant \frac{1}{2}\right)$
(c) Find the expected values of X and Y.

31. Suppose X and Y are random variables with joint density function

$$f(x, y) = \begin{cases} 0.1e^{-(0.5x + 0.2y)} & \text{if } x \geqslant 0,\ y \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Verify that f is indeed a joint density function.
(b) Find the following probabilities.
(i) $P(Y \geqslant 1)$ (ii) $P(X \leqslant 2, Y \leqslant 4)$
(c) Find the expected values of X and Y.

32. (a) A lamp has two bulbs, each of a type with average lifetime 1000 hours. Assuming that we can model the probability of failure of a bulb by an exponential density function with mean $\mu = 1000$, find the probability that both of the lamp's bulbs fail within 1000 hours.
(b) Another lamp has just one bulb of the same type as in part (a). If one bulb burns out and is replaced by a bulb of the same type, find the probability that the two bulbs fail within a total of 1000 hours.

T **33.** Suppose that X and Y are independent random variables, where X is normally distributed with mean 45 and standard deviation 0.5 and Y is normally distributed with mean 20 and standard deviation 0.1. Evaluate a double integral numerically to find the given probability correct to three decimal places.
(a) $P(40 \leqslant X \leqslant 50, 20 \leqslant Y \leqslant 25)$
(b) $P(4(X - 45)^2 + 100(Y - 20)^2 \leqslant 2)$

34. Xavier and Yolanda both have classes that end at noon and they agree to meet every day after class. They arrive at the coffee shop independently. Xavier's arrival time is X and Yolanda's arrival time is Y, where X and Y are measured in minutes after noon. The individual density functions are

$$f_1(x) = \begin{cases} e^{-x} & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad f_2(y) = \begin{cases} \frac{1}{50}y & \text{if } 0 \leqslant y \leqslant 10 \\ 0 & \text{otherwise} \end{cases}$$

(Xavier arrives sometime after noon and is more likely to arrive promptly than late. Yolanda always arrives by 12:10 PM and is more likely to arrive late than promptly.) After Yolanda arrives, she'll wait for up to half an hour for Xavier, but he won't wait for her. Find the probability that they meet.

35. When studying the spread of an epidemic, we assume that the probability that an infected individual will spread the disease to an uninfected individual is a function of the distance between them. Consider a circular city of radius 10 kilometers in which the population is uniformly distributed. For an uninfected individual at a fixed point $A(x_0, y_0)$, assume that the probability function is given by

$$f(P) = \tfrac{1}{20}[20 - d(P, A)]$$

where $d(P, A)$ denotes the distance between points P and A.
(a) Suppose the exposure of a person to the disease is the sum of the probabilities of catching the disease from all members of the population. Assume that the infected people are uniformly distributed throughout the city, with k infected individuals per square kilometer. Find a double integral that represents the exposure of a person residing at A.
(b) Evaluate the integral for the case in which A is the center of the city and for the case in which A is located on the edge of the city. Where would you prefer to live?

15.5 Surface Area

In Section 16.6 we will deal with areas of more general surfaces, called parametric surfaces, and so this section may be omitted if that later section will be covered.

In this section we apply double integrals to the problem of computing the area of a surface. In Section 8.2 we found the area of a very special type of surface—a surface of revolution—by the methods of single-variable calculus. Here we compute the area of a surface with equation $z = f(x, y)$, the graph of a function of two variables.

Let S be a surface with equation $z = f(x, y)$, where f has continuous partial derivatives. For simplicity in deriving the surface area formula, we assume that $f(x, y) \geqslant 0$ and the domain D of f is a rectangle. We divide D into small rectangles R_{ij} with area $\Delta A = \Delta x\, \Delta y$. If (x_i, y_j) is the corner of R_{ij} closest to the origin, let $P_{ij}(x_i, y_j, f(x_i, y_j))$ be

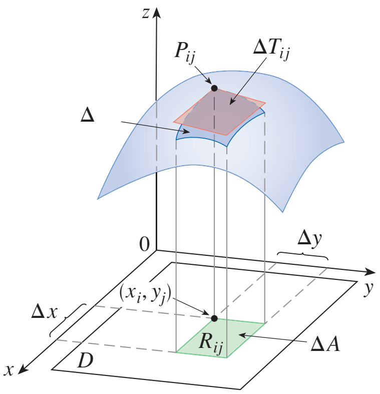


FIGURE 1

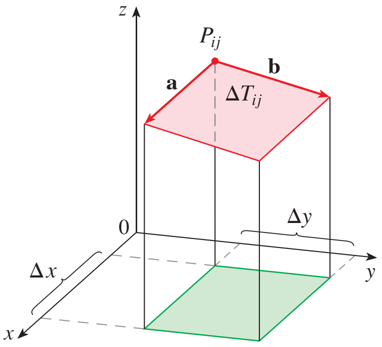


FIGURE 2

the point on S directly above it (see Figure 1). The tangent plane to S at P_{ij} is an approximation to S near P_{ij}. So the area ΔT_{ij} of the part of this tangent plane (a parallelogram) that lies directly above R_{ij} is an approximation to the area ΔS_{ij} of the part of S that lies directly above R_{ij}. Thus the sum $\Sigma\Sigma\, \Delta T_{ij}$ is an approximation to the total area of S, and this approximation appears to improve as the number of rectangles increases. Therefore we define the **surface area** of S to be

$$\boxed{1} \qquad A(S) = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$

To find a formula that is more convenient than Equation 1 for computational purposes, we let **a** and **b** be the vectors that start at P_{ij} and lie along the sides of the parallelogram with area ΔT_{ij}. (See Figure 2.) Then $\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}|$. Recall from Section 14.3 that $f_x(x_i, y_j)$ and $f_y(x_i, y_j)$ are the slopes of the tangent lines through P_{ij} in the directions of **a** and **b**. Therefore

$$\mathbf{a} = \Delta x\, \mathbf{i} + f_x(x_i, y_j)\, \Delta x\, \mathbf{k}$$

$$\mathbf{b} = \Delta y\, \mathbf{j} + f_y(x_i, y_j)\, \Delta y\, \mathbf{k}$$

and

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & f_x(x_i, y_j)\, \Delta x \\ 0 & \Delta y & f_y(x_i, y_j)\, \Delta y \end{vmatrix} \\ &= -f_x(x_i, y_j)\, \Delta x\, \Delta y\, \mathbf{i} - f_y(x_i, y_j)\, \Delta x\, \Delta y\, \mathbf{j} + \Delta x\, \Delta y\, \mathbf{k} \\ &= [-f_x(x_i, y_j)\mathbf{i} - f_y(x_i, y_j)\mathbf{j} + \mathbf{k}]\, \Delta A \end{aligned}$$

Thus $$\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}| = \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\; \Delta A$$

From Definition 1 we then have

$$\begin{aligned} A(S) &= \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij} \\ &= \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\; \Delta A \end{aligned}$$

and by the definition of a double integral we get the following formula.

$\boxed{2}$ The area of the surface with equation $z = f(x, y)$, $(x, y) \in D$, where f_x and f_y are continuous, is

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\; dA$$

We will verify in Section 16.6 that this formula is consistent with our previous formula for the area of a surface of revolution. If we use the alternative notation for partial derivatives, we can rewrite Formula 2 as follows:

$$\boxed{3} \qquad \boxed{A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\; dA}$$

Notice the similarity between the surface area formula in Equation 3 and the arc length formula from Section 8.1:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dx$$

EXAMPLE 1 Find the surface area of the part of the surface $z = x^2 + 2y + 2$ that lies above the triangular region T in the xy-plane with vertices (0, 0), (1, 0), and (1, 1).

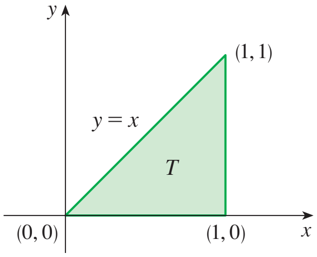


FIGURE 3

SOLUTION The region T is shown in Figure 3 and is described by

$$T = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x\}$$

Using Formula 2 with $f(x, y) = x^2 + 2y + 2$, we get

$$A = \iint_T \sqrt{(2x)^2 + (2)^2 + 1}\, dA = \int_0^1 \int_0^x \sqrt{4x^2 + 5}\, dy\, dx$$

$$= \int_0^1 x\sqrt{4x^2 + 5}\, dx = \tfrac{1}{8} \cdot \tfrac{2}{3}(4x^2 + 5)^{3/2}\Big]_0^1 = \tfrac{1}{12}\left(27 - 5\sqrt{5}\right)$$

Figure 4 shows the portion of the surface whose area we have just computed. ■

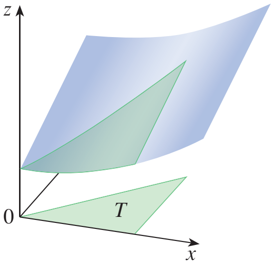


FIGURE 4

EXAMPLE 2 Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

SOLUTION The plane intersects the paraboloid in the circle $x^2 + y^2 = 9$, $z = 9$. Therefore the given surface lies above the disk D with center the origin and radius 3. (See Figure 5.) Using Formula 3, we have

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA$$

$$= \iint_D \sqrt{1 + 4(x^2 + y^2)}\, dA$$

Converting to polar coordinates, we obtain

$$A = \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^3 \tfrac{1}{8}\sqrt{1 + 4r^2}\,(8r)\, dr$$

$$= 2\pi\left(\tfrac{1}{8}\right)\tfrac{2}{3}(1 + 4r^2)^{3/2}\Big]_0^3 = \frac{\pi}{6}\left(37\sqrt{37} - 1\right)$$ ■

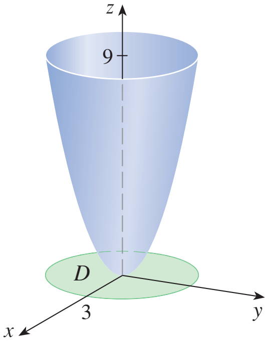
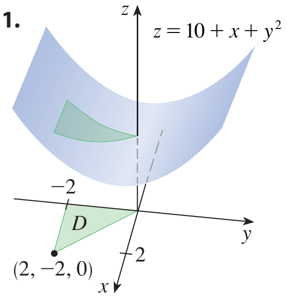


FIGURE 5

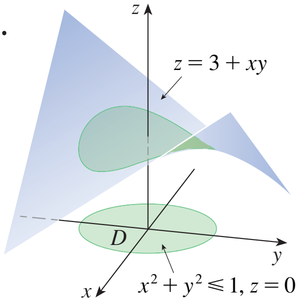
15.5 Exercises

1–2 Find the area of the indicated part of the surface (above the region D).

1.



2.



3–14 Find the area of the surface.

3. The part of the plane $5x + 3y - z + 6 = 0$ that lies above the rectangle $[1, 4] \times [2, 6]$
4. The part of the plane $6x + 4y + 2z = 1$ that lies inside the cylinder $x^2 + y^2 = 25$
5. The part of the plane $3x + 2y + z = 6$ that lies in the first octant
6. The part of the surface $2y + 4z - x^2 = 5$ that lies above the triangle with vertices (0, 0), (2, 0), and (2, 4)
7. The part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the plane $z = -2$

8. The part of the cylinder $x^2 + z^2 = 4$ that lies above the square with vertices (0, 0), (1, 0), (0, 1), and (1, 1)

9. The part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

10. The surface $z = \frac{2}{3}(x^{3/2} + y^{3/2}),\ 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1$

11. The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$

12. The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 1$

13. The part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies within the cylinder $x^2 + y^2 = ax$ and above the xy-plane

14. The part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$

T **15–16** Find the area of the surface correct to four decimal places by first simplifying an expression for area to one in terms of a single integral, and then evaluating the integral numerically.

15. The part of the surface $z = 1/(1 + x^2 + y^2)$ that lies above the disk $x^2 + y^2 \leqslant 1$

16. The part of the surface $z = \cos(x^2 + y^2)$ that lies inside the cylinder $x^2 + y^2 = 1$

---

17. (a) Use the Midpoint Rule for double integrals (see Section 15.1) with four squares to estimate the surface area of the portion of the paraboloid $z = x^2 + y^2$ that lies above the square $[0, 1] \times [0, 1]$.

T (b) Use a computer algebra system to approximate the surface area in part (a) to four decimal places. Compare with the answer to part (a).

18. (a) Use the Midpoint Rule for double integrals with $m = n = 2$ to estimate the area of the surface $z = xy + x^2 + y^2,\ 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 2$.

T (b) Use a computer algebra system to approximate the surface area in part (a) to four decimal places. Compare with the answer to part (a).

T **19.** Use a computer algebra system to find the exact area of the surface $z = 1 + 2x + 3y + 4y^2$, $1 \leqslant x \leqslant 4$, $0 \leqslant y \leqslant 1$.

T **20.** Use a computer algebra system to find the exact area of the surface

$$z = 1 + x + y + x^2 \qquad -2 \leqslant x \leqslant 1 \quad -1 \leqslant y \leqslant 1$$

Illustrate by graphing the surface.

T **21.** Use a computer algebra system to find, correct to four decimal places, the area of the part of the surface $z = 1 + x^2y^2$ that lies above the disk $x^2 + y^2 \leqslant 1$.

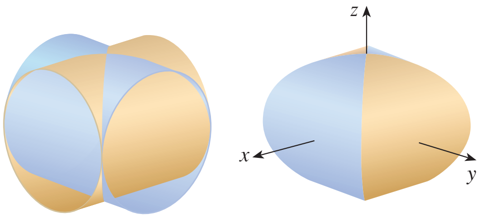
T **22.** Use a computer algebra system to find, correct to four decimal places, the area of the part of the surface $z = (1 + x^2)/(1 + y^2)$ that lies above the square $|x| + |y| \leqslant 1$. Illustrate by graphing this part of the surface.

23. Show that the area of the part of the plane $z = ax + by + c$ that projects onto a region D in the xy-plane with area $A(D)$ is $\sqrt{a^2 + b^2 + 1}\,A(D)$.

24. If you attempt to use Formula 2 to find the area of the top half of the sphere $x^2 + y^2 + z^2 = a^2$, you have a slight problem because the double integral is improper. In fact, the integrand has an infinite discontinuity at every point of the boundary circle $x^2 + y^2 = a^2$. However, the integral can be computed as the limit of the integral over the disk $x^2 + y^2 \leqslant t^2$ as $t \to a^-$. Use this method to show that the area of a sphere of radius a is $4\pi a^2$.

25. Find the area of the finite part of the paraboloid $y = x^2 + z^2$ cut off by the plane $y = 25$. [*Hint:* Project the surface onto the xz-plane.]

26. The figure shows the surface created when the cylinder $y^2 + z^2 = 1$ intersects the cylinder $x^2 + z^2 = 1$. Find the area of this surface.



15.6 Triple Integrals

Just as we defined single integrals for functions of one variable and double integrals for functions of two variables, so we can define triple integrals for functions of three variables.

Triple Integrals over Rectangular Boxes

Let's first deal with the simplest case where f is defined on a rectangular box:

$$\boxed{1} \qquad B = \{(x, y, z) \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d,\ r \leqslant z \leqslant s\}$$

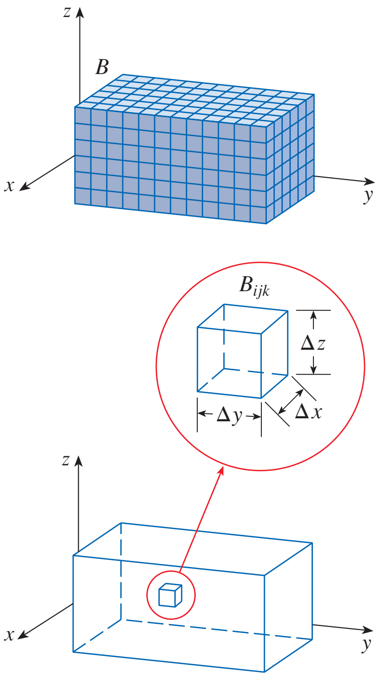


FIGURE 1

The first step is to divide B into sub-boxes. We do this by dividing the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal width Δx, dividing $[c, d]$ into m subintervals of width Δy, and dividing $[r, s]$ into n subintervals of width Δz. The planes through the endpoints of these subintervals parallel to the coordinate planes divide the box B into lmn sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

which are shown in Figure 1. Each sub-box has volume $\Delta V = \Delta x \, \Delta y \, \Delta z$.

Then we form the **triple Riemann sum**

$$\boxed{2} \qquad \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \, \Delta V$$

where the sample point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is in B_{ijk}. By analogy with the definition of a double integral (15.1.5), we define the triple integral as the limit of the triple Riemann sums in (2).

$\boxed{3}$ **Definition** The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z) \, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \, \Delta V$$

if this limit exists.

Again, the triple integral always exists if f is continuous. We can choose the sample point to be any point in the sub-box, but if we choose it to be the point (x_i, y_j, z_k) we get a simpler-looking expression for the triple integral:

$$\iiint_B f(x, y, z) \, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i, y_j, z_k) \, \Delta V$$

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

$\boxed{4}$ **Fubini's Theorem for Triple Integrals** If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z) \, dV = \int_r^s \int_c^d \int_a^b f(x, y, z) \, dx \, dy \, dz$$

The iterated integral on the right side of Fubini's Theorem means that we integrate first with respect to x (keeping y and z fixed), then we integrate with respect to y (keeping z fixed), and finally we integrate with respect to z. There are five other possible orders in which we can integrate, all of which give the same value. For instance, if we integrate with respect to y, then z, and then x, we have

$$\iiint_B f(x, y, z) \, dV = \int_a^b \int_r^s \int_c^d f(x, y, z) \, dy \, dz \, dx$$

EXAMPLE 1 Evaluate the triple integral $\iiint_B xyz^2\,dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

SOLUTION We could use any of the six possible orders of integration. If we choose to integrate with respect to x, then y, and then z, we obtain

$$\iiint_B xyz^2\,dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\,dx\,dy\,dz = \int_0^3 \int_{-1}^2 \left[\frac{x^2yz^2}{2}\right]_{x=0}^{x=1} dy\,dz$$

$$= \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\,dy\,dz = \int_0^3 \left[\frac{y^2z^2}{4}\right]_{y=-1}^{y=2} dz$$

$$= \int_0^3 \frac{3z^2}{4}\,dz = \frac{z^3}{4}\Big]_0^3 = \frac{27}{4}$$

■

Triple Integrals over General Regions

Now we define the **triple integral over a general bounded region *E*** in three-dimensional space (a solid) by much the same procedure that we used for double integrals (15.2.2). We enclose E in a box B of the type given by Equation 1. Then we define F so that it agrees with f on E but is 0 for points in B that are outside E. By definition,

$$\iiint_E f(x, y, z)\,dV = \iiint_B F(x, y, z)\,dV$$

This integral exists if f is continuous and the boundary of E is "reasonably smooth." The triple integral has essentially the same properties as the double integral (Properties 5–8 in Section 15.2).

We restrict our attention to continuous functions f and to certain simple types of regions. A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

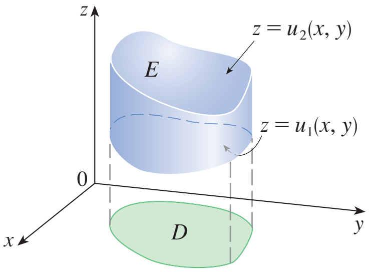


FIGURE 2
A type 1 solid region

5 $$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

where D is the projection of E onto the xy-plane as shown in Figure 2. Notice that the upper boundary of the solid E is the surface with equation $z = u_2(x, y)$, while the lower boundary is the surface $z = u_1(x, y)$.

By the same sort of argument that led to (15.2.3), it can be shown that if E is a type 1 region given by Equation 5, then

6 $$\boxed{\iiint_E f(x, y, z)\,dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\,dz\right] dA}$$

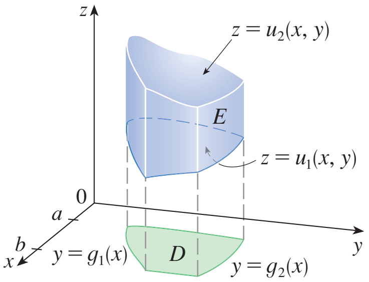


FIGURE 3
A type 1 solid region where the projection D is a type I plane region

The meaning of the inner integral on the right side of Equation 6 is that x and y are held fixed, and therefore $u_1(x, y)$ and $u_2(x, y)$ are regarded as constants, while $f(x, y, z)$ is integrated with respect to z.

In particular, if the projection D of E onto the xy-plane is a type I plane region (as in Figure 3), then

$$E = \{(x, y, z) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x),\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

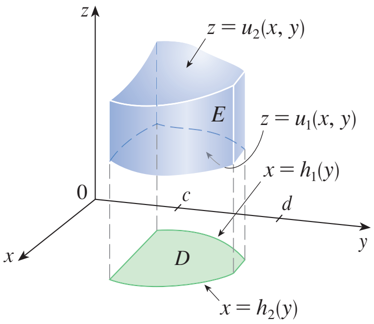


FIGURE 4
A type 1 solid region with a type II projection

and Equation 6 becomes

7
$$\boxed{\iiint\limits_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dy\, dx}$$

If, on the other hand, D is a type II plane region (as in Figure 4), then

$$E = \{(x, y, z) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y),\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

and Equation 6 becomes

8
$$\boxed{\iiint\limits_E f(x, y, z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dx\, dy}$$

EXAMPLE 2 Evaluate $\iiint_E z\, dV$ where E is the solid in the first octant bounded by the surface $z = 12xy$ and the planes $y = x$, $x = 1$.

SOLUTION When we set up a triple integral it's wise to draw *two* diagrams: one of the solid region E (Figure 5) and, for a type 1 region, one of its projection D onto the xy-plane (Figure 6). The lower boundary of the solid E is the plane $z = 0$ and the upper boundary is the surface $z = 12xy$, so we use $u_1(x, y) = 0$ and $u_2(x, y) = 12xy$ in Formula 7. Notice that the projection of E onto the xy-plane is the triangular region shown in Figure 6, and we have

9
$$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x, 0 \leqslant z \leqslant 12xy\}$$

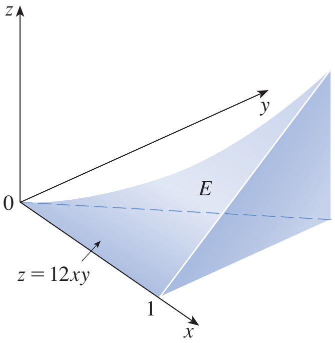


FIGURE 5

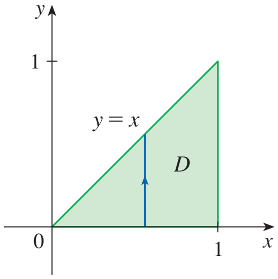


FIGURE 6

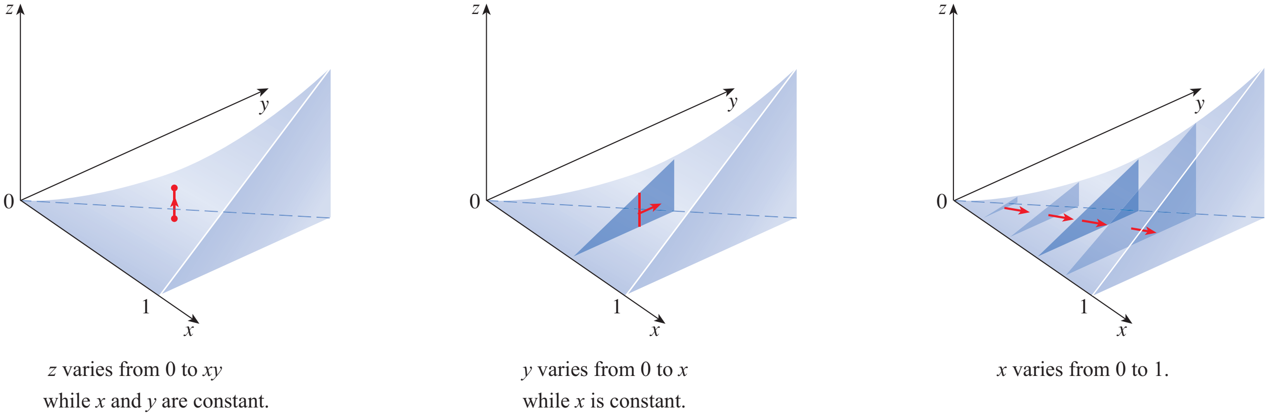
This description of E as a type 1 region enables us to evaluate the integral as follows:

$$\iiint\limits_E z\, dV = \int_0^1 \int_0^x \int_0^{12xy} z\, dz\, dy\, dx = \int_0^1 \int_0^x \left[\frac{z^2}{2}\right]_{z=0}^{z=12xy} dy\, dx$$

$$= \tfrac{1}{2} \int_0^1 \int_0^x (12xy)^2\, dy\, dx = 72 \int_0^1 \int_0^x x^2 y^2\, dy\, dx$$

$$= 72 \int_0^1 \left[x^2 \frac{y^3}{3}\right]_{y=0}^{y=x} dx = 24 \int_0^1 x^5\, dx = 24\left[\frac{x^6}{6}\right]_{x=0}^{x=1} = 4 \qquad \blacksquare$$

Figure 7 shows how the solid E of Example 2 is swept out by the iterated triple integral if we integrate first with respect to z, then y, then x.



z varies from 0 to xy while x and y are constant.

y varies from 0 to x while x is constant.

x varies from 0 to 1.

FIGURE 7

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D,\ u_1(y, z) \leqslant x \leqslant u_2(y, z)\}$$

where, this time, D is the projection of E onto the yz-plane (see Figure 8). The back surface is $x = u_1(y, z)$, the front surface is $x = u_2(y, z)$, and we have

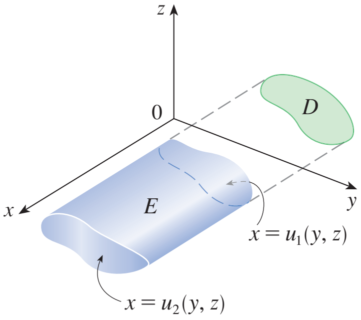


FIGURE 8
A type 2 region

$$\boxed{10} \qquad \iiint\limits_E f(x, y, z)\, dV = \iint\limits_D \left[\int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z)\, dx \right] dA$$

Finally, a **type 3** region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \leqslant y \leqslant u_2(x, z)\}$$

where D is the projection of E onto the xz-plane, $y = u_1(x, z)$ is the left surface, and $y = u_2(x, z)$ is the right surface (see Figure 9). For this type of region we have

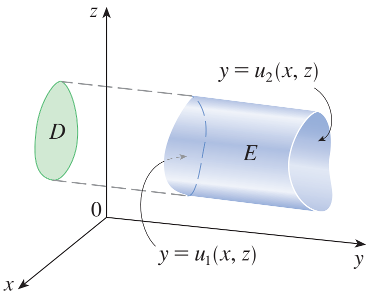


FIGURE 9
A type 3 region

$$\boxed{11} \qquad \iiint\limits_E f(x, y, z)\, dV = \iint\limits_D \left[\int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z)\, dy \right] dA$$

In each of Equations 10 and 11 there may be two possible expressions for the integral depending on whether D is a type I or type II plane region (and corresponding to Equations 7 and 8).

EXAMPLE 3 Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

SOLUTION The solid E is shown in Figure 10. If we regard it as a type 1 region, then we need to consider its projection D_1 onto the xy-plane, which is the parabolic region shown in Figures 10 and 11. (The trace of $y = x^2 + z^2$ in the plane $z = 0$ is the parabola $y = x^2$.)

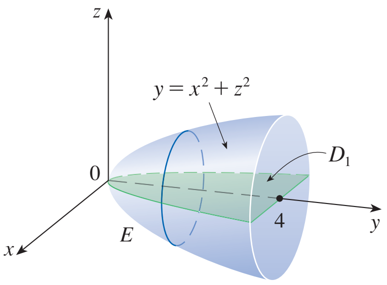


FIGURE 10
Region of integration

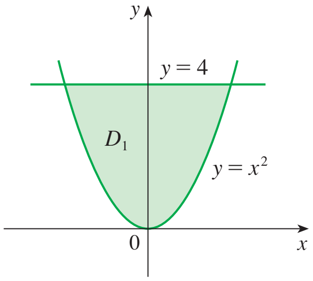


FIGURE 11
Projection onto the xy-plane

From $y = x^2 + z^2$ we obtain $z = \pm\sqrt{y - x^2}$, so the lower boundary surface of E is $z = -\sqrt{y - x^2}$ and the upper surface is $z = \sqrt{y - x^2}$. Therefore the description of E as a type 1 region is

$$E = \left\{(x, y, z) \mid -2 \leqslant x \leqslant 2,\ x^2 \leqslant y \leqslant 4,\ -\sqrt{y - x^2} \leqslant z \leqslant \sqrt{y - x^2}\right\}$$

and so we obtain

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \int_{-2}^{2} \int_{x^2}^{4} \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx$$

Although this expression is correct, it is extremely difficult to evaluate. So let's instead consider E as a region of a different type. If we regard E as a type 3 region, then we need to consider its projection D_3 onto the xz-plane, which is the disk $x^2 + z^2 \leqslant 4$ shown in Figures 12 and 13. (The trace of $y = x^2 + z^2$ in the plane $y = 4$ is the circle $x^2 + z^2 = 4$.)

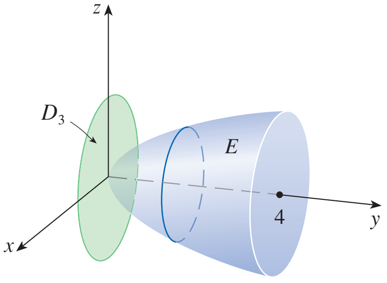


FIGURE 12
Region of integration

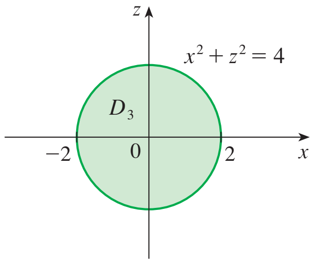


FIGURE 13
Projection onto the xz-plane

Then the left boundary of E is the paraboloid $y = x^2 + z^2$ and the right boundary is the plane $y = 4$, so taking $u_1(x, z) = x^2 + z^2$ and $u_2(x, z) = 4$ in Equation 11, we have

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \iint_{D_3} \left[\int_{x^2+z^2}^{4} \sqrt{x^2 + z^2}\, dy\right] dA = \iint_{D_3} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dA$$

⊘ The most difficult step in evaluating a triple integral is setting up an expression for the region of integration (such as Equation 9 in Example 2). Remember that the limits of integration in the inner integral contain at most two variables, the limits of integration in the middle integral contain at most one variable, and the limits of integration in the outer integral must be constants.

Although this integral could be written as

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dz\, dx$$

it's easier to convert to polar coordinates in the xz-plane: $x = r\cos\theta$, $z = r\sin\theta$. This gives

$$\begin{aligned}\iiint_E \sqrt{x^2 + z^2}\, dV &= \iint_{D_3} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dA\\ &= \int_0^{2\pi}\int_0^2 (4 - r^2) r\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^2 (4r^2 - r^4)\, dr\\ &= 2\pi\left[\frac{4r^3}{3} - \frac{r^5}{5}\right]_0^2 = \frac{128\pi}{15}\end{aligned}$$

■

■ Changing the Order of Integration

Fubini's Theorem for Triple Integrals allows us to express a triple integral as an iterated integral, and there are six different orders of integration in which we can do this. Given an iterated integral, it may be advantageous to change the order of integration because evaluating an iterated integral in one order may be simpler than in another. In the next example we investigate equivalent iterated integrals using different orders of integration.

EXAMPLE 4 Express the iterated integral $\int_0^1\int_0^{x^2}\int_0^y f(x, y, z)\, dz\, dy\, dx$ as a triple integral and then rewrite it as an iterated integral in the following orders.
(a) Integrate first with respect to x, then z, and then y.
(b) Integrate first with respect to y, then x, and then z.

SOLUTION We can write

$$\int_0^1\int_0^{x^2}\int_0^y f(x, y, z)\, dz\, dy\, dx = \iiint_E f(x, y, z)\, dV$$

where $E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x^2, 0 \leqslant z \leqslant y\}$. From this description of E as a type 1 region we see that E lies between the lower surface $z = 0$ and the upper surface $z = y$, and its projection onto the xy-plane is $\{(x, y) \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x^2\}$, as shown in Figures 14 and 15. So E is the solid enclosed by the planes $z = 0$, $x = 1$, $y = z$ and the parabolic cylinder $y = x^2$ (or $x = \sqrt{y}$).

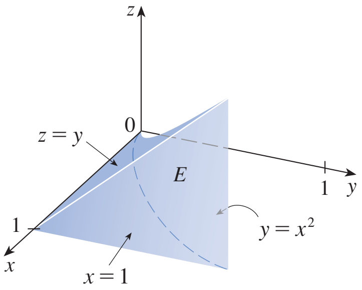


FIGURE 14 The solid E

Using Figure 14 as a guide, we can write projections onto the three coordinate planes as follows (see Figure 15):

$$\text{onto the } xy\text{-plane:}\quad D_1 = \{(x, y) \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x^2\}$$
$$= \{(x, y) \mid 0 \leqslant y \leqslant 1, \sqrt{y} \leqslant x \leqslant 1\}$$

$$\text{onto the } yz\text{-plane:}\quad D_2 = \{(y, z) \mid 0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant y\}$$
$$= \{(y, z) \mid 0 \leqslant z \leqslant 1, z \leqslant y \leqslant 1\}$$

$$\text{onto the } xz\text{-plane:}\quad D_3 = \{(x, z) \mid 0 \leqslant x \leqslant 1, 0 \leqslant z \leqslant x^2\}$$
$$= \{(x, z) \mid 0 \leqslant z \leqslant 1, \sqrt{z} \leqslant x \leqslant 1\}$$

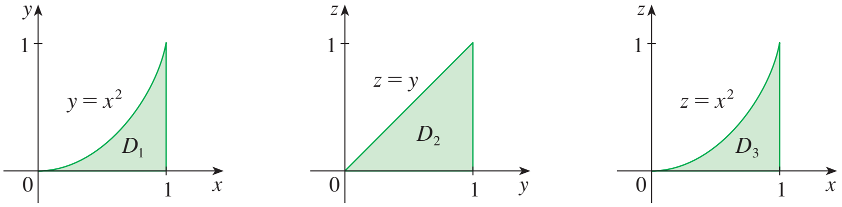


FIGURE 15
Projections of E

(a) In order to integrate first with respect to x, then z, and then y, we need to consider E as a type 2 region where the back boundary is the surface $x = \sqrt{y}$ and the front boundary is the plane $x = 1$; the projection onto the yz-plane is D_2. We describe E by

$$E = \left\{(x, y, z) \mid 0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant y, \sqrt{y} \leqslant x \leqslant 1\right\}$$

and then
$$\iiint\limits_E f(x, y, z)\, dV = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x, y, z)\, dx\, dz\, dy$$

(b) In order to integrate first with respect to y, then x, and then z, we need to consider E as a type 3 region where the left boundary is the plane $y = z$ and the right boundary is the surface $y = x^2$. The projection onto the xz-plane is D_3 and

$$E = \left\{(x, y, z) \mid 0 \leqslant z \leqslant 1, \sqrt{z} \leqslant x \leqslant 1, z \leqslant y \leqslant x^2\right\}$$

Thus
$$\iiint\limits_E f(x, y, z)\, dV = \int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f(x, y, z)\, dy\, dx\, dz$$ ■

Applications of Triple Integrals

Recall that if $f(x) \geqslant 0$, then the single integral $\int_a^b f(x)\, dx$ represents the area under the curve $y = f(x)$ from a to b, and if $f(x, y) \geqslant 0$, then the double integral $\iint_D f(x, y)\, dA$ represents the volume under the surface $z = f(x, y)$ and above D. The corresponding interpretation of a triple integral $\iiint_E f(x, y, z)\, dV$, where $f(x, y, z) \geqslant 0$, is not very useful because it would be the "hypervolume" of a four-dimensional object and, of course, that is very difficult to visualize. (Remember that E is just the *domain* of the function f; the graph of f lies in four-dimensional space.) Nonetheless, the triple integral $\iiint_E f(x, y, z)\, dV$ can be interpreted in different ways in different physical situations, depending on the physical interpretations of x, y, z, and $f(x, y, z)$.

Let's begin with the special case where $f(x, y, z) = 1$ for all points in E. Then the triple integral does represent the volume of E:

12
$$\boxed{V(E) = \iiint\limits_E dV}$$

For example, you can see this in the case of a type 1 region by putting $f(x, y, z) = 1$ in Formula 6:

$$\iiint\limits_E 1\, dV = \iint\limits_D \left[\int_{u_1(x, y)}^{u_2(x, y)} dz\right] dA = \iint\limits_D [u_2(x, y) - u_1(x, y)]\, dA$$

and from Section 15.2 we know this represents the volume that lies between the surfaces $z = u_1(x, y)$ and $z = u_2(x, y)$.

EXAMPLE 5 Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

SOLUTION The tetrahedron T and its projection D onto the xy-plane are shown in Figures 16 and 17. The lower boundary of T is the plane $z = 0$ and the upper boundary is the plane $x + 2y + z = 2$, that is, $z = 2 - x - 2y$.

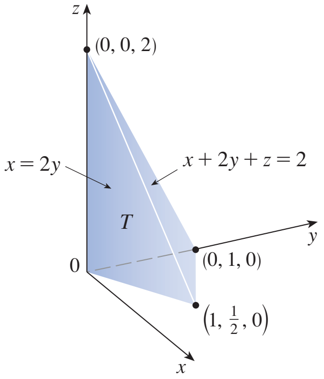


FIGURE 16

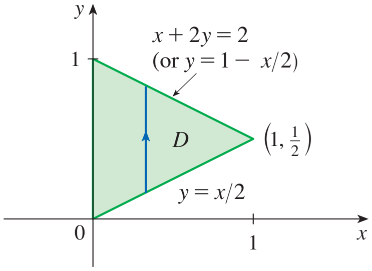


FIGURE 17

Therefore we have

$$V(T) = \iiint_T dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz\, dy\, dx$$
$$= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx = \tfrac{1}{3}$$

by the same calculation as in Example 15.2.4.

(Notice that it is not necessary to use triple integrals to compute volumes. They simply give an alternative method for setting up the calculation.) ■

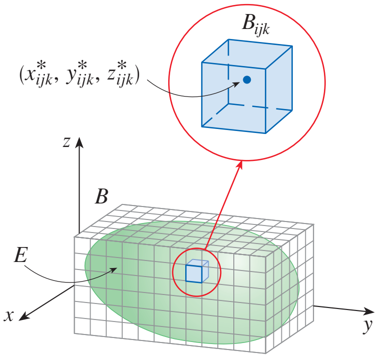


FIGURE 18
The mass of each sub-box B_{ijk} is approximated by $\rho(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V$

All the applications of double integrals in Section 15.4 can be extended to triple integrals using analogous reasoning. For example, suppose that a solid object occupying a region E has density $\rho(x, y, z)$, in units of mass per unit volume, at each point (x, y, z) in E. To find the total mass m of E we divide a rectangular box B containing E into sub-boxes B_{ijk} of the same size (as in Figure 18), and consider $\rho(x, y, z)$ to be 0 outside E. If we choose a point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ in B_{ijk}, then the mass of the part of E that occupies B_{ijk} is approximately $\rho(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V$, where ΔV is the volume of B_{ijk}. We get an approximation to the total mass by adding the (approximate) masses of all the sub-boxes, and if we increase the number of sub-boxes, we obtain the total mass m of E as the limiting value of the approximations:

$$\boxed{13} \qquad m = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} \rho(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V = \iiint_E \rho(x, y, z)\, dV$$

Similarly, the **moments** of E about the three coordinate planes are

$$\boxed{14} \qquad M_{yz} = \iiint_E x\,\rho(x, y, z)\, dV \qquad M_{xz} = \iiint_E y\,\rho(x, y, z)\, dV$$
$$M_{xy} = \iiint_E z\,\rho(x, y, z)\, dV$$

The **center of mass** is located at the point $(\bar{x}, \bar{y}, \bar{z})$, where

$$\boxed{15} \qquad \bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m}$$

If the density is constant, the center of mass of the solid is called the **centroid** of E. The **moments of inertia** about the three coordinate axes are

$$\boxed{16} \quad I_x = \iiint_E (y^2 + z^2)\,\rho(x, y, z)\,dV \qquad I_y = \iiint_E (x^2 + z^2)\,\rho(x, y, z)\,dV$$

$$I_z = \iiint_E (x^2 + y^2)\,\rho(x, y, z)\,dV$$

As in Section 15.4, the total **electric charge** on a solid object occupying a region E and having charge density $\sigma(x, y, z)$ is

$$Q = \iiint_E \sigma(x, y, z)\,dV$$

If we have three continuous random variables X, Y, and Z, their **joint density function** is a function of three variables such that the probability that (X, Y, Z) lies in E is

$$P((X, Y, Z) \in E) = \iiint_E f(x, y, z)\,dV$$

In particular,

$$P(a \leqslant X \leqslant b,\ c \leqslant Y \leqslant d,\ r \leqslant Z \leqslant s) = \int_a^b \int_c^d \int_r^s f(x, y, z)\,dz\,dy\,dx$$

The joint density function satisfies

$$f(x, y, z) \geqslant 0 \qquad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)\,dz\,dy\,dx = 1$$

EXAMPLE 6 Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$, and $x = 1$.

SOLUTION The solid E and its projection onto the xy-plane are shown in Figure 19. The lower and upper surfaces of E are the planes $z = 0$ and $z = x$, so we describe E as a type 1 region:

$$E = \{(x, y, z) \mid -1 \leqslant y \leqslant 1,\ y^2 \leqslant x \leqslant 1,\ 0 \leqslant z \leqslant x\}$$

Then, if the density is $\rho(x, y, z) = \rho$, the mass is

$$m = \iiint_E \rho\,dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x \rho\,dz\,dx\,dy$$

$$= \rho \int_{-1}^{1} \int_{y^2}^{1} x\,dx\,dy = \rho \int_{-1}^{1} \left[\frac{x^2}{2}\right]_{x=y^2}^{x=1} dy$$

$$= \frac{\rho}{2} \int_{-1}^{1} (1 - y^4)\,dy = \rho \int_0^1 (1 - y^4)\,dy$$

$$= \rho\left[y - \frac{y^5}{5}\right]_0^1 = \frac{4\rho}{5}$$

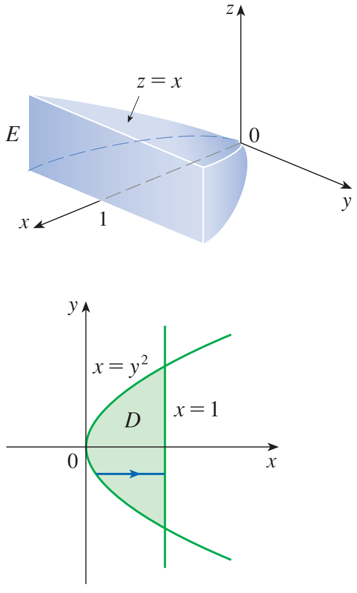


FIGURE 19

Because of the symmetry of E and ρ about the xz-plane, we can immediately say that $M_{xz} = 0$ and therefore $\bar{y} = 0$. The other moments are

$$\begin{aligned}
M_{yz} &= \iiint_E x\rho \, dV = \int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} x\rho \, dz\, dx\, dy \\
&= \rho \int_{-1}^{1}\int_{y^2}^{1} x^2 \, dx\, dy = \rho \int_{-1}^{1} \left[\frac{x^3}{3}\right]_{x=y^2}^{x=1} dy \\
&= \frac{2\rho}{3}\int_0^1 (1 - y^6)\, dy = \frac{2\rho}{3}\left[y - \frac{y^7}{7}\right]_0^1 = \frac{4\rho}{7}
\end{aligned}$$

$$\begin{aligned}
M_{xy} &= \iiint_E z\rho \, dV = \int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} z\rho \, dz\, dx\, dy \\
&= \rho \int_{-1}^{1}\int_{y^2}^{1} \left[\frac{z^2}{2}\right]_{z=0}^{z=x} dx\, dy = \frac{\rho}{2}\int_{-1}^{1}\int_{y^2}^{1} x^2 \, dx\, dy \\
&= \frac{\rho}{3}\int_0^1 (1 - y^6)\, dy = \frac{2\rho}{7}
\end{aligned}$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right) = \left(\tfrac{5}{7}, 0, \tfrac{5}{14}\right)$$

■

15.6 Exercises

1. Evaluate the integral in Example 1, integrating first with respect to y, then z, and then x.

2. Evaluate the integral $\iiint_E (xy + z^2)\, dV$, where

$$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 1,\ 0 \leqslant z \leqslant 3\}$$

using three different orders of integration.

3–8 Evaluate the iterated integral.

3. $\int_0^2 \int_0^{z^2} \int_0^{y-z} (2x - y)\, dx\, dy\, dz$

4. $\int_0^1 \int_y^{2y} \int_0^{x+y} 6xy\, dz\, dx\, dy$

5. $\int_1^2 \int_0^{2z} \int_0^{\ln x} xe^{-y}\, dy\, dx\, dz$

6. $\int_0^{\pi/2} \int_0^{2x} \int_0^{x+z} \cos(x - 2y + z)\, dy\, dz\, dx$

7. $\int_1^3 \int_{-1}^{2} \int_{-y}^{z} \frac{z}{y}\, dx\, dz\, dy$

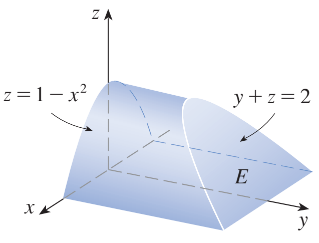
8. $\int_0^1 \int_0^1 \int_0^{2-x^2-y^2} xye^z\, dz\, dy\, dx$

9–12

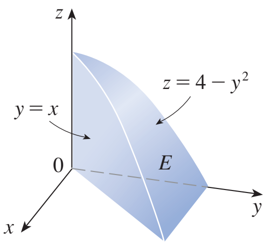
(a) Express the triple integral $\iiint_E f(x, y, z)\, dV$ as an iterated integral for the given function f and solid region E.

(b) Evaluate the iterated integral.

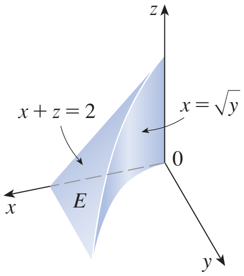
9. $f(x, y, z) = x$



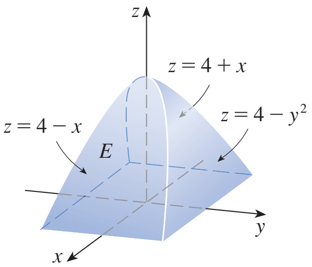
10. $f(x, y, z) = xy$



11. $f(x, y, z) = x + y$



12. $f(x, y, z) = 2$



13–22 Evaluate the triple integral.

13. $\iiint_E y\,dV$, where
$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 3, 0 \leqslant y \leqslant x, x - y \leqslant z \leqslant x + y\}$

14. $\iiint_E e^{z/y}\,dV$, where
$E = \{(x, y, z) \mid 0 \leqslant y \leqslant 1, y \leqslant x \leqslant 1, 0 \leqslant z \leqslant xy\}$

15. $\iiint_E (1/x^3)\,dV$, where
$E = \{(x, y, z) \mid 0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant y^2, 1 \leqslant x \leqslant z + 1\}$

16. $\iiint_E \sin y\,dV$, where E lies below the plane $z = x$ and above the triangular region with vertices $(0, 0, 0)$, $(\pi, 0, 0)$, and $(0, \pi, 0)$

17. $\iiint_E 6xy\,dV$, where E lies under the plane $z = 1 + x + y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$

18. $\iiint_E (x - y)\,dV$, where E is enclosed by the surfaces $z = x^2 - 1$, $z = 1 - x^2$, $y = 0$, and $y = 2$

19. $\iiint_T y^2\,dV$, where T is the solid tetrahedron with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$

20. $\iiint_T xz\,dV$, where T is the solid tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, and $(0, 0, 1)$

21. $\iiint_E x\,dV$, where E is bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$

22. $\iiint_E z\,dV$, where E is bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $y = 3x$, and $z = 0$ in the first octant

23–26 Use a triple integral to find the volume of the given solid.

23. The tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$

24. The solid enclosed by the paraboloids $y = x^2 + z^2$ and $y = 8 - x^2 - z^2$

25. The solid enclosed by the cylinder $y = x^2$ and the planes $z = 0$ and $y + z = 1$

26. The solid enclosed by the cylinder $x^2 + z^2 = 4$ and the planes $y = -1$ and $y + z = 4$

27. (a) Express the volume of the wedge in the first octant that is cut from the cylinder $y^2 + z^2 = 1$ by the planes $y = x$ and $x = 1$ as a triple integral.
(b) Use either the Table of Integrals (on Reference Pages 6–10) or a computer algebra system to find the exact value of the triple integral in part (a).

28–30 Midpoint Rule for Triple Integrals In the *Midpoint Rule for triple integrals* we use a triple Riemann sum to approximate a triple integral over a box B, where $f(x, y, z)$ is evaluated at the center $(\bar{x}_i, \bar{y}_j, \bar{z}_k)$ of the box B_{ijk}. Use the Midpoint Rule to estimate the value of the integral. Divide B into eight sub-boxes of equal size.

28. $\iiint_B \sqrt{x^2 + y^2 + z^2}\,dV$, where
$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 4,\ 0 \leqslant y \leqslant 4, 0 \leqslant z \leqslant 4\}$

29. $\iiint_B \cos(xyz)\,dV$, where
$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant 1\}$

30. $\iiint_B \sqrt{x}\,e^{xyz}\,dV$, where
$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 4,\ 0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant 2\}$

31–32 Sketch the solid whose volume is given by the iterated integral.

31. $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\,dz\,dx$

32. $\displaystyle\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\,dz\,dy$

33–36 Express the integral $\iiint_E f(x, y, z)\,dV$ as an iterated integral in six different ways, where E is the solid bounded by the given surfaces.

33. $y = 4 - x^2 - 4z^2, \quad y = 0$

34. $y^2 + z^2 = 9, \quad x = -2, \quad x = 2$

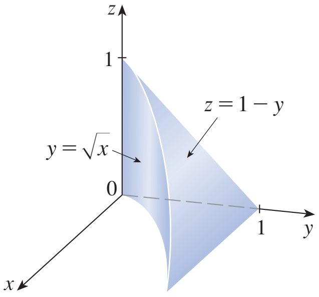
35. $y = x^2, \quad z = 0, \quad y + 2z = 4$

36. $x = 2, \quad y = 2, \quad z = 0, \quad x + y - 2z = 2$

37. The figure shows the region of integration for the integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx$$

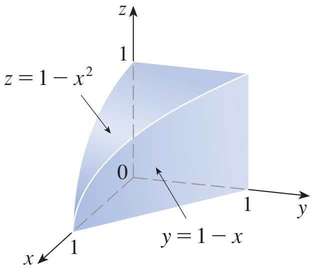
Rewrite this integral as an equivalent iterated integral in the five other orders.



38. The figure shows the region of integration for the integral

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z)\, dy\, dz\, dx$$

Rewrite this integral as an equivalent iterated integral in the five other orders.



39–40 Write five other iterated integrals that are equal to the given iterated integral.

39. $\int_0^1 \int_y^1 \int_0^y f(x, y, z)\, dz\, dx\, dy$ **40.** $\int_0^1 \int_y^1 \int_0^z f(x, y, z)\, dx\, dz\, dy$

41–42 Evaluate the triple integral using only geometric interpretation and symmetry.

41. $\iiint_C (4 + 5x^2yz^2)\, dV$, where C is the cylindrical region $x^2 + y^2 \leqslant 4,\ -2 \leqslant z \leqslant 2$

42. $\iiint_B (z^3 + \sin y + 3)\, dV$, where B is the unit ball $x^2 + y^2 + z^2 \leqslant 1$

43–46 Find the mass and center of mass of the solid E with the given density function ρ.

43. E lies above the xy-plane and below the paraboloid $z = 1 - x^2 - y^2$; $\rho(x, y, z) = 3$

44. E is bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $x + z = 1$, $x = 0$, and $z = 0$; $\rho(x, y, z) = 4$

45. E is the cube given by $0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant a,\ 0 \leqslant z \leqslant a$; $\rho(x, y, z) = x^2 + y^2 + z^2$

46. E is the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$; $\rho(x, y, z) = y$

47–50 Assume that the solid has constant density k.

47. Find the moments of inertia for a cube with side length L if one vertex is located at the origin and three edges lie along the coordinate axes.

48. Find the moments of inertia for a rectangular brick with dimensions a, b, and c and mass M if the center of the brick is situated at the origin and the edges are parallel to the coordinate axes.

49. Find the moment of inertia about the z-axis of the solid cylinder $x^2 + y^2 \leqslant a^2,\ 0 \leqslant z \leqslant h$.

50. Find the moment of inertia about the z-axis of the solid cone $\sqrt{x^2 + y^2} \leqslant z \leqslant h$.

51–52 Set up, but do not evaluate, integral expressions for (a) the mass, (b) the center of mass, and (c) the moment of inertia about the z-axis.

51. The solid of Exercise 25; $\rho(x, y, z) = \sqrt{x^2 + y^2}$

52. The hemisphere $x^2 + y^2 + z^2 \leqslant 1,\ z \geqslant 0$; $\rho(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

T **53.** Let E be the solid in the first octant bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y = z$, $x = 0$, and $z = 0$ with the density function $\rho(x, y, z) = 1 + x + y + z$. Use a computer algebra system to find the exact values of the following quantities for E.

(a) The mass
(b) The center of mass
(c) The moment of inertia about the z-axis

T **54.** If E is the solid of Exercise 22 with density function $\rho(x, y, z) = x^2 + y^2$, find the following quantities, correct to three decimal places.

(a) The mass
(b) The center of mass
(c) The moment of inertia about the z-axis

55. The joint density function for random variables X, Y, and Z is $f(x, y, z) = Cxyz$ if $0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 2$, and $f(x, y, z) = 0$ otherwise.

(a) Find the value of the constant C.
(b) Find $P(X \leqslant 1, Y \leqslant 1, Z \leqslant 1)$.
(c) Find $P(X + Y + Z \leqslant 1)$.

56. Suppose X, Y, and Z are random variables with joint density function $f(x, y, z) = Ce^{-(0.5x+0.2y+0.1z)}$ if $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$, and $f(x, y, z) = 0$ otherwise.
  (a) Find the value of the constant C.
  (b) Find $P(X \leqslant 1, Y \leqslant 1)$.
  (c) Find $P(X \leqslant 1, Y \leqslant 1, Z \leqslant 1)$.

57–58 Average Value The *average value* of a function $f(x, y, z)$ over a solid region E is defined to be

$$f_{\text{avg}} = \frac{1}{V(E)} \iiint_E f(x, y, z)\, dV$$

where $V(E)$ is the volume of E. For instance, if ρ is a density function, then ρ_{avg} is the average density of E.

57. Find the average value of the function $f(x, y, z) = xyz$ over the cube with side length L that lies in the first octant with one vertex at the origin and edges parallel to the coordinate axes.

58. Find the average height of the points in the solid hemisphere $x^2 + y^2 + z^2 \leqslant 1$, $z \geqslant 0$.

---

59. (a) Find the region E for which the triple integral

$$\iiint_E (1 - x^2 - 2y^2 - 3z^2)\, dV$$

is a maximum.

T (b) Use a computer algebra system to calculate the exact maximum value of the triple integral in part (a).

DISCOVERY PROJECT | VOLUMES OF HYPERSPHERES

In this project we find formulas for the volume enclosed by a hypersphere in n-dimensional space. The hypersphere in $\mathbb{R}^n$ of radius r centered at the origin has equation

$$x_1^{\,2} + x_2^{\,2} + x_3^{\,2} + \cdots + x_n^{\,2} = r^2$$

Let $V_n(r)$ denote the volume enclosed by this hypersphere. A hypersphere in $\mathbb{R}^2$ is a circle and in $\mathbb{R}^3$, a sphere.

1. Use a double integral and trigonometric substitution, together with Formula 64 in the Table of Integrals, to find the area enclosed by a circle of radius r in $\mathbb{R}^2$.
2. Use a triple integral and trigonometric substitution to find the volume $V_3(r)$ enclosed by a sphere with radius r in $\mathbb{R}^3$.
3. Use a quadruple integral to find the (4-dimensional) volume $V_4(r)$ enclosed by the hypersphere of radius r in $\mathbb{R}^4$. (Use only trigonometric substitution and the reduction formulas for $\int \sin^n x\, dx$ or $\int \cos^n x\, dx$.)
4. Use an n-tuple integral to find the volume $V_n(r)$ enclosed by a hypersphere of radius r in $\mathbb{R}^n$. [*Hint:* The formulas are different for n even and n odd.]
5. Show that the volume $V_n(1)$ enclosed by the unit hypersphere in $\mathbb{R}^n$ approaches zero as n increases.

15.7 Triple Integrals in Cylindrical Coordinates

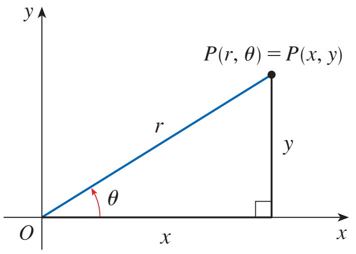


FIGURE 1

In plane geometry the polar coordinate system is used to give a convenient description of certain curves and regions. (See Section 10.3.) Figure 1 enables us to recall the connection between polar and Cartesian coordinates. If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ), then, from the figure,

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \qquad\qquad \tan\theta = \frac{y}{x}$$

In three dimensions there is a coordinate system, called *cylindrical coordinates*, that is similar to polar coordinates and gives convenient descriptions of some commonly occurring surfaces and solids. As we will see, some triple integrals are much easier to evaluate in cylindrical coordinates.

Cylindrical Coordinates

In the **cylindrical coordinate system**, a point P in three-dimensional space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P. (See Figure 2.)

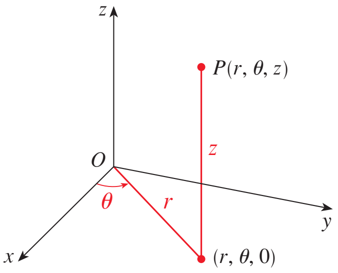


FIGURE 2
The cylindrical coordinates of a point

To convert from cylindrical to rectangular coordinates, we use the equations

1
$$\boxed{x = r\cos\theta \qquad y = r\sin\theta \qquad z = z}$$

whereas to convert from rectangular to cylindrical coordinates, we use

2
$$\boxed{r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z}$$

EXAMPLE 1

(a) Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.
(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

SOLUTION

(a) The point with cylindrical coordinates $(2, 2\pi/3, 1)$ is plotted in Figure 3. From Equations 1, its rectangular coordinates are

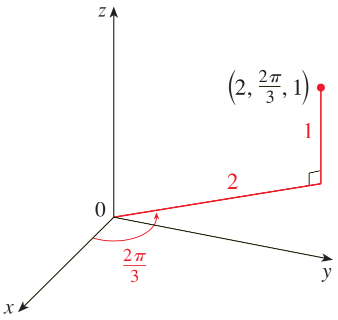


FIGURE 3

$$x = 2\cos\frac{2\pi}{3} = 2\left(-\frac{1}{2}\right) = -1$$

$$y = 2\sin\frac{2\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$z = 1$$

So the point is $(-1, \sqrt{3}, 1)$ in rectangular coordinates.

(b) From Equations 2 and noting that θ is in quadrant IV of the xy-plane, we have

$$r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$$

$$\tan\theta = \frac{-3}{3} = -1 \qquad \text{so} \qquad \theta = \frac{7\pi}{4} + 2n\pi$$

$$z = -7$$

Therefore one set of cylindrical coordinates is $(3\sqrt{2}, 7\pi/4, -7)$. Another is $(3\sqrt{2}, -\pi/4, -7)$. As with polar coordinates, there are infinitely many choices. ■

Cylindrical coordinates are useful in problems that involve symmetry about an axis, and the z-axis is chosen to coincide with this axis of symmetry. For instance, the axis of the circular cylinder with Cartesian equation $x^2 + y^2 = c^2$ is the z-axis. In cylindrical coordinates this cylinder has the very simple equation $r = c$. (See Figure 4.) This is the reason for the name “cylindrical” coordinates. The graph of the equation $\theta = c$ is a vertical plane through the origin (see Figure 5), and the graph of the equation $z = c$ is a horizontal plane (see Figure 6).

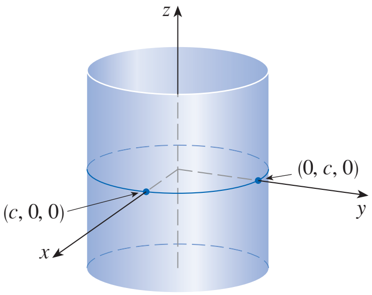


FIGURE 4
$r = c$, a cylinder

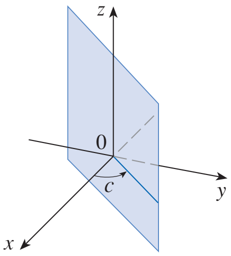


FIGURE 5
$\theta = c$, a vertical plane

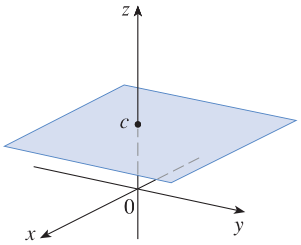


FIGURE 6
$z = c$, a horizontal plane

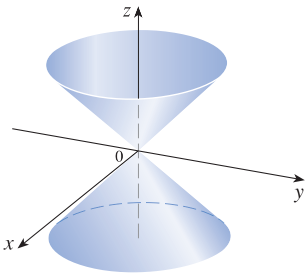


FIGURE 7
$z = r$, a cone

EXAMPLE 2 Describe the surface whose equation in cylindrical coordinates is $z = r$.

SOLUTION The equation says that the z-value, or height, of each point on the surface is the same as r, the distance from the point to the z-axis. Because θ doesn’t appear, it can vary. So any horizontal trace in the plane $z = k$ $(k > 0)$ is a circle of radius k. These traces suggest that the surface is a cone. This prediction can be confirmed by converting the equation into rectangular coordinates. From the first equation in (2) we have

$$z^2 = r^2 = x^2 + y^2$$

We recognize the equation $z^2 = x^2 + y^2$ (by comparison with Table 1 in Section 12.6) as being a circular cone whose axis is the z-axis (see Figure 7). ■

Triple Integrals in Cylindrical Coordinates

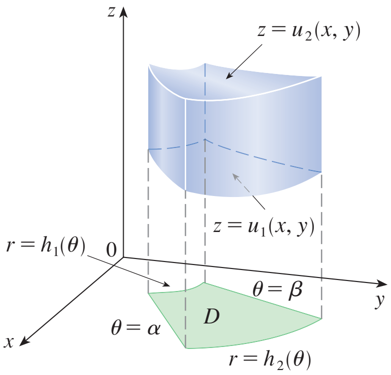


FIGURE 8

Suppose that E is a type 1 region whose projection D onto the xy-plane is conveniently described in polar coordinates (see Figure 8). In particular, suppose that f is continuous and

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

where D is given in polar coordinates by

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

We know from Equation 15.6.6 that

$$\boxed{3} \qquad \iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

But we also know how to evaluate double integrals in polar coordinates. In fact, combining Equation 3 with Equation 15.3.3, we obtain

$$\boxed{\boxed{4} \qquad \iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta}$$

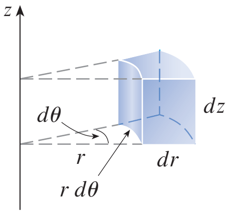


FIGURE 9

Volume element in cylindrical coordinates: $dV = r\,dz\,dr\,d\theta$

Formula 4 is the **formula for triple integration in cylindrical coordinates**. It says that we convert a triple integral from rectangular to cylindrical coordinates by writing $x = r\cos\theta$, $y = r\sin\theta$, leaving z as it is, using the appropriate limits of integration for z, r, and θ, and replacing dV by $r\,dz\,dr\,d\theta$. (Figure 9 shows how to remember this.) It is worthwhile to use this formula when E is a solid region easily described in cylindrical coordinates, and especially when the function $f(x, y, z)$ involves the expression $x^2 + y^2$.

EXAMPLE 3 Evaluate $\iiint_E x^2\,dV$, where E is the solid that lies under the paraboloid $z = 4 - x^2 - y^2$ and above the xy-plane (see Figure 10).

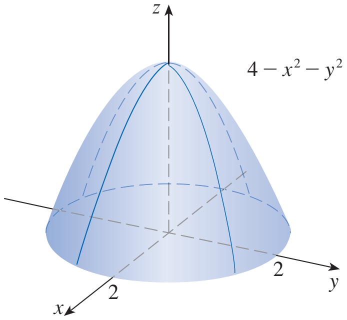


FIGURE 10

SOLUTION Because E is symmetric about the z-axis, we use cylindrical coordinates. In addition, cylindrical coordinates are appropriate because the paraboloid $z = 4 - x^2 - y^2 = 4 - (x^2 + y^2)$ is easily expressed in cylindrical coordinates as $z = 4 - r^2$. The paraboloid intersects the xy-plane in the circle $r^2 = 4$ or, equivalently, $r = 2$, so the projection of E onto the xy-plane is the disk $r \leqslant 2$. Thus the region E is given by

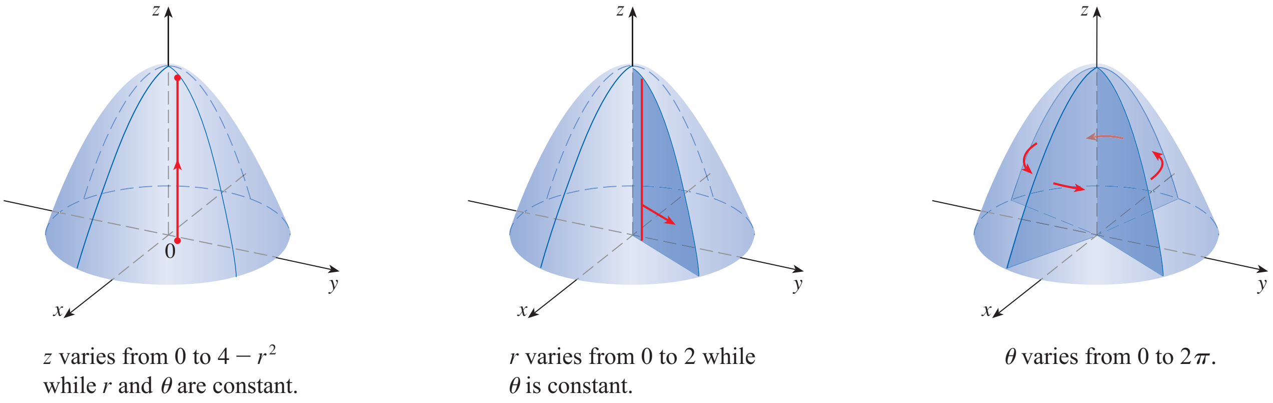
$$\{(r, \theta, z) \mid 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant r \leqslant 2,\ 0 \leqslant z \leqslant 4 - r^2\}$$

and from Formula 4 we have

$$\begin{aligned}\iiint\limits_E x^2\,dV &= \int_0^{2\pi}\int_0^2\int_0^{4-r^2} (r\cos\theta)^2\,r\,dz\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^2 (r^3\cos^2\theta)](4 - r^2)\,dr\,d\theta\\ &= \int_0^{2\pi}\cos^2\theta\,d\theta\int_0^2 (4r^3 - r^5)\,dr\\ &= \tfrac{1}{2}\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{2\pi}\Big[r^4 - \tfrac{1}{6}r^6\Big]_0^2\\ &= \tfrac{1}{2}(2\pi)\big(16 - \tfrac{32}{3}\big) = \tfrac{16}{3}\pi\end{aligned}$$

■

Figure 11 shows how the solid E in Example 3 is swept out by the iterated triple integral if we integrate first with respect to z, then r, then θ.



z varies from 0 to $4 - r^2$ while r and θ are constant.

r varies from 0 to 2 while θ is constant.

θ varies from 0 to 2π.

FIGURE 11

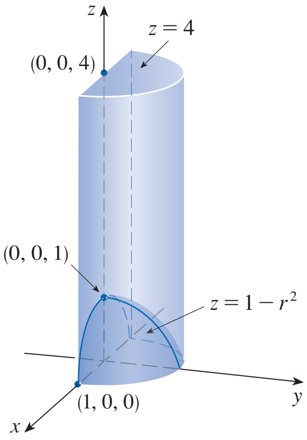


FIGURE 12

EXAMPLE 4 A solid E lies within the cylinder $x^2+y^2=1$ to the right of the xz-plane, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$. (See Figure 12.) The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

SOLUTION In cylindrical coordinates the cylinder is $r=1$ and the paraboloid is $z=1-r^2$, so we can write

$$E=\{(r,\theta,z)\mid 0\leqslant\theta\leqslant\pi,\ 0\leqslant r\leqslant 1,\ 1-r^2\leqslant z\leqslant 4\}$$

Since the density at (x, y, z) is proportional to the distance from the z-axis, the density function is

$$\rho(x,y,z)=K\sqrt{x^2+y^2}=Kr$$

where K is the proportionality constant. Therefore, from Formula 15.6.13, the mass of E is

$$\begin{aligned} m&=\iiint_E K\sqrt{x^2+y^2}\,dV=\int_0^{\pi}\int_0^1\int_{1-r^2}^4 (Kr)\,r\,dz\,dr\,d\theta\\ &=\int_0^{\pi}\int_0^1 Kr^2[4-(1-r^2)]\,dr\,d\theta=K\int_0^{\pi}d\theta\int_0^1(3r^2+r^4)\,dr\\ &=\pi K\left[r^3+\frac{r^5}{5}\right]_0^1=\frac{6\pi K}{5}\end{aligned}$$

■

EXAMPLE 5 Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$.

SOLUTION This iterated integral is a triple integral over the solid region

$$E=\left\{(x,y,z)\mid -2\leqslant x\leqslant 2,\ -\sqrt{4-x^2}\leqslant y\leqslant\sqrt{4-x^2},\ \sqrt{x^2+y^2}\leqslant z\leqslant 2\right\}$$

and the projection of E onto the xy-plane is the disk $x^2+y^2\leqslant 4$. The lower surface of E is the cone $z=\sqrt{x^2+y^2}$ and its upper surface is the plane $z=2$. (See Figure 13.) This region has a much simpler description in cylindrical coordinates:

$$E=\{(r,\theta,z)\mid 0\leqslant\theta\leqslant 2\pi,\ 0\leqslant r\leqslant 2,\ r\leqslant z\leqslant 2\}$$

Therefore we have

$$\begin{aligned}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx&=\iiint_E(x^2+y^2)\,dV\\ &=\int_0^{2\pi}\int_0^2\int_r^2 r^2\,r\,dz\,dr\,d\theta\\ &=\int_0^{2\pi}d\theta\int_0^2 r^3(2-r)\,dr\\ &=2\pi\left[\tfrac{1}{2}r^4-\tfrac{1}{5}r^5\right]_0^2=\tfrac{16}{5}\pi\end{aligned}$$

■

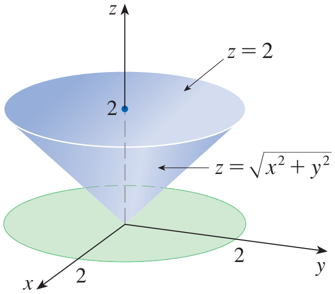


FIGURE 13

15.7 Exercises

1–2 Plot the point whose cylindrical coordinates are given. Then find the rectangular coordinates of the point.

1. (a) $(5, \pi/2, 2)$
(b) $(6, -\pi/4, -3)$

2. (a) $(2, 5\pi/6, 1)$
(b) $(8, -2\pi/3, 5)$

3–4 Change from rectangular to cylindrical coordinates.

3. (a) $(4, 4, -3)$
(b) $(5\sqrt{3}, -5, \sqrt{3})$

4. (a) $(0, -2, 9)$
(b) $(-1, \sqrt{3}, 6)$

5–6 Describe in words the surface whose equation is given.

5. $r = 2$

6. $\theta = \pi/6$

7–8 Identify the surface whose equation is given.

7. $r^2 + z^2 = 4$

8. $r = 2\sin\theta$

9–10 Write the equations in cylindrical coordinates.

9. (a) $x^2 - x + y^2 + z^2 = 1$
(b) $z = x^2 - y^2$

10. (a) $2x^2 + 2y^2 - z^2 = 4$
(b) $2x - y + z = 1$

11–12 Sketch the solid described by the given inequalities.

11. $r^2 \leqslant z \leqslant 8 - r^2$

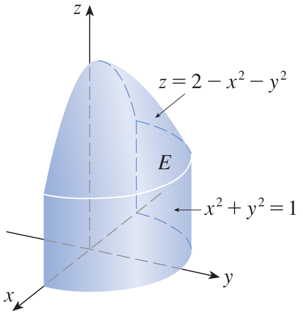
12. $0 \leqslant \theta \leqslant \pi/2, \quad r \leqslant z \leqslant 2$

13. A cylindrical shell is 20 cm long, with inner radius 6 cm and outer radius 7 cm. Write inequalities that describe the shell in an appropriate coordinate system. Explain how you have positioned the coordinate system with respect to the shell.

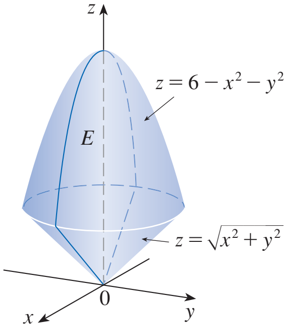
14. Use graphing software to draw the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 5 - x^2 - y^2$.

15–16
(a) Express the triple integral $\iiint_E f(x, y, z)\, dV$ as an iterated integral in cylindrical coordinates for the given function f and solid region E.
(b) Evaluate the iterated integral.

15. $f(x, y, z) = x^2 + y^2$



16. $f(x, y, z) = xy$



17–18 Sketch the solid whose volume is given by the integral and evaluate the integral.

17. $\int_{\pi/2}^{3\pi/2} \int_0^3 \int_{r^2}^9 r\, dz\, dr\, d\theta$

18. $\int_0^2 \int_0^{2\pi} \int_0^r r\, dz\, d\theta\, dr$

19–30 Use cylindrical coordinates.

19. Evaluate $\iiint_E \sqrt{x^2 + y^2}\, dV$, where E is the region that lies inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.

20. Evaluate $\iiint_E z\, dV$, where E is enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

21. Evaluate $\iiint_E (x + y + z)\, dV$, where E is the solid in the first octant that lies under the paraboloid $z = 4 - x^2 - y^2$.

22. Evaluate $\iiint_E (x - y)\, dV$, where E is the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 16$, above the xy-plane, and below the plane $z = y + 4$.

23. Evaluate $\iiint_E x^2\, dV$, where E is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$, and below the cone $z^2 = 4x^2 + 4y^2$.

24. Find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

25. Find the volume of the solid that is enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 2$.

26. Find the volume of the solid that lies between the paraboloid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 2$.

27. (a) Find the volume of the region E that lies between the paraboloid $z = 24 - x^2 - y^2$ and the cone $z = 2\sqrt{x^2 + y^2}$.
(b) Find the centroid of E (the center of mass in the case where the density is constant).

28. (a) Find the volume of the solid that the cylinder $r = a\cos\theta$ cuts out of the sphere of radius a centered at the origin.

(b) Illustrate the solid of part (a) by graphing the sphere and the cylinder on the same screen.

29. Find the mass and center of mass of the solid S bounded by the paraboloid $z = 4x^2 + 4y^2$ and the plane $z = a$ $(a > 0)$ if S has constant density K.

30. Find the mass of a ball B given by $x^2 + y^2 + z^2 \leqslant a^2$ if the density at any point is proportional to its distance from the z-axis.

31–32 Evaluate the integral by changing to cylindrical coordinates.

31. $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\,dz\,dx\,dy$

32. $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$

---

33. When studying the formation of mountain ranges, geologists estimate the amount of work required to lift a mountain from sea level. Consider a mountain that is essentially in the shape of a right circular cone. Suppose that the weight density of the material in the vicinity of a point P is $g(P)$ and the height is $h(P)$.

(a) Find a definite integral that represents the total work done in forming the mountain.

(b) Assume that Mount Fuji in Japan is in the shape of a right circular cone with radius 19,000 m, height 3800 m, and density a constant 3200 kg/m^3. How much work was done in forming Mount Fuji if the land was initially at sea level?

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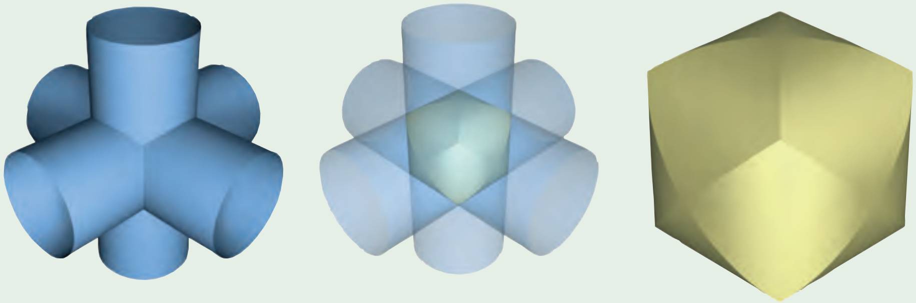
DISCOVERY PROJECT | THE INTERSECTION OF THREE CYLINDERS

The figure shows the solid enclosed by three circular cylinders with the same diameter that intersect at right angles. In this project we compute its volume and determine how its shape



1. Sketch carefully the solid enclosed by the three cylinders $x^2 + y^2 = 1$, $x^2 + z^2 = 1$, and $y^2 + z^2 = 1$. Indicate the positions of the coordinate axes and label the faces with the equations of the corresponding cylinders.

2. Find the volume of the solid in Problem 1.

3. Use graphing software to draw the edges of the solid.

4. What happens to the solid in Problem 1 if the radius of the first cylinder is different from 1? Illustrate with a hand-drawn sketch or a computer graph.

5. If the first cylinder is $x^2 + y^2 = a^2$, where $a < 1$, set up, but do not evaluate, a double integral for the volume of the solid. What if $a > 1$?

15.8 Triple Integrals in Spherical Coordinates

Another useful coordinate system in three dimensions is the *spherical coordinate system.* It simplifies the evaluation of triple integrals over regions bounded by spheres or cones.

Spherical Coordinates

The **spherical coordinates** (ρ, θ, ϕ) of a point P in space are shown in Figure 1, where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP. Note that

$$\rho \geqslant 0 \qquad 0 \leqslant \phi \leqslant \pi$$

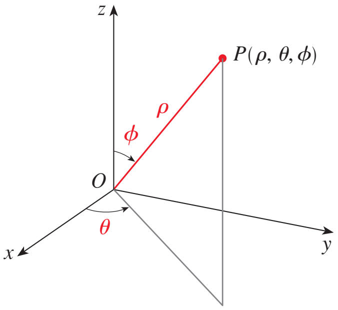


FIGURE 1
The spherical coordinates of a point

The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point. For example, the sphere with center the origin and radius c has the simple equation $\rho = c$ (see Figure 2): this is the reason for the name "spherical" coordinates. The graph of the equation $\theta = c$ is a vertical half-plane (see Figure 3), and the equation $\phi = c$ represents a half-cone with the z-axis as its axis (see Figure 4).

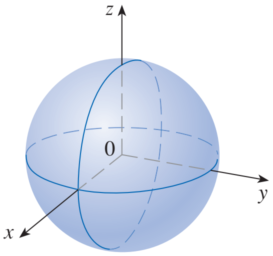


FIGURE 2 $\rho = c$, a sphere

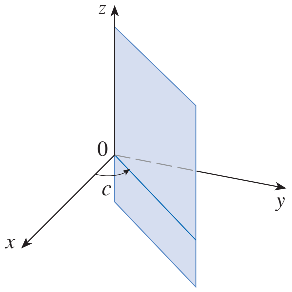


FIGURE 3 $\theta = c$, a half-plane

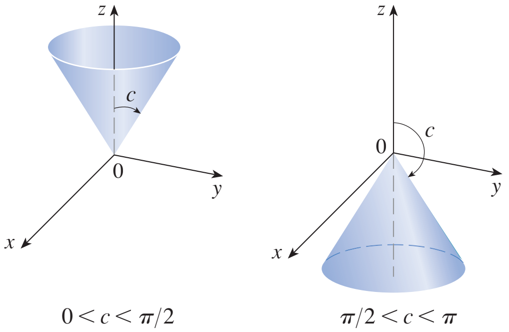


FIGURE 4 $\phi = c$, a half-cone

The relationship between rectangular and spherical coordinates can be seen from Figure 5. From triangles OPQ and OPP' we have

$$z = \rho \cos \phi \qquad r = \rho \sin \phi$$

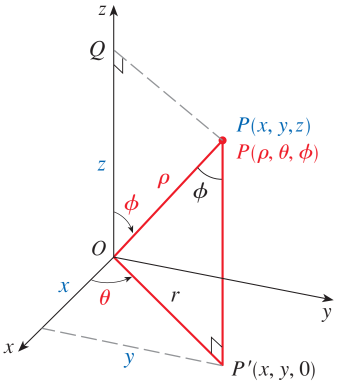


FIGURE 5

But $x = r \cos \theta$ and $y = r \sin \theta$, so to convert from spherical to rectangular coordinates, we use the equations

1 $$\boxed{x = \rho \sin \phi \, \cos \theta \qquad y = \rho \sin \phi \, \sin \theta \qquad z = \rho \cos \phi}$$

Also, the distance formula shows that

2 $$\boxed{\rho^2 = x^2 + y^2 + z^2}$$

We use this equation in converting from rectangular to spherical coordinates.

EXAMPLE 1 The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

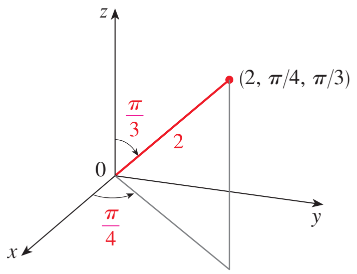


FIGURE 6

SOLUTION We plot the point in Figure 6. From Equations 1 we have

$$x = \rho \sin\phi \cos\theta = 2 \sin\frac{\pi}{3} \cos\frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$y = \rho \sin\phi \sin\theta = 2 \sin\frac{\pi}{3} \sin\frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$z = \rho \cos\phi = 2 \cos\frac{\pi}{3} = 2\left(\tfrac{1}{2}\right) = 1$$

Thus the point $(2, \pi/4, \pi/3)$ is $\left(\sqrt{3/2}, \sqrt{3/2}, 1\right)$ in rectangular coordinates. ■

WARNING There is not universal agreement on the notation for spherical coordinates. Most books on physics reverse the meanings of θ and ϕ and use r in place of ρ.

EXAMPLE 2 The point $\left(0, 2\sqrt{3}, -2\right)$ is given in rectangular coordinates. Find spherical coordinates for this point.

SOLUTION From Equation 2 we have $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0 + 12 + 4} = 4$ and so Equations 1 give

$$\cos\phi = \frac{z}{\rho} = \frac{-2}{4} = -\frac{1}{2} \qquad \phi = \frac{2\pi}{3}$$

$$\cos\theta = \frac{x}{\rho \sin\phi} = 0 \qquad \theta = \frac{\pi}{2}$$

(Note that $\theta \neq 3\pi/2$ because $y = 2\sqrt{3} > 0$.) Therefore spherical coordinates of the given point are $(4, \pi/2, 2\pi/3)$. ■

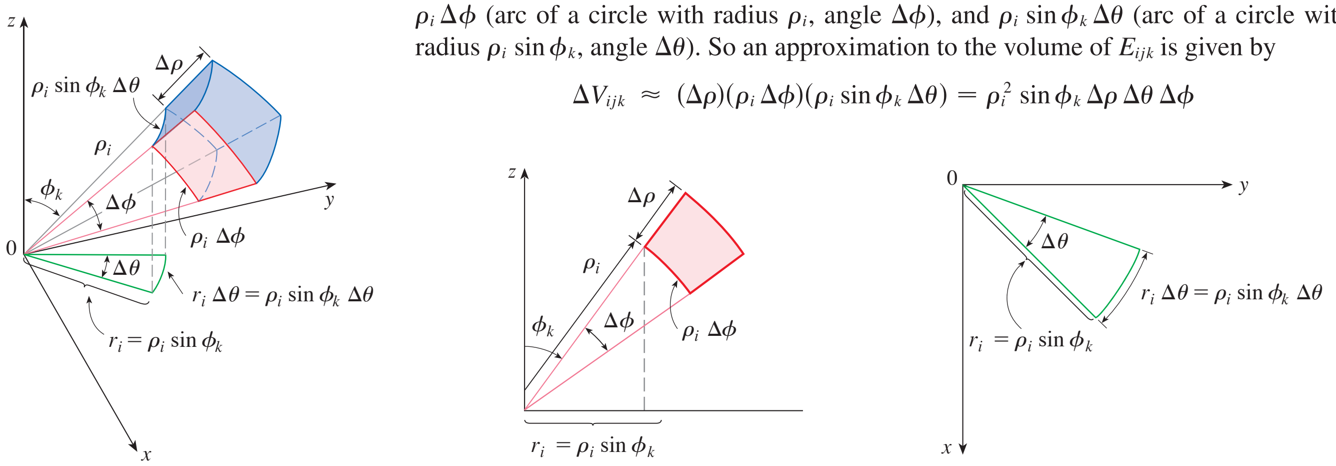
Triple Integrals in Spherical Coordinates

In the spherical coordinate system the counterpart of a rectangular box is a **spherical wedge**

$$E = \left\{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b,\ \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d\right\}$$

where $a \geqslant 0$ and $\beta - \alpha \leqslant 2\pi$, and $d - c \leqslant \pi$. Although we defined triple integrals by dividing solids into small boxes, it can be shown that dividing a solid into small spherical wedges always gives the same result. So we divide E into smaller spherical wedges E_{ijk} by means of equally spaced spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$, and half-cones $\phi = \phi_k$. Figure 7 shows that E_{ijk} is approximately a rectangular box with dimensions $\Delta\rho$, $\rho_i\,\Delta\phi$ (arc of a circle with radius ρ_i, angle $\Delta\phi$), and $\rho_i \sin\phi_k\,\Delta\theta$ (arc of a circle with radius $\rho_i \sin\phi_k$, angle $\Delta\theta$). So an approximation to the volume of E_{ijk} is given by

$$\Delta V_{ijk} \approx (\Delta\rho)(\rho_i\,\Delta\phi)(\rho_i \sin\phi_k\,\Delta\theta) = \rho_i^2 \sin\phi_k\,\Delta\rho\,\Delta\theta\,\Delta\phi$$



(a) A spherical wedge (b) Side view (c) Top view

FIGURE 7

In fact, it can be shown, with the aid of the Mean Value Theorem (Exercise 51), that the volume of E_{ijk} is given exactly by

$$\Delta V_{ijk} = \tilde{\rho}_i^2 \sin \tilde{\phi}_k \, \Delta\rho \, \Delta\theta \, \Delta\phi$$

where $(\tilde{\rho}_i, \tilde{\theta}_j, \tilde{\phi}_k)$ is some point in E_{ijk}. Let $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ be the rectangular coordinates of this point. Then

$$\iiint\limits_E f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V_{ijk}$$

$$= \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\tilde{\rho}_i \sin \tilde{\phi}_k \cos \tilde{\theta}_j, \tilde{\rho}_i \sin \tilde{\phi}_k \sin \tilde{\theta}_j, \tilde{\rho}_i \cos \tilde{\phi}_k)\, \tilde{\rho}_i^2 \sin \tilde{\phi}_k \, \Delta\rho \, \Delta\theta \, \Delta\phi$$

But this sum is a Riemann sum for the function

$$F(\rho, \theta, \phi) = f(\rho \sin\phi \, \cos\theta, \rho \sin\phi \, \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi$$

Consequently, we have arrived at the following **formula for triple integration in spherical coordinates**.

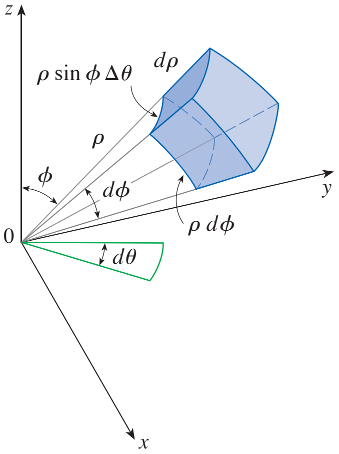


FIGURE 8
Volume element in spherical coordinates: $dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

3 $$\iiint\limits_E f(x, y, z)\, dV$$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \, \cos\theta, \rho \sin\phi \, \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b, \ \alpha \leqslant \theta \leqslant \beta, \ c \leqslant \phi \leqslant d\}$$

Formula 3 says that we convert a triple integral from rectangular coordinates to spherical coordinates by writing

$$x = \rho \sin\phi \, \cos\theta \qquad y = \rho \sin\phi \, \sin\theta \qquad z = \rho \cos\phi$$

using the appropriate limits of integration and replacing dV by $\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$. This is illustrated in Figure 8.

This formula can be extended to include more general spherical regions such as

$$E = \{(\rho, \theta, \phi) \mid \alpha \leqslant \theta \leqslant \beta, \ c \leqslant \phi \leqslant d, \ g_1(\theta, \phi) \leqslant \rho \leqslant g_2(\theta, \phi)\}$$

In this case the formula is the same as in (3) except that the limits of integration for ρ are $g_1(\theta, \phi)$ and $g_2(\theta, \phi)$.

Usually, spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region of integration.

EXAMPLE 3 Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, where B is the unit ball:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant 1\}$$

SOLUTION Since the boundary of B is a sphere, we use spherical coordinates:

$$B = \{(\rho, \theta, \phi) \mid 0 \leqslant \rho \leqslant 1, \ 0 \leqslant \theta \leqslant 2\pi, \ 0 \leqslant \phi \leqslant \pi\}$$

In addition, spherical coordinates are appropriate because

$$x^2 + y^2 + z^2 = \rho^2$$

Thus (3) gives

$$\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\,dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} e^{(\rho^2)^{3/2}}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi}\sin\phi\,d\phi\,\int_0^{2\pi}d\theta\,\int_0^{1}\rho^2 e^{\rho^3}\,d\rho$$

$$= \Big[-\cos\phi\Big]_0^{\pi}(2\pi)\Big[\tfrac{1}{3}e^{\rho^3}\Big]_0^{1} = \tfrac{4}{3}\pi(e-1)$$

■

NOTE It would have been extremely awkward to evaluate the integral in Example 3 without spherical coordinates. In rectangular coordinates the iterated integral would have been

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}}\,dz\,dy\,dx$$

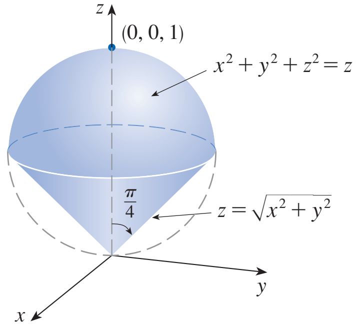


FIGURE 9

EXAMPLE 4 Use spherical coordinates to find the volume of the solid that lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$. (See Figure 9.)

SOLUTION Notice that the sphere passes through the origin and has center $\left(0, 0, \frac{1}{2}\right)$. We write the equation of the sphere in spherical coordinates as

$$\rho^2 = \rho\cos\phi \qquad \text{or} \qquad \rho = \cos\phi$$

The equation of the cone can be written as

$$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\,\cos^2\theta + \rho^2\sin^2\phi\,\sin^2\theta} = \rho\sin\phi$$

This gives $\sin\phi = \cos\phi$, or $\phi = \pi/4$. Therefore the description of the solid E in spherical coordinates is

$$E = \left\{(\rho,\theta,\phi) \mid 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \phi \leqslant \pi/4,\ 0 \leqslant \rho \leqslant \cos\phi\right\}$$

Figure 10 shows how E is swept out if we integrate first with respect to ρ, then ϕ, and then θ. The volume of E is

$$V(E) = \iiint_E dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}d\theta\int_0^{\pi/4}\sin\phi\left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=\cos\phi}d\phi$$

$$= \frac{2\pi}{3}\int_0^{\pi/4}\sin\phi\,\cos^3\phi\,d\phi = \frac{2\pi}{3}\left[-\frac{\cos^4\phi}{4}\right]_0^{\pi/4} = \frac{\pi}{8}$$

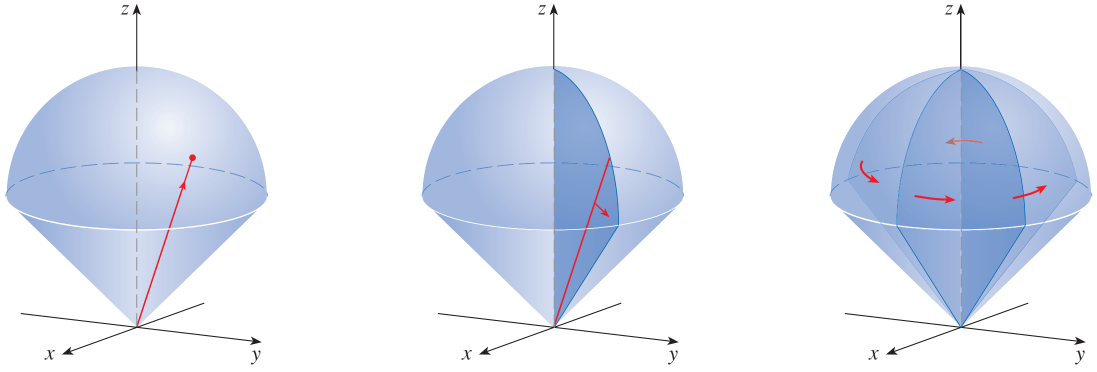


FIGURE 10

ρ varies from 0 to $\cos\phi$ while ϕ and θ are constant.

ϕ varies from 0 to $\pi/4$ while θ is constant.

θ varies from 0 to 2π.

■

15.8 Exercises

1–2 Plot the point whose spherical coordinates are given. Then find the rectangular coordinates of the point.

1. (a) $(2, 3\pi/4, \pi/2)$ (b) $(4, -\pi/3, \pi/4)$

2. (a) $(5, \pi/2, \pi/3)$ (b) $(6, 0, 5\pi/6)$

3–4 Change from rectangular to spherical coordinates.

3. (a) $(3, 3, 0)$ (b) $\left(1, -\sqrt{3}, 2\sqrt{3}\right)$

4. (a) $(0, 4, -4)$ (b) $\left(-2, 2, 2\sqrt{6}\right)$

5–6 Describe in words the surface whose equation is given.

5. $\phi = 3\pi/4$

6. $\rho^2 - 3\rho + 2 = 0$

7–8 Identify the surface whose equation is given.

7. $\rho \cos \phi = 1$

8. $\rho = \cos \phi$

9–10 Write the equation in spherical coordinates.

9. (a) $x^2 + y^2 + z^2 = 9$ (b) $x^2 - y^2 - z^2 = 1$

10. (a) $z = x^2 + y^2$ (b) $z = x^2 - y^2$

11–14 Sketch the solid described by the given inequalities.

11. $\rho \leqslant 1, \quad 0 \leqslant \phi \leqslant \pi/6, \quad 0 \leqslant \theta \leqslant \pi$

12. $1 \leqslant \rho \leqslant 2, \quad \pi/2 \leqslant \phi \leqslant \pi$

13. $1 \leqslant \rho \leqslant 3, \quad 0 \leqslant \phi \leqslant \pi/2, \quad \pi \leqslant \theta \leqslant 3\pi/2$

14. $\rho \leqslant 2, \quad \rho \leqslant \csc \phi$

15. A solid lies inside the sphere $x^2 + y^2 + z^2 = 4z$ and outside the cone $z = \sqrt{x^2 + y^2}$. Write a description of the solid in terms of inequalities involving spherical coordinates.

16. (a) Find inequalities that describe a hollow ball with diameter 30 cm and thickness 0.5 cm. Explain how you have positioned the coordinate system that you have chosen.
(b) Suppose the ball is cut in half. Write inequalities that describe one of the halves.

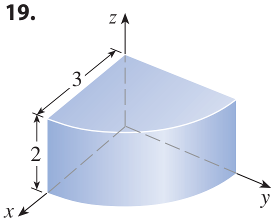
17–18 Sketch the solid whose volume is given by the integral and evaluate the integral.

17. $\displaystyle\int_0^{\pi/6}\int_0^{\pi/2}\int_0^{3} \rho^2 \sin\phi \; d\rho\, d\theta\, d\phi$

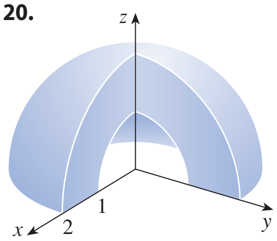
18. $\displaystyle\int_0^{\pi/4}\int_0^{2\pi}\int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$

19–20 Set up the triple integral of an arbitrary continuous function $f(x, y, z)$ in cylindrical or spherical coordinates over the solid shown.

19.

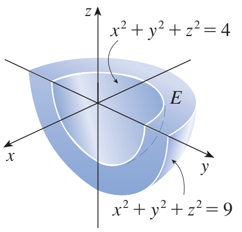


20.

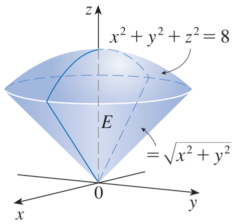


21–22
(a) Express the triple integral $\iiint_E f(x, y, z)\, dV$ as an iterated integral in spherical coordinates for the given function f and solid region E.
(b) Evaluate the iterated integral.

21. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$



22. $f(x, y, z) = xy$



23–36 Use spherical coordinates.

23. Evaluate $\iiint_B (x^2 + y^2 + z^2)^2\, dV$, where B is the ball with center the origin and radius 5.

24. Evaluate $\iiint_E y^2 z^2\, dV$, where E lies above the cone $\phi = \pi/3$ and below the sphere $\rho = 1$.

25. Evaluate $\iiint_E (x^2 + y^2)\, dV$, where E lies between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$.

26. Evaluate $\iiint_E y^2\, dV$, where E is the solid hemisphere $x^2 + y^2 + z^2 \leqslant 9$, $y \geqslant 0$.

27. Evaluate $\iiint_E xe^{x^2+y^2+z^2}\, dV$, where E is the portion of the unit ball $x^2 + y^2 + z^2 \leqslant 1$ that lies in the first octant.

28. Evaluate $\iiint_E \sqrt{x^2 + y^2 + z^2}\, dV$, where E lies above the cone $z = \sqrt{x^2 + y^2}$ and between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$.

29. Find the volume of the part of the ball $\rho \leqslant a$ that lies between the cones $\phi = \pi/6$ and $\phi = \pi/3$.

30. Find the average distance from a point in a ball of radius a to its center.

31. (a) Find the volume of the solid that lies above the cone $\phi = \pi/3$ and below the sphere $\rho = 4\cos\phi$.
(b) Find the centroid of the solid in part (a).

32. Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane, and below the cone $z = \sqrt{x^2 + y^2}$.

33. (a) Find the centroid of the solid in Example 4. (Assume constant density K.)
(b) Find the moment of inertia about the z-axis for this solid.

34. Let H be a solid hemisphere of radius a whose density at any point is proportional to its distance from the center of the base.
(a) Find the mass of H.
(b) Find the center of mass of H.
(c) Find the moment of inertia of H about its axis.

35. (a) Find the centroid of a solid homogeneous hemisphere of radius a.
(b) Find the moment of inertia of the solid in part (a) about a diameter of its base.

36. Find the mass and center of mass of a solid hemisphere of radius a if the density at any point is proportional to its distance from the base.

37–42 Use cylindrical or spherical coordinates, whichever seems more appropriate.

37. Find the volume and centroid of the solid E that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

38. Find the volume of the smaller wedge cut from a sphere of radius a by two planes that intersect along a diameter at an angle of $\pi/6$.

39. A solid cylinder with constant density has base radius a and height h.
(a) Find the moment of inertia of the cylinder about its axis.
(b) Find the moment of inertia of the cylinder about a diameter of its base.

40. A solid right circular cone with constant density has base radius a and height h.
(a) Find the moment of inertia of the cone about its axis.
(b) Find the moment of inertia of the cone about a diameter of its base.

41. Evaluate $\iiint_E z\,dV$, where E lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 2y$. Use either the Table of Integrals (on Reference Pages 6–10) or a computer algebra system to evaluate the integral.

42. (a) Find the volume enclosed by the torus $\rho = \sin\phi$.
(b) Use graphing software to draw the torus.

43–45 Evaluate the integral by changing to spherical coordinates.

43. $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx$

44. $\displaystyle\int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} (x^2z + y^2z + z^3)\,dz\,dx\,dy$

45. $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}} (x^2 + y^2 + z^2)^{3/2}\,dz\,dy\,dx$

46. A model for the density δ of the earth's atmosphere near its surface is

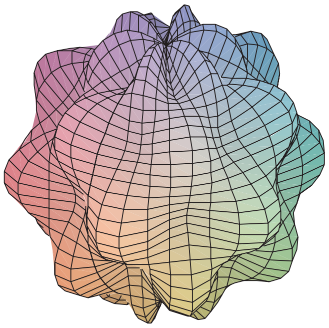
$$\delta = 619.09 - 0.000097\rho$$

where ρ (the distance from the center of the earth) is measured in meters and δ is measured in kilograms per cubic meter. If we take the surface of the earth to be a sphere with radius 6370 km, then this model is a reasonable one for $6.370 \times 10^6 \leqslant \rho \leqslant 6.375 \times 10^6$. Use this model to estimate the mass of the atmosphere between the ground and an altitude of 5 km.

47. Use graphing software to draw a silo consisting of a cylinder with radius 3 and height 10 surmounted by a hemisphere.

48. The latitude and longitude of a point P in the Northern Hemisphere are related to spherical coordinates ρ, θ, ϕ as follows. We take the origin to be the center of the earth and the positive z-axis to pass through the North Pole. The positive x-axis passes through the point where the prime meridian (the meridian through Greenwich, England) intersects the equator. Then the latitude of P is $\alpha = 90° - \phi°$ and the longitude is $\beta = 360° - \theta°$. Find the great-circle distance from Los Angeles (lat. 34.06° N, long. 118.25° W) to Montréal (lat. 45.50° N, long. 73.60° W). Take the radius of the earth to be 6370 km. (A *great circle* is the circle of intersection of a sphere and a plane through the center of the sphere.)

49. The surfaces $\rho = 1 + \frac{1}{5}\sin m\theta \sin n\phi$ have been used as models for tumors. The "bumpy sphere" with $m = 6$ and $n = 5$ is shown. Use a computer algebra system to find the volume it encloses.



50. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{x^2+y^2+z^2}\,e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz = 2\pi$$

(The improper triple integral is defined as the limit of a triple integral over a solid sphere as the radius of the sphere increases indefinitely.)

51. (a) Use cylindrical coordinates to show that the volume of the solid bounded above by the sphere $r^2 + z^2 = a^2$ and below by the cone $z = r\cot\phi_0$ (or $\phi = \phi_0$), where $0 < \phi_0 < \pi/2$, is

$$V = \frac{2\pi a^3}{3}(1 - \cos\phi_0)$$

(b) Deduce that the volume of the spherical wedge given by $\rho_1 \leqslant \rho \leqslant \rho_2$, $\theta_1 \leqslant \theta \leqslant \theta_2$, $\phi_1 \leqslant \phi \leqslant \phi_2$ is

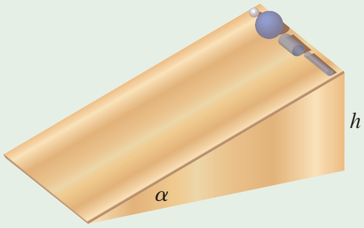
$$\Delta V = \frac{\rho_2^3 - \rho_1^3}{3}(\cos\phi_1 - \cos\phi_2)(\theta_2 - \theta_1)$$

(c) Use the Mean Value Theorem to show that the volume in part (b) can be written as

$$\Delta V = \tilde{\rho}^2 \sin\tilde{\phi}\,\Delta\rho\,\Delta\theta\,\Delta\phi$$

where $\tilde{\rho}$ lies between ρ_1 and ρ_2, $\tilde{\phi}$ lies between ϕ_1 and ϕ_2, $\Delta\rho = \rho_2 - \rho_1$, $\Delta\theta = \theta_2 - \theta_1$, and $\Delta\phi = \phi_2 - \phi_1$.

APPLIED PROJECT ROLLER DERBY



Suppose that a solid ball (a marble), a hollow ball (a squash ball), a solid cylinder (a steel bar), and a hollow cylinder (a lead pipe) roll down a slope. Which of these objects reaches the bottom first? (Make a guess before proceeding.)

To answer this question, we consider a ball or cylinder with mass m, radius r, and moment of inertia I (about the axis of rotation). If the vertical drop is h, then the potential energy at the top is mgh. Suppose the object reaches the bottom with velocity v and angular velocity ω, so $v = \omega r$. The kinetic energy at the bottom consists of two parts: $\frac{1}{2}mv^2$ from translation (moving down the slope) and $\frac{1}{2}I\omega^2$ from rotation. If we assume that energy loss from rolling friction is negligible, then conservation of energy gives

$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

1. Show that

$$v^2 = \frac{2gh}{1 + I^*} \qquad \text{where } I^* = \frac{I}{mr^2}$$

2. If $y(t)$ is the vertical distance traveled at time t, then the same reasoning as used in Problem 1 shows that $v^2 = 2gy/(1 + I^*)$ at any time t. Use this result to show that y satisfies the differential equation

$$\frac{dy}{dt} = \sqrt{\frac{2g}{1 + I^*}}\,(\sin\alpha)\sqrt{y}$$

where α is the angle of inclination of the plane.

3. By solving the differential equation in Problem 2, show that the total travel time is

$$T = \sqrt{\frac{2h(1 + I^*)}{g\sin^2\alpha}}$$

This shows that the object with the smallest value of I^* wins the race.

4. Show that $I^* = \frac{1}{2}$ for a solid cylinder and $I^* = 1$ for a hollow cylinder.

5. Calculate I^* for a partly hollow ball with inner radius a and outer radius r. Express your answer in terms of $b = a/r$. What happens as $a \to 0$ and as $a \to r$?

6. Show that $I^* = \frac{2}{5}$ for a solid ball and $I^* = \frac{2}{3}$ for a hollow ball. Thus the objects finish in the following order: solid ball, solid cylinder, hollow ball, hollow cylinder.

15.9 Change of Variables in Multiple Integrals

In one-dimensional calculus we often use a change of variable (a substitution) to simplify an integral. By reversing the roles of x and u, we can write the Substitution Rule (5.5.6) as

$$\boxed{1} \qquad \int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du$$

where $x = g(u)$ and $a = g(c)$, $b = g(d)$. Another way of writing Formula 1 is as follows:

$$\boxed{2} \qquad \int_a^b f(x)\,dx = \int_c^d f(x(u))\,\frac{dx}{du}\,du$$

A change of variables can also be useful in evaluating double and triple integrals.

Change of Variables in Double Integrals

We have already seen an example of a change of variables for double integrals: conversion to polar coordinates. The new variables r and θ are related to the old variables x and y by the equations

$$x = r\cos\theta \qquad y = r\sin\theta$$

and the change of variables formula (15.3.2) can be written as

$$\iint\limits_R f(x, y)\,dA = \iint\limits_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

where S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane.

More generally, we consider a change of variables that is given by a **transformation** T from the uv-plane to the xy-plane:

$$T(u, v) = (x, y)$$

where x and y are related to u and v by the equations

$$\boxed{3} \qquad x = g(u, v) \qquad y = h(u, v)$$

or, as we sometimes write,

$$x = x(u, v) \qquad y = y(u, v)$$

We usually assume that T is a $\boldsymbol{C^1}$ **transformation**, which means that g and h have continuous first-order partial derivatives.

A transformation T is really just a function whose domain and range are both subsets of $\mathbb{R}^2$. If $T(u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is called the **image** of the point (u_1, v_1). If no two points have the same image, T is called **one-to-one**. Figure 1 shows the effect of a transformation T on a region S in the uv-plane. T transforms S into a region R in the xy-plane called the **image of** $\boldsymbol{S}$, consisting of the images of all points in S.

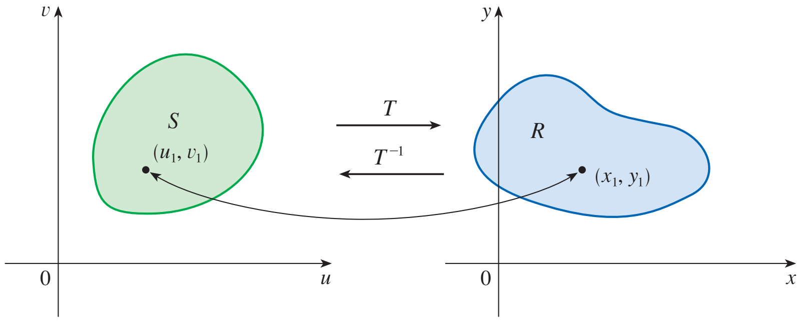


FIGURE 1

If T is a one-to-one transformation, then it has an **inverse transformation** T^{-1} from the xy-plane to the uv-plane and it may be possible to solve Equations 3 for u and v in terms of x and y:

$$u = G(x, y) \qquad v = H(x, y)$$

EXAMPLE 1 A transformation is defined by the equations

$$x = u^2 - v^2 \qquad y = 2uv$$

Find the image of the square $S = \{(u, v) \mid 0 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 1\}$.

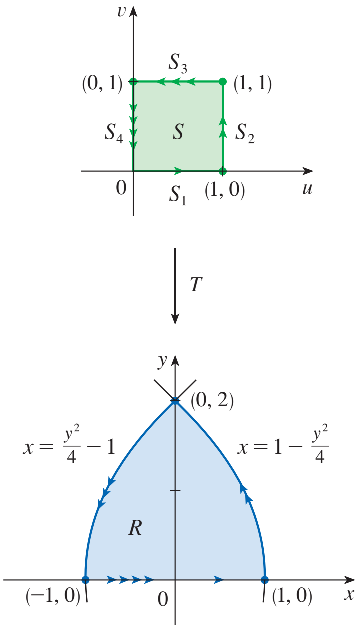


FIGURE 2

SOLUTION The transformation maps the boundary of S into the boundary of the image. So we begin by finding the images of the sides of S. The first side, S_1, is given by $v = 0$ $(0 \leqslant u \leqslant 1)$. (See Figure 2.) From the given equations we have $x = u^2$, $y = 0$, and so $0 \leqslant x \leqslant 1$. Thus S_1 is mapped onto the line segment from $(0, 0)$ to $(1, 0)$ in the xy-plane. The second side, S_2, is $u = 1$ $(0 \leqslant v \leqslant 1)$ and, putting $u = 1$ in the given equations, we get

$$x = 1 - v^2 \qquad y = 2v$$

Eliminating v, we obtain

$$\boxed{4} \qquad x = 1 - \frac{y^2}{4} \qquad 0 \leqslant x \leqslant 1$$

which is part of a parabola. Similarly, S_3 is given by $v = 1$ $(0 \leqslant u \leqslant 1)$, whose image is the parabolic arc

$$\boxed{5} \qquad x = \frac{y^2}{4} - 1 \qquad -1 \leqslant x \leqslant 0$$

Finally, S_4 is given by $u = 0$ $(0 \leqslant v \leqslant 1)$ whose image is $x = -v^2$, $y = 0$, that is, $-1 \leqslant x \leqslant 0$. (Notice that as we move around the square in the counterclockwise direction, we also move around the parabolic region in the counterclockwise direction.) The image of S is the region R (shown in Figure 2) bounded by the x-axis and the parabolas given by Equations 4 and 5. ■

Now let's see how a change of variables affects a double integral. We start with a small rectangle S in the uv-plane whose lower left corner is the point (u_0, v_0) and whose dimensions are Δu and Δv. (See Figure 3.)

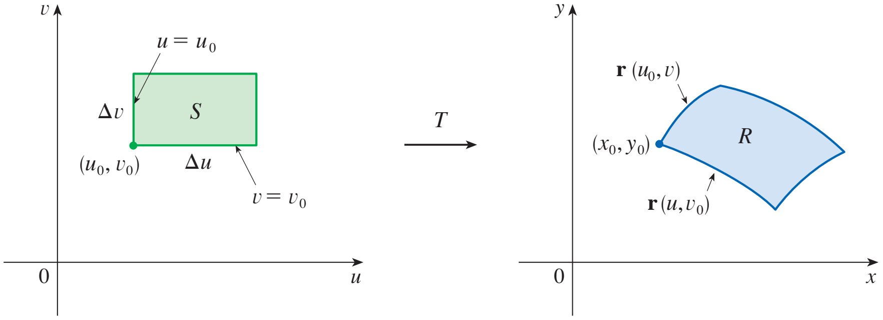


FIGURE 3

The image of S is a region R in the xy-plane, one of whose boundary points is $(x_0, y_0) = T(u_0, v_0)$. The vector

$$\mathbf{r}(u, v) = g(u, v)\,\mathbf{i} + h(u, v)\,\mathbf{j}$$

is the position vector of the image of the point (u, v). The equation of the lower side of S is $v = v_0$, whose image curve is given by the vector function $\mathbf{r}(u, v_0)$. The tangent vector at (x_0, y_0) to this image curve is

$$\mathbf{r}_u = g_u(u_0, v_0)\,\mathbf{i} + h_u(u_0, v_0)\,\mathbf{j} = \frac{\partial x}{\partial u}\,\mathbf{i} + \frac{\partial y}{\partial u}\,\mathbf{j}$$

Similarly, the tangent vector at (x_0, y_0) to the image curve of the left side of S (namely, $u = u_0$) is

$$\mathbf{r}_v = g_v(u_0, v_0)\,\mathbf{i} + h_v(u_0, v_0)\,\mathbf{j} = \frac{\partial x}{\partial v}\,\mathbf{i} + \frac{\partial y}{\partial v}\,\mathbf{j}$$

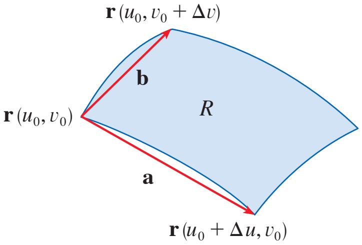

$\mathbf{r}(u_0, v_0 + \Delta v)$
b
R
$\mathbf{r}(u_0, v_0)$
a
$\mathbf{r}(u_0 + \Delta u, v_0)$

FIGURE 4

We can approximate the image region $R = T(S)$ by a parallelogram determined by the secant vectors

$$\mathbf{a} = \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \qquad \mathbf{b} = \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0)$$

shown in Figure 4. But

$$\mathbf{r}_u = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u}$$

and so $$\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \Delta u\,\mathbf{r}_u$$

Similarly $$\mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \Delta v\,\mathbf{r}_v$$

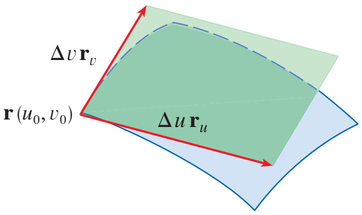


FIGURE 5

This means that we can approximate R by a parallelogram determined by the vectors $\Delta u\,\mathbf{r}_u$ and $\Delta v\,\mathbf{r}_v$. (See Figure 5.) Therefore we can approximate the area of R by the area of this parallelogram, which, from Section 12.4, is

$$\boxed{6} \qquad |(\Delta u\,\mathbf{r}_u) \times (\Delta v\,\mathbf{r}_v)| = |\mathbf{r}_u \times \mathbf{r}_v|\,\Delta u\,\Delta v$$

Computing the cross product, we obtain

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

The determinant that arises in this calculation is called the *Jacobian* of the transformation and is given a special notation.

The Jacobian is named after the German mathematician Carl Gustav Jacob Jacobi (1804–1851). Although the French mathematician Cauchy first used these special determinants involving partial derivatives, Jacobi developed them into a method for evaluating multiple integrals.

7 Definition The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

With this notation we can use Equation 6 to give an approximation to the area ΔA of R:

$$\boxed{8} \qquad \Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u\,\Delta v$$

where the Jacobian is evaluated at (u_0, v_0).

Next we divide a region S in the uv-plane into rectangles S_{ij} and call their images in the xy-plane R_{ij}. (See Figure 6.)

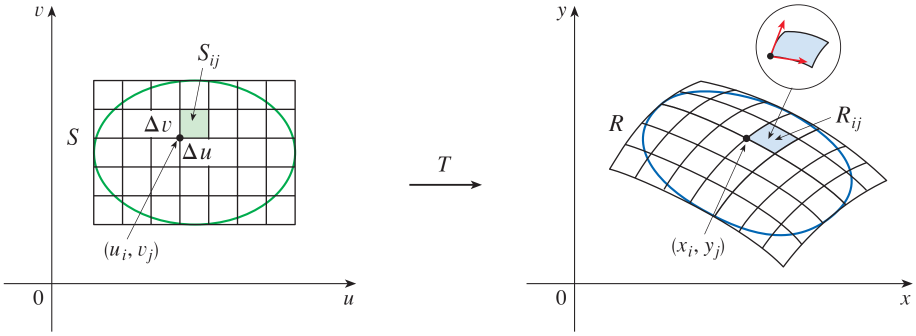


FIGURE 6

Applying the approximation (8) to each R_{ij}, we approximate the double integral of f over R as follows:

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A$$

$$\approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(g(u_i, v_j), h(u_i, v_j)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u\, \Delta v$$

where the Jacobian is evaluated at (u_i, v_j). Notice that this double sum is a Riemann sum for the integral

$$\iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

The foregoing argument suggests that the following theorem is true. (A full proof is given in books on advanced calculus.)

$\boxed{9}$ **Change of Variables in a Double Integral** Suppose that T is a C^1 transformation whose Jacobian is nonzero and that T maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Theorem 9 says that we change from an integral in x and y to an integral in u and v by expressing x and y in terms of u and v and writing

$$dA = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Notice the similarity between Theorem 9 and the one-dimensional formula in Equation 2. Instead of the derivative dx/du, we have the absolute value of the Jacobian, that is, $|\partial(x, y)/\partial(u, v)|$.

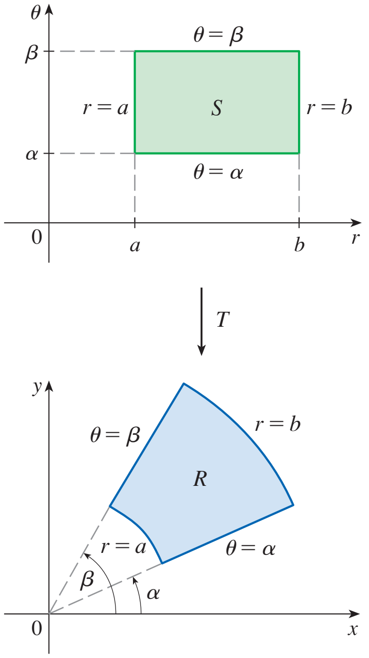


FIGURE 7
The polar coordinate transformation

As a first illustration of Theorem 9, we show that the formula for integration in polar coordinates is just a special case. Here the transformation T from the $r\theta$-plane to the xy-plane is given by

$$x = g(r, \theta) = r\cos\theta \qquad y = h(r, \theta) = r\sin\theta$$

and the geometry of the transformation is shown in Figure 7: T maps an ordinary rectangle in the $r\theta$-plane to a polar rectangle in the xy-plane. The Jacobian of T is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\[2ex] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r > 0$$

Thus Theorem 9 gives

$$\begin{aligned} \iint_R f(x, y)\, dx\, dy &= \iint_S f(r\cos\theta, r\sin\theta) \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| dr\, d\theta \\ &= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta \end{aligned}$$

which is the same as Formula 15.3.2.

EXAMPLE 2 Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\, dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geqslant 0$.

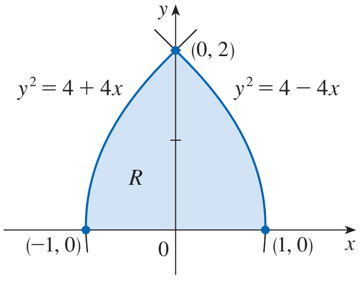


FIGURE 8

SOLUTION The region R is pictured in Figure 8. It is the region from Example 1 (see Figure 2); in that example we discovered that $T(S) = R$, where S is the square $[0, 1] \times [0, 1]$. Indeed, the reason for making the change of variables to evaluate the integral is that S is a much simpler region than R. First we need to compute the Jacobian:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 > 0$$

Therefore, by Theorem 9,

$$\begin{aligned} \iint_R y\, dA &= \iint_S 2uv \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dA = \int_0^1 \int_0^1 (2uv)4(u^2 + v^2)\, du\, dv \\ &= 8 \int_0^1 \int_0^1 (u^3v + uv^3)\, du\, dv = 8 \int_0^1 \left[\tfrac{1}{4}u^4v + \tfrac{1}{2}u^2v^3 \right]_{u=0}^{u=1} dv \\ &= \int_0^1 (2v + 4v^3)\, dv = \left[v^2 + v^4 \right]_0^1 = 2 \end{aligned}$$

■

NOTE Example 2 was not a very difficult problem to solve because we were given a suitable change of variables. If we are not supplied with a transformation, then the first step is to think of an appropriate change of variables. If $f(x, y)$ is difficult to integrate, then the form of $f(x, y)$ may suggest a transformation. If the region of integration R is awkward, then the transformation should be chosen so that the corresponding region S in the uv-plane has a convenient description.

EXAMPLE 3 Evaluate the integral $\iint_R e^{(x+y)/(x-y)}\,dA$, where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$, and $(0, -1)$.

SOLUTION Since it isn't easy to integrate $e^{(x+y)/(x-y)}$, we make a change of variables suggested by the form of this function:

$$u = x + y \qquad v = x - y \tag{10}$$

These equations define a transformation T^{-1} from the xy-plane to the uv-plane. Theorem 9 talks about a transformation T from the uv-plane to the xy-plane. It is obtained by solving Equations 10 for x and y:

$$x = \tfrac{1}{2}(u + v) \qquad y = \tfrac{1}{2}(u - v) \tag{11}$$

The Jacobian of T is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

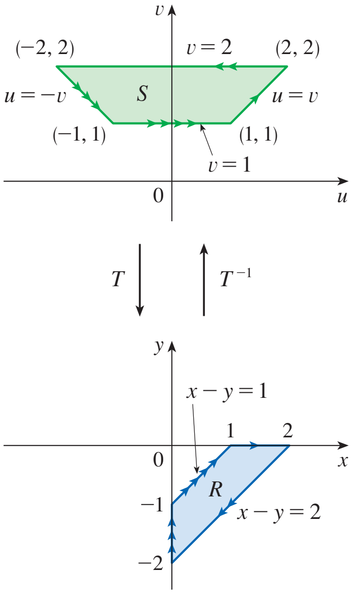


FIGURE 9

To find the region S in the uv-plane corresponding to R, we note that the sides of R lie on the lines

$$y = 0 \qquad x - y = 2 \qquad x = 0 \qquad x - y = 1$$

and, from either Equations 10 or Equations 11, the image lines in the uv-plane are

$$u = v \qquad v = 2 \qquad u = -v \qquad v = 1$$

Thus the region S is the trapezoidal region with vertices $(1, 1)$, $(2, 2)$, $(-2, 2)$, and $(-1, 1)$ shown in Figure 9. Since

$$S = \{(u, v) \mid 1 \leqslant v \leqslant 2,\ -v \leqslant u \leqslant v\}$$

Theorem 9 gives

$$\begin{aligned} \iint_R e^{(x+y)/(x-y)}\,dA &= \iint_S e^{u/v} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\,dv \\ &= \int_1^2 \int_{-v}^{v} e^{u/v}\left(\tfrac{1}{2}\right) du\,dv = \frac{1}{2}\int_1^2 \Big[ve^{u/v}\Big]_{u=-v}^{u=v} dv \\ &= \frac{1}{2}\int_1^2 (e - e^{-1})v\,dv = \tfrac{3}{4}(e - e^{-1}) \end{aligned}$$

Change of Variables in Triple Integrals

There is a similar change of variables formula for triple integrals. Let T be a one-to-one transformation that maps a region S in uvw-space onto a region R in xyz-space by means of the equations

$$x = g(u, v, w) \qquad y = h(u, v, w) \qquad z = k(u, v, w)$$

The **Jacobian** of T is the following 3×3 determinant:

$$\boxed{12} \qquad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

Under hypotheses similar to those in Theorem 9, we have the following formula for triple integrals:

$$\boxed{\boxed{13} \quad \iiint\limits_R f(x, y, z)\, dV = \iiint\limits_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw}$$

EXAMPLE 4 Use Formula 13 to derive the formula for triple integration in spherical coordinates.

SOLUTION Here the change of variables is given by

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

We compute the Jacobian as follows:

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \sin\phi \cos\theta & -\rho \sin\phi \sin\theta & \rho \cos\phi \cos\theta \\ \sin\phi \sin\theta & \rho \sin\phi \cos\theta & \rho \cos\phi \sin\theta \\ \cos\phi & 0 & -\rho \sin\phi \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} -\rho \sin\phi \sin\theta & \rho \cos\phi \cos\theta \\ \rho \sin\phi \cos\theta & \rho \cos\phi \sin\theta \end{vmatrix} - \rho \sin\phi \begin{vmatrix} \sin\phi \cos\theta & -\rho \sin\phi \sin\theta \\ \sin\phi \sin\theta & \rho \sin\phi \cos\theta \end{vmatrix}$$

$$= \cos\phi\,(-\rho^2 \sin\phi \cos\phi \sin^2\theta - \rho^2 \sin\phi \cos\phi \cos^2\theta)$$

$$- \rho \sin\phi\,(\rho \sin^2\phi \cos^2\theta + \rho \sin^2\phi \sin^2\theta)$$

$$= -\rho^2 \sin\phi \cos^2\phi - \rho^2 \sin\phi \sin^2\phi = -\rho^2 \sin\phi$$

Since $0 \leqslant \phi \leqslant \pi$, we have $\sin\phi \geqslant 0$. Therefore

$$\left| \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} \right| = |-\rho^2 \sin\phi| = \rho^2 \sin\phi$$

and Formula 13 gives

$$\iiint\limits_R f(x, y, z)\, dV = \iiint\limits_S f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

which is equivalent to Formula 15.8.3. ■

15.9 Exercises

1. Match the given transformation with the image (labeled I–VI) of the set $S = \{(u, v) \mid 0 \leqslant u \leqslant 1, 0 \leqslant v \leqslant 1\}$ under the transformation. Give reasons for your choices.

(a) $x = u + v$, $y = u - v$

(b) $x = u - v$, $y = uv$

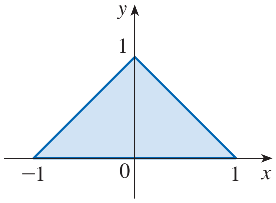
(c) $x = u \cos v$, $y = u \sin v$

(d) $x = u - v$, $y = u + v^2$

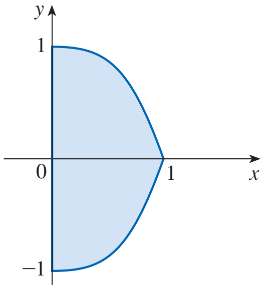
(e) $x = u + v$, $y = 2v$

(f) $x = uv$, $y = u^3 - v^3$

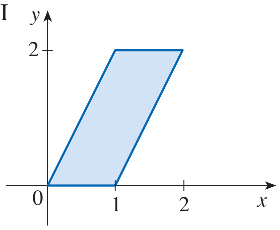
I



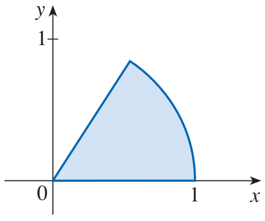
II



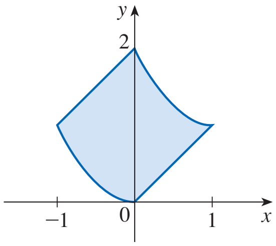
III



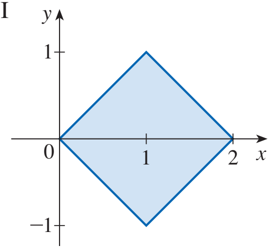
IV



V



VI



2–6 Find the image of the set S under the given transformation.

2. $S = \{(u, v) \mid 0 \leqslant u \leqslant 1, 0 \leqslant v \leqslant 2\}$;
$x = u + v, y = -v$

3. $S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\}$;
$x = 2u + 3v,\ y = u - v$

4. S is the square bounded by the lines $u = 0$, $u = 1$, $v = 0$, $v = 1$; $x = v,\ y = u(1 + v^2)$

5. S is the triangular region with vertices $(0, 0)$, $(1, 1)$, $(0, 1)$; $x = u^2,\ y = v$

6. S is the disk given by $u^2 + v^2 \leqslant 1$; $x = au,\ y = bv$

7–10 A region R in the xy-plane is given. Find equations for a transformation T that maps a rectangular region S in the uv-plane onto R, where the sides of S are parallel to the u- and v-axes.

7. R is bounded by $y = 2x - 1$, $y = 2x + 1$, $y = 1 - x$, $y = 3 - x$

8. R is the parallelogram with vertices $(0, 0)$, $(4, 3)$, $(2, 4)$, $(-2, 1)$

9. R lies between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$ in the first quadrant

10. R is bounded by the hyperbolas $y = 1/x$, $y = 4/x$ and the lines $y = x$, $y = 4x$ in the first quadrant

11–16 Find the Jacobian of the transformation.

11. $x = 2u + v, \quad y = 4u - v$

12. $x = u^2 + uv, \quad y = uv^2$

13. $x = s \cos t, \quad y = s \sin t$

14. $x = pe^q, \quad y = qe^p$

15. $x = uv, \quad y = vw, \quad z = wu$

16. $x = u + vw, \quad y = v + wu, \quad z = w + uv$

17–22 Use the given transformation to evaluate the integral.

17. $\iint_R (x - 3y)\, dA$, where R is the triangular region with vertices $(0, 0)$, $(2, 1)$, and $(1, 2)$; $\quad x = 2u + v,\ y = u + 2v$

18. $\iint_R (4x + 8y)\, dA$, where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$; $x = \frac{1}{4}(u + v),\ y = \frac{1}{4}(v - 3u)$

19. $\iint_R x^2\, dA$, where R is the region bounded by the ellipse $9x^2 + 4y^2 = 36$; $\quad x = 2u,\ y = 3v$

20. $\iint_R (x^2 - xy + y^2)\, dA$, where R is the region bounded by the ellipse $x^2 - xy + y^2 = 2$;
$x = \sqrt{2}\, u - \sqrt{2/3}\, v,\ y = \sqrt{2}\, u + \sqrt{2/3}\, v$

21. $\iint_R xy\, dA$, where R is the region in the first quadrant bounded by the lines $y = x$ and $y = 3x$ and the hyperbolas $xy = 1$, $xy = 3$; $\quad x = u/v,\ y = v$

22. $\iint_R y^2\, dA$, where R is the region bounded by the curves $xy = 1$, $xy = 2$, $xy^2 = 1$, $xy^2 = 2$; $\quad u = xy,\ v = xy^2$. Illustrate by using a graphing calculator or computer to draw R.

23. (a) Evaluate $\iiint_E dV$, where E is the solid enclosed by the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Use the transformation $x = au,\ y = bv,\ z = cw$.

(b) The earth is not a perfect sphere; rotation has resulted in flattening at the poles. So the shape can be approximated

by an ellipsoid with $a = b = 6378$ km and $c = 6356$ km. Use part (a) to estimate the volume of the earth.
(c) If the solid of part (a) has constant density k, find its moment of inertia about the z-axis.

24. An important problem in thermodynamics is to find the work done by an ideal Carnot engine. A cycle consists of alternating expansion and compression of gas in a piston. The work done by the engine is equal to the area of the region R enclosed by two isothermal curves $xy = a$, $xy = b$ and two adiabatic curves $xy^{1.4} = c$, $xy^{1.4} = d$, where $0 < a < b$ and $0 < c < d$. Compute the work done by determining the area of R.

25–30 Evaluate the integral by making an appropriate change of variables.

25. $\displaystyle\iint_R \frac{x - 2y}{3x - y}\,dA$, where R is the parallelogram enclosed by the lines $x - 2y = 0$, $x - 2y = 4$, $3x - y = 1$, and $3x - y = 8$

26. $\iint_R (x + y)e^{x^2 - y^2}\,dA$, where R is the rectangle enclosed by the lines $x - y = 0$, $x - y = 2$, $x + y = 0$, and $x + y = 3$

27. $\displaystyle\iint_R \cos\left(\frac{y - x}{y + x}\right) dA$, where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, 2)$, and $(0, 1)$

28. $\iint_R \sin(9x^2 + 4y^2)\,dA$, where R is the region in the first quadrant bounded by the ellipse $9x^2 + 4y^2 = 1$

29. $\iint_R e^{x+y}\,dA$, where R is given by the inequality $|x| + |y| \leqslant 1$

30. $\displaystyle\iint_R \frac{y}{x}\,dA$, where R is the region enclosed by the lines $x + y = 1$, $x + y = 3$, $y = 2x$, $y = x/2$

---

31. Let f be continuous on $[0, 1]$ and let R be the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. Show that

$$\iint_R f(x + y)\,dA = \int_0^1 u f(u)\,du$$

15 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. Suppose f is a continuous function defined on a rectangle $R = [a, b] \times [c, d]$.
(a) Write an expression for a double Riemann sum of f. If $f(x, y) \geqslant 0$, what does the sum represent?
(b) Write the definition of $\iint_R f(x, y)\,dA$ as a limit.
(c) What is the geometric interpretation of $\iint_R f(x, y)\,dA$ if $f(x, y) \geqslant 0$? What if f takes on both positive and negative values?
(d) How do you evaluate $\iint_R f(x, y)\,dA$?
(e) What does the Midpoint Rule for double integrals say?
(f) Write an expression for the average value of f.

2. (a) How do you define $\iint_D f(x, y)\,dA$ if D is a bounded region that is not a rectangle?
(b) What is a type I region? How do you evaluate $\iint_D f(x, y)\,dA$ if D is a type I region?
(c) What is a type II region? How do you evaluate $\iint_D f(x, y)\,dA$ if D is a type II region?
(d) What properties do double integrals have?

3. How do you change from rectangular coordinates to polar coordinates in a double integral? Why would you want to make the change?

4. If a lamina occupies a plane region D and has density function $\rho(x, y)$, write expressions for each of the following in terms of double integrals.
(a) The mass
(b) The moments about the axes
(c) The center of mass
(d) The moments of inertia about the axes and the origin

5. Let f be a joint density function of a pair of continuous random variables X and Y.
(a) Write a double integral for the probability that X lies between a and b and Y lies between c and d.
(b) What properties does f possess?
(c) What are the expected values of X and Y?

6. Write an expression for the area of a surface with equation $z = f(x, y)$, $(x, y) \in D$.

7. (a) Write the definition of the triple integral of f over a rectangular box B.
(b) How do you evaluate $\iiint_B f(x, y, z)\,dV$?
(c) How do you define $\iiint_E f(x, y, z)\,dV$ if E is a bounded solid region that is not a box?
(d) What is a type 1 solid region? How do you evaluate $\iiint_E f(x, y, z)\,dV$ if E is such a region?
(e) What is a type 2 solid region? How do you evaluate $\iiint_E f(x, y, z)\,dV$ if E is such a region?
(f) What is a type 3 solid region? How do you evaluate $\iiint_E f(x, y, z)\,dV$ if E is such a region?

8. Suppose a solid object occupies the region E and has density function $\rho(x, y, z)$. Write expressions for each of the following.
(a) The mass
(b) The moments about the coordinate planes
(c) The coordinates of the center of mass
(d) The moments of inertia about the axes

9. (a) How do you change from rectangular coordinates to cylindrical coordinates in a triple integral?
(b) How do you change from rectangular coordinates to spherical coordinates in a triple integral?
(c) In what situations would you change to cylindrical or spherical coordinates?

10. (a) If a transformation T is given by

$$x = g(u, v) \qquad y = h(u, v)$$

what is the Jacobian of T?
(b) How do you change variables in a double integral?
(c) How do you change variables in a triple integral?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. $\int_{-1}^{2}\int_{0}^{6} x^2 \sin(x-y)\,dx\,dy = \int_{0}^{6}\int_{-1}^{2} x^2\sin(x-y)\,dy\,dx$

2. $\int_0^1\int_0^x \sqrt{x+y^2}\,dy\,dx = \int_0^x\int_0^1 \sqrt{x+y^2}\,dx\,dy$

3. $\int_1^2\int_3^4 x^2e^y\,dy\,dx = \int_1^2 x^2\,dx \int_3^4 e^y\,dy$

4. $\int_{-1}^{1}\int_0^1 e^{x^2+y^2}\sin y\,dx\,dy = 0$

5. If f is continuous on $[0, 1]$, then

$$\int_0^1\int_0^1 f(x)f(y)\,dy\,dx = \left[\int_0^1 f(x)\,dx\right]^2$$

6. $\int_1^4\int_0^1 (x^2+\sqrt{y})\sin(x^2y^2)\,dx\,dy \leqslant 9$

7. If D is the disk given by $x^2+y^2 \leqslant 4$, then

$$\iint_D \sqrt{4-x^2-y^2}\,dA = \tfrac{16}{3}\pi$$

8. The integral $\iiint_E kr^3\,dz\,dr\,d\theta$ represents the moment of inertia about the z-axis of a solid E with constant density k.

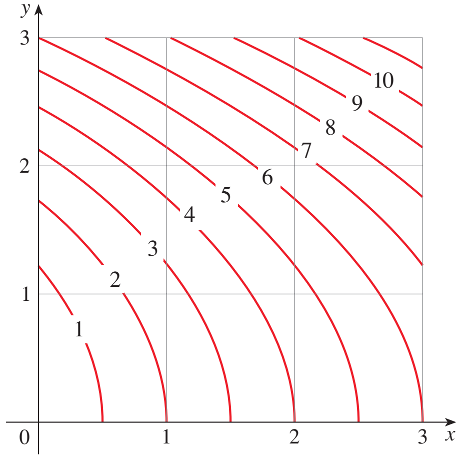
9. The integral

$$\int_0^{2\pi}\int_0^2\int_r^2 dz\,dr\,d\theta$$

represents the volume enclosed by the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 2$.

EXERCISES

1. A contour map is shown for a function f on the square $R = [0, 3] \times [0, 3]$. Use a Riemann sum with nine terms to estimate the value of $\iint_R f(x, y)\,dA$. Take the sample points to be the upper right corners of the squares.



2. Use the Midpoint Rule to estimate the integral in Exercise 1.

3–8 Calculate the iterated integral.

3. $\int_1^2\int_0^2 (y+2xe^y)\,dx\,dy$

4. $\int_0^1\int_0^1 ye^{xy}\,dx\,dy$

5. $\int_0^1\int_0^x \cos(x^2)\,dy\,dx$

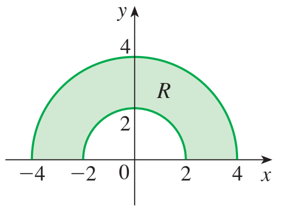
6. $\int_0^1\int_x^{e^x} 3xy^2\,dy\,dx$

7. $\int_0^\pi\int_0^1\int_0^{\sqrt{1-y^2}} y\sin x\,dz\,dy\,dx$

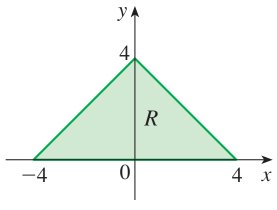
8. $\int_0^1\int_0^y\int_x^1 6xyz\,dz\,dx\,dy$

9–10 Write $\iint_R f(x, y)\,dA$ as an iterated integral, where R is the region shown and f is an arbitrary continuous function on R.

9.



10.



11. The cylindrical coordinates of a point are $(2\sqrt{3}, \pi/3, 2)$. Find the rectangular and spherical coordinates of the point.

12. The rectangular coordinates of a point are $(2, 2, -1)$. Find the cylindrical and spherical coordinates of the point.

13. The spherical coordinates of a point are $(8, \pi/4, \pi/6)$. Find the rectangular and cylindrical coordinates of the point.

14. Identify the surfaces whose equations are given.
(a) $\theta = \pi/4$ (b) $\phi = \pi/4$

15. Write the equation in cylindrical coordinates and in spherical coordinates.
(a) $x^2 + y^2 + z^2 = 4$ (b) $x^2 + y^2 = 4$

16. Sketch the solid consisting of all points with spherical coordinates (ρ, θ, ϕ) such that $0 \leqslant \theta \leqslant \pi/2$, $0 \leqslant \phi \leqslant \pi/6$, and $0 \leqslant \rho \leqslant 2\cos\phi$.

17. Describe the region whose area is given by the integral

$$\int_0^{\pi/2} \int_0^{\sin 2\theta} r\, dr\, d\theta$$

18. Describe the solid whose volume is given by the integral

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

and evaluate the integral.

19–20 Calculate the iterated integral by first reversing the order of integration.

19. $\displaystyle\int_0^1 \int_x^1 \cos(y^2)\, dy\, dx$ **20.** $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3}\, dx\, dy$

21–34 Calculate the value of the multiple integral.

21. $\iint_R ye^{xy}\, dA$, where $R = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 3\}$

22. $\iint_D xy\, dA$, where $D = \{(x, y) \mid 0 \leqslant y \leqslant 1,\ y^2 \leqslant x \leqslant y + 2\}$

23. $\displaystyle\iint_D \frac{y}{1 + x^2}\, dA$,
where D is bounded by $y = \sqrt{x}$, $y = 0$, $x = 1$

24. $\displaystyle\iint_D \frac{1}{1 + x^2}\, dA$, where D is the triangular region with vertices $(0, 0)$, $(1, 1)$, and $(0, 1)$

25. $\iint_D y\, dA$, where D is the region in the first quadrant bounded by the parabolas $x = y^2$ and $x = 8 - y^2$

26. $\iint_D y\, dA$, where D is the region in the first quadrant that lies above the hyperbola $xy = 1$ and the line $y = x$ and below the line $y = 2$

27. $\iint_D (x^2 + y^2)^{3/2}\, dA$, where D is the region in the first quadrant bounded by the lines $y = 0$ and $y = \sqrt{3}\,x$ and the circle $x^2 + y^2 = 9$

28. $\iint_D x\, dA$, where D is the region in the first quadrant that lies between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$

29. $\iiint_E xy\, dV$, where

$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 3,\ 0 \leqslant y \leqslant x,\ 0 \leqslant z \leqslant x + y\}$

30. $\iiint_T xy\, dV$, where T is the solid tetrahedron with vertices $(0, 0, 0)$, $\left(\tfrac{1}{3}, 0, 0\right)$, $(0, 1, 0)$, and $(0, 0, 1)$

31. $\iiint_E y^2z^2\, dV$, where E is bounded by the paraboloid $x = 1 - y^2 - z^2$ and the plane $x = 0$

32. $\iiint_E z\, dV$, where E is bounded by the planes $y = 0$, $z = 0$, $x + y = 2$ and the cylinder $y^2 + z^2 = 1$ in the first octant

33. $\iiint_E yz\, dV$, where E lies above the plane $z = 0$, below the plane $z = y$, and inside the cylinder $x^2 + y^2 = 4$

34. $\iiint_H z^3\sqrt{x^2 + y^2 + z^2}\, dV$, where H is the solid hemisphere that lies above the xy-plane and has center the origin and radius 1

35–40 Find the volume of the given solid.

35. Under the paraboloid $z = x^2 + 4y^2$ and above the rectangle $R = [0, 2] \times [1, 4]$

36. Under the surface $z = x^2y$ and above the triangle in the xy-plane with vertices $(1, 0)$, $(2, 1)$, and $(4, 0)$

37. The solid tetrahedron with vertices $(0, 0, 0)$, $(0, 0, 1)$, $(0, 2, 0)$, and $(2, 2, 0)$

38. Bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 3$

39. One of the wedges cut from the cylinder $x^2 + 9y^2 = a^2$ by the planes $z = 0$ and $z = mx$

40. Above the paraboloid $z = x^2 + y^2$ and below the half-cone $z = \sqrt{x^2 + y^2}$

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41. Consider a lamina that occupies the region D bounded by the parabola $x = 1 - y^2$ and the coordinate axes in the first quadrant with density function $\rho(x, y) = y$.
(a) Find the mass of the lamina.
(b) Find the center of mass.
(c) Find the moments of inertia and radii of gyration about the x- and y-axes.

42. A lamina occupies the part of the disk $x^2 + y^2 \leqslant a^2$ that lies in the first quadrant.
(a) Find the centroid of the lamina.
(b) Find the center of mass of the lamina if the density function is $\rho(x, y) = xy^2$.

43. (a) Find the centroid of a solid right circular cone with height h and base radius a. (Place the cone so that its base is in the xy-plane with center the origin and its axis along the positive z-axis.)
(b) If the cone has density function $\rho(x, y, z) = \sqrt{x^2 + y^2}$, find the moment of inertia of the cone about its axis (the z-axis).

44. Find the area of the part of the cone $z^2 = a^2(x^2 + y^2)$ between the planes $z = 1$ and $z = 2$.

45. Find the area of the part of the surface $z = x^2 + y$ that lies above the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 2)$.

T **46.** Use a computer algebra system to graph the surface $z = x\sin y$, $-3 \leqslant x \leqslant 3$, $-\pi \leqslant y \leqslant \pi$, and find its surface area correct to four decimal places.

47. Use polar coordinates to evaluate

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^3 + xy^2)\,dy\,dx$$

48. Use spherical coordinates to evaluate

$$\int_{-2}^{2} \int_0^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2 + y^2 + z^2}\,dz\,dx\,dy$$

49. If D is the region bounded by the curves $y = 1 - x^2$ and $y = e^x$, find the approximate value of the integral $\iint_D y^2\,dA$. (Use a graph to estimate the points of intersection of the curves.)

T **50.** Use a computer algebra system to find the center of mass of the solid tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 2, 0)$, $(0, 0, 3)$ and density function $\rho(x, y, z) = x^2 + y^2 + z^2$.

51. The joint density function for random variables X and Y is

$$f(x, y) = \begin{cases} C(x + y) & \text{if } 0 \leqslant x \leqslant 3,\ 0 \leqslant y \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant C.
(b) Find $P(X \leqslant 2, Y \geqslant 1)$.
(c) Find $P(X + Y \leqslant 1)$.

52. A lamp has three bulbs, each of a type with average lifetime 800 hours. If we model the probability of failure of a bulb by an exponential density function with mean 800, find the probability that all three bulbs fail within a total of 1000 hours.

53. Rewrite the integral

$$\int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y} f(x, y, z)\,dz\,dy\,dx$$

as an iterated integral in the order $dx\,dy\,dz$.

54. Give five other iterated integrals that are equal to

$$\int_0^2 \int_0^{y^3} \int_0^{y^2} f(x, y, z)\,dz\,dx\,dy$$

55. Use the transformation $u = x - y$, $v = x + y$ to evaluate

$$\iint_R \frac{x - y}{x + y}\,dA$$

where R is the square with vertices $(0, 2)$, $(1, 1)$, $(2, 2)$, and $(1, 3)$.

56. Use the transformation $x = u^2$, $y = v^2$, $z = w^2$ to find the volume of the region bounded by the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ and the coordinate planes.

57. Use the change of variables formula and an appropriate transformation to evaluate $\iint_R xy\,dA$, where R is the square with vertices $(0, 0)$, $(1, 1)$, $(2, 0)$, and $(1, -1)$.

58. (a) Evaluate

$$\iint_D \frac{1}{(x^2 + y^2)^{n/2}}\,dA$$

where n is an integer and D is the region bounded by the circles with center the origin and radii r and R, $0 < r < R$.

(b) For what values of n does the integral in part (a) have a limit as $r \to 0^+$?

(c) Find

$$\iiint_E \frac{1}{(x^2 + y^2 + z^2)^{n/2}}\,dV$$

where E is the region bounded by the spheres with center the origin and radii r and R, $0 < r < R$.

(d) For what values of n does the integral in part (c) have a limit as $r \to 0^+$?

Problems Plus

1. If $[\![x]\!]$ denotes the greatest integer in x, evaluate the integral

$$\iint_R [\![x+y]\!]\,dA$$

where $R = \{(x, y) \mid 1 \leqslant x \leqslant 3,\ 2 \leqslant y \leqslant 5\}$.

2. Evaluate the integral

$$\int_0^1 \int_0^1 e^{\max\{x^2,\, y^2\}}\,dy\,dx$$

where $\max\{x^2, y^2\}$ means the larger of the numbers x^2 and y^2.

3. Find the average value of the function $f(x) = \int_x^1 \cos(t^2)\,dt$ on the interval $[0, 1]$.

4. Show that

$$\int_0^2 \int_0^x 2e^{x^2-y^2}\,dy\,dx = \int_0^2 \int_y^{4-y} e^{xy}\,dx\,dy$$

5. The double integral $\displaystyle\int_0^1 \int_0^1 \frac{1}{1-xy}\,dx\,dy$ is an improper integral and could be defined as the limit of double integrals over the rectangle $[0, t] \times [0, t]$ as $t \to 1^-$. But if we expand the integrand as a geometric series, we can express the integral as the sum of an infinite series. Show that

$$\int_0^1 \int_0^1 \frac{1}{1-xy}\,dx\,dy = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

6. Leonhard Euler was able to find the exact sum of the series in Problem 5. In 1736 he proved that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

In this problem we ask you to prove this fact by evaluating the double integral in Problem 5. Start by making the change of variables

$$x = \frac{u-v}{\sqrt{2}} \qquad y = \frac{u+v}{\sqrt{2}}$$

This gives a rotation about the origin through the angle $\pi/4$. You will need to sketch the corresponding region in the uv-plane.

[*Hint:* If, in evaluating the integral, you encounter either of the expressions $(1 - \sin\theta)/\cos\theta$ or $(\cos\theta)/(1 + \sin\theta)$, you might like to use the identity $\cos\theta = \sin((\pi/2) - \theta)$ and the corresponding identity for $\sin\theta$.]

7. (a) Show that

$$\int_0^1 \int_0^1 \int_0^1 \frac{1}{1-xyz}\,dx\,dy\,dz = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

(Nobody has ever been able to find the exact value of the sum of this series.)

(b) Show that

$$\int_0^1 \int_0^1 \int_0^1 \frac{1}{1+xyz}\,dx\,dy\,dz = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$$

Use this equation to evaluate the triple integral correct to two decimal places.

8. Show that

$$\int_0^\infty \frac{\arctan \pi x - \arctan x}{x}\,dx = \frac{\pi}{2}\ln \pi$$

by first expressing the integral as an iterated integral.

9. (a) Show that when Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

is written in cylindrical coordinates, it becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

(b) Show that when Laplace's equation is written in spherical coordinates, it becomes

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{\cot \phi}{\rho^2}\frac{\partial u}{\partial \phi} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} = 0$$

10. (a) A lamina has constant density ρ and takes the shape of a disk with center the origin and radius R. Use Newton's Law of Gravitation (see Section 13.4) to show that the magnitude of the force of attraction that the lamina exerts on a body with mass m located at the point $(0, 0, d)$ on the positive z-axis is

$$F = 2\pi G m \rho d\left(\frac{1}{d} - \frac{1}{\sqrt{R^2 + d^2}}\right)$$

[*Hint:* Divide the disk as in Figure 15.3.4 and first compute the vertical component of the force exerted by the polar subrectangle R_{ij}.]

(b) Show that the magnitude of the force of attraction of a lamina with density ρ that occupies an entire plane on an object with mass m located at a distance d from the plane is

$$F = 2\pi G m \rho$$

Notice that this expression does not depend on d.

11. If f is continuous, show that

$$\int_0^x \int_0^y \int_0^z f(t)\,dt\,dz\,dy = \tfrac{1}{2}\int_0^x (x - t)^2 f(t)\,dt$$

12. Evaluate $\displaystyle\lim_{n\to\infty} n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n^2} \frac{1}{\sqrt{n^2 + ni + j}}$.

13. The plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad a > 0, \quad b > 0, \quad c > 0$$

cuts the solid ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1$$

into two pieces. Find the volume of the smaller piece.

Vector fields can be used to model such diverse phenomena as gravity, electricity and magnetism, and fluid flow. For instance, a hurricane can be modeled by a function that describes the velocity vectors at each point in space. We can then use vector calculus to calculate quantities such as the circulation, the twisting (curl), the flow (flux), or the expansions and compressions (divergence) of the wind, as well as relationships between these quantities.

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16

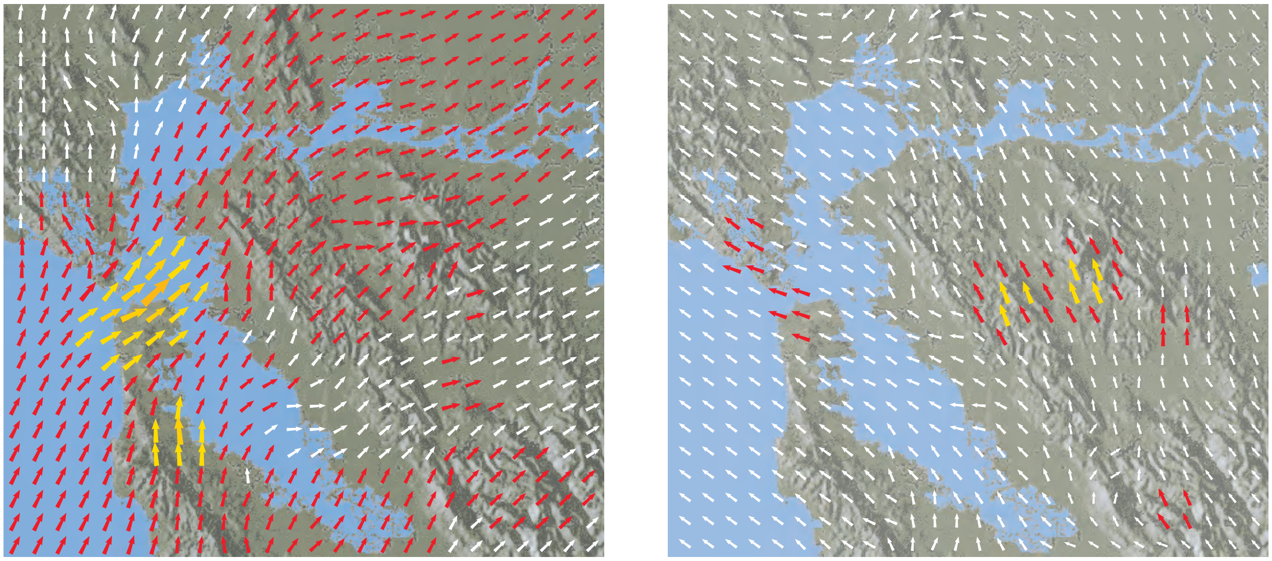
Vector Calculus

IN THIS CHAPTER WE STUDY the calculus of vector fields. (These are functions that assign vectors to points in space.) In particular we define line integrals (which can be used to find the work done by a force field in moving an object along a curve). Then we define surface integrals (which can be used to find the rate of fluid flow across a surface). The connections between these new types of integrals and the single, double, and triple integrals that we have already met are given by the higher-dimensional versions of the Fundamental Theorem of Calculus: Green's Theorem, Stokes' Theorem, and the Divergence Theorem.

16.1 Vector Fields

Vector Fields in $\mathbb{R}^2$ and $\mathbb{R}^3$

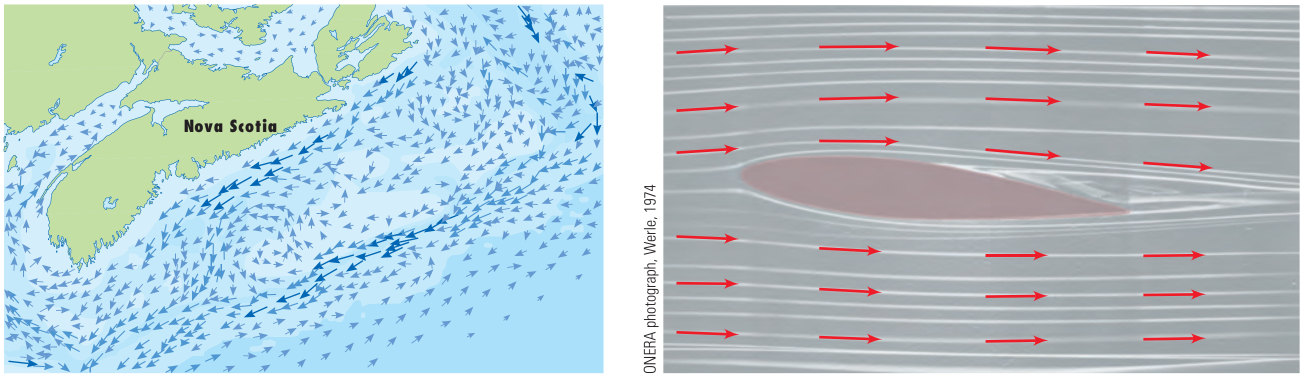
The vectors in Figure 1 are air velocity vectors that indicate the wind speed and direction at points 10 m above the surface elevation in the San Francisco Bay area. We see at a glance from the largest arrows in part (a) that the greatest wind speeds at that time occurred as the winds entered the bay across the Golden Gate Bridge. Part (b) shows the very different wind pattern 12 hours earlier. Associated with every point in the air we can imagine a wind velocity vector. This is an example of a *velocity vector field.*



(a) 6:00 PM (b) 6:00 AM

FIGURE 1 Velocity vector fields showing San Francisco Bay wind patterns on a particular spring day

Other examples of velocity vector fields are illustrated in Figure 2: ocean currents and flow past an airfoil.



(a) Ocean currents off the coast of Nova Scotia (b) Airflow past an inclined airfoil

FIGURE 2
Velocity vector fields

Another type of vector field, called a *force field,* associates a force vector with each point in a region. An example is the gravitational force field that we will look at in Example 4.

In general, a vector field is a function whose domain is a set of points in $\mathbb{R}^2$ (or $\mathbb{R}^3$) and whose range is a set of vectors in V_2 (or V_3).

1 Definition Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

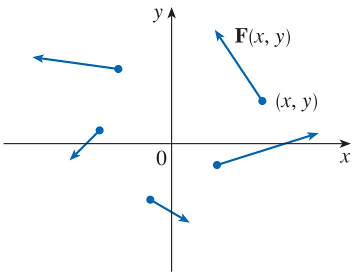


FIGURE 3
Vector field on $\mathbb{R}^2$

The best way to picture a vector field is to draw the arrow representing the vector $\mathbf{F}(x, y)$ starting at the point (x, y). Of course, it's impossible to do this for all points (x, y), but we can form a reasonable impression of $\mathbf{F}$ by drawing vectors for a few representative points in D as in Figure 3. Since $\mathbf{F}(x, y)$ is a two-dimensional vector, we can write it in terms of its **component functions** P and Q as follows:

$$\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j} = \langle P(x, y), Q(x, y)\rangle$$

or, for short,

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$$

Notice that P and Q are scalar functions of two variables and are sometimes called **scalar fields** to distinguish them from vector fields.

2 Definition Let E be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

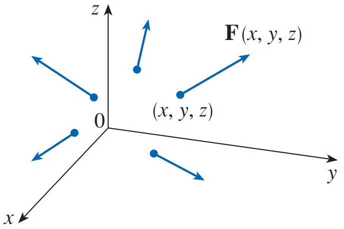


FIGURE 4
Vector field on $\mathbb{R}^3$

A vector field $\mathbf{F}$ on $\mathbb{R}^3$ is pictured in Figure 4. We can express it in terms of its component functions P, Q, and R as

$$\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

As with the vector functions in Section 13.1, we can define continuity of vector fields and show that $\mathbf{F}$ is continuous if and only if its component functions P, Q, and R are continuous.

We sometimes identify a point (x, y, z) with its position vector $\mathbf{x} = \langle x, y, z\rangle$ and write $\mathbf{F}(\mathbf{x})$ instead of $\mathbf{F}(x, y, z)$. Then $\mathbf{F}$ becomes a function that assigns a vector $\mathbf{F}(\mathbf{x})$ to a vector $\mathbf{x}$.

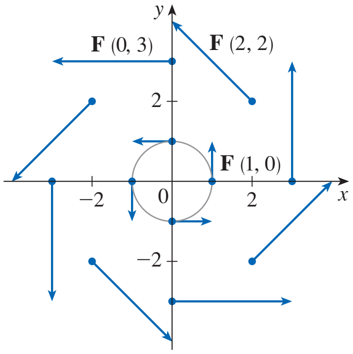


FIGURE 5
$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$

EXAMPLE 1 A vector field on $\mathbb{R}^2$ is defined by $\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$. Describe $\mathbf{F}$ by sketching some of the vectors $\mathbf{F}(x, y)$ as in Figure 3.

SOLUTION Since $\mathbf{F}(1, 0) = \mathbf{j}$, we draw the vector $\mathbf{j} = \langle 0, 1\rangle$ starting at the point $(1, 0)$ in Figure 5. Since $\mathbf{F}(0, 1) = -\mathbf{i}$, we draw the vector $\langle -1, 0\rangle$ with starting point $(0, 1)$. Continuing in this way, we calculate several other representative values of $\mathbf{F}(x, y)$ in the table and draw the corresponding vectors to represent the vector field in Figure 5.

(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$
$(1, 0)$	$\langle 0, 1\rangle$	$(-1, 0)$	$\langle 0, -1\rangle$
$(2, 2)$	$\langle -2, 2\rangle$	$(-2, -2)$	$\langle 2, -2\rangle$
$(3, 0)$	$\langle 0, 3\rangle$	$(-3, 0)$	$\langle 0, -3\rangle$
$(0, 1)$	$\langle -1, 0\rangle$	$(0, -1)$	$\langle 1, 0\rangle$
$(-2, 2)$	$\langle -2, -2\rangle$	$(2, -2)$	$\langle 2, 2\rangle$
$(0, 3)$	$\langle -3, 0\rangle$	$(0, -3)$	$\langle 3, 0\rangle$

It appears from Figure 5 that each arrow is tangent to a circle with center the origin. To confirm this, we take the dot product of the position vector $\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j}$ with the vector $\mathbf{F}(\mathbf{x}) = \mathbf{F}(x, y)$:

$$\mathbf{x} \cdot \mathbf{F}(\mathbf{x}) = (x\,\mathbf{i} + y\,\mathbf{j}) \cdot (-y\,\mathbf{i} + x\,\mathbf{j}) = -xy + yx = 0$$

This shows that $\mathbf{F}(x, y)$ is perpendicular to the position vector $\langle x, y\rangle$ and is therefore tangent to a circle with center the origin and radius $|\mathbf{x}| = \sqrt{x^2 + y^2}$. Notice also that

$$|\mathbf{F}(x, y)| = \sqrt{(-y)^2 + x^2} = \sqrt{x^2 + y^2} = |\mathbf{x}|$$

so the magnitude of the vector $\mathbf{F}(x, y)$ is equal to the radius of the circle. ■

Some graphing software is capable of plotting vector fields in two or three dimensions. The results give a better impression of the vector field than is possible by hand because a computer can plot a large number of representative vectors. Figure 6 shows a computer plot of the vector field in Example 1; Figures 7 and 8 show two other vector fields. Notice that the software scales the lengths of the vectors so they are not too long and yet are proportional to their true lengths.

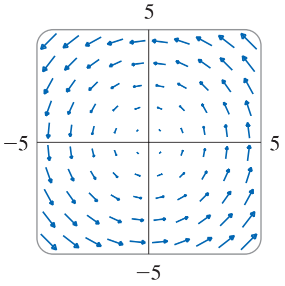


FIGURE 6
$\mathbf{F}(x, y) = \langle -y, x\rangle$

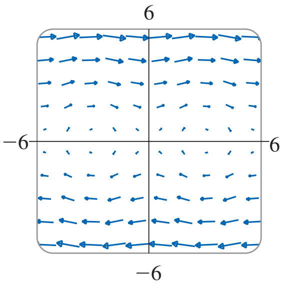


FIGURE 7
$\mathbf{F}(x, y) = \langle y, \sin x\rangle$

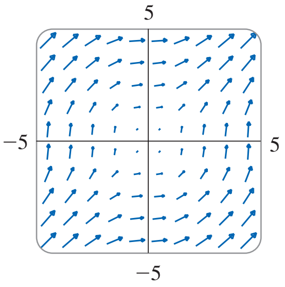


FIGURE 8
$\mathbf{F}(x, y) = \langle \ln(1 + y^2), \ln(1 + x^2)\rangle$

EXAMPLE 2 Sketch the vector field on $\mathbb{R}^3$ given by $\mathbf{F}(x, y, z) = z\,\mathbf{k}$.

SOLUTION A sketch is shown in Figure 9. Notice that all vectors are vertical and point upward above the xy-plane or downward below it. The magnitude increases with distance from the xy-plane.

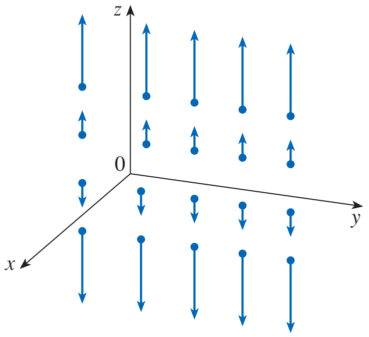


FIGURE 9
$\mathbf{F}(x, y, z) = z\,\mathbf{k}$

■

We were able to draw the vector field in Example 2 by hand because of its particularly simple formula. Most three-dimensional vector fields, however, are virtually impossible to sketch by hand and so we need to resort to computer software. Examples are shown in Figures 10, 11, and 12. Notice that the vector fields in Figures 10 and 11 have similar formulas, but all the vectors in Figure 11 point in the general direction of the negative y-axis because their y-components are all -2. If the vector field in Figure 12 represents a velocity field, then a particle would be swept upward and would spiral around the z-axis in the clockwise direction as viewed from above.

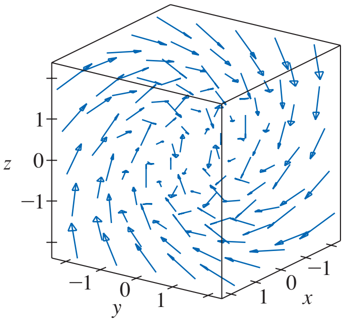


FIGURE 10

$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$

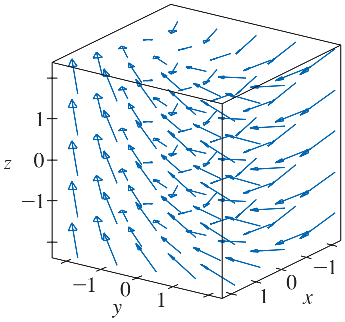


FIGURE 11

$\mathbf{F}(x, y, z) = y\,\mathbf{i} - 2\,\mathbf{j} + x\,\mathbf{k}$

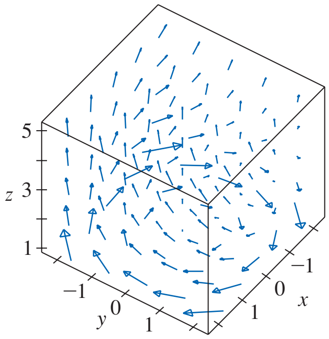


FIGURE 12

$\mathbf{F}(x, y, z) = \dfrac{y}{z}\,\mathbf{i} - \dfrac{x}{z}\,\mathbf{j} + \dfrac{z}{4}\,\mathbf{k}$

EXAMPLE 3 Imagine a fluid flowing steadily along a pipe and let $\mathbf{V}(x, y, z)$ be the velocity vector at a point (x, y, z). Then $\mathbf{V}$ assigns a vector to each point (x, y, z) in a certain domain E (the interior of the pipe) and so $\mathbf{V}$ is a vector field on $\mathbb{R}^3$ called a **velocity field**. A possible velocity field is illustrated in Figure 13. The speed at any given point is indicated by the length of the arrow.

Velocity fields also occur in other areas of physics. For instance, the vector field in Example 1 could be used as the velocity field describing the counterclockwise rotation of a wheel. We have seen other examples of velocity fields in Figures 1 and 2. ■

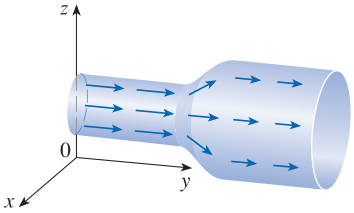


FIGURE 13

Velocity field in fluid flow

EXAMPLE 4 Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects with masses m and M is

$$|\mathbf{F}| = \frac{mMG}{r^2}$$

where r is the distance between the objects and G is the gravitational constant. (This is an example of an inverse square law; see Section 1.2.) Let's assume that the object with mass M is located at the origin in $\mathbb{R}^3$. (For instance, M could be the mass of the earth and the origin would be at its center.) Let the position vector of the object with mass m be $\mathbf{x} = \langle x, y, z\rangle$. Then $r = |\mathbf{x}|$, so $r^2 = |\mathbf{x}|^2$. The gravitational force exerted on this second object acts toward the origin, and the unit vector in this direction is

$$-\frac{\mathbf{x}}{|\mathbf{x}|}$$

Therefore the gravitational force acting on the object at $\mathbf{x} = \langle x, y, z\rangle$ is

$$\boxed{3} \qquad \mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

[Physicists often use the notation $\mathbf{r}$ instead of $\mathbf{x}$ for the position vector, so you may see Formula 3 written in the form $\mathbf{F} = -(mMG/r^3)\mathbf{r}$.] The function given by Equation 3 is an example of a vector field, called the **gravitational field**, because it associates a vector [the force $\mathbf{F}(\mathbf{x})$] with every point $\mathbf{x}$ in space.

Formula 3 is a compact way of writing the gravitational field, but we can also write it in terms of its component functions by using the facts that $\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $|\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$:

$$\mathbf{F}(x, y, z) = \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{k}$$

The gravitational field $\mathbf{F}$ is pictured in Figure 14.

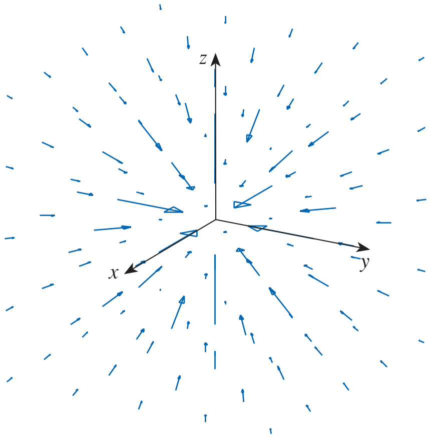


FIGURE 14
Gravitational force field

■

EXAMPLE 5 Suppose an electric charge Q is located at the origin. According to Coulomb's Law, the electric force $\mathbf{F}(\mathbf{x})$ exerted by this charge on a charge q located at a point (x, y, z) with position vector $\mathbf{x} = \langle x, y, z\rangle$ is

$$\boxed{4} \qquad \mathbf{F}(\mathbf{x}) = \frac{\varepsilon q Q}{|\mathbf{x}|^3}\,\mathbf{x}$$

where ε is a constant (that depends on the units used). For like charges, we have $qQ > 0$ and the force is repulsive; for unlike charges, we have $qQ < 0$ and the force is attractive. Notice the similarity between Formulas 3 and 4. Both vector fields are examples of **force fields**.

Instead of considering the electric force $\mathbf{F}$, physicists often consider the force per unit charge:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{q}\,\mathbf{F}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\,\mathbf{x}$$

Then $\mathbf{E}$ is a vector field on $\mathbb{R}^3$ called the **electric field** of Q. ■

Gradient Fields

If f is a scalar function of two variables, recall from Section 14.6 that its gradient ∇f (or grad f) is defined by

$$\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}$$

Therefore ∇f is really a vector field on $\mathbb{R}^2$ and is called a **gradient vector field**. Likewise, if f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x, y, z) = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}$$

EXAMPLE 6 Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f. How are they related?

SOLUTION The gradient vector field is given by

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = 2xy\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$$

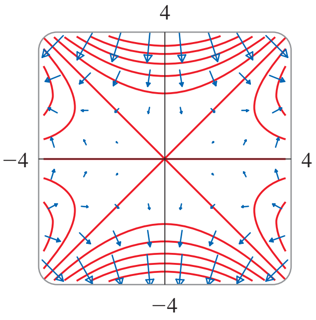


FIGURE 15

Figure 15 shows a contour map of f with the gradient vector field. Notice that the gradient vectors are perpendicular to the level curves, as we would expect from Section 14.6. Notice also that the gradient vectors are long where the level curves are close to each other and short where the curves are farther apart. That's because the length of the gradient vector is the value of the directional derivative of f and closely spaced level curves indicate a steep graph. ■

A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a **potential function** for $\mathbf{F}$.

Not all vector fields are conservative, but such fields do arise frequently in physics. For example, the gravitational field $\mathbf{F}$ in Example 4 is conservative because if we define

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

then

$$\begin{aligned}\nabla f(x, y, z) &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\\ &= \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k}\\ &= \mathbf{F}(x, y, z)\end{aligned}$$

In Sections 16.3 and 16.5 we will learn how to tell whether or not a given vector field is conservative.

16.1 Exercises

1–12 Sketch the vector field $\mathbf{F}$ by drawing a diagram like Figure 5 or Figure 9.

1. $\mathbf{F}(x, y) = \mathbf{i} + \frac{1}{2}\mathbf{j}$

2. $\mathbf{F}(x, y) = 2\,\mathbf{i} - \mathbf{j}$

3. $\mathbf{F}(x, y) = \mathbf{i} + \frac{1}{2}y\,\mathbf{j}$

4. $\mathbf{F}(x, y) = x\,\mathbf{i} + \frac{1}{2}y\,\mathbf{j}$

5. $\mathbf{F}(x, y) = -\frac{1}{2}\mathbf{i} + (y - x)\,\mathbf{j}$

6. $\mathbf{F}(x, y) = y\,\mathbf{i} + (x + y)\,\mathbf{j}$

7. $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

8. $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

9. $\mathbf{F}(x, y, z) = \mathbf{i}$

10. $\mathbf{F}(x, y, z) = z\,\mathbf{i}$

11. $\mathbf{F}(x, y, z) = -y\,\mathbf{i}$

12. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{k}$

13–18 Match the vector fields $\mathbf{F}$ with the plots labeled I–VI. Give reasons for your choices.

13. $\mathbf{F}(x, y) = \langle x, -y \rangle$

14. $\mathbf{F}(x, y) = \langle y, x - y \rangle$

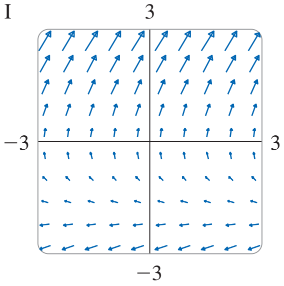
15. $\mathbf{F}(x, y) = \langle y, y + 2 \rangle$

16. $\mathbf{F}(x, y) = \langle y, 2x \rangle$

17. $\mathbf{F}(x, y) = \langle \sin y, \cos x \rangle$

18. $\mathbf{F}(x, y) = \langle \cos(x + y), x \rangle$

I



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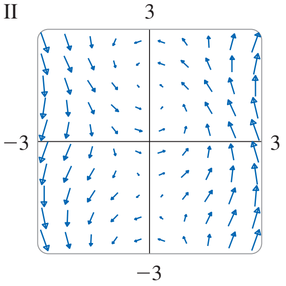

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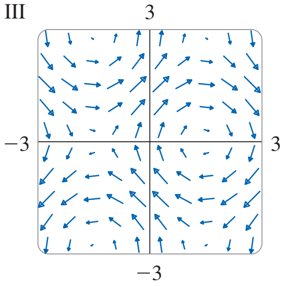

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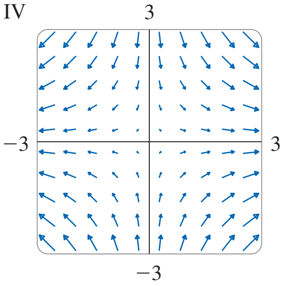

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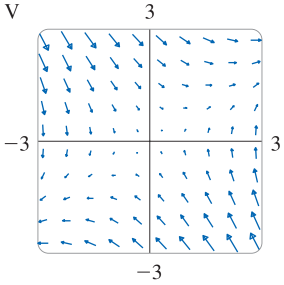
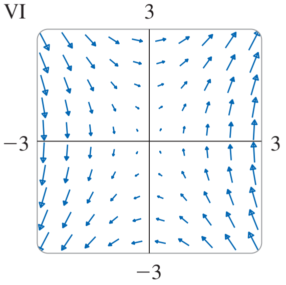


VI



19–22 Match the vector fields $\mathbf{F}$ on $\mathbb{R}^3$ with the plots labeled I–IV. Give reasons for your choices.

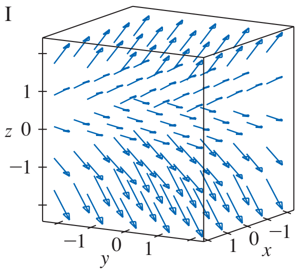
19. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$

20. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$

21. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 3\,\mathbf{k}$

22. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

I



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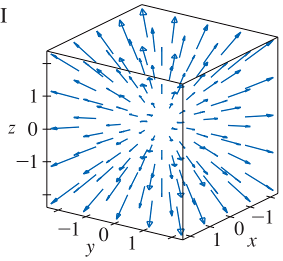

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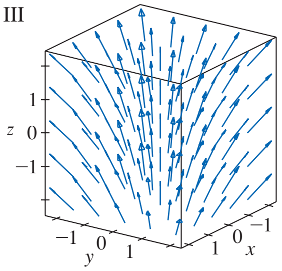

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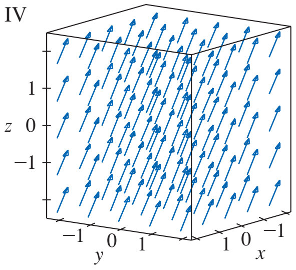


23. Use graphing software to plot the vector field

$$\mathbf{F}(x, y) = (y^2 - 2xy)\,\mathbf{i} + (3xy - 6x^2)\,\mathbf{j}$$

Explain the appearance by finding the set of points (x, y) such that $\mathbf{F}(x, y) = \mathbf{0}$.

24. Let $\mathbf{F}(\mathbf{x}) = (r^2 - 2r)\mathbf{x}$, where $\mathbf{x} = \langle x, y \rangle$ and $r = |\mathbf{x}|$. Use graphing software to plot this vector field in various domains until you can see what is happening. Describe the appearance of the plot and explain it by finding the points where $\mathbf{F}(\mathbf{x}) = \mathbf{0}$.

25–28 Find the gradient vector field ∇f of f.

25. $f(x, y) = y \sin(xy)$

26. $f(s, t) = \sqrt{2s + 3t}$

27. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

28. $f(x, y, z) = x^2 y e^{y/z}$

29–30 Find the gradient vector field ∇f of f and sketch it.

29. $f(x, y) = \frac{1}{2}(x - y)^2$

30. $f(x, y) = \frac{1}{2}(x^2 - y^2)$

31–34 Match the functions f with the plots of their gradient vector fields labeled I–IV. Give reasons for your choices.

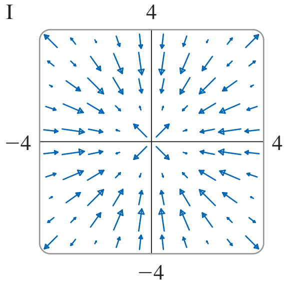
31. $f(x, y) = x^2 + y^2$

32. $f(x, y) = x(x + y)$

33. $f(x, y) = (x + y)^2$

34. $f(x, y) = \sin\sqrt{x^2 + y^2}$

I



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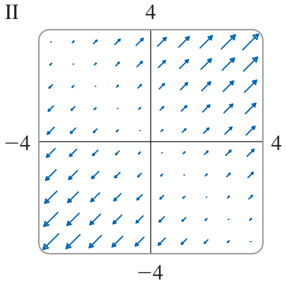

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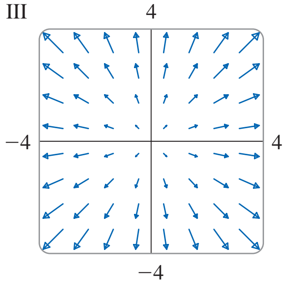

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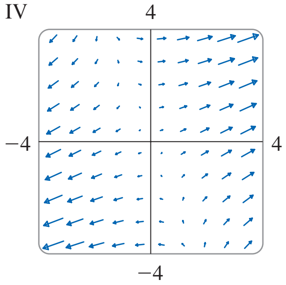


35–36 Plot the gradient vector field of f together with a contour map of f. Explain how they are related to each other.

35. $f(x, y) = \ln(1 + x^2 + 2y^2)$

36. $f(x, y) = \cos x - 2 \sin y$

37. A particle moves in a velocity field $\mathbf{V}(x, y) = \langle x^2, x + y^2 \rangle$. If it is at position $(2, 1)$ at time $t = 3$, estimate its location at time $t = 3.01$.

38. At time $t = 1$, a particle is located at position $(1, 3)$. If it moves in a velocity field

$$\mathbf{F}(x, y) = \langle xy - 2, y^2 - 10 \rangle$$

find its approximate location at time $t = 1.05$.

39–40 Flow Lines The *flow lines* (or *streamlines*) of a vector field are the paths followed by a particle whose velocity field is the given vector field. Thus the vectors in a vector field are tangent to the flow lines.

39. (a) Use a sketch of the vector field $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$ to draw some flow lines. From your sketches, can you guess the equations of the flow lines?

(b) If parametric equations of a flow line are $x = x(t)$, $y = y(t)$, explain why these functions satisfy the differential equations $dx/dt = x$ and $dy/dt = -y$. Then solve the differential equations to find an equation of the flow line that passes through the point $(1, 1)$.

40. (a) Sketch the vector field $\mathbf{F}(x, y) = \mathbf{i} + x\,\mathbf{j}$ and then sketch some flow lines. What shape do these flow lines appear to have?

(b) If parametric equations of the flow lines are $x = x(t)$, $y = y(t)$, what differential equations do these functions satisfy? Deduce that $dy/dx = x$.

(c) If a particle starts at the origin in the velocity field given by $\mathbf{F}$, find an equation of the path it follows.

16.2 Line Integrals

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval $[a, b]$, we integrate over a curve C. Such integrals are called *line integrals*, although "curve integrals" would be better terminology. They were invented in the early 19th century to solve problems involving fluid flow, forces, electricity, and magnetism.

Line Integrals in the Plane

We start with a plane curve C given by the parametric equations

$$\boxed{1} \qquad x = x(t) \qquad y = y(t) \qquad a \leqslant t \leqslant b$$

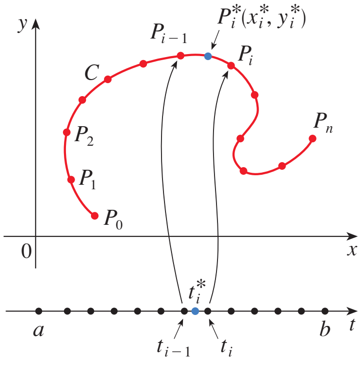


FIGURE 1

or, equivalently, by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, and we assume that C is a smooth curve. [This means that $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$. See Section 13.3.] If we divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width and we let $x_i = x(t_i)$ and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$. (See Figure 1.) We choose any point $P_i^*(x_i^*, y_i^*)$ in the ith subarc. (This corresponds to a point t_i^* in $[t_{i-1}, t_i]$.) Now if f is any function of two variables whose domain includes the curve C, we evaluate f at the point (x_i^*, y_i^*), multiply by the length Δs_i of the subarc, and form the sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

which is similar to a Riemann sum. Then we take the limit of these sums and make the following definition by analogy with a single integral.

2 Definition If f is defined on a smooth curve C given by Equations 1, then the **line integral of f along C** is

$$\int_C f(x, y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

if this limit exists.

In Section 10.2 we found that the length of C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

A similar type of argument can be used to show that if f is a continuous function, then the limit in Definition 2 always exists and the following formula can be used to evaluate the line integral:

$$\boxed{3} \qquad \int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

The arc length function s is discussed in Section 13.3.

If $s(t)$ is the length of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$, then

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

(See Equation 13.3.7.) So the way to remember Formula 3 is to express everything in terms of the parameter t: use the parametric equations to express x and y in terms of t and write ds as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

NOTE In the special case where C is the line segment that joins $(a, 0)$ to $(b, 0)$, using x as the parameter, we can write the parametric equations of C as follows: $x = x$, $y = 0$, $a \leqslant x \leqslant b$. Formula 3 then becomes

$$\int_C f(x, y)\, ds = \int_a^b f(x, 0)\, dx$$

and so the line integral reduces to an ordinary single integral in this case.

Just as for an ordinary single integral, we can interpret the line integral of a *positive* function as an area. In fact, if $f(x, y) \geqslant 0$, $\int_C f(x, y)\, ds$ represents the area of one side of the "fence" or "curtain" in Figure 2, whose base is C and whose height above the point (x, y) is $f(x, y)$.

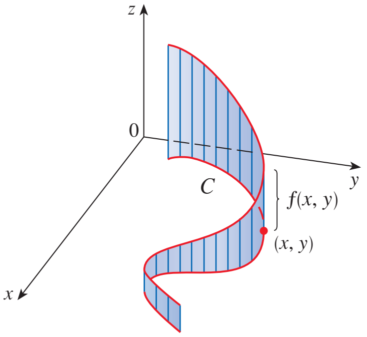


FIGURE 2

EXAMPLE 1 Evaluate $\int_C (2 + x^2 y)\, ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.

SOLUTION In order to use Formula 3, we first need parametric equations to represent C. Recall that the unit circle can be parametrized by means of the equations

$$x = \cos t \qquad y = \sin t$$

and the upper half of the circle is described by the parameter interval $0 \leqslant t \leqslant \pi$. (See Figure 3.) Therefore Formula 3 gives

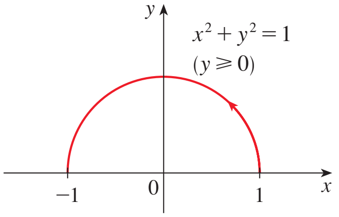


FIGURE 3

$$\begin{aligned}
\int_C (2 + x^2 y)\, ds &= \int_0^\pi (2 + \cos^2 t \sin t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_0^\pi (2 + \cos^2 t \sin t)\sqrt{\sin^2 t + \cos^2 t}\, dt \\
&= \int_0^\pi (2 + \cos^2 t \sin t)\, dt = \left[2t - \frac{\cos^3 t}{3}\right]_0^\pi \\
&= 2\pi + \tfrac{2}{3}
\end{aligned}$$

■

Suppose now that C is a **piecewise-smooth curve**; that is, C is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, where, as illustrated in Figure 4, the initial point of C_{i+1} is the terminal point of C_i. Then we define the integral of f along C as the sum of the integrals of f along each of the smooth pieces of C:

$$\int_C f(x, y)\, ds = \int_{C_1} f(x, y)\, ds + \int_{C_2} f(x, y)\, ds + \cdots + \int_{C_n} f(x, y)\, ds$$

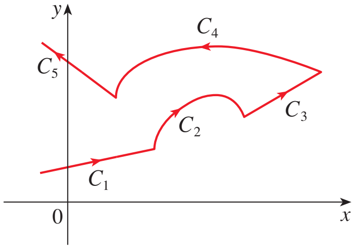


FIGURE 4
A piecewise-smooth curve

EXAMPLE 2 Evaluate $\int_C 2x\, ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ followed by the vertical line segment C_2 from $(1, 1)$ to $(1, 2)$.

SOLUTION The curve C is shown in Figure 5. C_1 is the graph of a function of x, so we can choose x as the parameter and the equations for C_1 become

$$x = x \qquad y = x^2 \qquad 0 \leqslant x \leqslant 1$$

Therefore

$$\begin{aligned}
\int_{C_1} 2x\, ds &= \int_0^1 2x\sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_0^1 2x\sqrt{1 + 4x^2}\, dx \\
&= \tfrac{1}{4} \cdot \tfrac{2}{3}(1 + 4x^2)^{3/2}\Big]_0^1 = \frac{5\sqrt{5} - 1}{6}
\end{aligned}$$

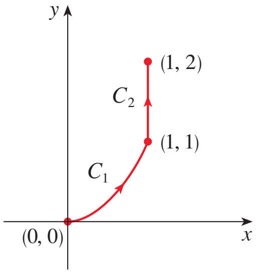


FIGURE 5
$C = C_1 \cup C_2$

On C_2 we choose y as the parameter, so the equations of C_2 are

$$x = 1 \qquad y = y \qquad 1 \leqslant y \leqslant 2$$

and $$\int_{C_2} 2x\,ds = \int_1^2 2(1)\sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\,dy = \int_1^2 2\,dy = 2$$

Thus $$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds = \frac{5\sqrt{5} - 1}{6} + 2$$ ■

Any physical interpretation of a line integral $\int_C f(x, y)\,ds$ depends on the physical interpretation of the function f. Suppose that $\rho(x, y)$ represents the linear density at a point (x, y) of a thin wire shaped like a curve C (see Example 3.7.2). Then the mass of the part of the wire from P_{i-1} to P_i in Figure 1 is approximately $\rho(x_i^*, y_i^*)\,\Delta s_i$ and so the total mass of the wire is approximately $\Sigma\, \rho(x_i^*, y_i^*)\,\Delta s_i$. By taking more and more points on the curve, we obtain the **mass** m of the wire as the limiting value of these approximations:

$$m = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*, y_i^*)\,\Delta s_i = \int_C \rho(x, y)\,ds$$

[For example, if $f(x, y) = 2 + x^2y$ represents the density of a semicircular wire, then the integral in Example 1 would represent the mass of the wire.] The **center of mass** of the wire with density function ρ is located at the point $(\bar{x}, \bar{y})$, where

$$\boxed{4} \qquad \bar{x} = \frac{1}{m}\int_C x\,\rho(x, y)\,ds \qquad \bar{y} = \frac{1}{m}\int_C y\,\rho(x, y)\,ds$$

Other physical interpretations of line integrals will be discussed later in this chapter.

EXAMPLE 3 A wire takes the shape of the semicircle $x^2 + y^2 = 1$, $y \geqslant 0$, and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

SOLUTION As in Example 1 we use the parametrization $x = \cos t$, $y = \sin t$, $0 \leqslant t \leqslant \pi$, and find that $ds = dt$. The linear density is

$$\rho(x, y) = k(1 - y)$$

where k is a constant, and so the mass of the wire is

$$m = \int_C k(1 - y)\,ds = \int_0^\pi k(1 - \sin t)\,dt = k\Big[t + \cos t\Big]_0^\pi = k(\pi - 2)$$

From Equations 4 we have

$$\begin{aligned}
\bar{y} &= \frac{1}{m}\int_C y\,\rho(x, y)\,ds = \frac{1}{k(\pi - 2)}\int_C y\,k(1 - y)\,ds \\
&= \frac{1}{\pi - 2}\int_0^\pi (\sin t - \sin^2 t)\,dt = \frac{1}{\pi - 2}\Big[-\cos t - \tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\Big]_0^\pi \\
&= \frac{4 - \pi}{2(\pi - 2)}
\end{aligned}$$

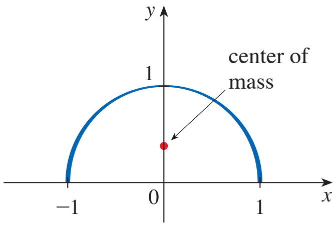


FIGURE 6

By symmetry we see that $\bar{x} = 0$, so the center of mass is

$$\left(0, \frac{4 - \pi}{2(\pi - 2)}\right) \approx (0, 0.38)$$

See Figure 6. ■

■ Line Integrals with Respect to *x* or *y*

Two other types of line integrals are obtained by replacing Δs_i by either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$ in Definition 2. They are called the **line integrals of *f* along *C* with respect to *x* and *y***:

$$\boxed{5} \qquad \int_C f(x, y)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta x_i$$

$$\boxed{6} \qquad \int_C f(x, y)\, dy = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta y_i$$

When we want to distinguish the original line integral $\int_C f(x, y)\, ds$ from those in Equations 5 and 6, we call it the **line integral with respect to arc length**.

The following formulas say that line integrals with respect to x and y can also be evaluated by expressing everything in terms of t: $x = x(t)$, $y = y(t)$, $dx = x'(t)\, dt$, $dy = y'(t)\, dt$.

$$\boxed{7} \qquad \boxed{\begin{aligned} \int_C f(x, y)\, dx &= \int_a^b f(x(t), y(t))\, x'(t)\, dt \\ \int_C f(x, y)\, dy &= \int_a^b f(x(t), y(t))\, y'(t)\, dt \end{aligned}}$$

We will see throughout this chapter that line integrals with respect to x and y frequently occur together (see, for instance, Equation 14). When this happens, it's customary to abbreviate by writing

$$\int_C P(x, y)\, dx + \int_C Q(x, y)\, dy = \int_C P(x, y)\, dx + Q(x, y)\, dy$$

When we are setting up a line integral, sometimes the most difficult thing is to think of a parametric representation for a curve whose geometric description is given. In particular, we often need to parametrize a line segment, so it's useful to remember that a vector representation of the line segment that starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ is given by

$$\boxed{8} \qquad \boxed{\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\,\mathbf{r}_1 \qquad 0 \leqslant t \leqslant 1}$$

(See Equation 12.5.4.)

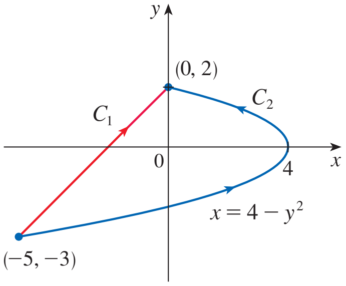


FIGURE 7

EXAMPLE 4 Evaluate $\int_C y^2\, dx + x\, dy$ for two different paths C.

(a) $C = C_1$ is the line segment from $(-5, -3)$ to $(0, 2)$.

(b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$.

(See Figure 7.)

SOLUTION

(a) A parametric representation for the line segment is

$$x = 5t - 5 \qquad y = 5t - 3 \qquad 0 \leqslant t \leqslant 1$$

(Use Equation 8 with $\mathbf{r}_0 = \langle -5, -3 \rangle$ and $\mathbf{r}_1 = \langle 0, 2 \rangle$.) Then $dx = 5\,dt$, $dy = 5\,dt$, and Formulas 7 give

$$\begin{aligned}\int_{C_1} y^2\,dx + x\,dy &= \int_0^1 (5t-3)^2(5\,dt) + (5t-5)(5\,dt)\\ &= 5\int_0^1 (25t^2 - 25t + 4)\,dt\\ &= 5\left[\frac{25t^3}{3} - \frac{25t^2}{2} + 4t\right]_0^1 = -\frac{5}{6}\end{aligned}$$

(b) Since the parabola is given as a function of y, let's take y as the parameter and write C_2 as

$$x = 4 - y^2 \qquad y = y \qquad -3 \leqslant y \leqslant 2$$

Then $dx = -2y\,dy$ and by Formulas 7 we have

$$\begin{aligned}\int_{C_2} y^2\,dx + x\,dy &= \int_{-3}^{2} y^2(-2y)\,dy + (4 - y^2)\,dy\\ &= \int_{-3}^{2} (-2y^3 - y^2 + 4)\,dy\\ &= \left[-\frac{y^4}{2} - \frac{y^3}{3} + 4y\right]_{-3}^{2} = 40\tfrac{5}{6}\end{aligned}$$ ■

Notice that we got different answers in parts (a) and (b) of Example 4 even though the two curves had the same endpoints. Thus, in general, the value of a line integral depends not just on the endpoints of the curve but also on the path. (But see Section 16.3 for conditions under which the integral is independent of the path.)

Notice also that the answers in Example 4 depend on the direction, or orientation, of the curve. If $-C_1$ denotes the line segment from $(0, 2)$ to $(-5, -3)$, you can verify, using the parametrization

$$x = -5t \qquad y = 2 - 5t \qquad 0 \leqslant t \leqslant 1$$

that
$$\int_{-C_1} y^2\,dx + x\,dy = \tfrac{5}{6}$$

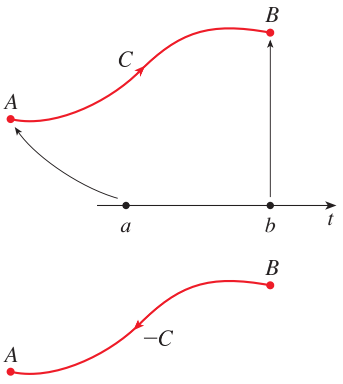


FIGURE 8

In general, a given parametrization $x = x(t)$, $y = y(t)$, $a \leqslant t \leqslant b$, determines an **orientation** of a curve C, with the positive direction corresponding to increasing values of the parameter t. (See Figure 8, where the initial point A corresponds to the parameter value a and the terminal point B corresponds to $t = b$.)

If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation (from initial point B to terminal point A in Figure 8), then we have

$$\int_{-C} f(x, y)\,dx = -\int_C f(x, y)\,dx \qquad \int_{-C} f(x, y)\,dy = -\int_C f(x, y)\,dy$$

But if we integrate with respect to arc length, the value of the line integral does *not* change when we reverse the orientation of the curve:

$$\int_{-C} f(x, y)\,ds = \int_C f(x, y)\,ds$$

This is because Δs_i is always positive, whereas Δx_i and Δy_i change sign when we reverse the orientation of C.

Line Integrals in Space

We now suppose that C is a smooth space curve given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \leqslant t \leqslant b$$

or by a vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$. If f is a function of three variables that is continuous on some region containing C, then we define the **line integral of f along C** (with respect to arc length) in a manner similar to that for plane curves:

$$\int_C f(x, y, z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta s_i$$

We evaluate it using a formula similar to Formula 3:

$$\boxed{9} \quad \int_C f(x, y, z)\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Observe that the integrals in both Formulas 3 and 9 can be written in the more compact vector notation

$$\int_a^b f(\mathbf{r}(t))\,|\,\mathbf{r}'(t)\,|\,dt$$

For the special case $f(x, y, z) = 1$, we get

$$\int_C ds = \int_a^b |\,\mathbf{r}'(t)\,|\,dt = L$$

where L is the length of the curve C (see Formula 13.3.3).

Line integrals along C with respect to x, y, and z can also be defined. For example,

$$\int_C f(x, y, z)\,dz = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta z_i$$

$$= \int_a^b f(x(t), y(t), z(t))\,z'(t)\,dt$$

Therefore, as with line integrals in the plane, we evaluate integrals of the form

$$\boxed{10} \quad \int_C P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz$$

by expressing everything (x, y, z, dx, dy, dz) in terms of the parameter t.

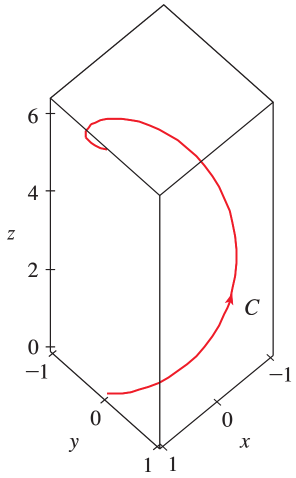


FIGURE 9

EXAMPLE 5 Evaluate $\int_C y \sin z\,ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leqslant t \leqslant 2\pi$. (See Figure 9.)

SOLUTION Formula 9 gives

$$\int_C y \sin z\,ds = \int_0^{2\pi} (\sin t) \sin t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \int_0^{2\pi} \sin^2 t\sqrt{\sin^2 t + \cos^2 t + 1}\,dt = \sqrt{2}\int_0^{2\pi} \tfrac{1}{2}(1 - \cos 2t)\,dt$$

$$= \frac{\sqrt{2}}{2}\Big[t - \tfrac{1}{2}\sin 2t\Big]_0^{2\pi} = \sqrt{2}\,\pi \qquad \blacksquare$$

EXAMPLE 6 Evaluate $\int_C y\,dx + z\,dy + x\,dz$, where C consists of the line segment C_1 from (2, 0, 0) to (3, 4, 5), followed by the vertical line segment C_2 from (3, 4, 5) to (3, 4, 0).

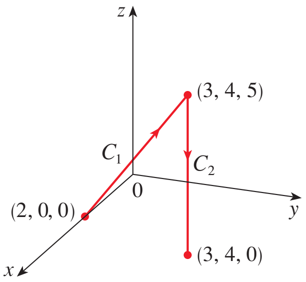


FIGURE 10

SOLUTION The curve C is shown in Figure 10. Using Equation 8, we write C_1 as

$$\mathbf{r}(t) = (1 - t)\langle 2, 0, 0\rangle + t\langle 3, 4, 5\rangle = \langle 2 + t, 4t, 5t\rangle$$

or, in parametric form, as

$$x = 2 + t \qquad y = 4t \qquad z = 5t \qquad 0 \leqslant t \leqslant 1$$

Thus

$$\int_{C_1} y\,dx + z\,dy + x\,dz = \int_0^1 (4t)\,dt + (5t)4\,dt + (2 + t)5\,dt$$

$$= \int_0^1 (10 + 29t)\,dt = 10t + 29\frac{t^2}{2}\Big]_0^1 = 24.5$$

Likewise, C_2 can be written in the form

$$\mathbf{r}(t) = (1 - t)\langle 3, 4, 5\rangle + t\langle 3, 4, 0\rangle = \langle 3, 4, 5 - 5t\rangle$$

or $$x = 3 \qquad y = 4 \qquad z = 5 - 5t \qquad 0 \leqslant t \leqslant 1$$

Then $dx = 0 = dy$, so

$$\int_{C_2} y\,dx + z\,dy + x\,dz = \int_0^1 3(-5)\,dt = -15$$

Adding the values of these integrals, we obtain

$$\int_C y\,dx + z\,dy + x\,dz = 24.5 - 15 = 9.5$$ ■

■ Line Integrals of Vector Fields; Work

Recall from Section 6.4 that the work done by a variable force $f(x)$ in moving a particle from a to b along the x-axis is $W = \int_a^b f(x)\,dx$. Then in Section 12.3 we found that the work done by a constant force $\mathbf{F}$ in moving an object from a point P to another point Q in space is $W = \mathbf{F} \cdot \mathbf{D}$, where $\mathbf{D} = \overrightarrow{PQ}$ is the displacement vector.

Now suppose that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a continuous force field on $\mathbb{R}^3$, such as the gravitational field of Example 16.1.4 or the electric force field of Example 16.1.5. (A force field on $\mathbb{R}^2$ could be regarded as a special case where $R = 0$ and P and Q depend only on x and y.) We wish to compute the work done by this force in moving a particle along a smooth curve C. See Figure 11.

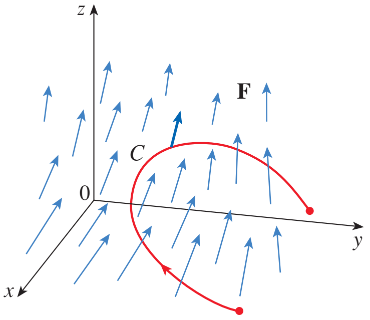


FIGURE 11

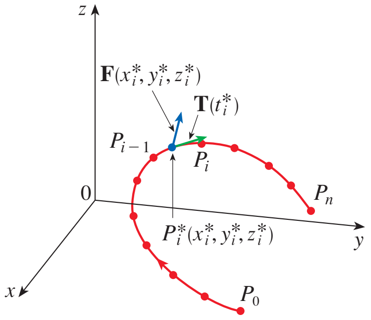

z
$\mathbf{F}(x_i^*, y_i^*, z_i^*)$
$\mathbf{T}(t_i^*)$
P_{i-1}
P_i
0
P_n
$P_i^*(x_i^*, y_i^*, z_i^*)$
y
x
P_0

FIGURE 12

To find the work done by **F** in moving a particle along C, we divide C into subarcs $P_{i-1}P_i$ with lengths Δs_i by dividing the parameter interval $[a, b]$ into subintervals of equal width. (See Figure 1 for the two-dimensional case or Figure 12 for the three-dimensional case.) Choose a point $P_i^*(x_i^*, y_i^*, z_i^*)$ on the ith subarc corresponding to the parameter value t_i^*. If Δs_i is small, then as the particle moves from P_{i-1} to P_i along the curve, it proceeds approximately in the direction of $\mathbf{T}(t_i^*)$, the unit tangent vector at P_i^*. Thus the work done by the force **F** in moving the particle from P_{i-1} to P_i is approximately

$$\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot [\Delta s_i \, \mathbf{T}(t_i^*)] = [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(t_i^*)] \, \Delta s_i$$

and the total work done in moving the particle along C is approximately

11
$$\sum_{i=1}^{n} [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(x_i^*, y_i^*, z_i^*)] \, \Delta s_i$$

where $\mathbf{T}(x, y, z)$ is the unit tangent vector at the point (x, y, z) on C. Intuitively, we see that these approximations ought to become better as n becomes larger. Therefore we define the **work** W done by the force field **F** as the limit of the Riemann sums in (11), namely,

12
$$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z) \, ds = \int_C \mathbf{F} \cdot \mathbf{T} \, ds$$

Equation 12 says that *work is the line integral with respect to arc length of the tangential component of the force.*

If the curve C is given by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, then $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$, so using Equation 9 we can rewrite Equation 12 in the form

$$W = \int_a^b \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)| \, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$

This integral is often abbreviated as $\int_C \mathbf{F} \cdot d\mathbf{r}$ and occurs in other areas of physics as well. Therefore we make the following definition for the line integral of *any* continuous vector field.

13 Definition Let **F** be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Then the **line integral of F along *C*** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt = \int_C \mathbf{F} \cdot \mathbf{T} \, ds$$

When using Definition 13, bear in mind that $\mathbf{F}(\mathbf{r}(t))$ is just an abbreviation for the vector field $\mathbf{F}(x(t), y(t), z(t))$, so we evaluate $\mathbf{F}(\mathbf{r}(t))$ simply by putting $x = x(t)$, $y = y(t)$, and $z = z(t)$ in the expression for $\mathbf{F}(x, y, z)$. Notice also that we can formally write $d\mathbf{r} = \mathbf{r}'(t) \, dt$.

Figure 13 shows the force field and the curve in Example 7. The work done is negative because the field impedes movement along the curve.

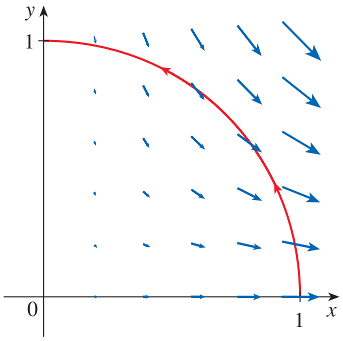


FIGURE 13

EXAMPLE 7 Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \leqslant t \leqslant \pi/2$.

SOLUTION Since $x = \cos t$ and $y = \sin t$, we have

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t\,\mathbf{i} - \cos t\,\sin t\,\mathbf{j}$$

and
$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

Therefore the work done is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^{\pi/2} (-\cos^2 t\,\sin t - \cos^2 t\,\sin t)\,dt$$

$$= \int_0^{\pi/2} (-2\cos^2 t\,\sin t)\,dt = 2\,\frac{\cos^3 t}{3}\Bigg]_0^{\pi/2} = -\frac{2}{3} \qquad \blacksquare$$

NOTE Even though $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\,ds$ and integrals with respect to arc length are unchanged when orientation is reversed, it is still true that

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

because the unit tangent vector $\mathbf{T}$ is replaced by its negative when C is replaced by $-C$.

Figure 14 shows the twisted cubic C in Example 8 and some typical vectors acting at three points on C.

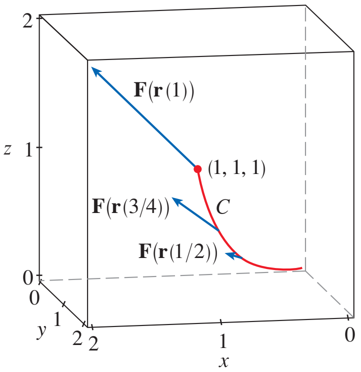


FIGURE 14

EXAMPLE 8 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \leqslant t \leqslant 1$$

SOLUTION We have

$$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(t)) = t^3\,\mathbf{i} + t^5\,\mathbf{j} + t^4\,\mathbf{k}$$

Thus
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

$$= \int_0^1 (t^3 + 5t^6)\,dt = \frac{t^4}{4} + \frac{5t^7}{7}\Bigg]_0^1 = \frac{27}{28} \qquad \blacksquare$$

Finally, we note the connection between line integrals of vector fields and line integrals of scalar fields. Suppose the vector field $\mathbf{F}$ on $\mathbb{R}^3$ is given in component form by the equation $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$. We use Definition 13 to compute its line integral along C:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

$$= \int_a^b (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot (x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k})\,dt$$

$$= \int_a^b \Big[P(x(t), y(t), z(t))\,x'(t) + Q(x(t), y(t), z(t))\,y'(t) + R(x(t), y(t), z(t))\,z'(t)\Big]\,dt$$

But this last integral is precisely the line integral in (10). Therefore we have

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz \qquad \text{where } \mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}}$$

For example, the integral $\int_C y\,dx + z\,dy + x\,dz$ in Example 6 could be expressed as $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$$

A similar result holds for vector fields $\mathbf{F}$ on $\mathbb{R}^2$:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy \tag{14}$$

where $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$.

16.2 Exercises

1–8 Evaluate the line integral, where C is the given plane curve.

1. $\int_C y\,ds$, C: $x = t^2$, $y = 2t$, $0 \leqslant t \leqslant 3$

2. $\int_C (x/y)\,ds$, C: $x = t^3$, $y = t^4$, $1 \leqslant t \leqslant 2$

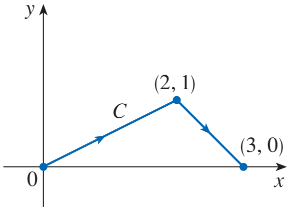
3. $\int_C xy^4\,ds$, C is the right half of the circle $x^2 + y^2 = 16$

4. $\int_C xe^y\,ds$, C is the line segment from $(2, 0)$ to $(5, 4)$

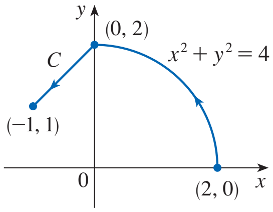
5. $\int_C (x^2y + \sin x)\,dy$,
C is the arc of the parabola $y = x^2$ from $(0, 0)$ to (π, π^2)

6. $\int_C e^x\,dx$,
C is the arc of the curve $x = y^3$ from $(-1, -1)$ to $(1, 1)$

7. $\int_C (x + 2y)\,dx + x^2\,dy$



8. $\int_C x^2\,dx + y^2\,dy$



9–18 Evaluate the line integral, where C is the given space curve.

9. $\int_C x^2y\,ds$,
C: $x = \cos t$, $y = \sin t$, $z = t$, $0 \leqslant t \leqslant \pi/2$

10. $\int_C y^2z\,ds$,
C is the line segment from $(3, 1, 2)$ to $(1, 2, 5)$

11. $\int_C xe^{yz}\,ds$,
C is the line segment from $(0, 0, 0)$ to $(1, 2, 3)$

12. $\int_C (x^2 + y^2 + z^2)\,ds$,
C: $x = t$, $y = \cos 2t$, $z = \sin 2t$, $0 \leqslant t \leqslant 2\pi$

13. $\int_C xye^{yz}\,dy$, C: $x = t$, $y = t^2$, $z = t^3$, $0 \leqslant t \leqslant 1$

14. $\int_C ye^z\,dz + x \ln x\,dy - y\,dx$,
C: $x = e^t$, $y = 2t$, $z = \ln t$, $1 \leqslant t \leqslant 2$

15. $\int_C z\,dx + xy\,dy + y^2\,dz$,
C: $x = \sin t$, $y = \cos t$, $z = \tan t$, $-\pi/4 \leqslant t \leqslant \pi/4$

16. $\int_C y\,dx + z\,dy + x\,dz$,
C: $x = \sqrt{t}$, $y = t$, $z = t^2$, $1 \leqslant t \leqslant 4$

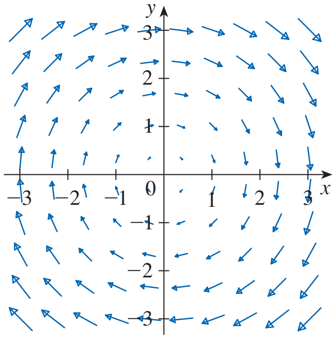
17. $\int_C z^2\,dx + x^2\,dy + y^2\,dz$,
C is the line segment from $(1, 0, 0)$ to $(4, 1, 2)$

18. $\int_C (y + z)\,dx + (x + z)\,dy + (x + y)\,dz$,
C consists of line segments from $(0, 0, 0)$ to $(1, 0, 1)$ and from $(1, 0, 1)$ to $(0, 1, 2)$

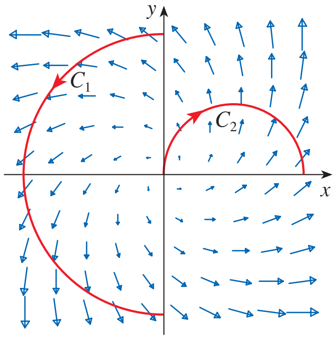
19. Let **F** be the vector field shown in the figure.

(a) If C_1 is the vertical line segment from $(-3, -3)$ to $(-3, 3)$, determine whether $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ is positive, negative, or zero.

(b) If C_2 is the counterclockwise-oriented circle with radius 3 and center the origin, determine whether $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ is positive, negative, or zero.



20. The figure shows a vector field **F** and two curves C_1 and C_2. Are the line integrals of **F** over C_1 and C_2 positive, negative, or zero? Explain.



21–24 Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the vector function $\mathbf{r}(t)$.

21. $\mathbf{F}(x, y) = xy^2\,\mathbf{i} - x^2\,\mathbf{j}$,
$\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j}, \quad 0 \leqslant t \leqslant 1$

22. $\mathbf{F}(x, y, z) = (x + y^2)\,\mathbf{i} + xz\,\mathbf{j} + (y + z)\,\mathbf{k}$,
$\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} - 2t\,\mathbf{k}, \quad 0 \leqslant t \leqslant 2$

23. $\mathbf{F}(x, y, z) = \sin x\,\mathbf{i} + \cos y\,\mathbf{j} + xz\,\mathbf{k}$,
$\mathbf{r}(t) = t^3\,\mathbf{i} - t^2\,\mathbf{j} + t\,\mathbf{k}, \quad 0 \leqslant t \leqslant 1$

24. $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + z^3\,\mathbf{j} + y\,\mathbf{k}$,
$\mathbf{r}(t) = e^t\,\mathbf{i} + e^{2t}\,\mathbf{j} + e^{-t}\,\mathbf{k}, \quad -1 \leqslant t \leqslant 1$

25–28 Use a calculator or computer to evaluate the line integral correct to four decimal places.

25. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \sqrt{x + y}\,\mathbf{i} + (y/x)\,\mathbf{j}$ and $\mathbf{r}(t) = \sin^2 t\,\mathbf{i} + \sin t \cos t\,\mathbf{j}, \;\; \pi/6 \leqslant t \leqslant \pi/3$

26. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = yze^x\,\mathbf{i} + zxe^y\,\mathbf{j} + xye^z\,\mathbf{k}$ and $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \tan t\,\mathbf{k}, \;\; 0 \leqslant t \leqslant \pi/4$

27. $\int_C xy \arctan z\, ds$, where C has parametric equations $x = t^2,\ y = t^3,\ z = \sqrt{t}, \;\; 1 \leqslant t \leqslant 2$

28. $\int_C z \ln(x + y)\, ds$, where C has parametric equations $x = 1 + 3t,\ y = 2 + t^2,\ z = t^4, \;\; -1 \leqslant t \leqslant 1$

29–30 Use a graph of the vector field **F** and the curve C to guess whether the line integral of **F** over C is positive, negative, or zero. Then evaluate the line integral.

29. $\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + xy\,\mathbf{j}$,
C is the arc of the circle $x^2 + y^2 = 4$ traversed counterclockwise from $(2, 0)$ to $(0, -2)$

30. $\mathbf{F}(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\,\mathbf{j}$,
C is the parabola $y = 1 + x^2$ from $(-1, 2)$ to $(1, 2)$

---

31. (a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = e^{x-1}\,\mathbf{i} + xy\,\mathbf{j}$ and C is given by $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j}, \;\; 0 \leqslant t \leqslant 1$.

(b) Illustrate part (a) by graphing C and the vectors from the vector field corresponding to $t = 0,\ 1/\sqrt{2}$, and 1 (as in Figure 14).

32. (a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x\,\mathbf{i} - z\,\mathbf{j} + y\,\mathbf{k}$ and C is given by $\mathbf{r}(t) = 2t\,\mathbf{i} + 3t\,\mathbf{j} - t^2\,\mathbf{k}, \;\; -1 \leqslant t \leqslant 1$.

(b) Illustrate part (a) by graphing C and the vectors from the vector field corresponding to $t = \pm 1$ and $\pm\frac{1}{2}$ (as in Figure 14).

33. Use a computer algebra system to find the exact value of $\int_C x^3 y^2 z\, ds$, where C is the curve with parametric equations $x = e^{-t} \cos 4t,\ y = e^{-t} \sin 4t,\ z = e^{-t}, \;\; 0 \leqslant t \leqslant 2\pi$.

34. (a) Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} + xy\,\mathbf{j}$ on a particle that moves once around the circle $x^2 + y^2 = 4$ oriented in the counterclockwise direction.

(b) Graph the force field and circle on the same screen. Use the graph to explain your answer to part (a).

35. A thin wire is bent into the shape of a semicircle $x^2 + y^2 = 4,\ x \geqslant 0$. If the linear density is a constant k, find the mass and center of mass of the wire.

36. A thin wire has the shape of the first-quadrant portion of the circle with center the origin and radius a. If the density function is $\rho(x, y) = kxy$, find the mass and center of mass of the wire.

37. (a) Write the formulas similar to Equations 4 for the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of a thin wire in the shape of a space curve C if the wire has density function $\rho(x, y, z)$.

(b) Find the center of mass of a wire in the shape of the helix $x = 2 \sin t$, $y = 2 \cos t$, $z = 3t$, $0 \leqslant t \leqslant 2\pi$, if the density is a constant k.

38. Find the mass and center of mass of a wire in the shape of the helix $x = t$, $y = \cos t$, $z = \sin t$, $0 \leqslant t \leqslant 2\pi$, if the density at any point is equal to the square of the distance from the origin.

39. If a wire with linear density $\rho(x, y)$ lies along a plane curve C, its **moments of inertia** about the x- and y-axes are defined as

$$I_x = \int_C y^2 \rho(x, y)\, ds \qquad I_y = \int_C x^2 \rho(x, y)\, ds$$

Find the moments of inertia for the wire in Example 3.

40. If a wire with linear density $\rho(x, y, z)$ lies along a space curve C, its **moments of inertia** about the x-, y-, and z-axes are defined as

$$I_x = \int_C (y^2 + z^2)\rho(x, y, z)\, ds$$

$$I_y = \int_C (x^2 + z^2)\rho(x, y, z)\, ds$$

$$I_z = \int_C (x^2 + y^2)\rho(x, y, z)\, ds$$

Find the moments of inertia for the wire in Exercise 37(b).

41. Find the work done by the force field

$$\mathbf{F}(x, y) = x\,\mathbf{i} + (y + 2)\,\mathbf{j}$$

in moving an object along an arch of the cycloid

$$\mathbf{r}(t) = (t - \sin t)\,\mathbf{i} + (1 - \cos t)\,\mathbf{j} \qquad 0 \leqslant t \leqslant 2\pi$$

42. Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} + ye^x\,\mathbf{j}$ on a particle that moves along the parabola $x = y^2 + 1$ from $(1, 0)$ to $(2, 1)$.

43. Find the work done by the force field

$$\mathbf{F}(x, y, z) = \langle x - y^2, y - z^2, z - x^2 \rangle$$

on a particle that moves along the line segment from $(0, 0, 1)$ to $(2, 1, 0)$.

44. The force exerted by an electric charge at the origin on a charged particle at a point (x, y, z) with position vector $\mathbf{r} = \langle x, y, z \rangle$ is $\mathbf{F}(\mathbf{r}) = K\mathbf{r}/|\mathbf{r}|^3$ where K is a constant. (See Example 16.1.5.) Find the work done as the particle moves along a straight line from $(2, 0, 0)$ to $(2, 1, 5)$.

45. The position of an object with mass m at time t is $\mathbf{r}(t) = at^2\,\mathbf{i} + bt^3\,\mathbf{j}$, $0 \leqslant t \leqslant 1$.
(a) What is the force acting on the object at time t?
(b) What is the work done by the force during the time interval $0 \leqslant t \leqslant 1$?

46. An object with mass m moves with position function $\mathbf{r}(t) = a \sin t\,\mathbf{i} + b \cos t\,\mathbf{j} + ct\,\mathbf{k}$, $0 \leqslant t \leqslant \pi/2$. Find the work done on the object during this time period.

47. A 72.5-kg man carries a 11-kg can of paint up a helical staircase that encircles a silo with a radius of 6 m. If the silo is 27 m high and the man makes exactly three complete revolutions climbing to the top, how much work is done by the man against gravity?

48. Suppose there is a hole in the can of paint in Exercise 47 and 4 kg of paint leaks steadily out of the can during the man's ascent. How much work is done?

49. (a) Show that a constant force field does zero work on a particle that moves once uniformly around the circle $x^2 + y^2 = 1$.
(b) Is this also true for a force field $\mathbf{F}(\mathbf{x}) = k\mathbf{x}$, where k is a constant and $\mathbf{x} = \langle x, y \rangle$?

50. The base of a circular fence with radius 10 m is given by $x = 10 \cos t$, $y = 10 \sin t$. The height of the fence at position (x, y) is given by the function $h(x, y) = 4 + 0.01(x^2 - y^2)$, so the height varies from 3 m to 5 m. Suppose that 1 L of paint covers 100 m^2. Sketch the fence and determine how much paint you will need if you paint both sides of the fence.

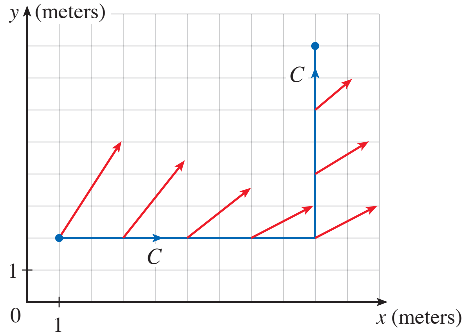
51. If C is a smooth curve given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, and $\mathbf{v}$ is a constant vector, show that

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \mathbf{v} \cdot [\mathbf{r}(b) - \mathbf{r}(a)]$$

52. If C is a smooth curve given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, show that

$$\int_C \mathbf{r} \cdot d\mathbf{r} = \tfrac{1}{2}\left[|\mathbf{r}(b)|^2 - |\mathbf{r}(a)|^2\right]$$

53. An object moves along the curve C shown in the figure from $(1, 2)$ to $(9, 8)$. The lengths of the vectors in the force field $\mathbf{F}$ are measured in newtons by the scales on the axes. Estimate the work done by $\mathbf{F}$ on the object.

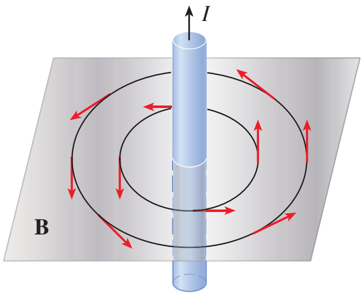


54. Experiments show that a steady current I in a long wire produces a magnetic field $\mathbf{B}$ that is tangent to any circle that lies in the plane perpendicular to the wire and whose center is the axis of the wire (as in the figure). *Ampère's Law* relates the electric current to its magnetic effects and states that

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$$

where I is the net current that passes through any surface bounded by a closed curve C, and μ_0 is a constant called the permeability of free space. By taking C to be a circle with radius r, show that the magnitude $B = |\mathbf{B}|$ of the magnetic field at a distance r from the center of the wire is

$$B = \frac{\mu_0 I}{2\pi r}$$



16.3 The Fundamental Theorem for Line Integrals

Recall from Section 5.3 that Part 2 of the Fundamental Theorem of Calculus can be written as

$$\boxed{1} \qquad \int_a^b F'(x)\, dx = F(b) - F(a)$$

where F' is continuous on $[a, b]$. Equation 1 says that to evaluate the definite integral of F' on $[a, b]$, we need only know the values of F at a and b, the endpoints of the interval. In this section we formulate a similar result for line integrals.

The Fundamental Theorem for Line Integrals

If we think of the gradient vector ∇f of a function f of two or three variables as a sort of derivative of f, then the following theorem can be regarded as a version of the Fundamental Theorem for line integrals.

$\boxed{2}$ **Theorem** Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

NOTE 1 Theorem 2 says that we can evaluate the line integral of a conservative vector field (the gradient vector field of the potential function f) simply by knowing the value of f at the endpoints of C. In fact, Theorem 2 says that the line integral of ∇f is the net change in f. If f is a function of two variables and C is a plane curve with initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$, as in Figure 1(a), then Theorem 2 becomes

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2) - f(x_1, y_1)$$

If f is a function of three variables and C is a space curve joining the point $A(x_1, y_1, z_1)$ to the point $B(x_2, y_2, z_2)$, as in Figure 1(b), then we have

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

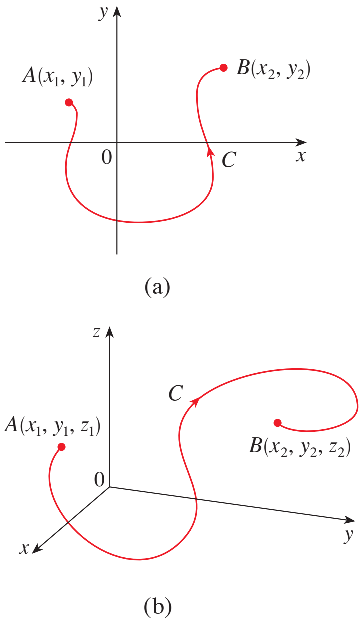


FIGURE 1

NOTE 2 Under the hypotheses of Theorem 2, if C_1 and C_2 are smooth curves with the same initial points and the same terminal points, then we can conclude that

$$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$$

We prove Theorem 2 for the case where f is a function of three variables.

PROOF OF THEOREM 2 Using Definition 16.2.13, we have

$$\begin{aligned}
\int_C \nabla f \cdot d\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt \\
&= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\, dt \qquad \text{(by the Chain Rule)} \\
&= f(\mathbf{r}(b)) - f(\mathbf{r}(a))
\end{aligned}$$

The last step follows from the Fundamental Theorem of Calculus (Equation 1). ■

NOTE 3 Although we have proved Theorem 2 for smooth curves, it is also true for piecewise-smooth curves. This can be seen by subdividing C into a finite number of smooth curves and adding the resulting integrals.

EXAMPLE 1 Find the work done by the gravitational field

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

in moving a particle with mass m from the point (3, 4, 12) to the point (2, 2, 0) along a piecewise-smooth curve C. (See Example 16.1.4.)

SOLUTION From Section 16.1 we know that $\mathbf{F}$ is a conservative vector field and, in fact, $\mathbf{F} = \nabla f$, where

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

Therefore, by Theorem 2, the work done is

$$\begin{aligned}
W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} \\
&= f(2, 2, 0) - f(3, 4, 12) \\
&= \frac{mMG}{\sqrt{2^2 + 2^2}} - \frac{mMG}{\sqrt{3^2 + 4^2 + 12^2}} = mMG\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right)
\end{aligned}$$

■

Independence of Path

Suppose C_1 and C_2 are two piecewise-smooth curves (which are called **paths**) that have the same initial point A and terminal point B. We know from Example 16.2.4 that, in general, $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. But in Note 2 we observed that

$$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$$

whenever ∇f is continuous (see Figure 2). In other words, the line integral of a *conservative* vector field depends only on the initial point and terminal point of a curve.

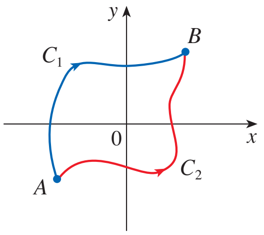


FIGURE 2

$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$

In general, if $\mathbf{F}$ is a continuous vector field with domain D, we say that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 in D that have the same initial points and the same terminal points. With this terminology we can say that *line integrals of conservative vector fields are independent of path.*

A curve is called **closed** if its terminal point coincides with its initial point, that is, $\mathbf{r}(b) = \mathbf{r}(a)$. (See Figure 3.) If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D and C is any closed path in D, we can choose any two points A and B on C and regard C as being composed of the path C_1 from A to B followed by the path C_2 from B to A. (See Figure 4.) Then

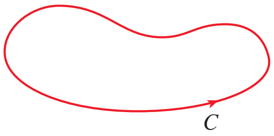


FIGURE 3

A closed curve

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

since C_1 and $-C_2$ have the same initial and terminal points.

Conversely, if it is true that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ whenever C is a closed path in D, then we demonstrate independence of path as follows. Take any two paths C_1 and C_2 from A to B in D and define C to be the curve consisting of C_1 followed by $-C_2$. Then

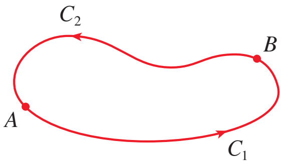


FIGURE 4

$$0 = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

and so $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. Thus we have proved the following theorem.

3 Theorem $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

Since we know that the line integral of any conservative vector field $\mathbf{F}$ is independent of path, it follows that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path. The physical interpretation is that the work done by a conservative force field (such as the gravitational or electric field in Section 16.1) as it moves an object around a closed path is 0.

The following theorem says that the *only* vector fields that are independent of path are conservative. It is stated and proved for plane curves, but there is a similar version for space curves. We assume that D is **open**, which means that for every point P in D there is a disk with center P that lies entirely in D. (So D doesn't contain any of its boundary points.) In addition, we assume that D is **connected**: this means that any two points in D can be joined by a path that lies in D.

4 Theorem Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

PROOF Let $A(a, b)$ be a fixed point in D. We construct the desired potential function f by defining

$$f(x, y) = \int_{(a, b)}^{(x, y)} \mathbf{F} \cdot d\mathbf{r}$$

for any point (x, y) in D. Since $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, it does not matter which path C from (a, b) to (x, y) is used to evaluate $f(x, y)$. Since D is open, there exists a disk contained in D with center (x, y). Choose any point (x_1, y) in the disk with $x_1 < x$ and let C consist of any path C_1 from (a, b) to (x_1, y) followed by the horizontal

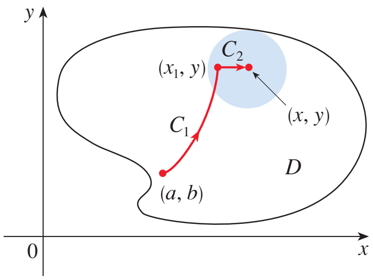


FIGURE 5

line segment C_2 from (x_1, y) to (x, y). (See Figure 5.) Then

$$f(x, y) = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{(a,b)}^{(x_1,y)} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

Notice that the first of these integrals does not depend on x, so

$$\frac{\partial}{\partial x} f(x, y) = 0 + \frac{\partial}{\partial x} \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

If we write $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, then

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} P\,dx + Q\,dy$$

On C_2, y is constant, so $dy = 0$. Using t as the parameter, where $x_1 \leqslant t \leqslant x$, we have

$$\frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x} \int_{C_2} P\,dx + Q\,dy = \frac{\partial}{\partial x} \int_{x_1}^{x} P(t, y)\,dt = P(x, y)$$

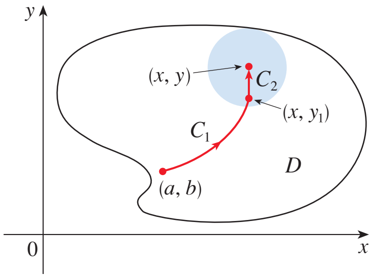


FIGURE 6

by Part 1 of the Fundamental Theorem of Calculus (see Section 5.3). A similar argument, using a vertical line segment (see Figure 6), shows that

$$\frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y} \int_{C_2} P\,dx + Q\,dy = \frac{\partial}{\partial y} \int_{y_1}^{y} Q(x, t)\,dt = Q(x, y)$$

Thus $$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} = \nabla f$$

which says that $\mathbf{F}$ is conservative. ■

Conservative Vector Fields and Potential Functions

The question remains: how can we determine whether or not a vector field $\mathbf{F}$ is conservative? And if we know that a field $\mathbf{F}$ is conservative, how can we find a potential function f?

Suppose it is known that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is conservative, where P and Q have continuous first-order partial derivatives. Then there is a function f such that $\mathbf{F} = \nabla f$, that is,

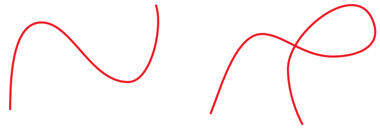
$$P = \frac{\partial f}{\partial x} \qquad \text{and} \qquad Q = \frac{\partial f}{\partial y}$$

Therefore, by Clairaut's Theorem,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial Q}{\partial x}$$

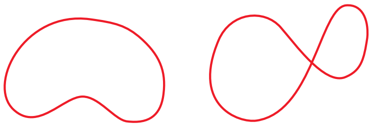
5 Theorem If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$



simple, not closed

not simple, not closed



simple, closed

not simple, closed

FIGURE 7
Types of curves

The converse of Theorem 5 is true only for a special type of region. To explain this, we first need the concept of a **simple curve**, which is a curve that doesn't intersect itself anywhere between its endpoints. [See Figure 7; $\mathbf{r}(a) = \mathbf{r}(b)$ for a simple closed curve, but $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ when $a < t_1 < t_2 < b$.]

In Theorem 4 we needed an open connected region. For the next theorem we need a stronger condition. A **simply-connected region** in the plane is a connected region D such that every simple closed curve in D encloses only points that are in D. Notice from Figure 8 that, intuitively speaking, a simply-connected region contains no hole and can't consist of two separate pieces.

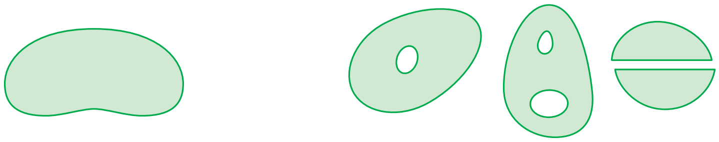


FIGURE 8 simply-connected region regions that are not simply-connected

In terms of simply-connected regions, we can now state a partial converse to Theorem 5 that gives a convenient method for verifying that a vector field on $\mathbb{R}^2$ is conservative. The proof will be sketched in Section 16.4 as a consequence of Green's Theorem.

6 Theorem Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

EXAMPLE 2 Determine whether or not the given vector field is conservative.

(a) $\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + (x - 2)\,\mathbf{j}$

(b) $\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$

SOLUTION

(a) Let $P(x, y) = x - y$ and $Q(x, y) = x - 2$. Then

$$\frac{\partial P}{\partial y} = -1 \qquad \frac{\partial Q}{\partial x} = 1$$

Since $\partial P/\partial y \neq \partial Q/\partial x$, $\mathbf{F}$ is not conservative by Theorem 5.

(b) Let $P(x, y) = 3 + 2xy$ and $Q(x, y) = x^2 - 3y^2$. Then

$$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$$

Also, the domain of $\mathbf{F}$ is the entire plane ($D = \mathbb{R}^2$), which is open and simply-connected. Therefore we can apply Theorem 6 and conclude that $\mathbf{F}$ is conservative. ■

In Example 2(b), Theorem 6 told us that $\mathbf{F}$ is conservative, but it did not tell us how to find the (potential) function f such that $\mathbf{F} = \nabla f$. The proof of Theorem 4 gives us a clue as to how to find f. We use "partial integration" as in the following example.

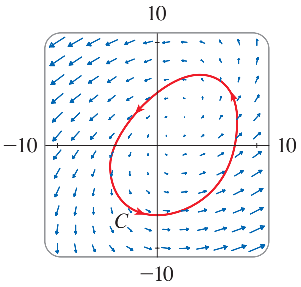


FIGURE 9

Figures 9 and 10 show the vector fields in Examples 2(a) and 2(b), respectively. The vectors in Figure 9 that start on the closed curve C all appear to point in roughly the same direction as C. So it looks as if $\int_C \mathbf{F} \cdot d\mathbf{r} > 0$ and therefore $\mathbf{F}$ is not conservative. The calculation in Example 2(a) confirms this impression. Some of the vectors near the curves C_1 and C_2 in Figure 10 point in approximately the same direction as the curves, whereas others point in the opposite direction. So it appears plausible that line integrals around all closed paths are 0. Example 2(b) shows that $\mathbf{F}$ is indeed conservative.

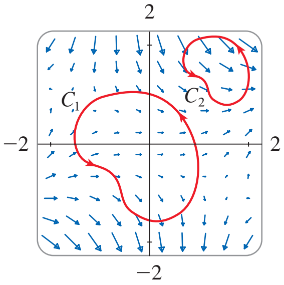


FIGURE 10

EXAMPLE 3 If $\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.

SOLUTION From Example 2(b) we know that $\mathbf{F}$ is conservative and so there exists a function f with $\nabla f = \mathbf{F}$, that is,

$$\boxed{7} \qquad f_x(x, y) = 3 + 2xy$$

$$\boxed{8} \qquad f_y(x, y) = x^2 - 3y^2$$

Integrating (7) with respect to x, we obtain

$$\boxed{9} \qquad f(x, y) = 3x + x^2y + g(y)$$

Notice that the constant of integration is a constant with respect to x, that is, a function of y, which we have called $g(y)$. Next we differentiate both sides of (9) with respect to y:

$$\boxed{10} \qquad f_y(x, y) = x^2 + g'(y)$$

Comparing (8) and (10), we see that

$$g'(y) = -3y^2$$

Integrating with respect to y, we have

$$g(y) = -y^3 + K$$

where K is a constant. Putting this in (9), we have

$$f(x, y) = 3x + x^2y - y^3 + K$$

as the desired potential function. ■

EXAMPLE 4 Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$$

and C is the curve given by

$$\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j} \qquad 0 \leqslant t \leqslant \pi$$

SOLUTION 1 From Example 2(b) we know that $\mathbf{F}$ is conservative, so we can use Theorem 2. In Example 3 we found that a potential function for $\mathbf{F}$ is $f(x, y) = 3x + x^2y - y^3$ (choosing $K = 0$). According to Theorem 2, we need to know only the initial and terminal points of C, namely, $\mathbf{r}(0) = (0, 1)$ and $\mathbf{r}(\pi) = (0, -e^\pi)$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(0, -e^\pi) - f(0, 1) = e^{3\pi} - (-1) = e^{3\pi} + 1$$

This method is much shorter than the straightforward method for evaluating line integrals that we learned in Section 16.2.

SOLUTION 2 Because $\mathbf{F}$ is conservative, we know that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path. Let's replace the curve C with another (simpler) curve C_1 that has the same initial point

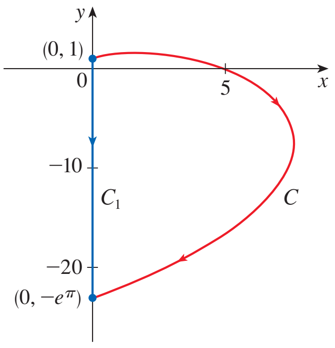


FIGURE 11

and the same terminal point as C. Let C_1 be the straight line segment from (0, 1) to (0, $-e^\pi$) as shown in Figure 11. Then C_1 is represented by

$$\mathbf{r}(t) = -t\,\mathbf{j} \qquad -1 \leqslant t \leqslant e^\pi$$

and

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{e^\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

$$= \int_{-1}^{e^\pi} (3\,\mathbf{i} - 3t^2\mathbf{j}) \cdot (-\mathbf{j})\,dt$$

$$= \int_{-1}^{e^\pi} 3t^2\,dt = t^3\Big]_{-1}^{e^\pi} = e^{3\pi} + 1$$ ■

A criterion for determining whether or not a vector field **F** on $\mathbb{R}^3$ is conservative is given in Section 16.5. Meanwhile, the next example shows that the technique for finding the potential function is much the same as for vector fields on $\mathbb{R}^2$.

EXAMPLE 5 If $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + (2xy + e^{3z})\,\mathbf{j} + 3ye^{3z}\,\mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

SOLUTION If there is such a function f, then

$$\boxed{11} \qquad f_x(x, y, z) = y^2$$

$$\boxed{12} \qquad f_y(x, y, z) = 2xy + e^{3z}$$

$$\boxed{13} \qquad f_z(x, y, z) = 3ye^{3z}$$

Integrating (11) with respect to x, we get

$$\boxed{14} \qquad f(x, y, z) = xy^2 + g(y, z)$$

where $g(y, z)$ is a constant with respect to x. Then differentiating (14) with respect to y, we have

$$f_y(x, y, z) = 2xy + g_y(y, z)$$

and comparison with (12) gives

$$g_y(y, z) = e^{3z}$$

Thus $g(y, z) = ye^{3z} + h(z)$ and we rewrite (14) as

$$f(x, y, z) = xy^2 + ye^{3z} + h(z)$$

Finally, differentiating with respect to z and comparing with (13), we obtain $h'(z) = 0$ and therefore $h(z) = K$, a constant. The desired function is

$$f(x, y, z) = xy^2 + ye^{3z} + K$$

It is easily verified that $\nabla f = \mathbf{F}$. ■

Conservation of Energy

Let's apply the ideas of this chapter to a continuous force field **F** that moves an object along a path C given by $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, where $\mathbf{r}(a) = A$ is the initial point and $\mathbf{r}(b) = B$ is the terminal point of C. According to Newton's Second Law of Motion (see Section 13.4), the force $\mathbf{F}(\mathbf{r}(t))$ at a point on C is related to the acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$ by the equation

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$$

So the work done by the force on the object is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_a^b m\mathbf{r}''(t) \cdot \mathbf{r}'(t)\, dt$$

$$= \frac{m}{2} \int_a^b \frac{d}{dt} [\mathbf{r}'(t) \cdot \mathbf{r}'(t)]\, dt \qquad \text{(Theorem 13.2.3, Formula 4)}$$

$$= \frac{m}{2} \int_a^b \frac{d}{dt} |\mathbf{r}'(t)|^2\, dt = \frac{m}{2} \Big[|\mathbf{r}'(t)|^2 \Big]_a^b \qquad \text{(Fundamental Theorem of Calculus)}$$

$$= \frac{m}{2} \left(|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2 \right)$$

Therefore

$$\boxed{15} \qquad W = \tfrac{1}{2} m |\mathbf{v}(b)|^2 - \tfrac{1}{2} m |\mathbf{v}(a)|^2$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity.

The quantity $\frac{1}{2} m |\mathbf{v}(t)|^2$, that is, half the mass times the square of the speed, is called the **kinetic energy** of the object. Therefore we can rewrite Equation 15 as

$$\boxed{16} \qquad W = K(B) - K(A)$$

which says that the work done by the force field along C is equal to the change in kinetic energy at the endpoints of C.

Now let's further assume that $\mathbf{F}$ is a conservative force field; that is, we can write $\mathbf{F} = \nabla f$. In physics, the **potential energy** of an object at the point (x, y, z) is defined as $P(x, y, z) = -f(x, y, z)$, so we have $\mathbf{F} = -\nabla P$. Then by Theorem 2 we have

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C \nabla P \cdot d\mathbf{r} = -[P(\mathbf{r}(b)) - P(\mathbf{r}(a))] = P(A) - P(B)$$

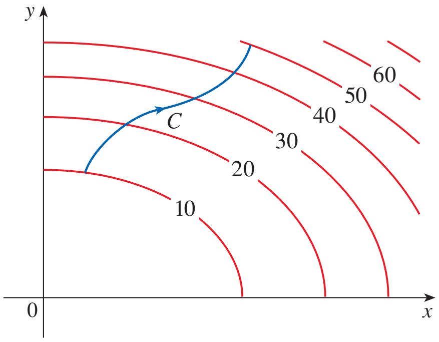
Comparing this equation with Equation 16, we see that

$$P(A) + K(A) = P(B) + K(B)$$

which says that if an object moves from one point A to another point B under the influence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant. This is called the **Law of Conservation of Energy** and it is the reason the vector field is called *conservative*.

16.3 Exercises

1. The figure shows a curve C and a contour map of a function f whose gradient is continuous. Find $\int_C \nabla f \cdot d\mathbf{r}$.



2. A table of values of a function f with continuous gradient is given. Find $\int_C \nabla f \cdot d\mathbf{r}$, where C has parametric equations

$$x = t^2 + 1 \qquad y = t^3 + t \qquad 0 \leqslant t \leqslant 1$$

x \ y	0	1	2
0	1	6	4
1	3	5	7
2	8	2	9

3–10 Determine whether or not **F** is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

3. $\mathbf{F}(x, y) = (xy + y^2)\,\mathbf{i} + (x^2 + 2xy)\,\mathbf{j}$

4. $\mathbf{F}(x, y) = (y^2 - 2x)\,\mathbf{i} + 2xy\,\mathbf{j}$

5. $\mathbf{F}(x, y) = y^2 e^{xy}\,\mathbf{i} + (1 + xy)e^{xy}\,\mathbf{j}$

6. $\mathbf{F}(x, y) = ye^x\,\mathbf{i} + (e^x + e^y)\,\mathbf{j}$

7. $\mathbf{F}(x, y) = (ye^x + \sin y)\,\mathbf{i} + (e^x + x\cos y)\,\mathbf{j}$

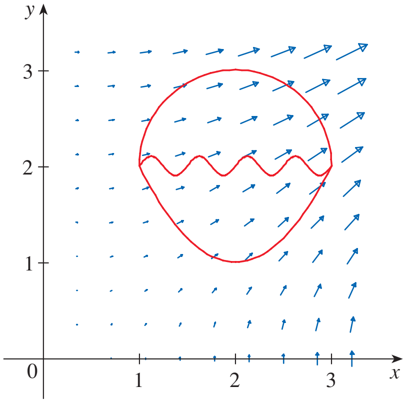
8. $\mathbf{F}(x, y) = (2xy + y^{-2})\,\mathbf{i} + (x^2 - 2xy^{-3})\,\mathbf{j}, \quad y > 0$

9. $\mathbf{F}(x, y) = (y^2\cos x + \cos y)\,\mathbf{i} + (2y\sin x - x\sin y)\,\mathbf{j}$

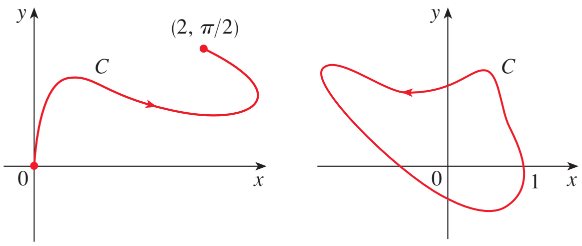
10. $\mathbf{F}(x, y) = (\ln y + y/x)\,\mathbf{i} + (\ln x + x/y)\,\mathbf{j}$

---

11. The figure shows the vector field $\mathbf{F}(x, y) = \langle 2xy, x^2\rangle$ and three curves that start at (1, 2) and end at (3, 2).
(a) Explain why $\int_C \mathbf{F}\cdot d\mathbf{r}$ has the same value for all three curves.
(b) What is this common value?

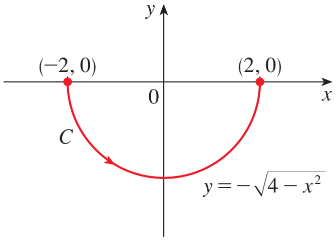


12. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ for the vector field $\mathbf{F}(x, y) = 2xy\,\mathbf{i} + (x^2 + \sin y)\,\mathbf{j}$ and the curve C shown.



(a) (b)

13. Let $\mathbf{F}(x, y) = (3x^2 + y^2)\,\mathbf{i} + 2xy\,\mathbf{j}$ and let C be the curve shown.



(a) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ directly.
(b) Show that **F** is conservative and find a function f such that $\mathbf{F} = \nabla f$.
(c) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ using Theorem 2.
(d) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ by first replacing C by a simpler curve that has the same initial and terminal points.

14–15 A vector field **F** and a curve C are given.
(a) Show that **F** is conservative and find a potential function f.
(b) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ using Theorem 2.
(c) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ by first replacing C with a line segment that has the same initial and terminal points.

14. $\mathbf{F}(x, y) = \langle \sin y + e^x, x\cos y\rangle$,
C: $x = t,\ y = t(3 - t),\ 0 \leqslant t \leqslant 3$

15. $\mathbf{F}(x, y) = \langle ye^{xy}, xe^{xy}\rangle$,
C: $x = \sin\dfrac{\pi}{2}t,\ y = e^{t-1}(1 - \cos \pi t),\ 0 \leqslant t \leqslant 1$

---

16. Evaluate $\int_C \nabla f\cdot d\mathbf{r}$, where $f(x, y, z) = xy^2z + x^2$ and C is the curve $x = t^2,\ y = e^{t^2-1},\ z = t^2 + t,\ -1 \leqslant t \leqslant 1$.

17–24 (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ along the given curve C.

17. $\mathbf{F}(x, y) = \langle 2x, 4y\rangle$,
C is the arc of the parabola $x = y^2$ from $(4, -2)$ to $(1, 1)$

18. $\mathbf{F}(x, y) = (3 + 2xy^2)\,\mathbf{i} + 2x^2y\,\mathbf{j}$,
C is the arc of the hyperbola $y = 1/x$ from $(1, 1)$ to $\left(4, \frac{1}{4}\right)$

19. $\mathbf{F}(x, y) = x^2y^3\,\mathbf{i} + x^3y^2\,\mathbf{j}$,
C: $\mathbf{r}(t) = \langle t^3 - 2t, t^3 + 2t\rangle, \quad 0 \leqslant t \leqslant 1$

20. $\mathbf{F}(x, y) = (1 + xy)e^{xy}\,\mathbf{i} + x^2e^{xy}\,\mathbf{j}$,
C: $\mathbf{r}(t) = \cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}, \quad 0 \leqslant t \leqslant \pi/2$

21. $\mathbf{F}(x, y, z) = 2xy\,\mathbf{i} + (x^2 + 2yz)\,\mathbf{j} + y^2\,\mathbf{k}$,
C is the line segment from $(2, -3, 1)$ to $(-5, 1, 2)$

22. $\mathbf{F}(x, y, z) = (y^2z + 2xz^2)\,\mathbf{i} + 2xyz\,\mathbf{j} + (xy^2 + 2x^2z)\,\mathbf{k}$,
C: $x = \sqrt{t},\ y = t + 1,\ z = t^2, \quad 0 \leqslant t \leqslant 1$

23. $\mathbf{F}(x, y, z) = yze^{xz}\,\mathbf{i} + e^{xz}\,\mathbf{j} + xye^{xz}\,\mathbf{k}$,
C: $\mathbf{r}(t) = (t^2 + 1)\,\mathbf{i} + (t^2 - 1)\,\mathbf{j} + (t^2 - 2t)\,\mathbf{k}$,
$0 \leqslant t \leqslant 2$

24. $\mathbf{F}(x, y, z) = \sin y\,\mathbf{i} + (x\cos y + \cos z)\,\mathbf{j} - y\sin z\,\mathbf{k}$,
C: $\mathbf{r}(t) = \sin t\,\mathbf{i} + t\,\mathbf{j} + 2t\,\mathbf{k}$, $0 \leqslant t \leqslant \pi/2$

25–26 Show that the line integral is independent of path and evaluate the integral.

25. $\int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})\,dy$,
C is any path from $(1, 0)$ to $(2, 1)$

26. $\int_C \sin y\,dx + (x\cos y - \sin y)\,dy$,
C is any path from $(2, 0)$ to $(1, \pi)$

27. Suppose you're asked to determine the curve that requires the least work for a force field **F** to move a particle from one point to another point. You decide to check first whether **F** is conservative, and indeed it turns out that it is. How would you reply to the request?

28. Suppose an experiment determines that the amount of work required for a force field **F** to move a particle from the point $(1, 2)$ to the point $(5, -3)$ along a curve C_1 is 1.2 J and the work done by **F** in moving the particle along another curve C_2 between the same two points is 1.4 J. What can you say about **F**? Why?

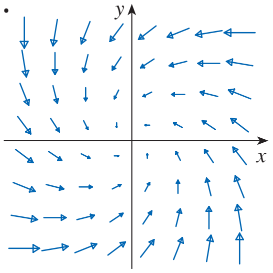
29–30 Find the work done by the force field **F** in moving an object from P to Q.

29. $\mathbf{F}(x, y) = x^3\,\mathbf{i} + y^3\,\mathbf{j}$; $P(1, 0)$, $Q(2, 2)$

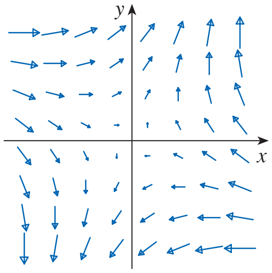
30. $\mathbf{F}(x, y) = (2x + y)\,\mathbf{i} + x\,\mathbf{j}$; $P(1, 1)$, $Q(4, 3)$

31–32 Is the vector field shown in the figure conservative? Explain.

31.



32.



33. If $\mathbf{F}(x, y) = \sin y\,\mathbf{i} + (1 + x\cos y)\,\mathbf{j}$, use a plot to guess whether **F** is conservative. Then determine whether your guess is correct.

34. Let $\mathbf{F} = \nabla f$, where $f(x, y) = \sin(x - 2y)$. Find curves C_1 and C_2 that are not closed and satisfy the equation.

(a) $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$ (b) $\displaystyle\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1$

35. Show that if the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is conservative and P, Q, R have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

36. Use Exercise 35 to show that the line integral $\int_C y\,dx + x\,dy + xyz\,dz$ is not independent of path.

37–40 Determine whether or not the given set is (a) open, (b) connected, and (c) simply-connected.

37. $\{(x, y) \mid 0 < y < 3\}$

38. $\{(x, y) \mid 1 < |x| < 2\}$

39. $\{(x, y) \mid 1 \leqslant x^2 + y^2 \leqslant 4,\ y \geqslant 0\}$

40. $\{(x, y) \mid (x, y) \neq (2, 3)\}$

41. Let $\mathbf{F}(x, y) = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2 + y^2}$.

(a) Show that $\partial P/\partial y = \partial Q/\partial x$.

(b) Show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path. [*Hint:* Compute $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where C_1 and C_2 are the upper and lower halves of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$.] Does this contradict Theorem 6?

42. Inverse Square Fields Suppose that **F** is an *inverse square force field*, that is,

$$\mathbf{F}(\mathbf{r}) = \frac{c\mathbf{r}}{|\mathbf{r}|^3}$$

for some constant c, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

(a) Find the work done by **F** in moving an object from a point P_1 along a path to a point P_2 in terms of the distances d_1 and d_2 from these points to the origin.

(b) An example of an inverse square field is the gravitational field $\mathbf{F} = -(mMG)\mathbf{r}/|\mathbf{r}|^3$ discussed in Example 16.1.4. Use part (a) to find the work done by the gravitational field when the earth moves from aphelion (at a maximum distance of 1.52×10^8 km from the sun) to perihelion (at a minimum distance of 1.47×10^8 km). (Use the values $m = 5.97 \times 10^{24}$ kg, $M = 1.99 \times 10^{30}$ kg, and $G = 6.67 \times 10^{-11}\ \mathrm{N{\cdot}m^2/kg^2}$.)

(c) Another example of an inverse square field is the electric force field $\mathbf{F} = \varepsilon qQ\mathbf{r}/|\mathbf{r}|^3$ discussed in Example 16.1.5. Suppose that an electron with a charge of -1.6×10^{-19} C is located at the origin. A positive unit charge is positioned a distance 10^{-12} m from the electron and moves to a position half that distance from the electron. Use part (a) to find the work done by the electric force field. (Use the value $\varepsilon = 8.985 \times 10^9$.)

16.4 Green's Theorem

Green's Theorem gives the relationship between a line integral around a simple closed curve and a double integral over the plane region bounded by the curve.

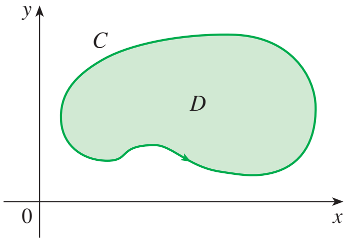


FIGURE 1

Green's Theorem

Let C be a simple closed curve and let D be the region bounded by C, as in Figure 1. (We assume that D consists of all points inside C as well as all points on C.) In stating Green's Theorem we use the convention that the **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C. Thus if C is given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, then the region D is always on the left as the point $\mathbf{r}(t)$ traverses C. (See Figure 2.)

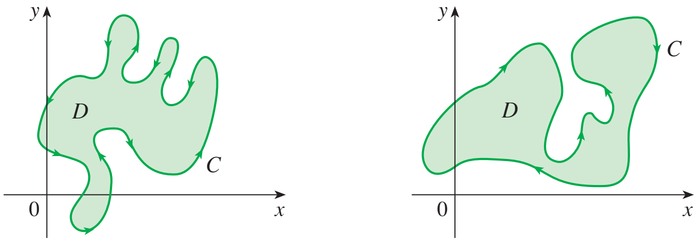


FIGURE 2

(a) Positive orientation (b) Negative orientation

Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Recall that the left side of this equation is another way of writing $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$.

NOTE The notation

$$\oint_C P\,dx + Q\,dy \qquad \text{or} \qquad \circlearrowleft\!\!\!\!\int_C P\,dx + Q\,dy$$

is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve C. Another notation for the positively oriented boundary curve of D is ∂D, so the equation in Green's Theorem can be written as

$$\boxed{1} \qquad \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial D} P\,dx + Q\,dy$$

Green's Theorem should be regarded as the counterpart of the Fundamental Theorem of Calculus for double integrals. Compare Equation 1 with the statement of the Fundamental Theorem of Calculus, Part 2, in the following equation:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

In both cases there is an integral involving derivatives (F', $\partial Q/\partial x$, and $\partial P/\partial y$) on the left side of the equation. And in both cases the right side involves the values of the original functions (F, Q, and P) only on the *boundary* of the domain. (In the one-dimensional case, the domain is an interval $[a, b]$ whose boundary consists of just two points, a and b.)

Green's Theorem is not easy to prove in general, but we can give a proof for the special case where the region is both type I and type II (see Section 15.2). Let's call such regions **simple regions**.

George Green

Green's Theorem is named after the self-taught English scientist George Green (1793–1841). He worked full-time in his father's bakery from the age of nine and taught himself mathematics from library books. In 1828 he published privately *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*, but only 100 copies were printed and most of those went to his friends. This pamphlet contained a theorem that is equivalent to what we know as Green's Theorem, but it didn't become widely known at that time. Finally, at age 40, Green entered Cambridge University as an undergraduate but died four years after graduation. In 1846 William Thomson (Lord Kelvin) located a copy of Green's essay, realized its significance, and had it reprinted. Green was the first person to try to formulate a mathematical theory of electricity and magnetism. His work was the basis for the subsequent electromagnetic theories of Thomson, Stokes, Rayleigh, and Maxwell.

PROOF OF GREEN'S THEOREM FOR THE CASE IN WHICH D IS A SIMPLE REGION

Notice that Green's Theorem will be proved if we can show that

$$\boxed{2} \qquad \int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA$$

and

$$\boxed{3} \qquad \int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA$$

We prove Equation 2 by expressing D as a type I region:

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous functions. This enables us to compute the double integral on the right side of Equation 2 as follows:

$$\boxed{4} \qquad \iint_D \frac{\partial P}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x, y)\,dy\,dx = \int_a^b [P(x, g_2(x)) - P(x, g_1(x))]\,dx$$

where the last step follows from the Fundamental Theorem of Calculus.

Now we compute the left side of Equation 2 by breaking up C as the union of the four curves C_1, C_2, C_3, and C_4 shown in Figure 3. On C_1 we take x as the parameter and write the parametric equations as $x = x$, $y = g_1(x)$, $a \leqslant x \leqslant b$. Thus

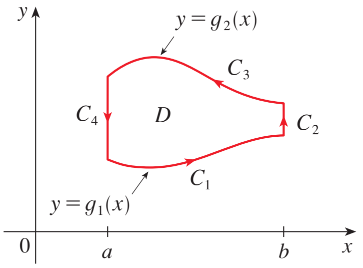


FIGURE 3

$$\int_{C_1} P(x, y)\,dx = \int_a^b P(x, g_1(x))\,dx$$

Observe that C_3 goes from right to left but $-C_3$ goes from left to right, so we can write the parametric equations of $-C_3$ as $x = x$, $y = g_2(x)$, $a \leqslant x \leqslant b$. Therefore

$$\int_{C_3} P(x, y)\,dx = -\int_{-C_3} P(x, y)\,dx = -\int_a^b P(x, g_2(x))\,dx$$

On C_2 or C_4 (either of which might reduce to just a single point), x is constant, so $dx = 0$ and

$$\int_{C_2} P(x, y)\,dx = 0 = \int_{C_4} P(x, y)\,dx$$

Hence

$$\begin{aligned}\int_C P(x, y)\,dx &= \int_{C_1} P(x, y)\,dx + \int_{C_2} P(x, y)\,dx + \int_{C_3} P(x, y)\,dx + \int_{C_4} P(x, y)\,dx \\ &= \int_a^b P(x, g_1(x))\,dx - \int_a^b P(x, g_2(x))\,dx\end{aligned}$$

Comparing this expression with the one in Equation 4, we see that

$$\int_C P(x, y)\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA$$

Equation 3 can be proved in much the same way by expressing D as a type II region (see Exercise 34). Then, by adding Equations 2 and 3, we obtain Green's Theorem. ■

EXAMPLE 1 Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from $(0, 0)$ to $(1, 0)$, from $(1, 0)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$.

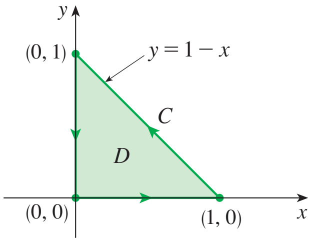


FIGURE 4

SOLUTION Although the given line integral could be evaluated as usual by the methods of Section 16.2, that would involve setting up three separate integrals along the three sides of the triangle, so let's use Green's Theorem instead. Notice that the region D enclosed by C is simple and C has positive orientation (see Figure 4). If we let $P(x, y) = x^4$ and $Q(x, y) = xy$, then we have

$$\int_C x^4\,dx + xy\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_0^{1-x} (y - 0)\,dy\,dx$$

$$= \int_0^1 \left[\tfrac{1}{2}y^2\right]_{y=0}^{y=1-x} dx = \tfrac{1}{2}\int_0^1 (1 - x)^2\,dx$$

$$= -\tfrac{1}{6}(1 - x)^3\Big]_0^1 = \tfrac{1}{6}$$ ■

EXAMPLE 2 Evaluate $\oint_C (3y - e^{\sin x})\,dx + \left(7x + \sqrt{y^4 + 1}\right) dy$, where C is the circle $x^2 + y^2 = 9$.

SOLUTION The region D bounded by C is the disk $x^2 + y^2 \leqslant 9$, so let's change to polar coordinates after applying Green's Theorem:

$$\oint_C (3y - e^{\sin x})\,dx + \left(7x + \sqrt{y^4 + 1}\right) dy$$

$$= \iint_D \left[\frac{\partial}{\partial x}\left(7x + \sqrt{y^4 + 1}\right) - \frac{\partial}{\partial y}(3y - e^{\sin x})\right] dA$$

$$= \int_0^{2\pi} \int_0^3 (7 - 3)\,r\,dr\,d\theta = 4\int_0^{2\pi} d\theta \int_0^3 r\,dr = 36\pi$$ ■

Instead of using polar coordinates, we could simply use the fact that D is a disk of radius 3 and write

$$\iint_D 4\,dA = 4 \cdot \pi(3)^2 = 36\pi$$

In Examples 1 and 2 we found that the double integral was easier to evaluate than the line integral. (Try setting up the line integral in Example 2 and you'll soon be convinced!) But sometimes it's easier to evaluate the line integral, and Green's Theorem is used in the reverse direction. For instance, if it is known that $P(x, y) = Q(x, y) = 0$ on the curve C, then Green's Theorem gives

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy = 0$$

no matter what values P and Q assume in the region D.

Finding Areas with Green's Theorem

Another application of the reverse direction of Green's Theorem is in computing areas. Since the area of D is $\iint_D 1\,dA$, we wish to choose P and Q so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

There are several possibilities:

$$P(x, y) = 0 \qquad P(x, y) = -y \qquad P(x, y) = -\tfrac{1}{2}y$$
$$Q(x, y) = x \qquad Q(x, y) = 0 \qquad Q(x, y) = \tfrac{1}{2}x$$

Then Green's Theorem gives the following formulas for the area of D:

$$\boxed{5} \qquad \boxed{A = \oint_C x\,dy = -\oint_C y\,dx = \tfrac{1}{2}\oint_C x\,dy - y\,dx}$$

EXAMPLE 3 Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

SOLUTION The ellipse has parametric equations $x = a\cos t$ and $y = b\sin t$, where $0 \leqslant t \leqslant 2\pi$. Using the third formula in Equation 5, we have

$$\begin{aligned} A &= \tfrac{1}{2}\int_C x\,dy - y\,dx \\ &= \tfrac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt \\ &= \frac{ab}{2}\int_0^{2\pi} dt = \pi ab \end{aligned}$$

■

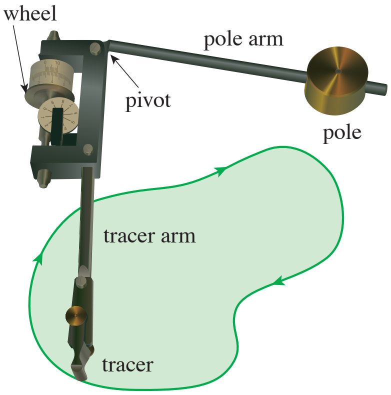


FIGURE 5

A Keuffel and Esser polar planimeter

Formula 5 can be used to explain how planimeters work. A **planimeter** is an ingenious mechanical instrument invented in the 19th century for measuring the area of a region by tracing its boundary curve. For instance, a biologist could use one of these devices to measure the surface area of a leaf or bird wing.

Figure 5 shows the operation of a polar planimeter: the pole is fixed and, as the tracer is moved along the boundary curve of the region, the wheel partly slides and partly rolls perpendicular to the tracer arm. The planimeter measures the distance that the wheel rolls and this is proportional to the area of the enclosed region. The explanation as a consequence of Formula 5 can be found in the following articles:

- R. W. Gatterman, "The planimeter as an example of Green's Theorem" *Amer. Math. Monthly,* Vol. 88 (1981), pp. 701–4.
- Tanya Leise, "As the planimeter wheel turns" *College Math. Journal,* Vol. 38 (2007), pp. 24–31.

Extended Versions of Green's Theorem

Although we have proved Green's Theorem only for the case where D is simple, we can now extend it to the case where D is a finite union of simple regions. For example, if D is the region shown in Figure 6, then we can write $D = D_1 \cup D_2$, where D_1 and D_2 are both simple. The boundary of D_1 is $C_1 \cup C_3$ and the boundary of D_2 is $C_2 \cup (-C_3)$ so, applying Green's Theorem to D_1 and D_2 separately, we get

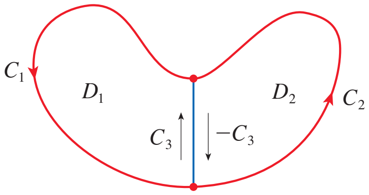


FIGURE 6

$$\int_{C_1 \cup C_3} P\,dx + Q\,dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\int_{C_2 \cup (-C_3)} P\,dx + Q\,dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

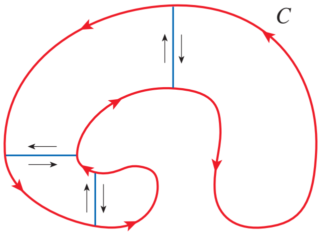


FIGURE 7

If we add these two equations, the line integrals along C_3 and $-C_3$ cancel, so we get

$$\int_{C_1 \cup C_2} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

which is Green's Theorem for $D = D_1 \cup D_2$, since its boundary is $C = C_1 \cup C_2$.

The same sort of argument allows us to establish Green's Theorem for any finite union of nonoverlapping simple regions (see Figure 7).

EXAMPLE 4 Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

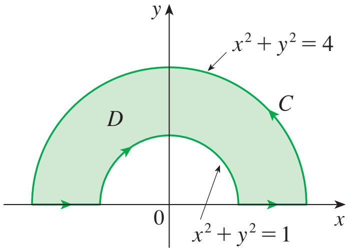


FIGURE 8

SOLUTION Notice that although D is not simple, the y-axis divides it into two simple regions (see Figure 8). In polar coordinates we can write

$$D = \{(r, \theta) \mid 1 \leqslant r \leqslant 2,\ 0 \leqslant \theta \leqslant \pi\}$$

Therefore Green's Theorem gives

$$\begin{aligned}
\oint_C y^2\,dx + 3xy\,dy &= \iint_D \left[\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2) \right] dA \\
&= \iint_D y\,dA = \int_0^\pi \int_1^2 (r \sin\theta)\, r\,dr\,d\theta \\
&= \int_0^\pi \sin\theta\,d\theta \int_1^2 r^2\,dr = \Big[-\cos\theta\Big]_0^\pi \Big[\tfrac{1}{3}r^3\Big]_1^2 = \tfrac{14}{3}
\end{aligned}$$

■

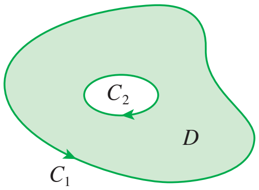


FIGURE 9

Green's Theorem can be extended to apply to regions with holes, that is, regions that are not simply-connected. Observe that the boundary C of the region D in Figure 9 consists of two simple closed curves C_1 and C_2. We assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed. Thus the positive direction is counterclockwise for the outer curve C_1 but clockwise for the inner curve C_2. If we divide D into two regions D' and D'' by means of the lines shown in Figure 10 and then apply Green's Theorem to each of D' and D'', we get

$$\begin{aligned}
\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA &= \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\
&= \int_{\partial D'} P\,dx + Q\,dy + \int_{\partial D''} P\,dx + Q\,dy
\end{aligned}$$

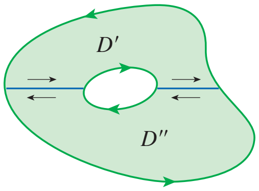


FIGURE 10

Since the line integrals along the common boundary lines are in opposite directions, they cancel and we get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy = \int_C P\,dx + Q\,dy$$

which is Green's Theorem for the region D.

EXAMPLE 5 If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

SOLUTION Since C is an *arbitrary* closed path that encloses the origin, it's difficult to compute the given integral directly. So let's consider a counterclockwise-oriented circle C'

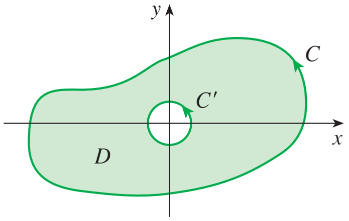


FIGURE 11

with center the origin and radius a, where a is chosen to be small enough that C' lies inside C. (See Figure 11.) Let D be the region bounded by C and C'. Then its positively oriented boundary is $C \cup (-C')$ and so the general version of Green's Theorem gives

$$\begin{aligned}\int_C P\,dx + Q\,dy + \int_{-C'} P\,dx + Q\,dy &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\ &= \iint_D \left[\frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2}\right] dA = 0\end{aligned}$$

Therefore $$\int_C P\,dx + Q\,dy = \int_{C'} P\,dx + Q\,dy$$

that is, $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}$$

We now easily compute this last integral using the parametrization given by $\mathbf{r}(t) = a \cos t\,\mathbf{i} + a \sin t\,\mathbf{j}$, $0 \leqslant t \leqslant 2\pi$. Thus

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\ &= \int_0^{2\pi} \frac{(-a \sin t)(-a \sin t) + (a \cos t)(a \cos t)}{a^2 \cos^2 t + a^2 \sin^2 t}\,dt = \int_0^{2\pi} dt = 2\pi\end{aligned}$$ ■

We end this section by using Green's Theorem to discuss a result that was stated in the preceding section.

SKETCH OF PROOF OF THEOREM 16.3.6 We're assuming that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is a vector field on an open simply-connected region D, that P and Q have continuous first-order partial derivatives, and that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

If C is any simple closed path in D and R is the region that C encloses, then Green's Theorem gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_R 0\,dA = 0$$

A curve that is not simple crosses itself at one or more points and can be broken up into a number of simple curves. We have shown that the line integrals of $\mathbf{F}$ around these simple curves are all 0 and, adding these integrals, we see that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C. Therefore $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D by Theorem 16.3.3. It follows that $\mathbf{F}$ is a conservative vector field. ■

16.4 Exercises

1–4 Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem.

1. $\oint_C y^2\,dx + x^2 y\,dy$,
C is the rectangle with vertices (0, 0), (5, 0), (5, 4), and (0, 4)

2. $\oint_C y\,dx - x\,dy$,
C is the circle with center the origin and radius 4

3. $\oint_C xy\,dx + x^2 y^3\,dy$,
C is the triangle with vertices (0, 0), (1, 0), and (1, 2)

4. $\oint_C x^2y^2\,dx + xy\,dy$, C consists of the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ and the line segments from $(1, 1)$ to $(0, 1)$ and from $(0, 1)$ to $(0, 0)$

5–12 Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

5. $\int_C ye^x\,dx + 2e^x\,dy$,
C is the rectangle with vertices $(0, 0)$, $(3, 0)$, $(3, 4)$, and $(0, 4)$

6. $\int_C \ln(xy)\,dx + (y/x)\,dy$,
C is the rectangle with vertices $(1, 1)$, $(1, 4)$, $(2, 4)$, and $(2, 1)$

7. $\int_C x^2y^2\,dx + y\tan^{-1}y\,dy$,
C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$

8. $\int_C (x^2 + y^2)\,dx + (x^2 - y^2)\,dy$,
C is the triangle with vertices $(0, 0)$, $(2, 1)$, and $(0, 1)$

9. $\int_C \left(y + e^{\sqrt{x}}\right)dx + (2x + \cos y^2)\,dy$,
C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$

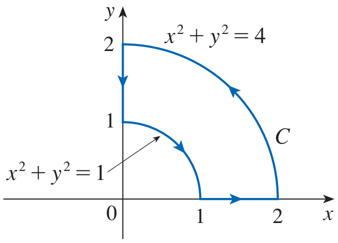
10. $\int_C y^4\,dx + 2xy^3\,dy$, C is the ellipse $x^2 + 2y^2 = 2$

11. $\int_C y^3\,dx - x^3\,dy$, C is the circle $x^2 + y^2 = 4$

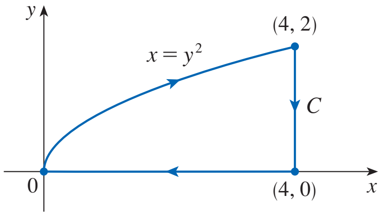
12. $\int_C (1 - y^3)\,dx + (x^3 + e^{y^2})\,dy$, C is the boundary of the region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$

13–18 Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. (Check the orientation of the curve before applying the theorem.)

13. $\int_C (3 + e^{x^2})\,dx + (\tan^{-1}y + 3x^2)\,dy$



14. $\int_C (x^{2/3} + y^2)\,dx + (y^{4/3} - x^2)\,dy$



15. $\mathbf{F}(x, y) = \langle y\cos x - xy\sin x,\ xy + x\cos x\rangle$,
C is the triangle from $(0, 0)$ to $(0, 4)$ to $(2, 0)$ to $(0, 0)$

16. $\mathbf{F}(x, y) = \langle e^{-x} + y^2,\ e^{-y} + x^2\rangle$,
C consists of the arc of the curve $y = \cos x$ from $(-\pi/2, 0)$ to $(\pi/2, 0)$ and the line segment from $(\pi/2, 0)$ to $(-\pi/2, 0)$

17. $\mathbf{F}(x, y) = \langle y - \cos y,\ x\sin y\rangle$,
C is the circle $(x - 3)^2 + (y + 4)^2 = 4$ oriented clockwise

18. $\mathbf{F}(x, y) = \left\langle \sqrt{x^2 + 1},\ \tan^{-1}x\right\rangle$, C is the triangle from $(0, 0)$ to $(1, 1)$ to $(0, 1)$ to $(0, 0)$

T **19–20** Verify Green's Theorem by using a computer algebra system to evaluate both the line integral and the double integral.

19. $P(x, y) = x^3y^4$, $Q(x, y) = x^5y^4$,
C consists of the line segment from $(-\pi/2, 0)$ to $(\pi/2, 0)$ followed by the arc of the curve $y = \cos x$ from $(\pi/2, 0)$ to $(-\pi/2, 0)$

20. $P(x, y) = 2x - x^3y^5$, $Q(x, y) = x^3y^8$,
C is the ellipse $4x^2 + y^2 = 4$

---

21. Use Green's Theorem to find the work done by the force $\mathbf{F}(x, y) = x(x + y)\,\mathbf{i} + xy^2\,\mathbf{j}$ in moving a particle from the origin along the x-axis to $(1, 0)$, then along the line segment to $(0, 1)$, and then back to the origin along the y-axis.

22. A particle starts at the origin, moves along the x-axis to $(5, 0)$, then along the quarter-circle $x^2 + y^2 = 25$, $x \geqslant 0$, $y \geqslant 0$ to the point $(0, 5)$, and then down the y-axis back to the origin. Use Green's Theorem to find the work done on this particle by the force field $\mathbf{F}(x, y) = \left\langle \sin x,\ \sin y + xy^2 + \tfrac{1}{3}x^3\right\rangle$.

23. Use one of the formulas in (5) to find the area under one arch of the cycloid $x = t - \sin t$, $y = 1 - \cos t$.

24. If a circle C with radius 1 rolls along the outside of the circle $x^2 + y^2 = 16$, a fixed point P on C traces out a curve called an *epicycloid,* with parametric equations $x = 5\cos t - \cos 5t$, $y = 5\sin t - \sin 5t$. Graph the epicycloid and use (5) to find the area it encloses.

25. (a) If C is the line segment connecting the point (x_1, y_1) to the point (x_2, y_2), show that

$$\int_C x\,dy - y\,dx = x_1y_2 - x_2y_1$$

(b) If the vertices of a polygon, in counterclockwise order, are $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, show that the area of the polygon is

$$A = \tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)]$$

(c) Find the area of the pentagon with vertices $(0, 0)$, $(2, 1)$, $(1, 3)$, $(0, 2)$, and $(-1, 1)$.

26. Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's Theorem to prove that the coordinates of the centroid $(\bar{x}, \bar{y})$ of D are

$$\bar{x} = \frac{1}{2A} \oint_C x^2\,dy \qquad \bar{y} = -\frac{1}{2A} \oint_C y^2\,dx$$

where A is the area of D.

27. Use Exercise 26 to find the centroid of a quarter-circular region of radius a.

28. Use Exercise 26 to find the centroid of the triangle with vertices $(0, 0)$, $(a, 0)$, and (a, b), where $a > 0$ and $b > 0$.

29. A plane lamina with constant density $\rho(x, y) = \rho$ occupies a region in the xy-plane bounded by a simple closed path C. Show that its moments of inertia about the axes are

$$I_x = -\frac{\rho}{3} \oint_C y^3\,dx \qquad I_y = \frac{\rho}{3} \oint_C x^3\,dy$$

(See Section 15.4.)

30. Use Exercise 29 to find the moment of inertia of a circular disk of radius a with constant density ρ about a diameter. (Compare with Example 15.4.4.)

31. Use the method of Example 5 to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y) = \frac{2xy\,\mathbf{i} + (y^2 - x^2)\,\mathbf{j}}{(x^2 + y^2)^2}$$

and C is any positively oriented simple closed curve that encloses the origin.

32. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle x^2 + y, 3x - y^2 \rangle$ and C is the positively oriented boundary curve of a region D that has area 6.

33. If $\mathbf{F}$ is the vector field of Example 5, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every simple closed path that does not pass through or enclose the origin.

34. Complete the proof of the special case of Green's Theorem by proving Equation 3.

35. Use Green's Theorem to prove the change of variables formula for a double integral (Formula 15.9.9) for the case where $f(x, y) = 1$:

$$\iint_R dx\,dy = \iint_S \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\,dv$$

Here R is the region in the xy-plane that corresponds to the region S in the uv-plane under the transformation given by $x = g(u, v)$, $y = h(u, v)$.

[*Hint:* Note that the left side is $A(R)$ and apply the first part of Equation 5. Convert the line integral over ∂R to a line integral over ∂S and apply Green's Theorem in the uv-plane.]

16.5 Curl and Divergence

In this section we define two operations that can be performed on vector fields and that play a basic role in the applications of vector calculus to fluid flow and electricity and magnetism. Each operation resembles differentiation, but one produces a vector field whereas the other produces a scalar field.

Curl

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\boxed{\boxed{1} \qquad \operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}}$$

As an aid to our memory, let's rewrite Equation 1 using operator notation. We introduce the vector differential operator ∇ ("del") as

$$\nabla = \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}$$

It has meaning when it operates on a scalar function to produce the gradient of f:

$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

If we think of ∇ as a vector with components $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, we can also consider the formal cross product of ∇ with the vector field $\mathbf{F}$ as follows:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$= \text{curl } \mathbf{F}$$

So the easiest way to remember Definition 1 is by means of the symbolic expression

$$\boxed{\text{curl } \mathbf{F} = \nabla \times \mathbf{F}} \tag{2}$$

EXAMPLE 1 If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find curl $\mathbf{F}$.

SOLUTION Using Equation 2, we have

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)\right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right]\mathbf{k}$$

$$= (-2y - xy)\,\mathbf{i} - (0 - x)\,\mathbf{j} + (yz - 0)\,\mathbf{k}$$

$$= -y(2 + x)\,\mathbf{i} + x\,\mathbf{j} + yz\,\mathbf{k}$$ ■

T Most computer algebra systems have commands that compute the curl and divergence of vector fields. If you have access to a CAS, use these commands to check the answers to the examples and exercises in this section.

Recall that the gradient of a function f of three variables is a vector field on $\mathbb{R}^3$ and so we can compute its curl. The following theorem says that the curl of a gradient vector field is $\mathbf{0}$.

3 Theorem If f is a function of three variables that has continuous second-order partial derivatives, then

$$\text{curl}(\nabla f) = \mathbf{0}$$

Notice the similarity to what we know from Section 12.4: $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for every three-dimensional vector **a**.

PROOF We have

$$\operatorname{curl}(\nabla f) = \nabla \times (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y\, \partial z} - \frac{\partial^2 f}{\partial z\, \partial y} \right) \mathbf{i} + \left(\frac{\partial^2 f}{\partial z\, \partial x} - \frac{\partial^2 f}{\partial x\, \partial z} \right) \mathbf{j} + \left(\frac{\partial^2 f}{\partial x\, \partial y} - \frac{\partial^2 f}{\partial y\, \partial x} \right) \mathbf{k}$$

$$= 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k} = \mathbf{0}$$

by Clairaut's Theorem. ■

Since a conservative vector field is one for which $\mathbf{F} = \nabla f$, Theorem 3 can be rephrased as follows:

Compare this with Exercise 16.3.35.

If **F** is conservative, then curl $\mathbf{F} = \mathbf{0}$.

This gives us a way of verifying that a vector field is *not* conservative.

EXAMPLE 2 Show that the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ is not conservative.

SOLUTION In Example 1 we showed that

$$\operatorname{curl} \mathbf{F} = -y(2 + x)\,\mathbf{i} + x\,\mathbf{j} + yz\,\mathbf{k}$$

This shows that curl $\mathbf{F} \neq \mathbf{0}$ and so, by the remarks preceding this example, **F** is not conservative. ■

The converse of Theorem 3 is not true in general, but the following theorem says the converse is true if **F** is defined everywhere. (More generally it is true if the domain is simply-connected, that is, "has no hole.") Theorem 4 is the three-dimensional version of Theorem 16.3.6. Its proof requires Stokes' Theorem and is sketched at the end of Section 16.8.

4 Theorem If **F** is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then **F** is a conservative vector field.

EXAMPLE 3

(a) Show that

$$\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$$

is a conservative vector field.

(b) Find a function f such that $\mathbf{F} = \nabla f$.

SOLUTION

(a) We compute the curl of $\mathbf{F}$:

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= (6xyz^2 - 6xyz^2)\,\mathbf{i} - (3y^2z^2 - 3y^2z^2)\,\mathbf{j} + (2yz^3 - 2yz^3)\,\mathbf{k}$$

$$= \mathbf{0}$$

Since curl $\mathbf{F} = \mathbf{0}$ and the domain of $\mathbf{F}$ is $\mathbb{R}^3$, $\mathbf{F}$ is a conservative vector field by Theorem 4.

(b) The technique for finding f was given in Section 16.3. We have

$$\boxed{5} \qquad f_x(x, y, z) = y^2z^3$$

$$\boxed{6} \qquad f_y(x, y, z) = 2xyz^3$$

$$\boxed{7} \qquad f_z(x, y, z) = 3xy^2z^2$$

Integrating (5) with respect to x, we obtain

$$\boxed{8} \qquad f(x, y, z) = xy^2z^3 + g(y, z)$$

Differentiating (8) with respect to y, we get $f_y(x, y, z) = 2xyz^3 + g_y(y, z)$, so comparison with (6) gives $g_y(y, z) = 0$. Thus $g(y, z) = h(z)$ and

$$f_z(x, y, z) = 3xy^2z^2 + h'(z)$$

Then (7) gives $h'(z) = 0$. Therefore

$$f(x, y, z) = xy^2z^3 + K$$ ■

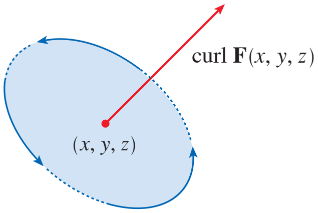
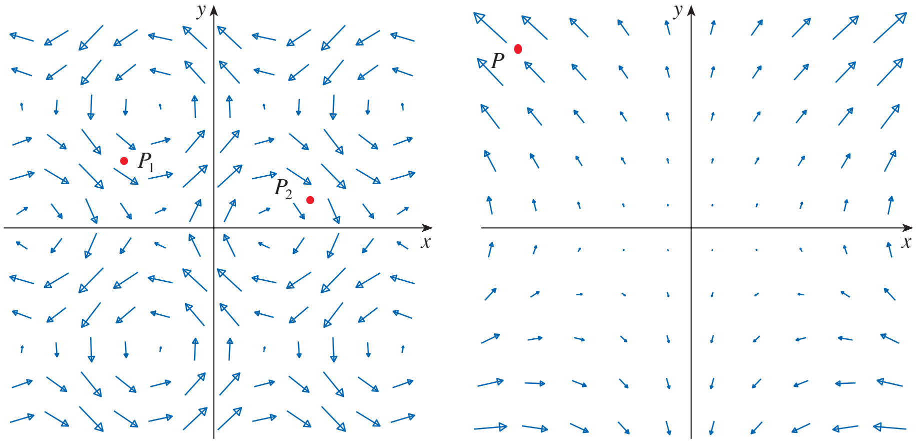


FIGURE 1

The reason for the name *curl* is that the curl vector is associated with rotations. One connection is explained in Exercise 39. Another occurs when $\mathbf{F}$ represents the velocity field in fluid flow (see Example 16.1.3). In Section 16.8 we show that particles near (x, y, z) in the fluid tend to rotate about the axis that points in the direction of curl $\mathbf{F}(x, y, z)$, following the right-hand rule, and the length of this curl vector is a measure of how quickly the particles move around the axis (see Figure 1). If curl $\mathbf{F} = \mathbf{0}$ at a point P, then the fluid is free from rotations at P and $\mathbf{F}$ is called **irrotational** at P. In this case, a tiny paddle wheel moves with the fluid but doesn't rotate about its axis. If curl $\mathbf{F} \neq \mathbf{0}$, the paddle wheel rotates about its axis.

As an illustration, each vector field $\mathbf{F}$ in Figure 2 represents the velocity field of a fluid. In Figure 2(a), curl $\mathbf{F} \neq \mathbf{0}$ at most points, including P_1 and P_2. A tiny paddle wheel placed at P_1 would rotate counterclockwise about its axis (the fluid near P_1 flows roughly in the same direction but with greater velocity on one side of the point than on the other), so the curl vector at P_1 points in the direction of $\mathbf{k}$. Similarly, a paddle wheel at P_2 would rotate clockwise and the curl vector there points in the direction of $-\mathbf{k}$. In Figure 2(b), curl $\mathbf{F} = \mathbf{0}$ everywhere. A paddle wheel placed at P moves with the fluid but doesn't rotate about its axis.

In Section 16.8 we give a more detailed explanation of curl and its interpretation (as a consequence of Stokes' Theorem).



(a) $\mathbf{F}(x, y, z) = \sin y\,\mathbf{i} + \cos x\,\mathbf{j}$
curl $\mathbf{F}(x, y, z) = -(\sin x + \cos y)\,\mathbf{k}$

(b) $\mathbf{F}(x, y, z) = 2xy\,\mathbf{i} + (x^2 + y)\,\mathbf{j}$
curl $\mathbf{F}(x, y, z) = \mathbf{0}$

FIGURE 2 Velocity fields in fluid flow. (Only the part of **F** in the *xy*-plane is shown; the vector field looks the same in all horizontal planes because **F** is independent of *z* and the *z*-component is 0.)

Divergence

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence of F** is the function of three variables defined by

$$\boxed{\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}} \tag{9}$$

(If **F** is a vector field on $\mathbb{R}^2$, then div **F** is a function of two variables defined similarly to the three-variable case.) Observe that curl **F** is a vector field but div **F** is a scalar field. In terms of the gradient operator $\nabla = (\partial/\partial x)\,\mathbf{i} + (\partial/\partial y)\,\mathbf{j} + (\partial/\partial z)\,\mathbf{k}$, the divergence of **F** can be written symbolically as the dot product of ∇ and **F**:

$$\boxed{\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}} \tag{10}$$

EXAMPLE 4 If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find div **F**.

SOLUTION By the definition of divergence (Equation 9 or 10), we have

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) = z + xz$$ ■

If **F** is a vector field on $\mathbb{R}^3$, then curl **F** is also a vector field on $\mathbb{R}^3$. As such, we can compute its divergence. The next theorem shows that the result is 0.

11 Theorem If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\operatorname{div} \operatorname{curl} \mathbf{F} = 0$$

Note the analogy with the scalar triple product: $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.

PROOF Using the definitions of divergence and curl, we have

$$\begin{aligned}\text{div curl } \mathbf{F} &= \nabla \cdot (\nabla \times \mathbf{F})\\ &= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\\ &= \frac{\partial^2 R}{\partial x\,\partial y} - \frac{\partial^2 Q}{\partial x\,\partial z} + \frac{\partial^2 P}{\partial y\,\partial z} - \frac{\partial^2 R}{\partial y\,\partial x} + \frac{\partial^2 Q}{\partial z\,\partial x} - \frac{\partial^2 P}{\partial z\,\partial y}\\ &= 0\end{aligned}$$

because the terms cancel in pairs by Clairaut's Theorem. ■

EXAMPLE 5 Show that the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ can't be written as the curl of another vector field, that is, $\mathbf{F} \neq \text{curl } \mathbf{G}$ for any vector field $\mathbf{G}$.

SOLUTION In Example 4 we showed that

$$\text{div } \mathbf{F} = z + xz$$

and therefore div $\mathbf{F} \neq 0$. If it were true that $\mathbf{F} = \text{curl } \mathbf{G}$, then Theorem 11 would give

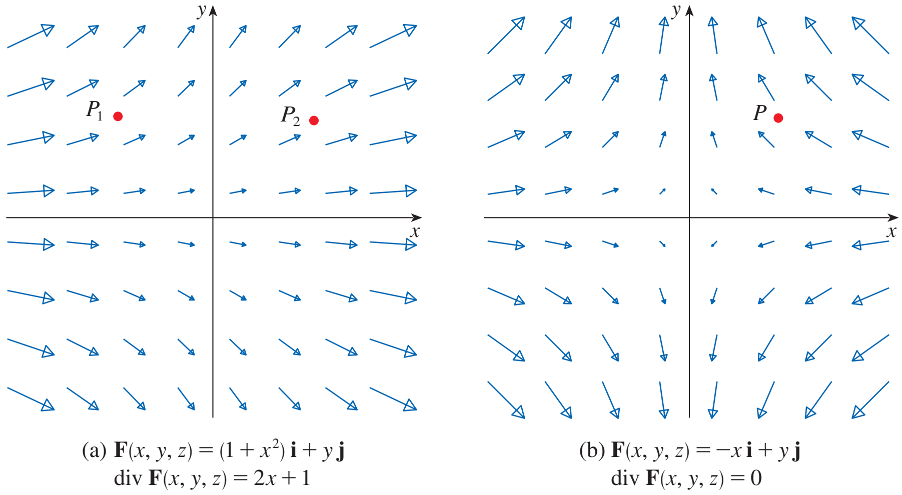
$$\text{div } \mathbf{F} = \text{div curl } \mathbf{G} = 0$$

which contradicts div $\mathbf{F} \neq 0$. Therefore $\mathbf{F}$ is not the curl of another vector field. ■

The reason for this interpretation of div **F** will be explained at the end of Section 16.9 as a consequence of the Divergence Theorem.

Again, the reason for the name *divergence* can be understood in the context of fluid flow. If $\mathbf{F}(x, y, z)$ is the velocity of a fluid (or gas), then div $\mathbf{F}(x, y, z)$ represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point (x, y, z) per unit volume. In other words, div $\mathbf{F}(x, y, z)$ measures the tendency of the fluid to diverge from the point (x, y, z). If div $\mathbf{F} = 0$, then $\mathbf{F}$ is said to be **incompressible**.

As an illustration, each vector field $\mathbf{F}$ in Figure 3 represents the velocity field of a fluid. In Figure 3(a), div $\mathbf{F} \neq 0$ in general. For instance, at the point P_1, div $\mathbf{F}$ is negative (the vectors that start near P_1 are shorter than those that end near P_1, so the net flow is inward there). At the point P_2, div $\mathbf{F}$ is positive (the vectors that start near P_2 are longer than those that end near P_2, so the net flow is outward there). In Figure 3(b), div $\mathbf{F} = 0$ everywhere (the vectors that start and end near any point P are about the same length).



(a) $\mathbf{F}(x, y, z) = (1 + x^2)\,\mathbf{i} + y\,\mathbf{j}$
div $\mathbf{F}(x, y, z) = 2x + 1$

(b) $\mathbf{F}(x, y, z) = -x\,\mathbf{i} + y\,\mathbf{j}$
div $\mathbf{F}(x, y, z) = 0$

FIGURE 3
Velocity fields in fluid flow. (Only the part of **F** in the xy-plane is shown; the vector field looks the same in all horizontal planes because **F** is independent of z and the z-component is 0.)

Another differential operator occurs when we compute the divergence of a gradient vector field ∇f. If f is a function of three variables, we have

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

and this expression occurs so often that we abbreviate it as $\nabla^2 f$. The operator

$$\nabla^2 = \nabla \cdot \nabla$$

is called the **Laplace operator** because of its relation to **Laplace's equation**

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

We can also apply the Laplace operator ∇^2 to a vector field

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

in terms of its components:

$$\nabla^2\mathbf{F} = \nabla^2 P\,\mathbf{i} + \nabla^2 Q\,\mathbf{j} + \nabla^2 R\,\mathbf{k}$$

Vector Forms of Green's Theorem

The curl and divergence operators allow us to rewrite Green's Theorem in versions that will be useful in our later work. We suppose that the plane region D, its boundary curve C, and the functions P and Q satisfy the hypotheses of Green's Theorem. Then we consider the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$. Its line integral is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy$$

and, regarding $\mathbf{F}$ as a vector field on $\mathbb{R}^3$ with third component 0, we have

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x, y) & Q(x, y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

Therefore

$$(\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

and we can now rewrite the equation in Green's Theorem in the vector form

12

$$\boxed{\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot \mathbf{T}\,ds = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\,dA}$$

Equation 12 expresses the line integral of the tangential component of $\mathbf{F}$ along C as the double integral of the vertical component of curl $\mathbf{F}$ over the region D enclosed by C. We now derive a similar formula involving the *normal* component of $\mathbf{F}$.

If C is given by the vector equation

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} \qquad a \leqslant t \leqslant b$$

then the unit tangent vector (see Section 13.2) is

$$\mathbf{T}(t) = \frac{x'(t)}{|\mathbf{r}'(t)|}\,\mathbf{i} + \frac{y'(t)}{|\mathbf{r}'(t)|}\,\mathbf{j}$$

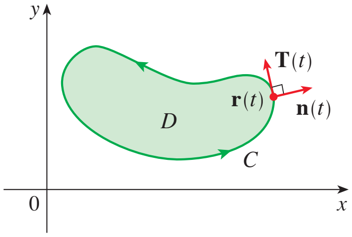


FIGURE 4

You can verify that the outward unit normal vector to C is given by

$$\mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|}\,\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\,\mathbf{j}$$

(See Figure 4.) Then, from Equation 16.2.3, we have

$$\begin{aligned}
\oint_C \mathbf{F}\cdot\mathbf{n}\,ds &= \int_a^b (\mathbf{F}\cdot\mathbf{n})(t)\,|\mathbf{r}'(t)|\,dt \\
&= \int_a^b \left[\frac{P(x(t), y(t))\,y'(t)}{|\mathbf{r}'(t)|} - \frac{Q(x(t), y(t))\,x'(t)}{|\mathbf{r}'(t)|}\right]|\mathbf{r}'(t)|\,dt \\
&= \int_a^b P(x(t), y(t))\,y'(t)\,dt - Q(x(t), y(t))\,x'(t)\,dt \\
&= \int_C P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA
\end{aligned}$$

by Green's Theorem. But the integrand in this double integral is just the divergence of **F**. So we have a second vector form of Green's Theorem.

$$\boxed{13} \qquad \oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}\mathbf{F}(x, y)\,dA$$

This version says that the line integral of the normal component of **F** along C is equal to the double integral of the divergence of **F** over the region D enclosed by C.

16.5 Exercises

1–8 Find (a) the curl and (b) the divergence of the vector field.

1. $\mathbf{F}(x, y, z) = xy^2z^2\,\mathbf{i} + x^2yz^2\,\mathbf{j} + x^2y^2z\,\mathbf{k}$

2. $\mathbf{F}(x, y, z) = x^3yz^2\,\mathbf{j} + y^4z^3\,\mathbf{k}$

3. $\mathbf{F}(x, y, z) = xye^z\,\mathbf{i} + yze^x\,\mathbf{k}$

4. $\mathbf{F}(x, y, z) = \sin yz\,\mathbf{i} + \sin zx\,\mathbf{j} + \sin xy\,\mathbf{k}$

5. $\mathbf{F}(x, y, z) = \dfrac{\sqrt{x}}{1+z}\,\mathbf{i} + \dfrac{\sqrt{y}}{1+x}\,\mathbf{j} + \dfrac{\sqrt{z}}{1+y}\,\mathbf{k}$

6. $\mathbf{F}(x, y, z) = \ln(2y + 3z)\,\mathbf{i} + \ln(x + 3z)\,\mathbf{j} + \ln(x + 2y)\,\mathbf{k}$

7. $\mathbf{F}(x, y, z) = \langle e^x \sin y, e^y \sin z, e^z \sin x\rangle$

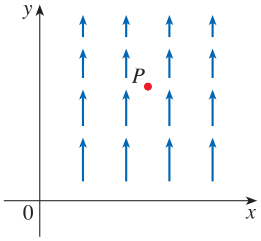
8. $\mathbf{F}(x, y, z) = \langle \arctan(xy), \arctan(yz), \arctan(zx)\rangle$

9–12 The vector field **F** is shown in the xy-plane and looks the same in all other horizontal planes. (In other words, **F** is independent of z and its z-component is 0.)

(a) Is div **F** positive, negative, or zero at P? Explain.

(b) Determine whether curl $\mathbf{F} = \mathbf{0}$. If not, in which direction does curl **F** point at P?

9.



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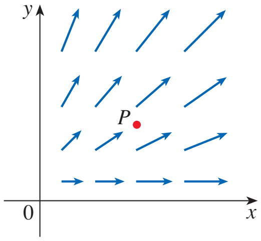
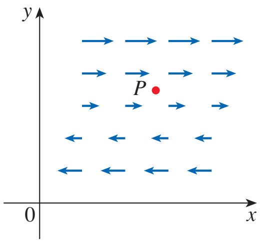
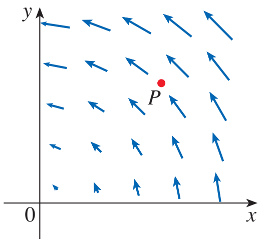


11.



12.



13. (a) Verify Formula 3 for $f(x, y, z) = \sin xyz$.
(b) Verify Formula 11 for $\mathbf{F}(x, y, z) = xyz^2\,\mathbf{i} + x^2yz^3\,\mathbf{j} + y^2\,\mathbf{k}$.

14. Let f be a scalar field and $\mathbf{F}$ a vector field. State whether each expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field.

(a) curl f
(b) grad f
(c) div $\mathbf{F}$
(d) curl(grad f)
(e) grad $\mathbf{F}$
(f) grad(div $\mathbf{F}$)
(g) div(grad f)
(h) grad(div f)
(i) curl(curl $\mathbf{F}$)
(j) div(div $\mathbf{F}$)
(k) (grad f) $\times$ (div $\mathbf{F}$)
(l) div(curl(grad f))

15–20 Determine whether or not the vector field is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

15. $\mathbf{F}(x, y, z) = \langle 2xy^3z^2, 3x^2y^2z^2, 2x^2y^3z \rangle$

16. $\mathbf{F}(x, y, z) = \langle yz, xz + y, xy - x \rangle$

17. $\mathbf{F}(x, y, z) = \langle \ln y, (x/y) + \ln z, y/z \rangle$

18. $\mathbf{F}(x, y, z) = yz \sin xy\,\mathbf{i} + xz \sin xy\,\mathbf{j} - \cos xy\,\mathbf{k}$

19. $\mathbf{F}(x, y, z) = yz^2e^{xz}\,\mathbf{i} + ze^{xz}\,\mathbf{j} + xyze^{xz}\,\mathbf{k}$

20. $\mathbf{F}(x, y, z) = e^z \cos x\,\mathbf{i} + e^y \cos z\,\mathbf{j} + (e^z \sin x - e^y \sin z)\,\mathbf{k}$

21. Is there a vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that curl $\mathbf{G} = \langle x \sin y, \cos y, z - xy \rangle$? Explain.

22. Is there a vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that curl $\mathbf{G} = \langle x, y, z \rangle$? Explain.

23. Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(x)\,\mathbf{i} + g(y)\,\mathbf{j} + h(z)\,\mathbf{k}$$

where f, g, h are differentiable functions, is irrotational.

24. Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(y, z)\,\mathbf{i} + g(x, z)\,\mathbf{j} + h(x, y)\,\mathbf{k}$$

is incompressible.

25–31 Prove the identity, assuming that the appropriate partial derivatives exist and are continuous. If f is a scalar field and $\mathbf{F}$, $\mathbf{G}$ are vector fields, then $f\mathbf{F}$, $\mathbf{F} \cdot \mathbf{G}$, and $\mathbf{F} \times \mathbf{G}$ are defined by

$$\begin{aligned}(f\mathbf{F})(x, y, z) &= f(x, y, z)\,\mathbf{F}(x, y, z)\\ (\mathbf{F} \cdot \mathbf{G})(x, y, z) &= \mathbf{F}(x, y, z) \cdot \mathbf{G}(x, y, z)\\ (\mathbf{F} \times \mathbf{G})(x, y, z) &= \mathbf{F}(x, y, z) \times \mathbf{G}(x, y, z)\end{aligned}$$

25. $\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}$

26. $\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl} \mathbf{F} + \operatorname{curl} \mathbf{G}$

27. $\operatorname{div}(f\mathbf{F}) = f \operatorname{div} \mathbf{F} + \mathbf{F} \cdot \nabla f$

28. $\operatorname{curl}(f\mathbf{F}) = f \operatorname{curl} \mathbf{F} + (\nabla f) \times \mathbf{F}$

29. $\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \operatorname{curl} \mathbf{F} - \mathbf{F} \cdot \operatorname{curl} \mathbf{G}$

30. $\operatorname{div}(\nabla f \times \nabla g) = 0$

31. $\operatorname{curl}(\operatorname{curl} \mathbf{F}) = \operatorname{grad}(\operatorname{div} \mathbf{F}) - \nabla^2\mathbf{F}$

32–34 Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $r = |\mathbf{r}|$.

32. Verify each identity.
(a) $\nabla \cdot \mathbf{r} = 3$
(b) $\nabla \cdot (r\mathbf{r}) = 4r$
(c) $\nabla^2 r^3 = 12r$

33. Verify each identity.
(a) $\nabla r = \mathbf{r}/r$
(b) $\nabla \times \mathbf{r} = \mathbf{0}$
(c) $\nabla(1/r) = -\mathbf{r}/r^3$
(d) $\nabla \ln r = \mathbf{r}/r^2$

34. If $\mathbf{F} = \mathbf{r}/r^p$, find div $\mathbf{F}$. Is there a value of p for which div $\mathbf{F} = 0$?

35. Use Green's Theorem in the form of Equation 13 to prove **Green's first identity**:

$$\iint_D f\nabla^2 g\, dA = \oint_C f(\nabla g) \cdot \mathbf{n}\, ds - \iint_D \nabla f \cdot \nabla g\, dA$$

where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous. (The quantity $\nabla g \cdot \mathbf{n} = D_{\mathbf{n}}\, g$ occurs in the line integral; it is the directional derivative in the direction of the normal vector $\mathbf{n}$ and is called the **normal derivative** of g.)

36. Use Green's first identity (Exercise 35) to prove **Green's second identity**:

$$\iint_D (f\nabla^2 g - g\nabla^2 f)\, dA = \oint_C (f\nabla g - g\nabla f) \cdot \mathbf{n}\, ds$$

where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous.

37. Recall from Section 14.3 that a function g is called *harmonic* on D if it satisfies Laplace's equation, that is, $\nabla^2 g = 0$ on D. Use Green's first identity (with the same hypotheses as in Exercise 35) to show that if g is harmonic on D, then $\oint_C D_{\mathbf{n}}\, g\, ds = 0$. Here $D_{\mathbf{n}}\, g$ is the normal derivative of g defined in Exercise 35.

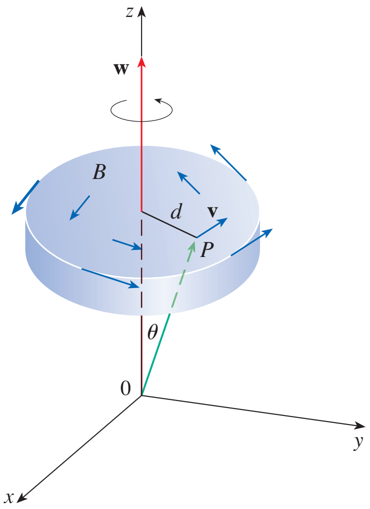
38. Use Green's first identity to show that if f is harmonic on D, and if $f(x, y) = 0$ on the boundary curve C, then $\iint_D |\nabla f|^2\, dA = 0$. (Assume the same hypotheses as in Exercise 35.)

39. This exercise demonstrates a connection between the curl vector and rotations. Let B be a rigid body rotating about the z-axis. The rotation can be described by the vector $\mathbf{w} = \omega\mathbf{k}$, where ω is the angular speed of B, that is, the tangential speed of any point P in B divided by the distance d from the axis of rotation. Let $\mathbf{r} = \langle x, y, z\rangle$ be the position vector of P.

(a) By considering the angle θ in the figure, show that the velocity field of B is given by $\mathbf{v} = \mathbf{w} \times \mathbf{r}$.

(b) Show that $\mathbf{v} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j}$.

(c) Show that $\operatorname{curl} \mathbf{v} = 2\mathbf{w}$.



40. Maxwell's equations relating the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ as they vary with time in a region containing no charge and no current can be stated as follows:

$$\operatorname{div} \mathbf{E} = 0 \qquad \operatorname{div} \mathbf{H} = 0$$

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \qquad \operatorname{curl} \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

where c is the speed of light. Use these equations to prove the following:

(a) $\nabla \times (\nabla \times \mathbf{E}) = -\dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{E}}{\partial t^2}$

(b) $\nabla \times (\nabla \times \mathbf{H}) = -\dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{H}}{\partial t^2}$

(c) $\nabla^2 \mathbf{E} = \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{E}}{\partial t^2}$ [*Hint:* Use Exercise 31.]

(d) $\nabla^2 \mathbf{H} = \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{H}}{\partial t^2}$

41. We have seen that all vector fields of the form $\mathbf{F} = \nabla g$ satisfy the equation $\operatorname{curl} \mathbf{F} = \mathbf{0}$ and that all vector fields of the form $\mathbf{F} = \operatorname{curl} \mathbf{G}$ satisfy the equation $\operatorname{div} \mathbf{F} = 0$ (assuming continuity of the appropriate partial derivatives). This suggests the question: are there any equations that all functions of the form $f = \operatorname{div} \mathbf{G}$ must satisfy? Show that the answer to this question is "no" by proving that *every* continuous function f on $\mathbb{R}^3$ is the divergence of some vector field.

[*Hint:* Let $\mathbf{G}(x, y, z) = \langle g(x, y, z), 0, 0\rangle$, where $g(x, y, z) = \int_0^x f(t, y, z)\,dt$.]

16.6 Parametric Surfaces and Their Areas

So far we have considered special types of surfaces: cylinders, quadric surfaces, graphs of functions of two variables, and level surfaces of functions of three variables. Here we use vector functions to describe more general surfaces, called *parametric surfaces*, and compute their areas. Then we take the general surface area formula and see how it applies to special surfaces.

Parametric Surfaces

In much the same way that we describe a space curve by a vector function $\mathbf{r}(t)$ of a single parameter t, we can describe a surface by a vector function $\mathbf{r}(u, v)$ of two parameters u and v. We suppose that

$$\boxed{1} \qquad \mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

is a vector-valued function defined on a region D in the uv-plane. So x, y, and z, the component functions of $\mathbf{r}$, are functions of the two variables u and v with domain D. The set of all points (x, y, z) in $\mathbb{R}^3$ such that

$$\boxed{2} \qquad x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

and (u, v) varies throughout D, is called a **parametric surface** S and Equations 2 are called **parametric equations** of S. Each choice of u and v gives a point on S; by making all choices, we get all of S. In other words, the surface S is traced out by the tip of the position vector $\mathbf{r}(u, v)$ as (u, v) moves throughout the region D. (See Figure 1.)

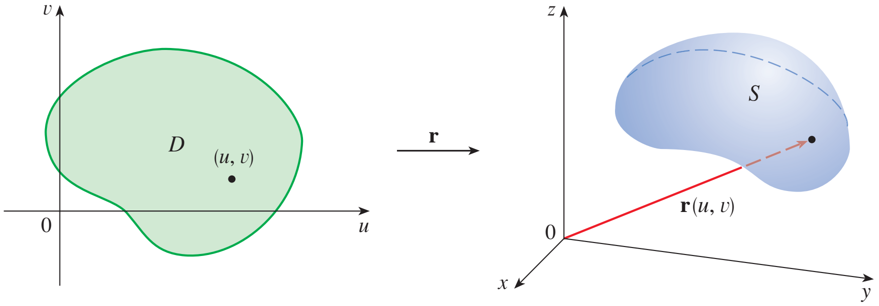


FIGURE 1
A parametric surface

EXAMPLE 1 Identify and sketch the surface with vector equation

$$\mathbf{r}(u, v) = 2 \cos u \, \mathbf{i} + v \, \mathbf{j} + 2 \sin u \, \mathbf{k}$$

SOLUTION The parametric equations for this surface are

$$x = 2 \cos u \qquad y = v \qquad z = 2 \sin u$$

So for any point (x, y, z) on the surface, we have

$$x^2 + z^2 = 4 \cos^2 u + 4 \sin^2 u = 4$$

This means that vertical cross-sections parallel to the xz-plane (that is, with y constant) are all circles with radius 2. Since $y = v$ and no restriction is placed on v, the surface is a circular cylinder with radius 2 whose axis is the y-axis (see Figure 2). ■

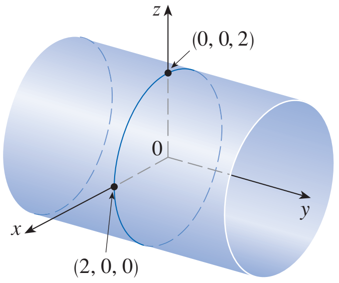


FIGURE 2

In Example 1 we placed no restrictions on the parameters u and v and so we obtained the entire cylinder. If, for instance, we restrict u and v by writing the parameter domain as

$$0 \leqslant u \leqslant \pi/2 \qquad 0 \leqslant v \leqslant 3$$

then $x \geqslant 0$, $z \geqslant 0$, $0 \leqslant y \leqslant 3$, and we get the quarter-cylinder with length 3 illustrated in Figure 3.

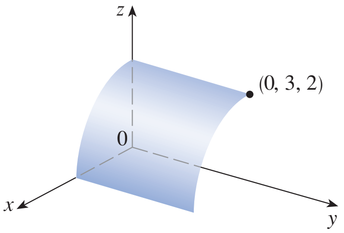


FIGURE 3

If a parametric surface S is given by a vector function $\mathbf{r}(u, v)$, then there are two useful families of curves that lie on S, one family with u constant and the other with v constant. These families correspond to vertical and horizontal lines in the uv-plane. If we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a curve C_1 lying on S. (See Figure 4.)

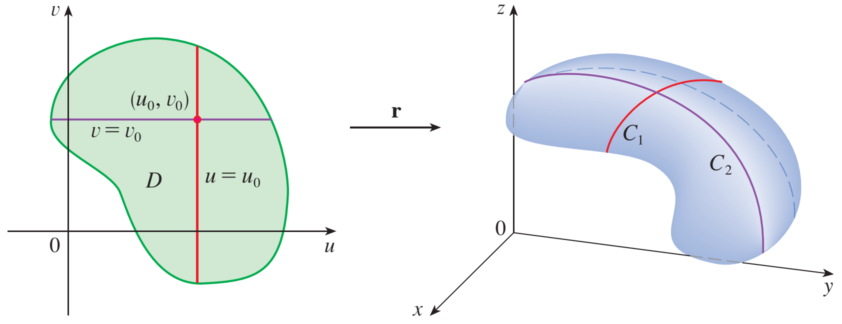


FIGURE 4

Similarly, if we keep v constant by putting $v = v_0$, we get a curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S. We call these curves **grid curves**. (In Example 1, for instance, the grid curves obtained by letting u be constant are horizontal lines, whereas the grid curves with v constant are circles.) In fact, when a computer graphs a parametric surface, it sometimes depicts the surface by plotting these grid curves, as we will see in the following example.

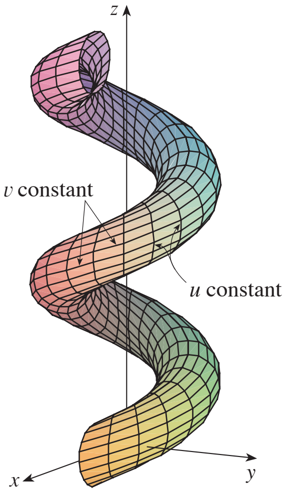


FIGURE 5

EXAMPLE 2 Use a computer to graph the surface

$$\mathbf{r}(u, v) = \langle (2 + \sin v) \cos u, (2 + \sin v) \sin u, u + \cos v \rangle$$

Which grid curves have u constant? Which have v constant?

SOLUTION We graph the portion of the surface with parameter domain $0 \leqslant u \leqslant 4\pi$, $0 \leqslant v \leqslant 2\pi$ in Figure 5. It has the appearance of a spiral tube. To identify the grid curves, we write the corresponding parametric equations:

$$x = (2 + \sin v) \cos u \qquad y = (2 + \sin v) \sin u \qquad z = u + \cos v$$

If v is constant, then $\sin v$ and $\cos v$ are constant, so the parametric equations resemble those of the helix in Example 13.1.4. Thus the grid curves with v constant are the spiral curves in Figure 5. We deduce that the grid curves with u constant must be the curves that look like circles in the figure. Further evidence for this assertion is that if u is kept constant, $u = u_0$, then the equation $z = u_0 + \cos v$ shows that the z-values vary from $u_0 - 1$ to $u_0 + 1$. ■

In Examples 1 and 2 we were given a vector equation and asked to graph the corresponding parametric surface. In the following examples, however, we are given the more challenging problem of finding a vector function to represent a given surface. In the rest of this chapter we will often need to do exactly that.

EXAMPLE 3 Find a vector function that represents the plane that passes through the point P_0 with position vector $\mathbf{r}_0$ and that contains two nonparallel vectors $\mathbf{a}$ and $\mathbf{b}$.

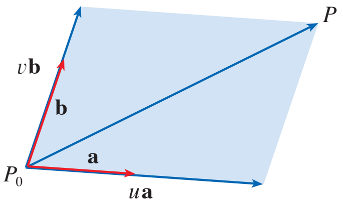


FIGURE 6

SOLUTION If P is any point in the plane, we can get from P_0 to P by moving a certain distance in the direction of $\mathbf{a}$ and another distance in the direction of $\mathbf{b}$. So there are scalars u and v such that $\overrightarrow{P_0P} = u\mathbf{a} + v\mathbf{b}$. (Figure 6 illustrates how this works, by means of the Parallelogram Law, for the case where u and v are positive. See also Exercise 12.2.46.) If $\mathbf{r}$ is the position vector of P, then

$$\mathbf{r} = \overrightarrow{OP_0} + \overrightarrow{P_0P} = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$$

So the vector equation of the plane can be written as

$$\mathbf{r}(u, v) = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$$

where u and v are real numbers.

If we write $\mathbf{r} = \langle x, y, z \rangle$, $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then we can write the parametric equations of the plane through the point (x_0, y_0, z_0) as follows:

$$x = x_0 + ua_1 + vb_1 \qquad y = y_0 + ua_2 + vb_2 \qquad z = z_0 + ua_3 + vb_3$$ ■

EXAMPLE 4 Find a parametric representation of the sphere

$$x^2 + y^2 + z^2 = a^2$$

SOLUTION The sphere has a simple representation $\rho = a$ in spherical coordinates, so let's choose the angles ϕ and θ in spherical coordinates as the parameters (see Section 15.8). Then, putting $\rho = a$ in the equations for conversion from spherical to rectangular coordinates (Equations 15.8.1), we obtain

$$x = a \sin \phi \cos \theta \qquad y = a \sin \phi \sin \theta \qquad z = a \cos \phi$$

as the parametric equations of the sphere. The corresponding vector equation is

$$\mathbf{r}(\phi, \theta) = a \sin \phi \cos \theta \, \mathbf{i} + a \sin \phi \sin \theta \, \mathbf{j} + a \cos \phi \, \mathbf{k}$$

We have $0 \leqslant \phi \leqslant \pi$ and $0 \leqslant \theta \leqslant 2\pi$, so the parameter domain is the rectangle $D = [0, \pi] \times [0, 2\pi]$. The grid curves with ϕ constant are the circles of constant latitude (including the equator). The grid curves with θ constant are the meridians (semicircles), which connect the north and south poles (see Figure 7).

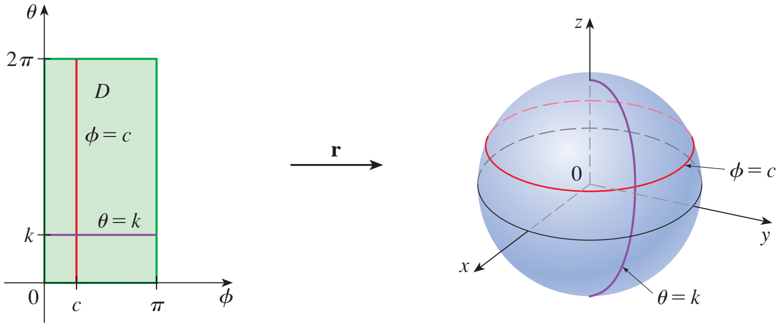


FIGURE 7

■

NOTE We saw in Example 4 that the grid curves for a sphere are curves of constant latitude or constant longitude. For a general parametric surface we are really making a map and the grid curves are similar to lines of latitude and longitude. Describing a point on a parametric surface (like the one in Figure 5) by giving specific values of u and v is like giving the latitude and longitude of a point.

One of the uses of parametric surfaces is in computer graphics. Figure 8 shows the result of trying to graph the sphere $x^2 + y^2 + z^2 = 1$ by solving the equation for z and graphing the top and bottom hemispheres separately. Part of the sphere appears to be missing because of the rectangular grid system used by the software. The much better picture in Figure 9 was produced by a computer using the parametric equations found in Example 4.

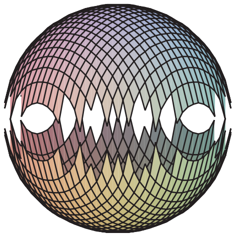


FIGURE 8

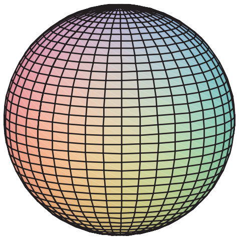


FIGURE 9

EXAMPLE 5 Find a parametric representation for the cylinder

$$x^2 + y^2 = 4 \qquad 0 \leqslant z \leqslant 1$$

SOLUTION The cylinder has a simple representation $r = 2$ in cylindrical coordinates, so we choose as parameters θ and z in cylindrical coordinates. Then the parametric equations of the cylinder are

$$x = 2 \cos \theta \qquad y = 2 \sin \theta \qquad z = z$$

where $0 \leqslant \theta \leqslant 2\pi$ and $0 \leqslant z \leqslant 1$. In vector notation,

$$\mathbf{r}(\theta, z) = 2 \cos \theta \, \mathbf{i} + 2 \sin \theta \, \mathbf{j} + z \, \mathbf{k}$$

and the vector function **r** maps the parameter domain

$$D = \{(\theta, z) \mid 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant z \leqslant 1\}$$

to a cylinder, as shown in Figure 10.

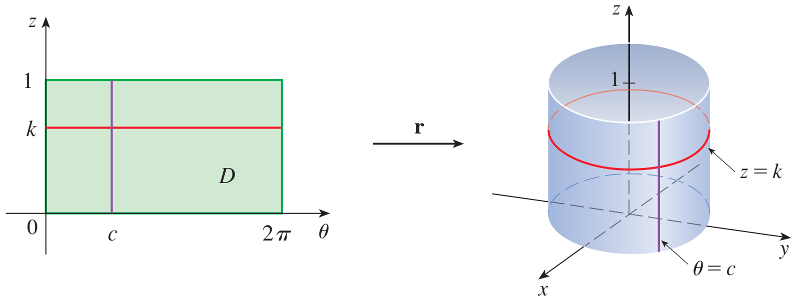


FIGURE 10 ■

EXAMPLE 6 Find a vector function that represents the elliptic paraboloid $z = x^2 + 2y^2$.

SOLUTION If we regard x and y as parameters, then the parametric equations are simply

$$x = x \qquad y = y \qquad z = x^2 + 2y^2$$

and the vector equation is

$$\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + (x^2 + 2y^2)\,\mathbf{k}$$ ■

In general, a surface given as the graph of a function of x and y, that is, with an equation of the form $z = f(x, y)$, can always be regarded as a parametric surface by taking x and y as parameters and writing the parametric equations as

$$x = x \qquad y = y \qquad z = f(x, y)$$

Parametric representations (also called parametrizations) of surfaces are not unique. The next example shows two ways to parametrize a cone.

EXAMPLE 7 Find a parametric representation for the surface $z = 2\sqrt{x^2 + y^2}$, that is, the top half of the cone $z^2 = 4x^2 + 4y^2$.

SOLUTION 1 One possible representation is obtained by choosing x and y as parameters:

$$x = x \qquad y = y \qquad z = 2\sqrt{x^2 + y^2}$$

So the vector equation is

$$\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + 2\sqrt{x^2 + y^2}\,\mathbf{k}$$

SOLUTION 2 Another representation results from choosing as parameters the polar coordinates r and θ. A point (x, y, z) on the cone satisfies $x = r\cos\theta$, $y = r\sin\theta$, and $z = 2\sqrt{x^2 + y^2} = 2r$. So a vector equation for the cone is

$$\mathbf{r}(r, \theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + 2r\,\mathbf{k}$$

where $r \geqslant 0$ and $0 \leqslant \theta \leqslant 2\pi$. ■

For some purposes the parametric representations in Solutions 1 and 2 of Example 7 are equally good, but Solution 2 might be preferable in certain situations. If we are interested only in the part of the cone that lies below the plane $z = 1$, for instance, all we have to do in Solution 2 is change the parameter domain to

$$D = \left\{(r, \theta) \mid 0 \leqslant r \leqslant \tfrac{1}{2}, 0 \leqslant \theta \leqslant 2\pi\right\}$$

Then the vector function $\mathbf{r}$ maps the region D to the half-cone shown in Figure 11.

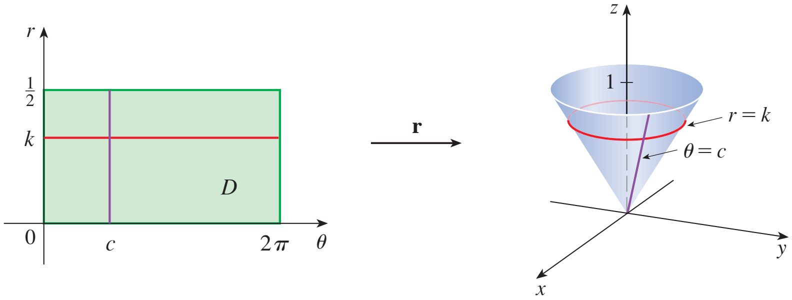


FIGURE 11

Surfaces of Revolution

Surfaces of revolution can be represented parametrically. For instance, let's consider the surface S obtained by rotating the curve $y = f(x)$, $a \leqslant x \leqslant b$, about the x-axis, where $f(x) \geqslant 0$. Let θ be the angle of rotation as shown in Figure 12. If (x, y, z) is a point on S, then

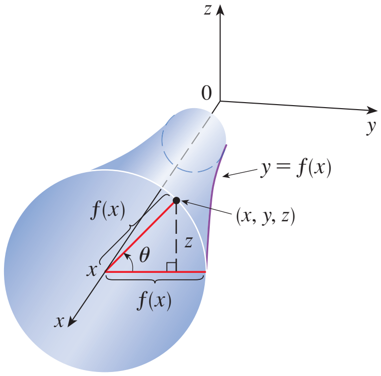


FIGURE 12

$$\boxed{3} \qquad x = x \qquad y = f(x)\cos\theta \qquad z = f(x)\sin\theta$$

Therefore we take x and θ as parameters and regard Equations 3 as parametric equations of S. The parameter domain is given by $a \leqslant x \leqslant b$, $0 \leqslant \theta \leqslant 2\pi$.

EXAMPLE 8 Find parametric equations for the surface generated by rotating the curve $y = \sin x$, $0 \leqslant x \leqslant 2\pi$, about the x-axis. Use these equations to graph the surface of revolution.

SOLUTION From Equations 3, the parametric equations are

$$x = x \qquad y = \sin x \cos\theta \qquad z = \sin x \sin\theta$$

and the parameter domain is $0 \leqslant x \leqslant 2\pi$, $0 \leqslant \theta \leqslant 2\pi$. Using a computer to plot these equations, we obtain the graph in Figure 13. ■

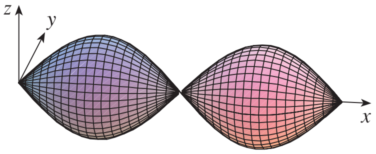


FIGURE 13

We can adapt Equations 3 to represent a surface obtained through revolution about the y- or z-axis (see Exercise 30).

Tangent Planes

We now find the tangent plane to a parametric surface S traced out by a vector function

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

at a point P_0 with position vector $\mathbf{r}(u_0, v_0)$. If we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a grid curve C_1

lying on S. (See Figure 14.) The tangent vector to C_1 at P_0 is obtained by taking the partial derivative of $\mathbf{r}$ with respect to v:

$$\boxed{4} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\,\mathbf{k}$$

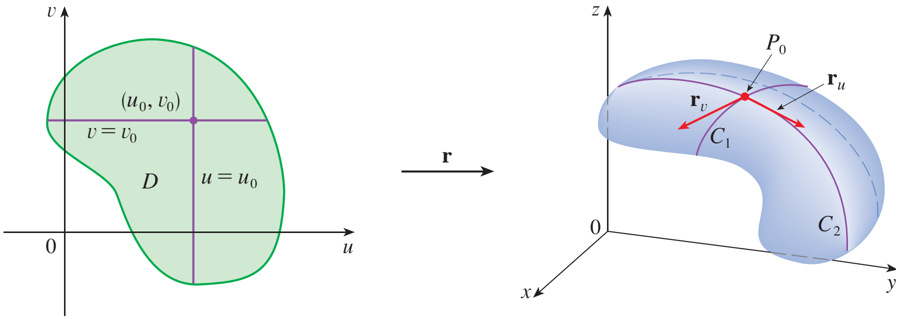


FIGURE 14

Similarly, if we keep v constant by putting $v = v_0$, we get a grid curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S, and its tangent vector at P_0 is

$$\boxed{5} \qquad \mathbf{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\,\mathbf{k}$$

If $\mathbf{r}_u \times \mathbf{r}_v$ is never $\mathbf{0}$, then the surface S is called **smooth** (it has no "corners"). For a smooth surface, the **tangent plane** is the plane that contains the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$, and the vector $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the tangent plane.

Figure 15 shows the self-intersecting surface in Example 9 and its tangent plane at (1, 1, 3).

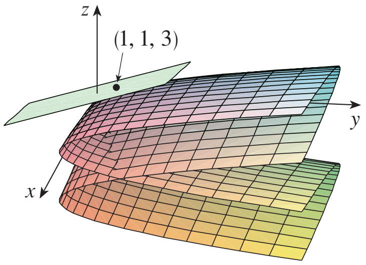


FIGURE 15

EXAMPLE 9 Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$, $z = u + 2v$ at the point $(1, 1, 3)$.

SOLUTION We first compute the tangent vectors:

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} = 2u\,\mathbf{i} + \mathbf{k}$$

$$\mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k} = 2v\,\mathbf{j} + 2\,\mathbf{k}$$

Thus a normal vector to the tangent plane is

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = -2v\,\mathbf{i} - 4u\,\mathbf{j} + 4uv\,\mathbf{k}$$

Notice that the point $(1, 1, 3)$ corresponds to the parameter values $u = 1$ and $v = 1$, so the normal vector there is

$$-2\,\mathbf{i} - 4\,\mathbf{j} + 4\,\mathbf{k}$$

Therefore an equation of the tangent plane at $(1, 1, 3)$ is

$$-2(x - 1) - 4(y - 1) + 4(z - 3) = 0$$

or

$$x + 2y - 2z + 3 = 0$$

■

Surface Area

Now we define the surface area of a general parametric surface given by Equation 1. For simplicity we start by considering a surface S whose parameter domain D is a rectangle, and we divide it into subrectangles R_{ij}. Let's choose (u_i^*, v_j^*) to be the lower left corner of R_{ij}. (See Figure 16.)

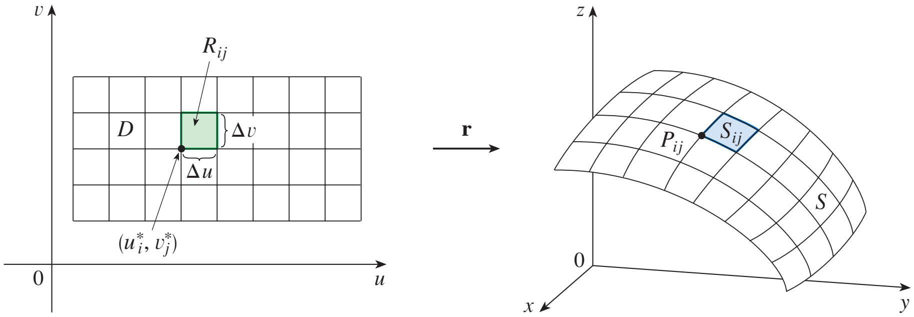
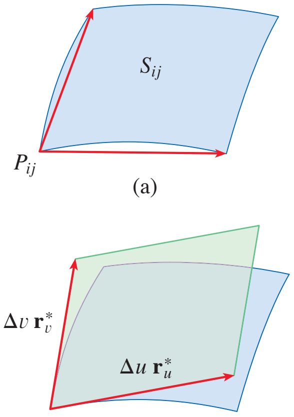


FIGURE 16
The image of the subrectangle R_{ij} is the patch S_{ij}.

The part S_{ij} of the surface S that corresponds to R_{ij} is called a *patch* and has the point P_{ij} with position vector $\mathbf{r}(u_i^*, v_j^*)$ as one of its corners. Let

$$\mathbf{r}_u^* = \mathbf{r}_u(u_i^*, v_j^*) \qquad \text{and} \qquad \mathbf{r}_v^* = \mathbf{r}_v(u_i^*, v_j^*)$$

be the tangent vectors at P_{ij} as given by Equations 5 and 4.



(a)

(b)

FIGURE 17
Approximating a patch by a parallelogram

Figure 17(a) shows how the two edges of the patch that meet at P_{ij} can be approximated by vectors. These vectors, in turn, can be approximated by the vectors $\Delta u\, \mathbf{r}_u^*$ and $\Delta v\, \mathbf{r}_v^*$ because partial derivatives can be approximated by difference quotients. So we approximate S_{ij} by the parallelogram determined by the vectors $\Delta u\, \mathbf{r}_u^*$ and $\Delta v\, \mathbf{r}_v^*$. This parallelogram is shown in Figure 17(b) and lies in the tangent plane to S at P_{ij}. The area of this parallelogram is

$$|(\Delta u\, \mathbf{r}_u^*) \times (\Delta v\, \mathbf{r}_v^*)| = |\mathbf{r}_u^* \times \mathbf{r}_v^*|\, \Delta u\, \Delta v$$

and so an approximation to the area of S is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} |\mathbf{r}_u^* \times \mathbf{r}_v^*|\, \Delta u\, \Delta v$$

Our intuition tells us that this approximation gets better as we increase the number of subrectangles, and we recognize the double sum as a Riemann sum for the double integral $\iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$. This motivates the following definition.

6 Definition If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$$

and S is covered just once as (u, v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

where $\qquad \mathbf{r}_u = \dfrac{\partial x}{\partial u}\,\mathbf{i} + \dfrac{\partial y}{\partial u}\,\mathbf{j} + \dfrac{\partial z}{\partial u}\,\mathbf{k} \qquad \mathbf{r}_v = \dfrac{\partial x}{\partial v}\,\mathbf{i} + \dfrac{\partial y}{\partial v}\,\mathbf{j} + \dfrac{\partial z}{\partial v}\,\mathbf{k}$

EXAMPLE 10 Find the surface area of a sphere of radius a.

SOLUTION In Example 4 we found the parametric representation

$$x = a\sin\phi\,\cos\theta \qquad y = a\sin\phi\,\sin\theta \qquad z = a\cos\phi$$

where the parameter domain is

$$D = \{(\phi, \theta) \mid 0 \leqslant \phi \leqslant \pi,\ 0 \leqslant \theta \leqslant 2\pi\}$$

We first compute the cross product of the tangent vectors:

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial \phi} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial z}{\partial \phi} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\,\cos\theta & a\cos\phi\,\sin\theta & -a\sin\phi \\ -a\sin\phi\,\sin\theta & a\sin\phi\,\cos\theta & 0 \end{vmatrix}$$

$$= a^2\sin^2\phi\,\cos\theta\,\mathbf{i} + a^2\sin^2\phi\,\sin\theta\,\mathbf{j} + a^2\sin\phi\,\cos\phi\,\mathbf{k}$$

Thus

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{a^4\sin^4\phi\,\cos^2\theta + a^4\sin^4\phi\,\sin^2\theta + a^4\sin^2\phi\,\cos^2\phi}$$

$$= \sqrt{a^4\sin^4\phi + a^4\sin^2\phi\,\cos^2\phi} = a^2\sqrt{\sin^2\phi} = a^2\sin\phi$$

since $\sin\phi \geqslant 0$ for $0 \leqslant \phi \leqslant \pi$. Therefore, by Definition 6, the area of the sphere is

$$A = \iint_D |\mathbf{r}_\phi \times \mathbf{r}_\theta|\,dA = \int_0^{2\pi}\int_0^{\pi} a^2\sin\phi\,d\phi\,d\theta$$

$$= a^2\int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\,d\phi = a^2(2\pi)2 = 4\pi a^2$$

■

■ Surface Area of the Graph of a Function

For the special case of a surface S with equation $z = f(x, y)$, where (x, y) lies in D and f has continuous partial derivatives, we take x and y as parameters. The parametric equations are

$$x = x \qquad y = y \qquad z = f(x, y)$$

so

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}\right)\mathbf{k}$$

and

$$\boxed{7} \qquad \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \dfrac{\partial f}{\partial x} \\ 0 & 1 & \dfrac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$$

Thus we have

$$\boxed{8} \qquad |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

Notice the similarity between the surface area formula in Equation 9 and the arc length formula

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

from Section 8.1.

and the surface area formula in Definition 6 becomes

$$\boxed{9} \qquad \boxed{A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA}$$

EXAMPLE 11 Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

SOLUTION The plane intersects the paraboloid in the circle $x^2 + y^2 = 9$, $z = 9$. Therefore the given surface lies above the disk D with center the origin and radius 3. (See Figure 18.) Using Formula 9, we have

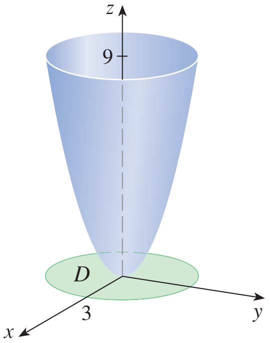


FIGURE 18

$$\begin{aligned} A &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\ &= \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA = \iint_D \sqrt{1 + 4(x^2 + y^2)}\, dA \end{aligned}$$

Converting to polar coordinates, we obtain

$$\begin{aligned} A &= \int_0^{2\pi}\!\int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^3 r\sqrt{1 + 4r^2}\, dr \\ &= 2\pi\left(\tfrac{1}{8}\right)\tfrac{2}{3}(1 + 4r^2)^{3/2}\Big]_0^3 = \frac{\pi}{6}\left(37\sqrt{37} - 1\right) \end{aligned}$$

■

The question remains whether our definition of surface area (6) is consistent with the surface area formula from single-variable calculus (8.2.4).

We consider the surface S obtained by rotating the curve $y = f(x)$, $a \leqslant x \leqslant b$, about the x-axis, where $f(x) \geqslant 0$ and f' is continuous. From Equations 3 we know that parametric equations of S are

$$x = x \qquad y = f(x)\cos\theta \qquad z = f(x)\sin\theta \qquad a \leqslant x \leqslant b \qquad 0 \leqslant \theta \leqslant 2\pi$$

To compute the surface area of S we need the tangent vectors

$$\begin{aligned} \mathbf{r}_x &= \mathbf{i} + f'(x)\cos\theta\, \mathbf{j} + f'(x)\sin\theta\, \mathbf{k} \\ \mathbf{r}_\theta &= -f(x)\sin\theta\, \mathbf{j} + f(x)\cos\theta\, \mathbf{k} \end{aligned}$$

Thus

$$\begin{aligned} \mathbf{r}_x \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(x)\cos\theta & f'(x)\sin\theta \\ 0 & -f(x)\sin\theta & f(x)\cos\theta \end{vmatrix} \\ &= f(x)f'(x)\,\mathbf{i} - f(x)\cos\theta\,\mathbf{j} - f(x)\sin\theta\,\mathbf{k} \end{aligned}$$

and so

$$\begin{aligned} |\mathbf{r}_x \times \mathbf{r}_\theta| &= \sqrt{[f(x)]^2[f'(x)]^2 + [f(x)]^2\cos^2\theta + [f(x)]^2\sin^2\theta} \\ &= \sqrt{[f(x)]^2[1 + [f'(x)]^2]} = f(x)\sqrt{1 + [f'(x)]^2} \end{aligned}$$

because $f(x) \geqslant 0$. Therefore the area of S is

$$\begin{aligned} A &= \iint_D |\mathbf{r}_x \times \mathbf{r}_\theta|\, dA \\ &= \int_0^{2\pi} \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx\, d\theta \\ &= 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx \end{aligned}$$

This is precisely the formula that was used to define the area of a surface of revolution in single-variable calculus (8.2.4).

16.6 Exercises

1–2 Determine whether the points P and Q lie on the given surface.

1. $\mathbf{r}(u, v) = \langle u + v, u - 2v, 3 + u - v \rangle$
$P(4, -5, 1)$, $Q(0, 4, 6)$

2. $\mathbf{r}(u, v) = \langle 1 + u - v, u + v^2, u^2 - v^2 \rangle$
$P(1, 2, 1)$, $Q(2, 3, 3)$

3–6 Identify the surface with the given vector equation.

3. $\mathbf{r}(u, v) = (u + v)\,\mathbf{i} + (3 - v)\,\mathbf{j} + (1 + 4u + 5v)\,\mathbf{k}$

4. $\mathbf{r}(u, v) = u^2\,\mathbf{i} + u\cos v\,\mathbf{j} + u\sin v\,\mathbf{k}$

5. $\mathbf{r}(s, t) = \langle s\cos t, s\sin t, s \rangle$

6. $\mathbf{r}(s, t) = \langle 3\cos t, s, \sin t \rangle, \quad -1 \leqslant s \leqslant 1$

7–12 Use a computer to graph the parametric surface. Indicate on the graph which grid curves have u constant and which have v constant.

7. $\mathbf{r}(u, v) = \langle u^2, v^2, u + v \rangle$,
$-1 \leqslant u \leqslant 1,\ -1 \leqslant v \leqslant 1$

8. $\mathbf{r}(u, v) = \langle u, v^3, -v \rangle$,
$-2 \leqslant u \leqslant 2,\ -2 \leqslant v \leqslant 2$

9. $\mathbf{r}(u, v) = \langle u^3, u\sin v, u\cos v \rangle$,
$-1 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 2\pi$

10. $\mathbf{r}(u, v) = \langle u, \sin(u + v), \sin v \rangle$,
$-\pi \leqslant u \leqslant \pi,\ -\pi \leqslant v \leqslant \pi$

11. $x = \sin v, \quad y = \cos u \sin 4v, \quad z = \sin 2u \sin 4v$,
$0 \leqslant u \leqslant 2\pi,\ -\pi/2 \leqslant v \leqslant \pi/2$

12. $x = \cos u, \quad y = \sin u \sin v, \quad z = \cos v$,
$0 \leqslant u \leqslant 2\pi,\ 0 \leqslant v \leqslant 2\pi$

13–18 Match the equations with the graphs labeled I–VI and give reasons for your answers. Determine which families of grid curves have u constant and which have v constant.

13. $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$

14. $\mathbf{r}(u, v) = uv^2\,\mathbf{i} + u^2v\,\mathbf{j} + (u^2 - v^2)\,\mathbf{k}$

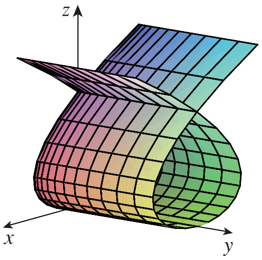
15. $\mathbf{r}(u, v) = (u^3 - u)\,\mathbf{i} + v^2\,\mathbf{j} + u^2\,\mathbf{k}$

16. $x = (1 - u)(3 + \cos v)\cos 4\pi u$,
$y = (1 - u)(3 + \cos v)\sin 4\pi u$,
$z = 3u + (1 - u)\sin v$

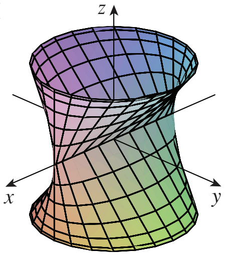
17. $x = \cos^3 u \cos^3 v, \quad y = \sin^3 u \cos^3 v, \quad z = \sin^3 v$

18. $x = \sin u, \quad y = \cos u \sin v, \quad z = \sin v$

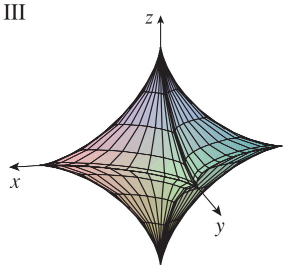
I



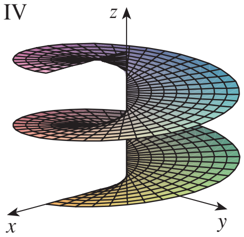
II



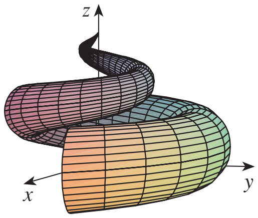
III



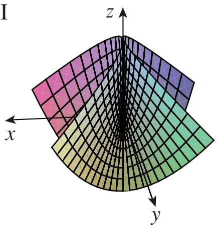
IV



V



VI



19–26 Find a parametric representation for the surface.

19. The plane through the origin that contains the vectors $\mathbf{i} - \mathbf{j}$ and $\mathbf{j} - \mathbf{k}$

20. The plane that passes through the point $(0, -1, 5)$ and contains the vectors $\langle 2, 1, 4 \rangle$ and $\langle -3, 2, 5 \rangle$

21. The part of the hyperboloid $4x^2 - 4y^2 - z^2 = 4$ that lies in front of the yz-plane

22. The part of the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ that lies to the left of the xz-plane

23. The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$

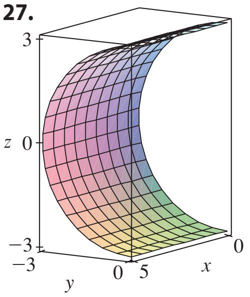
24. The part of the cylinder $x^2 + z^2 = 9$ that lies above the xy-plane and between the planes $y = -4$ and $y = 4$

25. The part of the sphere $x^2 + y^2 + z^2 = 36$ that lies between the planes $z = 0$ and $z = 3\sqrt{3}$

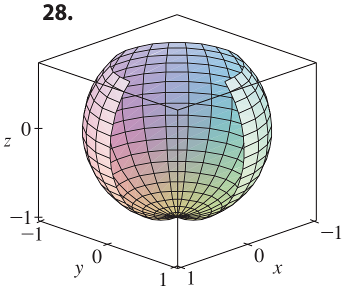
26. The part of the plane $z = x + 3$ that lies inside the cylinder $x^2 + y^2 = 1$

27–28 Use a computer to produce a graph that looks like the given one.

27.



28.



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29. Find parametric equations for the surface obtained by rotating the curve $y = 1/(1 + x^2)$, $-2 \leqslant x \leqslant 2$, about the x-axis and use them to graph the surface.

30. Find parametric equations for the surface obtained by rotating the curve $x = 1/y$, $y \geqslant 1$, about the y-axis and use them to graph the surface.

31. (a) What happens to the spiral tube in Example 2 (see Figure 5) if we replace $\cos u$ by $\sin u$ and $\sin u$ by $\cos u$?
(b) What happens if we replace $\cos u$ by $\cos 2u$ and $\sin u$ by $\sin 2u$?

32. The surface with parametric equations

$$x = 2\cos\theta + r\cos(\theta/2)$$
$$y = 2\sin\theta + r\cos(\theta/2)$$
$$z = r\sin(\theta/2)$$

where $-\frac{1}{2} \leqslant r \leqslant \frac{1}{2}$ and $0 \leqslant \theta \leqslant 2\pi$, is called a **Möbius strip**. Graph this surface with several viewpoints. What is unusual about it?

33–36 Find an equation of the tangent plane to the given parametric surface at the specified point.

33. $x = u + v, \quad y = 3u^2, \quad z = u - v; \quad (2, 3, 0)$

34. $x = u^2 + 1, \quad y = v^3 + 1, \quad z = u + v; \quad (5, 2, 3)$

35. $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}; \quad u = 1,\ v = \pi/3$

36. $\mathbf{r}(u, v) = \sin u\,\mathbf{i} + \cos u\sin v\,\mathbf{j} + \sin v\,\mathbf{k};$
$u = \pi/6,\ v = \pi/6$

37–38 Find an equation of the tangent plane to the given parametric surface at the specified point. Graph the surface and the tangent plane.

37. $\mathbf{r}(u, v) = u^2\,\mathbf{i} + 2u\sin v\,\mathbf{j} + u\cos v\,\mathbf{k}; \quad u = 1,\ v = 0$

38. $\mathbf{r}(u, v) = (1 - u^2 - v^2)\,\mathbf{i} - v\,\mathbf{j} - u\,\mathbf{k}; \quad (-1, -1, -1)$

39–50 Find the area of the surface.

39. The part of the plane $3x + 2y + z = 6$ that lies in the first octant

40. The part of the plane with vector equation $\mathbf{r}(u, v) = \langle u + v, 2 - 3u, 1 + u - v \rangle$ that is given by $0 \leqslant u \leqslant 2$, $-1 \leqslant v \leqslant 1$

41. The part of the plane $x + 2y + 3z = 1$ that lies inside the cylinder $x^2 + y^2 = 3$

42. The part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the plane $y = x$ and the cylinder $y = x^2$

43. The surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$

44. The part of the surface $z = 4 - 2x^2 + y$ that lies above the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$

45. The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$

46. The part of the surface $x = z^2 + y$ that lies between the planes $y = 0$, $y = 2$, $z = 0$, and $z = 2$

47. The part of the paraboloid $y = x^2 + z^2$ that lies within the cylinder $x^2 + z^2 = 16$

48. The helicoid (or spiral ramp) with vector equation $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi$

49. The surface with parametric equations $x = u^2$, $y = uv$, $z = \frac{1}{2}v^2$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant 2$

50. The part of the sphere $x^2 + y^2 + z^2 = b^2$ that lies inside the cylinder $x^2 + y^2 = a^2$, where $0 < a < b$

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51. If the equation of a surface S is $z = f(x, y)$, where $x^2 + y^2 \leqslant R^2$, and you know that $|f_x| \leqslant 1$ and $|f_y| \leqslant 1$, what can you say about $A(S)$?

52–53 Find the area of the surface correct to four decimal places by first simplifying an expression for area to one in terms of a single integral and then evaluating the integral numerically.

52. The part of the surface $z = \cos(x^2 + y^2)$ that lies inside the cylinder $x^2 + y^2 = 1$

53. The part of the surface $z = \ln(x^2 + y^2 + 2)$ that lies above the disk $x^2 + y^2 \leqslant 1$

---

54. Use a computer algebra system to find, to four decimal places, the area of the part of the surface $z = (1 + x^2)/(1 + y^2)$ that lies above the square $|x| + |y| \leqslant 1$. Illustrate by graphing this part of the surface.

55. (a) Use the Midpoint Rule for double integrals (see Section 15.1) with six squares to estimate the area of the surface $z = 1/(1 + x^2 + y^2)$, $0 \leqslant x \leqslant 6$, $0 \leqslant y \leqslant 4$.

(b) Use a computer algebra system to approximate the surface area in part (a) to four decimal places. Compare with the answer to part (a).

56. Use a computer algebra system to find the area of the surface with vector equation

$$\mathbf{r}(u, v) = \langle \cos^3 u \cos^3 v, \sin^3 u \cos^3 v, \sin^3 v \rangle$$

$0 \leqslant u \leqslant \pi$, $0 \leqslant v \leqslant 2\pi$. State your answer correct to four decimal places.

57. Use a computer algebra system to find the exact area of the surface $z = 1 + 2x + 3y + 4y^2$, $1 \leqslant x \leqslant 4$, $0 \leqslant y \leqslant 1$.

58. (a) Set up, but do not evaluate, a double integral for the area of the surface with parametric equations $x = au \cos v$, $y = bu \sin v$, $z = u^2$, $0 \leqslant u \leqslant 2$, $0 \leqslant v \leqslant 2\pi$.

(b) Eliminate the parameters to show that the surface is an elliptic paraboloid and set up another double integral for the surface area.

(c) Use the parametric equations in part (a) with $a = 2$ and $b = 3$ to graph the surface.

(d) For the case $a = 2$, $b = 3$, use a computer algebra system to find the surface area correct to four decimal places.

59. (a) Show that the parametric equations $x = a \sin u \cos v$, $y = b \sin u \sin v$, $z = c \cos u$, $0 \leqslant u \leqslant \pi$, $0 \leqslant v \leqslant 2\pi$, represent an ellipsoid.

(b) Use the parametric equations in part (a) to graph the ellipsoid for the case $a = 1$, $b = 2$, $c = 3$.

(c) Set up, but do not evaluate, a double integral for the surface area of the ellipsoid in part (b).

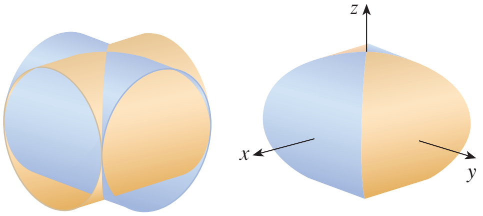
60. (a) Show that the parametric equations $x = a \cosh u \cos v$, $y = b \cosh u \sin v$, $z = c \sinh u$, represent a hyperboloid of one sheet.

(b) Use the parametric equations in part (a) to graph the hyperboloid for the case $a = 1$, $b = 2$, $c = 3$.

(c) Set up, but do not evaluate, a double integral for the surface area of the part of the hyperboloid in part (b) that lies between the planes $z = -3$ and $z = 3$.

61. Find the area of the part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$.

62. The figure shows the surface created when the cylinder $y^2 + z^2 = 1$ intersects the cylinder $x^2 + z^2 = 1$. Find the area of this surface.

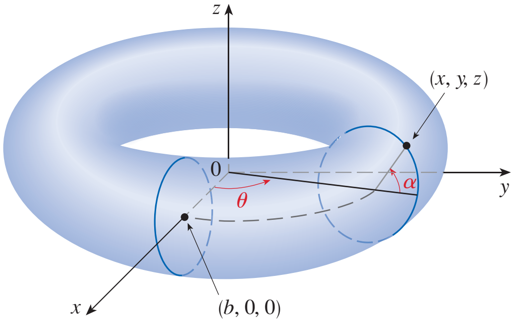


63. Find the area of the part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies inside the cylinder $x^2 + y^2 = ax$.

64. (a) Find a parametric representation for the torus obtained by rotating about the z-axis the circle in the xz-plane with center $(b, 0, 0)$ and radius $a < b$. [*Hint:* Take as parameters the angles θ and α shown in the figure.]

(b) Use the parametric equations found in part (a) to graph the torus for several values of a and b.

(c) Use the parametric representation from part (a) to find the surface area of the torus.



16.7 Surface Integrals

The relationship between surface integrals and surface area is much the same as the relationship between line integrals and arc length. Suppose f is a function of three variables whose domain includes a surface S. We will define the surface integral of f over S in such a way that, in the case where $f(x, y, z) = 1$, the value of the surface integral is equal to the surface area of S. We start with parametric surfaces and then deal with the special case where S is the graph of a function of two variables.

Parametric Surfaces

Suppose that a surface S has a vector equation

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$$

We first assume that the parameter domain D is a rectangle and we divide it into subrectangles R_{ij} with dimensions Δu and Δv. Then the surface S is divided into corresponding patches S_{ij} as in Figure 1.

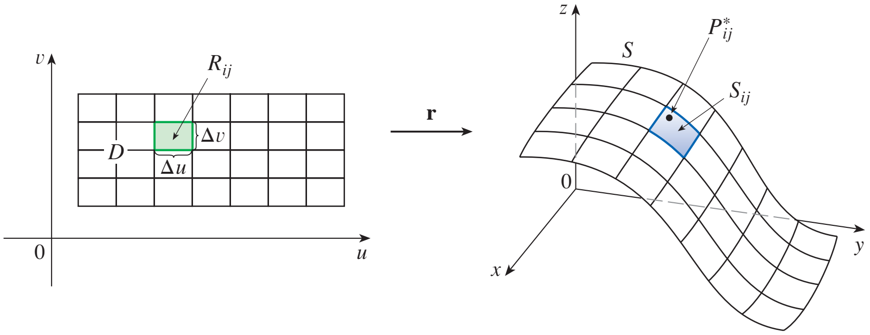


FIGURE 1

We evaluate f at a point P_{ij}^* in each patch, multiply by the area ΔS_{ij} of the patch, and form the Riemann sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\,\Delta S_{ij}$$

Then we take the limit as the number of patches increases and define the **surface integral of f over the surface S** as

$$\boxed{1} \qquad \boxed{\iint_S f(x, y, z)\,dS = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\,\Delta S_{ij}}$$

Notice the analogy with the definition of a line integral (16.2.2) and also the analogy with the definition of a double integral (15.1.5).

To evaluate the surface integral in Equation 1 we approximate the patch area ΔS_{ij} by the area of an approximating parallelogram in the tangent plane. In our discussion of surface area in Section 16.6 we made the approximation

$$\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v|\,\Delta u\,\Delta v$$

where $$\mathbf{r}_u = \frac{\partial x}{\partial u}\,\mathbf{i} + \frac{\partial y}{\partial u}\,\mathbf{j} + \frac{\partial z}{\partial u}\,\mathbf{k} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\,\mathbf{i} + \frac{\partial y}{\partial v}\,\mathbf{j} + \frac{\partial z}{\partial v}\,\mathbf{k}$$

are the tangent vectors at a corner of S_{ij}. If the components are continuous and $\mathbf{r}_u$ and $\mathbf{r}_v$ are nonzero and nonparallel in the interior of D, it can be shown from Definition 1, even when D is not a rectangle, that

We assume that the surface is covered only once as (u, v) ranges throughout D. The value of the surface integral does not depend on the parametrization that is used.

$$\boxed{2} \qquad \boxed{\iint_S f(x, y, z)\,dS = \iint_D f(\mathbf{r}(u, v))\,|\mathbf{r}_u \times \mathbf{r}_v|\,dA}$$

This should be compared with the formula for a line integral:

$$\int_C f(x, y, z)\, ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\, dt$$

Observe also that

$$\iint_S 1\, dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA = A(S)$$

Formula 2 allows us to compute a surface integral by converting it into a double integral over the parameter domain D. When using this formula, remember that $f(\mathbf{r}(u, v))$ is evaluated by writing $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ in the formula for $f(x, y, z)$.

EXAMPLE 1 Compute the surface integral $\iint_S x^2\, dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

SOLUTION As in Example 16.6.4, we use the parametric representation

$$x = \sin\phi\, \cos\theta \qquad y = \sin\phi\, \sin\theta \qquad z = \cos\phi \qquad 0 \leqslant \phi \leqslant \pi \qquad 0 \leqslant \theta \leqslant 2\pi$$

that is, $$\mathbf{r}(\phi, \theta) = \sin\phi\, \cos\theta\, \mathbf{i} + \sin\phi\, \sin\theta\, \mathbf{j} + \cos\phi\, \mathbf{k}$$

As in Example 16.6.10, we can compute that

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin\phi$$

Therefore, by Formula 2,

$$\begin{aligned}\iint_S x^2\, dS &= \iint_D (\sin\phi\, \cos\theta)^2\, |\mathbf{r}_\phi \times \mathbf{r}_\theta|\, dA \\ &= \int_0^{2\pi} \int_0^{\pi} \sin^2\phi\, \cos^2\theta\, \sin\phi\, d\phi\, d\theta = \int_0^{2\pi} \cos^2\theta\, d\theta \int_0^{\pi} \sin^3\phi\, d\phi \\ &= \int_0^{2\pi} \tfrac{1}{2}(1 + \cos 2\theta)\, d\theta \int_0^{\pi} (\sin\phi - \sin\phi\, \cos^2\phi)\, d\phi \\ &= \tfrac{1}{2}\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{2\pi} \Big[-\cos\phi + \tfrac{1}{3}\cos^3\phi\Big]_0^{\pi} = \frac{4\pi}{3}\end{aligned}$$ ■

Here we use the identities

$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta)$$

$$\sin^2\phi = 1 - \cos^2\phi$$

Instead, we could use Formulas 64 and 67 in the Table of Integrals.

Surface integrals have applications similar to those for the integrals we have previously considered. For example, if a thin sheet (say, of aluminum foil) has the shape of a surface S and the density (mass per unit area) at the point (x, y, z) is $\rho(x, y, z)$, then the total **mass** of the sheet is

$$m = \iint_S \rho(x, y, z)\, dS$$

and the **center of mass** is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{m}\iint_S x\,\rho(x, y, z)\, dS \qquad \bar{y} = \frac{1}{m}\iint_S y\,\rho(x, y, z)\, dS \qquad \bar{z} = \frac{1}{m}\iint_S z\,\rho(x, y, z)\, dS$$

Moments of inertia can also be defined as before (see Exercise 41).

Graphs of Functions

Any surface S with equation $z = g(x, y)$ can be regarded as a parametric surface with parametric equations

$$x = x \qquad y = y \qquad z = g(x, y)$$

and so we have $\quad \mathbf{r}_x = \mathbf{i} + \left(\dfrac{\partial g}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\dfrac{\partial g}{\partial y}\right)\mathbf{k}$

Thus

$$\boxed{3} \qquad \mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}$$

and

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

Therefore, in this case, Formula 2 becomes

$$\boxed{4} \qquad \iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

Similar formulas apply when it is more convenient to project S onto the yz-plane or xz-plane. For instance, if S is a surface with equation $y = h(x, z)$ and D is its projection onto the xz-plane, then

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, h(x, z), z) \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\, dA$$

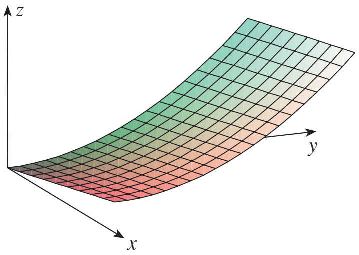


FIGURE 2

EXAMPLE 2 Evaluate $\iint_S y\, dS$, where S is the surface $z = x + y^2$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2$. (See Figure 2.)

SOLUTION Since

$$\frac{\partial z}{\partial x} = 1 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 2y$$

Formula 4 gives

$$\begin{aligned}\iint_S y\, dS &= \iint_D y \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\ &= \int_0^1 \int_0^2 y\sqrt{1 + 1 + 4y^2}\, dy\, dx \\ &= \int_0^1 dx\, \sqrt{2} \int_0^2 y\sqrt{1 + 2y^2}\, dy \\ &= \sqrt{2}\left(\tfrac{1}{4}\right)\tfrac{2}{3}(1 + 2y^2)^{3/2}\Big]_0^2 = \frac{13\sqrt{2}}{3}\end{aligned}$$ ■

If S is a piecewise-smooth surface, that is, a finite union of smooth surfaces $S_1, S_2, \ldots, S_n$ that intersect only along their boundaries, then the surface integral of f over S is defined by

$$\iint_S f(x, y, z)\, dS = \iint_{S_1} f(x, y, z)\, dS + \cdots + \iint_{S_n} f(x, y, z)\, dS$$

EXAMPLE 3 Evaluate $\iint_S z\, dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2 + y^2 = 1$, whose bottom S_2 is the disk $x^2 + y^2 \leqslant 1$ in the plane $z = 0$, and whose top S_3 is the part of the plane $z = 1 + x$ that lies above S_2.

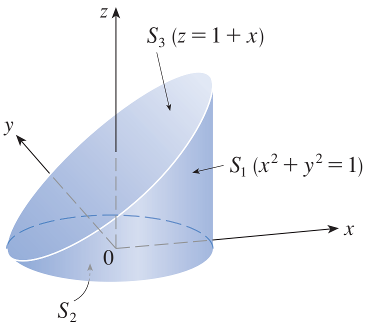


FIGURE 3

SOLUTION The surface S is shown in Figure 3. (We have changed the usual position of the axes to get a better look at S.) For S_1 we use θ and z as parameters (see Example 16.6.5) and write its parametric equations as

$$x = \cos\theta \qquad y = \sin\theta \qquad z = z$$

where

$$0 \leqslant \theta \leqslant 2\pi \qquad \text{and} \qquad 0 \leqslant z \leqslant 1 + x = 1 + \cos\theta$$

Therefore

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

and

$$|\mathbf{r}_\theta \times \mathbf{r}_z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

Thus the surface integral over S_1 is

$$\begin{aligned} \iint_{S_1} z\,dS &= \iint_D z\,|\mathbf{r}_\theta \times \mathbf{r}_z|\,dA \\ &= \int_0^{2\pi}\int_0^{1+\cos\theta} z\,dz\,d\theta = \int_0^{2\pi} \tfrac{1}{2}(1+\cos\theta)^2\,d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi}\left[1 + 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right]d\theta \\ &= \tfrac{1}{2}\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{3\pi}{2} \end{aligned}$$

Since S_2 lies in the plane $z = 0$, we have

$$\iint_{S_2} z\,dS = \iint_{S_2} 0\,dS = 0$$

The top surface S_3 lies above the unit disk D and is part of the plane $z = 1 + x$. So, taking $g(x, y) = 1 + x$ in Formula 4 and converting to polar coordinates, we have

$$\begin{aligned} \iint_{S_3} z\,dS &= \iint_D (1 + x)\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA \\ &= \int_0^{2\pi}\int_0^1 (1 + r\cos\theta)\sqrt{1 + 1 + 0}\;r\,dr\,d\theta \\ &= \sqrt{2}\int_0^{2\pi}\int_0^1 (r + r^2\cos\theta)\,dr\,d\theta = \sqrt{2}\int_0^{2\pi}\left(\tfrac{1}{2} + \tfrac{1}{3}\cos\theta\right)d\theta \\ &= \sqrt{2}\left[\frac{\theta}{2} + \frac{\sin\theta}{3}\right]_0^{2\pi} = \sqrt{2}\,\pi \end{aligned}$$

Therefore

$$\begin{aligned} \iint_S z\,dS &= \iint_{S_1} z\,dS + \iint_{S_2} z\,dS + \iint_{S_3} z\,dS \\ &= \frac{3\pi}{2} + 0 + \sqrt{2}\,\pi = \left(\tfrac{3}{2} + \sqrt{2}\right)\pi \end{aligned}$$

■

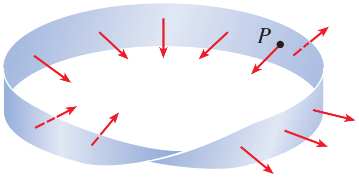


FIGURE 4
A Möbius strip

Oriented Surfaces

To define surface integrals of vector fields, we need to rule out nonorientable surfaces such as the Möbius strip shown in Figure 4. [It is named after the German geometer August Möbius (1790–1868).] You can construct one for yourself by taking a long rectangular strip of paper, giving it a half-twist, and taping the short edges together as in Figure 5. If an ant were to crawl along the Möbius strip starting at a point P, it would end up on the "other side" of the strip (that is, with its upper side pointing in the opposite direction). Then, if the ant continued to crawl in the same direction, it would end up back at the same point P without ever having crossed an edge. (If you have constructed a Möbius strip, try drawing a pencil line down the middle.) Therefore a Möbius strip really has only one side. You can graph the Möbius strip using the parametric equations in Exercise 16.6.32.

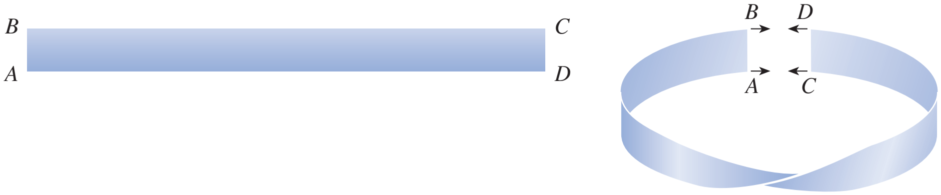


FIGURE 5
Constructing a Möbius strip

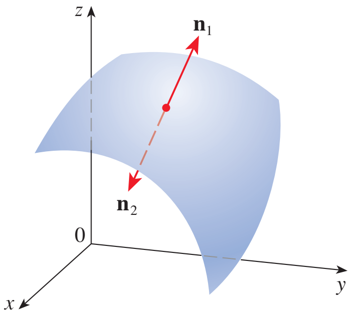


FIGURE 6

From now on we consider only orientable (two-sided) surfaces. We start with a surface S that has a tangent plane at every point (x, y, z) on S (except at any boundary point). There are two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2 = -\mathbf{n}_1$ at (x, y, z). (See Figure 6.)

If it is possible to choose a unit normal vector $\mathbf{n}$ at every such point (x, y, z) so that $\mathbf{n}$ varies continuously over S, then S is called an **oriented surface** and the given choice of $\mathbf{n}$ provides S with an **orientation**. For any orientable surface, there are two possible orientations (see Figure 7).

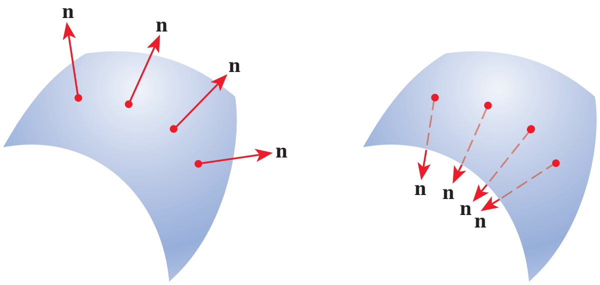


FIGURE 7
The two orientations
of an orientable surface

For a surface $z = g(x, y)$ given as the graph of g, we use Equation 3 to associate with the surface a natural orientation given by the unit normal vector

$$\boxed{5} \qquad \mathbf{n} = \frac{-\dfrac{\partial g}{\partial x}\mathbf{i} - \dfrac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial g}{\partial x}\right)^2 + \left(\dfrac{\partial g}{\partial y}\right)^2}}$$

Since the $\mathbf{k}$-component is positive, this gives the *upward* orientation of the surface.

If S is a smooth orientable surface given in parametric form by a vector function $\mathbf{r}(u, v)$, then it is automatically supplied with the orientation of the unit normal vector

$$\boxed{6} \qquad \mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

and the opposite orientation is given by $-\mathbf{n}$. For instance, in Example 16.6.4 we found the parametric representation

$$\mathbf{r}(\phi, \theta) = a \sin\phi \cos\theta\, \mathbf{i} + a \sin\phi \sin\theta\, \mathbf{j} + a \cos\phi\, \mathbf{k}$$

for the sphere $x^2 + y^2 + z^2 = a^2$. Then in Example 16.6.10 we found that

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2 \sin^2\phi \cos\theta\, \mathbf{i} + a^2 \sin^2\phi \sin\theta\, \mathbf{j} + a^2 \sin\phi \cos\phi\, \mathbf{k}$$

and $$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2 \sin\phi$$

So the orientation induced by $\mathbf{r}(\phi, \theta)$ is defined by the unit normal vector

$$\mathbf{n} = \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} = \sin\phi \cos\theta\, \mathbf{i} + \sin\phi \sin\theta\, \mathbf{j} + \cos\phi\, \mathbf{k} = \frac{1}{a}\mathbf{r}(\phi, \theta)$$

Observe that **n** points in the same direction as the position vector, that is, outward from the sphere (see Figure 8). The opposite (inward) orientation would have been obtained (see Figure 9) if we had reversed the order of the parameters because $\mathbf{r}_\theta \times \mathbf{r}_\phi = -\mathbf{r}_\phi \times \mathbf{r}_\theta$.

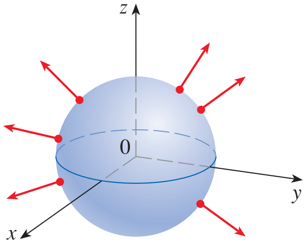


FIGURE 8
Positive orientation

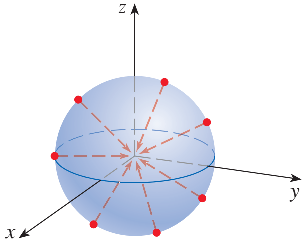


FIGURE 9
Negative orientation

For a **closed surface**, that is, a surface that is the boundary of a solid region E, the convention is that the **positive orientation** is the one for which the normal vectors point *outward* from E, and inward-pointing normals give the negative orientation (see Figures 8 and 9).

Surface Integrals of Vector Fields; Flux

Suppose that S is an oriented surface with unit normal vector **n**, and imagine a fluid with density $\rho(x, y, z)$ and velocity field $\mathbf{v}(x, y, z)$ flowing through S. (Think of S as an imaginary surface that doesn't impede the fluid flow, like a fishing net across a stream.) Then the rate of flow (mass per unit time) per unit area is given by the vector field $\rho\mathbf{v}$. (See Figure 10.)

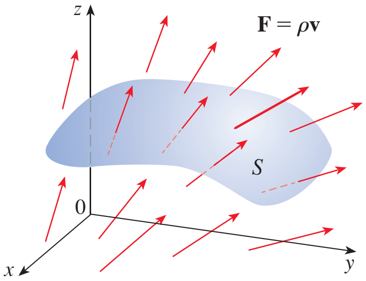


FIGURE 10

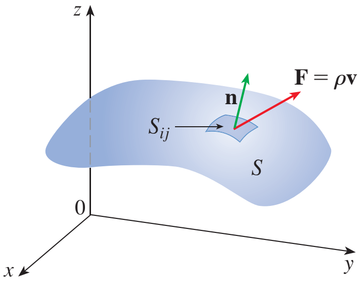


FIGURE 11

If we divide S into small patches S_{ij}, as in Figure 11 (compare with Figure 1), then S_{ij} is nearly planar and so we can approximate the mass of fluid per unit time crossing S_{ij} in the direction of the normal $\mathbf{n}$ by the quantity

$$(\rho\mathbf{v} \cdot \mathbf{n})A(S_{ij})$$

where ρ, $\mathbf{v}$, and $\mathbf{n}$ are evaluated at some point on S_{ij}. (Recall that the component of the vector $\rho\mathbf{v}$ in the direction of the unit vector $\mathbf{n}$ is $\rho\mathbf{v} \cdot \mathbf{n}$.) By summing these quantities and taking the limit we get, according to Definition 1, the surface integral of the function $\rho\mathbf{v} \cdot \mathbf{n}$ over S:

$$\boxed{7} \qquad \iint_S \rho\mathbf{v} \cdot \mathbf{n}\, dS = \iint_S \rho(x, y, z)\mathbf{v}(x, y, z) \cdot \mathbf{n}(x, y, z)\, dS$$

and this is interpreted physically as the rate of flow through S.

If we write $\mathbf{F} = \rho\mathbf{v}$, then $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ and the integral given in Equation 7 becomes

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

A surface integral of this form occurs frequently in physics, even when $\mathbf{F}$ is not $\rho\mathbf{v}$, and is called the *surface integral* (or *flux integral*) of $\mathbf{F}$ over S.

8 Definition If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the **surface integral of F over S** is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

This integral is also called the **flux** of $\mathbf{F}$ across S.

In words, Definition 8 says that the surface integral of a vector field over S is equal to the surface integral of its normal component over S (as previously defined).

If S is given by a vector function $\mathbf{r}(u, v)$, then $\mathbf{n}$ is given by Equation 6, and from Definition 8 and Equation 2 we have

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\, dS \\ &= \iint_D \left[\mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\right] |\mathbf{r}_u \times \mathbf{r}_v|\, dA\end{aligned}$$

where D is the parameter domain. Thus we have

Compare Equation 9 to the similar expression for evaluating line integrals of vector fields in Definition 16.2.13:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$\boxed{9} \qquad \boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA}$$

Formula 9 assumes the orientation of S induced by $\mathbf{r}_u \times \mathbf{r}_v$, as in Equation 6. For the opposite orientation, we multiply by -1.

EXAMPLE 4 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

Figure 12 shows the vector field **F** in Example 4 at points on the unit sphere.

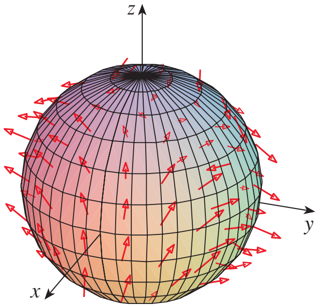


FIGURE 12

SOLUTION As in Example 1, we use the parametric representation

$$\mathbf{r}(\phi, \theta) = \sin\phi\,\cos\theta\,\mathbf{i} + \sin\phi\,\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k} \qquad 0 \leqslant \phi \leqslant \pi \qquad 0 \leqslant \theta \leqslant 2\pi$$

Then $$\mathbf{F}(\mathbf{r}(\phi, \theta)) = \cos\phi\,\mathbf{i} + \sin\phi\,\sin\theta\,\mathbf{j} + \sin\phi\,\cos\theta\,\mathbf{k}$$

and, from Example 16.6.10,

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2\!\phi\,\cos\theta\,\mathbf{i} + \sin^2\!\phi\,\sin\theta\,\mathbf{j} + \sin\phi\,\cos\phi\,\mathbf{k}$$

(You can check that these vectors correspond to the outward orientation of the sphere.) Therefore

$$\mathbf{F}(\mathbf{r}(\phi, \theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = \cos\phi\,\sin^2\!\phi\,\cos\theta + \sin^3\!\phi\,\sin^2\!\theta + \sin^2\!\phi\,\cos\phi\,\cos\theta$$

and, by Formula 9, the flux is

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\,dA \\
&= \int_0^{2\pi}\!\int_0^{\pi} (2\sin^2\!\phi\,\cos\phi\,\cos\theta + \sin^3\!\phi\,\sin^2\!\theta)\,d\phi\,d\theta \\
&= 2\int_0^{\pi} \sin^2\!\phi\,\cos\phi\,d\phi \int_0^{2\pi} \cos\theta\,d\theta + \int_0^{\pi} \sin^3\!\phi\,d\phi \int_0^{2\pi} \sin^2\!\theta\,d\theta \\
&= 0 + \int_0^{\pi} \sin^3\!\phi\,d\phi \int_0^{2\pi} \sin^2\!\theta\,d\theta \qquad \left(\text{since } \int_0^{2\pi} \cos\theta\,d\theta = 0\right) \\
&= \frac{4\pi}{3}
\end{aligned}$$

by the same calculation as in Example 1. ■

If, for instance, the vector field in Example 4 is a velocity field describing the flow of a fluid with density 1, then the answer, $4\pi/3$, represents the rate of flow through the unit sphere in units of mass per unit time.

In the case of a surface S given by a graph $z = g(x, y)$, we can think of x and y as parameters and use Equation 3 to write

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot \left(-\frac{\partial g}{\partial x}\,\mathbf{i} - \frac{\partial g}{\partial y}\,\mathbf{j} + \mathbf{k}\right)$$

Thus Formula 9 becomes

10
$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA}$$

This formula assumes the upward orientation of S; for a downward orientation we multiply by -1. Similar formulas can be worked out if S is given by $y = h(x, z)$ or $x = k(y, z)$. (See Exercises 37 and 38.)

EXAMPLE 5 Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

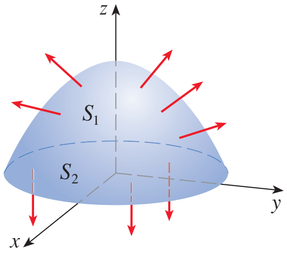


FIGURE 13

SOLUTION S consists of a parabolic top surface S_1 and a circular bottom surface S_2. (See Figure 13.) Since S is a closed surface, we use the convention of positive (outward) orientation. This means that S_1 is oriented upward and we can use Equation 10 with D being the projection of S_1 onto the xy-plane, namely, the disk $x^2 + y^2 \leqslant 1$. Since

$$P(x, y, z) = y \qquad Q(x, y, z) = x \qquad R(x, y, z) = z = 1 - x^2 - y^2$$

on S_1 and

$$\frac{\partial g}{\partial x} = -2x \qquad \frac{\partial g}{\partial y} = -2y$$

we have

$$\begin{aligned}
\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA \\
&= \iint_D [-y(-2x) - x(-2y) + 1 - x^2 - y^2]\, dA \\
&= \iint_D (1 + 4xy - x^2 - y^2)\, dA \\
&= \int_0^{2\pi}\int_0^1 (1 + 4r^2\cos\theta\,\sin\theta - r^2)\, r\, dr\, d\theta \\
&= \int_0^{2\pi}\int_0^1 (r - r^3 + 4r^3\cos\theta\,\sin\theta)\, dr\, d\theta \\
&= \int_0^{2\pi} \left(\tfrac{1}{4} + \cos\theta\,\sin\theta\right) d\theta = \tfrac{1}{4}(2\pi) + 0 = \frac{\pi}{2}
\end{aligned}$$

The disk S_2 is oriented downward, so its unit normal vector is $\mathbf{n} = -\mathbf{k}$ and we have

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot (-\mathbf{k})\, dS = \iint_D (-z)\, dA = \iint_D 0\, dA = 0$$

since $z = 0$ on S_2. Finally, we compute, by definition, $\iint_S \mathbf{F} \cdot d\mathbf{S}$ as the sum of the surface integrals of $\mathbf{F}$ over the pieces S_1 and S_2:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2} + 0 = \frac{\pi}{2}$$ ■

Although we motivated the surface integral of a vector field using the example of fluid flow, this concept also arises in other physical situations. For instance, if $\mathbf{E}$ is an electric field (see Example 16.1.5), then the surface integral

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

is called the **electric flux** of $\mathbf{E}$ through the surface S. One of the important laws of electrostatics is **Gauss's Law**, which says that the net charge enclosed by a closed surface S is

$$\boxed{11} \qquad Q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S}$$

where ε_0 is a constant (called the permittivity of free space) that depends on the units used. (In the SI system, $\varepsilon_0 \approx 8.8542 \times 10^{-12}\ \mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$.) Therefore, if the vector field **F** in Example 4 represents an electric field, we can conclude that the charge enclosed by S is $Q = \frac{4}{3}\pi\varepsilon_0$.

Another application of surface integrals occurs in the study of heat flow. Suppose the temperature at a point (x, y, z) in a body is $u(x, y, z)$. Then the **heat flow** is defined as the vector field

$$\mathbf{F} = -K\,\nabla u$$

where K is an experimentally determined constant called the **conductivity** of the substance. The rate of heat flow across the surface S in the body is then given by the surface integral

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -K \iint_S \nabla u \cdot d\mathbf{S}$$

EXAMPLE 6 The temperature u in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.

SOLUTION Taking the center of the ball to be at the origin, we have

$$u(x, y, z) = C(x^2 + y^2 + z^2)$$

where C is the proportionality constant. Then the heat flow is

$$\mathbf{F}(x, y, z) = -K\,\nabla u = -KC(2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k})$$

where K is the conductivity of the metal. Instead of using the usual parametrization of the sphere as in Example 4, we observe that the outward unit normal to the sphere $x^2 + y^2 + z^2 = a^2$ at the point (x, y, z) is

$$\mathbf{n} = \frac{1}{a}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$$

and so

$$\mathbf{F}\cdot\mathbf{n} = -\frac{2KC}{a}(x^2 + y^2 + z^2)$$

But on S we have $x^2 + y^2 + z^2 = a^2$, so $\mathbf{F}\cdot\mathbf{n} = -2aKC$. Therefore the rate of heat flow across S is

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = -2aKC \iint_S dS \\ &= -2aKCA(S) = -2aKC(4\pi a^2) = -8KC\pi a^3\end{aligned}$$

■

16.7 Exercises

1. Let S be the surface of the box enclosed by the planes $x = \pm 1$, $y = \pm 1$, $z = \pm 1$. Approximate $\iint_S \cos(x + 2y + 3z)\,dS$ by using a Riemann sum as in Definition 1, taking the patches S_{ij} to be the squares that are the faces of the box S and the points P_{ij}^* to be the centers of the squares.

2. A surface S consists of the cylinder $x^2 + y^2 = 1$, $-1 \leqslant z \leqslant 1$, together with its top and bottom disks. Suppose you know that f is a continuous function with

$$f(\pm 1, 0, 0) = 2 \qquad f(0, \pm 1, 0) = 3 \qquad f(0, 0, \pm 1) = 4$$

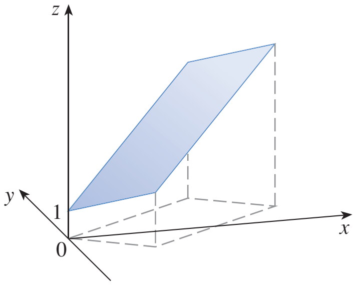
Estimate the value of $\iint_S f(x, y, z)\, dS$ by using a Riemann sum, taking the patches S_{ij} to be four quarter-cylinders and the top and bottom disks.

3. Let H be the hemisphere $x^2 + y^2 + z^2 = 50$, $z \geqslant 0$, and suppose f is a continuous function with $f(3, 4, 5) = 7$, $f(3, -4, 5) = 8$, $f(-3, 4, 5) = 9$, and $f(-3, -4, 5) = 12$. By dividing H into four patches, estimate the value of $\iint_H f(x, y, z)\, dS$.

4. Suppose that $f(x, y, z) = g\left(\sqrt{x^2 + y^2 + z^2}\right)$, where g is a function of one variable such that $g(2) = -5$. Evaluate $\iint_S f(x, y, z)\, dS$, where S is the sphere $x^2 + y^2 + z^2 = 4$.

5–20 Evaluate the surface integral.

5. $\iint_S (x + y + z)\, dS$,
S is the parallelogram with parametric equations $x = u + v$, $y = u - v$, $z = 1 + 2u + v$, $0 \leqslant u \leqslant 2$, $0 \leqslant v \leqslant 1$



6. $\iint_S xyz\, dS$,
S is the cone with parametric equations $x = u \cos v$, $y = u \sin v$, $z = u$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi/2$

7. $\iint_S y\, dS$, S is the helicoid with vector equation $\mathbf{r}(u, v) = \langle u \cos v, u \sin v, v \rangle$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi$

8. $\iint_S (x^2 + y^2)\, dS$,
S is the surface with vector equation
$\mathbf{r}(u, v) = \langle 2uv, u^2 - v^2, u^2 + v^2 \rangle$, $u^2 + v^2 \leqslant 1$

9. $\iint_S x^2 yz\, dS$, S is the part of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $[0, 3] \times [0, 2]$

10. $\iint_S xz\, dS$, S is the part of the plane $2x + 2y + z = 4$ that lies in the first octant

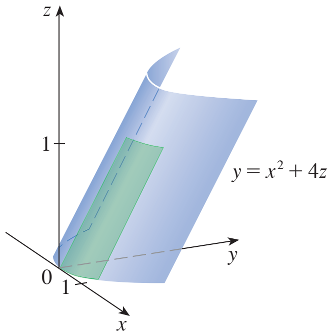
11. $\iint_S x\, dS$,
S is the triangular region with vertices $(1, 0, 0)$, $(0, -2, 0)$, and $(0, 0, 4)$

12. $\iint_S y\, dS$,
S is the surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$

13. $\iint_S z^2\, dS$,
S is the part of the paraboloid $x = y^2 + z^2$ given by $0 \leqslant x \leqslant 1$

14. $\iint_S y^2 z^2\, dS$,
S is the part of the cone $y = \sqrt{x^2 + z^2}$ given by $0 \leqslant y \leqslant 5$

15. $\iint_S x\, dS$,
S is the surface $y = x^2 + 4z$, $0 \leqslant x \leqslant 1$, $0 \leqslant z \leqslant 1$

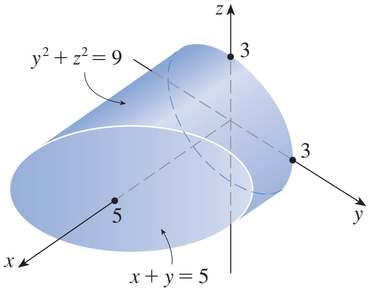


16. $\iint_S y^2\, dS$,
S is the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies above the cone $z = \sqrt{x^2 + y^2}$

17. $\iint_S (x^2 z + y^2 z)\, dS$,
S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geqslant 0$

18. $\iint_S (x + y + z)\, dS$,
S is the part of the half-cylinder $x^2 + z^2 = 1$, $z \geqslant 0$, that lies between the planes $y = 0$ and $y = 2$

19. $\iint_S xz\, dS$,
S is the boundary of the region enclosed by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$ and $x + y = 5$



20. $\iint_S (x^2 + y^2 + z^2)\, dS$,
S is the part of the cylinder $x^2 + y^2 = 9$ between the planes $z = 0$ and $z = 2$, together with its top and bottom disks

---

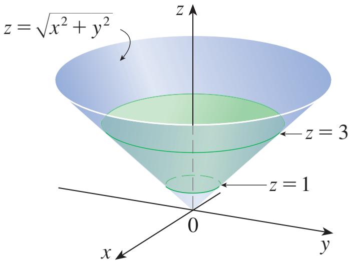
21–32 Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the given vector field $\mathbf{F}$ and the oriented surface S. In other words, find the flux of $\mathbf{F}$ across S. For closed surfaces, use the positive (outward) orientation.

21. $\mathbf{F}(x, y, z) = ze^{xy}\, \mathbf{i} - 3ze^{xy}\, \mathbf{j} + xy\, \mathbf{k}$,
S is the parallelogram of Exercise 5 with upward orientation

22. $\mathbf{F}(x, y, z) = z\, \mathbf{i} + y\, \mathbf{j} + x\, \mathbf{k}$,
S is the helicoid of Exercise 7 with upward orientation

23. $\mathbf{F}(x, y, z) = xy\, \mathbf{i} + yz\, \mathbf{j} + zx\, \mathbf{k}$, S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, and has upward orientation

24. $\mathbf{F}(x, y, z) = -x\,\mathbf{i} - y\,\mathbf{j} + z^3\,\mathbf{k}$, S is the part of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 3$ with downward orientation



25. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z^2\,\mathbf{k}$, S is the sphere with radius 1 and center the origin

26. $\mathbf{F}(x, y, z) = y\,\mathbf{i} - x\,\mathbf{j} + 2z\,\mathbf{k}$, S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geqslant 0$, oriented downward

27. $\mathbf{F}(x, y, z) = y\,\mathbf{j} - z\,\mathbf{k}$,
S consists of the paraboloid $y = x^2 + z^2$, $0 \leqslant y \leqslant 1$, and the disk $x^2 + z^2 \leqslant 1$, $y = 1$

28. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k}$, S is the surface $z = x \sin y$, $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant \pi$, with upward orientation

29. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}$,
S is the cube with vertices $(\pm 1, \pm 1, \pm 1)$

30. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 5\,\mathbf{k}$, S is the boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $x + y = 2$

31. $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + y^2\mathbf{j} + z^2\,\mathbf{k}$, S is the boundary of the solid half-cylinder $0 \leqslant z \leqslant \sqrt{1 - y^2}$, $0 \leqslant x \leqslant 2$

32. $\mathbf{F}(x, y, z) = y\,\mathbf{i} + (z - y)\,\mathbf{j} + x\,\mathbf{k}$,
S is the surface of the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

T **33.** Use a computer algebra system to evaluate $\iint_S (x^2 + y^2 + z^2)\, dS$ correct to four decimal places, where S is the surface $z = xe^y$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.

T **34.** Use a computer algebra system to find the exact value of $\iint_S xyz\, dS$, where S is the surface $z = x^2y^2$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2$.

T **35.** Use a computer algebra system to find the value of $\iint_S x^2y^2z^2\, dS$ correct to four decimal places, where S is the part of the paraboloid $z = 3 - 2x^2 - y^2$ that lies above the xy-plane.

T **36.** Use a computer algebra system to find the flux of

$$\mathbf{F}(x, y, z) = \sin(xyz)\,\mathbf{i} + x^2y\,\mathbf{j} + z^2e^{x/5}\,\mathbf{k}$$

across the part of the cylinder $4y^2 + z^2 = 4$ that lies above the xy-plane and between the planes $x = -2$ and $x = 2$ with upward orientation. Illustrate by graphing the cylinder and the vector field on the same screen.

37. Find a formula for $\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to Formula 10 for the case where S is given by $y = h(x, z)$ and $\mathbf{n}$ is the unit normal that points toward the left (when the axes are drawn in the usual way).

38. Find a formula for $\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to Formula 10 for the case where S is given by $x = k(y, z)$ and $\mathbf{n}$ is the unit normal that points forward (that is, toward the viewer when the axes are drawn in the usual way).

39. Find the center of mass of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geqslant 0$, if it has constant density.

40. Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2 + y^2}$, $1 \leqslant z \leqslant 4$, if its density function is $\rho(x, y, z) = 10 - z$.

41. (a) Give an integral expression for the moment of inertia I_z about the z-axis of a thin sheet in the shape of a surface S if the density function is ρ.
(b) Find the moment of inertia about the z-axis of the funnel in Exercise 40.

42. Let S be the part of the sphere $x^2 + y^2 + z^2 = 25$ that lies above the plane $z = 4$. If S has constant density k, find (a) the center of mass and (b) the moment of inertia about the z-axis.

43. A fluid has density 870 kg/m^3 and flows with velocity $\mathbf{v} = z\,\mathbf{i} + y^2\,\mathbf{j} + x^2\,\mathbf{k}$, where x, y, and z are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the cylinder $x^2 + y^2 = 4$, $0 \leqslant z \leqslant 1$.

44. Seawater has density 1025 kg/m^3 and flows in a velocity field $\mathbf{v} = y\,\mathbf{i} + x\,\mathbf{j}$, where x, y, and z are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geqslant 0$.

45. Use Gauss's Law to find the charge contained in the solid hemisphere $x^2 + y^2 + z^2 \leqslant a^2$, $z \geqslant 0$, if the electric field is

$$\mathbf{E}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}$$

46. Use Gauss's Law to find the charge enclosed by the cube with vertices $(\pm 1, \pm 1, \pm 1)$ if the electric field is

$$\mathbf{E}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

47. The temperature at the point (x, y, z) in a substance with conductivity $K = 6.5$ is $u(x, y, z) = 2y^2 + 2z^2$. Find the rate of heat flow inward across the cylindrical surface $y^2 + z^2 = 6$, $0 \leqslant x \leqslant 4$.

48. The temperature at a point in a ball with conductivity K is inversely proportional to the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.

49. Let $\mathbf{F}$ be an inverse square field, that is, $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\,\mathbf{r}\,|^3$ for some constant c, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Show that the flux of $\mathbf{F}$ across a sphere S with center the origin is independent of the radius of S.

16.8 Stokes' Theorem

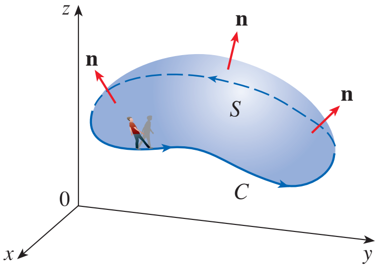


FIGURE 1

Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem. Whereas Green's Theorem relates a double integral over a plane region D to a line integral around its plane boundary curve, Stokes' Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (which is a space curve). Figure 1 shows an oriented surface with unit normal vector $\mathbf{n}$. The orientation of S induces the **positive orientation of the boundary curve C** shown in the figure. This means that if you walk in the positive direction around C with your head pointing in the direction of $\mathbf{n}$, then the surface will always be on your left.

Stokes' Theorem Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

Since

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds \qquad \text{and} \qquad \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS$$

Stokes' Theorem says that the line integral around the boundary curve of S of the tangential component of $\mathbf{F}$ is equal to the surface integral over S of the normal component of the curl of $\mathbf{F}$.

The positively oriented boundary curve of the oriented surface S is often written as ∂S, so Stokes' Theorem can be expressed as

$$\boxed{1} \qquad \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

There is an analogy among Stokes' Theorem, Green's Theorem, and the Fundamental Theorem of Calculus. As before, there is an integral involving derivatives on the left side of Equation 1 (recall that curl $\mathbf{F}$ is a sort of derivative of $\mathbf{F}$) and the right side involves the values of $\mathbf{F}$ only on the *boundary* of S.

In fact, in the special case where the surface S is flat and lies in the xy-plane with upward orientation, the unit normal is $\mathbf{k}$, the surface integral becomes a double integral, and Stokes' Theorem becomes

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\, dA$$

This is precisely the vector form of Green's Theorem given in Equation 16.5.12. Thus we see that Green's Theorem is really a special case of Stokes' Theorem.

Although Stokes' Theorem is too difficult for us to prove in its full generality, we can give a proof when S is a graph and $\mathbf{F}$, S, and C are well behaved.

PROOF OF A SPECIAL CASE OF STOKES' THEOREM We assume that the equation of S is $z = g(x, y)$, $(x, y) \in D$, where g has continuous second-order partial derivatives and D is a simple plane region whose boundary curve C_1 corresponds to C. If the orientation

George Stokes

Stokes' Theorem is named after the Irish mathematical physicist Sir George Stokes (1819–1903). Stokes was a professor at Cambridge University (in fact he held the same position as Newton, Lucasian Professor of Mathematics) and was especially noted for his studies of fluid flow and light. What we call Stokes' Theorem was actually discovered by the Scottish physicist Sir William Thomson (1824–1907, known as Lord Kelvin). Stokes learned of this theorem in a letter from Thomson in 1850 and asked students to prove it on an examination at Cambridge University in 1854. We don't know if any of those students was able to do so.

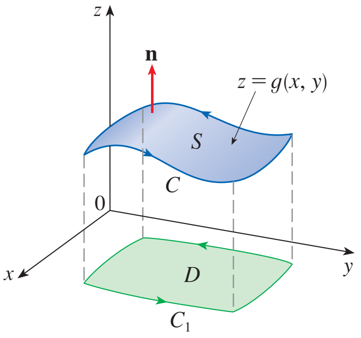


FIGURE 2

of S is upward, then the positive orientation of C corresponds to the positive orientation of C_1. (See Figure 2.) We are also given that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, where the partial derivatives of P, Q, and R are continuous.

Since S is a graph of a function, we can apply Formula 16.7.10 with $\mathbf{F}$ replaced by curl $\mathbf{F}$. The result is

$$\boxed{2}\quad \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$$
$$= \iint_D \left[-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\frac{\partial z}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\frac{\partial z}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] dA$$

where the partial derivatives of P, Q, and R are evaluated at $(x, y, g(x, y))$. If

$$x = x(t) \qquad y = y(t) \qquad a \leqslant t \leqslant b$$

is a parametric representation of C_1, then a parametric representation of C is

$$x = x(t) \qquad y = y(t) \qquad z = g(x(t), y(t)) \qquad a \leqslant t \leqslant b$$

This allows us, with the aid of the Chain Rule, to evaluate the line integral as follows:

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right) dt \\
&= \int_a^b \left[P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right)\right] dt \\
&= \int_a^b \left[\left(P + R\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \left(Q + R\frac{\partial z}{\partial y}\right)\frac{dy}{dt}\right] dt \\
&= \int_{C_1} \left(P + R\frac{\partial z}{\partial x}\right) dx + \left(Q + R\frac{\partial z}{\partial y}\right) dy \\
&= \iint_D \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x}\right)\right] dA
\end{aligned}$$

where we have used Green's Theorem in the last step. Then, using the Chain Rule again and remembering that P, Q, and R are functions of x, y, and z and that z is itself a function of x and y, we get

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_D \Bigg[&\left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + R\frac{\partial^2 z}{\partial x\,\partial y}\right) \\
&- \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + R\frac{\partial^2 z}{\partial y\,\partial x}\right)\Bigg] dA
\end{aligned}$$

Four of the terms in this double integral cancel and the remaining six terms can be arranged to coincide with the right side of Equation 2. Therefore

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$$

■

EXAMPLE 1 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient C to be counterclockwise when viewed from above.)

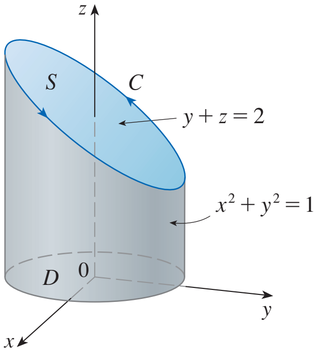


FIGURE 3

SOLUTION The curve C (an ellipse) is shown in Figure 3. Although $\int_C \mathbf{F} \cdot d\mathbf{r}$ could be evaluated directly, it's easier to use Stokes' Theorem. We first compute

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1 + 2y)\,\mathbf{k}$$

Stokes' Theorem allows us to choose any (oriented, piecewise-smooth) surface with boundary curve C. Among the many possible such surfaces, the most convenient choice is the elliptical region S in the plane $y + z = 2$ that is bounded by C. If we orient S upward, then C has the induced positive orientation. The projection D of S onto the xy-plane is the disk $x^2 + y^2 \leqslant 1$ and so using Equation 16.7.10 with $z = g(x, y) = 2 - y$, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} = \iint_D (1 + 2y)\,dA$$

$$= \int_0^{2\pi}\!\int_0^1 (1 + 2r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} + 2\frac{r^3}{3}\sin\theta\right]_0^1 d\theta = \int_0^{2\pi} \left(\tfrac{1}{2} + \tfrac{2}{3}\sin\theta\right) d\theta$$

$$= \tfrac{1}{2}(2\pi) + 0 = \pi$$

■

NOTE Stokes' Theorem allows us to compute a surface integral simply by knowing the values of $\mathbf{F}$ on the boundary curve C. This means that if we have another oriented surface with the same boundary curve C, then we get exactly the same value for the surface integral. In general, if S_1 and S_2 are oriented surfaces with the same oriented boundary curve C and both satisfy the hypotheses of Stokes' Theorem, then

$$\boxed{3} \qquad \iint_{S_1} \text{curl}\,\mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \text{curl}\,\mathbf{F} \cdot d\mathbf{S}$$

This fact is useful when it is difficult to integrate over one surface but easy to integrate over the other.

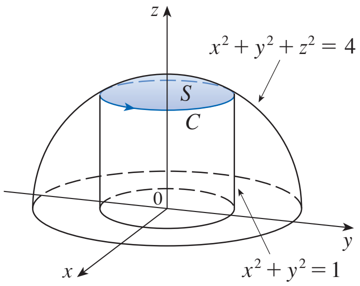


FIGURE 4

EXAMPLE 2 Use Stokes' Theorem to compute the integral $\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + xy\,\mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane. (See Figure 4.)

SOLUTION 1 To find the boundary curve C we solve the equations $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$. Subtracting, we get $z^2 = 3$ and so $z = \sqrt{3}$ (since $z > 0$). Thus C is the circle given by the equations $x^2 + y^2 = 1$, $z = \sqrt{3}$. A vector equation of C is

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \sqrt{3}\,\mathbf{k} \qquad 0 \leqslant t \leqslant 2\pi$$

so

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

Also, we have

$$\mathbf{F}(\mathbf{r}(t)) = \sqrt{3}\cos t\,\mathbf{i} + \sqrt{3}\sin t\,\mathbf{j} + \cos t\sin t\,\mathbf{k}$$

Therefore, by Stokes' Theorem,

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$$

$$= \int_0^{2\pi} \left(-\sqrt{3}\cos t\sin t + \sqrt{3}\sin t\cos t\right)dt = \sqrt{3}\int_0^{2\pi} 0\,dt = 0$$

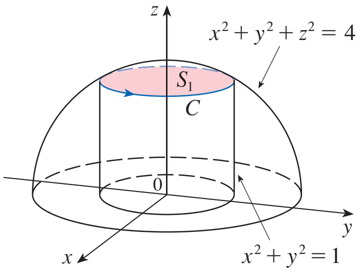


FIGURE 5

SOLUTION 2 Let S_1 be the disk in the plane $z = \sqrt{3}$ inside the cylinder $x^2 + y^2 = 1$, as shown in Figure 5. Since S_1 and S have the same boundary curve C, it follows by Stokes' Theorem that

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$$

Because S_1 is part of a horizontal plane, its upward normal is $\mathbf{k}$. We calculate that curl $\mathbf{F} = (x - y)\,\mathbf{i} + (x - y)\,\mathbf{j}$, so

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS$$

$$= \iint_{S_1} [(x - y)\,\mathbf{i} + (x - y)\,\mathbf{j}]\cdot\mathbf{k}\,dS = \iint_{S_1} 0\,dS = 0 \qquad \blacksquare$$

We now use Stokes' Theorem to shed some light on the meaning of the curl vector. Suppose that C is an oriented closed curve and $\mathbf{v}$ represents the velocity field in fluid flow. Consider the line integral

$$\int_C \mathbf{v}\cdot d\mathbf{r} = \int_C \mathbf{v}\cdot\mathbf{T}\,ds$$

and recall that $\mathbf{v}\cdot\mathbf{T}$ is the component of $\mathbf{v}$ in the direction of the unit tangent vector $\mathbf{T}$. This means that the closer the direction of $\mathbf{v}$ is to the direction of $\mathbf{T}$, the larger the value of $\mathbf{v}\cdot\mathbf{T}$. (Recall that if $\mathbf{v}$ and $\mathbf{T}$ point in generally opposite directions, then $\mathbf{v}\cdot\mathbf{T}$ is negative.) Thus $\int_C \mathbf{v}\cdot d\mathbf{r}$ is a measure of the tendency of the fluid to move around C in the same direction as the orientation of C, and is called the **circulation** of $\mathbf{v}$ around C. (See Figure 6.)

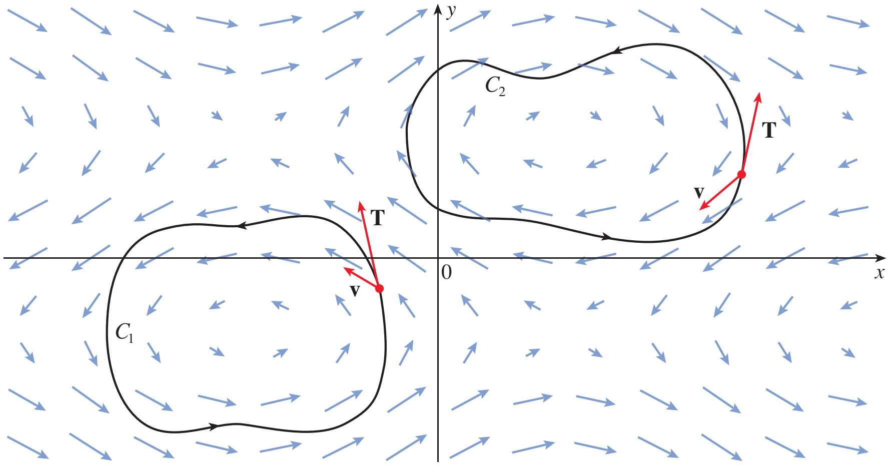


FIGURE 6

$\int_{C_1} \mathbf{v}\cdot d\mathbf{r} > 0$, positive circulation

$\int_{C_2} \mathbf{v}\cdot d\mathbf{r} < 0$, negative circulation

Now let $P_0(x_0, y_0, z_0)$ be a point in the fluid and let S_a be a small disk with radius a and center P_0. Then $(\text{curl } \mathbf{F})(P) \approx (\text{curl } \mathbf{F})(P_0)$ for all points P on S_a because curl $\mathbf{F}$ is continuous. Thus, by Stokes' Theorem, we get the following approximation to the circulation around the boundary circle C_a:

$$\begin{aligned}\int_{C_a} \mathbf{v} \cdot d\mathbf{r} &= \iint_{S_a} \text{curl } \mathbf{v} \cdot d\mathbf{S} = \iint_{S_a} \text{curl } \mathbf{v} \cdot \mathbf{n} \, dS \\ &\approx \iint_{S_a} \text{curl } \mathbf{v}(P_0) \cdot \mathbf{n}(P_0) \, dS = \text{curl } \mathbf{v}(P_0) \cdot \mathbf{n}(P_0) \pi a^2\end{aligned}$$

Imagine a tiny paddle wheel placed in the fluid at a point P, as in Figure 7; the paddle wheel rotates fastest when its axis is parallel to curl **v**.

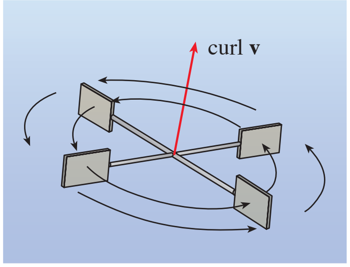


FIGURE 7

This approximation becomes better as $a \to 0$ and we have

$$\boxed{4} \qquad \text{curl } \mathbf{v}(P_0) \cdot \mathbf{n}(P_0) = \lim_{a \to 0} \frac{1}{\pi a^2} \int_{C_a} \mathbf{v} \cdot d\mathbf{r}$$

Equation 4 gives the relationship between the curl and the circulation. It shows that curl $\mathbf{v} \cdot \mathbf{n}$ is a measure of the rotating effect of the fluid about the axis $\mathbf{n}$. The curling effect is greatest about the axis parallel to curl **v**.

Finally, we mention that Stokes' Theorem can be used to prove Theorem 16.5.4 (which states that if curl $\mathbf{F} = \mathbf{0}$ on all of $\mathbb{R}^3$, then $\mathbf{F}$ is conservative). From our previous work (Theorems 16.3.3 and 16.3.4), we know that $\mathbf{F}$ is conservative if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C. Given C, suppose we can find an orientable surface S whose boundary is C. (This can be done, but the proof requires advanced techniques.) Then Stokes' Theorem gives

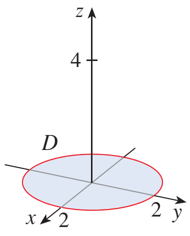
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{0} \cdot d\mathbf{S} = 0$$

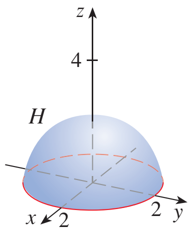
A curve that is not simple can be broken into a number of simple curves, and the integrals around these simple curves are all 0. Adding these integrals, we obtain $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C.

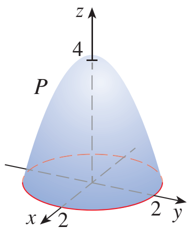
16.8 Exercises

1. A disk D, a hemisphere H, and a portion P of a paraboloid are shown. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Explain why this statement is true:

$$\iint_D \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_H \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_P \text{curl } \mathbf{F} \cdot d\mathbf{S}$$







2–6 Use Stokes' Theorem to evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$.

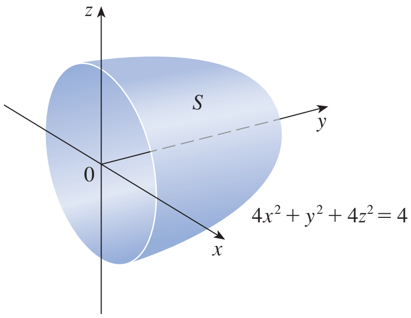
2. $\mathbf{F}(x, y, z) = x^2 \sin z \, \mathbf{i} + y^2 \, \mathbf{j} + xy \, \mathbf{k}$,
S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane, oriented upward

3. $\mathbf{F}(x, y, z) = ze^y \, \mathbf{i} + x \cos y \, \mathbf{j} + xz \sin y \, \mathbf{k}$,
S is the hemisphere $x^2 + y^2 + z^2 = 16$, $y \geqslant 0$, oriented in the direction of the positive y-axis

4. $\mathbf{F}(x, y, z) = \tan^{-1}(x^2yz^2) \, \mathbf{i} + x^2y \, \mathbf{j} + x^2z^2 \, \mathbf{k}$,
S is the cone $x = \sqrt{y^2 + z^2}$, $0 \leqslant x \leqslant 2$, oriented in the direction of the positive x-axis

5. $\mathbf{F}(x, y, z) = xyz \, \mathbf{i} + xy \, \mathbf{j} + x^2yz \, \mathbf{k}$,
S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outward

6. $\mathbf{F}(x, y, z) = e^{xy}\,\mathbf{i} + e^{xz}\,\mathbf{j} + x^2z\,\mathbf{k}$,
S is the half of the ellipsoid $4x^2 + y^2 + 4z^2 = 4$ that lies to the right of the xz-plane, oriented in the direction of the positive y-axis

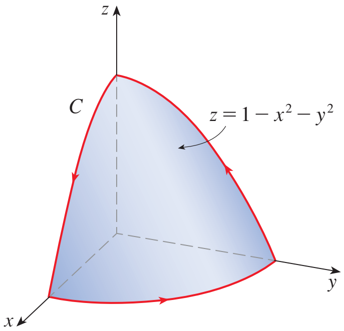


7–14 Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. In each case C is oriented counterclockwise as viewed from above, unless otherwise stated.

7. $\mathbf{F}(x, y, z) = (x + y^2)\,\mathbf{i} + (y + z^2)\,\mathbf{j} + (z + x^2)\,\mathbf{k}$,
C is the triangle with vertices (1, 0, 0), (0, 1, 0), and (0, 0, 1)

8. $\mathbf{F}(x, y, z) = \mathbf{i} + (x + yz)\,\mathbf{j} + \left(xy - \sqrt{z}\right)\mathbf{k}$,
C is the boundary of the part of the plane $3x + 2y + z = 1$ in the first octant

9. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$,
C is the boundary of the part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant

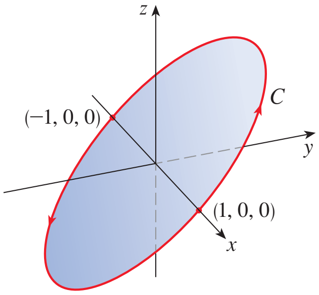


10. $\mathbf{F}(x, y, z) = 2y\,\mathbf{i} + xz\,\mathbf{j} + (x + y)\,\mathbf{k}$,
C is the curve of intersection of the plane $z = y + 2$ and the cylinder $x^2 + y^2 = 1$

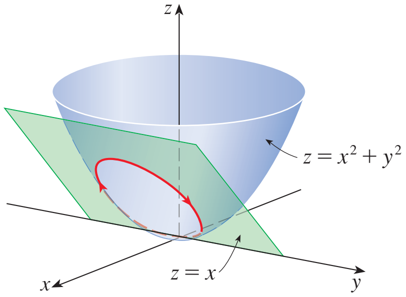
11. $\mathbf{F}(x, y, z) = \langle -yx^2, xy^2, e^{xy} \rangle$, C is the circle in the xy-plane of radius 2 centered at the origin

12. $\mathbf{F}(x, y, z) = ze^x\,\mathbf{i} + (z - y^3)\,\mathbf{j} + (x - z^3)\,\mathbf{k}$,
C is the circle $y^2 + z^2 = 4$, $x = 3$, oriented clockwise as viewed from the origin

13. $\mathbf{F}(x, y, z) = x^2y\,\mathbf{i} + x^3\,\mathbf{j} + e^z \tan^{-1}z\,\mathbf{k}$,
C is the curve with parametric equations $x = \cos t$, $y = \sin t$, $z = \sin t$, $0 \leqslant t \leqslant 2\pi$



14. $\mathbf{F}(x, y, z) = \langle x^3 - z, xy, y + z^2 \rangle$, C is the curve of intersection of the paraboloid $z = x^2 + y^2$ and the plane $z = x$



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15. (a) Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = x^2z\,\mathbf{i} + xy^2\,\mathbf{j} + z^2\,\mathbf{k}$$

and C is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$, oriented counterclockwise as viewed from above.

(b) Graph both the plane and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).

(c) Find parametric equations for C and use them to graph C.

16. (a) Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x^2y\,\mathbf{i} + \frac{1}{3}x^3\,\mathbf{j} + xy\,\mathbf{k}$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

(b) Graph both the hyperbolic paraboloid and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).

(c) Find parametric equations for C and use them to graph C.

17–19 Verify that Stokes' Theorem is true for the given vector field $\mathbf{F}$ and surface S.

17. $\mathbf{F}(x, y, z) = -y\,\mathbf{i} + x\,\mathbf{j} - 2\,\mathbf{k}$,
S is the cone $z^2 = x^2 + y^2$, $0 \leqslant z \leqslant 4$, oriented downward

18. $\mathbf{F}(x, y, z) = -2yz\,\mathbf{i} + y\,\mathbf{j} + 3x\,\mathbf{k}$,
S is the part of the paraboloid $z = 5 - x^2 - y^2$ that lies above the plane $z = 1$, oriented upward

19. $\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$,
S is the hemisphere $x^2 + y^2 + z^2 = 1$, $y \geqslant 0$, oriented in the direction of the positive y-axis

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20. Let C be a simple closed smooth curve that lies in the plane $x + y + z = 1$. Show that the line integral

$$\int_C z\,dx - 2x\,dy + 3y\,dz$$

depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane.

21. A particle moves along line segments from the origin to the points $(1, 0, 0)$, $(1, 2, 1)$, $(0, 2, 1)$, and back to the origin under the influence of the force field

$$\mathbf{F}(x, y, z) = z^2\,\mathbf{i} + 2xy\,\mathbf{j} + 4y^2\,\mathbf{k}$$

Find the work done.

22. Evaluate

$$\int_C (y + \sin x)\,dx + (z^2 + \cos y)\,dy + x^3\,dz$$

where C is the curve $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle$, $0 \leqslant t \leqslant 2\pi$. [*Hint:* Observe that C lies on the surface $z = 2xy$.]

23. If S is a sphere and $\mathbf{F}$ satisfies the hypotheses of Stokes' Theorem, show that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

24. Suppose S and C satisfy the hypotheses of Stokes' Theorem and f, g have continuous second-order partial derivatives. Use Exercises 26 and 28 in Section 16.5 to show the following.

(a) $\int_C (f\,\nabla g) \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot d\mathbf{S}$

(b) $\int_C (f\,\nabla f) \cdot d\mathbf{r} = 0$

(c) $\int_C (f\,\nabla g + g\,\nabla f) \cdot d\mathbf{r} = 0$

16.9 The Divergence Theorem

In Section 16.5 we rewrote Green's Theorem in a vector version as

$$\int_C \mathbf{F} \cdot \mathbf{n}\,ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\,dA$$

where C is the positively oriented boundary curve of the plane region D. If we were seeking to extend this theorem to vector fields on $\mathbb{R}^3$, we might make the guess that

$$\boxed{1} \qquad \iint_S \mathbf{F} \cdot \mathbf{n}\,dS = \iiint_E \operatorname{div} \mathbf{F}(x, y, z)\,dV$$

where S is the boundary surface of the solid region E. It turns out that Equation 1 is true, under appropriate hypotheses, and is called the Divergence Theorem. Notice its similarity to Green's Theorem and Stokes' Theorem in that it relates the integral of a derivative of a function ($\operatorname{div} \mathbf{F}$ in this case) over a region to the integral of the original function $\mathbf{F}$ over the boundary of the region.

At this stage you may wish to review the various types of regions over which we were able to evaluate triple integrals in Section 15.6. We state and prove the Divergence Theorem for regions E that are simultaneously of types 1, 2, and 3 and we call such regions **simple solid regions**. (For instance, regions bounded by ellipsoids or rectangular boxes are simple solid regions.) The boundary of E is a closed surface, and we use the convention, introduced in Section 16.7, that the positive orientation is outward; that is, the unit normal vector $\mathbf{n}$ is directed outward from E.

The Divergence Theorem is sometimes called Gauss's Theorem after the great German mathematician Karl Friedrich Gauss (1777–1855), who discovered this theorem during his investigation of electrostatics. In Eastern Europe the Divergence Theorem is known as Ostrogradsky's Theorem after the Russian mathematician Mikhail Ostrogradsky (1801–1862), who published this result in 1826.

The Divergence Theorem Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\,dV$$

Thus the Divergence Theorem states that, under the given conditions, the flux of **F** across the boundary surface of E is equal to the triple integral of the divergence of **F** over E.

PROOF Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$. Then

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

so

$$\iiint_E \operatorname{div} \mathbf{F}\,dV = \iiint_E \frac{\partial P}{\partial x}\,dV + \iiint_E \frac{\partial Q}{\partial y}\,dV + \iiint_E \frac{\partial R}{\partial z}\,dV$$

If **n** is the unit outward normal of S, then the surface integral on the left side of the Divergence Theorem is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\,dS = \iint_S (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot \mathbf{n}\,dS$$

$$= \iint_S P\,\mathbf{i} \cdot \mathbf{n}\,dS + \iint_S Q\,\mathbf{j} \cdot \mathbf{n}\,dS + \iint_S R\,\mathbf{k} \cdot \mathbf{n}\,dS$$

Therefore, to prove the Divergence Theorem, it suffices to prove the following three equations:

$$\boxed{2} \qquad \iint_S P\,\mathbf{i} \cdot \mathbf{n}\,dS = \iiint_E \frac{\partial P}{\partial x}\,dV$$

$$\boxed{3} \qquad \iint_S Q\,\mathbf{j} \cdot \mathbf{n}\,dS = \iiint_E \frac{\partial Q}{\partial y}\,dV$$

$$\boxed{4} \qquad \iint_S R\,\mathbf{k} \cdot \mathbf{n}\,dS = \iiint_E \frac{\partial R}{\partial z}\,dV$$

To prove Equation 4 we use the fact that E is a type 1 region:

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

where D is the projection of E onto the xy-plane. By Equation 15.6.6, we have

$$\iiint_E \frac{\partial R}{\partial z}\,dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} \frac{\partial R}{\partial z}(x, y, z)\,dz \right] dA$$

and therefore, by the Fundamental Theorem of Calculus,

$$\boxed{5} \qquad \iiint_E \frac{\partial R}{\partial z}\,dV = \iint_D \left[R\big(x, y, u_2(x, y)\big) - R\big(x, y, u_1(x, y)\big) \right] dA$$

The boundary surface S consists of three pieces: the bottom surface S_1, the top surface S_2, and possibly a vertical surface S_3, which lies above the boundary curve of D. (See Figure 1. It might happen that S_3 doesn't appear, as in the case of a sphere.) Notice that on S_3 we have $\mathbf{k} \cdot \mathbf{n} = 0$, because **k** is vertical and **n** is horizontal, and so

$$\iint_{S_3} R\,\mathbf{k} \cdot \mathbf{n}\,dS = \iint_{S_3} 0\,dS = 0$$

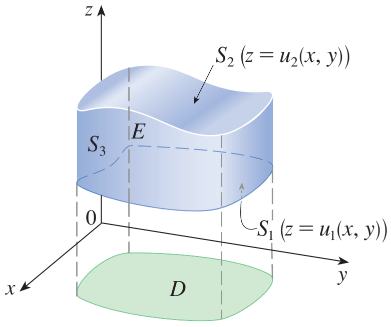


FIGURE 1

Thus, regardless of whether there is a vertical surface, we can write

$$\boxed{6} \qquad \iint_S R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iint_{S_1} R\,\mathbf{k}\cdot\mathbf{n}\,dS + \iint_{S_2} R\,\mathbf{k}\cdot\mathbf{n}\,dS$$

The equation of S_2 is $z = u_2(x, y)$, $(x, y) \in D$, and the outward normal $\mathbf{n}$ points upward, so from Equation 16.7.10 (with $\mathbf{F}$ replaced by $R\,\mathbf{k}$) we have

$$\iint_{S_2} R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iint_D R(x, y, u_2(x, y))\,dA$$

On S_1 we have $z = u_1(x, y)$, but here the outward normal $\mathbf{n}$ points downward, so we multiply by -1:

$$\iint_{S_1} R\,\mathbf{k}\cdot\mathbf{n}\,dS = -\iint_D R(x, y, u_1(x, y))\,dA$$

Therefore Equation 6 gives

$$\iint_S R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iint_D \left[R(x, y, u_2(x, y)) - R(x, y, u_1(x, y))\right] dA$$

Comparison with Equation 5 shows that

$$\iint_S R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iiint_E \frac{\partial R}{\partial z}\,dV$$

Notice that the method of proof of the Divergence Theorem is very similar to that of Green's Theorem.

Equations 2 and 3 are proved in a similar manner using the expressions for E as a type 2 or type 3 region, respectively. ■

EXAMPLE 1 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

SOLUTION First we compute the divergence of $\mathbf{F}$:

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1$$

The solution in Example 1 should be compared with the solution in Example 16.7.4.

The unit sphere S is the boundary of the unit ball B given by $x^2 + y^2 + z^2 \leqslant 1$. Thus the Divergence Theorem gives the flux as

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_B \operatorname{div}\mathbf{F}\,dV = \iiint_B 1\,dV = V(B) = \tfrac{4}{3}\pi(1)^3 = \frac{4\pi}{3} \qquad ■$$

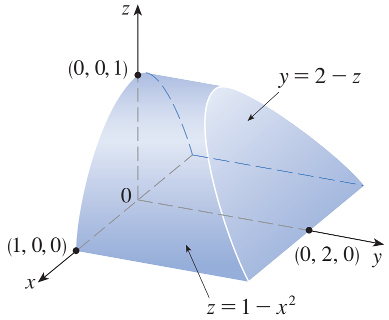


FIGURE 2

EXAMPLE 2 Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy\,\mathbf{i} + \left(y^2 + e^{xz^2}\right)\mathbf{j} + \sin(xy)\,\mathbf{k}$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$. (See Figure 2.)

SOLUTION It would be extremely difficult to evaluate the given surface integral directly. (We would have to evaluate four surface integrals corresponding to the four pieces of S.) Furthermore, the divergence of $\mathbf{F}$ is much less complicated than $\mathbf{F}$ itself:

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(y^2 + e^{xz^2}\right) + \frac{\partial}{\partial z}(\sin xy) = y + 2y = 3y$$

Therefore we use the Divergence Theorem to transform the given surface integral into a triple integral. The easiest way to evaluate the triple integral is to express E as a type 3 region:

$$E = \left\{(x, y, z) \mid -1 \leqslant x \leqslant 1,\ 0 \leqslant z \leqslant 1 - x^2,\ 0 \leqslant y \leqslant 2 - z\right\}$$

Then we have

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E 3y\, dV \\
&= 3 \int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} y\, dy\, dz\, dx = 3 \int_{-1}^{1} \int_0^{1-x^2} \frac{(2-z)^2}{2}\, dz\, dx \\
&= \frac{3}{2} \int_{-1}^{1} \left[-\frac{(2-z)^3}{3}\right]_0^{1-x^2} dx = -\tfrac{1}{2} \int_{-1}^{1} \left[(x^2+1)^3 - 8\right] dx \\
&= -\int_0^1 (x^6 + 3x^4 + 3x^2 - 7)\, dx = \frac{184}{35}
\end{aligned}$$

■

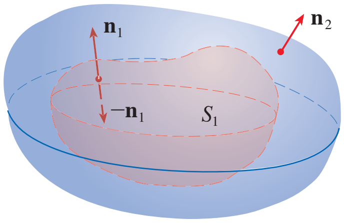


FIGURE 3

Although we have proved the Divergence Theorem only for simple solid regions, it can be proved for regions that are finite unions of simple solid regions. (The procedure is similar to the one we used in Section 16.4 to extend Green's Theorem.)

For example, let's consider the region E that lies between the closed surfaces S_1 and S_2, where S_1 lies inside S_2. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be outward normals of S_1 and S_2. Then the boundary surface of E is $S = S_1 \cup S_2$ and its normal $\mathbf{n}$ is given by $\mathbf{n} = -\mathbf{n}_1$ on S_1 and $\mathbf{n} = \mathbf{n}_2$ on S_2. (See Figure 3.) Applying the Divergence Theorem to S, we get

$$\begin{aligned}
\boxed{7} \qquad \iiint_E \operatorname{div} \mathbf{F}\, dV &= \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS \\
&= \iint_{S_1} \mathbf{F} \cdot (-\mathbf{n}_1)\, dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS \\
&= -\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}
\end{aligned}$$

EXAMPLE 3 In Example 16.1.5 we considered the electric field

$$\mathbf{E}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\, \mathbf{x}$$

where the electric charge Q is located at the origin and $\mathbf{x} = \langle x, y, z \rangle$ is a position vector. Use the Divergence Theorem to show that the electric flux of $\mathbf{E}$ through any closed surface S that encloses the origin is

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = 4\pi\varepsilon Q$$

SOLUTION The difficulty is that we don't have an explicit equation for S because it is *any* closed surface enclosing the origin. Let S_1 be a sphere centered at the origin with radius a, where a is chosen to be small enough so that S_1 is contained within S. Let E be the region that lies between S_1 and S. Then Equation 7 gives

$$\boxed{8} \qquad \iiint_E \operatorname{div} \mathbf{E}\, dV = -\iint_{S_1} \mathbf{E} \cdot d\mathbf{S} + \iint_S \mathbf{E} \cdot dS$$

You can verify that div $\mathbf{E} = 0$. (See Exercise 25.) Therefore from (8) we have

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot d\mathbf{S}$$

The point of this calculation is that we can compute the surface integral over S_1 because S_1 is a sphere. The normal vector at $\mathbf{x}$ is $\mathbf{x}/|\mathbf{x}|$. Therefore

$$\mathbf{E} \cdot \mathbf{n} = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x} \cdot \left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) = \frac{\varepsilon Q}{|\mathbf{x}|^4}\mathbf{x} \cdot \mathbf{x} = \frac{\varepsilon Q}{|\mathbf{x}|^2} = \frac{\varepsilon Q}{a^2}$$

since the equation of S_1 is $|\mathbf{x}| = a$. Thus we have

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathbf{n}\, dS = \frac{\varepsilon Q}{a^2}\iint_{S_1} dS = \frac{\varepsilon Q}{a^2}A(S_1) = \frac{\varepsilon Q}{a^2}4\pi a^2 = 4\pi\varepsilon Q$$

This shows that the electric flux of $\mathbf{E}$ is $4\pi\varepsilon Q$ through *any* closed surface S that contains the origin. [This is a special case of Gauss's Law (Equation 16.7.11) for a single charge. The relationship between ε and ε_0 is $\varepsilon = 1/(4\pi\varepsilon_0)$.] ■

Another application of the Divergence Theorem occurs in fluid flow. Let $\mathbf{v}(x, y, z)$ be the velocity field of a fluid with constant density ρ. Then $\mathbf{F} = \rho\mathbf{v}$ is the rate of flow per unit area. If $P_0(x_0, y_0, z_0)$ is a point in the fluid and B_a is a ball with center P_0 and very small radius a, then div $\mathbf{F}(P) \approx$ div $\mathbf{F}(P_0)$ for all points P in B_a since div $\mathbf{F}$ is continuous. We approximate the flux over the boundary sphere S_a as follows:

$$\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} = \iiint_{B_a} \operatorname{div} \mathbf{F}\, dV \approx \iiint_{B_a} \operatorname{div} \mathbf{F}(P_0)\, dV = \operatorname{div} \mathbf{F}(P_0)V(B_a)$$

This approximation becomes better as $a \to 0$ and suggests that

$$\boxed{9} \qquad \operatorname{div} \mathbf{F}(P_0) = \lim_{a \to 0} \frac{1}{V(B_a)} \iint_{S_a} \mathbf{F} \cdot d\mathbf{S}$$

Equation 9 says that div $\mathbf{F}(P_0)$ is the net rate of outward flux per unit volume at P_0. (This is the reason for the name *divergence.*) If div $\mathbf{F}(P) > 0$, the net flow is outward near P and P is called a **source**. If div $\mathbf{F}(P) < 0$, the net flow is inward near P and P is called a **sink**.

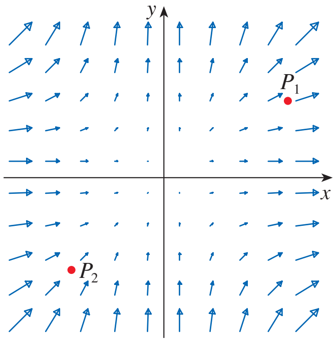


FIGURE 4

The vector field $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$

For the vector field in Figure 4, it appears that the vectors that end near P_1 are shorter than the vectors that start near P_1. Thus the net flow is outward near P_1, so div $\mathbf{F}(P_1) > 0$ and P_1 is a source. Near P_2, on the other hand, the incoming arrows are longer than the outgoing arrows. Here the net flow is inward, so div $\mathbf{F}(P_2) < 0$ and P_2 is a sink. We can use the formula for $\mathbf{F}$ to confirm this impression. Since $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$, we have div $\mathbf{F} = 2x + 2y$, which is positive when $y > -x$. So the points above the line $y = -x$ are sources and those below are sinks.

16.9 Exercises

1–4 Verify that the Divergence Theorem is true for the vector field **F** on the region E.

1. $\mathbf{F}(x, y, z) = 3x\,\mathbf{i} + xy\,\mathbf{j} + 2xz\,\mathbf{k}$,
E is the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$

2. $\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2yz\,\mathbf{j} + 4z^2\,\mathbf{k}$,
E is the solid enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 9$

3. $\mathbf{F}(x, y, z) = \langle z, y, x\rangle$,
E is the solid ball $x^2 + y^2 + z^2 \leqslant 16$

4. $\mathbf{F}(x, y, z) = \langle x^2, -y, z\rangle$,
E is the solid cylinder $y^2 + z^2 \leqslant 9$, $0 \leqslant x \leqslant 2$

5–17 Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$; that is, calculate the flux of **F** across S.

5. $\mathbf{F}(x, y, z) = xye^z\,\mathbf{i} + xy^2z^3\,\mathbf{j} - ye^z\,\mathbf{k}$,
S is the surface of the box bounded by the coordinate planes and the planes $x = 3$, $y = 2$, and $z = 1$

6. $\mathbf{F}(x, y, z) = x^2yz\,\mathbf{i} + xy^2z\,\mathbf{j} + xyz^2\,\mathbf{k}$,
S is the surface of the box enclosed by the planes $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$, and $z = c$, where a, b, and c are positive numbers

7. $\mathbf{F}(x, y, z) = 3xy^2\,\mathbf{i} + xe^z\,\mathbf{j} + z^3\,\mathbf{k}$,
S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$

8. $\mathbf{F}(x, y, z) = (x^3 + y^3)\,\mathbf{i} + (y^3 + z^3)\,\mathbf{j} + (z^3 + x^3)\,\mathbf{k}$,
S is the sphere with center the origin and radius 2

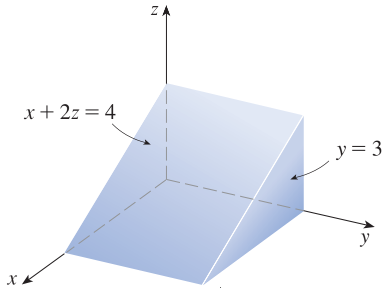
9. $\mathbf{F}(x, y, z) = xe^y\,\mathbf{i} + (z - e^y)\,\mathbf{j} - xy\,\mathbf{k}$,
S is the ellipsoid $x^2 + 2y^2 + 3z^2 = 4$

10. $\mathbf{F}(x, y, z) = e^y\tan z\,\mathbf{i} + x^2y\,\mathbf{j} + e^x\cos y\,\mathbf{k}$,
S is the surface of the solid that lies above the xy-plane and below the surface $z = 2 - x - y^3$, $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$

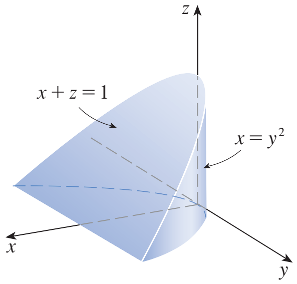
11. $\mathbf{F}(x, y, z) = (2x^3 + y^3)\,\mathbf{i} + (y^3 + z^3)\,\mathbf{j} + 3y^2z\,\mathbf{k}$,
S is the surface of the solid bounded by the paraboloid $z = 1 - x^2 - y^2$ and the xy-plane

12. $\mathbf{F}(x, y, z) = (xy + 2xz)\,\mathbf{i} + (x^2 + y^2)\,\mathbf{j} + (xy - z^2)\,\mathbf{k}$,
S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = y - 2$ and $z = 0$

13. $\mathbf{F}(x, y, z) = x^2z\,\mathbf{i} + xz^3\,\mathbf{j} + y\ln(x + 1)\,\mathbf{k}$,
S is the surface of the solid bounded by the planes $x + 2z = 4$, $y = 3$, $x = 0$, $y = 0$, and $z = 0$



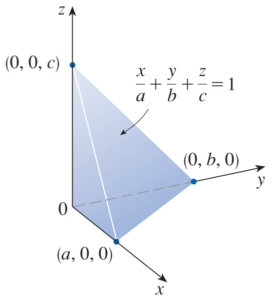
14. $\mathbf{F}(x, y, z) = (xy - z^2)\,\mathbf{i} + x^3\sqrt{z}\,\mathbf{j} + (xy + z^2)\,\mathbf{k}$,
S is the surface of the solid bounded by the cylinder $x = y^2$ and the planes $x + z = 1$ and $z = 0$



15. $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + zx\,\mathbf{k}$,
S is the surface of the tetrahedron enclosed by the coordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where a, b, and c are positive numbers



16. $\mathbf{F} = |\mathbf{r}|^2\mathbf{r}$, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$,
S is the sphere with radius R and center the origin

17. $\mathbf{F} = |\mathbf{r}|\,\mathbf{r}$, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$,
S consists of the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and the disk $x^2 + y^2 \leqslant 1$ in the xy-plane

T **18.** Plot the vector field

$$\mathbf{F}(x, y, z) = \sin x \cos^2 y\,\mathbf{i} + \sin^3 y \cos^4 z\,\mathbf{j} + \sin^5 z \cos^6 x\,\mathbf{k}$$

in the cube cut from the first octant by the planes $x = \pi/2$, $y = \pi/2$, and $z = \pi/2$. Then use a computer algebra system to compute the flux across the surface of the cube.

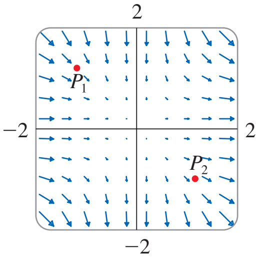
19. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = z^2 x\,\mathbf{i} + \left(\tfrac{1}{3}y^3 + \tan^{-1}z\right)\mathbf{j} + (x^2 z + y^2)\,\mathbf{k}$$

and S is the top half of the sphere $x^2 + y^2 + z^2 = 1$. [*Hint:* Note that S is not a closed surface. First compute integrals over S_1 and S_2, where S_1 is the disk $x^2 + y^2 \leqslant 1$, oriented downward, and $S_2 = S \cup S_1$.]

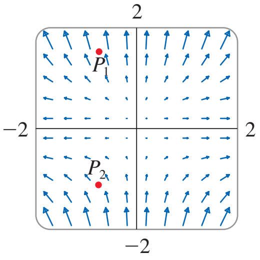
20. Let $\mathbf{F}(x, y, z) = z \tan^{-1}(y^2)\,\mathbf{i} + z^3 \ln(x^2 + 1)\,\mathbf{j} + z\,\mathbf{k}$. Find the flux of $\mathbf{F}$ across the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$ and is oriented upward.

21. A vector field $\mathbf{F}$ is shown. Use the interpretation of divergence derived in this section to determine whether the points P_1 and P_2 are sources or sinks.



22. (a) Are the points P_1 and P_2 sources or sinks for the vector field $\mathbf{F}$ shown in the figure? Give an explanation based solely on the picture.

(b) Given that $\mathbf{F}(x, y) = \langle x, y^2 \rangle$, use the definition of divergence to verify your answer to part (a).



23–24 Plot the vector field and guess where $\operatorname{div} \mathbf{F} > 0$ and where $\operatorname{div} \mathbf{F} < 0$. Then calculate $\operatorname{div} \mathbf{F}$ to check your guess.

23. $\mathbf{F}(x, y) = \langle xy, x + y^2 \rangle$ **24.** $\mathbf{F}(x, y) = \langle x^2, y^2 \rangle$

25. Verify that $\operatorname{div} \mathbf{E} = 0$ for the electric field $\mathbf{E}(\mathbf{x}) = \dfrac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x}$.

26. Use the Divergence Theorem to evaluate

$$\iint_S (2x + 2y + z^2)\, dS$$

where S is the sphere $x^2 + y^2 + z^2 = 1$.

27–32 Prove each identity, assuming that S and E satisfy the conditions of the Divergence Theorem and the scalar functions and components of the vector fields have continuous second-order partial derivatives.

27. $\displaystyle\iint_S \mathbf{a} \cdot \mathbf{n}\, dS = 0$, where $\mathbf{a}$ is a constant vector

28. $\displaystyle V(E) = \frac{1}{3}\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

29. $\displaystyle\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$

30. $\displaystyle\iint_S D_{\mathbf{n}} f\, dS = \iiint_E \nabla^2 f\, dV$

31. $\displaystyle\iint_S (f\nabla g) \cdot \mathbf{n}\, dS = \iiint_E (f\nabla^2 g + \nabla f \cdot \nabla g)\, dV$

32. $\displaystyle\iint_S (f\nabla g - g\nabla f) \cdot \mathbf{n}\, dS = \iiint_E (f\nabla^2 g - g\nabla^2 f)\, dV$

33. Suppose S and E satisfy the conditions of the Divergence Theorem and f is a scalar function with continuous partial derivatives. Prove that

$$\iint_S f\mathbf{n}\, dS = \iiint_E \nabla f\, dV$$

These surface and triple integrals of vector functions are vectors defined by integrating each component function. [*Hint:* Start by applying the Divergence Theorem to $\mathbf{F} = f\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant vector.]

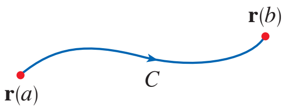
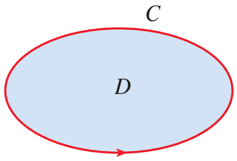
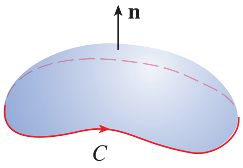
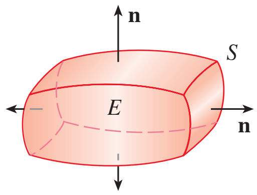
34. A solid occupies a region E with surface S and is immersed in a liquid with constant density ρ. We set up a coordinate system so that the xy-plane coincides with the surface of the liquid, and positive values of z are measured downward into the liquid. Then the pressure at depth z is $p = \rho g z$, where g is the acceleration due to gravity (see Section 8.3). The total buoyant force on the solid due to the pressure distribution is given by the surface integral

$$\mathbf{F} = -\iint_S p\mathbf{n}\, dS$$

where $\mathbf{n}$ is the outer unit normal. Use the result of Exercise 33 to show that $\mathbf{F} = -W\mathbf{k}$, where W is the weight of the liquid displaced by the solid. (Note that $\mathbf{F}$ is directed upward because z is directed downward.) The result is *Archimedes' Principle:* the buoyant force on an object equals the weight of the displaced liquid.

16.10 Summary

The main results of this chapter are all higher-dimensional versions of the Fundamental Theorem of Calculus. To help you remember them, we collect them together here (without hypotheses) so that you can see more easily their essential similarity. Notice that in each case we have an integral of a "derivative" over a region on the left side, and the right side involves the values of the original function only on the *boundary* of the region.

Curves and their boundaries (endpoints)		
Fundamental Theorem of Calculus	$\displaystyle\int_a^b F'(x)\,dx = F(b) - F(a)$	
Fundamental Theorem for Line Integrals	$\displaystyle\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$	
Surfaces and their boundaries		
Green's Theorem	$\displaystyle\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$	
Stokes' Theorem	$\displaystyle\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$	
Solids and their boundaries		
Divergence Theorem	$\displaystyle\iiint_E \operatorname{div} \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$	

16 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. What is a vector field? Give three examples that have physical meaning.

2. (a) What is a conservative vector field?
(b) What is a potential function?

3. (a) Write the definition of the line integral of a scalar function f along a smooth curve C with respect to arc length.
(b) How do you evaluate such a line integral?
(c) Write expressions for the mass and center of mass of a thin wire shaped like a curve C if the wire has linear density function $\rho(x, y)$.
(d) Write the definitions of the line integrals along C of a scalar function f with respect to x, y, and z.
(e) How do you evaluate these line integrals?

4. (a) Define the line integral of a vector field $\mathbf{F}$ along a smooth curve C given by a vector function $\mathbf{r}(t)$.
(b) If $\mathbf{F}$ is a force field, what does this line integral represent?
(c) If $\mathbf{F} = \langle P, Q, R \rangle$, what is the connection between the line integral of $\mathbf{F}$ and the line integrals of the component functions P, Q, and R?

5. State the Fundamental Theorem for Line Integrals.

6. (a) What does it mean to say that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path?
(b) If you know that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, what can you say about $\mathbf{F}$?

7. State Green's Theorem.

8. Write expressions for the area enclosed by a curve C in terms of line integrals around C.

9. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$.
(a) Define curl $\mathbf{F}$. (b) Define div $\mathbf{F}$.
(c) If $\mathbf{F}$ is a velocity field in fluid flow, what are the physical interpretations of curl $\mathbf{F}$ and div $\mathbf{F}$?

10. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, how do you determine whether $\mathbf{F}$ is conservative? What if $\mathbf{F}$ is a vector field on $\mathbb{R}^3$?

11. (a) What is a parametric surface? What are its grid curves?
(b) Write an expression for the area of a parametric surface.
(c) What is the area of a surface given by an equation $z = g(x, y)$?

12. (a) Write the definition of the surface integral of a scalar function f over a surface S.
(b) How do you evaluate such an integral if S is a parametric surface given by a vector function $\mathbf{r}(u, v)$?
(c) What if S is given by an equation $z = g(x, y)$?
(d) If a thin sheet has the shape of a surface S, and the density at (x, y, z) is $\rho(x, y, z)$, write expressions for the mass and center of mass of the sheet.

13. (a) What is an oriented surface? Give an example of a non-orientable surface.
(b) Define the surface integral (or flux) of a vector field $\mathbf{F}$ over an oriented surface S with unit normal vector $\mathbf{n}$.
(c) How do you evaluate such an integral if S is a parametric surface given by a vector function $\mathbf{r}(u, v)$?
(d) What if S is given by an equation $z = g(x, y)$?

14. State Stokes' Theorem.

15. State the Divergence Theorem.

16. In what ways are the Fundamental Theorem for Line Integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem similar?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If $\mathbf{F}$ is a vector field, then div $\mathbf{F}$ is a vector field.

2. If $\mathbf{F}$ is a vector field, then curl $\mathbf{F}$ is a vector field.

3. If f has continuous partial derivatives of all orders on $\mathbb{R}^3$, then $\operatorname{div}(\operatorname{curl} \nabla f) = 0$.

4. If f has continuous partial derivatives on $\mathbb{R}^3$ and C is any circle, then $\int_C \nabla f \cdot d\mathbf{r} = 0$.

5. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ and $P_y = Q_x$ in an open region D, then $\mathbf{F}$ is conservative.

6. $\int_{-C} f(x, y)\, ds = -\int_C f(x, y)\, ds$

7. If $\mathbf{F}$ and $\mathbf{G}$ are vector fields and $\operatorname{div}\mathbf{F} = \operatorname{div}\mathbf{G}$, then $\mathbf{F} = \mathbf{G}$.

8. The work done by a conservative force field in moving a particle around a closed path is zero.

9. If $\mathbf{F}$ and $\mathbf{G}$ are vector fields, then

$$\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl} \mathbf{F} + \operatorname{curl} \mathbf{G}$$

10. If $\mathbf{F}$ and $\mathbf{G}$ are vector fields, then

$$\operatorname{curl}(\mathbf{F} \cdot \mathbf{G}) = \operatorname{curl} \mathbf{F} \cdot \operatorname{curl} \mathbf{G}$$

11. If S is a sphere and $\mathbf{F}$ is a constant vector field, then $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$.

12. There is a vector field $\mathbf{F}$ such that

$$\operatorname{curl} \mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

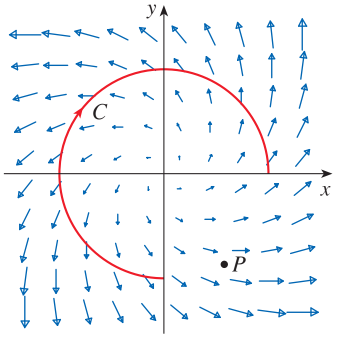
13. The area of the region bounded by the positively oriented, piecewise smooth, simple closed curve C is $A = \oint_C y\,dx$.

EXERCISES

1. A vector field $\mathbf{F}$, a curve C, and a point P are shown.

(a) Is $\int_C \mathbf{F} \cdot d\mathbf{r}$ positive, negative, or zero? Explain.

(b) Is $\operatorname{div} \mathbf{F}(P)$ positive, negative, or zero? Explain.



2–9 Evaluate the line integral.

2. $\int_C x\,ds$,
C is the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$

3. $\int_C yz \cos x\,ds$,
C: $x = t,\ \ y = 3\cos t,\ \ z = 3\sin t,\ \ 0 \leqslant t \leqslant \pi$

4. $\int_C y\,dx + (x + y^2)\,dy$, C is the ellipse $4x^2 + 9y^2 = 36$ with counterclockwise orientation

5. $\int_C y^3\,dx + x^2\,dy$, C is the arc of the parabola $x = 1 - y^2$ from $(0, -1)$ to $(0, 1)$

6. $\int_C \sqrt{xy}\,dx + e^y\,dy + xz\,dz$,
C is given by $\mathbf{r}(t) = t^4\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$, $0 \leqslant t \leqslant 1$

7. $\int_C xy\,dx + y^2\,dy + yz\,dz$,
C is the line segment from $(1, 0, -1)$, to $(3, 4, 2)$

8. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = xy\,\mathbf{i} + x^2\,\mathbf{j}$ and C is given by $\mathbf{r}(t) = \sin t\,\mathbf{i} + (1 + t)\,\mathbf{j}$, $0 \leqslant t \leqslant \pi$

9. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = e^z\,\mathbf{i} + xz\,\mathbf{j} + (x + y)\,\mathbf{k}$ and C is given by $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} - t\,\mathbf{k}$, $0 \leqslant t \leqslant 1$

10. Find the work done by the force field

$$\mathbf{F}(x, y, z) = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$$

in moving a particle from the point $(3, 0, 0)$ to the point $(0, \pi/2, 3)$ along each path.

(a) A straight line

(b) The helix $x = 3\cos t,\ \ y = t,\ \ z = 3\sin t$

11–12 Show that $\mathbf{F}$ is a conservative vector field. Then find a function f such that $\mathbf{F} = \nabla f$.

11. $\mathbf{F}(x, y) = (1 + xy)e^{xy}\,\mathbf{i} + (e^y + x^2 e^{xy})\,\mathbf{j}$

12. $\mathbf{F}(x, y, z) = \sin y\,\mathbf{i} + x\cos y\,\mathbf{j} - \sin z\,\mathbf{k}$

13–14 Show that $\mathbf{F}$ is conservative and use this fact to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve.

13. $\mathbf{F}(x, y) = (4x^3y^2 - 2xy^3)\,\mathbf{i} + (2x^4y - 3x^2y^2 + 4y^3)\,\mathbf{j}$,
C: $\mathbf{r}(t) = (t + \sin \pi t)\,\mathbf{i} + (2t + \cos \pi t)\,\mathbf{j}$, $0 \leqslant t \leqslant 1$

14. $\mathbf{F}(x, y, z) = e^y\,\mathbf{i} + (xe^y + e^z)\,\mathbf{j} + ye^z\,\mathbf{k}$,
C is the line segment from $(0, 2, 0)$ to $(4, 0, 3)$

15. Verify that Green's Theorem is true for the line integral $\int_C xy^2\,dx - x^2y\,dy$, where C consists of the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$ and the line segment from $(1, 1)$ to $(-1, 1)$.

16. Use Green's Theorem to evaluate

$$\int_C \sqrt{1 + x^3}\,dx + 2xy\,dy$$

where C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$.

17. Use Green's Theorem to evaluate $\int_C x^2y\,dx - xy^2\,dy$, where C is the circle $x^2 + y^2 = 4$ with counterclockwise orientation.

18. Find $\operatorname{curl} \mathbf{F}$ and $\operatorname{div} \mathbf{F}$ if

$$\mathbf{F}(x, y, z) = e^{-x}\sin y\,\mathbf{i} + e^{-y}\sin z\,\mathbf{j} + e^{-z}\sin x\,\mathbf{k}$$

19. Show that there is no vector field $\mathbf{G}$ such that

$$\operatorname{curl} \mathbf{G} = 2x\,\mathbf{i} + 3yz\,\mathbf{j} - xz^2\,\mathbf{k}$$

20. If $\mathbf{F}$ and $\mathbf{G}$ are vector fields whose component functions have continuous first partial derivatives, show that

$$\operatorname{curl}(\mathbf{F} \times \mathbf{G}) = \mathbf{F} \operatorname{div} \mathbf{G} - \mathbf{G} \operatorname{div} \mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$$

21. If C is any piecewise-smooth simple closed plane curve and f and g are differentiable functions, show that $\int_C f(x)\,dx + g(y)\,dy = 0$.

22. If f and g are twice differentiable functions, show that

$$\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$$

23. If f is a harmonic function, that is, $\nabla^2 f = 0$, show that the line integral $\int f_y\,dx - f_x\,dy$ is independent of path in any simple region D.

24. (a) Sketch the curve C with parametric equations

$$x = \cos t \qquad y = \sin t \qquad z = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

(b) Find $\int_C 2xe^{2y}\,dx + (2x^2e^{2y} + 2y\cot z)\,dy - y^2\csc^2 z\,dz$.

25. Find the area of the part of the surface $z = x^2 + 2y$ that lies above the triangle with vertices (0, 0), (1, 0), and (1, 2).

26. (a) Find an equation of the tangent plane at the point $(4, -2, 1)$ to the parametric surface S given by

$$\mathbf{r}(u, v) = v^2\,\mathbf{i} - uv\,\mathbf{j} + u^2\,\mathbf{k}$$
$$0 \leqslant u \leqslant 3,\ -3 \leqslant v \leqslant 3$$

(b) Graph the surface S and the tangent plane found in part (a).

(c) Set up, but do not evaluate, an integral for the surface area of S.

(d) If

$$\mathbf{F}(x, y, z) = \frac{z^2}{1 + x^2}\,\mathbf{i} + \frac{x^2}{1 + y^2}\,\mathbf{j} + \frac{y^2}{1 + z^2}\,\mathbf{k}$$

use a computer algebra system to find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ correct to four decimal places.

27–30 Evaluate the surface integral.

27. $\iint_S z\,dS$, where S is the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 4$

28. $\iint_S (x^2z + y^2z)\,dS$, where S is the part of the plane $z = 4 + x + y$ that lies inside the cylinder $x^2 + y^2 = 4$

29. $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz\,\mathbf{i} - 2y\,\mathbf{j} + 3x\,\mathbf{k}$ and S is the sphere $x^2 + y^2 + z^2 = 4$ with outward orientation

30. $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + xy\,\mathbf{j} + z\,\mathbf{k}$ and S is the part of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$ with upward orientation

31. Verify that Stokes' Theorem is true for the vector field $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$, where S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane and S has upward orientation.

32. Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^2yz\,\mathbf{i} + yz^2\,\mathbf{j} + z^3e^{xy}\,\mathbf{k}$, S is the part of the sphere $x^2 + y^2 + z^2 = 5$ that lies above the plane $z = 1$, and S is oriented upward.

33. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the triangle with vertices (1, 0, 0), (0, 1, 0), and (0, 0, 1), oriented counterclockwise as viewed from above.

34. Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$ and S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 2$.

35. Verify that the Divergence Theorem is true for the vector field $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, where E is the unit ball $x^2 + y^2 + z^2 \leqslant 1$.

36. Compute the outward flux of

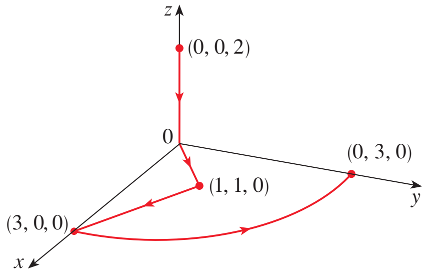
$$\mathbf{F}(x, y, z) = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$$

through the ellipsoid $4x^2 + 9y^2 + 6z^2 = 36$.

37. Let

$$\mathbf{F}(x, y, z) = (3x^2yz - 3y)\,\mathbf{i} + (x^3z - 3x)\,\mathbf{j} + (x^3y + 2z)\,\mathbf{k}$$

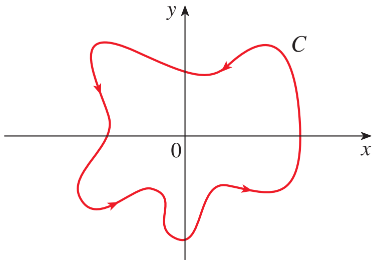
Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve with initial point (0, 0, 2) and terminal point (0, 3, 0) shown in the figure.



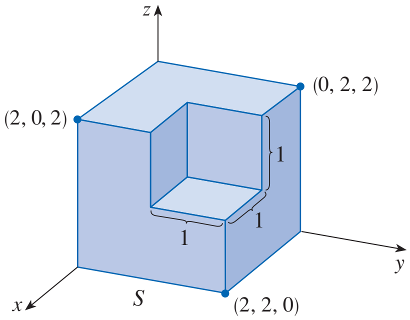
38. Let

$$\mathbf{F}(x, y) = \frac{(2x^3 + 2xy^2 - 2y)\,\mathbf{i} + (2y^3 + 2x^2y + 2x)\,\mathbf{j}}{x^2 + y^2}$$

Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is shown in the figure.



39. Find $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and S is the outwardly oriented surface shown in the figure (the boundary surface of a cube with a unit corner cube removed).



40. If the components of $\mathbf{F}$ have continuous second partial derivatives and S is the boundary surface of a simple solid region, show that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

41. If $\mathbf{a}$ is a constant vector, $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, and S is an oriented, smooth surface with a simple, closed, smooth, positively oriented boundary curve C, show that

$$\iint_S 2\mathbf{a} \cdot d\mathbf{S} = \int_C (\mathbf{a} \times \mathbf{r}) \cdot d\mathbf{r}$$

Problems Plus

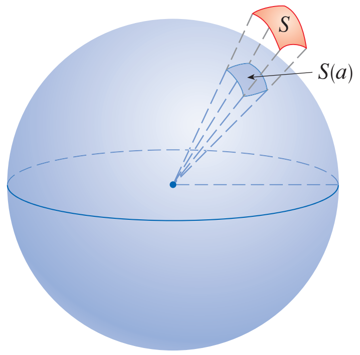


FIGURE FOR PROBLEM 1

1. Let S be a smooth parametric surface and let P be a point such that each line that starts at P intersects S at most once. The **solid angle** $\Omega(S)$ subtended by S at P is the set of lines starting at P and passing through S. Let $S(a)$ be the intersection of $\Omega(S)$ with the surface of the sphere with center P and radius a. Then the measure of the solid angle (in *steradians*) is defined to be

$$|\Omega(S)| = \frac{\text{area of } S(a)}{a^2}$$

Apply the Divergence Theorem to the part of $\Omega(S)$ between $S(a)$ and S to show that

$$|\Omega(S)| = \iint_S \frac{\mathbf{r} \cdot \mathbf{n}}{r^3}\, dS$$

where $\mathbf{r}$ is the radius vector from P to any point on S, $r = |\mathbf{r}|$, and the unit normal vector $\mathbf{n}$ is directed away from P.

This shows that the definition of the measure of a solid angle is independent of the radius a of the sphere. Thus the measure of the solid angle is equal to the area subtended on a *unit* sphere. (Note the analogy with the definition of radian measure.) The total solid angle subtended by a sphere at its center is thus 4π steradians.

2. Find the positively oriented simple closed curve C for which the value of the line integral

$$\int_C (y^3 - y)\, dx - 2x^3\, dy$$

is a maximum.

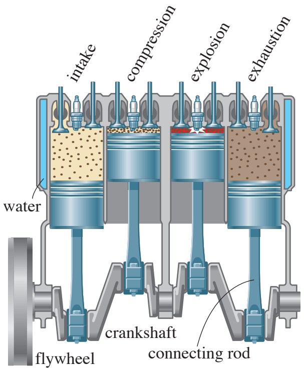
3. Let C be a simple closed piecewise-smooth space curve that lies in a plane with unit normal vector $\mathbf{n} = \langle a, b, c \rangle$ and has positive orientation with respect to $\mathbf{n}$. Show that the plane area enclosed by C is

$$\tfrac{1}{2} \int_C (bz - cy)\, dx + (cx - az)\, dy + (ay - bx)\, dz$$

4. Investigate the shape of the surface with parametric equations $x = \sin u$, $y = \sin v$, $z = \sin(u + v)$. Start by graphing the surface from several points of view. Explain the appearance of the graphs by determining the traces in the horizontal planes $z = 0$, $z = \pm 1$, and $z = \pm\frac{1}{2}$.

5. Prove the following identity:

$$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times \operatorname{curl} \mathbf{G} + \mathbf{G} \times \operatorname{curl} \mathbf{F}$$



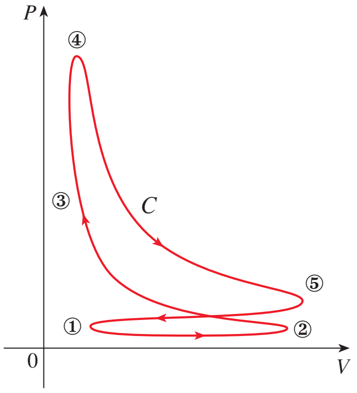


FIGURE FOR PROBLEM 6

6. The figure depicts the sequence of events in each cylinder of a four-cylinder internal combustion engine. Each piston moves up and down and is connected by a pivoted arm to a rotating crankshaft. Let $P(t)$ and $V(t)$ be the pressure and volume within a cylinder at time t, where $a \leqslant t \leqslant b$ gives the time required for a complete cycle. The graph shows how P and V vary through one cycle of a four-stroke engine.

During the intake stroke (from ① to ②) a mixture of air and gasoline at atmospheric pressure is drawn into a cylinder through the intake valve as the piston moves downward. Then the piston rapidly compresses the mix with the valves closed in the compression stroke (from ② to ③) during which the pressure rises and the volume decreases. At ③ the sparkplug ignites the fuel, raising the temperature and pressure at almost constant volume to ④. Then, with valves closed, the rapid expansion forces the piston downward during the power stroke (from ④ to ⑤). The exhaust valve opens, temperature and pressure drop, and mechanical energy stored in a rotating flywheel pushes the piston upward, forcing the waste products out of the exhaust valve in the exhaust stroke. The exhaust valve closes and the intake valve opens. We're now back at ① and the cycle starts again.

(a) Show that the work done on the piston during one cycle of a four-stroke engine is $W = \int_C P\,dV$, where C is the curve in the PV-plane shown in the figure.

[*Hint:* Let $x(t)$ be the distance from the piston to the top of the cylinder and note that the force on the piston is $\mathbf{F} = AP(t)\,\mathbf{i}$, where A is the area of the top of the piston. Then $W = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}$, where C_1 is given by $\mathbf{r}(t) = x(t)\,\mathbf{i}$, $a \leqslant t \leqslant b$. An alternative approach is to work directly with Riemann sums.]

(b) Use Formula 16.4.5 to show that the work is the difference of the areas enclosed by the two loops of C.

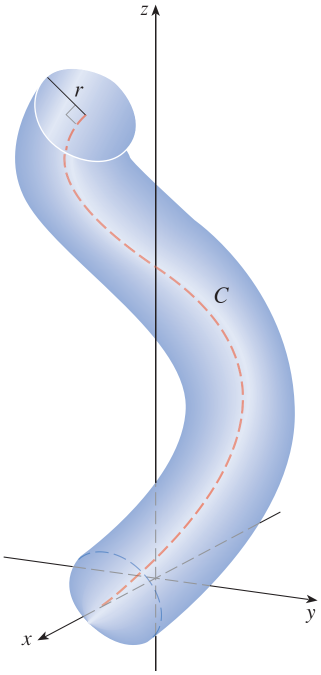


FIGURE FOR PROBLEM 7

7. The set of all points within a perpendicular distance r from a smooth simple curve C in $\mathbb{R}^3$ form a "tube," which we denote by Tube(C, r); see the figure at the left. (We assume that r is small enough that the tube does not intersect itself.) It may seem that the volume of such a tube would depend on the twists and turns of C, but in this problem you will find a formula for the volume of Tube(C, r) which, perhaps surprisingly, depends only on r and the length of C. We assume that C is parameterized with respect to arc length s as $\mathbf{r}(s)$, where $a \leqslant s \leqslant b$, so the arc length of C is $L = b - a$.

(a) Show that the surface of Tube(C, q) is parameterized by

$$\mathbf{X}(u, v) = \mathbf{r}(u) + q\cos v\,\mathbf{N}(u) + q\sin v\,\mathbf{B}(u) \qquad a \leqslant u \leqslant b,\ 0 \leqslant v \leqslant 2\pi$$

where $\mathbf{N}$ and $\mathbf{B}$ are the unit normal and binormal vectors for C.

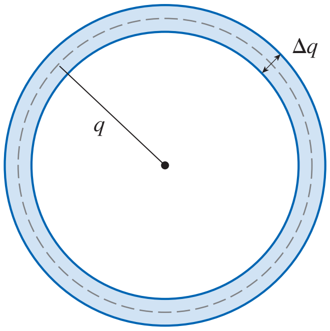
(b) Use the Frenet-Serret Formulas (Exercises 13.3.71–72) and the Pythagorean Theorem for vectors (Exercise 12.3.66) to show that

$$|\mathbf{X}_u(u, v) \times \mathbf{X}_v(u, v)| = q[1 - \kappa(u)\,q\cos v]$$

and so the surface area of Tube(C, q) is

$$S(q) = \int_a^b \int_0^{2\pi} |\mathbf{X}_u(u, v) \times \mathbf{X}_v(u, v)|\,dv\,du = 2\pi qL$$

(c) Consider a thin tubular shell of radius q and thickness Δq along C, a cross-section of which is shown in the figure.



Observe that the volume of the shell is approximately $\Delta q\,S(q)$ and conclude that the volume of Tube(C, r) is

$$\int_0^r S(q)\,dq = \pi r^2 L$$

(d) Find the volume of a tube of radius $r = 0.2$ around the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \leqslant t \leqslant 4\pi$.

(e) Find the volume of the torus in Example 8.3.7.

Source: Adapted from A. Gray, *Tubes*, 2nd ed. (Basel; Boston: Birkhäuser, 2004).

Appendixes

- **A** Numbers, Inequalities, and Absolute Values
- **B** Coordinate Geometry and Lines
- **C** Graphs of Second-Degree Equations
- **D** Trigonometry
- **E** Sigma Notation
- **F** Proofs of Theorems
- **G** The Logarithm Defined as an Integral
- **H** Answers to Odd-Numbered Exercises

A Numbers, Inequalities, and Absolute Values

Calculus is based on the real number system. We start with the **integers**:

$$\ldots, \quad -3, \quad -2, \quad -1, \quad 0, \quad 1, \quad 2, \quad 3, \quad 4, \quad \ldots$$

Then we construct the **rational numbers**, which are ratios of integers. Thus any rational number r can be expressed as

$$r = \frac{m}{n} \qquad \text{where } m \text{ and } n \text{ are integers and } n \neq 0$$

Examples are

$$\tfrac{1}{2} \qquad -\tfrac{3}{7} \qquad 46 = \tfrac{46}{1} \qquad 0.17 = \tfrac{17}{100}$$

(Recall that division by 0 is always ruled out, so expressions like $\frac{3}{0}$ and $\frac{0}{0}$ are undefined.) Some real numbers, such as $\sqrt{2}$, can't be expressed as a ratio of integers and are therefore called **irrational numbers**. It can be shown, with varying degrees of difficulty, that the following are also irrational numbers:

$$\sqrt{3} \qquad \sqrt{5} \qquad \sqrt[3]{2} \qquad \pi \qquad \sin 1^\circ \qquad \log_{10} 2$$

The set of all real numbers is usually denoted by the symbol $\mathbb{R}$. When we use the word *number* without qualification, we mean "real number."

Every number has a decimal representation. If the number is rational, then the corresponding decimal is repeating. For example,

$$\tfrac{1}{2} = 0.5000\ldots = 0.5\overline{0} \qquad \tfrac{2}{3} = 0.66666\ldots = 0.\overline{6}$$

$$\tfrac{157}{495} = 0.317171717\ldots = 0.3\overline{17} \qquad \tfrac{9}{7} = 1.285714285714\ldots = 1.\overline{285714}$$

(The bar indicates that the sequence of digits repeats forever.) On the other hand, if the number is irrational, the decimal is nonrepeating:

$$\sqrt{2} = 1.414213562373095\ldots \qquad \pi = 3.141592653589793\ldots$$

If we stop the decimal expansion of any number at a certain place, we get an approximation to the number. For instance, we can write

$$\pi \approx 3.14159265$$

where the symbol $\approx$ is read "is approximately equal to." The more decimal places we retain, the better the approximation we get.

The real numbers can be represented by points on a line as in Figure 1. The positive direction (to the right) is indicated by an arrow. We choose an arbitrary reference point O, called the **origin**, which corresponds to the real number 0. Given any convenient unit of measurement, each positive number x is represented by the point on the line a distance of x units to the right of the origin, and each negative number $-x$ is represented by the point x units to the left of the origin. Thus every real number is represented by a point on the line, and every point P on the line corresponds to exactly one real number. The number associated with the point P is called the **coordinate** of P and the line is then called a

coordinate line, or a **real number line**, or simply a **real line**. Often we identify the point with its coordinate and think of a number as being a point on the real line.

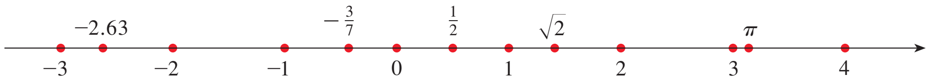


FIGURE 1

The real numbers are ordered. We say *a is less than b* and write $a < b$ if $b - a$ is a positive number. Geometrically this means that a lies to the left of b on the number line. (Equivalently, we say *b is greater than a* and write $b > a$.) The symbol $a \leqslant b$ (or $b \geqslant a$) means that either $a < b$ or $a = b$ and is read "a is less than or equal to b." For instance, the following are true inequalities:

$$7 < 7.4 < 7.5 \qquad -3 > -\pi \qquad \sqrt{2} < 2 \qquad \sqrt{2} \leqslant 2 \qquad 2 \leqslant 2$$

In what follows we need to use *set notation*. A **set** is a collection of objects, and these objects are called the **elements** of the set. If S is a set, the notation $a \in S$ means that a is an element of S, and $a \notin S$ means that a is not an element of S. For example, if Z represents the set of integers, then $-3 \in Z$ but $\pi \notin Z$. If S and T are sets, then their **union** $S \cup T$ is the set consisting of all elements that are in S or T (or in both S and T). The **intersection** of S and T is the set $S \cap T$ consisting of all elements that are in both S and T. In other words, $S \cap T$ is the common part of S and T. The empty set, denoted by $\varnothing$, is the set that contains no element.

Some sets can be described by listing their elements between braces. For instance, the set A consisting of all positive integers less than 7 can be written as

$$A = \{1, 2, 3, 4, 5, 6\}$$

We could also write A in *set-builder notation* as

$$A = \{x \mid x \text{ is an integer and } 0 < x < 7\}$$

which is read "A is the set of x such that x is an integer and $0 < x < 7$."

Intervals

Certain sets of real numbers, called **intervals**, occur frequently in calculus and correspond geometrically to line segments. For example, if $a < b$, the **open interval** from a to b consists of all numbers between a and b and is denoted by the symbol (a, b). Using set-builder notation, we can write

$$(a, b) = \{x \mid a < x < b\}$$

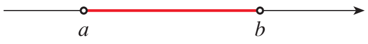


FIGURE 2
Open interval (a, b)

Notice that the endpoints of the interval—namely, a and b—are excluded. This is indicated by the round brackets ( ) and by the open dots in Figure 2. The **closed interval** from a to b is the set

$$[a, b] = \{x \mid a \leqslant x \leqslant b\}$$

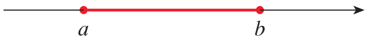


FIGURE 3
Closed interval $[a, b]$

Here the endpoints of the interval are included. This is indicated by the square brackets [ ] and by the solid dots in Figure 3. It is also possible to include only one endpoint in an interval, as shown in Table 1.

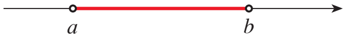
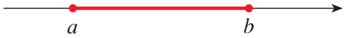
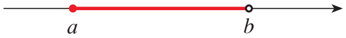
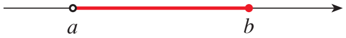
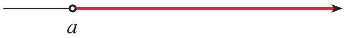
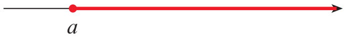
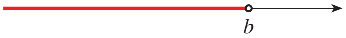
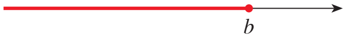
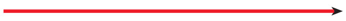
We also need to consider infinite intervals such as

$$(a, \infty) = \{x \mid x > a\}$$

This does not mean that ∞ ("infinity") is a number. The notation (a, ∞) stands for the set of all numbers that are greater than a, so the symbol ∞ simply indicates that the interval extends indefinitely far in the positive direction.

Table 1 lists the nine possible types of intervals. When these intervals are discussed, it is always assumed that a is less than b.

1 Table of Intervals

Notation	Set description	Picture
(a, b)	$\{x \mid a < x < b\}$	
$[a, b]$	$\{x \mid a \leqslant x \leqslant b\}$	
$[a, b)$	$\{x \mid a \leqslant x < b\}$	
$(a, b]$	$\{x \mid a < x \leqslant b\}$	
(a, ∞)	$\{x \mid x > a\}$	
$[a, \infty)$	$\{x \mid x \geqslant a\}$	
$(-\infty, b)$	$\{x \mid x < b\}$	
$(-\infty, b]$	$\{x \mid x \leqslant b\}$	
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

Inequalities

When working with inequalities, note the following rules.

2 Rules for Inequalities

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$ and $c < d$, then $a + c < b + d$.
3. If $a < b$ and $c > 0$, then $ac < bc$.
4. If $a < b$ and $c < 0$, then $ac > bc$.
5. If $0 < a < b$, then $1/a > 1/b$.

Rule 1 says that we can add any number to both sides of an inequality, and Rule 2 says that two inequalities can be added. However, we have to be careful with multiplication. Rule 3 says that we can multiply both sides of an inequality by a *positive* number, but Rule 4 says that if we multiply both sides of an inequality by a negative number, then we reverse the direction of the inequality. For example, if we take the inequality $3 < 5$ and multiply by 2, we get $6 < 10$, but if we multiply by -2, we get $-6 > -10$. Finally, Rule 5 says that if we take reciprocals, then we reverse the direction of an inequality (provided the numbers are positive).

EXAMPLE 1 Solve the inequality $1 + x < 7x + 5$.

SOLUTION The given inequality is satisfied by some values of x but not by others. To *solve* an inequality means to determine the set of numbers x for which the inequality is true. This is called the *solution set.*

First we subtract 1 from each side of the inequality (using Rule 1 with $c = -1$):

$$x < 7x + 4$$

Then we subtract $7x$ from both sides (Rule 1 with $c = -7x$):

$$-6x < 4$$

Now we divide both sides by -6 $\left(\text{Rule 4 with } c = -\frac{1}{6}\right)$:

$$x > -\tfrac{4}{6} = -\tfrac{2}{3}$$

These steps can all be reversed, so the solution set consists of all numbers greater than $-\frac{2}{3}$. In other words, the solution of the inequality is the interval $\left(-\frac{2}{3}, \infty\right)$. ■

EXAMPLE 2 Solve the inequalities $4 \leqslant 3x - 2 < 13$.

SOLUTION Here the solution set consists of all values of x that satisfy both inequalities. Using the rules given in (2), we see that the following inequalities are equivalent:

$$\begin{aligned} &4 \leqslant 3x - 2 < 13 \\ &6 \leqslant 3x < 15 \qquad \text{(add 2)} \\ &2 \leqslant x < 5 \qquad \text{(divide by 3)} \end{aligned}$$

Therefore the solution set is $[2, 5)$. ■

EXAMPLE 3 Solve the inequality $x^2 - 5x + 6 \leqslant 0$.

SOLUTION First we factor the left side:

$$(x - 2)(x - 3) \leqslant 0$$

We know that the corresponding equation $(x - 2)(x - 3) = 0$ has the solutions 2 and 3. The numbers 2 and 3 divide the real line into three intervals:

$$(-\infty, 2) \qquad (2, 3) \qquad (3, \infty)$$

On each of these intervals we determine the signs of the factors. For instance,

$$x \in (-\infty, 2) \quad \Rightarrow \quad x < 2 \quad \Rightarrow \quad x - 2 < 0$$

Then we record these signs in the following chart:

Interval	$x - 2$	$x - 3$	$(x - 2)(x - 3)$
$x < 2$	$-$	$-$	$+$
$2 < x < 3$	$+$	$-$	$-$
$x > 3$	$+$	$+$	$+$

A visual method for solving Example 3 is to graph the parabola $y = x^2 - 5x + 6$ (as in Figure 4; see Appendix C) and observe that the curve lies on or below the x-axis when $2 \leqslant x \leqslant 3$.

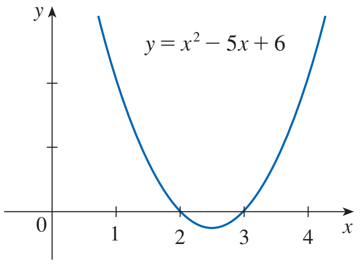


FIGURE 4

Another method for obtaining the information in the chart is to use *test values*. For instance, if we use the test value $x = 1$ for the interval $(-\infty, 2)$, then substitution in $x^2 - 5x + 6$ gives

$$1^2 - 5(1) + 6 = 2$$

The polynomial $x^2 - 5x + 6$ doesn't change sign inside any of the three intervals, so we conclude that it is positive on $(-\infty, 2)$.

Then we read from the chart that $(x - 2)(x - 3)$ is negative when $2 < x < 3$. Thus the solution of the inequality $(x - 2)(x - 3) \leqslant 0$ is

$$\{x \mid 2 \leqslant x \leqslant 3\} = [2, 3]$$

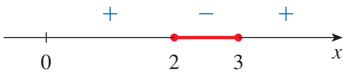


FIGURE 5

Notice that we have included the endpoints 2 and 3 because we are looking for values of x such that the product is either negative or zero. The solution is illustrated in Figure 5. ■

EXAMPLE 4 Solve $x^3 + 3x^2 > 4x$.

SOLUTION First we take all nonzero terms to one side of the inequality sign and factor the resulting expression:

$$x^3 + 3x^2 - 4x > 0 \qquad \text{or} \qquad x(x - 1)(x + 4) > 0$$

As in Example 3 we solve the corresponding equation $x(x - 1)(x + 4) = 0$ and use the solutions $x = -4$, $x = 0$, and $x = 1$ to divide the real line into four intervals $(-\infty, -4)$, $(-4, 0)$, $(0, 1)$, and $(1, \infty)$. On each interval the product keeps a constant sign, which we list in the following chart:

Interval	x	$x - 1$	$x + 4$	$x(x - 1)(x + 4)$
$x < -4$	−	−	−	−
$-4 < x < 0$	−	−	+	+
$0 < x < 1$	+	−	+	−
$x > 1$	+	+	+	+

Then we read from the chart that the solution set is

$$\{x \mid -4 < x < 0 \text{ or } x > 1\} = (-4, 0) \cup (1, \infty)$$

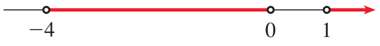


FIGURE 6

The solution is illustrated in Figure 6. ■

■ Absolute Value

The **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line. Distances are always positive or 0, so we have

$$|a| \geqslant 0 \qquad \text{for every number } a$$

For example,

$$|3| = 3 \qquad |-3| = 3 \qquad |0| = 0 \qquad |\sqrt{2} - 1| = \sqrt{2} - 1 \qquad |3 - \pi| = \pi - 3$$

In general, we have

Remember that if a is negative, then $-a$ is positive.

3

$$\boxed{\begin{aligned} |a| &= a \qquad \text{if } a \geqslant 0 \\ |a| &= -a \qquad \text{if } a < 0 \end{aligned}}$$

EXAMPLE 5 Express $|3x - 2|$ without using the absolute-value symbol.

SOLUTION

$$|3x - 2| = \begin{cases} 3x - 2 & \text{if } 3x - 2 \geqslant 0 \\ -(3x - 2) & \text{if } 3x - 2 < 0 \end{cases}$$

$$= \begin{cases} 3x - 2 & \text{if } x \geqslant \frac{2}{3} \\ 2 - 3x & \text{if } x < \frac{2}{3} \end{cases}$$

■

Recall that the symbol $\sqrt{}$ means "the positive square root of." Thus $\sqrt{r} = s$ means $s^2 = r$ and $s \geqslant 0$. Therefore the equation $\sqrt{a^2} = a$ is not always true. It is true only when $a \geqslant 0$. If $a < 0$, then $-a > 0$, so we have $\sqrt{a^2} = -a$. In view of (3), we then have the equation

4 $$\boxed{\sqrt{a^2} = |a|}$$

which is true for all values of a.

Hints for the proofs of the following properties are given in the exercises.

5 **Properties of Absolute Values** Suppose a and b are any real numbers and n is an integer. Then

1. $|ab| = |a||b|$ **2.** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$ $(b \neq 0)$ **3.** $|a^n| = |a|^n$

For solving equations or inequalities involving absolute values, it's often very helpful to use the following statements.

6 Suppose $a > 0$. Then

4. $|x| = a$ if and only if $x = \pm a$

5. $|x| < a$ if and only if $-a < x < a$

6. $|x| > a$ if and only if $x > a$ or $x < -a$

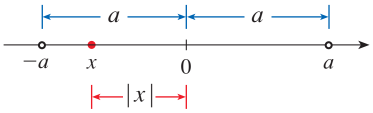


FIGURE 7

For instance, the inequality $|x| < a$ says that the distance from x to the origin is less than a, and you can see from Figure 7 that this is true if and only if x lies between $-a$ and a.

If a and b are any real numbers, then the distance between a and b is the absolute value of the difference, namely, $|a - b|$, which is also equal to $|b - a|$. (See Figure 8.)

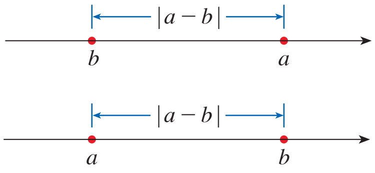


FIGURE 8
Length of a line segment $= |a - b|$

EXAMPLE 6 Solve $|2x - 5| = 3$.

SOLUTION By Property 4 of (6), $|2x - 5| = 3$ is equivalent to

$$2x - 5 = 3 \qquad \text{or} \qquad 2x - 5 = -3$$

So $2x = 8$ or $2x = 2$. Thus $x = 4$ or $x = 1$. ■

EXAMPLE 7 Solve $|x - 5| < 2$.

SOLUTION 1 By Property 5 of (6), $|x - 5| < 2$ is equivalent to

$$-2 < x - 5 < 2$$

Therefore, adding 5 to each side, we have

$$3 < x < 7$$

and the solution set is the open interval $(3, 7)$.

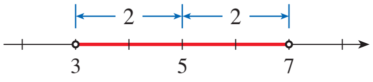


FIGURE 9

SOLUTION 2 Geometrically the solution set consists of all numbers x whose distance from 5 is less than 2. From Figure 9 we see that this is the interval $(3, 7)$. ■

EXAMPLE 8 Solve $|3x + 2| \geqslant 4$.

SOLUTION By Properties 4 and 6 of (6), $|3x + 2| \geqslant 4$ is equivalent to

$$3x + 2 \geqslant 4 \qquad \text{or} \qquad 3x + 2 \leqslant -4$$

In the first case $3x \geqslant 2$, which gives $x \geqslant \frac{2}{3}$. In the second case $3x \leqslant -6$, which gives $x \leqslant -2$. So the solution set is

$$\left\{x \mid x \leqslant -2 \text{ or } x \geqslant \tfrac{2}{3}\right\} = (-\infty, -2] \cup \left[\tfrac{2}{3}, \infty\right)$$ ■

Another important property of absolute value, called the Triangle Inequality, is used frequently in calculus and throughout mathematics in general.

7 **The Triangle Inequality** If a and b are any real numbers, then

$$|a + b| \leqslant |a| + |b|$$

Observe that if the numbers a and b are both positive or both negative, then the two sides in the Triangle Inequality are actually equal. But if a and b have opposite signs, the left side involves a subtraction and the right side does not. This makes the Triangle Inequality seem reasonable, but we can prove it as follows.

Notice that

$$-|a| \leqslant a \leqslant |a|$$

is always true because a equals either $|a|$ or $-|a|$. The corresponding statement for b is

$$-|b| \leqslant b \leqslant |b|$$

Adding these inequalities, we get

$$-(|a| + |b|) \leqslant a + b \leqslant |a| + |b|$$

If we now apply Properties 4 and 5 (with x replaced by $a + b$ and a by $|a| + |b|$), we obtain

$$|a + b| \leqslant |a| + |b|$$

which is what we wanted to show.

EXAMPLE 9 If $|x - 4| < 0.1$ and $|y - 7| < 0.2$, use the Triangle Inequality to estimate $|(x + y) - 11|$.

SOLUTION In order to use the given information, we use the Triangle Inequality with $a = x - 4$ and $b = y - 7$:

$$\begin{aligned} |(x + y) - 11| &= |(x - 4) + (y - 7)| \\ &\leqslant |x - 4| + |y - 7| \\ &< 0.1 + 0.2 = 0.3 \end{aligned}$$

Thus $$|(x + y) - 11| < 0.3$$ ■

A Exercises

1–12 Rewrite the expression without using the absolute-value symbol.

1. $|5 - 23|$

2. $|5| - |-23|$

3. $|-\pi|$

4. $|\pi - 2|$

5. $|\sqrt{5} - 5|$

6. $||-2| - |-3||$

7. $|x - 2|$ if $x < 2$

8. $|x - 2|$ if $x > 2$

9. $|x + 1|$

10. $|2x - 1|$

11. $|x^2 + 1|$

12. $|1 - 2x^2|$

13–38 Solve the inequality in terms of intervals and illustrate the solution set on the real number line.

13. $2x + 7 > 3$

14. $3x - 11 < 4$

15. $1 - x \leqslant 2$

16. $4 - 3x \geqslant 6$

17. $2x + 1 < 5x - 8$

18. $1 + 5x > 5 - 3x$

19. $-1 < 2x - 5 < 7$

20. $1 < 3x + 4 \leqslant 16$

21. $0 \leqslant 1 - x < 1$

22. $-5 \leqslant 3 - 2x \leqslant 9$

23. $4x < 2x + 1 \leqslant 3x + 2$

24. $2x - 3 < x + 4 < 3x - 2$

25. $(x - 1)(x - 2) > 0$

26. $(2x + 3)(x - 1) \geqslant 0$

27. $2x^2 + x \leqslant 1$

28. $x^2 < 2x + 8$

29. $x^2 + x + 1 > 0$

30. $x^2 + x > 1$

31. $x^2 < 3$

32. $x^2 \geqslant 5$

33. $x^3 - x^2 \leqslant 0$

34. $(x + 1)(x - 2)(x + 3) \geqslant 0$

35. $x^3 > x$

36. $x^3 + 3x < 4x^2$

37. $\dfrac{1}{x} < 4$

38. $-3 < \dfrac{1}{x} \leqslant 1$

39. The relationship between the Celsius and Fahrenheit temperature scales is given by $C = \frac{5}{9}(F - 32)$, where C is the temperature in degrees Celsius and F is the temperature in degrees Fahrenheit. What interval on the Celsius scale corresponds to the temperature range $50 \leqslant F \leqslant 95$?

40. Use the relationship between C and F given in Exercise 39 to find the interval on the Fahrenheit scale corresponding to the temperature range $20 \leqslant C \leqslant 30$.

41. As dry air moves upward, it expands and in so doing cools at a rate of about 1°C for each 100-meter rise, up to about 12 km.

(a) If the ground temperature is 20°C, write a formula for the temperature at height h.

(b) What range of temperature can be expected if a plane takes off and reaches a maximum height of 5 km?

42. If a ball is thrown upward from the top of a building 30 m high with an initial velocity of 10 m/s, then the height h above the ground t seconds later will be

$$h = 30 + 10t - 5t^2$$

During what time interval will the ball be at least 15 m above the ground?

43–46 Solve the equation for x.

43. $|2x| = 3$

44. $|3x + 5| = 1$

45. $|x + 3| = |2x + 1|$

46. $\left|\dfrac{2x - 1}{x + 1}\right| = 3$

47–56 Solve the inequality.

47. $|x| < 3$

48. $|x| \geqslant 3$

49. $|x - 4| < 1$

50. $|x - 6| < 0.1$

51. $|x + 5| \geqslant 2$

52. $|x + 1| \geqslant 3$

53. $|2x - 3| \leqslant 0.4$

54. $|5x - 2| < 6$

55. $1 \leqslant |x| \leqslant 4$

56. $0 < |x - 5| < \frac{1}{2}$

57–58 Solve for x, assuming a, b, and c are positive constants.

57. $a(bx - c) \geqslant bc$

58. $a \leqslant bx + c < 2a$

59–60 Solve for x, assuming a, b, and c are negative constants.

59. $ax + b < c$

60. $\dfrac{ax + b}{c} \leqslant b$

61. Suppose that $|x - 2| < 0.01$ and $|y - 3| < 0.04$. Use the Triangle Inequality to show that $|(x + y) - 5| < 0.05$.

62. Show that if $|x + 3| < \frac{1}{2}$, then $|4x + 13| < 3$.

63. Show that if $a < b$, then $a < \dfrac{a + b}{2} < b$.

64. Use Rule 3 to prove Rule 5 of (2).

65. Prove that $|ab| = |a||b|$. [*Hint:* Use Equation 4.]

66. Prove that $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$.

67. Show that if $0 < a < b$, then $a^2 < b^2$.

68. Prove that $|x - y| \geqslant |x| - |y|$. [*Hint:* Use the Triangle Inequality with $a = x - y$ and $b = y$.]

69. Show that the sum, difference, and product of rational numbers are rational numbers.

70. (a) Is the sum of two irrational numbers always an irrational number?
(b) Is the product of two irrational numbers always an irrational number?

B Coordinate Geometry and Lines

Just as the points on a line can be identified with real numbers by assigning them coordinates, as described in Appendix A, so the points in a plane can be identified with ordered pairs of real numbers. We start by drawing two perpendicular coordinate lines that intersect at the origin O on each line. Usually one line is horizontal with positive direction to the right and is called the x-axis; the other line is vertical with positive direction upward and is called the y-axis.

Any point P in the plane can be located by a unique ordered pair of numbers as follows. Draw lines through P perpendicular to the x- and y-axes. These lines intersect the axes in points with coordinates a and b as shown in Figure 1. Then the point P is assigned the ordered pair (a, b). The first number a is called the ***x*-coordinate** of P; the second number b is called the ***y*-coordinate** of P. We say that P is the point with coordinates (a, b), and we denote the point by the symbol $P(a, b)$. Several points are labeled with their coordinates in Figure 2.

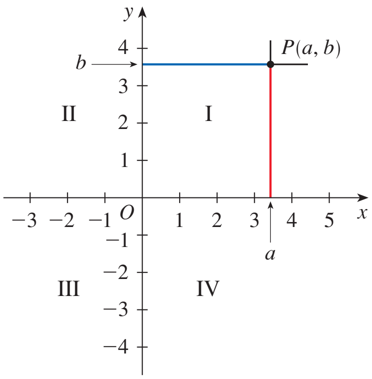


FIGURE 1

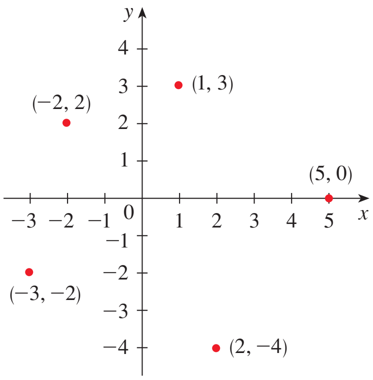


FIGURE 2

By reversing the preceding process we can start with an ordered pair (a, b) and arrive at the corresponding point P. Often we identify the point P with the ordered pair (a, b) and refer to "the point (a, b)." [Although the notation used for an open interval (a, b) is the same as the notation used for a point (a, b), you will be able to tell from the context which meaning is intended.]

This coordinate system is called the **rectangular coordinate system** or the **Cartesian coordinate system** in honor of the French mathematician René Descartes (1596–1650), even though another Frenchman, Pierre Fermat (1601–1665), invented the principles of analytic geometry at about the same time as Descartes. The plane supplied with this coordinate system is called the **coordinate plane** or the **Cartesian plane** and is denoted by $\mathbb{R}^2$.

The x- and y-axes are called the **coordinate axes** and divide the Cartesian plane into four quadrants, which are labeled I, II, III, and IV in Figure 1. Notice that the first quadrant consists of those points whose x- and y-coordinates are both positive.

EXAMPLE 1 Describe and sketch the regions given by the following sets.

(a) $\{(x, y) \mid x \geqslant 0\}$ (b) $\{(x, y) \mid y = 1\}$ (c) $\{(x, y) \mid |y| < 1\}$

SOLUTION

(a) The points whose x-coordinates are 0 or positive lie on the y-axis or to the right of it as indicated by the shaded region in Figure 3(a).

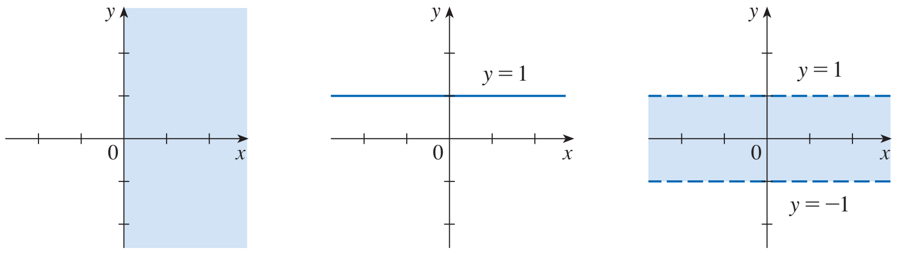


FIGURE 3

(a) $x \geqslant 0$ (b) $y = 1$ (c) $|y| < 1$

(b) The set of all points with y-coordinate 1 is a horizontal line one unit above the x-axis [see Figure 3(b)].

(c) Recall from Appendix A that

$$|y| < 1 \qquad \text{if and only if} \qquad -1 < y < 1$$

The given region consists of those points in the plane whose y-coordinates lie between -1 and 1. Thus the region consists of all points that lie between (but not on) the horizontal lines $y = 1$ and $y = -1$. [These lines are shown as dashed lines in Figure 3(c) to indicate that the points on these lines don't lie in the set.] ■

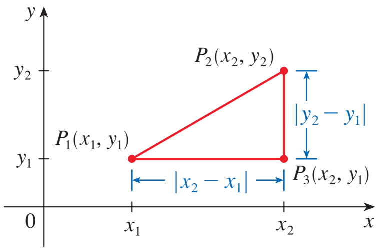


FIGURE 4

Recall from Appendix A that the distance between points a and b on a number line is $|a - b| = |b - a|$. Thus the distance between points $P_1(x_1, y_1)$ and $P_3(x_2, y_1)$ on a horizontal line must be $|x_2 - x_1|$ and the distance between $P_2(x_2, y_2)$ and $P_3(x_2, y_1)$ on a vertical line must be $|y_2 - y_1|$. (See Figure 4.)

To find the distance $|P_1P_2|$ between any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, we note that triangle $P_1P_2P_3$ in Figure 4 is a right triangle, and so by the Pythagorean Theorem we have

$$\begin{aligned} |P_1P_2| &= \sqrt{|P_1P_3|^2 + |P_2P_3|^2} = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \end{aligned}$$

1 Distance Formula The distance between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLE 2 The distance between $(1, -2)$ and $(5, 3)$ is

$$\sqrt{(5 - 1)^2 + [3 - (-2)]^2} = \sqrt{4^2 + 5^2} = \sqrt{41}$$ ■

Lines

We want to find an equation of a given line L; such an equation is satisfied by the coordinates of the points on L and by no other point. To find the equation of L we use its *slope,* which is a measure of the steepness of the line.

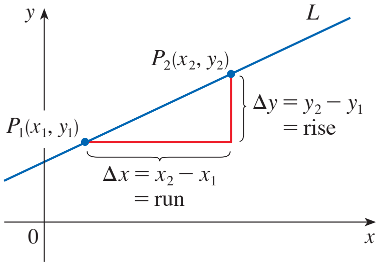


FIGURE 5

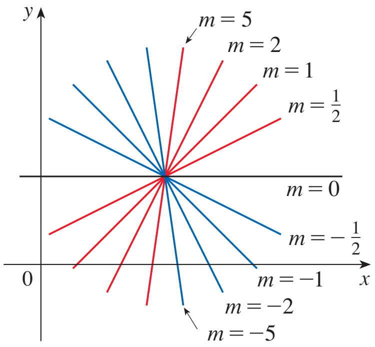


FIGURE 6

2 Definition The **slope** of a nonvertical line that passes through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.

Thus the slope of a line is the ratio of the change in y, Δy, to the change in x, Δx. (See Figure 5.) The slope is therefore the rate of change of y with respect to x. The fact that the line is straight means that the rate of change is constant.

Figure 6 shows several lines labeled with their slopes. Notice that lines with positive slope slant upward to the right, whereas lines with negative slope slant downward to the right. Notice also that the steepest lines are the ones for which the absolute value of the slope is largest, and a horizontal line has slope 0.

Now let's find an equation of the line that passes through a given point $P_1(x_1, y_1)$ and has slope m. A point $P(x, y)$ with $x \neq x_1$ lies on this line if and only if the slope of the line through P_1 and P is equal to m; that is,

$$\frac{y - y_1}{x - x_1} = m$$

This equation can be rewritten in the form

$$y - y_1 = m(x - x_1)$$

and we observe that this equation is also satisfied when $x = x_1$ and $y = y_1$. Therefore it is an equation of the given line.

3 Point-Slope Form of the Equation of a Line An equation of the line passing through the point $P_1(x_1, y_1)$ and having slope m is

$$y - y_1 = m(x - x_1)$$

EXAMPLE 3 Find an equation of the line through $(1, -7)$ with slope $-\frac{1}{2}$.

SOLUTION Using (3) with $m = -\frac{1}{2}$, $x_1 = 1$, and $y_1 = -7$, we obtain an equation of the line as

$$y + 7 = -\tfrac{1}{2}(x - 1)$$

which we can rewrite as

$$2y + 14 = -x + 1 \qquad \text{or} \qquad x + 2y + 13 = 0$$ ■

EXAMPLE 4 Find an equation of the line through the points $(-1, 2)$ and $(3, -4)$.

SOLUTION By Definition 2 the slope of the line is

$$m = \frac{-4 - 2}{3 - (-1)} = -\frac{3}{2}$$

Using the point-slope form with $x_1 = -1$ and $y_1 = 2$, we obtain

$$y - 2 = -\tfrac{3}{2}(x + 1)$$

which simplifies to $$3x + 2y = 1$$ ■

Suppose a nonvertical line has slope m and y-intercept b. (See Figure 7.) This means it intersects the y-axis at the point $(0, b)$, so the point-slope form of the equation of the line, with $x_1 = 0$ and $y_1 = b$, becomes

$$y - b = m(x - 0)$$

This simplifies as follows.

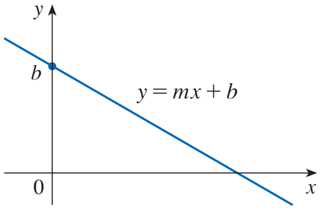


FIGURE 7

4 **Slope-Intercept Form of the Equation of a Line** An equation of the line with slope m and y-intercept b is

$$y = mx + b$$

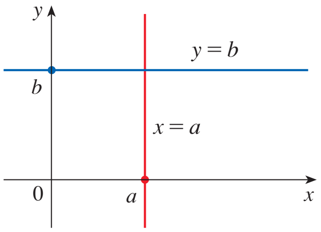


FIGURE 8

In particular, if a line is horizontal, its slope is $m = 0$, so its equation is $y = b$, where b is the y-intercept (see Figure 8). A vertical line does not have a slope, but we can write its equation as $x = a$, where a is the x-intercept, because the x-coordinate of every point on the line is a.

Observe that the equation of every line can be written in the form

5 $$\boxed{Ax + By + C = 0}$$

because a vertical line has the equation $x = a$ or $x - a = 0$ ($A = 1$, $B = 0$, $C = -a$) and a nonvertical line has the equation $y = mx + b$ or $-mx + y - b = 0$ ($A = -m$, $B = 1$, $C = -b$). Conversely, if we start with a general first-degree equation, that is, an equation of the form (5), where A, B, and C are constants and A and B are not both 0, then we can show that it is the equation of a line. If $B = 0$, the equation becomes $Ax + C = 0$ or $x = -C/A$, which represents a vertical line with x-intercept $-C/A$. If $B \neq 0$, the equation can be rewritten by solving for y:

$$y = -\frac{A}{B}x - \frac{C}{B}$$

and we recognize this as being the slope-intercept form of the equation of a line ($m = -A/B$, $b = -C/B$). Therefore an equation of the form (5) is called a **linear equation** or the **general equation of a line**. For brevity, we often refer to "the line $Ax + By + C = 0$" instead of "the line whose equation is $Ax + By + C = 0$."

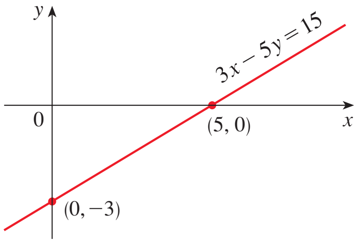


FIGURE 9

EXAMPLE 5 Sketch the graph of the equation $3x - 5y = 15$.

SOLUTION Since the equation is linear, its graph is a line. To draw the graph, we can simply find two points on the line. It's easiest to find the intercepts. Substituting $y = 0$ (the equation of the x-axis) in the given equation, we get $3x = 15$, so $x = 5$ is the x-intercept. Substituting $x = 0$ in the equation, we see that the y-intercept is -3. This allows us to sketch the graph as in Figure 9. ■

EXAMPLE 6 Graph the inequality $x + 2y > 5$.

SOLUTION We are asked to sketch the graph of the set $\{(x, y) \mid x + 2y > 5\}$ and we begin by solving the inequality for y:

$$x + 2y > 5$$
$$2y > -x + 5$$
$$y > -\tfrac{1}{2}x + \tfrac{5}{2}$$

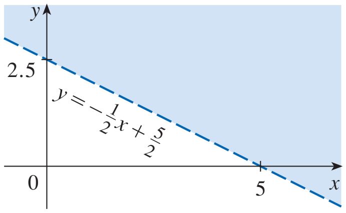


FIGURE 10

Compare this inequality with the equation $y = -\frac{1}{2}x + \frac{5}{2}$, which represents a line with slope $-\frac{1}{2}$ and y-intercept $\frac{5}{2}$. We see that the given graph consists of points whose y-coordinates are *larger* than those on the line $y = -\frac{1}{2}x + \frac{5}{2}$. Thus the graph is the region that lies *above* the line, as illustrated in Figure 10. ■

Parallel and Perpendicular Lines

Slopes can be used to show that lines are parallel or perpendicular. The following facts are proved, for instance, in *Precalculus: Mathematics for Calculus,* Seventh Edition, by Stewart, Redlin, and Watson (Boston, 2016).

6 Parallel and Perpendicular Lines

1. Two nonvertical lines are parallel if and only if they have the same slope.
2. Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1m_2 = -1$; that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$

EXAMPLE 7 Find an equation of the line through the point (5, 2) that is parallel to the line $4x + 6y + 5 = 0$.

SOLUTION The given line can be written in the form

$$y = -\tfrac{2}{3}x - \tfrac{5}{6}$$

which is in slope-intercept form with $m = -\frac{2}{3}$. Parallel lines have the same slope, so the required line has slope $-\frac{2}{3}$ and its equation in point-slope form is

$$y - 2 = -\tfrac{2}{3}(x - 5)$$

We can write this equation as $2x + 3y = 16$. ■

EXAMPLE 8 Show that the lines $2x + 3y = 1$ and $6x - 4y - 1 = 0$ are perpendicular.

SOLUTION The equations can be written as

$$y = -\tfrac{2}{3}x + \tfrac{1}{3} \qquad \text{and} \qquad y = \tfrac{3}{2}x - \tfrac{1}{4}$$

from which we see that the slopes are

$$m_1 = -\tfrac{2}{3} \qquad \text{and} \qquad m_2 = \tfrac{3}{2}$$

Since $m_1 m_2 = -1$, the lines are perpendicular. ■

B Exercises

1–6 Find the distance between the points.

1. $(1, 1)$, $(4, 5)$

2. $(1, -3)$, $(5, 7)$

3. $(6, -2)$, $(-1, 3)$

4. $(1, -6)$, $(-1, -3)$

5. $(2, 5)$, $(4, -7)$

6. (a, b), (b, a)

7–10 Find the slope of the line through P and Q.

7. $P(1, 5)$, $Q(4, 11)$

8. $P(-1, 6)$, $Q(4, -3)$

9. $P(-3, 3)$, $Q(-1, -6)$

10. $P(-1, -4)$, $Q(6, 0)$

11. Show that the triangle with vertices $A(0, 2)$, $B(-3, -1)$, and $C(-4, 3)$ is isosceles.

12. (a) Show that the triangle with vertices $A(6, -7)$, $B(11, -3)$, and $C(2, -2)$ is a right triangle using the converse of the Pythagorean Theorem.
(b) Use slopes to show that ABC is a right triangle.
(c) Find the area of the triangle.

13. Show that the points $(-2, 9)$, $(4, 6)$, $(1, 0)$, and $(-5, 3)$ are the vertices of a square.

14. (a) Show that the points $A(-1, 3)$, $B(3, 11)$, and $C(5, 15)$ are collinear (lie on the same line) by showing that $|AB| + |BC| = |AC|$.
(b) Use slopes to show that A, B, and C are collinear.

15. Show that $A(1, 1)$, $B(7, 4)$, $C(5, 10)$, and $D(-1, 7)$ are vertices of a parallelogram.

16. Show that $A(1, 1)$, $B(11, 3)$, $C(10, 8)$, and $D(0, 6)$ are vertices of a rectangle.

17–20 Sketch the graph of the equation.

17. $x = 3$

18. $y = -2$

19. $xy = 0$

20. $|y| = 1$

21–36 Find an equation of the line that satisfies the given conditions.

21. Through $(2, -3)$, slope 6

22. Through $(-1, 4)$, slope -3

23. Through $(1, 7)$, slope $\frac{2}{3}$

24. Through $(-3, -5)$, slope $-\frac{7}{2}$

25. Through $(2, 1)$ and $(1, 6)$

26. Through $(-1, -2)$ and $(4, 3)$

27. Slope 3, y-intercept -2

28. Slope $\frac{2}{5}$, y-intercept 4

29. x-intercept 1, y-intercept -3

30. x-intercept -8, y-intercept 6

31. Through $(4, 5)$, parallel to the x-axis

32. Through $(4, 5)$, parallel to the y-axis

33. Through $(1, -6)$, parallel to the line $x + 2y = 6$

34. y-intercept 6, parallel to the line $2x + 3y + 4 = 0$

35. Through $(-1, -2)$, perpendicular to the line $2x + 5y + 8 = 0$

36. Through $\left(\frac{1}{2}, -\frac{2}{3}\right)$, perpendicular to the line $4x - 8y = 1$

37–42 Find the slope and y-intercept of the line and draw its graph.

37. $x + 3y = 0$

38. $2x - 5y = 0$

39. $y = -2$

40. $2x - 3y + 6 = 0$

41. $3x - 4y = 12$

42. $4x + 5y = 10$

43–52 Sketch the region in the xy-plane.

43. $\{(x, y) \mid x < 0\}$

44. $\{(x, y) \mid y > 0\}$

45. $\{(x, y) \mid xy < 0\}$

46. $\{(x, y) \mid x \geqslant 1 \text{ and } y < 3\}$

47. $\{(x, y) \mid |x| \leqslant 2\}$

48. $\{(x, y) \mid |x| < 3 \text{ and } |y| < 2\}$

49. $\{(x, y) \mid 0 \leqslant y \leqslant 4 \text{ and } x \leqslant 2\}$

50. $\{(x, y) \mid y > 2x - 1\}$

51. $\{(x, y) \mid 1 + x \leqslant y \leqslant 1 - 2x\}$

52. $\{(x, y) \mid -x \leqslant y < \frac{1}{2}(x + 3)\}$

---

53. Find a point on the y-axis that is equidistant from $(5, -5)$ and $(1, 1)$.

54. Show that the midpoint of the line segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

55. Find the midpoint of the line segment joining the given points.
(a) $(1, 3)$ and $(7, 15)$ (b) $(-1, 6)$ and $(8, -12)$

56. Find the lengths of the medians of the triangle with vertices $A(1, 0)$, $B(3, 6)$, and $C(8, 2)$. (A median is a line segment from a vertex to the midpoint of the opposite side.)

57. Show that the lines $2x - y = 4$ and $6x - 2y = 10$ are not parallel and find their point of intersection.

58. Show that the lines

$$3x - 5y + 19 = 0 \qquad \text{and} \qquad 10x + 6y - 50 = 0$$

are perpendicular and find their point of intersection.

59. Find an equation of the perpendicular bisector of the line segment joining the points $A(1, 4)$ and $B(7, -2)$.

60. (a) Find equations for the sides of the triangle with vertices $P(1, 0)$, $Q(3, 4)$, and $R(-1, 6)$.
(b) Find equations for the medians of this triangle. Where do they intersect?

61. (a) Show that if the x- and y-intercepts of a line are nonzero numbers a and b, then the equation of the line can be put in the form

$$\frac{x}{a} + \frac{y}{b} = 1$$

This equation is called the **two-intercept form** of an equation of a line.
(b) Use part (a) to find an equation of the line whose x-intercept is 6 and whose y-intercept is -8.

62. A car leaves Detroit at 2:00 PM, traveling at a constant speed west along I-96. It passes Ann Arbor, 65 km from Detroit, at 2:50 PM.
(a) Express the distance traveled in terms of the time elapsed.
(b) Draw the graph of the equation in part (a).
(c) What is the slope of this line? What does it represent?

C | Graphs of Second-Degree Equations

In Appendix B we saw that a first-degree, or linear, equation $Ax + By + C = 0$ represents a line. In this section we discuss second-degree equations such as

$$x^2 + y^2 = 1 \qquad y = x^2 + 1 \qquad \frac{x^2}{9} + \frac{y^2}{4} = 1 \qquad x^2 - y^2 = 1$$

which represent a circle, a parabola, an ellipse, and a hyperbola, respectively.

The graph of such an equation in x and y is the set of all points (x, y) that satisfy the equation; it gives a visual representation of the equation. Conversely, given a curve in the xy-plane, we may have to find an equation that represents it, that is, an equation satisfied by the coordinates of the points on the curve and by no other point. This is the other half of the basic principle of analytic geometry as formulated by Descartes and Fermat. The idea is that if a geometric curve can be represented by an algebraic equation, then the rules of algebra can be used to analyze the geometric problem.

Circles

As an example of this type of problem, let's find an equation of the circle with radius r and center (h, k). By definition, the circle is the set of all points $P(x, y)$ whose distance

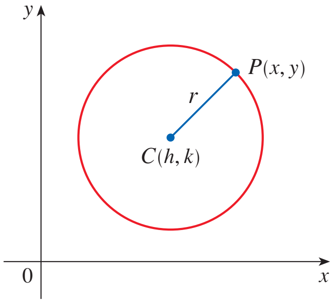


FIGURE 1

from the center $C(h, k)$ is r. (See Figure 1.) Thus P is on the circle if and only if $|PC| = r$. From the distance formula, we have

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

or equivalently, squaring both sides, we get

$$(x-h)^2 + (y-k)^2 = r^2$$

This is the desired equation.

1 **Equation of a Circle** An equation of the circle with center (h, k) and radius r is

$$(x-h)^2 + (y-k)^2 = r^2$$

In particular, if the center is the origin $(0, 0)$, the equation is

$$x^2 + y^2 = r^2$$

EXAMPLE 1 Find an equation of the circle with radius 3 and center $(2, -5)$.

SOLUTION From Equation 1 with $r = 3$, $h = 2$, and $k = -5$, we obtain

$$(x-2)^2 + (y+5)^2 = 9$$

■

EXAMPLE 2 Sketch the graph of the equation $x^2 + y^2 + 2x - 6y + 7 = 0$ by first showing that it represents a circle and then finding its center and radius.

SOLUTION We first group the x-terms and y-terms as follows:

$$(x^2 + 2x) + (y^2 - 6y) = -7$$

Then we complete the square within each grouping, adding the appropriate constants (the squares of half the coefficients of x and y) to both sides of the equation:

$$(x^2 + 2x + 1) + (y^2 - 6y + 9) = -7 + 1 + 9$$

or

$$(x+1)^2 + (y-3)^2 = 3$$

Comparing this equation with the standard equation of a circle (1), we see that $h = -1$, $k = 3$, and $r = \sqrt{3}$, so the given equation represents a circle with center $(-1, 3)$ and radius $\sqrt{3}$. It is sketched in Figure 2.

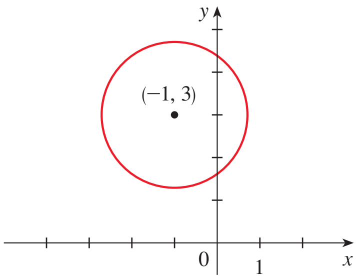


FIGURE 2
$x^2 + y^2 + 2x - 6y + 7 = 0$

■

Parabolas

The geometric properties of parabolas are reviewed in Section 10.5. Here we regard a parabola as a graph of an equation of the form $y = ax^2 + bx + c$.

EXAMPLE 3 Draw the graph of the parabola $y = x^2$.

SOLUTION We set up a table of values, plot points, and join them by a smooth curve to obtain the graph in Figure 3.

x	$y = x^2$
0	0
$\pm\frac{1}{2}$	$\frac{1}{4}$
± 1	1
± 2	4
± 3	9

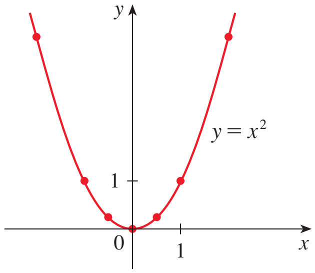


FIGURE 3 ■

Figure 4 shows the graphs of several parabolas with equations of the form $y = ax^2$ for various values of the number a. In each case the *vertex*—the point where the parabola changes direction—is the origin. We see that the parabola $y = ax^2$ opens upward if $a > 0$ and downward if $a < 0$ (as in Figure 5).

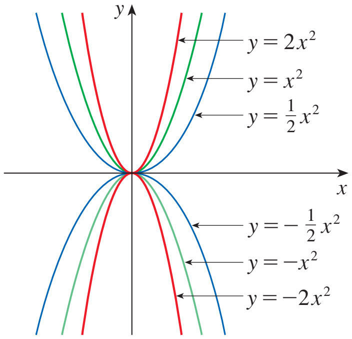
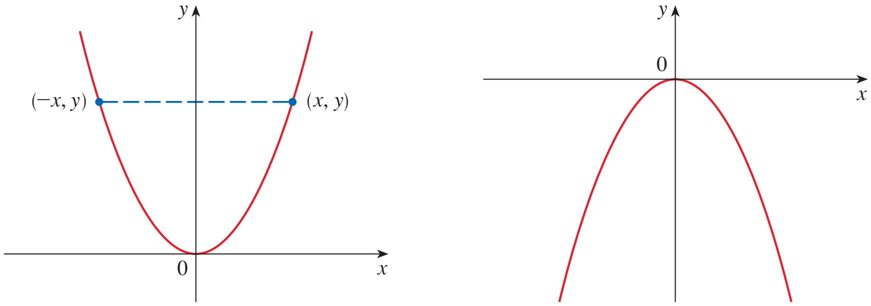


FIGURE 4



(a) $y = ax^2,\ a > 0$ (b) $y = ax^2,\ a < 0$

FIGURE 5

Notice that if (x, y) satisfies $y = ax^2$, then so does $(-x, y)$. This corresponds to the geometric fact that if the right half of the graph is reflected about the y-axis, then the left half of the graph is obtained. We say that the graph is **symmetric with respect to the *y*-axis**.

> The graph of an equation is symmetric with respect to the y-axis if the equation is unchanged when x is replaced by $-x$.

If we interchange x and y in the equation $y = ax^2$, the result is $x = ay^2$, which also represents a parabola. (Interchanging x and y amounts to reflecting about the diagonal line $y = x$.) The parabola $x = ay^2$ opens to the right if $a > 0$ and to the left if $a < 0$. (See

Figure 6.) This time the parabola is symmetric with respect to the x-axis because if (x, y) satisfies $x = ay^2$, then so does $(x, -y)$.

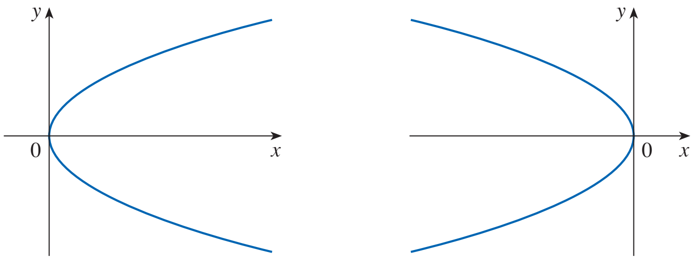


FIGURE 6

(a) $x = ay^2$, $a > 0$

(b) $x = ay^2$, $a < 0$

The graph of an equation is symmetric with respect to the x-axis if the equation is unchanged when y is replaced by $-y$.

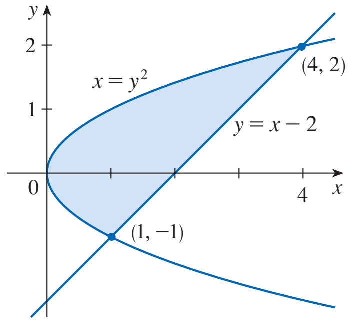


FIGURE 7

EXAMPLE 4 Sketch the region bounded by the parabola $x = y^2$ and the line $y = x - 2$.

SOLUTION First we find the points of intersection by solving the two equations simultaneously. Substituting $x = y + 2$ into the equation $x = y^2$, we get $y + 2 = y^2$, which gives

$$0 = y^2 - y - 2 = (y - 2)(y + 1)$$

so $y = 2$ or -1. Thus the points of intersection are $(4, 2)$ and $(1, -1)$, and we draw the line $y = x - 2$ passing through these points. We then sketch the parabola $x = y^2$ by referring to Figure 6(a) and having the parabola pass through $(4, 2)$ and $(1, -1)$. The region bounded by $x = y^2$ and $y = x - 2$ means the finite region whose boundaries are these curves. It is sketched in Figure 7. ■

Ellipses

The curve with equation

2
$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

where a and b are positive numbers, is called an **ellipse** in standard position. (Geometric properties of ellipses are discussed in Section 10.5.) Observe that Equation 2 is unchanged if x is replaced by $-x$ or y is replaced by $-y$, so the ellipse is symmetric with respect to both axes. As a further aid to sketching the ellipse, we find its intercepts.

The ***x*-intercepts** of a graph are the x-coordinates of the points where the graph intersects the x-axis. They are found by setting $y = 0$ in the equation of the graph.

The ***y*-intercepts** are the y-coordinates of the points where the graph intersects the y-axis. They are found by setting $x = 0$ in its equation.

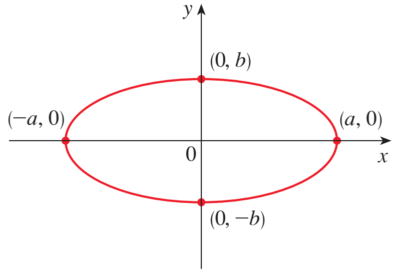


FIGURE 8

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

If we set $y = 0$ in Equation 2, we get $x^2 = a^2$ and so the x-intercepts are $\pm a$. Setting $x = 0$, we get $y^2 = b^2$, so the y-intercepts are $\pm b$. Using this information, together with symmetry, we sketch the ellipse in Figure 8. If $a = b$, the ellipse is a circle with radius a.

EXAMPLE 5 Sketch the graph of $9x^2 + 16y^2 = 144$.

SOLUTION We divide both sides of the equation by 144:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The equation is now in the standard form for an ellipse (2), so we have $a^2 = 16$, $b^2 = 9$, $a = 4$, and $b = 3$. The x-intercepts are ± 4; the y-intercepts are ± 3. The graph is sketched in Figure 9.

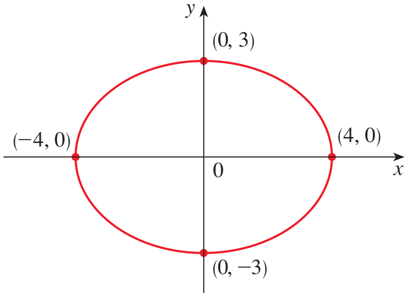


FIGURE 9

$9x^2 + 16y^2 = 144$

■

Hyperbolas

The curve with equation

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1} \tag{3}$$

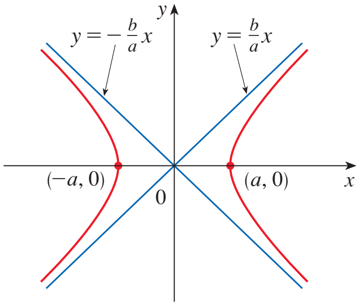


FIGURE 10

The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

is called a **hyperbola** in standard position. Again, Equation 3 is unchanged when x is replaced by $-x$ or y is replaced by $-y$, so the hyperbola is symmetric with respect to both axes. To find the x-intercepts we set $y = 0$ and obtain $x^2 = a^2$ and $x = \pm a$. However, if we put $x = 0$ in Equation 3, we get $y^2 = -b^2$, which is impossible, so there is no y-intercept. In fact, from Equation 3 we obtain

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geqslant 1$$

which shows that $x^2 \geqslant a^2$ and so $|x| = \sqrt{x^2} \geqslant a$. Therefore we have $x \geqslant a$ or $x \leqslant -a$. This means that the hyperbola consists of two parts, called its *branches*. It is sketched in Figure 10.

In drawing a hyperbola it is useful to draw first its *asymptotes*, which are the lines $y = (b/a)x$ and $y = -(b/a)x$ shown in Figure 10. Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes. This involves the idea of a limit, which is discussed in Chapter 2. (See also Exercise 4.5.77.)

By interchanging the roles of x and y we get an equation of the form

$$\boxed{\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1}$$

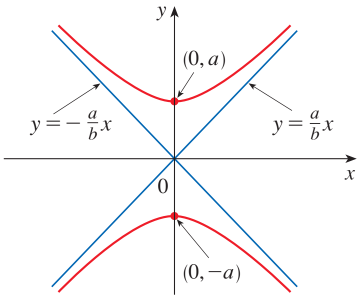


FIGURE 11

The hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

which also represents a hyperbola and is sketched in Figure 11.

EXAMPLE 6 Sketch the curve $9x^2 - 4y^2 = 36$.

SOLUTION Dividing both sides by 36, we obtain

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

which is the standard form of the equation of a hyperbola (Equation 3). Since $a^2 = 4$, the x-intercepts are ± 2. Since $b^2 = 9$, we have $b = 3$ and the asymptotes are $y = \pm\frac{3}{2}x$. The hyperbola is sketched in Figure 12.

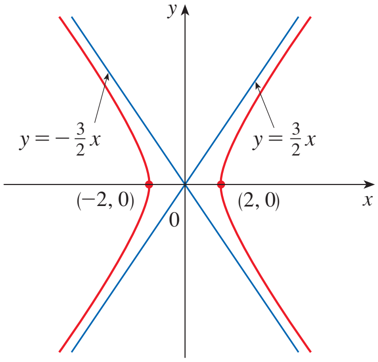
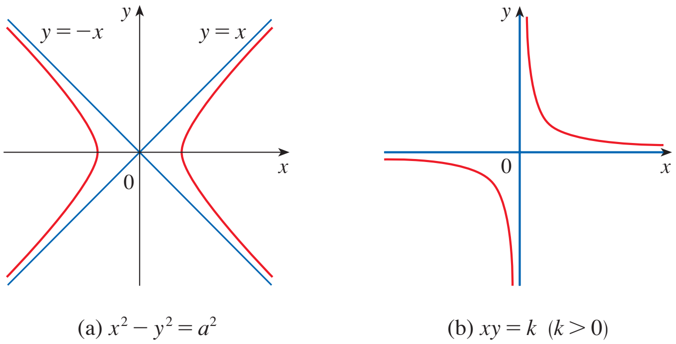


FIGURE 12
The hyperbola $9x^2 - 4y^2 = 36$

■

If $b = a$, a hyperbola has the equation $x^2 - y^2 = a^2$ (or $y^2 - x^2 = a^2$) and is called an *equilateral hyperbola* [see Figure 13(a)]. Its asymptotes are $y = \pm x$, which are perpendicular. If an equilateral hyperbola is rotated by 45°, the asymptotes become the x- and y-axes, and it can be shown that the new equation of the hyperbola is $xy = k$, where k is a constant [see Figure 13(b)].



(a) $x^2 - y^2 = a^2$

(b) $xy = k$ $(k > 0)$

FIGURE 13
Equilateral hyperbolas

■ Shifted Conics

Recall that an equation of the circle with center the origin and radius r is $x^2 + y^2 = r^2$, but if the center is the point (h, k), then the equation of the circle becomes

$$(x - h)^2 + (y - k)^2 = r^2$$

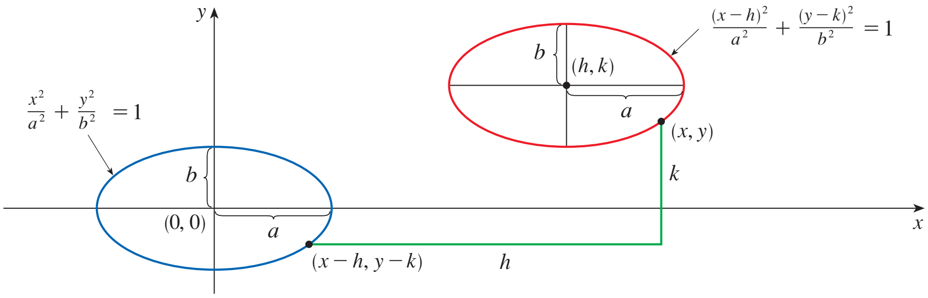
Similarly, if we take the ellipse with equation

$$\boxed{4} \qquad \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

and translate it (shift it) so that its center is the point (h, k), then its equation becomes

$$\boxed{5} \qquad \boxed{\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1}$$

(See Figure 14.)



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
y
x
b
a
(0, 0)
(x − h, y − k)
h
b
(h, k)
a
(x, y)
k

FIGURE 14

Notice that in shifting the ellipse, we replaced x by $x - h$ and y by $y - k$ in Equation 4 to obtain Equation 5. We use the same procedure to shift the parabola $y = ax^2$ so that its vertex (the origin) becomes the point (h, k) as in Figure 15. Replacing x by $x - h$ and y by $y - k$, we see that the new equation is

$$y - k = a(x - h)^2 \qquad \text{or} \qquad y = a(x - h)^2 + k$$

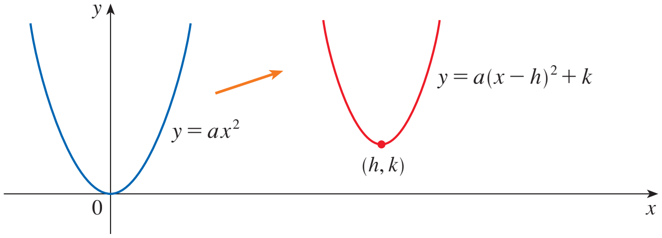


FIGURE 15

EXAMPLE 7 Sketch the graph of the equation $y = 2x^2 - 4x + 1$.

SOLUTION First we complete the square:

$$y = 2(x^2 - 2x) + 1 = 2(x - 1)^2 - 1$$

In this form we see that the equation represents the parabola obtained by shifting $y = 2x^2$ so that its vertex is at the point $(1, -1)$. The graph is sketched in Figure 16.

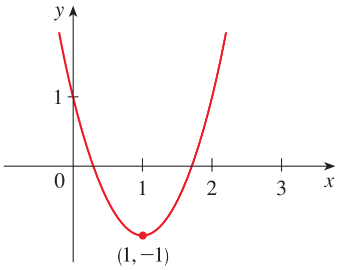


FIGURE 16
$y = 2x^2 - 4x + 1$

■

EXAMPLE 8 Sketch the curve $x = 1 - y^2$.

SOLUTION This time we start with the parabola $x = -y^2$ (as in Figure 6 with $a = -1$) and shift one unit to the right to get the graph of $x = 1 - y^2$. (See Figure 17.)

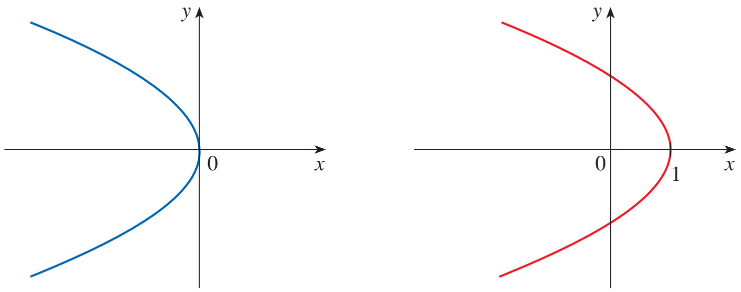


FIGURE 17

(a) $x = -y^2$

(b) $x = 1 - y^2$

■

C Exercises

1–4 Find an equation of a circle that satisfies the given conditions.

1. Center $(3, -1)$, radius 5
2. Center $(-2, -8)$, radius 10
3. Center at the origin, passes through $(4, 7)$
4. Center $(-1, 5)$, passes through $(-4, -6)$

5–9 Show that the equation represents a circle and find the center and radius.

5. $x^2 + y^2 - 4x + 10y + 13 = 0$
6. $x^2 + y^2 + 6y + 2 = 0$
7. $x^2 + y^2 + x = 0$
8. $16x^2 + 16y^2 + 8x + 32y + 1 = 0$
9. $2x^2 + 2y^2 - x + y = 1$

---

10. Under what condition on the coefficients a, b, and c does the equation $x^2 + y^2 + ax + by + c = 0$ represent a circle? When that condition is satisfied, find the center and radius of the circle.

11–32 Identify the type of curve and sketch the graph. Do not plot points. Just use the standard graphs given in Figures 5, 6, 8, 10, and 11 and shift if necessary.

11. $y = -x^2$
12. $y^2 - x^2 = 1$
13. $x^2 + 4y^2 = 16$
14. $x = -2y^2$
15. $16x^2 - 25y^2 = 400$
16. $25x^2 + 4y^2 = 100$
17. $4x^2 + y^2 = 1$
18. $y = x^2 + 2$
19. $x = y^2 - 1$
20. $9x^2 - 25y^2 = 225$
21. $9y^2 - x^2 = 9$
22. $2x^2 + 5y^2 = 10$
23. $xy = 4$
24. $y = x^2 + 2x$
25. $9(x - 1)^2 + 4(y - 2)^2 = 36$
26. $16x^2 + 9y^2 - 36y = 108$
27. $y = x^2 - 6x + 13$
28. $x^2 - y^2 - 4x + 3 = 0$
29. $x = 4 - y^2$
30. $y^2 - 2x + 6y + 5 = 0$
31. $x^2 + 4y^2 - 6x + 5 = 0$
32. $4x^2 + 9y^2 - 16x + 54y + 61 = 0$

33–34 Sketch the region bounded by the curves.

33. $y = 3x, \quad y = x^2$
34. $y = 4 - x^2, \quad x - 2y = 2$

---

35. Find an equation of the parabola with vertex $(1, -1)$ that passes through the points $(-1, 3)$ and $(3, 3)$.
36. Find an equation of the ellipse with center at the origin that passes through the points $\left(1, -10\sqrt{2}/3\right)$ and $\left(-2, 5\sqrt{5}/3\right)$.

37–40 Sketch the graph of the set.

37. $\{(x, y) \mid x^2 + y^2 \leqslant 1\}$
38. $\{(x, y) \mid x^2 + y^2 > 4\}$
39. $\{(x, y) \mid y \geqslant x^2 - 1\}$
40. $\{(x, y) \mid x^2 + 4y^2 \leqslant 4\}$

D Trigonometry

Angles

Angles can be measured in degrees or in radians (abbreviated as rad). The angle given by a complete revolution contains 360°, which is the same as 2π rad. Therefore

$$\boxed{\pi \text{ rad} = 180^\circ} \tag{1}$$

and

$$1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ \approx 57.3^\circ \qquad 1^\circ = \frac{\pi}{180} \text{ rad} \approx 0.017 \text{ rad} \tag{2}$$

EXAMPLE 1

(a) Find the radian measure of 60°. (b) Express $5\pi/4$ rad in degrees.

SOLUTION

(a) From Equation 1 or 2 we see that to convert from degrees to radians we multiply by $\pi/180$. Therefore

$$60^\circ = 60\left(\frac{\pi}{180}\right) = \frac{\pi}{3} \text{ rad}$$

(b) To convert from radians to degrees we multiply by $180/\pi$. Thus

$$\frac{5\pi}{4} \text{ rad} = \frac{5\pi}{4}\left(\frac{180}{\pi}\right) = 225^\circ$$ ■

In calculus we use radians to measure angles except when otherwise indicated. The following table gives the correspondence between degree and radian measures of some common angles.

Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

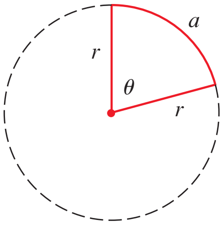


FIGURE 1

Figure 1 shows a sector of a circle with central angle θ and radius r subtending an arc with length a. Since the length of the arc is proportional to the size of the angle, and since the entire circle has circumference $2\pi r$ and central angle 2π, we have

$$\frac{\theta}{2\pi} = \frac{a}{2\pi r}$$

Solving this equation for θ and for a, we obtain

$$\boxed{\theta = \frac{a}{r}} \qquad \boxed{a = r\theta} \tag{3}$$

Remember that Equations 3 are valid only when θ is measured in radians.

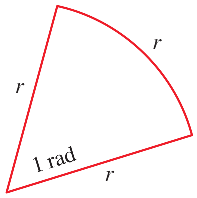


FIGURE 2

In particular, putting $a = r$ in Equation 3, we see that an angle of 1 rad is the angle subtended at the center of a circle by an arc equal in length to the radius of the circle (see Figure 2).

EXAMPLE 2

(a) If the radius of a circle is 5 cm, what angle is subtended by an arc of 6 cm?
(b) If a circle has radius 3 cm, what is the length of an arc subtended by a central angle of $3\pi/8$ rad?

SOLUTION

(a) Using Equation 3 with $a = 6$ and $r = 5$, we see that the angle is

$$\theta = \tfrac{6}{5} = 1.2 \text{ rad}$$

(b) With $r = 3$ cm and $\theta = 3\pi/8$ rad, the arc length is

$$a = r\theta = 3\left(\frac{3\pi}{8}\right) = \frac{9\pi}{8} \text{ cm}$$

■

The **standard position** of an angle occurs when we place its vertex at the origin of a coordinate system and its initial side on the positive x-axis as in Figure 3. A **positive** angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side. Likewise, **negative** angles are obtained by clockwise rotation as in Figure 4.

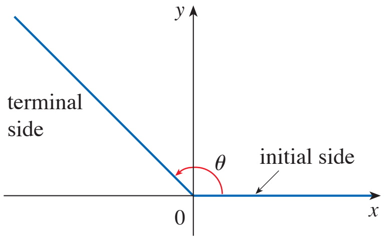


FIGURE 3 $\theta \geqslant 0$

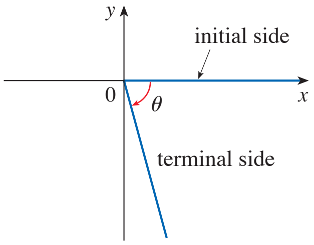


FIGURE 4 $\theta < 0$

Figure 5 shows several examples of angles in standard position. Notice that different angles can have the same terminal side. For instance, the angles $3\pi/4$, $-5\pi/4$, and $11\pi/4$ have the same initial and terminal sides because

$$\frac{3\pi}{4} - 2\pi = -\frac{5\pi}{4} \qquad \frac{3\pi}{4} + 2\pi = \frac{11\pi}{4}$$

and 2π rad represents a complete revolution.

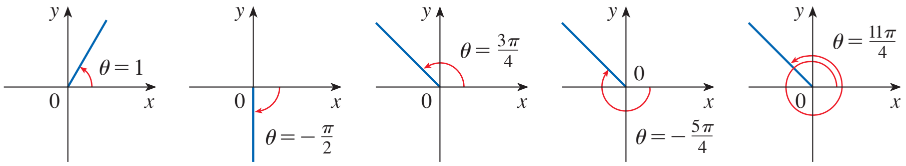


FIGURE 5
Angles in standard position

The Trigonometric Functions

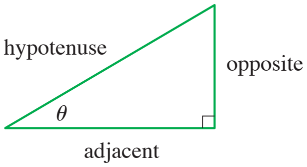


FIGURE 6

For an acute angle θ the six trigonometric functions are defined as ratios of lengths of sides of a right triangle as follows (see Figure 6).

$\boxed{4}$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

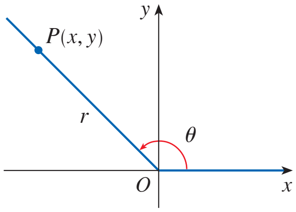


FIGURE 7

This definition doesn't apply to obtuse or negative angles, so for a general angle θ in standard position we let $P(x, y)$ be any point on the terminal side of θ and we let r be the distance $|OP|$ as in Figure 7. Then we define

$\boxed{5}$

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Since division by 0 is not defined, $\tan\theta$ and $\sec\theta$ are undefined when $x = 0$ and $\csc\theta$ and $\cot\theta$ are undefined when $y = 0$. Notice that the definitions in (4) and (5) are consistent when θ is an acute angle.

If we put $r = 1$ in Definition 5 and draw a unit circle with center the origin and label θ as in Figure 8, then the coordinates of P are $(\cos\theta, \sin\theta)$.

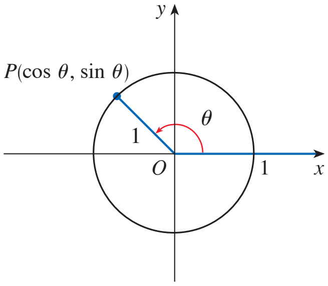


FIGURE 8

If θ is a number, the convention is that $\sin\theta$ means the sine of the angle whose *radian* measure is θ. For example, the expression $\sin 3$ implies that we are dealing with an angle of 3 rad. When finding a calculator approximation to this number, we must remember to set our calculator in radian mode, and then we obtain

$$\sin 3 \approx 0.14112$$

If we want to know the sine of the angle 3° we would write $\sin 3°$ and, with our calculator in degree mode, we find that

$$\sin 3° \approx 0.05234$$

The exact trigonometric ratios for certain angles can be read from the triangles in Figure 9. For instance,

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

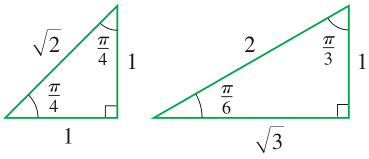


FIGURE 9

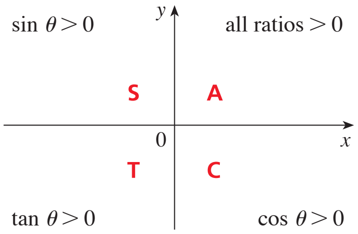


FIGURE 10

The signs of the trigonometric functions for angles in each of the four quadrants can be remembered by means of the rule "All Students Take Calculus" shown in Figure 10.

EXAMPLE 3 Find the exact trigonometric ratios for $\theta = 2\pi/3$.

SOLUTION From Figure 11 we see that a point on the terminal line for $\theta = 2\pi/3$ is $P(-1, \sqrt{3})$. Therefore, taking

$$x = -1 \qquad y = \sqrt{3} \qquad r = 2$$

in the definitions of the trigonometric ratios, we have

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{2\pi}{3} = -\frac{1}{2} \qquad \tan\frac{2\pi}{3} = -\sqrt{3}$$

$$\csc\frac{2\pi}{3} = \frac{2}{\sqrt{3}} \qquad \sec\frac{2\pi}{3} = -2 \qquad \cot\frac{2\pi}{3} = -\frac{1}{\sqrt{3}}$$ ■

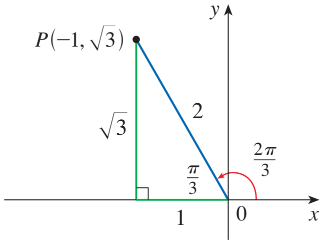


FIGURE 11

The following table gives some values of $\sin\theta$ and $\cos\theta$ found by the method of Example 3.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1

EXAMPLE 4 If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.

SOLUTION Since $\cos\theta = \frac{2}{5}$, we can label the hypotenuse as having length 5 and the adjacent side as having length 2 in Figure 12. If the opposite side has length x, then the Pythagorean Theorem gives $x^2 + 4 = 25$ and so $x^2 = 21$, $x = \sqrt{21}$. We can now use the diagram to write the other five trigonometric functions:

$$\sin\theta = \frac{\sqrt{21}}{5} \qquad \tan\theta = \frac{\sqrt{21}}{2}$$

$$\csc\theta = \frac{5}{\sqrt{21}} \qquad \sec\theta = \frac{5}{2} \qquad \cot\theta = \frac{2}{\sqrt{21}}$$ ■

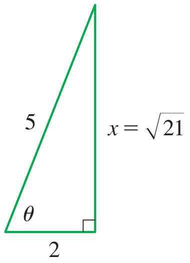


FIGURE 12

EXAMPLE 5 Use a calculator to approximate the value of x in Figure 13.

SOLUTION From the diagram we see that

$$\tan 40° = \frac{16}{x}$$

Therefore

$$x = \frac{16}{\tan 40°} \approx 19.07$$ ■

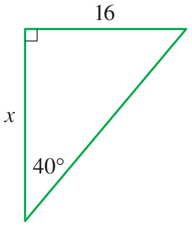


FIGURE 13

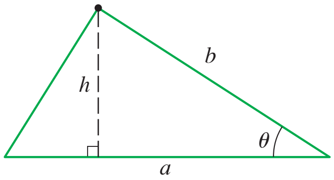


FIGURE 14

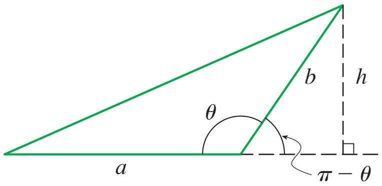


FIGURE 15

If θ is an acute angle, then the height of the triangle in Figure 14 is $h = b \sin \theta$, so the area $\mathcal{A}$ of the triangle is

$$\mathcal{A} = \tfrac{1}{2}(\text{base})(\text{height}) = \tfrac{1}{2}ab \sin \theta$$

If θ is obtuse, as in Figure 15, then the height is $h = b \sin(\pi - \theta) = b \sin \theta$ (see Exercise 44). Thus in either case, the area of a triangle with sides of lengths a and b and with included angle θ is

6 $$\mathcal{A} = \tfrac{1}{2}ab \sin \theta$$

EXAMPLE 6 Find the area of an equilateral triangle with sides of length a.

SOLUTION Each angle of an equilateral triangle has measure $\pi/3$ and is included by sides of lengths a, so the area of the triangle is

$$\mathcal{A} = \frac{1}{2}a^2 \sin \frac{\pi}{3} = \frac{\sqrt{3}}{4}a^2$$ ■

Trigonometric Identities

A trigonometric identity is a relationship among the trigonometric functions. The most elementary are the following, which are immediate consequences of the definitions of the trigonometric functions.

7 $$\boxed{\begin{array}{ccc} \csc \theta = \dfrac{1}{\sin \theta} & \sec \theta = \dfrac{1}{\cos \theta} & \cot \theta = \dfrac{1}{\tan \theta} \\[2ex] \tan \theta = \dfrac{\sin \theta}{\cos \theta} & & \cot \theta = \dfrac{\cos \theta}{\sin \theta} \end{array}}$$

For the next identity we refer back to Figure 7. The distance formula (or, equivalently, the Pythagorean Theorem) tells us that $x^2 + y^2 = r^2$. Therefore

$$\sin^2\theta + \cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

We have therefore proved one of the most useful of all trigonometric identities:

8 $$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

If we now divide both sides of Equation 8 by $\cos^2\theta$ and use Equations 7, we get

9 $$\boxed{\tan^2\theta + 1 = \sec^2\theta}$$

Similarly, if we divide both sides of Equation 8 by $\sin^2\theta$, we get

10 $$\boxed{1 + \cot^2\theta = \csc^2\theta}$$

The identities

$$\boxed{\begin{aligned} \sin(-\theta) &= -\sin\theta \qquad \text{11a} \\ \cos(-\theta) &= \cos\theta \qquad \text{11b} \end{aligned}}$$

Odd functions and even functions are discussed in Section 1.1.

show that sine is an odd function and cosine is an even function. They are easily proved by drawing a diagram showing θ and $-\theta$ in standard position (see Exercise 39).

Since the angles θ and $\theta + 2\pi$ have the same terminal side, we have

$$\sin(\theta + 2\pi) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta \tag{12}$$

These identities show that the sine and cosine functions are periodic with period 2π.

The remaining trigonometric identities are all consequences of two basic identities called the **addition formulas**:

$$\boxed{\begin{aligned} \sin(x + y) &= \sin x \cos y + \cos x \sin y \qquad \text{13a} \\ \cos(x + y) &= \cos x \cos y - \sin x \sin y \qquad \text{13b} \end{aligned}}$$

The proofs of these addition formulas are outlined in Exercises 89, 90, and 91.

By substituting $-y$ for y in Equations 13a and 13b and using Equations 11a and 11b, we obtain the following **subtraction formulas**:

$$\boxed{\begin{aligned} \sin(x - y) &= \sin x \cos y - \cos x \sin y \qquad \text{14a} \\ \cos(x - y) &= \cos x \cos y + \sin x \sin y \qquad \text{14b} \end{aligned}}$$

Then, by dividing the formulas in Equations 13 or Equations 14, we obtain the corresponding formulas for $\tan(x \pm y)$:

$$\boxed{\begin{aligned} \tan(x + y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} \qquad \text{15a} \\[2ex] \tan(x - y) &= \frac{\tan x - \tan y}{1 + \tan x \tan y} \qquad \text{15b} \end{aligned}}$$

If we put $y = x$ in the addition formulas (13), we get the **double-angle formulas**:

$$\boxed{\begin{aligned} \sin 2x &= 2 \sin x \cos x \qquad \text{16a} \\ \cos 2x &= \cos^2 x - \sin^2 x \qquad \text{16b} \end{aligned}}$$

Then, by using the identity $\sin^2 x + \cos^2 x = 1$, we obtain the following alternate forms of the double-angle formulas for $\cos 2x$:

$$\boxed{\begin{aligned} \cos 2x &= 2\cos^2 x - 1 \qquad \text{17a} \\ \cos 2x &= 1 - 2\sin^2 x \qquad \text{17b} \end{aligned}}$$

If we now solve these equations for $\cos^2 x$ and $\sin^2 x$, we get the following **half-angle formulas**, which are useful in integral calculus:

18a $$\cos^2 x = \frac{1 + \cos 2x}{2}$$

18b $$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Finally, we state the **product identities**, which can be deduced from Equations 13 and 14:

19a $$\sin x \cos y = \tfrac{1}{2}[\sin(x + y) + \sin(x - y)]$$

19b $$\cos x \cos y = \tfrac{1}{2}[\cos(x + y) + \cos(x - y)]$$

19c $$\sin x \sin y = \tfrac{1}{2}[\cos(x - y) - \cos(x + y)]$$

There are many other trigonometric identities, but those we have stated are the ones used most often in calculus. If you forget any of the identities 14–19, remember that they can all be deduced from Equations 13a and 13b.

EXAMPLE 7 Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$.

SOLUTION Using the double-angle formula (16a), we rewrite the given equation as

$$\sin x = 2 \sin x \cos x \qquad \text{or} \qquad \sin x\,(1 - 2\cos x) = 0$$

Therefore there are two possibilities:

$$\begin{array}{lcl} \sin x = 0 & \qquad \text{or} \qquad & 1 - 2\cos x = 0 \\ x = 0, \pi, 2\pi & & \cos x = \tfrac{1}{2} \\ & & x = \dfrac{\pi}{3}, \dfrac{5\pi}{3} \end{array}$$

The given equation has five solutions: 0, $\pi/3$, π, $5\pi/3$, and 2π. ■

The Law of Sines and the Law of Cosines

The **Law of Sines** states that in any triangle the lengths of the sides are proportional to the sines of the corresponding opposite angles. We let A, B, C denote the vertices of the triangle as well as the angles at these vertices, as appropriate, and we follow the convention of labeling the lengths of the corresponding opposite sides as a, b, and c, as shown in Figure 16.

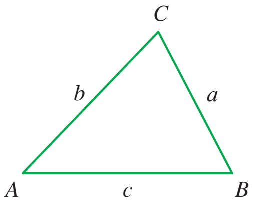


FIGURE 16

Law of Sines In any triangle ABC

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

PROOF By Formula 6 the area of the triangle ABC in Figure 16 is $\frac{1}{2}ab\sin C$. By the same formula the area is also $\frac{1}{2}ac\sin B$ and $\frac{1}{2}bc\sin A$. So

$$\tfrac{1}{2}bc\sin A = \tfrac{1}{2}ac\sin B = \tfrac{1}{2}ab\sin C$$

and multiplying by $2/abc$ gives the Law of Sines. ■

The **Law of Cosines** expresses the length of a side of a triangle in terms of the other two side lengths and the included angle.

Law of Cosines In any triangle ABC

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

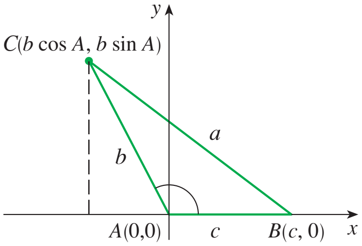


FIGURE 17

PROOF We prove the first formula; the remaining two are proved similarly. Place the triangle in the coordinate plane so that vertex A is at the origin, as shown in Figure 17. The coordinates of vertices B and C are $(c, 0)$ and $(b\cos A, b\sin A)$, respectively. (You should check that the coordinates of these points will be the same if we draw angle A as an acute angle.) By the distance formula we have

$$\begin{aligned} a^2 &= (b\cos A - c)^2 + (b\sin A - 0)^2 \\ &= b^2\cos^2 A - 2bc\cos A + c^2 + b^2\sin^2 A \\ &= b^2(\cos^2 A + \sin^2 A) - 2bc\cos A + c^2 \\ &= b^2 + c^2 - 2bc\cos A \qquad \text{(by Formula 8)} \end{aligned}$$ ■

The Law of Cosines can be used to prove the following area formula in which we need only know the side lengths of a triangle.

Heron's Formula The area $\mathcal{A}$ of any triangle ABC is given by

$$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a + b + c)$ is the *semiperimeter* of the triangle.

PROOF First, by the Law of Cosines,

$$\begin{aligned} 1 + \cos C = 1 + \frac{a^2 + b^2 - c^2}{2ab} &= \frac{2ab + a^2 + b^2 - c^2}{2ab} \\ &= \frac{(a+b)^2 - c^2}{2ab} = \frac{(a+b+c)(a+b-c)}{2ab} \end{aligned}$$

Similarly,

$$1 - \cos C = \frac{(c + a - b)(c - a + b)}{2ab}$$

Then, by Formula 6,

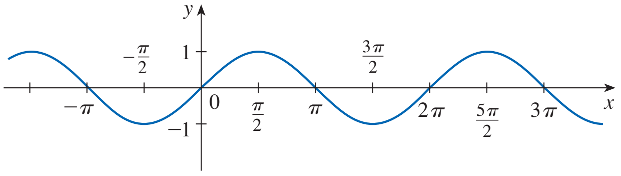
$$\begin{aligned}
\mathcal{A}^2 &= \tfrac{1}{4}a^2b^2 \sin^2\theta = \tfrac{1}{4}a^2b^2(1 - \cos^2\theta) \\
&= \tfrac{1}{4}a^2b^2(1 + \cos\theta)(1 - \cos\theta) \\
&= \tfrac{1}{4}a^2b^2 \frac{(a + b + c)(a + b - c)}{2ab} \frac{(c + a - b)(c - a + b)}{2ab} \\
&= \frac{(a + b + c)}{2} \frac{(a + b - c)}{2} \frac{(c + a - b)}{2} \frac{(c - a + b)}{2} \\
&= s(s - c)(s - b)(s - a)
\end{aligned}$$

Taking the square root of each side gives Heron's formula. ■

Graphs of the Trigonometric Functions

The graph of the function $f(x) = \sin x$, shown in Figure 18(a), is obtained by plotting points for $0 \leqslant x \leqslant 2\pi$ and then using the periodic nature of the function (from Equation 12) to complete the graph. Notice that the zeros of the sine function occur at the integer multiples of π, that is,

$$\sin x = 0 \qquad \text{whenever } x = n\pi, \quad n \text{ an integer}$$



(a) $f(x) = \sin x$

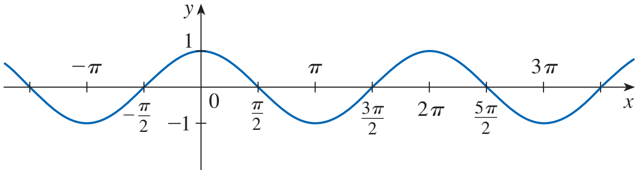


FIGURE 18 (b) $g(x) = \cos x$

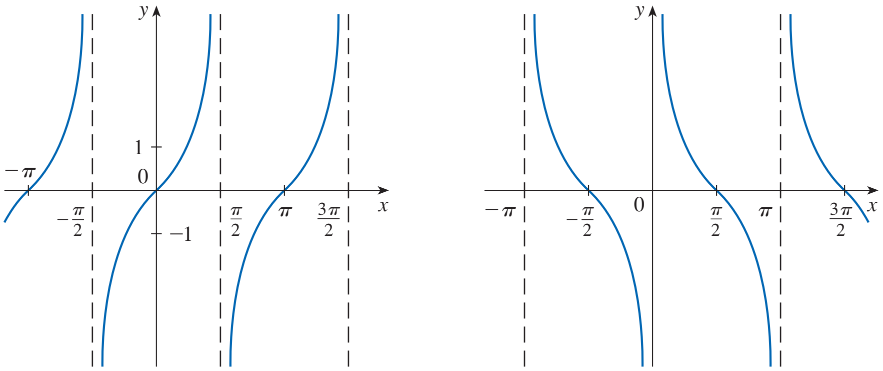
Because of the identity

$$\cos x = \sin\left(x + \frac{\pi}{2}\right)$$

(which can be verified using Equation 13a), the graph of cosine is obtained by shifting the graph of sine by an amount $\pi/2$ to the left [see Figure 18(b)]. Note that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$. Thus, for all values of x, we have

$$\boxed{-1 \leqslant \sin x \leqslant 1 \qquad -1 \leqslant \cos x \leqslant 1}$$

The graphs of the remaining four trigonometric functions are shown in Figure 19 and their domains are indicated there. Notice that tangent and cotangent have range $(-\infty, \infty)$, whereas cosecant and secant have range $(-\infty, -1] \cup [1, \infty)$. All four functions are periodic: tangent and cotangent have period π, whereas cosecant and secant have period 2π.



(a) $y = \tan x$ (b) $y = \cot x$

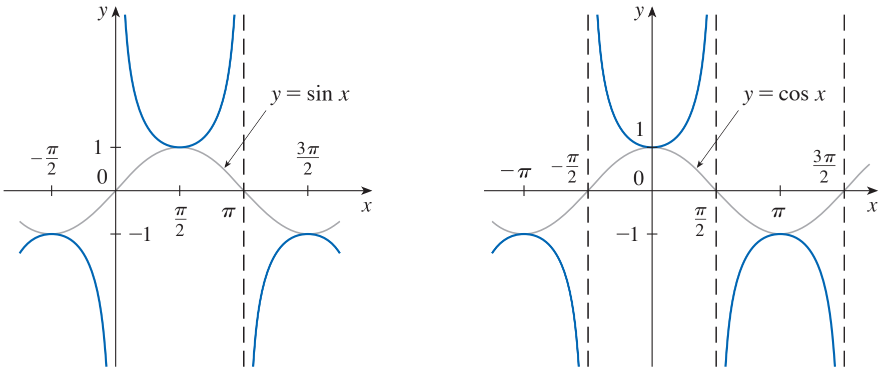


FIGURE 19 (c) $y = \csc x$ (d) $y = \sec x$

D Exercises

1–6 Convert from degrees to radians.

1. 210° **2.** 300° **3.** 9°

4. −315° **5.** 900° **6.** 36°

7–12 Convert from radians to degrees.

7. 4π **8.** $-\dfrac{7\pi}{2}$ **9.** $\dfrac{5\pi}{12}$

10. $\dfrac{8\pi}{3}$ **11.** $-\dfrac{3\pi}{8}$ **12.** 5

13. Find the length of a circular arc subtended by an angle of $\pi/12$ rad if the radius of the circle is 36 cm.

14. If a circle has radius 10 cm, find the length of the arc subtended by a central angle of 72°.

15. A circle has radius 1.5 m. What angle is subtended at the center of the circle by an arc 1 m long?

16. Find the radius of a circular sector with angle $3\pi/4$ and arc length 6 cm.

17–22 Draw, in standard position, the angle whose measure is given.

17. $315°$

18. $-150°$

19. $-\dfrac{3\pi}{4}$ rad

20. $\dfrac{7\pi}{3}$ rad

21. 2 rad

22. -3 rad

23–28 Find the exact trigonometric ratios for the angle whose radian measure is given.

23. $\dfrac{3\pi}{4}$

24. $\dfrac{4\pi}{3}$

25. $\dfrac{9\pi}{2}$

26. -5π

27. $\dfrac{5\pi}{6}$

28. $\dfrac{11\pi}{4}$

29–34 Find the remaining trigonometric ratios.

29. $\sin\theta = \dfrac{3}{5},\quad 0 < \theta < \dfrac{\pi}{2}$

30. $\tan\alpha = 2,\quad 0 < \alpha < \dfrac{\pi}{2}$

31. $\sec\phi = -1.5,\quad \dfrac{\pi}{2} < \phi < \pi$

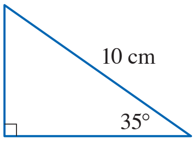
32. $\cos x = -\dfrac{1}{3},\quad \pi < x < \dfrac{3\pi}{2}$

33. $\cot\beta = 3,\quad \pi < \beta < 2\pi$

34. $\csc\theta = -\dfrac{4}{3},\quad \dfrac{3\pi}{2} < \theta < 2\pi$

35–38 Find, correct to five decimal places, the length of the side labeled x.

35.



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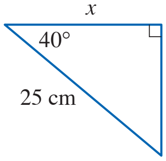

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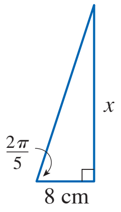

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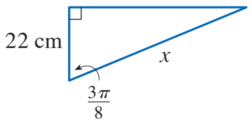


39–41 Prove each equation.

39. (a) Equation 11a (b) Equation 11b

40. (a) Equation 15a (b) Equation 15b

41. (a) Equation 19a (b) Equation 19b
(c) Equation 19c

42–58 Prove the identity.

42. $\cos\left(\dfrac{\pi}{2} - x\right) = \sin x$

43. $\sin\left(\dfrac{\pi}{2} + x\right) = \cos x$

44. $\sin(\pi - x) = \sin x$

45. $\sin\theta\ \cot\theta = \cos\theta$

46. $(\sin x + \cos x)^2 = 1 + \sin 2x$

47. $\sec y - \cos y = \tan y\ \sin y$

48. $\tan^2\alpha - \sin^2\alpha = \tan^2\alpha\ \sin^2\alpha$

49. $\cot^2\theta + \sec^2\theta = \tan^2\theta + \csc^2\theta$

50. $2\csc 2t = \sec t\ \csc t$

51. $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

52. $\dfrac{1}{1 - \sin\theta} + \dfrac{1}{1 + \sin\theta} = 2\sec^2\theta$

53. $\sin x\ \sin 2x + \cos x\ \cos 2x = \cos x$

54. $\sin^2 x - \sin^2 y = \sin(x + y)\ \sin(x - y)$

55. $\dfrac{\sin\phi}{1 - \cos\phi} = \csc\phi + \cot\phi$

56. $\tan x + \tan y = \dfrac{\sin(x + y)}{\cos x\ \cos y}$

57. $\sin 3\theta + \sin\theta = 2\sin 2\theta\ \cos\theta$

58. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

59–64 If $\sin x = \frac{1}{3}$ and $\sec y = \frac{5}{4}$, where x and y lie between 0 and $\pi/2$, evaluate the expression.

59. $\sin(x + y)$

60. $\cos(x + y)$

61. $\cos(x - y)$

62. $\sin(x - y)$

63. $\sin 2y$

64. $\cos 2y$

65–72 Find all values of x in the interval $[0, 2\pi]$ that satisfy the equation.

65. $2 \cos x - 1 = 0$

66. $3 \cot^2 x = 1$

67. $2 \sin^2 x = 1$

68. $|\tan x| = 1$

69. $\sin 2x = \cos x$

70. $2 \cos x + \sin 2x = 0$

71. $\sin x = \tan x$

72. $2 + \cos 2x = 3 \cos x$

73–76 Find all values of x in the interval $[0, 2\pi]$ that satisfy the inequality.

73. $\sin x \leqslant \frac{1}{2}$

74. $2 \cos x + 1 > 0$

75. $-1 < \tan x < 1$

76. $\sin x > \cos x$

---

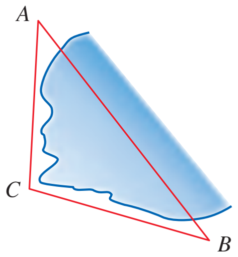
77. In triangle ABC, $\angle A = 50°$, $\angle B = 68°$, and $c = 230$. Use the Law of Sines to find the remaining side lengths and angles, correct to two decimal places.

78. In triangle ABC, $a = 3.0$, $b = 4.0$, and $\angle C = 53°$. Use the Law of Cosines to find c, correct to two decimal places.

79. In order to find the distance $|AB|$ across a small inlet, a point C was located as in the figure and the following measurements were recorded:

$$\angle C = 103° \qquad |AC| = 820 \text{ m} \qquad |BC| = 910 \text{ m}$$

Use the Law of Cosines to find the required distance.



80. In triangle ABC, $a = 100$, $c = 200$, and $\angle B = 160°$. Find b and $\angle A$, correct to two decimal places.

81. Find the area of triangle ABC, correct to five decimal places, if

$$|AB| = 10 \text{ cm} \qquad |BC| = 3 \text{ cm} \qquad \angle B = 107°$$

82. In triangle ABC, $a = 4$, $b = 5$, and $c = 7$. Find the area of the triangle.

83–88 Graph the function by starting with the graphs in Figures 18 and 19 and applying the transformations of Section 1.3 where appropriate.

83. $y = \cos\left(x - \dfrac{\pi}{3}\right)$

84. $y = \tan 2x$

85. $y = \dfrac{1}{3} \tan\left(x - \dfrac{\pi}{2}\right)$

86. $y = 1 + \sec x$

87. $y = |\sin x|$

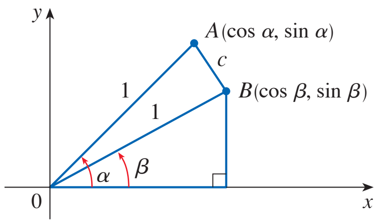
88. $y = 2 + \sin\left(x + \dfrac{\pi}{4}\right)$

---

89. Use the figure to prove the subtraction formula

$$\cos(\alpha - \beta) = \cos \alpha \ \cos \beta + \sin \alpha \ \sin \beta$$

[*Hint:* Compute c^2 in two ways (using the Law of Cosines and the distance formula) and compare the two expressions.]



90. Use the formula in Exercise 89 to prove the addition formula for cosine (13b).

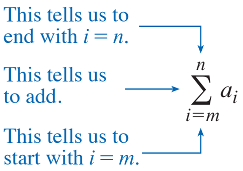
91. Use the addition formula for cosine and the identities

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta \qquad \sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$$

to prove the subtraction formula (14a) for the sine function.

E Sigma Notation

A convenient way of writing sums uses the Greek letter Σ (capital sigma, corresponding to our letter S) and is called **sigma notation**.


This tells us to end with $i = n$.
This tells us to add.
$\displaystyle\sum_{i=m}^{n} a_i$
This tells us to start with $i = m$.

1 Definition If $a_m, a_{m+1}, \ldots, a_n$ are real numbers and m and n are integers such that $m \leqslant n$, then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

With function notation, Definition 1 can be written as

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n-1) + f(n)$$

Thus the symbol $\Sigma_{i=m}^{n}$ indicates a summation in which the letter i (called the **index of summation**) takes on consecutive integer values beginning with m and ending with n, that is, $m, m+1, \ldots, n$. Other letters can also be used as the index of summation.

EXAMPLE 1

(a) $\displaystyle\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$

(b) $\displaystyle\sum_{i=3}^{n} i = 3 + 4 + 5 + \cdots + (n-1) + n$

(c) $\displaystyle\sum_{j=0}^{5} 2^j = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 63$

(d) $\displaystyle\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

(e) $\displaystyle\sum_{i=1}^{3} \frac{i-1}{i^2+3} = \frac{1-1}{1^2+3} + \frac{2-1}{2^2+3} + \frac{3-1}{3^2+3} = 0 + \frac{1}{7} + \frac{1}{6} = \frac{13}{42}$

(f) $\displaystyle\sum_{i=1}^{4} 2 = 2 + 2 + 2 + 2 = 8$ ■

EXAMPLE 2 Write the sum $2^3 + 3^3 + \cdots + n^3$ in sigma notation.

SOLUTION There is no unique way of writing a sum in sigma notation. We could write

$$2^3 + 3^3 + \cdots + n^3 = \sum_{i=2}^{n} i^3$$

or

$$2^3 + 3^3 + \cdots + n^3 = \sum_{j=1}^{n-1} (j+1)^3$$

or

$$2^3 + 3^3 + \cdots + n^3 = \sum_{k=0}^{n-2} (k+2)^3$$ ■

The following theorem gives three simple rules for working with sigma notation.

2 Theorem If c is any constant (that is, it does not depend on i), then

(a) $\displaystyle\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$ (b) $\displaystyle\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$

(c) $\displaystyle\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$

PROOF To see why these rules are true, all we have to do is write both sides in expanded form. Rule (a) is just the distributive property of real numbers:

$$ca_m + ca_{m+1} + \cdots + ca_n = c(a_m + a_{m+1} + \cdots + a_n)$$

Rule (b) follows from the associative and commutative properties:

$$\begin{aligned}(a_m + b_m) &+ (a_{m+1} + b_{m+1}) + \cdots + (a_n + b_n)\\ &= (a_m + a_{m+1} + \cdots + a_n) + (b_m + b_{m+1} + \cdots + b_n)\end{aligned}$$

Rule (c) is proved similarly. ■

EXAMPLE 3 Find $\displaystyle\sum_{i=1}^{n} 1$.

SOLUTION

$$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + \cdots + 1}_{n \text{ terms}} = n$$ ■

EXAMPLE 4 Prove the formula for the sum of the first n positive integers:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Mathematical induction is discussed in Principles of Problem Solving following Chapter 1.

SOLUTION This formula can be proved by mathematical induction or by the following method used by the German mathematician Karl Friedrich Gauss (1777–1855) when he was ten years old.

Write the sum S twice, once in the usual order and once in reverse order:

$$\begin{aligned}S &= 1 + 2 + 3 + \cdots + (n-1) + n\\ S &= n + (n-1) + (n-2) + \cdots + 2 + 1\end{aligned}$$

Adding all columns vertically, we get

$$2S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1)$$

On the right side there are n terms, each of which is $n + 1$, so

$$2S = n(n+1) \qquad \text{or} \qquad S = \frac{n(n+1)}{2}$$ ■

EXAMPLE 5 Prove the formula for the sum of the squares of the first n positive integers:

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

SOLUTION 1 Let S be the desired sum. We start with the *telescoping sum* (or collapsing sum):

Most terms cancel in pairs.

$$\begin{aligned}\sum_{i=1}^{n} [(1+i)^3 - i^3] &= (\cancel{2^3} - 1^3) + (\cancel{3^3} - \cancel{2^3}) + (\cancel{4^3} - \cancel{3^3}) + \cdots + [(n+1)^3 - \cancel{n^3}] \\ &= (n+1)^3 - 1^3 = n^3 + 3n^2 + 3n\end{aligned}$$

On the other hand, using Theorem 2 and Examples 3 and 4, we have

$$\begin{aligned}\sum_{i=1}^{n} [(1+i)^3 - i^3] &= \sum_{i=1}^{n} [3i^2 + 3i + 1] = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \\ &= 3S + 3\,\frac{n(n+1)}{2} + n = 3S + \tfrac{3}{2}n^2 + \tfrac{5}{2}n\end{aligned}$$

Thus we have

$$n^3 + 3n^2 + 3n = 3S + \tfrac{3}{2}n^2 + \tfrac{5}{2}n$$

Solving this equation for S, we obtain

$$3S = n^3 + \tfrac{3}{2}n^2 + \tfrac{1}{2}n$$

or

$$S = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$$

Principle of Mathematical Induction

Let S_n be a statement involving the positive integer n. Suppose that

1. S_1 is true.
2. If S_k is true, then S_{k+1} is true.

Then S_n is true for all positive integers n.

SOLUTION 2 Let S_n be the given formula.

1. S_1 is true because $\quad 1^2 = \dfrac{1(1+1)(2\cdot 1+1)}{6}$

2. Assume that S_k is true; that is,

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Then

$$\begin{aligned}1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 &= (1^2 + 2^2 + 3^2 + \cdots + k^2) + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= (k+1)\,\frac{k(2k+1) + 6(k+1)}{6} \\ &= (k+1)\,\frac{2k^2 + 7k + 6}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}\end{aligned}$$

So S_{k+1} is true.

By the Principle of Mathematical Induction, S_n is true for all n. ■

We list the results of Examples 3, 4, and 5 together with a similar result for cubes (see Exercises 37–40) as Theorem 3. These formulas are needed for finding areas and evaluating integrals in Chapter 5.

3 Theorem Let c be a constant and n a positive integer. Then

(a) $\displaystyle\sum_{i=1}^{n} 1 = n$

(b) $\displaystyle\sum_{i=1}^{n} c = nc$

(c) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(d) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(e) $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

EXAMPLE 6 Evaluate $\displaystyle\sum_{i=1}^{n} i(4i^2-3)$.

SOLUTION Using Theorems 2 and 3, we have

$$\begin{aligned}
\sum_{i=1}^{n} i(4i^2-3) &= \sum_{i=1}^{n} (4i^3-3i) = 4\sum_{i=1}^{n} i^3 - 3\sum_{i=1}^{n} i \\
&= 4\left[\frac{n(n+1)}{2}\right]^2 - 3\,\frac{n(n+1)}{2} \\
&= \frac{n(n+1)[2n(n+1)-3]}{2} \\
&= \frac{n(n+1)(2n^2+2n-3)}{2}
\end{aligned}$$

■

EXAMPLE 7 Find $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left[\left(\frac{i}{n}\right)^2+1\right]$.

The type of calculation in Example 7 arises in Chapter 5 when we compute areas.

SOLUTION

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left[\left(\frac{i}{n}\right)^2+1\right] &= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{3}{n^3}i^2+\frac{3}{n}\right] \\
&= \lim_{n\to\infty}\left[\frac{3}{n^3}\sum_{i=1}^{n} i^2 + \frac{3}{n}\sum_{i=1}^{n} 1\right] \\
&= \lim_{n\to\infty}\left[\frac{3}{n^3}\,\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}\cdot n\right] \\
&= \lim_{n\to\infty}\left[\frac{1}{2}\cdot\frac{n}{n}\cdot\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)+3\right] \\
&= \lim_{n\to\infty}\left[\frac{1}{2}\cdot 1\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+3\right] \\
&= \tfrac{1}{2}\cdot 1\cdot 1\cdot 2 + 3 = 4
\end{aligned}$$

■

E Exercises

1–10 Write the sum in expanded form.

1. $\displaystyle\sum_{i=1}^{5} \sqrt{i}$

2. $\displaystyle\sum_{i=1}^{6} \frac{1}{i+1}$

3. $\displaystyle\sum_{i=4}^{6} 3^i$

4. $\displaystyle\sum_{i=4}^{6} i^3$

5. $\displaystyle\sum_{k=0}^{4} \frac{2k-1}{2k+1}$

6. $\displaystyle\sum_{k=5}^{8} x^k$

7. $\displaystyle\sum_{i=1}^{n} i^{10}$

8. $\displaystyle\sum_{j=n}^{n+3} j^2$

9. $\displaystyle\sum_{j=0}^{n-1} (-1)^j$

10. $\displaystyle\sum_{i=1}^{n} f(x_i)\,\Delta x_i$

11–20 Write the sum in sigma notation.

11. $1 + 2 + 3 + 4 + \cdots + 10$

12. $\sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6} + \sqrt{7}$

13. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots + \frac{19}{20}$

14. $\frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + \cdots + \frac{23}{27}$

15. $2 + 4 + 6 + 8 + \cdots + 2n$

16. $1 + 3 + 5 + 7 + \cdots + (2n - 1)$

17. $1 + 2 + 4 + 8 + 16 + 32$

18. $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36}$

19. $x + x^2 + x^3 + \cdots + x^n$

20. $1 - x + x^2 - x^3 + \cdots + (-1)^n x^n$

21–35 Find the value of the sum.

21. $\displaystyle\sum_{i=4}^{8} (3i - 2)$

22. $\displaystyle\sum_{i=3}^{6} i(i + 2)$

23. $\displaystyle\sum_{j=1}^{6} 3^{j+1}$

24. $\displaystyle\sum_{k=0}^{8} \cos k\pi$

25. $\displaystyle\sum_{n=1}^{20} (-1)^n$

26. $\displaystyle\sum_{i=1}^{100} 4$

27. $\displaystyle\sum_{i=0}^{4} (2^i + i^2)$

28. $\displaystyle\sum_{i=-2}^{4} 2^{3-i}$

29. $\displaystyle\sum_{i=1}^{n} 2i$

30. $\displaystyle\sum_{i=1}^{n} (2 - 5i)$

31. $\displaystyle\sum_{i=1}^{n} (i^2 + 3i + 4)$

32. $\displaystyle\sum_{i=1}^{n} (3 + 2i)^2$

33. $\displaystyle\sum_{i=1}^{n} (i + 1)(i + 2)$

34. $\displaystyle\sum_{i=1}^{n} i(i + 1)(i + 2)$

35. $\displaystyle\sum_{i=1}^{n} (i^3 - i - 2)$

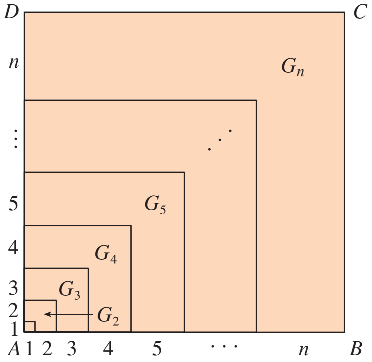
36. Find the number n such that $\displaystyle\sum_{i=1}^{n} i = 78$.

37. Prove formula (b) of Theorem 3.

38. Prove formula (e) of Theorem 3 using mathematical induction.

39. Prove formula (e) of Theorem 3 using a method similar to that of Example 5, Solution 1 [start with $(1 + i)^4 - i^4$].

40. Prove formula (e) of Theorem 3 using the following method published by Abu Bekr Mohammed ibn Alhusain Alkarchi in about AD 1010. The figure shows a square *ABCD* in which sides *AB* and *AD* have been divided into segments of lengths $1, 2, 3, \ldots, n$. Thus the side of the square has length $n(n + 1)/2$ so the area is $[n(n + 1)/2]^2$. But the area is also the sum of the areas of the n "gnomons" $G_1, G_2, \ldots, G_n$ shown in the figure. Show that the area of G_i is i^3 and conclude that formula (e) is true.



41. Evaluate each telescoping sum.

(a) $\displaystyle\sum_{i=1}^{n} [i^4 - (i - 1)^4]$

(b) $\displaystyle\sum_{i=1}^{100} (5^i - 5^{i-1})$

(c) $\displaystyle\sum_{i=3}^{99} \left(\frac{1}{i} - \frac{1}{i+1}\right)$

(d) $\displaystyle\sum_{i=1}^{n} (a_i - a_{i-1})$

42. Prove the generalized triangle inequality:

$$\left|\sum_{i=1}^{n} a_i\right| \leqslant \sum_{i=1}^{n} |a_i|$$

43–46 Find the limit.

43. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2$

44. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^3 + 1\right]$

45. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[\left(\frac{2i}{n}\right)^3 + 5\left(\frac{2i}{n}\right)\right]$

46. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left[\left(1+\frac{3i}{n}\right)^3-2\left(1+\frac{3i}{n}\right)\right]$

47. Prove the formula for the sum of a finite geometric series with first term a and common ratio $r \neq 1$:

$$\sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n-1)}{r-1}$$

48. Evaluate $\displaystyle\sum_{i=1}^{n}\frac{3}{2^{i-1}}$.

49. Evaluate $\displaystyle\sum_{i=1}^{n}(2i+2^i)$.

50. Evaluate $\displaystyle\sum_{i=1}^{m}\left[\sum_{j=1}^{n}(i+j)\right]$.

F Proofs of Theorems

In this appendix we present proofs of several theorems that are stated in the main body of the text. The sections in which they occur are indicated in the margin.

Section 2.3

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) = L \qquad \text{and} \qquad \lim_{x\to a} g(x) = M$$

exist. Then

1. $\displaystyle\lim_{x\to a}[f(x)+g(x)] = L + M$

2. $\displaystyle\lim_{x\to a}[f(x)-g(x)] = L - M$

3. $\displaystyle\lim_{x\to a}[cf(x)] = cL$

4. $\displaystyle\lim_{x\to a}[f(x)g(x)] = LM$

5. $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$

PROOF OF LAW 4 Let $\varepsilon > 0$ be given. We want to find $\delta > 0$ such that

$$\text{if} \quad 0 < |x-a| < \delta \quad \text{then} \quad |f(x)g(x) - LM| < \varepsilon$$

In order to get terms that contain $|f(x) - L|$ and $|g(x) - M|$, we add and subtract $Lg(x)$ as follows:

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\
&= |[f(x) - L]g(x) + L[g(x) - M]| \\
&\leqslant |[f(x) - L]g(x)| + |L[g(x) - M]| \qquad \text{(Triangle Inequality)} \\
&= |f(x) - L||g(x)| + |L||g(x) - M|
\end{aligned}$$

We want to make each of these terms less than $\varepsilon/2$.

Since $\lim_{x\to a} g(x) = M$, there is a number $\delta_1 > 0$ such that

$$\text{if} \quad 0 < |x-a| < \delta_1 \quad \text{then} \quad |g(x) - M| < \frac{\varepsilon}{2(1+|L|)}$$

Also, there is a number $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$, then

$$|g(x) - M| < 1$$

and therefore

$$|g(x)| = |g(x) - M + M| \leqslant |g(x) - M| + |M| < 1 + |M|$$

Since $\lim_{x \to a} f(x) = L$, there is a number $\delta_3 > 0$ such that

$$\text{if} \qquad 0 < |x - a| < \delta_3 \qquad \text{then} \qquad |f(x) - L| < \frac{\varepsilon}{2(1 + |M|)}$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. If $0 < |x - a| < \delta$, then we have $0 < |x - a| < \delta_1$, $0 < |x - a| < \delta_2$, and $0 < |x - a| < \delta_3$, so we can combine the inequalities to obtain

$$\begin{aligned} |f(x)g(x) - LM| &\leqslant |f(x) - L|\,|g(x)| + |L|\,|g(x) - M| \\ &< \frac{\varepsilon}{2(1 + |M|)}(1 + |M|) + |L| \frac{\varepsilon}{2(1 + |L|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

This shows that $\lim_{x \to a} [f(x)g(x)] = LM$. ■

PROOF OF LAW 3 If we take $g(x) = c$ in Law 4, we get

$$\begin{aligned} \lim_{x \to a} [cf(x)] = \lim_{x \to a} [g(x)f(x)] &= \lim_{x \to a} g(x) \cdot \lim_{x \to a} f(x) \\ &= \lim_{x \to a} c \cdot \lim_{x \to a} f(x) \\ &= c \lim_{x \to a} f(x) \qquad \text{(by Law 8)} \end{aligned}$$ ■

PROOF OF LAW 2 Using Law 1 and Law 3 with $c = -1$, we have

$$\begin{aligned} \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} [f(x) + (-1)g(x)] &= \lim_{x \to a} f(x) + \lim_{x \to a} (-1)g(x) \\ &= \lim_{x \to a} f(x) + (-1) \lim_{x \to a} g(x) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) \end{aligned}$$ ■

PROOF OF LAW 5 First let us show that

$$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}$$

To do this we must show that, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad \left| \frac{1}{g(x)} - \frac{1}{M} \right| < \varepsilon$$

Observe that
$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|Mg(x)|}$$

We know that we can make the numerator small. But we also need to know that the denominator is not small when x is near a. Since $\lim_{x \to a} g(x) = M$, there is a number $\delta_1 > 0$ such that, whenever $0 < |x - a| < \delta_1$, we have

$$|g(x) - M| < \frac{|M|}{2}$$

and therefore
$$\begin{aligned} |M| = |M - g(x) + g(x)| &\leqslant |M - g(x)| + |g(x)| \\ &< \frac{|M|}{2} + |g(x)| \end{aligned}$$

This shows that

$$\text{if} \qquad 0 < |x - a| < \delta_1 \qquad \text{then} \qquad |g(x)| > \frac{|M|}{2}$$

and so, for these values of x,

$$\frac{1}{|Mg(x)|} = \frac{1}{|M|\,|g(x)|} < \frac{1}{|M|} \cdot \frac{2}{|M|} = \frac{2}{M^2}$$

Also, there exists $\delta_2 > 0$ such that

$$\text{if} \qquad 0 < |x - a| < \delta_2 \qquad \text{then} \qquad |g(x) - M| < \frac{M^2}{2}\varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for $0 < |x - a| < \delta$, we have

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|Mg(x)|} < \frac{2}{M^2} \frac{M^2}{2} \varepsilon = \varepsilon$$

It follows that $\lim_{x \to a} 1/g(x) = 1/M$. Finally, using Law 4, we obtain

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \left(f(x) \cdot \frac{1}{g(x)} \right) = \lim_{x \to a} f(x) \lim_{x \to a} \frac{1}{g(x)} = L \cdot \frac{1}{M} = \frac{L}{M} \qquad ■$$

$\boxed{2}$ **Theorem** If $f(x) \leqslant g(x)$ for all x in an open interval that contains a (except possibly at a) and

$$\lim_{x \to a} f(x) = L \qquad \text{and} \qquad \lim_{x \to a} g(x) = M$$

then $L \leqslant M$.

PROOF We use the method of proof by contradiction. Suppose, if possible, that $L > M$. Law 2 of limits says that

$$\lim_{x \to a} [g(x) - f(x)] = M - L$$

Therefore, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad |[g(x) - f(x)] - (M - L)| < \varepsilon$$

In particular, taking $\varepsilon = L - M$ (noting that $L - M > 0$ by hypothesis), we have a number $\delta > 0$ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad |[g(x) - f(x)] - (M - L)| < L - M$$

Since $b \leqslant |b|$ for any number b, we have

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad [g(x) - f(x)] - (M - L) < L - M$$

which simplifies to

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad g(x) < f(x)$$

But this contradicts $f(x) \leqslant g(x)$. Thus the inequality $L > M$ must be false. Therefore $L \leqslant M$. ■

3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ for all x in an open interval that contains a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

PROOF Let $\varepsilon > 0$ be given. Since $\lim_{x \to a} f(x) = L$, there is a number $\delta_1 > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta_1 \quad \text{then} \quad |f(x) - L| < \varepsilon$$

that is,

$$\text{if} \quad 0 < |x - a| < \delta_1 \quad \text{then} \quad L - \varepsilon < f(x) < L + \varepsilon$$

Since $\lim_{x \to a} h(x) = L$, there is a number $\delta_2 > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta_2 \quad \text{then} \quad |h(x) - L| < \varepsilon$$

that is,

$$\text{if} \quad 0 < |x - a| < \delta_2 \quad \text{then} \quad L - \varepsilon < h(x) < L + \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x - a| < \delta$, then $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$, so

$$L - \varepsilon < f(x) \leqslant g(x) \leqslant h(x) < L + \varepsilon$$

In particular,

$$L - \varepsilon < g(x) < L + \varepsilon$$

and so $|g(x) - L| < \varepsilon$. Therefore $\lim_{x \to a} g(x) = L$. ■

Section 2.5

Theorem If f is a one-to-one continuous function defined on an interval (a, b), then its inverse function f^{-1} is also continuous.

PROOF First we show that if f is both one-to-one and continuous on (a, b), then it must be either increasing or decreasing on (a, b). If it were neither increasing nor decreasing, then there would exist numbers x_1, x_2, and x_3 in (a, b) with $x_1 < x_2 < x_3$ such that $f(x_2)$ does not lie between $f(x_1)$ and $f(x_3)$. There are two possibilities: either (1) $f(x_3)$ lies between $f(x_1)$ and $f(x_2)$ or (2) $f(x_1)$ lies between $f(x_2)$ and $f(x_3)$. (Draw a picture.) In case (1) we apply the Intermediate Value Theorem to the continuous function f to get a number c between x_1 and x_2 such that $f(c) = f(x_3)$. In case (2) the Intermediate Value Theorem gives a number c between x_2 and x_3 such that $f(c) = f(x_1)$. In either case we have contradicted the fact that f is one-to-one.

Let us assume, for the sake of definiteness, that f is increasing on (a, b). We take any number y_0 in the domain of f^{-1} and we let $f^{-1}(y_0) = x_0$; that is, x_0 is the number in (a, b) such that $f(x_0) = y_0$. To show that f^{-1} is continuous at y_0 we take any $\varepsilon > 0$ such that the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ is contained in the interval (a, b). Since f is increasing, it maps the numbers in the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ onto the numbers in the interval $(f(x_0 - \varepsilon), f(x_0 + \varepsilon))$ and f^{-1} reverses the correspondence. If we let δ denote the smaller of the numbers $\delta_1 = y_0 - f(x_0 - \varepsilon)$ and $\delta_2 = f(x_0 + \varepsilon) - y_0$, then the interval $(y_0 - \delta, y_0 + \delta)$ is contained in the interval $(f(x_0 - \varepsilon), f(x_0 + \varepsilon))$ and so is mapped into the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ by f^{-1}. (See the arrow diagram in Figure 1.) We have therefore found a number $\delta > 0$ such that

$$\text{if} \quad |y - y_0| < \delta \quad \text{then} \quad |f^{-1}(y) - f^{-1}(y_0)| < \varepsilon$$

This shows that $\lim_{y \to y_0} f^{-1}(y) = f^{-1}(y_0)$ and so f^{-1} is continuous at any number y_0 in its domain.

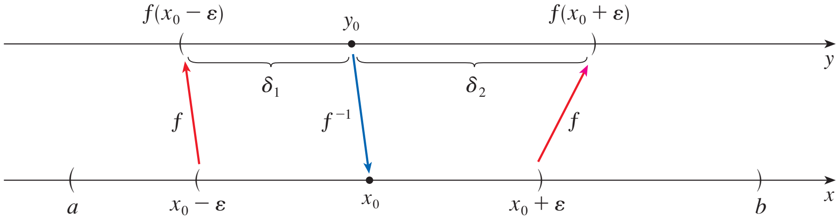


FIGURE 1 ■

8 Theorem If f is continuous at b and $\lim_{x \to a} g(x) = b$, then

$$\lim_{x \to a} f(g(x)) = f(b)$$

PROOF Let $\varepsilon > 0$ be given. We want to find a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(g(x)) - f(b)| < \varepsilon$$

Since f is continuous at b, we have

$$\lim_{y \to b} f(y) = f(b)$$

and so there exists $\delta_1 > 0$ such that

$$\text{if} \quad 0 < |y - b| < \delta_1 \quad \text{then} \quad |f(y) - f(b)| < \varepsilon$$

Since $\lim_{x \to a} g(x) = b$, there exists $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |g(x) - b| < \delta_1$$

Combining these two statements, we see that whenever $0 < |x - a| < \delta$ we have $|g(x) - b| < \delta_1$, which implies that $|f(g(x)) - f(b)| < \varepsilon$. Therefore we have proved that $\lim_{x \to a} f(g(x)) = f(b)$. ■

Section 3.3

The proof of the following result was promised when we proved that $\lim_{\theta \to 0} \dfrac{\sin \theta}{\theta} = 1$.

Theorem If $0 < \theta < \pi/2$, then $\theta \leqslant \tan \theta$.

PROOF Figure 2 shows a sector of a circle with center O, central angle θ, and radius 1. Then

$$|AD| = |OA| \tan \theta = \tan \theta$$

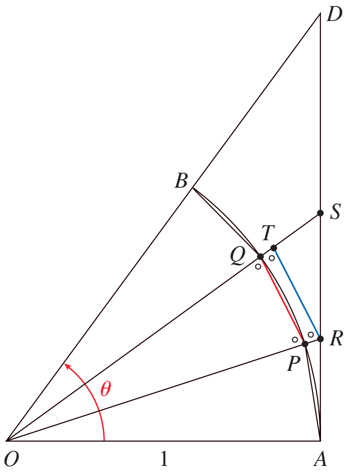


FIGURE 2

We approximate the arc AB by an inscribed polygonal path consisting of n equal line segments and we look at a typical segment PQ. We extend the lines OP and OQ to meet AD in the points R and S. Then we draw $RT \parallel PQ$ as in Figure 2. Observe that

$$\angle RTO = \angle PQO < 90^\circ$$

and so $\angle RTS > 90^\circ$. Therefore we have

$$|PQ| < |RT| < |RS|$$

If we add n such inequalities, we get

$$L_n < |AD| = \tan\theta$$

where L_n is the length of the inscribed polygonal path. Thus, by Theorem 2.3.2, we have

$$\lim_{n\to\infty} L_n \leqslant \tan\theta$$

But the arc length is defined in Equation 8.1.1 as the limit of the lengths of inscribed polygonal paths, so

$$\theta = \lim_{n\to\infty} L_n \leqslant \tan\theta$$ ■

Section 3.6

Theorem If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

PROOF Write the definition of derivative as in Equation 2.7.5:

$$(f^{-1})'(a) = \lim_{x\to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a}$$

If $f(b) = a$, then $f^{-1}(a) = b$. And if we let $y = f^{-1}(x)$, then $f(y) = x$. Since f is differentiable, it is continuous, so f^{-1} is continuous (see Section 2.5). Thus if $x \to a$, then $f^{-1}(x) \to f^{-1}(a)$, that is, $y \to b$. Therefore

$$\begin{aligned}(f^{-1})'(a) &= \lim_{x\to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a} = \lim_{x\to b} \frac{y - b}{f(y) - f(b)}\\ &= \lim_{y\to b} \frac{1}{\dfrac{f(y) - f(b)}{y - b}} = \frac{1}{\displaystyle\lim_{y\to b} \frac{f(y) - f(b)}{y - b}}\\ &= \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(a))}\end{aligned}$$ ■

Section 4.3

Concavity Test

(a) If $f''(x) > 0$ for on an interval I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for on an interval I, then the graph of f is concave downward on I.

PROOF OF (a) Let a be any number in I. We must show that the curve $y = f(x)$ lies above the tangent line at the point $(a, f(a))$. The equation of this tangent is

$$y = f(a) + f'(a)(x - a)$$

So we must show that

$$f(x) > f(a) + f'(a)(x - a)$$

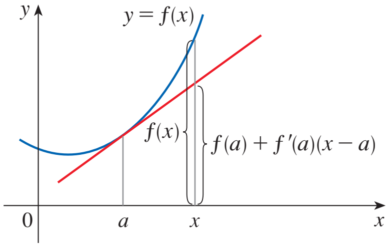


FIGURE 3

whenever $x \in I$ $(x \neq a)$. (See Figure 3.)

First let us take the case where $x > a$. Applying the Mean Value Theorem to f on the interval $[a, x]$, we get a number c, with $a < c < x$, such that

$$\boxed{1} \qquad f(x) - f(a) = f'(c)(x - a)$$

Since $f'' > 0$ on I, we know from the Increasing/Decreasing Test that f' is increasing on I. Thus, since $a < c$, we have

$$f'(a) < f'(c)$$

and so, multiplying this inequality by the positive number $x - a$, we get

$$\boxed{2} \qquad f'(a)(x - a) < f'(c)(x - a)$$

Now we add $f(a)$ to both sides of this inequality:

$$f(a) + f'(a)(x - a) < f(a) + f'(c)(x - a)$$

But from Equation 1 we have $f(x) = f(a) + f'(c)(x - a)$. So this inequality becomes

$$\boxed{3} \qquad f(x) > f(a) + f'(a)(x - a)$$

which is what we wanted to prove.

For the case where $x < a$ we have $f'(c) < f'(a)$, but multiplication by the negative number $x - a$ reverses the inequality, so we get (2) and (3) as before. ■

Section 4.4

In order to give the promised proof of l'Hospital's Rule, we first need a generalization of the Mean Value Theorem. The following theorem is named after another French mathematician, Augustin-Louis Cauchy (1789–1857).

See the biographical sketch of Cauchy in Section 2.4.

$\boxed{1}$ **Cauchy's Mean Value Theorem** Suppose that the functions f and g are continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$ for all x in (a, b). Then there is a number c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Notice that if we take the special case in which $g(x) = x$, then $g'(c) = 1$ and Theorem 1 is just the ordinary Mean Value Theorem. Furthermore, Theorem 1 can be proved in a similar manner. You can verify that all we have to do is change the function h given by Equation 4.2.4 to the function

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}\,[g(x) - g(a)]$$

and apply Rolle's Theorem as before.

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

PROOF OF L'HOSPITAL'S RULE We are assuming that $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$. Let

$$L = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

We must show that $\lim_{x \to a} f(x)/g(x) = L$. Define

$$F(x) = \begin{cases} f(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases} \qquad\qquad G(x) = \begin{cases} g(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$$

Then F is continuous on I since f is continuous on $\{x \in I \mid x \neq a\}$ and

$$\lim_{x \to a} F(x) = \lim_{x \to a} f(x) = 0 = F(a)$$

Likewise, G is continuous on I. Let $x \in I$ and $x > a$. Then F and G are continuous on $[a, x]$ and differentiable on (a, x) and $G' \neq 0$ there (since $F' = f'$ and $G' = g'$). Therefore, by Cauchy's Mean Value Theorem, there is a number y such that $a < y < x$ and

$$\frac{F'(y)}{G'(y)} = \frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F(x)}{G(x)}$$

Here we have used the fact that, by definition, $F(a) = 0$ and $G(a) = 0$. Now, if we let $x \to a^+$, then $y \to a^+$ (since $a < y < x$), so

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{F(x)}{G(x)} = \lim_{y\to a^+} \frac{F'(y)}{G'(y)} = \lim_{y\to a^+} \frac{f'(y)}{g'(y)} = L$$

A similar argument shows that the left-hand limit is also L. Therefore

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L$$

This proves l'Hospital's Rule for the case where a is finite.

If a is infinite, we let $t = 1/x$. Then $t \to 0^+$ as $x \to \infty$, so we have

$$\begin{aligned} \lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{t\to 0^+} \frac{f(1/t)}{g(1/t)} \\ &= \lim_{t\to 0^+} \frac{f'(1/t)(-1/t^2)}{g'(1/t)(-1/t^2)} \qquad \text{(by l'Hospital's Rule for finite } a\text{)} \\ &= \lim_{t\to 0^+} \frac{f'(1/t)}{g'(1/t)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} \end{aligned}$$

■

Section 11.1

7 **Theorem** If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

PROOF Let $\varepsilon > 0$ be given. Since f is continuous at L, we have $\lim_{x\to L} f(x) = f(L)$. Thus there exists $\delta > 0$ such that

1 if $0 < |x - L| < \delta$ then $|f(x) - f(L)| < \varepsilon$

Now, since $\lim_{n\to\infty} a_n = L$ and δ is a positive number, there exists an integer N such that

2 if $n > N$ then $|a_n - L| < \delta$

Combining (1) and (2), we have

$$\text{if} \quad n > N \quad \text{then} \quad |f(a_n) - f(L)| < \varepsilon$$

so by Definition 11.1.2, $\lim_{n\to\infty} f(a_n) = f(L)$. ■

Section 11.8

In order to prove Theorem 11.8.4, we first need the following results.

Theorem

1. If a power series $\Sigma c_n x^n$ converges when $x = b$ (where $b \neq 0$), then it converges whenever $|x| < |b|$.
2. If a power series $\Sigma c_n x^n$ diverges when $x = d$ (where $d \neq 0$), then it diverges whenever $|x| > |d|$.

PROOF OF 1 Suppose that $\Sigma c_n b^n$ converges. Then, by Theorem 11.2.6, we have $\lim_{n\to\infty} c_n b^n = 0$. According to Definition 11.1.2 with $\varepsilon = 1$, there is a positive integer N such that $|c_n b^n| < 1$ whenever $n \geqslant N$. Thus, for $n \geqslant N$, we have

$$|c_n x^n| = \left|\frac{c_n b^n x^n}{b^n}\right| = |c_n b^n| \left|\frac{x}{b}\right|^n < \left|\frac{x}{b}\right|^n$$

If $|x| < |b|$, then $|x/b| < 1$, so $\Sigma\,|x/b|^n$ is a convergent geometric series. Therefore, by the Direct Comparison Test, the series $\Sigma_{n=N}^{\infty}|c_n x^n|$ is convergent. Thus the series $\Sigma c_n x^n$ is absolutely convergent and therefore convergent. ■

PROOF OF 2 Suppose that $\Sigma c_n d^n$ diverges. If x is any number such that $|x| > |d|$, then $\Sigma c_n x^n$ cannot converge because, by part 1, the convergence of $\Sigma c_n x^n$ would imply the convergence of $\Sigma c_n d^n$. Therefore $\Sigma c_n x^n$ diverges whenever $|x| > |d|$. ■

Theorem For a power series $\Sigma c_n x^n$ there are only three possibilities:

(i) The series converges only when $x = 0$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x| < R$ and diverges if $|x| > R$.

PROOF Suppose that neither case (i) nor case (ii) is true. Then there are nonzero numbers b and d such that $\Sigma c_n x^n$ converges for $x = b$ and diverges for $x = d$. Therefore the set $S = \{x \mid \Sigma c_n x^n \text{ converges}\}$ is not empty. By the preceding theorem, the series diverges if $|x| > |d|$, so $|x| \leqslant |d|$ for all $x \in S$. This says that $|d|$ is an upper bound for the set S. Thus, by the Completeness Axiom (see Section 11.1), S has a least upper bound R. If $|x| > R$, then $x \notin S$, so $\Sigma c_n x^n$ diverges. If $|x| < R$, then $|x|$ is not an upper bound for S and so there exists $b \in S$ such that $b > |x|$. Since $b \in S$, $\Sigma c_n x^n$ converges, so by the preceding theorem $\Sigma c_n x^n$ converges. ■

We are now ready to prove Theorem 11.8.4.

4 Theorem For a power series $\Sigma\, c_n(x - a)^n$ there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

PROOF If we make the change of variable $u = x - a$, then the power series becomes $\Sigma c_n u^n$ and we can apply the preceding theorem to this series. In case (iii) we have convergence for $|u| < R$ and divergence for $|u| > R$. Thus we have convergence for $|x - a| < R$ and divergence for $|x - a| > R$. ■

Section 14.3

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

PROOF For small values of h, $h \neq 0$, consider the difference

$$\Delta(h) = [f(a + h, b + h) - f(a + h, b)] - [f(a, b + h) - f(a, b)]$$

Notice that if we let $g(x) = f(x, b + h) - f(x, b)$, then

$$\Delta(h) = g(a + h) - g(a)$$

By the Mean Value Theorem, there is a number c between a and $a + h$ such that

$$g(a + h) - g(a) = g'(c)h = h[f_x(c, b + h) - f_x(c, b)]$$

Applying the Mean Value Theorem again, this time to f_x, we get a number d between b and $b + h$ such that

$$f_x(c, b + h) - f_x(c, b) = f_{xy}(c, d)h$$

Combining these equations, we obtain

$$\Delta(h) = h^2 f_{xy}(c, d)$$

If $h \to 0$, then $(c, d) \to (a, b)$, so the continuity of f_{xy} at (a, b) gives

$$\lim_{h \to 0} \frac{\Delta(h)}{h^2} = \lim_{(c, d) \to (a, b)} f_{xy}(c, d) = f_{xy}(a, b)$$

Similarly, by writing

$$\Delta(h) = [f(a + h, b + h) - f(a, b + h)] - [f(a + h, b) - f(a, b)]$$

and using the Mean Value Theorem twice and the continuity of f_{yx} at (a, b), we obtain

$$\lim_{h \to 0} \frac{\Delta(h)}{h^2} = f_{yx}(a, b)$$

It follows that $f_{xy}(a, b) = f_{yx}(a, b)$. ■

Section 14.4

8 **Theorem** If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

PROOF Let

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

According to Definition 14.4.7, to prove that f is differentiable at (a, b) we have to show that we can write Δz in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \ \Delta y) \to (0, 0)$.

Referring to Figure 4, we write

$$\boxed{1}\quad \Delta z = [f(a + \Delta x, b + \Delta y) - f(a, b + \Delta y)] + [f(a, b + \Delta y) - f(a, b)]$$

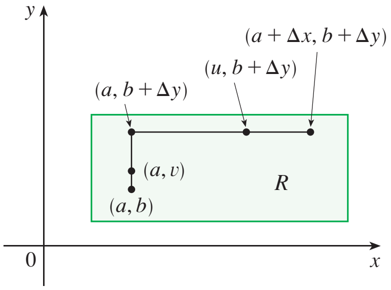


FIGURE 4

Observe that the function of a single variable

$$g(x) = f(x, b + \Delta y)$$

is defined on the interval $[a, a + \Delta x]$ and $g'(x) = f_x(x, b + \Delta y)$. If we apply the Mean Value Theorem to g, we get

$$g(a + \Delta x) - g(a) = g'(u)\,\Delta x$$

where u is some number between a and $a + \Delta x$. In terms of f, this equation becomes

$$f(a + \Delta x, b + \Delta y) - f(a, b + \Delta y) = f_x(u, b + \Delta y)\,\Delta x$$

This gives us an expression for the first part of the right side of Equation 1. For the second part we let $h(y) = f(a, y)$. Then h is a function of a single variable defined on the interval $[b, b + \Delta y]$ and $h'(y) = f_y(a, y)$. A second application of the Mean Value Theorem then gives

$$h(b + \Delta y) - h(b) = h'(v)\,\Delta y$$

where v is some number between b and $b + \Delta y$. In terms of f, this becomes

$$f(a, b + \Delta y) - f(a, b) = f_y(a, v)\,\Delta y$$

We now substitute these expressions into Equation 1 and obtain

$$\begin{aligned}\Delta z &= f_x(u, b + \Delta y)\,\Delta x + f_y(a, v)\,\Delta y \\ &= f_x(a, b)\,\Delta x + [f_x(u, b + \Delta y) - f_x(a, b)]\,\Delta x + f_y(a, b)\,\Delta y \\ &\qquad + [f_y(a, v) - f_y(a, b)]\,\Delta y \\ &= f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y\end{aligned}$$

where

$$\varepsilon_1 = f_x(u, b + \Delta y) - f_x(a, b)$$

$$\varepsilon_2 = f_y(a, v) - f_y(a, b)$$

Since $(u, b + \Delta y) \to (a, b)$ and $(a, v) \to (a, b)$ as $(\Delta x, \ \Delta y) \to (0, 0)$ and since f_x and f_y are continuous at (a, b), we see that $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \ \Delta y) \to (0, 0)$.

Therefore f is differentiable at (a, b). ■

G The Logarithm Defined as an Integral

Our treatment of exponential and logarithmic functions until now has relied on our intuition, which is based on numerical and visual evidence. (See Sections 1.4, 1.5, and 3.1.) Here we use the Fundamental Theorem of Calculus to give an alternative treatment that provides a surer footing for these functions.

Instead of starting with b^x and defining $\log_b x$ as its inverse, this time we start by defining $\ln x$ as an integral and then define the exponential function as its inverse. You should bear in mind that we do not use any of our previous definitions and results concerning exponential and logarithmic functions.

The Natural Logarithm

We first define $\ln x$ as an integral.

1 Definition The **natural logarithmic function** is the function defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad x > 0$$

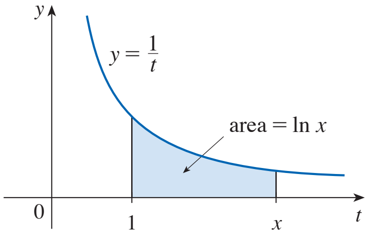


FIGURE 1

The existence of this function depends on the fact that the integral of a continuous function always exists. If $x > 1$, then $\ln x$ can be interpreted geometrically as the area under the hyperbola $y = 1/t$ from $t = 1$ to $t = x$. (See Figure 1.) For $x = 1$, we have

$$\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$$

For $0 < x < 1$, $$\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt < 0$$

and so $\ln x$ is the negative of the area shown in Figure 2.

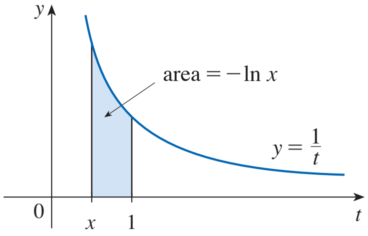


FIGURE 2

EXAMPLE 1

(a) By comparing areas, show that $\frac{1}{2} < \ln 2 < \frac{3}{4}$.

(b) Use the Midpoint Rule with $n = 10$ to estimate the value of $\ln 2$.

SOLUTION

(a) We can interpret $\ln 2$ as the area under the curve $y = 1/t$ from 1 to 2. From Figure 3 we see that this area is larger than the area of rectangle $BCDE$ and smaller than the area of trapezoid $ABCD$. Thus we have

$$\tfrac{1}{2} \cdot 1 < \ln 2 < 1 \cdot \tfrac{1}{2}\left(1 + \tfrac{1}{2}\right)$$

$$\tfrac{1}{2} < \ln 2 < \tfrac{3}{4}$$

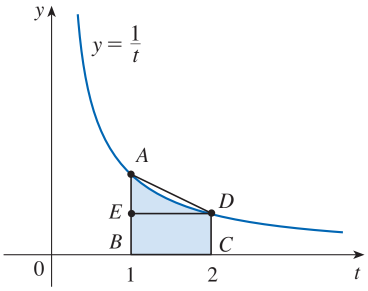


FIGURE 3

(b) If we use the Midpoint Rule with $f(t) = 1/t$, $n = 10$, and $\Delta t = 0.1$, we get

$$\ln 2 = \int_1^2 \frac{1}{t}\,dt \approx (0.1)[f(1.05) + f(1.15) + \cdots + f(1.95)]$$

$$= (0.1)\left(\frac{1}{1.05} + \frac{1}{1.15} + \cdots + \frac{1}{1.95}\right) \approx 0.693$$ ■

Notice that the integral that defines $\ln x$ is exactly the type of integral discussed in Part 1 of the Fundamental Theorem of Calculus (see Section 5.3). In fact, using that theorem, we have

$$\frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$

and so

2
$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

We now use this differentiation rule to prove the following properties of the logarithm function.

3 Laws of Logarithms If x and y are positive numbers and r is a rational number, then

1. $\ln(xy) = \ln x + \ln y$ **2.** $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$ **3.** $\ln(x^r) = r\ln x$

PROOF

1. Let $f(x) = \ln(ax)$, where a is a positive constant. Then, using Equation 2 and the Chain Rule, we have

$$f'(x) = \frac{1}{ax}\frac{d}{dx}(ax) = \frac{1}{ax}\cdot a = \frac{1}{x}$$

Therefore $f(x)$ and $\ln x$ have the same derivative and so they must differ by a constant:

$$\ln(ax) = \ln x + C$$

Putting $x = 1$ in this equation, we get $\ln a = \ln 1 + C = 0 + C = C$. Thus

$$\ln(ax) = \ln x + \ln a$$

If we now replace the constant a by any number y, we have

$$\ln(xy) = \ln x + \ln y$$

2. Using Law 1 with $x = 1/y$, we have

$$\ln\frac{1}{y} + \ln y = \ln\left(\frac{1}{y}\cdot y\right) = \ln 1 = 0$$

and so
$$\ln\frac{1}{y} = -\ln y$$

Using Law 1 again, we have

$$\ln\left(\frac{x}{y}\right) = \ln\left(x\cdot\frac{1}{y}\right) = \ln x + \ln\frac{1}{y} = \ln x - \ln y$$

The proof of Law 3 is left as an exercise. ■

In order to graph $y = \ln x$, we first determine its limits:

4

$$\boxed{\text{(a)} \lim_{x \to \infty} \ln x = \infty \qquad \text{(b)} \lim_{x \to 0^+} \ln x = -\infty}$$

PROOF

(a) Using Law 3 with $x = 2$ and $r = n$ (where n is any positive integer), we have $\ln(2^n) = n \ln 2$. Now $\ln 2 > 0$, so this shows that $\ln(2^n) \to \infty$ as $n \to \infty$. But $\ln x$ is an increasing function since its derivative $1/x > 0$. Therefore $\ln x \to \infty$ as $x \to \infty$.

(b) If we let $t = 1/x$, then $t \to \infty$ as $x \to 0^+$. Thus, using (a), we have

$$\lim_{x \to 0^+} \ln x = \lim_{t \to \infty} \ln\left(\frac{1}{t}\right) = \lim_{t \to \infty} (-\ln t) = -\infty \qquad ■$$

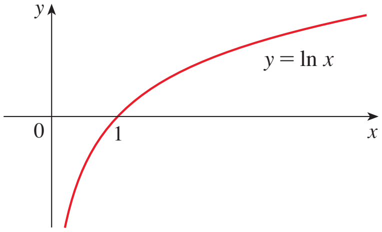


FIGURE 4

If $y = \ln x$, $x > 0$, then

$$\frac{dy}{dx} = \frac{1}{x} > 0 \qquad \text{and} \qquad \frac{d^2y}{dx^2} = -\frac{1}{x^2} < 0$$

which shows that $\ln x$ is increasing and concave downward on $(0, \infty)$. Putting this information together with (4), we draw the graph of $y = \ln x$ in Figure 4.

Since $\ln 1 = 0$ and $\ln x$ is an increasing continuous function that takes on arbitrarily large values, the Intermediate Value Theorem shows that there is a number where $\ln x$ takes on the value 1. (See Figure 5.) This important number is denoted by e.

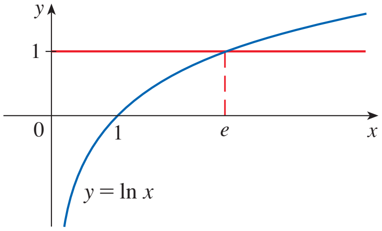


FIGURE 5

5 **Definition** e is the number such that $\ln e = 1$.

We will show (in Theorem 19) that this definition is consistent with our previous definition of e.

The Natural Exponential Function

Since ln is an increasing function, it is one-to-one and therefore has an inverse function, which we denote by exp. Thus, according to the definition of an inverse function,

$$f^{-1}(x) = y \iff f(y) = x$$

6

$$\boxed{\exp(x) = y \iff \ln y = x}$$

and the cancellation equations are

$$f^{-1}(f(x)) = x$$
$$f(f^{-1}(x)) = x$$

7

$$\boxed{\exp(\ln x) = x \qquad \text{and} \qquad \ln(\exp x) = x}$$

In particular, we have

$$\begin{aligned} \exp(0) &= 1 \quad \text{since} \quad \ln 1 = 0 \\ \exp(1) &= e \quad \text{since} \quad \ln e = 1 \end{aligned}$$

We obtain the graph of $y = \exp x$ by reflecting the graph of $y = \ln x$ about the line

$y = x$. (See Figure 6.) The domain of exp is the range of ln, that is, $(-\infty, \infty)$; the range of exp is the domain of ln, that is, $(0, \infty)$.

If r is any rational number, then the third law of logarithms gives

$$\ln(e^r) = r \ln e = r$$

Therefore, by (6), $$\exp(r) = e^r$$

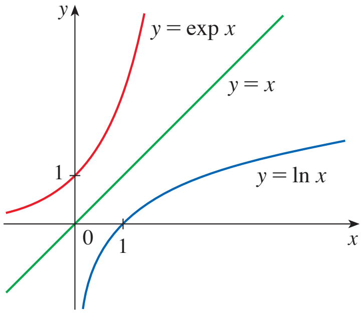


FIGURE 6

Thus $\exp(x) = e^x$ whenever x is a rational number. This leads us to define e^x, even for irrational values of x, by the equation

$$\boxed{e^x = \exp(x)}$$

In other words, for the reasons given, we define e^x to be the inverse of the function $\ln x$. In this notation (6) becomes

8 $$\boxed{e^x = y \iff \ln y = x}$$

and the cancellation equations (7) become

9 $$\boxed{e^{\ln x} = x \qquad x > 0}$$

10 $$\boxed{\ln(e^x) = x \qquad \text{for all } x}$$

The natural exponential function $f(x) = e^x$ is one of the most frequently occurring functions in calculus and its applications, so it is important to be familiar with its graph (Figure 7) and its properties (which follow from the fact that it is the inverse of the natural logarithmic function).

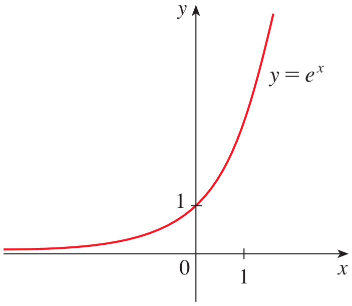


FIGURE 7
The natural exponential function

Properties of the Exponential Function The exponential function $f(x) = e^x$ is an increasing continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. Thus $e^x > 0$ for all x. Also

$$\lim_{x \to -\infty} e^x = 0 \qquad \lim_{x \to \infty} e^x = \infty$$

So the x-axis is a horizontal asymptote of $f(x) = e^x$.

We now verify that f has the other properties expected of an exponential function.

11 **Laws of Exponents** If x and y are real numbers and r is rational, then

1. $e^{x+y} = e^x e^y$ **2.** $e^{x-y} = \dfrac{e^x}{e^y}$ **3.** $(e^x)^r = e^{rx}$

PROOF OF LAW 1 Using the first law of logarithms and Equation 10, we have

$$\ln(e^x e^y) = \ln(e^x) + \ln(e^y) = x + y = \ln(e^{x+y})$$

Since ln is a one-to-one function, it follows that $e^x e^y = e^{x+y}$.

Laws 2 and 3 are proved similarly (see Exercises 6 and 7). As we will soon see, Law 3 actually holds when r is any real number. ■

We now prove the differentiation formula for e^x.

12

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

PROOF The function $y = e^x$ is differentiable because it is the inverse function of $y = \ln x$, which we know is differentiable with nonzero derivative. To find its derivative, we use the inverse function method. Let $y = e^x$. Then $\ln y = x$ and, differentiating this latter equation implicitly with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = y = e^x$$

■

General Exponential Functions

If $b > 0$ and r is any rational number, then by (9) and (11),

$$b^r = (e^{\ln b})^r = e^{r \ln b}$$

Therefore, even for irrational numbers x, we *define*

13

$$\boxed{b^x = e^{x \ln b}}$$

Thus, for instance,

$$2^{\sqrt{3}} = e^{\sqrt{3}\ln 2} \approx e^{1.20} \approx 3.32$$

The function $f(x) = b^x$ is called the **exponential function with base b**. Notice that b^x is positive for all x because e^x is positive for all x.

Definition 13 allows us to extend one of the laws of logarithms. We already know that $\ln(b^r) = r \ln b$ when r is rational. But if we now let r be *any* real number we have, from Definition 13,

$$\ln b^r = \ln(e^{r \ln b}) = r \ln b$$

Thus

14

$$\ln b^r = r \ln b \qquad \text{for any real number } r$$

The general laws of exponents follow from Definition 13 together with the laws of exponents for e^x.

15 **Laws of Exponents** If x and y are real numbers and $a, b > 0$, then

1. $b^{x+y} = b^x b^y$ **2.** $b^{x-y} = b^x/b^y$ **3.** $(b^x)^y = b^{xy}$ **4.** $(ab)^x = a^x b^x$

PROOF

1. Using Definition 13 and the laws of exponents for e^x, we have

$$\begin{aligned} b^{x+y} &= e^{(x+y)\ln b} = e^{x\ln b + y\ln b} \\ &= e^{x\ln b} e^{y\ln b} = b^x b^y \end{aligned}$$

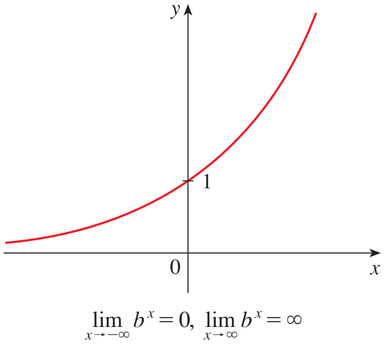
3. Using Equation 14 we obtain

$$(b^x)^y = e^{y\ln(b^x)} = e^{yx\ln b} = e^{xy\ln b} = b^{xy}$$

The remaining proofs are left as exercises. ■

The differentiation formula for exponential functions is also a consequence of Definition 13:

16 $$\boxed{\frac{d}{dx}(b^x) = b^x \ln b}$$



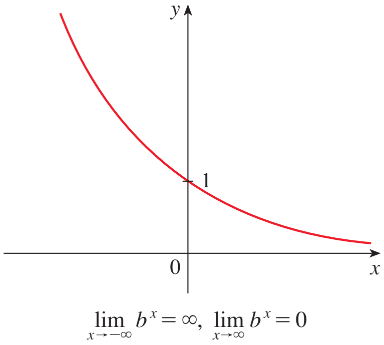
$\lim_{x\to-\infty} b^x = 0, \ \lim_{x\to\infty} b^x = \infty$

FIGURE 8 $y = b^x, \ b > 1$

PROOF

$$\frac{d}{dx}(b^x) = \frac{d}{dx}(e^{x\ln b}) = e^{x\ln b}\frac{d}{dx}(x\ln b) = b^x \ln b$$ ■

If $b > 1$, then $\ln b > 0$, so $(d/dx)\,b^x = b^x \ln b > 0$, which shows that $y = b^x$ is increasing (see Figure 8). If $0 < b < 1$, then $\ln b < 0$ and so $y = b^x$ is decreasing (see Figure 9).



$\lim_{x\to-\infty} b^x = \infty, \ \lim_{x\to\infty} b^x = 0$

FIGURE 9 $y = b^x, \ 0 < b < 1$

General Logarithmic Functions

If $b > 0$ and $b \neq 1$, then $f(x) = b^x$ is a one-to-one function. Its inverse function is called the **logarithmic function with base *b*** and is denoted by $\log_b$. Thus

17 $$\boxed{\log_b x = y \iff b^y = x}$$

In particular, we see that

$$\log_e x = \ln x$$

The laws of logarithms are similar to those for the natural logarithm and can be deduced from the laws of exponents (see Exercise 10).

To differentiate $y = \log_b x$, we write the equation as $b^y = x$. From Equation 14 we have $y \ln b = \ln x$, so

$$\log_b x = y = \frac{\ln x}{\ln b}$$

Since $\ln b$ is a constant, we can differentiate as follows:

$$\frac{d}{dx}(\log_b x) = \frac{d}{dx}\frac{\ln x}{\ln b} = \frac{1}{\ln b}\frac{d}{dx}(\ln x) = \frac{1}{x \ln b}$$

18 $$\boxed{\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}}$$

The Number e Expressed as a Limit

In this appendix we defined e as the number such that $\ln e = 1$. The next theorem shows that this is the same as the number e defined in Section 3.1 (see Equation 3.6.5).

19 $$\boxed{e = \lim_{x \to 0} (1 + x)^{1/x}}$$

PROOF Let $f(x) = \ln x$. Then $f'(x) = 1/x$, so $f'(1) = 1$. But, by the definition of derivative,

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1 + x) - f(1)}{x} \\ &= \lim_{x \to 0} \frac{\ln(1 + x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1 + x) = \lim_{x \to 0} \ln(1 + x)^{1/x} \end{aligned}$$

Because $f'(1) = 1$, we have

$$\lim_{x \to 0} \ln(1 + x)^{1/x} = 1$$

Then, by Theorem 2.5.8 and the continuity of the exponential function, we have

$$e = e^1 = e^{\lim_{x \to 0} \ln(1+x)^{1/x}} = \lim_{x \to 0} e^{\ln(1+x)^{1/x}} = \lim_{x \to 0} (1 + x)^{1/x}$$ ■

G Exercises

1. (a) By comparing areas, show that

$$\tfrac{1}{3} < \ln 1.5 < \tfrac{5}{12}$$

(b) Use the Midpoint Rule with $n = 10$ to estimate $\ln 1.5$.

2. Refer to Example 1.

(a) Find the equation of the tangent line to the curve $y = 1/t$ that is parallel to the secant line AD.

(b) Use part (a) to show that $\ln 2 > 0.66$.

3. By comparing areas, show that

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n - 1}$$

4. (a) By comparing areas, show that $\ln 2 < 1 < \ln 3$.

(b) Deduce that $2 < e < 3$.

5. Prove the third law of logarithms. [*Hint:* Start by showing that both sides of the equation have the same derivative.]

6. Prove the second law of exponents for e^x [see (11)].

7. Prove the third law of exponents for e^x [see (11)].

8. Prove the second law of exponents [see (15)].

9. Prove the fourth law of exponents [see (15)].

10. Deduce the following laws of logarithms from (15):

(a) $\log_b(xy) = \log_b x + \log_b y$

(b) $\log_b(x/y) = \log_b x - \log_b y$

(c) $\log_b(x^y) = y \log_b x$

H | Answers to Odd-Numbered Exercises

CHAPTER 1

EXERCISES 1.1 ■ PAGE 17

1. Yes

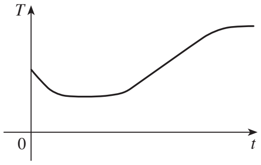
3. (a) 2, −2, 1, 2.5 (b) −4 (c) [−4, 4]
(d) [−4, 4], [−2, 3] (e) [0, 2]

5. [−85, 115] **7.** Yes **9.** No **11.** Yes **13.** No

15. No **17.** Yes, [−3, 2], [−3, −2) ∪ [−1, 3]

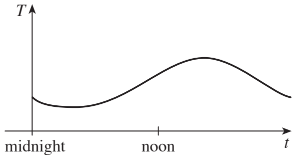
19. (a) 13.8°C (b) 1990 (c) 1910, 2000
(d) [13.5, 14.4]

21.



23. (a) 500 MW; 730 MW (b) 4 AM; noon; yes

25.



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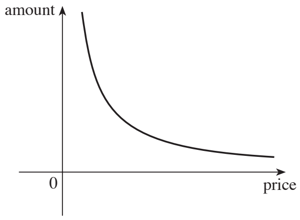

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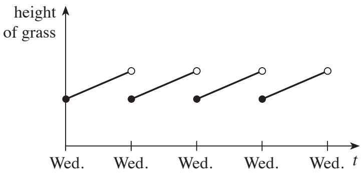
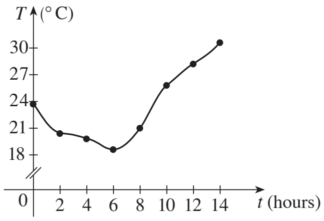


31. (a)



(b) 23°C

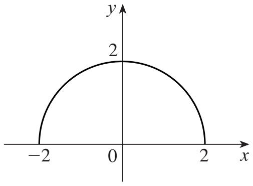
33. 12, 16, $3a^2 - a + 2$, $3a^2 + a + 2$, $3a^2 + 5a + 4$,
$6a^2 - 2a + 4$, $12a^2 - 2a + 2$, $3a^4 - a^2 + 2$,
$9a^4 - 6a^3 + 13a^2 - 4a + 4$, $3a^2 + 6ah + 3h^2 - a - h + 2$

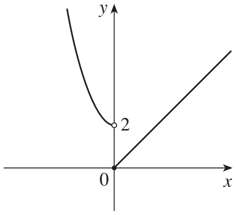
35. $-3 - h$ **37.** $-1/(ax)$

39. (−∞, −3) ∪ (−3, 3) ∪ (3, ∞) **41.** (−∞, ∞)

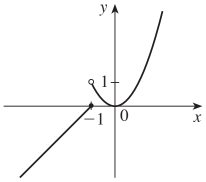
43. (−∞, 0) ∪ (5, ∞) **45.** [0, 4]

47. [−2, 2], [0, 2] **49.** 11, 0, 2





51. −2, 0, 4



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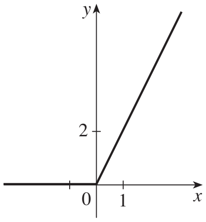

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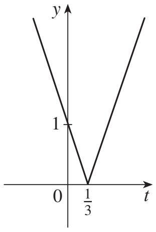

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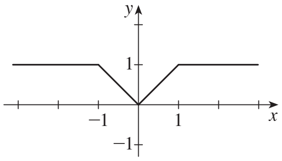


59. $f(x) = \frac{5}{2}x - \frac{11}{2},\ 1 \leqslant x \leqslant 5$ **61.** $f(x) = 1 - \sqrt{-x}$

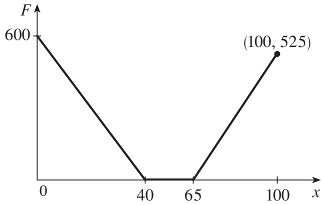
63. $f(x) = \begin{cases} -x + 3 & \text{if } 0 \leqslant x \leqslant 3 \\ 2x - 6 & \text{if } 3 < x \leqslant 5 \end{cases}$

65. $A(L) = 10L - L^2,\ 0 < L < 10$

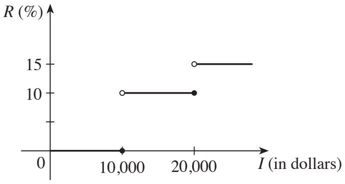
67. $A(x) = \sqrt{3}x^2/4,\ x > 0$ **69.** $S(x) = x^2 + (8/x),\ x > 0$

71. $V(x) = 4x^3 - 64x^2 + 240x,\ 0 < x < 6$

73. $F(x) = \begin{cases} 15(40 - x) & \text{if } 0 \leqslant x < 40 \\ 0 & \text{if } 40 \leqslant x \leqslant 65 \\ 15(x - 65) & \text{if } x > 65 \end{cases}$

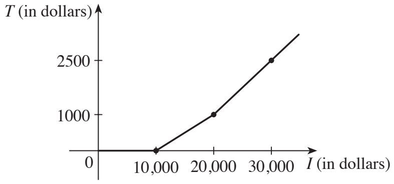


75. (a)



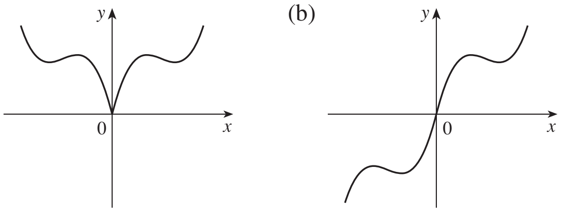
(b) \$400, \$1900

(c)



77. f is odd, g is even

79. (a) (b)



81. Odd **83.** Neither **85.** Even

87. Even; odd; neither (unless $f = 0$ or $g = 0$)

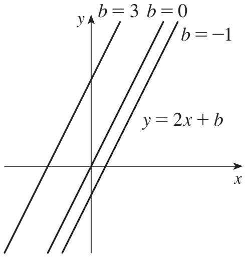
EXERCISES 1.2 ■ PAGE 33

1. (a) Polynomial, degree 3 (b) Trigonometric (c) Power (d) Exponential (e) Algebraic (f) Logarithmic

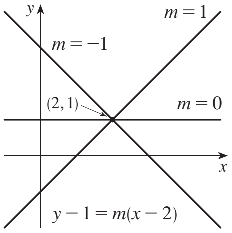
3. (a) h (b) f (c) g

5. $\{x \mid x \neq \pi/2 + 2n\pi\}$, n an integer

7. (a) $y = 2x + b$, where b is the y-intercept.

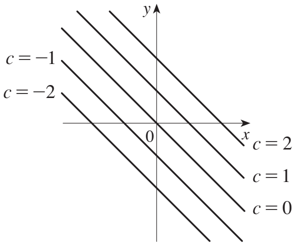


(b) $y = mx + 1 - 2m$, where m is the slope.



(c) $y = 2x - 3$

9. Their graphs have slope −1.

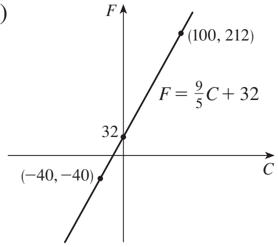


11. $f(x) = 2x^2 - 12x + 18$

13. $f(x) = -3x(x + 1)(x - 2)$

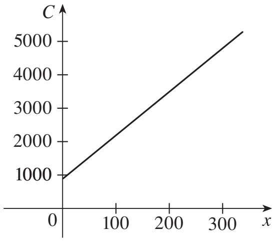
15. (a) 8.34, change in mg for every 1 year change
(b) 8.34 mg

17. (a)



(b) $\frac{9}{5}$, change in °F for every 1°C change; 32, Fahrenheit temperature corresponding to 0°C

19. (a) $C = 13x + 900$



(b) 13; cost (in dollars) of producing each additional chair
(c) 900; daily fixed costs

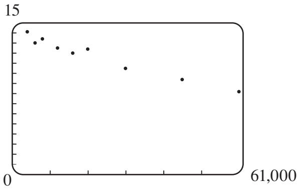
21. (a) $P = 0.1d + 1.05$ (b) 59.5 m

23. Four times brighter

25. (a) 8 (b) 4 (c) 605,000 W; 2,042,000 W; 9,454,000 W

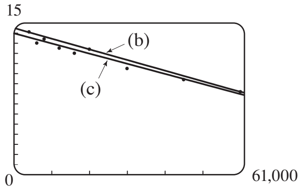
27. (a) Cosine (b) Linear

29. (a)



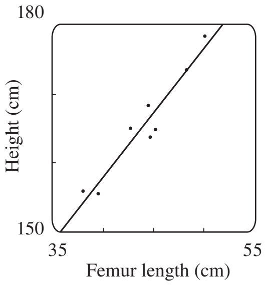
A linear model is appropriate.

(b) $y = -0.000105x + 14.521$

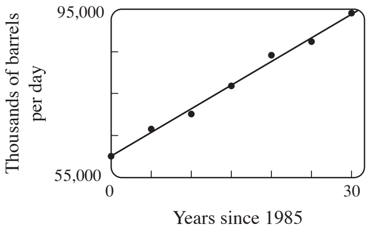


(c) $y = -0.00009979x + 13.951$
(d) About 11.5 per 100 population
(e) About 6% (f) No

31. (a) See the graph in part (b).
(b) $y = 1.88074x + 82.64974$



(c) 182.3 cm
33. (a) A linear model is appropriate. See the graph in part (b).
(b) $y = 1124.86x + 60{,}119.86$



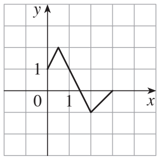
(c) In thousands of barrels per day: 79,242 and 96,115
35. (a) 2 (b) 334 m^2

EXERCISES 1.3 ■ PAGE 42

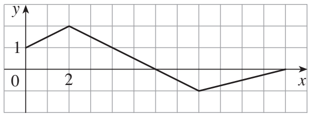
1. (a) $y = f(x) + 3$ (b) $y = f(x) - 3$ (c) $y = f(x - 3)$
(d) $y = f(x + 3)$ (e) $y = -f(x)$ (f) $y = f(-x)$
(g) $y = 3f(x)$ (h) $y = \frac{1}{3}f(x)$

3. (a) 3 (b) 1 (c) 4 (d) 5 (e) 2

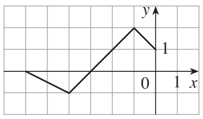
5. (a)



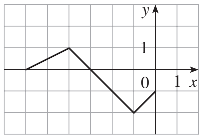
(b)



(c)

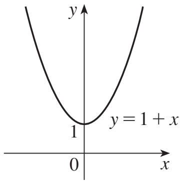


(d)

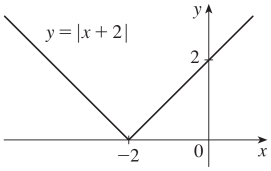


7. $y = -\sqrt{-x^2 - 5x - 4} - 1$

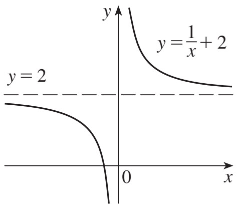
9.



11.



13.



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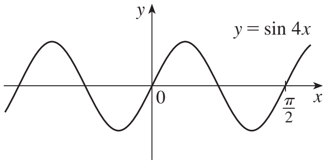

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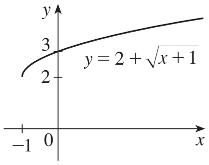

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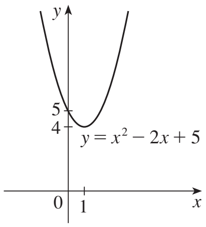

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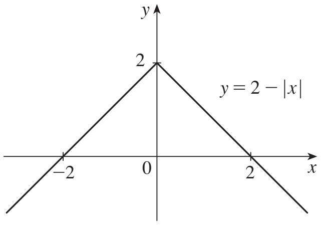

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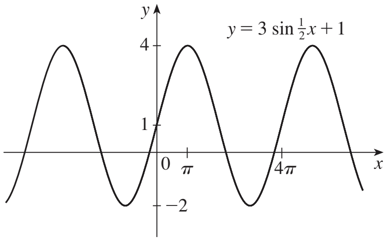

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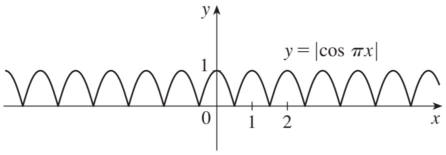
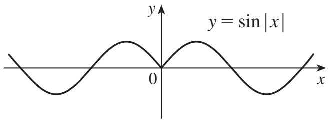


27. $L(t) = 12 + 2\sin\left[\dfrac{2\pi}{365}(t - 80)\right]$

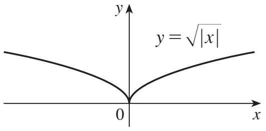
29. $D(t) = 5\cos[(\pi/6)(t - 6.75)] + 7$

31. (a) The portion of the graph of $y = f(x)$ to the right of the y-axis is reflected about the y-axis.

(b)



(c)



33. (a) $(f + g)(x) = \sqrt{25 - x^2} + \sqrt{x + 1}$, $[-1, 5]$

(b) $(f - g)(x) = \sqrt{25 - x^2} - \sqrt{x + 1}$, $[-1, 5]$

(c) $(fg)(x) = \sqrt{-x^3 - x^2 + 25x + 25}$, $[-1, 5]$

(d) $(f/g)(x) = \sqrt{\dfrac{25 - x^2}{x + 1}}$, $(-1, 5]$

35. (a) $(f \circ g)(x) = x + 5$, $(-\infty, \infty)$

(b) $(g \circ f)(x) = \sqrt[3]{x^3 + 5}$, $(-\infty, \infty)$

(c) $(f \circ f)(x) = (x^3 + 5)^3 + 5$, $(-\infty, \infty)$

(d) $(g \circ g)(x) = \sqrt[9]{x}$, $(-\infty, \infty)$

37. (a) $(f \circ g)(x) = \dfrac{1}{\sqrt{x + 1}}$, $(-1, \infty)$

(b) $(g \circ f)(x) = \dfrac{1}{\sqrt{x}} + 1$, $(0, \infty)$

(c) $(f \circ f)(x) = \sqrt[4]{x}$, $(0, \infty)$

(d) $(g \circ g)(x) = x + 2$, $(-\infty, \infty)$

39. (a) $(f \circ g)(x) = \dfrac{2}{\sin x}$, $\{x \mid x \neq n\pi\}$, n an integer

(b) $(g \circ f)(x) = \sin\left(\dfrac{2}{x}\right)$, $\{x \mid x \neq 0\}$

(c) $(f \circ f)(x) = x$, $\{x \mid x \neq 0\}$

(d) $(g \circ g)(x) = \sin(\sin x)$, $\mathbb{R}$

41. $(f \circ g \circ h)(x) = 3\sin(x^2) - 2$

43. $(f \circ g \circ h)(x) = \sqrt{x^6 + 4x^3 + 1}$

45. $g(x) = 2x + x^2$, $f(x) = x^4$

47. $g(x) = \sqrt[3]{x}$, $f(x) = x/(1 + x)$

49. $g(t) = t^2$, $f(t) = \sec t \tan t$

51. $h(x) = \sqrt{x}$, $g(x) = x - 1$, $f(x) = \sqrt{x}$

53. $h(t) = \cos t$, $g(t) = \sin t$, $f(t) = t^2$

55. (a) 6 (b) 5 (c) 5 (d) 3

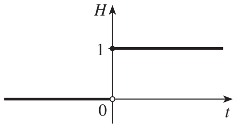
57. (a) 4 (b) 3 (c) 0 (d) Does not exist; $f(6) = 6$ is not in the domain of g. (e) 4 (f) -2

59. (a) $r(t) = 60t$ (b) $(A \circ r)(t) = 3600\pi t^2$; the area of the circle as a function of time

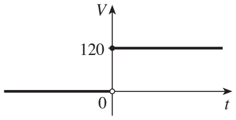
61. (a) $s = \sqrt{d^2 + 36}$ (b) $d = 30t$

(c) $(f \circ g)(t) = \sqrt{900t^2 + 36}$; the distance between the lighthouse and the ship as a function of the time elapsed since noon

63. (a)

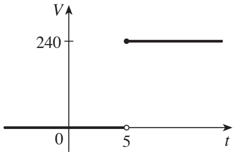


(b)



$V(t) = 120H(t)$

(c)



$V(t) = 240H(t - 5)$

65. Yes; $m_1 m_2$

67. (a) $f(x) = x^2 + 6$ (b) $g(x) = x^2 + x - 1$

69. Yes

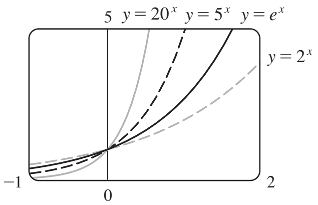
71. (d) $f(x) = \frac{1}{2}E(x) + \frac{1}{2}O(x)$, where
$E(x) = 2^x + 2^{-x} + (x - 3)^2 + (x + 3)^2$ and
$O(x) = 2^x - 2^{-x} + (x - 3)^2 - (x + 3)^2$

EXERCISES 1.4 ■ PAGE 52

1. (a) -1 (b) $3^{-6} = \frac{1}{729}$ (c) $x^{-5/4} = 1/(x\sqrt[4]{x})$ (d) x^2
(e) $b^5/9$ (f) $2x^6/(9y)$

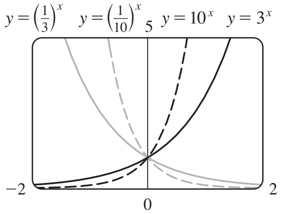
3. (a) $f(x) = b^x$, $b > 0$ (b) $\mathbb{R}$ (c) $(0, \infty)$
(d) See Figures 4(c), 4(b), and 4(a), respectively.

5.



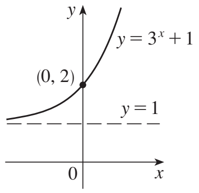
All approach 0 as $x \to -\infty$, all pass through $(0, 1)$, and all are increasing. The larger the base, the faster the rate of increase.

7.

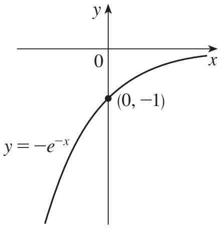


The functions with base greater than 1 are increasing and those with base less than 1 are decreasing. The latter are reflections of the former about the y-axis.

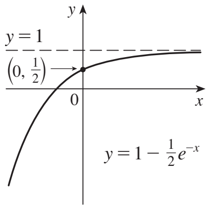
9.



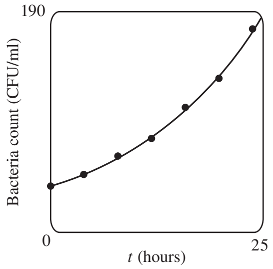
11.



13.



15. (a) $y = e^x - 2$ (b) $y = e^{x-2}$ (c) $y = -e^x$
(d) $y = e^{-x}$ (e) $y = -e^{-x}$
17. (a) $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ (b) $(-\infty, \infty)$
19. $f(x) = 3 \cdot 2^x$ **25.** At $x \approx 35.8$
27. (a) See graph in part (c).
(b) $f(t) = 36.89301 \cdot (1.06614)^t$
(c)



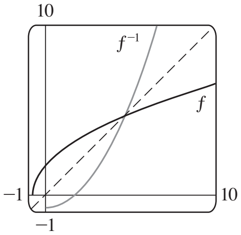
About 10.87 h

29. (a) 32,000 (b) $y = 500 \cdot 2^{2t}$ (c) ≈1260 (d) 3.82 h
31. 3.5 days
35. The minimum value is a, and the graph becomes flatter near the y-axis as a increases.

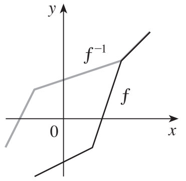
EXERCISES 1.5 ■ PAGE 64

1. (a) See Definition 1.
(b) It must pass the Horizontal Line Test.
3. No **5.** No **7.** Yes **9.** Yes **11.** Yes **13.** No

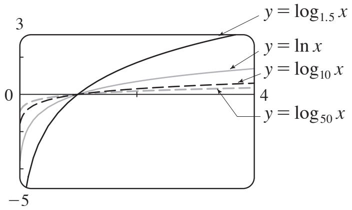
15. No **17.** (a) 6 (b) 3 **19.** 0
21. $F = \frac{9}{5}C + 32$; the Fahrenheit temperature as a function of the Celsius temperature; $[-273.15, \infty)$
23. $f^{-1}(x) = \sqrt{1-x}$ **25.** $g^{-1}(x) = (x-2)^2 - 1, x \geqslant 2$
27. $y = 1 - \ln x$ **29.** $y = \left(\sqrt[5]{x} - 2\right)^3$
31. $f^{-1}(x) = \frac{1}{4}(x^2 - 3), x \geqslant 0$



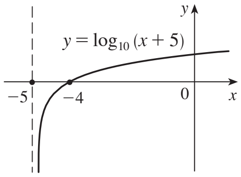
33.



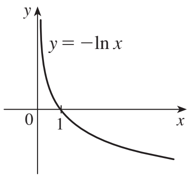
35. (a) $f^{-1}(x) = \sqrt{1 - x^2}, 0 \leqslant x \leqslant 1$; f^{-1} and f are the same function. (b) Quarter-circle in the first quadrant
37. (a) It's defined as the inverse of the exponential function with base b, that is, $\log_b x = y \iff b^y = x$.
(b) $(0, \infty)$ (c) $\mathbb{R}$ (d) See Figure 11.
39. (a) 4 (b) −4 (c) $\frac{1}{2}$
41. (a) 1 (b) −2 (c) −4
43. (a) $2\log_{10} x + 3\log_{10} y + \log_{10} z$
(b) $4\ln x - \frac{1}{2}\ln(x+2) - \frac{1}{2}\ln(x-2)$
45. (a) $\log_{10} 2$ (b) $\ln \dfrac{ac^3}{b^2}$
47. (a) 1.430677 (b) 0.917600
49.



All graphs approach $-\infty$ as $x \to 0^+$, all pass through (1, 0), and all are increasing. The larger the base, the slower the rate of increase.
51. About 335,544 km
53. (a)

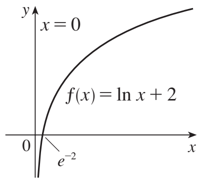


(b)



55. (a) $(0, \infty)$; $(-\infty, \infty)$ (b) e^{-2}

(c)



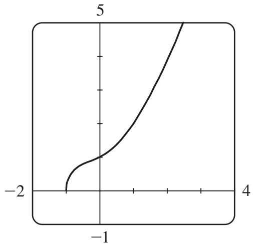
57. (a) $\frac{1}{4}(e^3 - 2) \approx 4.521$ (b) $\frac{1}{2}(3 + \ln 12) \approx 2.742$

59. (a) $\frac{1}{2}(1 + \sqrt{5}) \approx 1.618$ (b) $\dfrac{1}{2} - \dfrac{\ln 9}{2 \ln 5} \approx -0.183$

61. (a) $0 < x < 1$ (b) $x > \ln 5$

63. (a) $(\ln 3, \infty)$ (b) $f^{-1}(x) = \ln(e^x + 3)$; $\mathbb{R}$

65. The graph passes the Horizontal Line Test.



$f^{-1}(x) = -\frac{1}{6}\sqrt[3]{4}\left(\sqrt[3]{D - 27x^2 + 20} - \sqrt[3]{D + 27x^2 - 20} + \sqrt[3]{2}\right)$, where $D = 3\sqrt{3}\sqrt{27x^4 - 40x^2 + 16}$; two of the expressions are complex.

67. (a) $f^{-1}(n) = (3/\ln 2)\ln(n/100)$; the time elapsed when there are n bacteria (b) After about 26.9 h

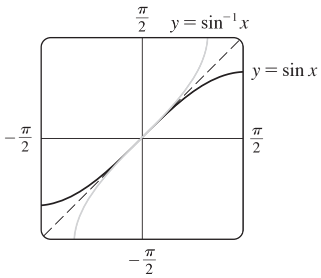
69. (a) π (b) $\pi/6$

71. (a) $\pi/4$ (b) $\pi/2$

73. (a) $5\pi/6$ (b) $\pi/3$

77. $x/\sqrt{1 + x^2}$

79.



The second graph is the reflection of the first graph about the line $y = x$.

81. $\left[-\frac{2}{3}, 0\right]$, $[-\pi/2, \pi/2]$

83. (a) Shifts downward; $g^{-1}(x) = f^{-1}(x) - c$

(b) $h^{-1}(x) = (1/c)f^{-1}(x)$

CHAPTER 1 REVIEW ■ PAGE 67

True-False Quiz

1. False **3.** False **5.** True **7.** False **9.** True

11. False **13.** False

Exercises

1. (a) 2.7 (b) 2.3, 5.6 (c) $[-6, 6]$ (d) $[-4, 4]$

(e) $[-4, 4]$ (f) No; it fails the Horizontal Line Test.

(g) Odd; its graph is symmetric about the origin.

3. $2a + h - 2$ **5.** $\left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$, $(-\infty, 0) \cup (0, \infty)$

7. $(-6, \infty)$, $(-\infty, \infty)$

9. (a) Shift the graph 5 units upward.

(b) Shift the graph 5 units to the left.

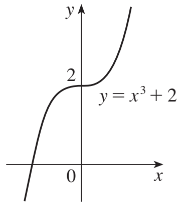
(c) Stretch the graph vertically by a factor of 2, then shift 1 unit upward.

(d) Shift the graph 2 units to the right and 2 units downward.

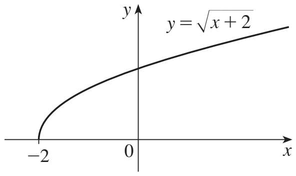
(e) Reflect the graph about the x-axis.

(f) Reflect the graph about the line $y = x$ (assuming f is one-to-one).

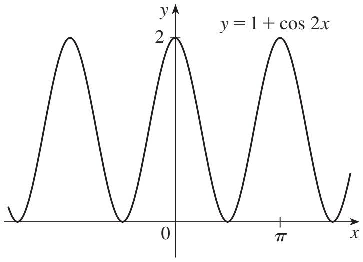
11.



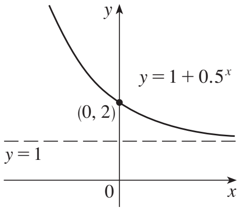
13.



15.



17.



19. (a) Neither (b) Odd (c) Even (d) Neither

(e) Even (f) Neither

21. (a) $(f \circ g)(x) = \ln(x^2 - 9)$, $(-\infty, -3) \cup (3, \infty)$

(b) $(g \circ f)(x) = (\ln x)^2 - 9$, $(0, \infty)$

(c) $(f \circ f)(x) = \ln \ln x$, $(1, \infty)$

(d) $(g \circ g)(x) = (x^2 - 9)^2 - 9$, $(-\infty, \infty)$

23. $y = 0.2441x - 413.3960$; about 82.1 years

25. 1

27. (a) $\ln x + \frac{1}{2}\ln(x + 1)$ (b) $\frac{1}{2}\log_2(x^2 + 1) - \frac{1}{2}\log_2(x - 1)$

29. (a) 25 (b) 3 (c) $\frac{4}{3}$

31. $\frac{1}{2}\ln 3 \approx 0.549$

33. $\ln(\ln 10) \approx 0.834$

35. $\pm 1/\sqrt{3} \approx \pm 0.577$

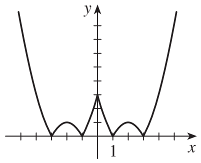
37. (a) 6.5 RNA copies/mL (b) $V(t) = 52.0(\frac{1}{2})^{t/8}$
(c) $t(V) = -8\log_2(V/52.0)$; the time required for the viral load to reach a given number V
(d) 37.6 days

PRINCIPLES OF PROBLEM SOLVING ■ PAGE 75

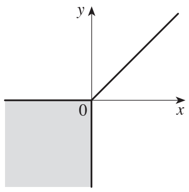
1. $a = 4\sqrt{h^2 - 16}/h$, where a is the length of the altitude and h is the length of the hypotenuse

3. $-\frac{2}{3}, \frac{4}{3}$

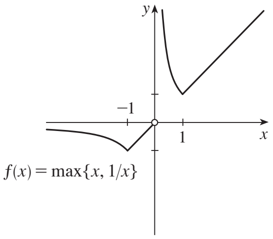
5.



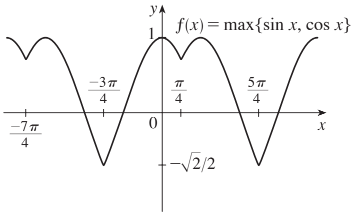
7.



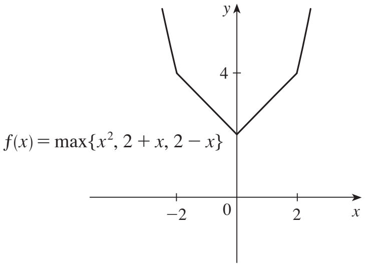
9. (a)



(b)



(c)



13. 0 **15.** $x \in \left[-1, 1 - \sqrt{3}\right) \cup \left(1 + \sqrt{3}, 3\right]$

17. 66.7 km/h **21.** $f_n(x) = x^{2^{n+1}}$

CHAPTER 2

EXERCISES 2.1 ■ PAGE 82

1. (a) $-44.4, -38.8, -27.8, -22.2, -16.\overline{6}$
(b) -33.3 (c) $-33\frac{1}{3}$

3. (a) (i) 2 (ii) 1.111111 (iii) 1.010101 (iv) 1.001001
(v) 0.666667 (vi) 0.909091 (vii) 0.990099
(viii) 0.999001 (b) 1 (c) $y = x - 3$

5. (a) (i) −40 m/s (ii) −39.4 m/s (iii) −39.3 m/s
(b) −39 m/s

7. (a) (i) 8.9 m/s (ii) 9.9 m/s (iii) 13.9 m/s
(iv) 14.9 m/s (b) 8.9 m/s

9. (a) 0, 1.7321, −1.0847, −2.7433, 4.3301, −2.8173, 0, −2.1651, −2.6061, −5, 3.4202; no (c) −31.4

EXERCISES 2.2 ■ PAGE 92

1. Yes

3. (a) $\lim_{x\to -3} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to −3 (but not equal to −3).
(b) $\lim_{x\to 4^+} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to 4 through values larger than 4.

5. (a) 2 (b) 1 (c) 4 (d) Does not exist (e) 3

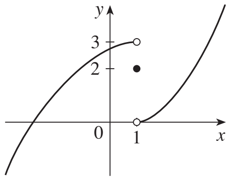
7. (a) 4 (b) 5 (c) 2, 4 (d) 4

9. (a) $-\infty$ (b) ∞ (c) ∞ (d) $-\infty$ (e) ∞
(f) $x = -7, x = -3, x = 0, x = 6$

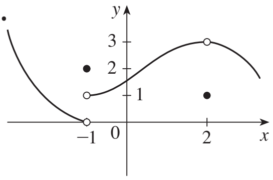
11. $\lim_{x\to a} f(x)$ exists for all a except $a = 0$.

13. (a) −1 (b) 1 (c) Does not exist

15.



17.

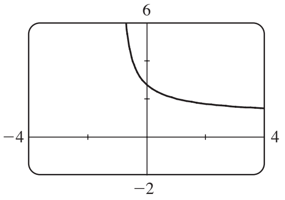


19. $\frac{1}{2}$ **21.** 5 **23.** 0.25 **25.** 1.5 **27.** 1

29. ∞ **31.** ∞ **33.** $-\infty$ **35.** $-\infty$ **37.** ∞

39. $-\infty$ **41.** $x = -2$ **43.** $-\infty$; ∞

45. (a) 2.71828 (b)



47. (a) 0.998000, 0.638259, 0.358484, 0.158680, 0.038851, 0.008928, 0.001465; 0
(b) 0.000572, −0.000614, −0.000907, −0.000978, −0.000993, −0.001000; −0.001

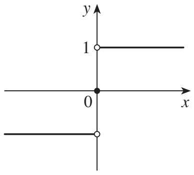
49. $x \approx \pm 0.90, \pm 2.24$; $x = \pm\sin^{-1}(\pi/4), \pm(\pi - \sin^{-1}(\pi/4))$

51. $m \to \infty$

EXERCISES 2.3 ■ PAGE 102

1. (a) -6 (b) -8 (c) 2 (d) -6
(e) Does not exist (f) 0
3. 75 **5.** 88 **7.** 5 **9.** $-\frac{1}{27}$ **11.** -13
13. 6 **15.** Does not exist **17.** $\frac{5}{7}$ **19.** $\frac{9}{2}$
21. -6 **23.** $\frac{1}{6}$ **25.** $-\frac{1}{9}$ **27.** 1 **29.** $\frac{1}{128}$
31. $-\frac{1}{2}$ **33.** $3x^2$ **35.** (a), (b) $\frac{2}{3}$ **39.** 7 **43.** 8
45. -4 **47.** Does not exist

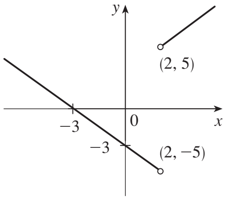
49. (a)



(b) (i) 1
(ii) -1
(iii) Does not exist
(iv) 1

51. (a) (i) 5 (ii) -5 (b) Does not exist

(c)



53. 7
55. (a) (i) -2 (ii) Does not exist (iii) -3
(b) (i) $n-1$ (ii) n (c) a is not an integer.
61. 8 **67.** 15; -1

EXERCISES 2.4 ■ PAGE 113

1. 0.1 (or any smaller positive number)
3. 1.44 (or any smaller positive number)
5. 0.4269 (or any smaller positive number)
7. 0.0219 (or any smaller positive number);
0.011 (or any smaller positive number)
9. (a) 0.01 (or any smaller positive number)

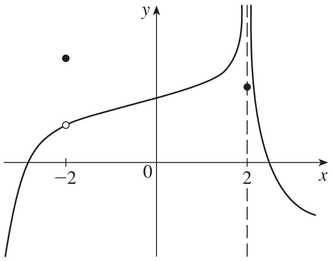
(b) $\lim_{x \to 2^+} \dfrac{1}{\ln(x-1)} = \infty$

11. (a) $\sqrt{1000/\pi}$ cm (b) Within approximately 0.0445 cm
(c) Radius; area; $\sqrt{1000/\pi}$; 1000; 5; ≈ 0.0445
13. (a) 0.025 (b) 0.0025
35. (a) 0.093 (b) $d = (B^{2/3} - 12)/(6B^{1/3}) - 1$, where
$B = 216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}$
41. Within 0.1

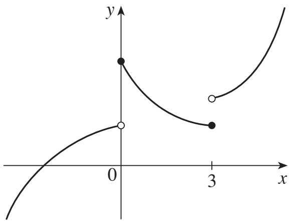
EXERCISES 2.5 ■ PAGE 124

1. $\lim_{x \to 4} f(x) = f(4)$
3. (a) $-4, -2, 2, 4$; $f(-4)$ is not defined and $\lim_{x \to a} f(x)$ does not exist for $a = -2$, 2, and 4
(b) -4, neither; -2, left; 2, right; 4, right
5. (a) 1 (b) 1, 3 (c) 3

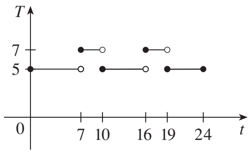
7.



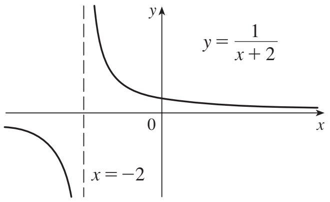
9.



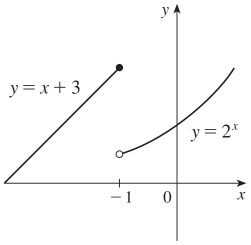
11. (a)



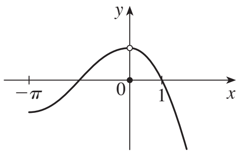
19. $f(-2)$ is undefined.



21. $\lim_{x \to -1} f(x)$ does not exist.

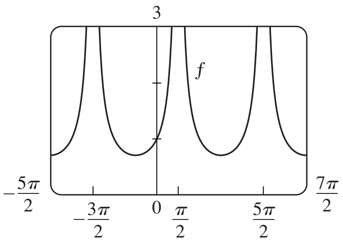


23. $\lim_{x \to 0} f(x) \neq f(0)$

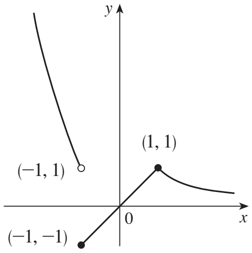


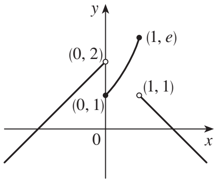
25. (b) Define $f(3) = \frac{1}{6}$. **27.** $(-\infty, \infty)$ **29.** $(-\infty, 0) \cup (0, \infty)$
31. $(-1, 1)$ **33.** $(-\infty, -1] \cup (0, \infty)$ **35.** 8 **37.** ln 2

39. $x = \dfrac{\pi}{2} + 2n\pi$, n any integer



43. -1, right **45.** 0, right; 1, left



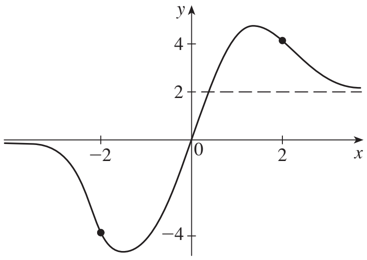


47. $\frac{2}{3}$ **49.** 4
51. (a) $g(x) = x^3 + x^2 + x + 1$ (b) $g(x) = x^2 + x$
59. (b) (0.86, 0.87) **61.** (b) 70.347 **71.** None

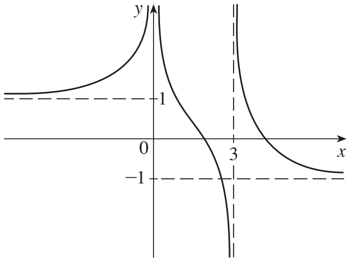
EXERCISES 2.6 ■ PAGE 137

1. (a) As x becomes large, $f(x)$ approaches 5.
(b) As x becomes large negative, $f(x)$ approaches 3.
3. (a) -2 (b) 2 (c) ∞ (d) $-\infty$
(e) $x = 1, x = 3, y = -2, y = 2$

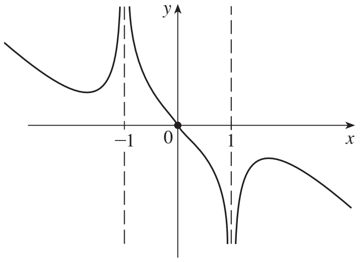
5.



7.

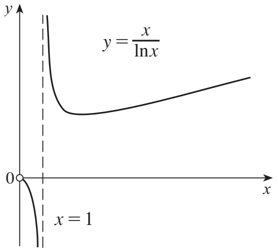


9.

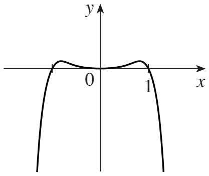


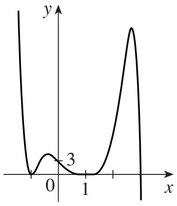
11. 0 **13.** $\frac{2}{5}$ **15.** $\frac{4}{5}$ **17.** 0 **19.** $-\frac{1}{3}$ **21.** -1
23. $\frac{\sqrt{3}}{4}$ **25.** -2 **27.** $-\infty$ **29.** 0 **31.** $\frac{1}{2}(a - b)$
33. $-\infty$ **35.** 0 **37.** $-\frac{1}{2}$ **39.** 0 **41.** ∞

43. (a) (i) 0 (ii) $-\infty$ (iii) ∞ (b) ∞
(c)

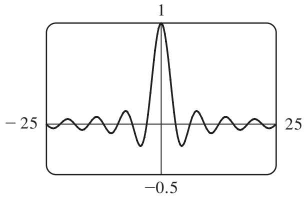


45. (a), (b) $-\frac{1}{2}$ **47.** $y = 4, x = -3$
49. $y = 2; x = -2, x = 1$ **51.** $x = 5$ **53.** $y = 3$
55. (a) 0 (b) $\pm\infty$
57. $f(x) = \dfrac{2 - x}{x^2(x - 3)}$ **59.** (a) $\frac{5}{4}$ (b) 5
61. $-\infty, -\infty$ **63.** $-\infty, \infty$

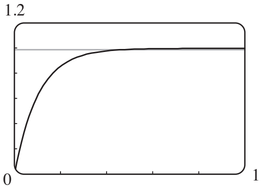




65. (a) 0 (b) An infinite number of times



67. 5
69. (a) v^* (b)

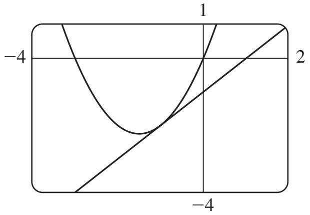


≈ 0.47 s

71. $N \geqslant 15$ **73.** $N \leqslant -9, N \leqslant -19$
75. (a) $x > 100$

EXERCISES 2.7 ■ PAGE 149

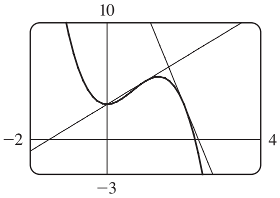
1. (a) $\dfrac{f(x) - f(3)}{x - 3}$ (b) $\displaystyle\lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}$
3. (a) 1 (b) $y = x - 1$ (c)



5. $y = 7x - 17$ **7.** $y = -5x + 6$

9. (a) $8a - 6a^2$ (b) $y = 2x + 3$, $y = -8x + 19$

(c)

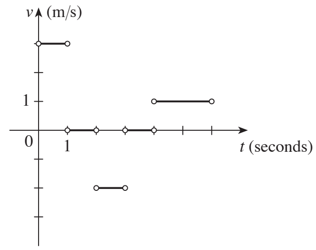


11. (a) 2.5 s (b) 24.5 m/s

13. $-2/a^3$ m/s; -2 m/s; $-\frac{1}{4}$ m/s; $-\frac{2}{27}$ m/s

15. (a) Right: $0 < t < 1$ and $4 < t < 6$; left: $2 < t < 3$; standing still: $1 < t < 2$ and $3 < t < 4$

(b)

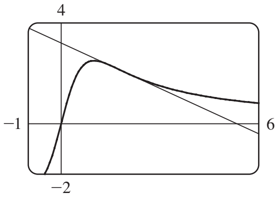


17. $g'(0)$, 0, $g'(4)$, $g'(2)$, $g'(-2)$

19. $\frac{2}{5}$ **21.** $\frac{5}{9}$ **23.** $4a - 5$

25. $-\dfrac{2a}{(a^2 + 1)^2}$ **27.** $y = -\frac{1}{2}x + 3$ **29.** $y = 3x - 1$

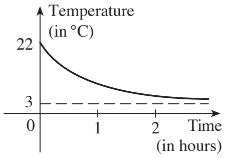
31. (a) $-\frac{3}{5}$; $y = -\frac{3}{5}x + \frac{16}{5}$ (b)



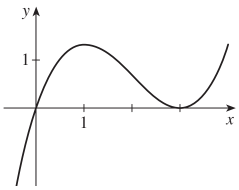
33. $f(2) = 3$; $f'(2) = 4$

35. 32 m/s; 32 m/s

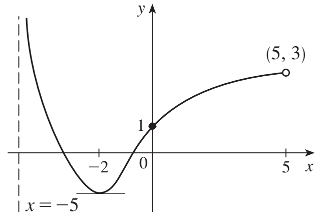
37. Greater (in magnitude)



39.



41.



43. $f(x) = \sqrt{x}$, $a = 9$ **45.** $f(x) = x^6$, $a = 2$

47. $f(x) = \tan x$, $a = \pi/4$

49. (a) (i) \$20.25/unit (ii) \$20.05/unit (b) \$20/unit

51. (a) The rate at which the cost is changing per ounce of gold produced; dollars per ounce

(b) When the 22nd kilogram of gold is produced, the cost of production is \$17/kg

(c) Decrease in the short term; increase in the long term

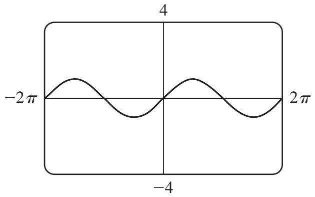
53. (a) The rate at which the oxygen solubility changes with respect to the water temperature; (mg/L)/°C

(b) $S'(16) \approx -0.25$; as the temperature increases past 16°C, the oxygen solubility is decreasing at a rate of 0.25 (mg/L)/°C.

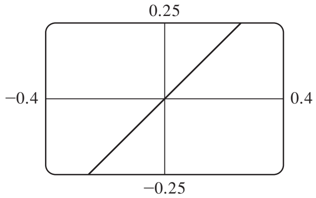
55. (a) In (g/dL)/h: (i) -0.015 (ii) -0.012 (iii) -0.012 (iv) -0.011 (b) -0.012 (g/dL)/h; After 2 hours, the BAC is decreasing at a rate of 0.012 (g/dL)/h.

57. Does not exist

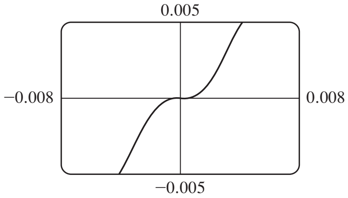
59. (a) Slope appears to be 1.



(b) Yes

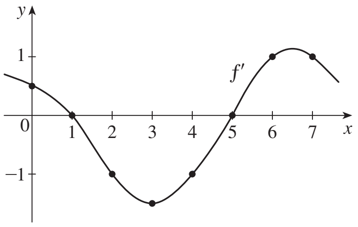


(c) Yes; 0



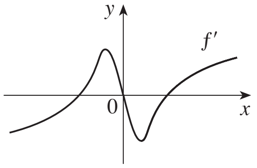
EXERCISES 2.8 ■ PAGE 161

1. (a) 0.5 (b) 0 (c) -1 (d) -1.5
(e) -1 (f) 0 (g) 1 (h) 1

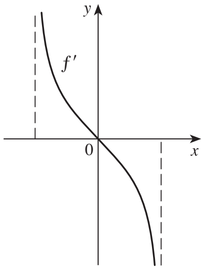


3. (a) II (b) IV (c) I (d) III

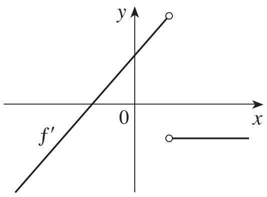
5.



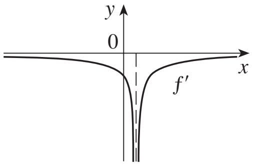
7.



9.

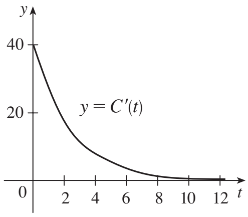


11.



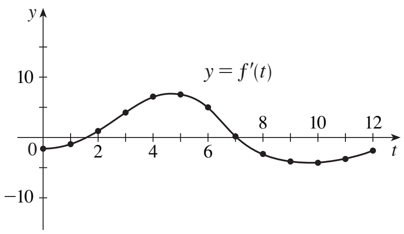
13. (a) The instantaneous rate of change of percentage of full capacity with respect to elapsed time in hours

(b)



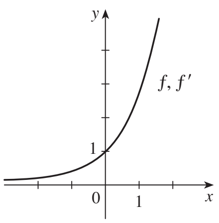
The rate of change of percentage of full capacity is decreasing and approaching 0.

15.



When $t \approx 5.25$

17.



$f'(x) = e^x$

19. (a) 0, 1, 2, 4 (b) $-1, -2, -4$ (c) $f'(x) = 2x$

21. $f'(x) = 3$, $\mathbb{R}$, $\mathbb{R}$ **23.** $f'(t) = 5t + 6$, $\mathbb{R}$, $\mathbb{R}$

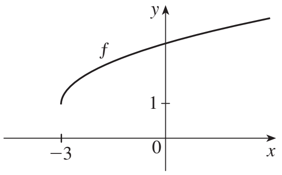
25. $A'(p) = 12p^2 + 3$, $\mathbb{R}$, $\mathbb{R}$

27. $f'(x) = -\dfrac{2x}{(x^2-4)^2}$, $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$, $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

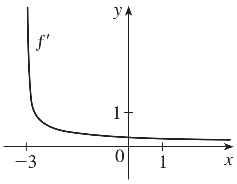
29. $g'(u) = -\dfrac{5}{(4u-1)^2}$, $\left(-\infty, \frac{1}{4}\right) \cup \left(\frac{1}{4}, \infty\right)$, $\left(-\infty, \frac{1}{4}\right) \cup \left(\frac{1}{4}, \infty\right)$

31. $f'(x) = -\dfrac{1}{2(1+x)^{3/2}}$, $(-1, \infty)$, $(-1, \infty)$

33. (a)



(b), (d)

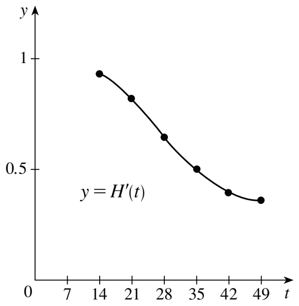


(c) $f'(x) = \dfrac{1}{2\sqrt{x+3}}$, $[-3, \infty)$, $(-3, \infty)$

35. (a) $f'(x) = 4x^3 + 2$

37.

t	14	21	28	35	42	49
$H'(t)$	0.57	0.43	0.33	0.29	0.14	0.5

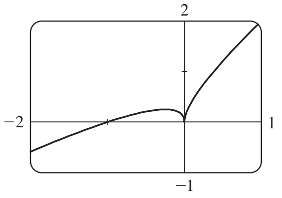


39. (a) The rate at which the percentage of electrical power produced by solar panels is changing, in percentage points per year. (b) On January 1, 2022, the percentage of electrical power produced by solar panels was increasing at a rate of 3.5 percentage points per year.

41. -4 (corner); 0 (discontinuity)

43. 1 (not defined); 5 (vertical tangent)

45.

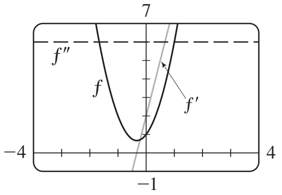


Differentiable at -1; not differentiable at 0

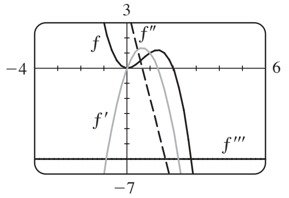
47. $f''(1)$ **49.** $a = f, b = f', c = f''$

51. a = acceleration, b = velocity, c = position

53. $6x + 2$; 6



55.

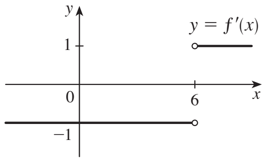


$f'(x) = 4x - 3x^2$,
$f''(x) = 4 - 6x$,
$f'''(x) = -6$,
$f^{(4)}(x) = 0$

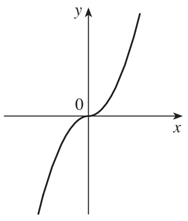
57. (a) $\frac{1}{3}a^{-2/3}$

59. $f'(x) = \begin{cases} -1 & \text{if } x < 6 \\ 1 & \text{if } x > 6 \end{cases}$

or $f'(x) = \dfrac{x-6}{|x-6|}$

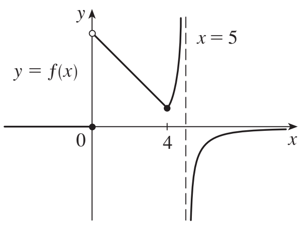


61. (a)



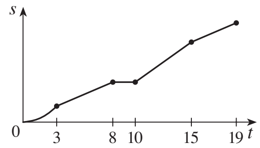
(b) All x
(c) $f'(x) = 2|x|$

65. (a) $-1, 1$ (b)

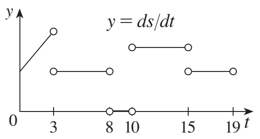


(c) 0, 5 (d) 0, 4, 5

67. (a)



(b)



CHAPTER 2 REVIEW ■ PAGE 167

True-False Quiz

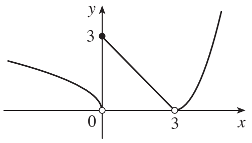
1. False **3.** True **5.** True **7.** False **9.** True
11. False **13.** False **15.** False **17.** True
19. False **21.** True **23.** True **25.** False

Exercises

1. (a) (i) 3 (ii) 0 (iii) Does not exist (iv) 2
(v) ∞ (vi) $-\infty$ (vii) 4 (viii) -1
(b) $y = 4, y = -1$ (c) $x = 0, x = 2$ (d) $-3, 0, 2, 4$

3. 1 **5.** $\frac{3}{2}$ **7.** 3 **9.** ∞ **11.** $\frac{5}{7}$ **13.** $\frac{1}{2}$
15. $-\infty$ **17.** 2 **19.** $\pi/2$ **21.** $x = 0, y = 0$ **23.** 1

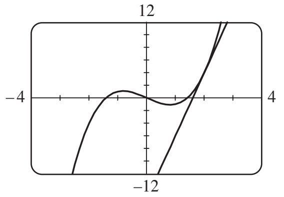
29. (a) (i) 3 (ii) 0 (iii) Does not exist
(iv) 0 (v) 0 (vi) 0
(b) At 0 and 3 (c)



31. $\mathbb{R}$ **35.** (a) -8 (b) $y = -8x + 17$

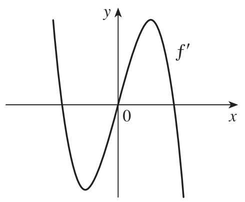
37. (a) (i) 3 m/s (ii) 2.75 m/s (iii) 2.625 m/s
(iv) 2.525 m/s (b) 2.5 m/s

39. (a) 10 (b) $y = 10x - 16$
(c)



41. (a) The rate at which the cost changes with respect to the interest rate; dollars/(percent per year)
(b) As the interest rate increases past 10%, the cost is increasing at a rate of \$1200/(percent per year).
(c) Always positive

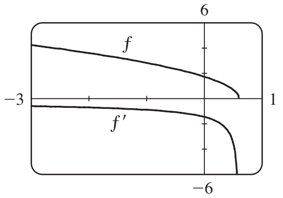
43.



45. $f'(x) = -4/x^3$, $(-\infty, 0) \cup (0, \infty)$

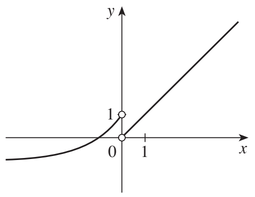
47. (a) $f'(x) = -\frac{5}{2}(3 - 5x)^{-1/2}$ (b) $\left(-\infty, \frac{3}{5}\right]$, $\left(-\infty, \frac{3}{5}\right)$

(c)



49. −4 (discontinuity), −1 (corner), 2 (discontinuity), 5 (vertical tangent)

51.



53. The rate at which the number of US \$20 bills in circulation is changing with respect to time; 0.28 billion bills per year

55. 0

PROBLEMS PLUS ■ PAGE 171

1. $\frac{2}{3}$ **3.** -4 **5.** (a) Does not exist (b) 1

7. (a) $(-\infty, 0) \cup [1, \infty)$, $(0, 2)$ (b) 1 **9.** $\frac{3}{4}$

11. (b) Yes (c) Yes; no

13. (a) 0 (b) 1 (c) $f'(x) = x^2 + 1$

CHAPTER 3

EXERCISES 3.1 ■ PAGE 181

1. (a) e is the number such that $\displaystyle\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.

(b) 0.99, 1.03; $2.7 < e < 2.8$

3. $g'(x) = 4$ **5.** $f'(x) = 75x^{74} - 1$

7. $f'(t) = -2e^t$ **9.** $W'(v) = -5.4v^{-4}$

11. $f'(x) = \frac{3}{2}x^{1/2} - 3x^{-4}$ **13.** $s'(t) = -\dfrac{1}{t^2} - \dfrac{2}{t^3}$

15. $y' = 2 + 1/(2\sqrt{x})$ **17.** $g'(x) = -\frac{1}{2}x^{-3/2} + \frac{1}{4}x^{-3/4}$

19. $f'(x) = 4x^3 + 9x^2$ **21.** $y' = 3e^x - \frac{4}{3}x^{-4/3}$

23. $f'(x) = 3 + 2x$ **25.** $G'(r) = \frac{3}{2}r^{-1/2} + \frac{3}{2}r^{1/2}$

27. $j'(x) = 2.4x^{1.4}$ **29.** $F'(z) = -\dfrac{2A}{z^3} - \dfrac{B}{z^2}$

31. $D'(t) = -\dfrac{3}{64t^4} - \dfrac{1}{4t^2}$

33. $P'(w) = 3\sqrt{w} - \frac{1}{2}w^{-1/2} - 2w^{-3/2}$

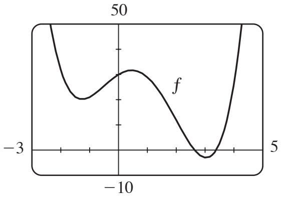
35. $dy/dx = 2tx + t^3$; $dy/dt = x^2 + 3t^2x$

37. $y = 4x - 1$ **39.** $y = \frac{1}{2}x + 2$

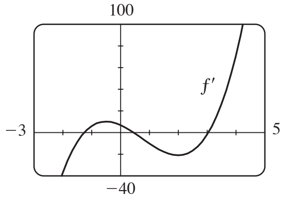
41. Tangent: $y = 2x + 2$; normal: $y = -\frac{1}{2}x + 2$

43. $y = 3x - 1$ **45.** $f'(x) = 4x^3 - 6x^2 + 2x$

47. (a)



(c) $4x^3 - 9x^2 - 12x + 7$



49. $f'(x) = 0.005x^4 - 0.06x^2$, $f''(x) = 0.02x^3 - 0.12x$

51. $f'(x) = 2 - \frac{15}{4}x^{-1/4}$, $f''(x) = \frac{15}{16}x^{-5/4}$

53. (a) $v(t) = 3t^2 - 3$, $a(t) = 6t$ (b) 12 m/s^2

(c) $a(1) = 6$ m/s^2

55. 4.198; at 12 years, the length of the fish is increasing at a rate of 4.198 cm/year

57. (a) $V = 5.3/P$

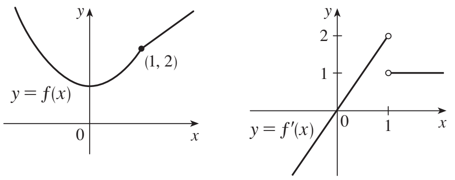
(b) −0.00212; instantaneous rate of change of the volume with respect to the pressure at 25°C; m^3/kPa

59. $(-3, 37)$, $(1, 5)$ **63.** $y = 3x - 3$, $y = 3x - 7$

65. $y = -2x + 3$ **67.** $(\pm 2, 4)$ **71.** $P(x) = x^2 - x + 3$

73. $y = \frac{3}{16}x^3 - \frac{9}{4}x + 3$

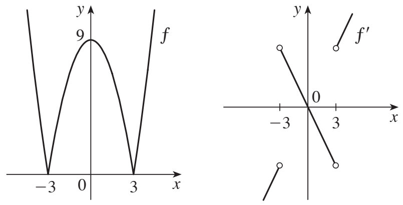
75. No



77. (a) Not differentiable at 3 or −3

$$f'(x) = \begin{cases} 2x & \text{if } |x| > 3 \\ -2x & \text{if } |x| < 3 \end{cases}$$

(b)



79. $y = 2x^2 - x$ **81.** $a = -\frac{1}{2}$, $b = 2$ **83.** $-\frac{1}{3}$

85. $m = 4$, $b = -4$ **87.** 1000 **89.** $\left(0, -\frac{1}{4}\right)$ **91.** 3; 1

EXERCISES 3.2 ■ PAGE 189

1. $1 - 2x + 6x^2 - 8x^3$ **3.** $y' = 24x^2 + 40x + 6$

5. $y' = e^x(x^3 + 3x^2)$ **7.** $f'(x) = e^x(3x^2 + x - 5)$

9. $y' = \dfrac{1 - x}{e^x}$ **11.** $g'(t) = \dfrac{-17}{(5t + 1)^2}$

13. $f'(t) = \dfrac{-10t^3 - 5}{(t^3 - t - 1)^2}$ **15.** $y' = \dfrac{3 - 2\sqrt{s}}{2s^{5/2}}$

17. $J'(u) = -\left(\dfrac{1}{u^2} + \dfrac{2}{u^3} + \dfrac{3}{u^4}\right)$

19. $H'(u) = 2u - 1$ **21.** $V'(t) = \dfrac{3t + 2e^t + 4te^t}{2\sqrt{t}}$

23. $y' = e^p\left(1 + \frac{3}{2}\sqrt{p} + p + p\sqrt{p}\right)$

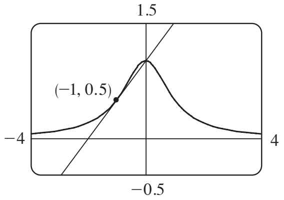
25. $f'(t) = \dfrac{-2t - 3}{3t^{2/3}(t - 3)^2}$ **27.** $f'(x) = \dfrac{xe^x(x^3 + 2e^x)}{(x^2 + e^x)^2}$

29. $f'(x) = \dfrac{2cx}{(x^2 + c)^2}$ **31.** $e^x(x^2 + 2x)$; $e^x(x^2 + 4x + 2)$

33. $\dfrac{-x^2 - 1}{(x^2 - 1)^2}$; $\dfrac{2x^3 + 6x}{(x^2 - 1)^3}$ **35.** $y = \frac{3}{4}x - \frac{1}{4}$

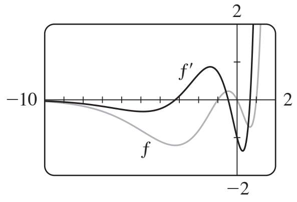
37. $y = -\frac{1}{3}x + \frac{5}{6}$; $y = 3x - \frac{5}{2}$

39. (a) $y = \frac{1}{2}x + 1$ (b)



41. (a) $e^x(x^3 + 3x^2 - x - 1)$

(b)



43. $\frac{1}{4}$ **45.** (a) -16 (b) $-\frac{20}{9}$ (c) 20 **47.** 7

49. $y = -2x + 18$ **51.** (a) 3 (b) $-\frac{7}{12}$

53. (a) $y' = xg'(x) + g(x)$ (b) $y' = \dfrac{g(x) - xg'(x)}{[g(x)]^2}$

(c) $y' = \dfrac{xg'(x) - g(x)}{x^2}$

55. Two, $\left(-2 \pm \sqrt{3}, \frac{1}{2}(1 \mp \sqrt{3})\right)$ **57.** 1

59. \$359.6 million/year

61. $\dfrac{0.0021}{(0.015 + [\mathrm{S}])^2}$;
the rate of change of the rate of an enzymatic reaction with respect to the concentration of a substrate S.

63. (c) $3e^{3x}$

65. $f'(x) = (x^2 + 2x)e^x$, $f''(x) = (x^2 + 4x + 2)e^x$, $f'''(x) = (x^2 + 6x + 6)e^x$, $f^{(4)}(x) = (x^2 + 8x + 12)e^x$, $f^{(5)}(x) = (x^2 + 10x + 20)e^x$; $f^{(n)}(x) = [x^2 + 2nx + n(n - 1)]e^x$

EXERCISES 3.3 ■ PAGE 197

1. $f'(x) = 3\cos x + 2\sin x$ **3.** $y' = 2x - \csc^2 x$

5. $h'(\theta) = \theta(\theta\cos\theta + 2\sin\theta)$

7. $y' = \sec\theta\,(\sec^2\theta + \tan^2\theta)$

9. $f'(\theta) = \theta\cos\theta - \cos^2\theta + \sin\theta + \sin^2\theta$

11. $H'(t) = -2\sin t\cos t$ **13.** $f'(\theta) = \dfrac{1}{1 + \cos\theta}$

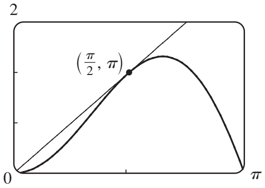
15. $y' = \dfrac{2 - \tan x + x\sec^2 x}{(2 - \tan x)^2}$

17. $f'(w) = \dfrac{2\sec w\tan w}{(1 - \sec w)^2}$ **19.** $y' = \dfrac{(t^2 + t)\cos t + \sin t}{(1 + t)^2}$

21. $f'(\theta) = \frac{1}{2}\sin 2\theta + \theta\cos 2\theta$

27. $y = x + 1$ **29.** $y = 2x + 1$

31. (a) $y = 2x$ (b) $\frac{3\pi}{2}$



33. (a) $\sec x\tan x - 1$

35. $\dfrac{\theta\cos\theta - \sin\theta}{\theta^2}$; $\dfrac{-\theta^2\sin\theta - 2\theta\cos\theta + 2\sin\theta}{\theta^3}$

37. (a) $f'(x) = (1 + \tan x)/\sec x$ (b) $f'(x) = \cos x + \sin x$

39. $(2n + 1)\pi \pm \frac{1}{3}\pi$, n an integer

41. (a) $v(t) = 8\cos t$, $a(t) = -8\sin t$
(b) $4\sqrt{3}, -4, -4\sqrt{3}$; to the left

43. 3 m/rad **45.** $\frac{5}{3}$ **47.** 3 **49.** 0 **51.** 2

53. $-\frac{3}{4}$ **55.** $\frac{1}{2}$ **57.** $-\frac{1}{4}$ **59.** $-\sqrt{2}$

61. $-\cos x$ **63.** $A = -\frac{3}{10}$, $B = -\frac{1}{10}$

65. (a) $\sec^2 x = \dfrac{1}{\cos^2 x}$ (b) $\sec x\tan x = \dfrac{\sin x}{\cos^2 x}$

(c) $\cos x - \sin x = \dfrac{\cot x - 1}{\csc x}$ **67.** 1

EXERCISES 3.4 ■ PAGE 206

1. $dy/dx = -12x^3(5 - x^4)^2$ **3.** $dy/dx = -\sin x\cos(\cos x)$

5. $dy/dx = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$ **7.** $f'(x) = 10x(2x^3 - 5x^2 + 4)^4(3x - 5)$

9. $f'(x) = \dfrac{5}{2\sqrt{5x + 1}}$ **11.** $g'(t) = \dfrac{-4}{(2t + 1)^3}$

13. $f'(\theta) = -2\theta\sin(\theta^2)$ **15.** $g'(x) = e^{x^2 - x}(2x - 1)$

17. $y' = xe^{-3x}(2 - 3x)$ **19.** $f'(t) = e^{at}(b \cos bt + a \sin bt)$

21. $F'(x) = 4(4x + 5)^2(x^2 - 2x + 5)^3(11x^2 - 4x + 5)$

23. $y' = \dfrac{1}{2\sqrt{x}\,(x + 1)^{3/2}}$ **25.** $y' = (\sec^2 \theta)\, e^{\tan \theta}$

27. $g'(u) = \dfrac{48u^2(u^3 - 1)^7}{(u^3 + 1)^9}$ **29.** $r'(t) = \dfrac{(\ln 10)10^{2\sqrt{t}}}{\sqrt{t}}$

31. $H'(r) = \dfrac{2(r^2 - 1)^2(r^2 + 3r + 5)}{(2r + 1)^6}$

33. $F'(t) = e^{t \sin 2t}(2t \cos 2t + \sin 2t)$

35. $G'(x) = -C(\ln 4)\dfrac{4^{C/x}}{x^2}$

37. $f'(x) = 2x \sin x \sin(1 - x^2) + \cos x \cos(1 - x^2)$

39. $F'(t) = \dfrac{t \sec^2\sqrt{1 + t^2}}{\sqrt{1 + t^2}}$

41. $y' = 4x \sin(x^2 + 1) \cos(x^2 + 1)$

43. $g'(x) = \dfrac{e^x}{(1 + e^x)^2} \cos\left(\dfrac{e^x}{1 + e^x}\right)$

45. $f'(t) = -\sec^2(\sec(\cos t)) \sec(\cos t) \tan(\cos t) \sin t$

47. $f'(x) = 4x \sin(x^2) \cos(x^2)e^{\sin^2(x^2)}$

49. $y' = -8x\,(\ln 3) \sin(x^2)\, 3^{\cos(x^2)}(3^{\cos(x^2)} - 1)^3$

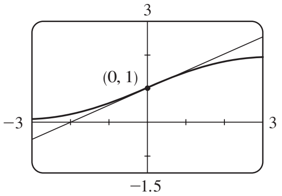
51. $y' = -\dfrac{\pi \cos(\tan \pi x) \sec^2(\pi x) \sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}}$

53. $y' = -3 \cos 3\theta \sin(\sin 3\theta)$;
$y'' = -9 \cos^2(3\theta) \cos(\sin 3\theta) + 9(\sin 3\theta) \sin(\sin 3\theta)$

55. $y' = \dfrac{-\sin x}{2\sqrt{\cos x}}$; $y'' = -\dfrac{1 + \cos^2 x}{4(\cos x)^{3/2}}$

57. $y = (\ln 2)x + 1$ **59.** $y = -x + \pi$

61. (a) $y = \frac{1}{2}x + 1$ (b)



63. (a) $f'(x) = \dfrac{2 - 2x^2}{\sqrt{2 - x^2}}$

65. $((\pi/2) + 2n\pi, 3)$, $((3\pi/2) + 2n\pi, -1)$, n an integer

67. 24 **69.** (a) 30 (b) 36

71. (a) $\frac{1}{4}$ (b) -2 (c) $-\frac{1}{2}$ **73.** $-\frac{1}{6}\sqrt{2}$

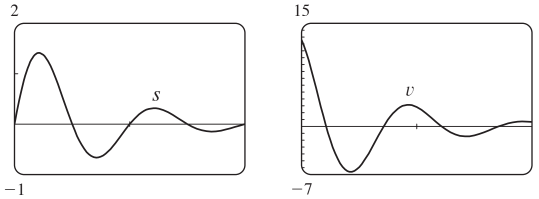
75. (a) $F'(x) = e^x f'(e^x)$ (b) $G'(x) = e^{f(x)} f'(x)$

77. 120 **79.** 96

83. $-2^{50} \cos 2x$ **85.** $v(t) = \frac{5}{2}\pi \cos(10\pi t)$ cm/s

87. (a) $\dfrac{dB}{dt} = \dfrac{7\pi}{54} \cos \dfrac{2\pi t}{5.4}$ (b) 0.16

89. $v(t) = 2e^{-1.5t}(2\pi \cos 2\pi t - 1.5 \sin 2\pi t)$



91. (a) 0.00075 (g/dL)/min (b) 0.00030 (g/dL)/min

93. dv/dt is the rate of change of velocity with respect to time; dv/ds is the rate of change of velocity with respect to displacement

EXERCISES 3.5 ■ PAGE 214

1. (a) $y' = \dfrac{10x}{3y^2}$ (b) $y = \sqrt[3]{5x^2 - 7}$, $y' = \dfrac{10x}{3(5x^2 - 7)^{2/3}}$

3. (a) $y' = -\sqrt{y}/\sqrt{x}$ (b) $y = \left(1 - \sqrt{x}\,\right)^2$, $y' = 1 - 1/\sqrt{x}$

5. $y' = \dfrac{2y - x}{y - 2x}$ **7.** $y' = -\dfrac{2x(2x^2 + y^2)}{y(2x^2 + 3y)}$

9. $y' = \dfrac{x(x + 2y)}{2x^2y + 4xy^2 + 2y^3 + x^2}$ **11.** $y' = \dfrac{2 - \cos x}{3 - \sin y}$

13. $y' = -\dfrac{\cos(x + y) + \sin x}{\cos(x + y) + \sin y}$ **15.** $y' = \dfrac{2x + y \sin x}{\cos x - 2y}$

17. $y' = -\dfrac{2e^y + ye^x}{2xe^y + e^x}$ **19.** $y' = \dfrac{1 - 8x^3\sqrt{x + y}}{8y^3\sqrt{x + y} - 1}$

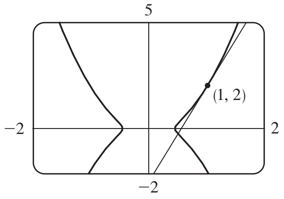
21. $y' = \dfrac{y(y - e^{x/y})}{y^2 - xe^{x/y}}$ **23.** $-\frac{16}{13}$

25. $x' = \dfrac{-2x^4y + x^3 - 6xy^2}{4x^3y^2 - 3x^2y + 2y^3}$ **27.** $y = x$

29. $y = \dfrac{1}{\sqrt{3}}x + 4$ **31.** $y = \frac{3}{4}x - \frac{1}{2}$ **33.** $y = x + \frac{1}{2}$

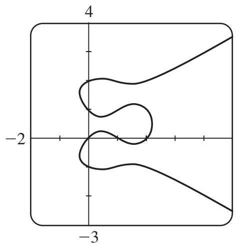
35. $y = -\frac{9}{13}x + \frac{40}{13}$

37. (a) $y = \frac{9}{2}x - \frac{5}{2}$ (b)



39. $-1/(4y^3)$ **41.** $\dfrac{\cos^2 y \cos x + \sin^2 x \sin y}{\cos^3 y}$ **43.** $1/e^2$

45. (a)

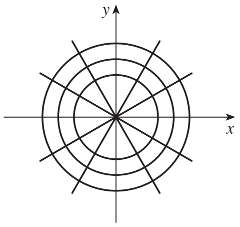


Eight; $x \approx 0.42, 1.58$

(b) $y = -x + 1$; $y = \frac{1}{3}x + 2$ (c) $1 \mp \frac{1}{3}\sqrt{3}$

47. $\left(\pm\frac{5}{4}\sqrt{3}, \pm\frac{5}{4}\right)$ **49.** $(x_0x/a^2) - (y_0y/b^2) = 1$

53.



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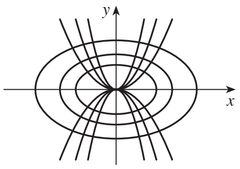


59. (a) $\dfrac{V^3(nb - V)}{PV^3 - n^2aV + 2n^3ab}$ (b) ≈ -4.04 L/atm

61. $(\pm\sqrt{3}, 0)$ **63.** $(-1, -1), (1, 1)$

65. $y' = \dfrac{y}{x + 2y^3}$; $y' = \dfrac{1}{3y^2 + 1}$

67. 2 units

EXERCISES 3.6 ■ PAGE 224

1. The differentiation formula is simplest.

3. $f'(x) = \dfrac{2x + 3}{x^2 + 3x + 5}$ **5.** $f'(x) = \dfrac{\cos(\ln x)}{x}$

7. $f'(x) = -\dfrac{1}{x}$ **9.** $g'(x) = \dfrac{1}{x} - 2$

11. $F'(t) = \ln t\left(\ln t \cos t + \dfrac{2\sin t}{t}\right)$

13. $y' = \dfrac{2x + 3}{(x^2 + 3x)\ln 8}$ **15.** $F'(s) = \dfrac{1}{s \ln s}$

17. $T'(z) = 2^z\left(\dfrac{1}{z \ln 2} + \ln z\right)$ **19.** $y' = \dfrac{-10x^4}{3 - 2x^5}$

21. $y' = \dfrac{-x}{1 + x}$ **23.** $h'(x) = e^{x^2}(2x^2 + 1)$

25. $y' = \dfrac{a}{x} - \ln b$

29. $y' = (2 + \ln x)/(2\sqrt{x})$; $y'' = -\ln x/(4x\sqrt{x})$

31. $y' = \tan x$; $y'' = \sec^2 x$

33. $f'(x) = \dfrac{2x - 1 - (x - 1)\ln(x - 1)}{(x - 1)[1 - \ln(x - 1)]^2}$;
$(1, 1 + e) \cup (1 + e, \infty)$

35. $f'(x) = \dfrac{2(x - 1)}{x(x - 2)}$; $(-\infty, 0) \cup (2, \infty)$ **37.** 2

39. $y = 3x - 9$ **41.** $\cos x + 1/x$ **43.** 7

45. $y' = (x^2 + 2)^2(x^4 + 4)^4\left(\dfrac{4x}{x^2 + 2} + \dfrac{16x^3}{x^4 + 4}\right)$

47. $y' = \sqrt{\dfrac{x - 1}{x^4 + 1}}\left(\dfrac{1}{2x - 2} - \dfrac{2x^3}{x^4 + 1}\right)$

49. $y' = x^x(1 + \ln x)$

51. $y' = x^{\sin x}\left(\dfrac{\sin x}{x} + \ln x \cos x\right)$

53. $y' = (\cos x)^x(-x \tan x + \ln \cos x)$

55. $y' = \dfrac{(2x^{\ln x})\ln x}{x}$ **57.** $y' = \dfrac{2x}{x^2 + y^2 - 2y}$

59. $f^{(n)}(x) = \dfrac{(-1)^{n-1}(n - 1)!}{(x - 1)^n}$ **63.** $f'(x) = \dfrac{5}{\sqrt{1 - 25x^2}}$

65. $y' = \dfrac{1}{2x\sqrt{x - 1}}$ **67.** $y' = \dfrac{2\tan^{-1}x}{1 + x^2}$

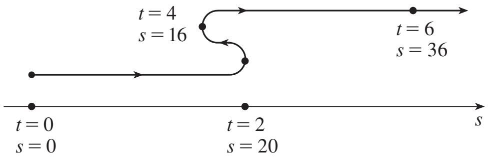
69. $h'(x) = \dfrac{\arcsin x}{x} + \dfrac{\ln x}{\sqrt{1 - x^2}}$ **71.** $f'(z) = \dfrac{2ze^{\arcsin(z^2)}}{\sqrt{1 - z^4}}$

73. $h'(t) = 0$ **75.** $y' = \sin^{-1}x$

77. $y' = \dfrac{a}{x^2 + a^2} + \dfrac{a}{x^2 - a^2}$ **79.** $1 - \dfrac{x \arcsin x}{\sqrt{1 - x^2}}$ **85.** $\frac{1}{2}$

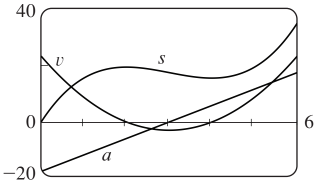
EXERCISES 3.7 ■ PAGE 235

1. (a) $3t^2 - 18t + 24$ (b) 9 m/s (c) $t = 2, 4$
(d) $0 \leqslant t < 2$, $t > 4$ (e) 44 m
(f)



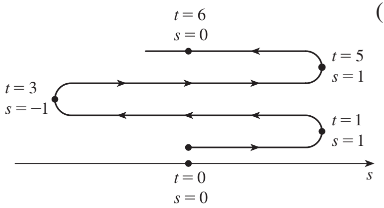
(g) $6t - 18$; -12 m/s^2

(h)



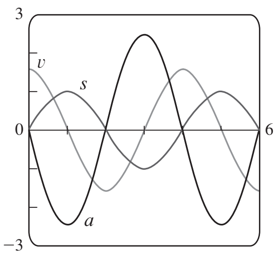
(i) Speeding up when $2 < t < 3$ and $t > 4$; slowing down when $0 \leqslant t < 2$ and $3 < t < 4$

3. (a) $(\pi/2)\cos(\pi t/2)$ (b) 0 m/s
(c) $t = 2n + 1$, n a nonnegative integer
(d) $0 < t < 1$, $3 < t < 5$, $7 < t < 9$, and so on (e) 6 m
(f)



(g) $(-\pi^2/4)\sin(\pi t/2)$; $-\pi^2/4$ m/s^2

(h)



(i) Speeding up when $1 < t < 2$, $3 < t < 4$, and $5 < t < 6$; slowing down when $0 < t < 1$, $2 < t < 3$, and $4 < t < 5$

5. (a) Speeding up when $0 < t < 1$ and $2 < t < 3$; slowing down when $1 < t < 2$
(b) Speeding up when $1 < t < 2$ and $3 < t < 4$; slowing down when $0 < t < 1$ and $2 < t < 3$

7. Traveling forward when $0 < t < 5$; traveling backward when $7 < t < 8$; not moving

9. (a) 4.9 m/s; -14.7 m/s (b) After 2.5 s (c) $32\frac{5}{8}$ m
(d) ≈ 5.08 s (e) ≈ -25.3 m/s

11. (a) 7.56 m/s (b) ≈ 6.24 m/s; ≈ -6.24 m/s

13. (a) 30 mm^2/mm; the rate at which the area is increasing with respect to side length as x reaches 15 mm
(b) $\Delta A \approx 2x\,\Delta x$

15. (a) (i) 5π (ii) 4.5π (iii) 4.1π
(b) 4π (c) $\Delta A \approx 2\pi r\,\Delta r$

17. (a) 160π cm^2/cm (b) 320π cm^2/cm (c) 480π cm^2/cm
The rate increases as the radius increases.

19. (a) 6 kg/m (b) 12 kg/m (c) 18 kg/m
At the right end; at the left end

21. (a) 4.75 A (b) 5 A; $t = \frac{2}{3}$ s

25. (a) $dV/dP = -C/P^2$ (b) At the beginning

27. $400(3^t)$; ≈ 6850 bacteria/h

29. (a) 16 million/year; 78.5 million/year
(b) $P(t) = at^3 + bt^2 + ct + d$, where $a \approx -0.0002849$, $b \approx 0.5224331$, $c \approx -6.395641$, $d \approx 1720.586$
(c) $P'(t) = 3at^2 + 2bt + c$
(d) 14.16 million/year (smaller); 71.72 million/year (smaller)
(e) $f'(t) = (1.43653 \times 10^9) \cdot (1.01395)^t \ln 1.01395$
(f) 26.25 million/year (larger); 60.28 million/year (smaller)
(g) $P'(85) \approx 76.24$ million/year, $f'(85) = 64.61$ million/year

31. (a) 0.926 cm/s; 0.694 cm/s; 0
(b) 0; -92.6 (cm/s)/cm; -185.2 (cm/s)/cm
(c) At the center; at the edge

33. (a) $C'(x) = 3 + 0.02x + 0.0006x^2$
(b) \$11/pair; the rate at which the cost is changing as the 100th pair of jeans is being produced; the cost of the 101st pair
(c) \$11.07

35. (a) $[xp'(x) - p(x)]/x^2$; the average productivity increases as new workers are added.

37. $\dfrac{dt}{dc} = \dfrac{3\sqrt{9c^2 - 8c} + 9c - 4}{\sqrt{9c^2 - 8c}\left(3c + \sqrt{9c^2 - 8c}\right)}$; the rate of change of duration of dialysis required with respect to the initial urea concentration

39. ≈ -0.2436 K/min

41. (a) 0 and 0 (b) $C = 0$
(c) (0, 0), (500, 50); it is possible for the species to coexist.

43. (a) $\dfrac{1}{D \ln 2}$; decreases
(b) $-\dfrac{1}{W \ln 2}$; difficulty decreases with increasing width; increases

EXERCISES 3.8 ■ PAGE 245

1. About 8.7 million

3. (a) $50e^{1.9803t}$ (b) $\approx 19{,}014$ (c) $\approx 37{,}653$ cells/h
(d) ≈ 4.30 h

5. (a) 1508 million, 1871 million (b) 2161 million
(c) 3972 million; wars in the first half of century, increased life expectancy in second half

7. (a) $Ce^{-0.0005t}$ (b) $-2000 \ln 0.9 \approx 211$ s

9. (a) $100 \times 2^{-t/30}$ mg (b) ≈ 9.92 mg (c) ≈ 199.3 years

11. ≈ 2500 years **13.** Yes; 12.5 billion years

15. (a) ≈ 58°C (b) $\approx$ 89 min

17. (a) $13.\overline{3}$°C (b) ≈ 67.74 min

19. (a) ≈ 64.5 kPa (b) ≈ 39.9 kPa

21. (a) (i) \$4362.47 (ii) \$4364.11 (iii) \$4365.49
(iv) \$4365.70 (v) \$4365.76 (vi) \$4365.77
(b) $dA/dt = 0.0175A$, $A(0) = 4000$

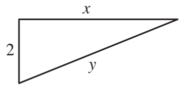
EXERCISES 3.9 ■ PAGE 251

1. (a) $dV/dt = 3x^2\,dx/dt$ (b) 2700 cm^3/s **3.** 48 cm^2/s

5. 128π cm^2/min **7.** $3/(25\pi)$ m/min

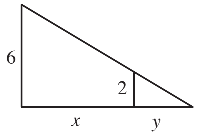
9. (a) $-\frac{3}{8}$ (b) $\frac{8}{3}$ **11.** -3.41 N/s

13. (a) The plane's altitude is 2 km and its speed is 800 km/h.
(b) The rate at which the distance from the plane to the station is increasing when the plane is 3 km from the station
(c)



(d) $y^2 = x^2 + 4$
(e) $800/3\sqrt{5}$ km/h

15. (a) The height of the pole (6 m), the height of the man (2 m), and the speed of the man (1.5 m/s)
(b) The rate at which the tip of the man's shadow is moving when he is 10 m from the pole
(c)



(d) $\dfrac{6}{2} = \dfrac{x + y}{y}$ (e) $\frac{9}{4}$ m/s

17. 78 km/h **19.** $8064/\sqrt{8{,}334{,}400} \approx 2.79$ m/s

21. -1.6 cm/min **23.** 9.8 m/s

25. $(10{,}000 + 800{,}000\pi/9) \approx 2.89 \times 10^5$ cm^3/min

27. $\frac{10}{3}$ cm/min **29.** $4/(3\pi) \approx 0.42$ m/min

31. $150\sqrt{3}$ cm^2/min **33.** ≈ 20.3 m/s **35.** $-\frac{1}{2}$ rad/s

37. 80 cm^3/min **39.** $\frac{107}{810} \approx 0.132$ Ω/s **41.** ≈ 87.2 km/h

43. $\sqrt{7}\,\pi/21 \approx 0.396$ m/min

45. (a) 120 ms/s (b) ≈ 0.107 rad/s

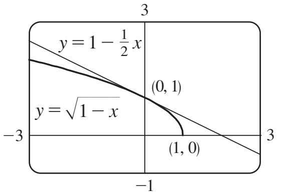
47. $\frac{10}{9}\pi$ km/min **49**. $1650/\sqrt{31} \approx 296$ km/h

51. $\frac{7}{4}\sqrt{15} \approx 6.78$ m/s

EXERCISES 3.10 ■ PAGE 258

1. $L(x) = 16x + 23$ **3.** $L(x) = \frac{1}{12}x + \frac{4}{3}$

5. $\sqrt{1 - x} \approx 1 - \frac{1}{2}x$;
$\sqrt{0.9} \approx 0.95$,
$\sqrt{0.99} \approx 0.995$



7. $-0.731 < x < 0.731$ **9.** $-0.368 < x < 0.677$

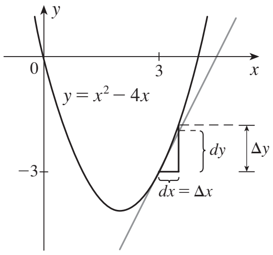
11. $dy = 5e^{5x}dx$ **13.** $dy = \dfrac{-1}{(1 + 3u)^2}\,du$

15. $dy = \dfrac{3 - 2x}{(x^2 - 3x)^2}\,dx$ **17.** $dy = \cot\theta\, d\theta$

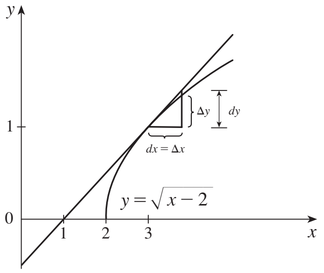
19. (a) $dy = \frac{1}{10}e^{x/10}\,dx$ (b) 0.01

21. (a) $dy = \dfrac{x}{\sqrt{3 + x^2}}\,dx$ (b) -0.05

23. $\Delta y = 1.25$, $dy = 1$



25. $\Delta y \approx 0.34$, $dy = 0.4$



27. $\Delta y \approx 0.1655$, $dy = 0.15$; $\Delta y \approx 0.0306$, $dy = 0.03$; yes
29. $\Delta y \approx -0.012539$, $dy = -0.0125$;
$\Delta y \approx -0.002502$, $dy = -0.0025$; yes
31. 15.968 **33.** $10.00\overline{3}$ **35.** 1.1
41. (a) 270 cm^3, 0.01, 1% (b) 36 cm^2, $0.00\overline{6}$, $0.\overline{6}$%
43. (a) $84/\pi \approx 27$ cm^2; $\frac{1}{84} \approx 0.012 = 1.2\%$
(b) $1764/\pi^2 \approx 179$ cm^3; $\frac{1}{56} \approx 0.018 = 1.8\%$
45. (a) $2\pi rh\,\Delta r$ (b) $\pi(\Delta r)^2h$
51. (a) 4.8, 5.2 (b) Too large

EXERCISES 3.11 ■ PAGE 266

1. (a) 0 (b) 1 **3.** (a) $\frac{13}{5}$ (b) $\frac{1}{2}(e^5 + e^{-5}) \approx 74.20995$

5. (a) 1 (b) 0 **7.** $\frac{13}{2}e^x - \frac{3}{2}e^{-x}$ **9.** $\dfrac{x^2 - 1}{2x}$

25. $\operatorname{sech} x = \frac{3}{5}$, $\sinh x = \frac{4}{3}$, $\operatorname{csch} x = \frac{3}{4}$, $\tanh x = \frac{4}{5}$, $\coth x = \frac{5}{4}$
27. (a) 1 (b) -1 (c) ∞ (d) $-\infty$ (e) 0 (f) 1
(g) ∞ (h) $-\infty$ (i) 0 (j) $\frac{1}{2}$
35. $f'(x) = 3\sinh 3x$ **37.** $h'(x) = 2x\cosh(x^2)$

39. $G'(t) = \dfrac{t^2 + 1}{2t^2}$ **41.** $f'(x) = \dfrac{\operatorname{sech}^2\sqrt{x}}{2\sqrt{x}}$

43. $y' = \operatorname{sech}^3x - \operatorname{sech} x\tanh^2x$

45. $g'(t) = \coth\sqrt{t^2 + 1} - \dfrac{t^2}{\sqrt{t^2 + 1}}\operatorname{csch}^2\sqrt{t^2 + 1}$

47. $f'(x) = \dfrac{-2}{\sqrt{1 + 4x^2}}$ **49.** $y' = \sec\theta$

51. $G'(u) = \dfrac{1}{\sqrt{1 + u^2}}$ **53.** $y' = \sinh^{-1}(x/3)$

59. (a) 0.3572 (b) 70.34°
61. (a) 1176 N; 164.50 m (b) 120 m; 164.13 m
63. (b) $y = 2\sinh 3x - 4\cosh 3x$
65. $\left(\ln\left(1 + \sqrt{2}\right), \sqrt{2}\right)$

CHAPTER 3 REVIEW ■ PAGE 269

True-False Quiz

1. True **3.** True **5.** False **7.** False **9.** True
11. True **13.** True **15.** True

Exercises

1. $4x^7(x + 1)^3(3x + 2)$ **3.** $\dfrac{3}{2}\sqrt{x} - \dfrac{1}{2\sqrt{x}} - \dfrac{1}{\sqrt{x^3}}$

5. $x(\pi x\cos\pi x + 2\sin\pi x)$

7. $\dfrac{8t^3}{(t^4 + 1)^2}$ **9.** $\dfrac{1 + \ln x}{x\ln x}$ **11.** $\dfrac{\cos\sqrt{x} - \sqrt{x}\sin\sqrt{x}}{2\sqrt{x}}$

13. $-\dfrac{e^{1/x}(1 + 2x)}{x^4}$ **15.** $\dfrac{2xy - \cos y}{1 - x\sin y - x^2}$

17. $\dfrac{1}{2\sqrt{\arctan x}\,(1 + x^2)}$ **19.** $\dfrac{1 - t^2}{(1 + t^2)^2}\sec^2\left(\dfrac{t}{1 + t^2}\right)$

21. $3^{x\ln x}(\ln 3)(1 + \ln x)$ **23.** $-(x - 1)^{-2}$

25. $\dfrac{2x - y\cos(xy)}{x\cos(xy) + 1}$ **27.** $\dfrac{2}{(1 + 2x)\ln 5}$

29. $\cot x - \sin x\cos x$ **31.** $\dfrac{4x}{1 + 16x^2} + \tan^{-1}(4x)$

33. $5\sec 5x$ **35.** $-6x\csc^2(3x^2 + 5)$

37. $\cos\left(\tan\sqrt{1 + x^3}\right)\left(\sec^2\sqrt{1 + x^3}\right)\dfrac{3x^2}{2\sqrt{1 + x^3}}$

39. $\dfrac{-5}{x^2 + 1}$ **41.** $\tan^{-1}x$ **43.** $2\cos\theta\tan(\sin\theta)\sec^2(\sin\theta)$

45. $\dfrac{(2 - x)^4(3x^2 - 55x - 52)}{2\sqrt{x + 1}(x + 3)^8}$ **47.** $2x^2\cosh(x^2) + \sinh(x^2)$

49. $3 \tanh 3x$ **51.** $\dfrac{\cosh x}{\sqrt{\sinh^2 x - 1}}$

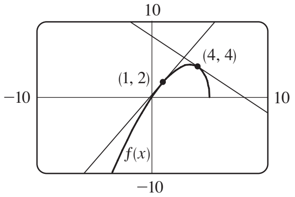
53. $\dfrac{-3 \sin\left(e^{\sqrt{\tan 3x}}\right) e^{\sqrt{\tan 3x}} \sec^2(3x)}{2\sqrt{\tan 3x}}$ **55.** $-\frac{4}{27}$

57. $-5x^4/y^{11}$ **61.** $y = 2\sqrt{3}x + 1 - \pi\sqrt{3}/3$

63. $y = 2x + 1$ **65.** $y = -x + 2;\ y = x + 2$

67. (a) $\dfrac{10 - 3x}{2\sqrt{5 - x}}$ (b) $y = \frac{7}{4}x + \frac{1}{4},\ y = -x + 8$

(c)



69. $\left(\pi/4, \sqrt{2}\right), \left(5\pi/4, -\sqrt{2}\right)$

73. (a) 4 (b) 6 (c) $\frac{7}{9}$ (d) 12

75. $x^2 g'(x) + 2xg(x)$ **77.** $2g(x)g'(x)$ **79.** $g'(e^x)e^x$

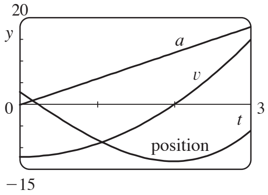
81. $g'(x)/g(x)$ **83.** $\dfrac{f'(x)[g(x)]^2 + g'(x)\,[f(x)]^2}{[f(x) + g(x)]^2}$

85. $f'(g(\sin 4x))g'(\sin 4x)(\cos 4x)(4)$

87. $(-3, 0)$ **89.** $y = -\frac{2}{3}x^2 + \frac{14}{3}x$

91. $v(t) = -Ae^{-ct}[\omega \sin(\omega t + \delta) + c \cos(\omega t + \delta)]$,
$a(t) = Ae^{-ct}[(c^2 - \omega^2) \cos(\omega t + \delta) + 2c\omega \sin(\omega t + \delta)]$

93. (a) $v(t) = 3t^2 - 12;\ a(t) = 6t$ (b) $t > 2;\ 0 \leqslant t < 2$
(c) 23 (d)



(e) $t > 2;\ 0 < t < 2$

95. 4 kg/m

97. (a) $200(3.24)^t$ (b) $\approx$22,040
(c) $\approx$25,910 cells/h (d) $(\ln 50)/(\ln 3.24) \approx 3.33$ h

99. (a) $C_0 e^{-kt}$ (b) $\approx$100 h **101.** $\frac{4}{3}$ cm^2/min

103. $117/\sqrt{666} \approx 4.53$ m/s **105.** 400 m/h

107. (a) $L(x) = 1 + x;\ \sqrt[3]{1 + 3x} \approx 1 + x;\ \sqrt[3]{1.03} \approx 1.01$
(b) $-0.235 < x < 0.401$

109. $12 + \frac{3}{2}\pi \approx 16.7$ cm^2 **111.** $\left[\dfrac{d}{dx}\sqrt[4]{x}\right]_{x=16} = \dfrac{1}{32}$

113. $\frac{1}{4}$ **115.** $\frac{1}{8}x^2$

PROBLEMS PLUS ■ PAGE 275

1. $\left(\pm\sqrt{3}/2, \frac{1}{4}\right)$ **5.** $3\sqrt{2}$ **11.** $\left(0, \frac{5}{4}\right)$

13. 3 lines; $(0, 2)$, $\left(\frac{4}{3}\sqrt{2}, \frac{2}{3}\right)$ and $\left(\frac{2}{3}\sqrt{2}, \frac{10}{3}\right)$, $\left(-\frac{4}{3}\sqrt{2}, \frac{2}{3}\right)$ and $\left(-\frac{2}{3}\sqrt{2}, \frac{10}{3}\right)$

15. (a) $4\pi\sqrt{3}/\sqrt{11}$ rad/s (b) $40\left(\cos\theta + \sqrt{8 + \cos^2\theta}\right)$ cm
(c) $-480\pi \sin\theta \left(1 + (\cos\theta/)\sqrt{8 + \cos^2\theta}\right)$ cm/s

19. $x_T \in (3, \infty),\ y_T \in (2, \infty),\ x_N \in \left(0, \frac{5}{3}\right),\ y_N \in \left(-\frac{5}{2}, 0\right)$

21. (b) (i) 53° (or 127°) (ii) 63° (or 117°)

23. R approaches the midpoint of the radius AO.

25. $-\sin a$ **27.** $2\sqrt{e}$ **31.** $(1, -2), (-1, 0)$

33. $\sqrt{29}/58$ **35.** $2 + \frac{375}{128}\pi \approx 11.204$ cm^3/min

CHAPTER 4

EXERCISES 4.1 ■ PAGE 286

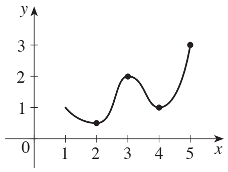
Abbreviations: abs, absolute; loc, local; max, maximum; min, minimum

1. Abs min: smallest function value on the entire domain of the function; loc min at c: smallest function value when x is near c

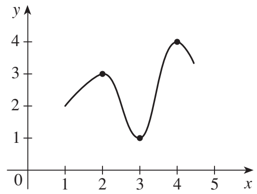
3. Abs max at s, abs min at r, loc max at c, loc min at b and r, neither a max nor a min at a and d

5. Abs max $f(4) = 5$, loc max $f(4) = 5$ and $f(6) = 4$, loc min $f(2) = 2$ and $f(1) = f(5) = 3$

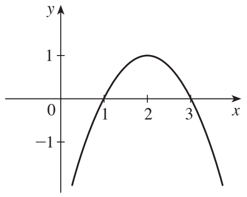
7.



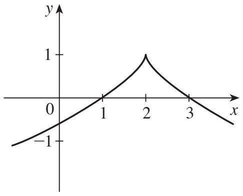
9.



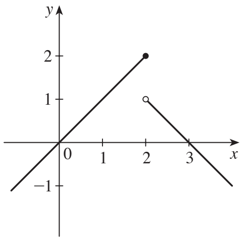
11. (a)



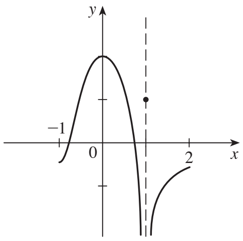
(b)



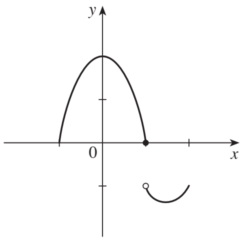
(c)



13. (a)



(b)

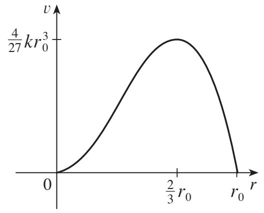


15. Abs max $f(-1) = 5$ **17.** Abs max $f(1) = 1$

19. Abs min $f(0) = 0$

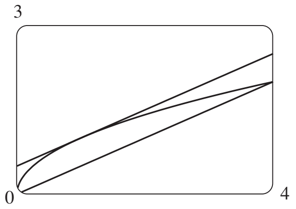
21. Abs max $f(\pi/2) = 1$; abs min $f(-\pi/2) = -1$

23. Abs max $f(2) = \ln 2$ **25.** Abs max $f(0) = 1$
27. Abs min $f(1) = -1$; loc min $f(0) = 0$ **29.** $-\frac{1}{6}$
31. $-4, 0, 2$ **33.** None **35.** $0, 2$ **37.** $-1, 2$
39. $0, \frac{4}{9}$ **41.** $0, \frac{8}{7}, 4$ **43.** $0, \frac{4}{3}, 4$
45. $n\pi$ (n an integer) **47.** $1/\sqrt{e}$ **49.** 10
51. $f(2) = 16, f(5) = 7$ **53.** $f(-1) = 8, f(2) = -19$
55. $f(-2) = 33, f(2) = -31$ **57.** $f(0.2) = 5.2, f(1) = 2$
59. $f(4) = 4 - \sqrt[3]{4}, f(\sqrt{3}/9) = -2\sqrt{3}/9$
61. $f(\pi/6) = \frac{3}{2}\sqrt{3}, f(\pi/2) = 0$
63. $f(e^{1/2}) = 1/(2e), f(\frac{1}{2}) = -4 \ln 2$
65. $f(1) = \ln 3, f(-\frac{1}{2}) = \ln \frac{3}{4}$
67. $f\left(\dfrac{a}{a+b}\right) = \dfrac{a^a b^b}{(a+b)^{a+b}}$
69. (a) 2.19, 1.81 (b) $\frac{6}{25}\sqrt{\frac{3}{5}} + 2, -\frac{6}{25}\sqrt{\frac{3}{5}} + 2$
71. (a) 0.32, 0.00 (b) $\frac{3}{16}\sqrt{3}, 0$
73. 0.0177 g/dL; 21.4 min **75.** ≈3.9665°C
77. About 4.1 months after Jan. 1
79. (a) $r = \frac{2}{3}r_0$ (b) $v = \frac{4}{27}kr_0^3$
(c)



EXERCISES 4.2 ■ PAGE 295

1. 1, 5
3. (a) g is continuous on $[0, 8]$ and differentiable on $(0, 8)$.
(b) 2.2, 6.4 (c) 3.7, 5.5
5. No **7.** Yes; ≈3.8
9. 1 **11.** π
13. f is not differentiable on $(-1, 1)$ **15.** 1
17. $3/\ln 4$ **19.** 1; yes

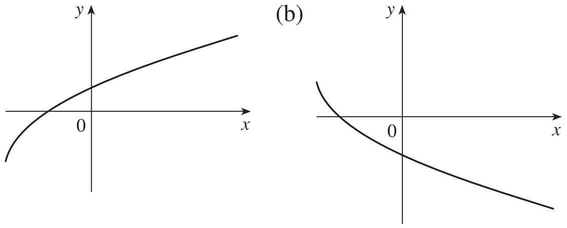


21. f is not continous at 3 **29.** 16 **31.** No **37.** No

EXERCISES 4.3 ■ PAGE 305

Abbreviations: CD, concave downward; CU, concave upward; dec, decreasing; inc, increasing; HA, horizontal asymptote; IP, inflection point; VA, vertical asymptote

1. (a) (1, 3), (4, 6) (b) (0, 1), (3, 4) (c) (0, 2)
(d) (2, 4), (4, 6) (e) (2, 3)
3. (a) I/D Test (b) Concavity Test
(c) Find points at which the concavity changes.
5. (a) Inc on (0,1), (3,5); dec on (1,3), (5,6)
(b) Loc max at $x = 1, x = 5$; loc min at $x = 3$
7. (a) 3, 5 (b) 2, 4, 6 (c) 1, 7
9. Inc on $(-\infty, 1), (4, \infty)$; dec on $(1, 4)$; loc max $f(1) = 6$; loc min $f(4) = -21$
11. Inc on $(2, \infty)$; dec on $(-\infty, 2)$; loc min $f(2) = -31$
13. Inc on $(-\infty, 4), (6, \infty)$; dec on $(4, 5), (5, 6)$; loc max $f(4) = 8$; loc min $f(6) = 12$
15. Inc on $(0, \pi/4), (5\pi/4, 2\pi)$; dec on $(\pi/4, 5\pi/4)$; loc max $f(\pi/4) = \sqrt{2}$; loc min $f(5\pi/4) = -\sqrt{2}$
17. CU on $(1, \infty)$; CD on $(-\infty, 1)$; IP $(1, -7)$
19. CU on $(0, \pi/4), (3\pi/4, \pi)$; CD on $(\pi/4, 3\pi/4)$; IP $(\pi/4, \frac{1}{2}), (3\pi/4, \frac{1}{2})$
21. CU on $(-\sqrt{5}, \sqrt{5})$; CD on $(-\infty, -\sqrt{5}), (\sqrt{5}, \infty)$; IP $(\pm\sqrt{5}, \ln 10)$
23. (a) Inc on $(-1, 0), (1, \infty)$; dec on $(-\infty, -1), (0, 1)$
(b) Loc max $f(0) = 3$; loc min $f(\pm 1) = 2$
(c) CU on $(-\infty, -\sqrt{3}/3), (\sqrt{3}/3, \infty)$;
CD on $(-\sqrt{3}/3, \sqrt{3}/3)$; IP $(\pm\sqrt{3}/3, \frac{22}{9})$
25. (a) Inc on $(1, \infty)$; dec on $(0, 1)$ (b) Loc min $f(1) = 0$
(c) CU on $(0, \infty)$; No IP
27. (a) Inc on $(-\frac{1}{2}, \infty)$; dec on $(-\infty, -\frac{1}{2})$
(b) Loc min $f(-\frac{1}{2}) = -\dfrac{1}{2e}$
(c) CU on $(-1, \infty)$; CD on $(-\infty, -1)$; IP $\left(-1, -\dfrac{1}{e^2}\right)$
29. Loc max $f(1) = 2$; loc min $f(0) = 1$ **31.** $(-3, \infty)$
33. (a) f has a local maximum at 2.
(b) f has a horizontal tangent at 6.
35. (a) (b)



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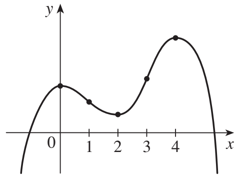

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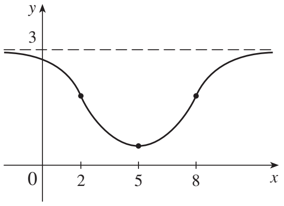

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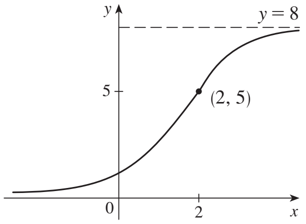
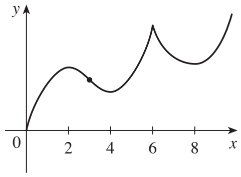
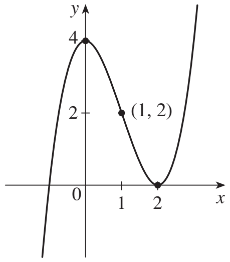


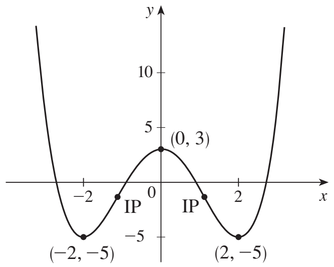
43. (a) Inc on $(0, 2)$, $(4, 6)$, $(8, \infty)$;
dec on $(2, 4)$, $(6, 8)$
(b) Loc max at $x = 2, 6$;
loc min at $x = 4, 8$
(c) CU on $(3, 6)$, $(6, \infty)$;
CD on $(0, 3)$ (d) 3
(e) See graph at right.



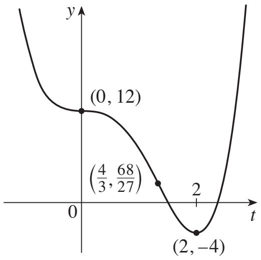
45. (a) Inc on $(-\infty, 0)$, $(2, \infty)$;
dec on $(0, 2)$
(b) Loc max $f(0) = 4$; loc min
$f(2) = 0$
(c) CU on $(1, \infty)$; CD on $(-\infty, 1)$;
IP $(1, 2)$
(d) See graph at right.



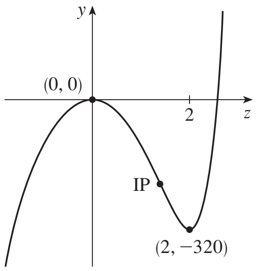
47. (a) Inc on $(-2, 0)$, $(2, \infty)$; dec on $(-\infty, -2)$, $(0, 2)$
(b) Loc max $f(0) = 3$; loc min $f(\pm 2) = -5$
(c) CU on $\left(-\infty, -\frac{2}{\sqrt{3}}\right), \left(\frac{2}{\sqrt{3}}, \infty\right)$; CD on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$;
IPs $\left(\pm\frac{2}{\sqrt{3}}, -\frac{13}{9}\right)$
(d)



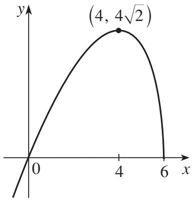
49. (a) Inc on $(2, \infty)$; dec on $(-\infty, 2)$
(b) Loc min $f(2) = -4$
(c) CU on $(-\infty, 0)$, $\left(\frac{4}{3}, \infty\right)$;
CD on $\left(0, \frac{4}{3}\right)$; IPs $(0, 12)$, $\left(\frac{4}{3}, \frac{68}{27}\right)$
(d) See graph at right.



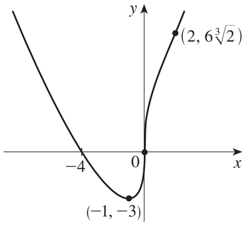
51. (a) Inc on $(-\infty, 0)$, $(2, \infty)$;
dec on $(0, 2)$
(b) Loc max $f(0) = 0$; loc min
$f(2) = -320$
(c) CU on $\left(\sqrt[5]{\frac{16}{3}}, \infty\right)$;
CD on $\left(-\infty, \sqrt[5]{\frac{16}{3}}\right)$;
IP $\left(\sqrt[5]{\frac{16}{3}}, -\frac{320}{3}\sqrt[5]{\frac{256}{9}}\right) \approx (1.398, -208.4)$
(d) See graph at right.



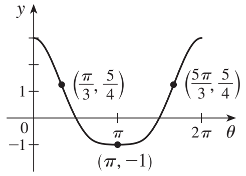
53. (a) Inc on $(-\infty, 4)$;
dec on $(4, 6)$
(b) Loc max $F(4) = 4\sqrt{2}$
(c) CD on $(-\infty, 6)$; No IP
(d) See graph at right.



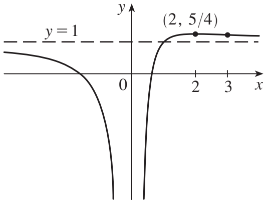
55. (a) Inc on $(-1, \infty)$;
dec on $(-\infty, -1)$
(b) Loc min $C(-1) = -3$
(c) CU on $(-\infty, 0)$, $(2, \infty)$;
CD on $(0, 2)$;
IPs $(0, 0)$, $\left(2, 6\sqrt[3]{2}\right)$
(d) See graph at right.



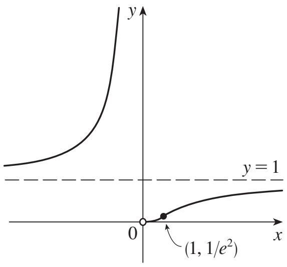
57. (a) Inc on $(\pi, 2\pi)$;
dec on $(0, \pi)$
(b) Loc min $f(\pi) = -1$
(c) CU on $(\pi/3, 5\pi/3)$;
CD on $(0, \pi/3)$, $(5\pi/3, 2\pi)$;
IPs $\left(\pi/3, \frac{5}{4}\right)$, $\left(5\pi/3, \frac{5}{4}\right)$
(d) See graph at right.



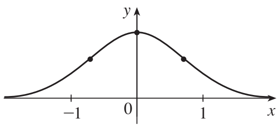
59. (a) VA $x = 0$; HA $y = 1$
(b) Inc on $(0, 2)$;
dec on $(-\infty, 0)$, $(2, \infty)$
(c) Loc max $f(2) = \frac{5}{4}$
(d) CU on $(3, \infty)$;
CD on $(-\infty, 0)$, $(0, 3)$; IP $\left(3, \frac{11}{9}\right)$
(e) See graph at right.



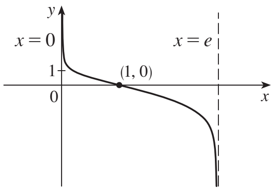
61. (a) VA $x = 0$; HA $y = 1$
(b) Inc on $(-\infty, 0)$, $(0, \infty)$
(c) None
(d) CU on $(-\infty, 0)$, $(0, 1)$;
CD on $(1, \infty)$;
IP $(1, 1/e^2)$
(e) See graph at right.



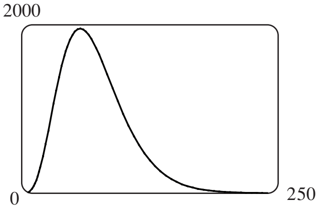
63. (a) HA $y = 0$
(b) Inc on $(-\infty, 0)$;
dec on $(0, \infty)$
(c) Loc max $f(0) = 1$
(d) CU on $\left(-\infty, -1/\sqrt{2}\right)$,
$\left(1/\sqrt{2}, \infty\right)$; CD on $\left(-1/\sqrt{2}, 1/\sqrt{2}\right)$; IPs $\left(\pm 1/\sqrt{2}, e^{-1/2}\right)$
(e) See graph at right.



65. (a) VAs $x = 0$, $x = e$
(b) Dec on $(0, e)$
(c) None
(d) CU on $(0, 1)$; CD on $(1, e)$;
IP $(1, 0)$
(e) See graph at right.



67. f is CU on $(-\infty, \infty)$ for all $c > 0$. As c increases, the minimum point gets farther away from the origin.
69. (a) Loc and abs max $f(1) = \sqrt{2}$, no min (b) $\frac{1}{4}(3 - \sqrt{17})$
71. (b) CD on $(0, 0.85)$, $(1.57, 2.29)$; CU on $(0.85, 1.57)$, $(2.29, \pi)$; IPs $(0.85, 0.74)$, $(1.57, 0)$, $(2.29, -0.74)$
73. CU on $(-\infty, -0.6)$, $(0.0, \infty)$; CD on $(-0.6, 0.0)$
75. (a) The rate of increase is initially very small, increases to a maximum at $t \approx 8$ h, then decreases toward 0.
(b) When $t = 8$ (c) CU on $(0, 8)$; CD on $(8, 18)$
(d) $(8, 350)$
77. If $D(t)$ is the size of the deficit as a function of time, then at the time of the speech $D'(t) > 0$, but $D''(t) < 0$.
79. $K(3) - K(2)$; CD
81. 28.57 min, when the rate of increase of drug level in the bloodstream is greatest; 85.71 min, when rate of decrease is greatest



83. $f(x) = \frac{1}{9}(2x^3 + 3x^2 - 12x + 7)$

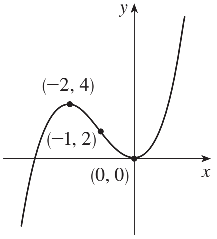
EXERCISES 4.4 ■ PAGE 316

1. (a) Indeterminate (b) 0 (c) 0
(d) ∞, $-\infty$, or does not exist (e) Indeterminate
3. (a) $-\infty$ (b) Indeterminate (c) ∞
5. $\frac{9}{4}$ **7.** 1 **9.** 6 **11.** $\frac{7}{3}$ **13.** $\sqrt{2}/2$ **15.** 2
17. $\frac{1}{4}$ **19.** 0 **21.** $-\infty$ **23.** $-\frac{1}{3}$ **25.** 3 **27.** 2
29. 1 **31.** 1 **33.** $1/\ln 3$ **35.** 0 **37.** 0
39. a/b **41.** $\frac{1}{24}$ **43.** π **45.** $\frac{5}{3}$ **47.** 0
49. $-2/\pi$ **51.** $\frac{1}{2}$ **53.** $\frac{1}{2}$ **55.** 0 **57.** 1 **59.** e^{-2}
61. $1/e$ **63.** 1 **65.** e^4 **67.** e^3 **69.** 0
71. e^2 **73.** $\frac{1}{4}$ **77.** 1
79. f has an absolute minimum for $c > 0$. As c increases, the minimum points get farther away from the origin.
83. (a) M; the population should approach its maximum size as time increases (b) $P_0 e^{kt}$; exponential
85. $\frac{16}{9}a$ **87.** $\frac{1}{2}$
89. (a) One possibility: $f(x) = 7/x^2$, $g(x) = 1/x^2$
(b) One possibility: $f(x) = 7 + (1/x^2)$, $g(x) = 1/x^2$
91. (a) 0

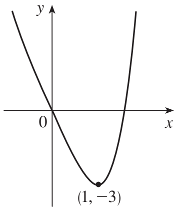
EXERCISES 4.5 ■ PAGE 327

Abbreviations: int, intercept; SA, slant asymptote

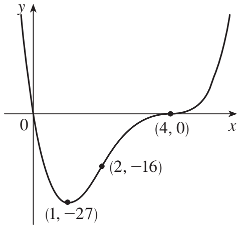
1. A. $\mathbb{R}$ B. y-int 0; x-int -3, 0
C. None D. None
E. Inc on $(-\infty, -2)$, $(0, \infty)$; dec on $(-2, 0)$
F. Loc max $f(-2) = 4$; loc min $f(0) = 0$
G. CU on $(-1, \infty)$; CD on $(-\infty, -1)$; IP $(-1, 2)$
H. See graph at right.



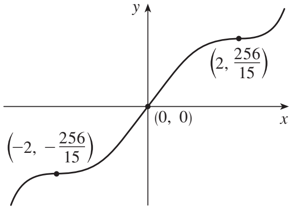
3. A. $\mathbb{R}$ B. y-int 0; x-int 0, $\sqrt[3]{4}$
C. None D. None
E. Inc on $(1, \infty)$; dec on $(-\infty, 1)$
F. Loc min $f(1) = -3$
G. CU on $(-\infty, \infty)$
H. See graph at right.



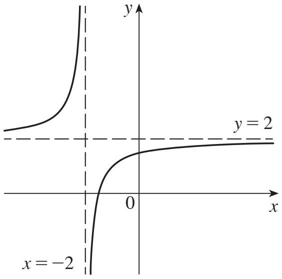
5. A. $\mathbb{R}$ B. y-int 0; x-int 0, 4
C. None D. None
E. Inc on $(1, \infty)$; dec on $(-\infty, 1)$
F. Loc min $f(1) = -27$
G. CU on $(-\infty, 2)$, $(4, \infty)$; CD on $(2, 4)$;
IPs $(2, -16)$, $(4, 0)$
H. See graph at right.



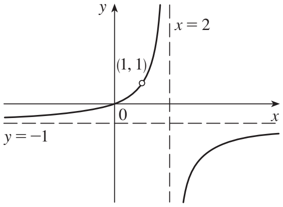
7. A. $\mathbb{R}$ B. y-int 0; x-int 0
C. About $(0, 0)$ D. None
E. Inc on $(-\infty, \infty)$
F. None
G. CU on $(-2, 0)$, $(2, \infty)$;
CD on $(-\infty, -2)$, $(0, 2)$;
IPs $\left(-2, -\frac{256}{15}\right)$, $(0, 0)$, $\left(2, \frac{256}{15}\right)$
H. See graph at right.



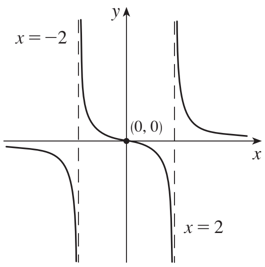
9. A. $(-\infty, -2) \cup (-2, \infty)$
B. y-int $\frac{3}{2}$; x-int $-\frac{3}{2}$
C. None D. VA $x = -2$, HA $y = 2$
E. Inc on $(-\infty, -2)$, $(-2, \infty)$
F. None
G. CU on $(-\infty, -2)$;
CD on $(-2, \infty)$
H. See graph at right.



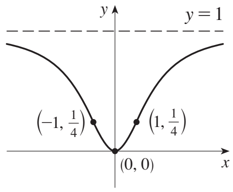
11. A. $(-\infty, 1) \cup (1, 2) \cup (2, \infty)$
B. y-int 0; x-int 0 C. None
D. VA $x = 2$; HA $y = -1$
E. Inc on $(-\infty, 1)$, $(1, 2)$, $(2, \infty)$
F. None
G. CU on $(-\infty, 1)$, $(1, 2)$;
CD on $(2, \infty)$
H. See graph at right.



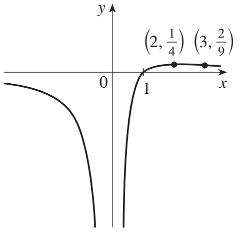
13. A. $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$ B. y-int 0; x-int 0
C. About $(0, 0)$ D. VA $x = \pm 2$; HA $y = 0$
E. Dec on $(-\infty, -2)$, $(-2, 2)$, $(2, \infty)$
F. No local extrema
G. CU on $(-2, 0)$, $(2, \infty)$;
CD on $(-\infty, -2)$, $(0, 2)$; IP $(0, 0)$
H. See graph at right.



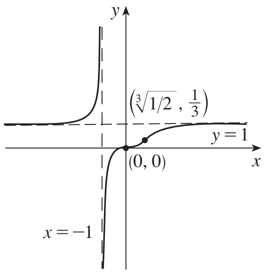
15. A. $\mathbb{R}$ B. y-int 0; x-int 0
C. About y-axis D. HA $y = 1$
E. Inc on $(0, \infty)$; dec on $(-\infty, 0)$
F. Loc min $f(0) = 0$
G. CU on $(-1, 1)$;
CD on $(-\infty, -1)$, $(1, \infty)$; IPs $\left(\pm 1, \frac{1}{4}\right)$
H. See graph at right.



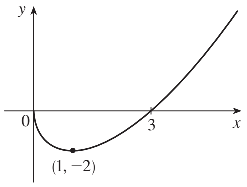
17. A. $(-\infty, 0) \cup (0, \infty)$ B. x-int 1
C. None D. VA $x = 0$; HA $y = 0$
E. Inc on $(0, 2)$;
dec on $(-\infty, 0)$, $(2, \infty)$
F. Loc max $f(2) = \frac{1}{4}$
G. CU on $(3, \infty)$;
CD on $(-\infty, 0)$, $(0, 3)$; IP $\left(3, \frac{2}{9}\right)$
H. See graph at right.



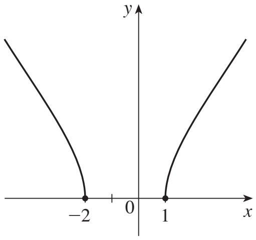
19. A. $(-\infty, -1) \cup (-1, \infty)$
B. y-int 0; x-int 0 C. None
D. VA $x = -1$; HA $y = 1$
E. Inc on $(-\infty, -1)$, $(-1, \infty)$;
F. None
G. CU on $(-\infty, -1)$, $\left(0, \sqrt[3]{\frac{1}{2}}\right)$;
CD on $(-1, 0)$, $\left(\sqrt[3]{\frac{1}{2}}, \infty\right)$;
IPs $(0, 0)$, $\left(\sqrt[3]{\frac{1}{2}}, \frac{1}{3}\right)$
H. See graph at right.



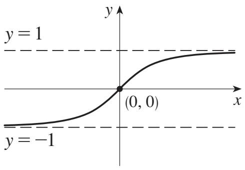
21. A. $[0, \infty)$ B. y-int 0; x-int 0, 3
C. None D. None
E. Inc on $(1, \infty)$; dec on $(0, 1)$
F. Loc min $f(1) = -2$
G. CU on $(0, \infty)$
H. See graph at right.



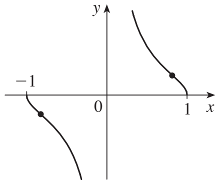
23. A. $(-\infty, -2] \cup [1, \infty)$
B. x-int -2, 1 C. None
D. None
E. Inc on $(1, \infty)$; dec on $(-\infty, -2)$
F. None
G. CD on $(-\infty, -2)$, $(1, \infty)$
H. See graph at right.



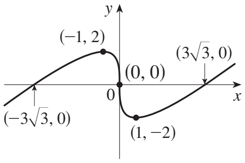
25. A. $\mathbb{R}$ B. y-int 0; x-int 0
C. About $(0, 0)$
D. HA $y = \pm 1$
E. Inc on $(-\infty, \infty)$ F. None
G. CU on $(-\infty, 0)$;
CD on $(0, \infty)$; IP $(0, 0)$
H. See graph at right.



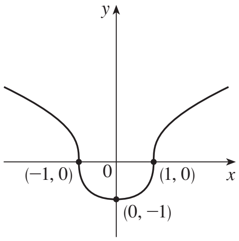
27. A. $[-1, 0) \cup (0, 1]$ B. x-int ± 1 C. About $(0, 0)$
D. VA $x = 0$
E. Dec on $(-1, 0)$, $(0, 1)$
F. None
G. CU on $\left(-1, -\sqrt{2/3}\right)$, $\left(0, \sqrt{2/3}\right)$;
CD on $\left(-\sqrt{2/3}, 0\right)$, $\left(\sqrt{2/3}, 1\right)$;
IPs $\left(\pm\sqrt{2/3}, \pm 1/\sqrt{2}\right)$
H. See graph at right.



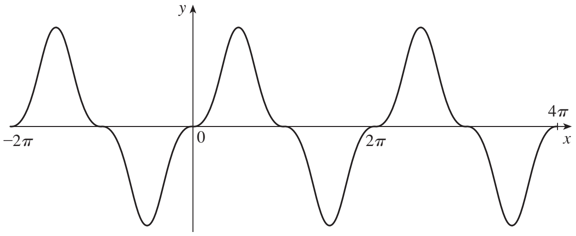
29. A. $\mathbb{R}$ B. y-int 0; x-int $\pm 3\sqrt{3}$, 0 C. About $(0, 0)$
D. None E. Inc on $(-\infty, -1)$, $(1, \infty)$; dec on $(-1, 1)$
F. Loc max $f(-1) = 2$;
loc min $f(1) = -2$
G. CU on $(0, \infty)$;
CD on $(-\infty, 0)$; IP $(0, 0)$
H. See graph at right.



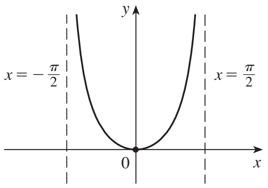
31. A. $\mathbb{R}$ B. y-int -1; x-int ± 1
C. About the y-axis D. None
E. Inc on $(0, \infty)$; dec on $(-\infty, 0)$
F. Loc min $f(0) = -1$
G. CU on $(-1, 1)$;
CD on $(-\infty, -1)$, $(1, \infty)$; IPs $(\pm 1, 0)$
H. See graph at right.



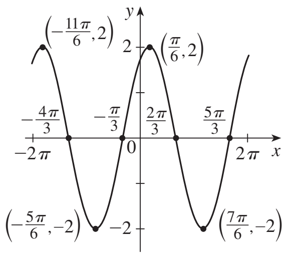
33. A. $\mathbb{R}$ B. y-int 0; x-int $n\pi$ (n an integer)
C. About $(0, 0)$, period 2π D. None
E–G answers for $0 \leqslant x \leqslant \pi$:
E. Inc on $(0, \pi/2)$; dec on $(\pi/2, \pi)$ F. Loc max $f(\pi/2) = 1$
G. Let $\alpha = \sin^{-1}\sqrt{2/3}$; CU on $(0, \alpha)$, $(\pi - \alpha, \pi)$;
CD on $(\alpha, \pi - \alpha)$; IPs at $x = 0$, π, α, $\pi - \alpha$
H.



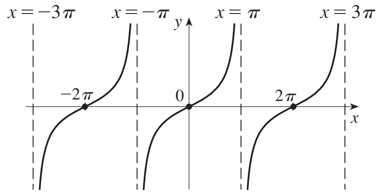
35. A. $(-\pi/2, \pi/2)$ B. y-int 0; x-int 0 C. About y-axis
D. VA $x = \pm\pi/2$
E. Inc on $(0, \pi/2)$;
dec on $(-\pi/2, 0)$
F. Loc min $f(0) = 0$
G. CU on $(-\pi/2, \pi/2)$
H. See graph at right.



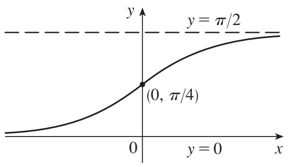
37. A. $[-2\pi, 2\pi]$
B. y-int $\sqrt{3}$; x-int $-4\pi/3$, $-\pi/3$, $2\pi/3$, $5\pi/3$
C. Period 2π D. None
E. Inc on $(-2\pi, -11\pi/6)$, $(-5\pi/6, \pi/6)$, $(7\pi/6, 2\pi)$;
dec on $(-11\pi/6, -5\pi/6)$, $(\pi/6, 7\pi/6)$
F. Loc max $f(-11\pi/6) = f(\pi/6) = 2$;
loc min $f(-5\pi/6) = f(7\pi/6) = -2$
G. CU on $(-4\pi/3, -\pi/3)$,
$(2\pi/3, 5\pi/3)$;
CD on $(-2\pi, -4\pi/3)$,
$(-\pi/3, 2\pi/3)$, $(5\pi/3, 2\pi)$;
IPs $(-4\pi/3, 0)$, $(-\pi/3, 0)$,
$(2\pi/3, 0)$, $(5\pi/3, 0)$
H. See graph at right.



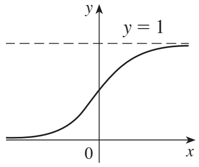
39. A. All reals except $(2n + 1)\pi$ (n an integer)
B. y-int 0; x-int $2n\pi$ C. About the origin, period 2π
D. VA $x = (2n + 1)\pi$ E. Inc on $((2n - 1)\pi, (2n + 1)\pi)$
F. None G. CU on $(2n\pi, (2n + 1)\pi)$;
CD on $((2n - 1)\pi, 2n\pi)$; IPs $(2n\pi, 0)$
H.



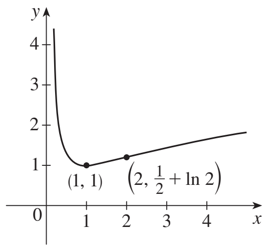
41. A. $\mathbb{R}$ B. y-int $\pi/4$
C. None
D. HA $y = 0$, $y = \pi/2$
E. Inc on $(-\infty, \infty)$ F. None
G. CU on $(-\infty, 0)$;
CD on $(0, \infty)$; IP $(0, \pi/4)$
H. See graph at right.



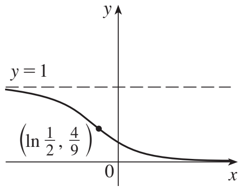
43. A. $\mathbb{R}$ B. y-int $\frac{1}{2}$ C. None
D. HA $y = 0$, $y = 1$
E. Inc on $\mathbb{R}$ F. None
G. CU on $(-\infty, 0)$;
CD on $(0, \infty)$; IP $\left(0, \frac{1}{2}\right)$
H. See graph at right.



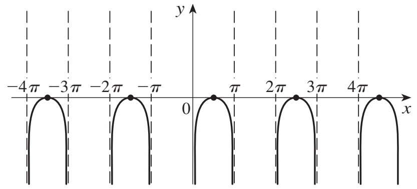
45. A. $(0, \infty)$ B. None
C. None D. VA $x = 0$
E. Inc on $(1, \infty)$; dec on $(0, 1)$
F. Loc min $f(1) = 1$
G. CU on $(0, 2)$; CD on $(2, \infty)$;
IP $\left(2, \frac{1}{2} + \ln 2\right)$
H. See graph at right.



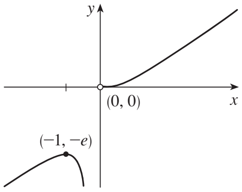
47. A. $\mathbb{R}$ B. y-int $\frac{1}{4}$
C. None
D. HA $y = 0$, $y = 1$
E. Dec on $\mathbb{R}$ F. None
G. CU on $\left(\ln \frac{1}{2}, \infty\right)$;
CD on $\left(-\infty, \ln \frac{1}{2}\right)$; IP $\left(\ln \frac{1}{2}, \frac{4}{9}\right)$
H. See graph at right.



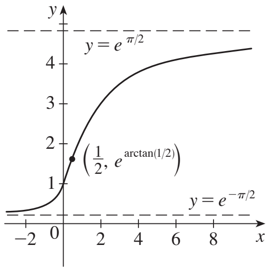
49. A. All x in $(2n\pi, (2n + 1)\pi)$ (n an integer)
B. x-int $\pi/2 + 2n\pi$ C. Period 2π D. VA $x = n\pi$
E. Inc on $(2n\pi, \pi/2 + 2n\pi)$; dec on $(\pi/2 + 2n\pi, (2n + 1)\pi)$
F. Loc max $f(\pi/2 + 2n\pi) = 0$ G. CD on $(2n\pi, (2n + 1)\pi)$
H.



51. A. $(-\infty, 0) \cup (0, \infty)$
B. None C. None
D. VA $x = 0$
E. Inc on $(-\infty, -1)$, $(0, \infty)$;
dec on $(-1, 0)$
F. Loc max $f(-1) = -e$
G. CU on $(0, \infty)$; CD on $(-\infty, 0)$
H. See graph at right.



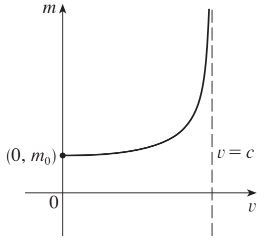
53. A. $\mathbb{R}$ B. y-int 1
C. None D. HA $y = e^{\pm\pi/2}$
E. Inc on $\mathbb{R}$ F. None
G. CU on $\left(-\infty, \frac{1}{2}\right)$; CD on $\left(\frac{1}{2}, \infty\right)$;
IP $\left(\frac{1}{2}, e^{\arctan(1/2)}\right)$
H. See graph at right.



55. (a) $(-\infty, 7]$; $(-\infty, 3) \cup (3, 7)$ (b) 3, 5
(c) $-1/\sqrt{3} \approx -0.58$ (d) HA $y = \sqrt{2}$

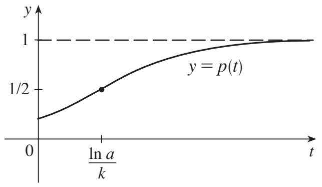
57. (a) $\mathbb{R}$; $(-\infty, 3) \cup (3, 7) \cup (7, \infty)$ (b) 3, 5, 7, 9 (c) -2
(d) HA $y = 1$, $y = 2$

59.

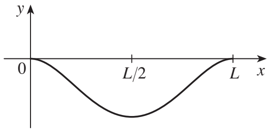


61. (a) When $t = (\ln a)/k$ (b) When $t = (\ln a)/k$

(c)

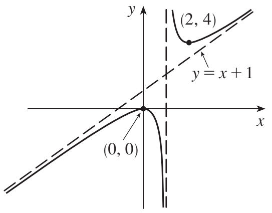


63.

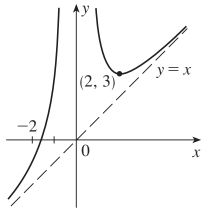


65. $y = x - 1$ **67.** $y = 2x - 3$

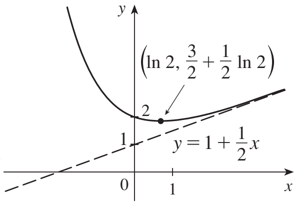
69. A. $(-\infty, 1) \cup (1, \infty)$
B. y-int 0; x-int 0
C. None
D. VA $x = 1$; SA $y = x + 1$
E. Inc on $(-\infty, 0)$, $(2, \infty)$;
dec on $(0, 1)$, $(1, 2)$
F. Loc max $f(0) = 0$;
loc min $f(2) = 4$
G. CU on $(1, \infty)$; CD on
$(-\infty, 1)$
H. See graph at right.



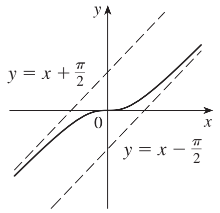
71. A. $(-\infty, 0) \cup (0, \infty)$
B. x-int $-\sqrt[3]{4}$ C. None
D. VA $x = 0$; SA $y = x$
E. Inc on $(-\infty, 0)$, $(2, \infty)$;
dec on $(0, 2)$
F. Loc min $f(2) = 3$
G. CU on $(-\infty, 0)$, $(0, \infty)$
H. See graph at right.



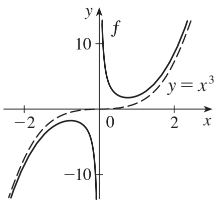
73. A. $\mathbb{R}$ B. y-int 2
C. None
D. SA $y = 1 + \frac{1}{2}x$
E. Inc on $(\ln 2, \infty)$;
dec on $(-\infty, \ln 2)$
F. Loc min $f(\ln 2) = \frac{3}{2} + \frac{1}{2}\ln 2$
G. CU on $(-\infty, \infty)$
H. See graph at right.



75.

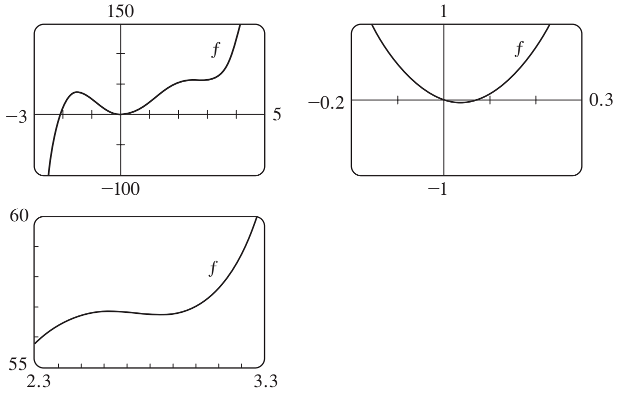


79. VA $x = 0$, asymptotic to $y = x^3$

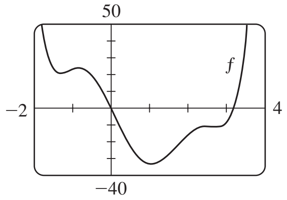


EXERCISES 4.6 ■ PAGE 334

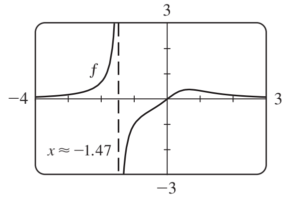
1. Inc on $(-\infty, -1.50)$, $(0.04, 2.62)$, $(2.84, \infty)$; dec on $(-1.50, 0.04)$, $(2.62, 2.84)$; loc max $f(-1.50) \approx 36.47$, $f(2.62) \approx 56.83$; loc min $f(0.04) \approx -0.04$, $f(2.84) \approx 56.73$; CU on $(-0.89, 1.15)$, $(2.74, \infty)$; CD on $(-\infty, -0.89)$, $(1.15, 2.74)$; IPs $(-0.89, 20.90)$, $(1.15, 26.57)$, $(2.74, 56.78)$



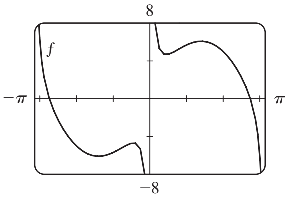
3. Inc on $(-1.31, -0.84)$, $(1.06, 2.50)$, $(2.75, \infty)$; dec on $(-\infty, -1.31)$, $(-0.84, 1.06)$, $(2.50, 2.75)$; loc max $f(-0.84) \approx 23.71$, $f(2.50) \approx -11.02$; loc min $f(-1.31) \approx 20.72$, $f(1.06) \approx -33.12$, $f(2.75) \approx -11.33$; CU on $(-\infty, -1.10)$, $(0.08, 1.72)$, $(2.64, \infty)$; CD on $(-1.10, 0.08)$, $(1.72, 2.64)$; IPs $(-1.10, 22.09)$, $(0.08, -3.88)$, $(1.72, -22.53)$, $(2.64, -11.18)$



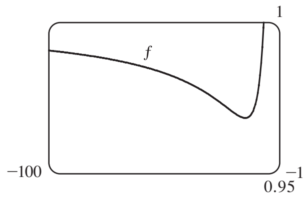
5. Inc on $(-\infty, -1.47)$, $(-1.47, 0.66)$; dec on $(0.66, \infty)$; loc max $f(0.66) \approx 0.38$; CU on $(-\infty, -1.47)$, $(-0.49, 0)$, $(1.10, \infty)$; CD on $(-1.47, -0.49)$, $(0, 1.10)$; IPs $(-0.49, -0.44)$, $(1.10, 0.31)$, $(0, 0)$

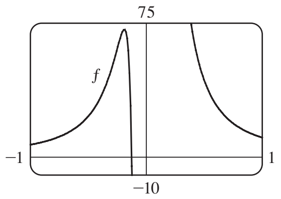


7. Inc on $(-1.40, -0.44)$, $(0.44, 1.40)$; dec on $(-\pi, -1.40)$, $(-0.44, 0)$, $(0, 0.44)$, $(1.40, \pi)$; loc max $f(-0.44) \approx -4.68$, $f(1.40) \approx 6.09$; loc min $f(-1.40) \approx -6.09$, $f(0.44) \approx 4.68$; CU on $(-\pi, -0.77)$, $(0, 0.77)$; CD on $(-0.77, 0)$, $(0.77, \pi)$; IPs $(-0.77, -5.22)$, $(0.77, 5.22)$

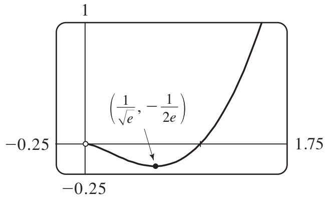


9. Inc on $\left(-8 - \sqrt{61}, -8 + \sqrt{61}\right)$; dec on $\left(-\infty, -8 - \sqrt{61}\right)$, $\left(-8 + \sqrt{61}, 0\right)$, $(0, \infty)$; CU on $\left(-12 - \sqrt{138}, -12 + \sqrt{138}\right)$, $(0, \infty)$; CD on $\left(-\infty, -12 - \sqrt{138}\right)$, $\left(-12 + \sqrt{138}, 0\right)$



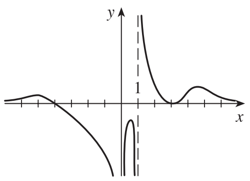


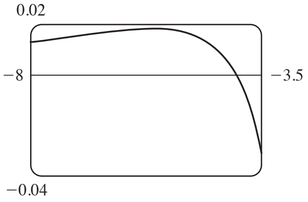
11. (a)

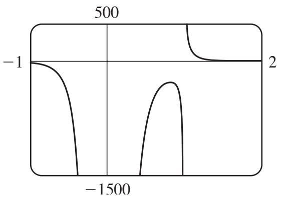


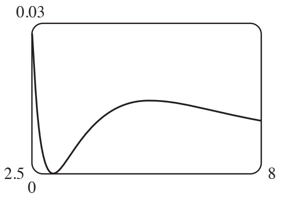
(b) $\lim_{x \to 0^+} f(x) = 0$
(c) Loc min $f\left(1/\sqrt{e}\right) = -1/(2e)$;
CD on $(0, e^{-3/2})$; CU on $(e^{-3/2}, \infty)$

13. Loc max $f(-5.6) \approx 0.018$, $f(0.82) \approx -281.5$, $f(5.2) \approx 0.0145$; loc min $f(3) = 0$







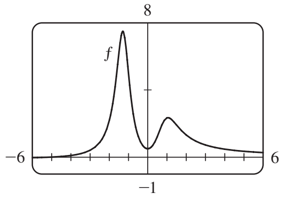


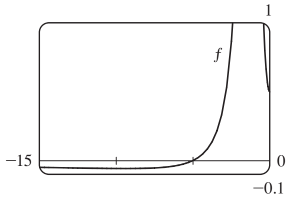
15. $f'(x) = -\dfrac{x(x+1)^2(x^3 + 18x^2 - 44x - 16)}{(x-2)^3(x-4)^5}$

$$f''(x) = 2\,\frac{(x+1)(x^6 + 36x^5 + 6x^4 - 628x^3 + 684x^2 + 672x + 64)}{(x-2)^4(x-4)^6}$$

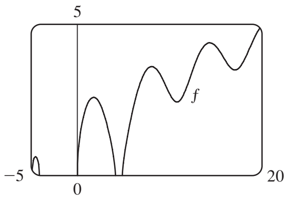
CU on $(-35.3, -5.0)$, $(-1, -0.5)$, $(-0.1, 2)$, $(2, 4)$, $(4, \infty)$; CD on $(-\infty, -35.3)$, $(-5.0, -1)$, $(-0.5, -0.1)$; IPs $(-35.3, -0.015)$, $(-5.0, -0.005)$, $(-1, 0)$, $(-0.5, 0.00001)$, $(-0.1, 0.0000066)$

17. Inc on $(-9.41, -1.29)$, $(0, 1.05)$; dec on $(-\infty, -9.41)$, $(-1.29, 0)$, $(1.05, \infty)$; loc max $f(-1.29) \approx 7.49$, $f(1.05) \approx 2.35$; loc min $f(-9.41) \approx -0.056$, $f(0) = 0.5$; CU on $(-13.81, -1.55)$, $(-1.03, 0.60)$, $(1.48, \infty)$; CD on $(-\infty, -13.81)$, $(-1.55, -1.03)$, $(0.60, 1.48)$; IPs $(-13.81, -0.05)$, $(-1.55, 5.64)$, $(-1.03, 5.39)$, $(0.60, 1.52)$, $(1.48, 1.93)$

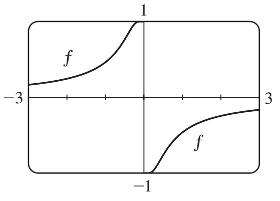




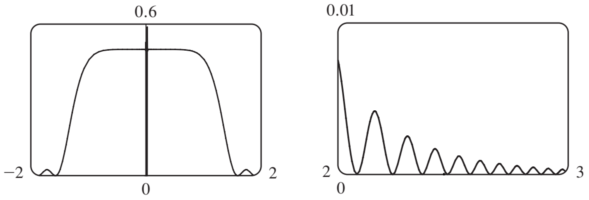
19. Inc on $(-4.91, -4.51)$, $(0, 1.77)$, $(4.91, 8.06)$, $(10.79, 14.34)$, $(17.08, 20)$; dec on $(-4.51, -4.10)$, $(1.77, 4.10)$, $(8.06, 10.79)$, $(14.34, 17.08)$; loc max $f(-4.51) \approx 0.62$, $f(1.77) \approx 2.58$, $f(8.06) \approx 3.60$, $f(14.34) \approx 4.39$; loc min $f(10.79) \approx 2.43$, $f(17.08) \approx 3.49$; CU on $(9.60, 12.25)$, $(15.81, 18.65)$; CD on $(-4.91, -4.10)$, $(0, 4.10)$, $(4.91, 9.60)$, $(12.25, 15.81)$, $(18.65, 20)$; IPs $(9.60, 2.95)$, $(12.25, 3.27)$, $(15.81, 3.91)$, $(18.65, 4.20)$



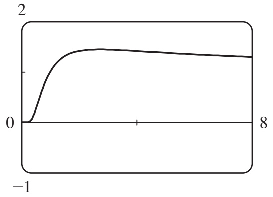
21. Inc on $(-\infty, 0)$, $(0, \infty)$; CU on $(-\infty, -0.42)$, $(0, 0.42)$; CD on $(-0.42, 0)$, $(0.42, \infty)$; IPs $(\mp 0.42, \pm 0.83)$



23.



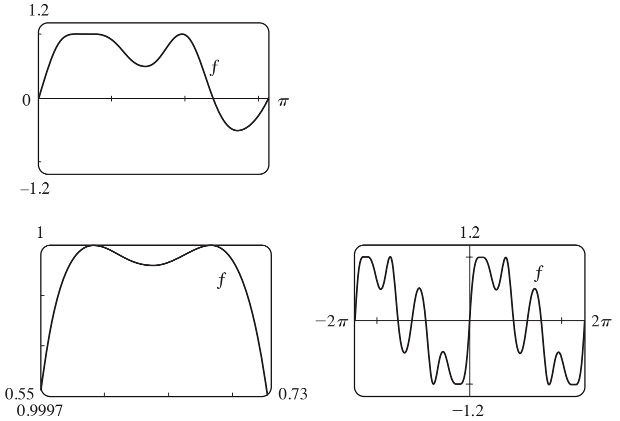
25. (a)



(b) $\lim_{x\to 0^+} x^{1/x} = 0$, $\lim_{x\to\infty} x^{1/x} = 1$

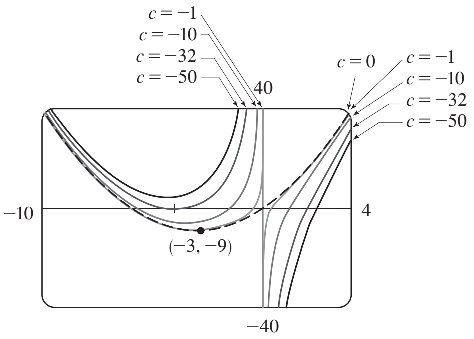
(c) Loc max $f(e) = e^{1/e}$ (d) IPs at $x \approx 0.58, 4.37$

27. Max $f(0.59) \approx 1$, $f(0.68) \approx 1$, $f(1.96) \approx 1$; min $f(0.64) \approx 0.99996$, $f(1.46) \approx 0.49$, $f(2.73) \approx -0.51$; IPs (0.61, 0.99998), (0.66, 0.99998), (1.17, 0.72), (1.75, 0.77), (2.28, 0.34)

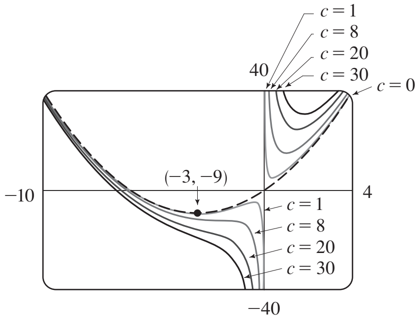


29. For $c < 0$, there is a loc min that moves toward $(-3, -9)$ as c increases. For $0 < c < 8$, there is a loc min that moves toward $(-3, -9)$ and a loc max that moves toward the origin as c decreases. For all $c > 0$, there is a first-quadrant loc min that moves toward the origin as c decreases. $c = 0$ is a transitional value that gives the graph of a parabola. For all nonzero c, the y-axis is a VA and there is an IP that moves toward the origin as $|c| \to 0$.

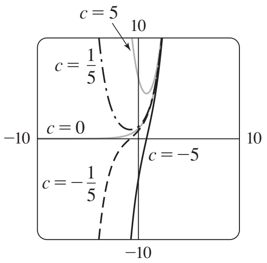
$c \leqslant 0$:



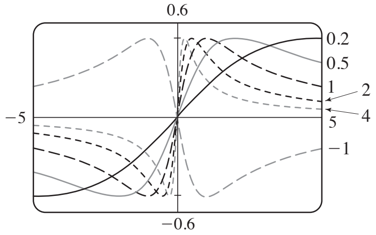
$c \geqslant 0$:



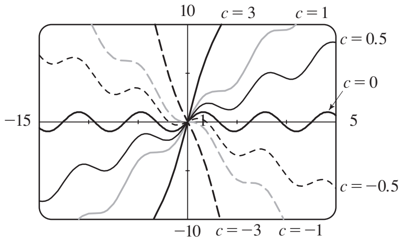
31. For $c < 0$, there is no extreme point and one IP, which decreases along the x-axis. For $c > 0$, there is no IP, and one minimum point.



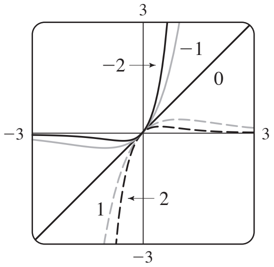
33. For $c > 0$, the maximum and minimum values are always $\pm\frac{1}{2}$, but the extreme points and IPs move closer to the y-axis as c increases. $c = 0$ is a transitional value: when c is replaced by $-c$, the curve is reflected in the x-axis.



35. For $|c| < 1$, the graph has loc max and min values; for $|c| \geqslant 1$ it does not. The function increases for $c \geqslant 1$ and decreases for $c \leqslant -1$. As c changes, the IPs move vertically but not horizontally.

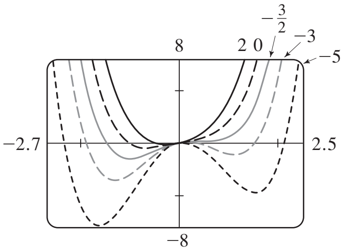


37.



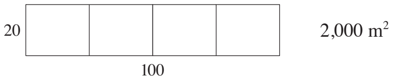
For $c > 0$, $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = -\infty$.
For $c < 0$, $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = 0$.
As $|c|$ increases, the max and min points and the IPs get closer to the origin.

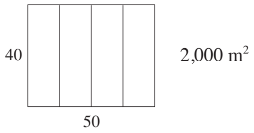
39. $c = 0$; $c = -1.5$

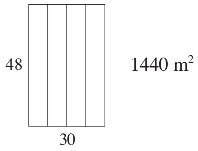


EXERCISES 4.7 ■ PAGE 342

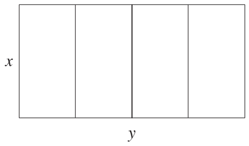
1. (a) 11, 12 (b) 11.5, 11.5 **3.** 10, 10 **5.** $\frac{9}{4}$
7. 25 m by 25 m **9.** $N = 1$
11. (a)







(b)



(c) $A = xy$ (d) $5x + 2y = 300$ (e) $A(x) = 150x - \frac{5}{2}x^2$
(f) 2250 m^2
13. 100 m by 150 m, middle fence parallel to short side
15. 20 m by 600 m **19.** 4000 cm^3 **21.** ≈\$163.54
23. 45 cm by 45 cm by 90 cm.
25. $\left(-\frac{6}{5}, \frac{3}{5}\right)$ **27.** $\left(-\frac{1}{3}, \pm\frac{4}{3}\sqrt{2}\right)$ **29.** Square, side $\sqrt{2}\,r$
31. $L/2$, $\sqrt{3}\,L/4$ **33.** Base $\sqrt{3}\,r$, height $3r/2$
37. $4\pi r^3/(3\sqrt{3})$ **39.** $\pi r^2(1 + \sqrt{5})$
41. 24 cm by 36 cm
43. (a) Use all of the wire for the square
(b) $40\sqrt{3}/(9 + 4\sqrt{3})$ m for the square
45. 30 cm **47.** $V = 2\pi R^3/(9\sqrt{3})$ **51.** $E^2/(4r)$
53. (a) $\frac{3}{2}s^2 \csc\theta\,(\csc\theta - \sqrt{3}\cot\theta)$ (b) $\cos^{-1}(1/\sqrt{3}) \approx 55°$
(c) $6s\left[h + s/(2\sqrt{2})\right]$
55. Row directly to B **57.** ≈4.85 km east of the refinery
59. $10\sqrt[3]{3}/(1 + \sqrt[3]{3}) \approx 5.91$ m from the stronger source
61. $(a^{2/3} + b^{2/3})^{3/2}$ **63.** $2\sqrt{6}$
65. (b) (i) \$342,491; \$342.49/unit; \$389.74/unit
(ii) 400 (iii) \$320/unit
67. (a) $p(x) = 19 - \frac{1}{3000}x$ (b) \$9.50
69. (a) $p(x) = 500 - \frac{1}{8}x$ (b) \$250 (c) \$310
75. 9.35 m **79.** $x = 15$ cm **81.** $\pi/6$
83. At a distance $5 - 2\sqrt{5} \approx 0.53$ from A **85.** $\frac{1}{2}(L + W)^2$
87. (a) About 5.1 km from B (b) C is close to B; C is close to D; $W/L = \sqrt{25 + x^2}/x$, where $x = |BC|$
(c) ≈1.07; no such value (d) $\sqrt{41}/4 \approx 1.6$

EXERCISES 4.8 ■ PAGE 354

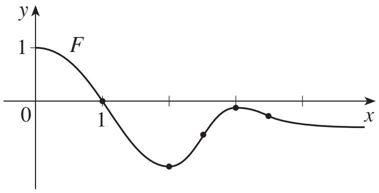
1. (a) $x_2 \approx 7.3$, $x_3 \approx 6.8$ (b) Yes
3. $\frac{9}{2}$ **5.** a, b, c **7.** 1.5215 **9.** −1.25
11. 2.94283096 **13.** (b) 2.630020 **15.** −1.914021
17. 1.934563 **19.** −1.257691, 0.653483
21. −1.428293, 2.027975
23. −1.69312029, −0.74466668, 1.26587094
25. 0.76682579 **27.** −0.87828292, 0.79177077
29. (b) 31.622777
35. (a) −1.293227, −0.441731, 0.507854 (b) −2.0212
37. (1.519855, 2.306964) **39.** (0.410245, 0.347810)
41. 0.76286%

EXERCISES 4.9 ■ PAGE 361

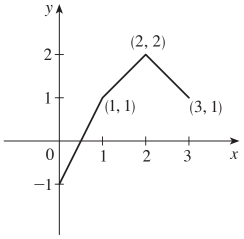
1. (a) $F(x) = 6x$ (b) $G(t) = t^3$
3. (a) $H(q) = \sin q$ (b) $F(x) = e^x$
5. $F(x) = 2x^2 + 7x + C$ **7.** $F(x) = \frac{1}{2}x^4 - \frac{2}{9}x^3 + \frac{5}{2}x^2 + C$
9. $F(x) = 4x^3 + 4x^2 + C$ **11.** $G(x) = 12x^{1/3} - \frac{3}{4}x^{8/3} + C$
13. $F(x) = 2x^{3/2} - \frac{3}{2}x^{4/3} + C$
15. $F(t) = \frac{4}{3}t^{3/2} - 8\sqrt{t} + 3t + C$
17. $F(x) = \dfrac{2}{5}\ln|x| + \dfrac{3}{x} + C$
19. $G(t) = 7e^t - e^3 t + C$

21. $F(\theta) = -2\cos\theta - 3\sec\theta + C$
23. $F(r) = 4\tan^{-1}r - \frac{5}{9}r^{9/5} + C$
25. $F(x) = 2^x/\ln 2 + 4\cosh x + C$
27. $F(x) = 2e^x - 3x^2 - 1$
29. $f(x) = 4x^3 + Cx + D$
31. $f(x) = \frac{1}{5}x^5 + 4x^3 - \frac{1}{2}x^2 + Cx + D$
33. $f(x) = \frac{1}{3}x^3 + 3e^x + Cx + D$
35. $f(t) = 2t^3 + \cos t + Ct^2 + Dt + E$
37. $f(x) = 2x^4 + \ln x - 5$
39. $f(t) = 4\arctan t - \pi$
41. $f(x) = 3x^{5/3} - 75$
43. $f(t) = \tan t + \sec t - 2 - \sqrt{2}$
45. $f(x) = -x^2 + 2x^3 - x^4 + 12x + 4$
47. $f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4$
49. $f(x) = 2x^2 + x^3 + 2x^4 + 2x + 3$
51. $f(x) = e^x + 2\sin x - \dfrac{2}{\pi}\left(e^{\pi/2} + 4\right)x + 2$
53. $f(x) = -\ln x + (\ln 2)x - \ln 2$
55. 8 **57.** b

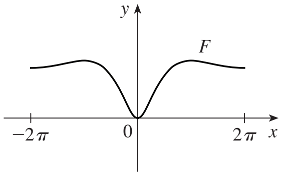
59.



61.



63.



65. $s(t) = 2\sin t - 4\cos t + 7$
67. $s(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t + 3$
69. $s(t) = -\sin t + \cos t + \dfrac{8}{\pi}t - 1$
71. (a) $s(t) = 450 - 4.9t^2$ (b) $\sqrt{450/4.9} \approx 9.58$ s
(c) $-9.8\sqrt{450/4.9} \approx -93.9$ m/s (d) About 9.09 s
75. 81.6 m **77.** \$742.08 **79.** $\frac{130}{11} \approx 11.8$ s
81. 1.79 m/s^2 **83.** 62,500 km/h^2 ≈ 4.82 m/s^2
85. (a) 101.0 km (b) 87.7 km (c) 21 min 50 s
(d) 172 km

CHAPTER 4 REVIEW ■ PAGE 364

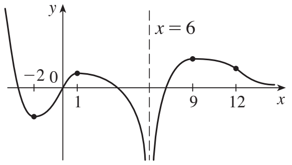
True-False Quiz

1. False **3.** False **5.** True **7.** False **9.** True
11. True **13.** False **15.** True **17.** True
19. True **21.** False

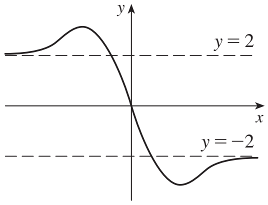
Exercises

1. Abs max $f(2) = f(5) = 18$, abs min $f(0) = -2$, loc max $f(2) = 18$, loc min $f(4) = 14$
3. Abs max $f(2) = \frac{2}{5}$, abs and loc min $f\left(-\frac{1}{3}\right) = -\frac{9}{2}$
5. Abs and loc max $f(\pi/6) = \pi/6 + \sqrt{3}$, abs min $f(-\pi) = -\pi - 2$, loc min $f(5\pi/6) = 5\pi/6 - \sqrt{3}$
7. 1 **9.** 4 **11.** 0 **13.** $\frac{1}{2}$

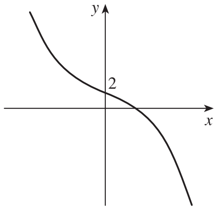
15.



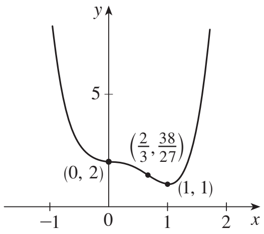
17.



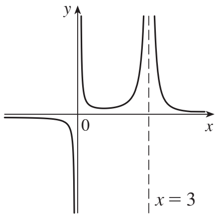
19. A. ℝ B. y-int 2
C. None D. None
E. Dec on $(-\infty, \infty)$ F. None
G. CU on $(-\infty, 0)$;
CD on $(0, \infty)$; IP $(0, 2)$
H. See graph at right.



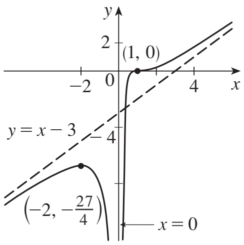
21. A. ℝ B. y-int 2
C. None D. None
E. Inc on $(1, \infty)$; dec on $(-\infty, 1)$
F. Loc min $f(1) = 1$
G. CU on $(-\infty, 0)$, $\left(\frac{2}{3}, \infty\right)$;
CD on $\left(0, \frac{2}{3}\right)$; IPs $(0, 2)$, $\left(\frac{2}{3}, \frac{38}{27}\right)$
H. See graph at right.



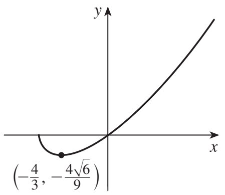
23. A. $(-\infty, 0) \cup (0, 3) \cup (3, \infty)$
B. None C. None
D. HA $y = 0$; VA $x = 0$, $x = 3$
E. Inc on $(1, 3)$;
dec on $(-\infty, 0)$, $(0, 1)$, $(3, \infty)$
F. Loc min $f(1) = \frac{1}{4}$
G. CU on $(0, 3)$, $(3, \infty)$;
CD on $(-\infty, 0)$
H. See graph at right.



25. A. $(-\infty, 0) \cup (0, \infty)$
B. x-int 1 C. None
D. VA $x = 0$; SA $y = x - 3$
E. Inc on $(-\infty, -2)$, $(0, \infty)$;
dec on $(-2, 0)$
F. Loc max $f(-2) = -\frac{27}{4}$
G. CU on $(1, \infty)$; CD on $(-\infty, 0)$, $(0, 1)$; IP $(1, 0)$
H. See graph at right.

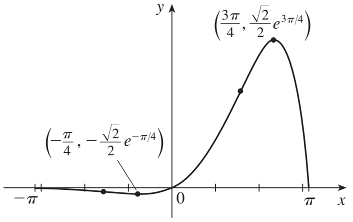


27. A. $[-2, \infty)$
B. y-int 0; x-int -2, 0
C. None D. None
E. Inc on $\left(-\frac{4}{3}, \infty\right)$, dec on $\left(-2, -\frac{4}{3}\right)$
F. Loc min $f\left(-\frac{4}{3}\right) = -\frac{4}{9}\sqrt{6}$
G. CU on $(-2, \infty)$
H. See graph at right.

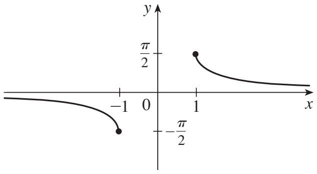


29. A. $[-\pi, \pi]$ B. y-int 0; x-int $-\pi$, 0, π
C. None D. None
E. Inc on $(-\pi/4, 3\pi/4)$; dec on $(-\pi, -\pi/4)$, $(3\pi/4, \pi)$
F. Loc max $f(3\pi/4) = \frac{1}{2}\sqrt{2}\,e^{3\pi/4}$,
loc min $f(-\pi/4) = -\frac{1}{2}\sqrt{2}\,e^{-\pi/4}$
G. CU on $(-\pi/2, \pi/2)$; CD on $(-\pi, -\pi/2)$, $(\pi/2, \pi)$;
IPs $(-\pi/2, -e^{-\pi/2})$, $(\pi/2, e^{\pi/2})$

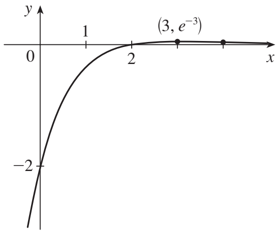
H.



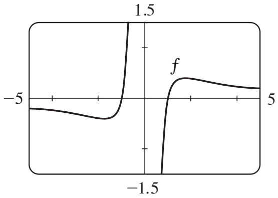
31. A. $(-\infty, -1] \cup [1, \infty)$
B. None C. About (0, 0)
D. HA $y = 0$
E. Dec on $(-\infty, -1)$, $(1, \infty)$
F. None
G. CU on $(1, \infty)$;
CD on $(-\infty, -1)$
H. See graph at right.



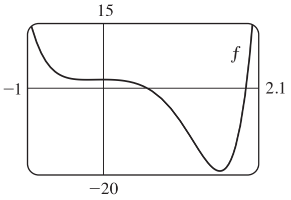
33. A. $\mathbb{R}$
B. y-int -2; x-int 2
C. None D. HA $y = 0$
E. Inc on $(-\infty, 3)$; dec on $(3, \infty)$
F. Loc max $f(3) = e^{-3}$
G. CU on $(4, \infty)$;
CD on $(-\infty, 4)$;
IP $(4, 2e^{-4})$
H. See graph at right.

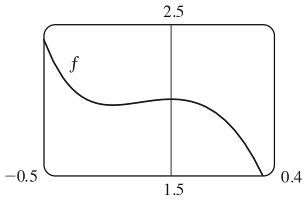


35. Inc on $\left(-\sqrt{3}, 0\right)$, $\left(0, \sqrt{3}\right)$;
dec on $\left(-\infty, -\sqrt{3}\right)$, $\left(\sqrt{3}, \infty\right)$;
loc max $f\left(\sqrt{3}\right) = \frac{2}{9}\sqrt{3}$,
loc min $f\left(-\sqrt{3}\right) = -\frac{2}{9}\sqrt{3}$;
CU on $\left(-\sqrt{6}, 0\right)$, $\left(\sqrt{6}, \infty\right)$;
CD on $\left(-\infty, -\sqrt{6}\right)$, $\left(0, \sqrt{6}\right)$;
IPs $\left(\sqrt{6}, \frac{5}{36}\sqrt{6}\right)$, $\left(-\sqrt{6}, -\frac{5}{36}\sqrt{6}\right)$

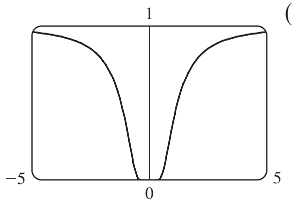


37. Inc on $(-0.23, 0)$, $(1.62, \infty)$; dec on $(-\infty, -0.23)$, $(0, 1.62)$;
loc max $f(0) = 2$; loc min $f(-0.23) \approx 1.96$, $f(1.62) \approx -19.2$;
CU on $(-\infty, -0.12)$, $(1.24, \infty)$;
CD on $(-0.12, 1.24)$; IPs $(-0.12, 1.98)$, $(1.24, -12.1)$





39.



$(\pm 0.82, 0.22)$; $\left(\pm\sqrt{2/3}, e^{-3/2}\right)$

41. Loc max at $x \approx -2.96$, -0.18, 3.01;
loc min at $x \approx -1.57$, 1.57; IP at $x \approx -2.16$, -0.75, 0.46, 2.21

43. For $c > -1$, f is periodic with period 2π and has local maxima at $2n\pi + \pi/2$, n an integer. For $c \leqslant -1$, f has no graph. For $-1 < c \leqslant 1$, f has vertical asymptotes. For $c > 1$, f is continuous on $\mathbb{R}$. As c increases, f moves upward and its oscillations become less pronounced.

49. (a) 0 (b) CU on $\mathbb{R}$ **53.** $3\sqrt{3}\,r^2$

55. $4/\sqrt{3}$ cm from D **57.** $L = C$ **59.** \$11.50

61. 1.297383 **63.** 1.16718557

65. $F(x) = \frac{8}{3}x^{3/2} - 2x^3 + 3x + C$

67. $F(t) = -2\cos t - 3e^t + C$

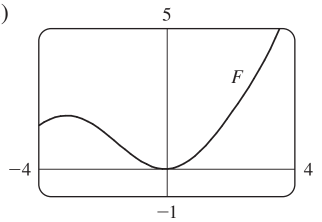
69. $f(t) = t^2 + 3\cos t + 2$

71. $f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + 1$

73. $s(t) = t^2 - \tan^{-1}t + 1$

75. (b) $0.1e^x - \cos x + 0.9$

(c)



77. No

79. (b) About 25.44 cm by 5.96 cm (c) $2\sqrt{300}$ cm by $2\sqrt{600}$ cm

85. $\tan^{-1}\left(-\frac{2}{\pi}\right) + 180° \approx 147.5°$

87. (a) $10\sqrt{2} \approx 14$ m

(b) $\frac{dI}{dt} = \frac{-60k(h-1)}{[(h-1)^2+400]^{-5/2}}$, where k is the constant of proportionality

PROBLEMS PLUS ■ PAGE 369

3. Abs max $f(-5) = e^{45}$, no abs min **7.** 24

9. $(-2, 4)$, $(2, -4)$ **13.** $(1+\sqrt{5})/2$ **15.** $(m/2, m^2/4)$

17. $a \leqslant e^{1/e}$

21. (a) $T_1 = D/c_1$, $T_2 = (2h\sec\theta)/c_1 + (D - 2h\tan\theta)/c_2$, $T_3 = \sqrt{4h^2+D^2}/c_1$

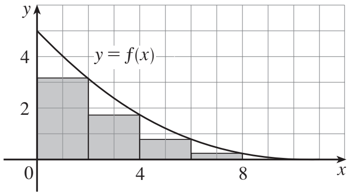
(c) $c_1 \approx 3.85$ km/s, $c_2 \approx 7.66$ km/s, $h \approx 0.42$ km

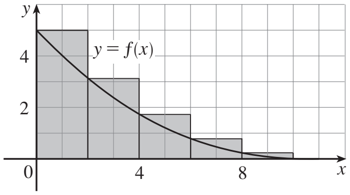
25. $3/(\sqrt[3]{2}-1) \approx 11\frac{1}{2}$ h

CHAPTER 5

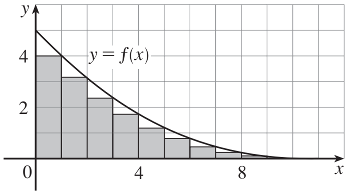
EXERCISES 5.1 ■ PAGE 381

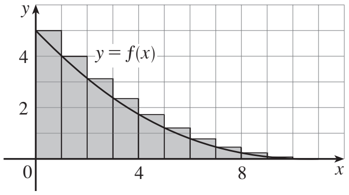
1. (a) Lower ≈ 12, upper ≈ 22



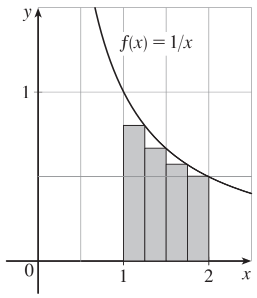


(b) Lower ≈ 14.4, upper ≈ 19.4

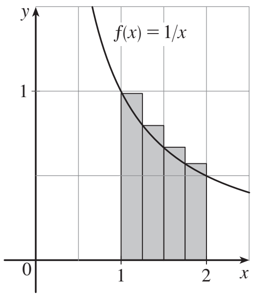




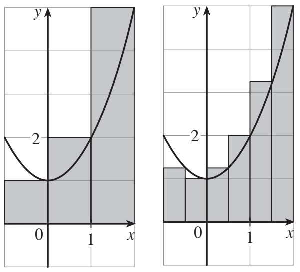
3. (a) 0.6345, underestimate



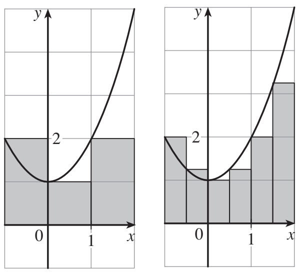
(b) 0.7595, overestimate



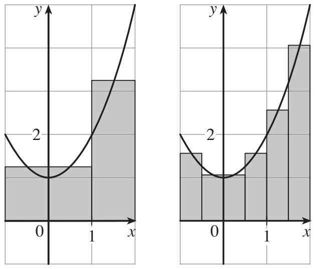
5. (a) 8, 6.875



(b) 5, 5.375

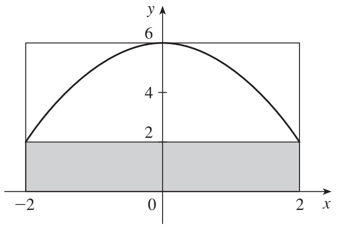


(c) 5.75, 5.9375

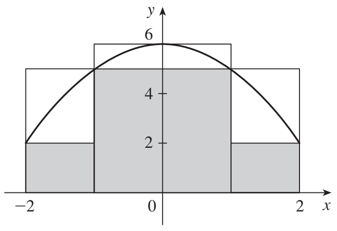


(d) M_6

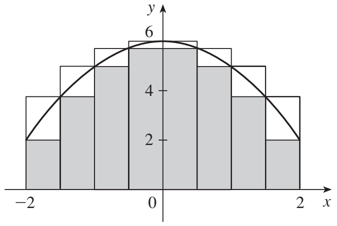
7. $n = 2$: upper $= 24$, lower $= 8$



$n = 4$: upper $= 22$, lower $= 14$



$n = 8$: upper $= 20.5$, lower $= 16.5$



9. 10.55 m, 13.65 m **11.** 63.2 L, 70 L **13.** 39 m

15. 7840 **17.** $\lim_{n\to\infty}\sum_{i=1}^{n}[2+\sin^2(\pi i/n)]\cdot\dfrac{\pi}{n}$

19. $\lim_{n\to\infty}\sum_{i=1}^{n}(1+4i/n)\sqrt{(1+4i/n)^3+8}\cdot\dfrac{4}{n}$

21. The region under the graph of $y=\dfrac{1}{1+x}$ from 0 to 2

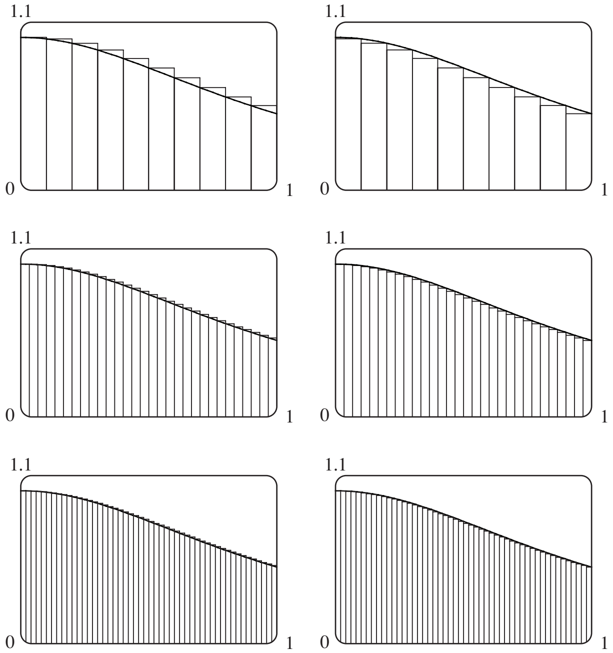
23. The region under the graph of $y=\tan x$ from 0 to $\pi/4$

25. (a) $L_n < A < R_n$

27. 0.2533, 0.2170, 0.2101, 0.2050; 0.2

29. (a) Left: 0.8100, 0.7937, 0.7904;
right: 0.7600, 0.7770, 0.7804

(b)



31. (a) $\lim_{n\to\infty}\dfrac{64}{n^6}\sum_{i=1}^{n}i^5$ (b) $\dfrac{n^2(n+1)^2(2n^2+2n-1)}{12}$

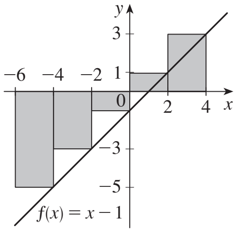
(c) $\frac{32}{3}$

33. $\sin b$, 1

EXERCISES 5.2 ■ PAGE 394

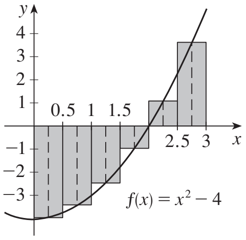
1. -10

The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the three rectangles below the x-axis; that is, the *net area* of the rectangles with respect to the x-axis.



3. $-\frac{49}{16}$

The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the four rectangles below the x-axis.



5. (a) 4 (b) 2 (c) 6

7. Lower $= -64$; upper $= 16$ **9.** 168

11. 10.2857 **13.** 0.3186 **15.** 0.3181, 0.3180

17.

n	R_n
5	1.933766
10	1.983524
50	1.999342
100	1.999836

The values of R_n appear to be approaching 2.

19. $\int_0^1 \dfrac{e^x}{1+x}\,dx$ **21.** $\int_2^7 (5x^3-4x)\,dx$

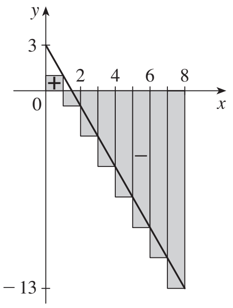
23. $-\frac{40}{3}$ **25.** $\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{4+(1+2i/n)}\cdot\dfrac{2}{n}$

27. 6 **29.** $\frac{57}{2}$ **31.** 208 **33.** $-\frac{3}{4}$

35. (a) 4 (b) 10 (c) -3 (d) 0 (e) 6 (f) -4

37. (a) 18

39. (a) -48 (b)



(c) -40

41. $\frac{35}{2}$ **43.** $\frac{25}{4}$ **45.** $3+\frac{9}{4}\pi$

49. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\sin\dfrac{5\pi i}{n}\right)\dfrac{\pi}{n}=\dfrac{2}{5}$ **51.** 0 **53.** 3

55. e^5-e^3 **57.** $\int_{-1}^{5} f(x)\,dx$ **59.** 122

61. B < E < A < D < C **63.** 15

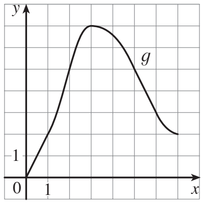
69. $0 \leqslant \int_0^1 x^3\,dx \leqslant 1$ **71.** $\dfrac{\pi}{12} \leqslant \int_{\pi/4}^{\pi/3}\tan x\,dx \leqslant \dfrac{\pi}{12}\sqrt{3}$

73. $0 \leqslant \int_0^2 xe^{-x}\,dx \leqslant 2/e$ **77.** $\int_1^2 \arctan x\,dx$
83. $\int_0^1 x^4\,dx$ **85.** $\frac{1}{2}$

EXERCISES 5.3 ■ PAGE 406

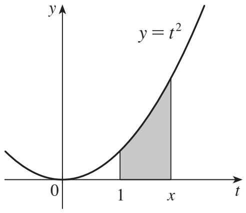
1. One process undoes what the other one does. See the Fundamental Theorem of Calculus.

3. (a) 0, 2, 5, 7, 3 (d)
(b) (0, 3)
(c) $x = 3$



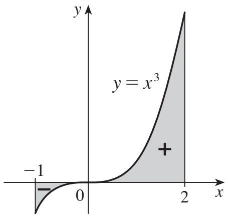
5. (a) $g(x) = 3x$

7.

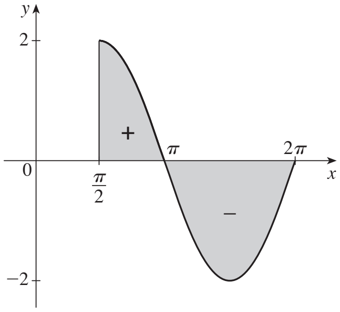


(a), (b) x^2

9. $g'(x) = \sqrt{x + x^3}$ **11.** $g'(w) = \sin(1 + w^3)$
13. $F'(x) = -\sqrt{1 + \sec x}$ **15.** $h'(x) = xe^x$
17. $y' = \dfrac{3(3x + 2)}{1 + (3x + 2)^3}$ **19.** $y' = -\frac{1}{2}\tan\sqrt{x}$
21. 3.75



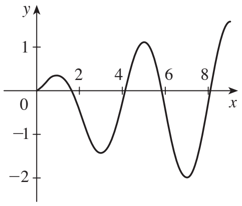
23. -2



25. $\frac{26}{3}$ **27.** 2 **29.** $\frac{52}{3}$ **31.** $\frac{512}{15}$ **33.** -1
35. $-\frac{37}{6}$ **37.** $\frac{82}{5}$ **39.** $8 + \ln 3$ **41.** 1
43. $\frac{15}{4}$ **45.** $\ln 2 + 7$ **47.** $\dfrac{1}{e + 1} + e - 1$

49. $4\pi/3$ **51.** $\dfrac{15}{\ln 2}$ **53.** 0 **55.** $\frac{16}{3}$
57. $\frac{32}{3}$ **59.** $\frac{243}{4}$ **61.** 2
63. The function $f(x) = x^{-4}$ is not continuous on the interval $[-2, 1]$, so FTC2 cannot be applied.
65. The function $f(\theta) = \sec\theta\,\tan\theta$ is not continuous on the interval $[\pi/3, \pi]$, so FTC2 cannot be applied.
67. $g'(x) = \dfrac{-2(4x^2 - 1)}{4x^2 + 1} + \dfrac{3(9x^2 - 1)}{9x^2 + 1}$
69. $F'(x) = 2xe^{x^4} - e^{x^2}$
71. $y' = \sin x \ln(1 + 2\cos x) + \cos x \ln(1 + 2\sin x)$
73. $(-4, 0)$ **75.** $y = e^4x - 2e^4$ **77.** 1 **79.** 29
81. (a) $-2\sqrt{n}$, $\sqrt{4n - 2}$, n an integer > 0
(b) $(0, 1)$, $(-\sqrt{4n - 1}, -\sqrt{4n - 3})$, and $(\sqrt{4n - 1}, \sqrt{4n + 1})$, n an integer > 0 (c) 0.74

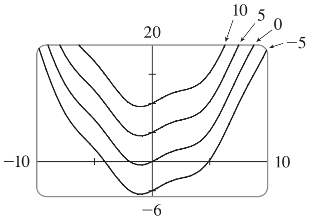
83. (a) Loc max at 1 and 5; loc min at 3 and 7
(b) $x = 9$
(c) $(\frac{1}{2}, 2)$, $(4, 6)$, $(8, 9)$
(d) See graph at right.



85. $\frac{7}{10}$ **93.** $f(x) = x^{3/2}$, $a = 9$
95. (b) Average expenditure over $[0, t]$; to minimize average expenditure

EXERCISES 5.4 ■ PAGE 415

5. $x^3 + 2x^2 + x + C$ **7.** $\frac{1}{2}x^2 + \sin x + C$
9. $\frac{1}{2.3}x^{2.3} + 2x^{3.5} + C$ **11.** $5x + \frac{2}{9}x^3 + \frac{3}{16}x^4 + C$
13. $\frac{2}{3}u^3 + \frac{9}{2}u^2 + 4u + C$ **15.** $\ln|x| + 2\sqrt{x} + x + C$
17. $e^x + \ln|x| + C$ **19.** $-\cos x + \cosh x + C$
21. $\theta + \tan\theta + C$ **23.** $-3\cot t + C$
25. $\sin x + \frac{1}{4}x^2 + C$

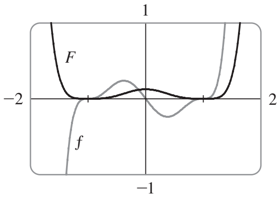


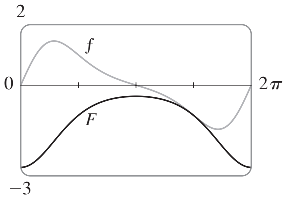
27. $-\frac{10}{3}$ **29.** 505.5 **31.** -2 **33.** $20 + \ln 3$
35. 36 **37.** $8/\sqrt{3}$ **39.** $\frac{55}{63}$ **41.** $\frac{3}{4} - 2\ln 2$
43. $2\sinh 2$ **45.** $1 + \pi/4$ **47.** $4\sqrt{3} - 6$
49. $\pi/3$ **51.** $\pi/6$ **53.** -3.5 **55.** ≈ 1.36 **57.** $\frac{4}{3}$
59. The increase in the child's weight (in kilograms) between the ages of 5 and 10
61. Number of liters of oil leaked in the first 2 hours (120 minutes)
63. Increase in revenue when production is increased from 1000 to 5000 units

65. Total number of heart beats during the first 30 min of exercise
67. Newton-meters (or joules) **69.** (a) $-\frac{3}{2}$ m (b) $\frac{41}{6}$ m
71. (a) $v(t) = \frac{1}{2}t^2 + 4t + 5$ m/s (b) $416\frac{2}{3}$ m
73. $46\frac{2}{3}$ kg **75.** 2.3 km **77.** \$58,000
79. 12.1 m/s
81. 5443 bacteria **83.** 332.6 gigawatt-hours

EXERCISES 5.5 ■ PAGE 425

1. $\frac{1}{2}\sin 2x + C$ **3.** $\frac{2}{9}(x^3 + 1)^{3/2} + C$
5. $\frac{1}{4}\ln|x^4 - 5| + C$ **7.** $2\sin\sqrt{t} + C$
9. $-\frac{1}{3}(1 - x^2)^{3/2} + C$ **11.** $-\frac{1}{4}e^{-t^4} + C$
13. $-(3/\pi)\cos(\pi t/3) + C$ **15.** $\frac{1}{4}\ln|4x + 7| + C$
17. $\ln|1 + \sin\theta| + C$ **19.** $-\frac{1}{4}\cos^4\theta + C$
21. $\dfrac{1}{1 - e^u} + C$ **23.** $\frac{2}{3}\sqrt{3ax + bx^3} + C$
25. $\frac{1}{3}(\ln x)^3 + C$ **27.** $\frac{1}{4}\tan^4\theta + C$
29. $\dfrac{1}{12}\left(x^2 + \dfrac{2}{x}\right)^6 + C$ **31.** $\dfrac{2}{15}(2 + 3e^r)^{5/2} + C$
33. $\ln|\tan\theta| + C$ **35.** $\frac{1}{3}(\arctan x)^3 + C$
37. $-\dfrac{1}{\ln 5}\cos(5^t) + C$ **39.** $\frac{1}{5}\sin(1 + 5t) + C$
41. $-\frac{2}{3}(\cot x)^{3/2} + C$ **43.** $\frac{1}{3}\sinh^3 x + C$
45. $-\ln(1 + \cos^2 x) + C$ **47.** $\ln|\sin x| + C$
49. $\ln|\sin^{-1}x| + C$ **51.** $\tan^{-1}x + \frac{1}{2}\ln(1 + x^2) + C$
53. $\frac{1}{40}(2x + 5)^{10} - \frac{5}{36}(2x + 5)^9 + C$
55. $\frac{1}{8}(x^2 - 1)^4 + C$ **57.** $-e^{\cos x} + C$





59. $2/\pi$ **61.** $\frac{45}{28}$ **63.** $2/\sqrt{3} - 1$ **65.** $e - \sqrt{e}$
67. 0 **69.** 3 **71.** $\frac{1}{3}(2\sqrt{2} - 1)a^3$ **73.** $\frac{16}{15}$ **75.** 2
77. $\ln(e + 1)$ **79.** $\frac{1}{6}$ **81.** $\sqrt{3} - \frac{1}{3}$ **83.** 6π
85. All three areas are equal. **87.** ≈ 4512 L
89. $\dfrac{5}{4\pi}\left(1 - \cos\dfrac{2\pi t}{5}\right)$ L
91. $C_0(1 - e^{-30r/V})$; the total amount of urea removed from the blood in the first 30 minutes of dialysis treatment
93. 5 **99.** $\pi^2/4$

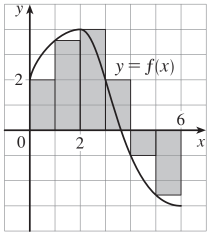
CHAPTER 5 REVIEW ■ PAGE 428

True-False Quiz

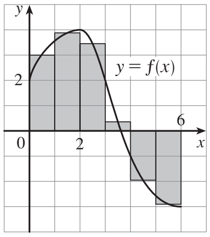
1. True **3.** True **5.** False **7.** True **9.** False
11. True **13.** False **15.** True **17.** False
19. False

Exercises

1. (a) 8



(b) 5.7



3. $\frac{1}{2} + \pi/4$ **5.** 3 **7.** f is c, f' is b, $\int_0^x f(t)\,dt$ is a.
9. 3, 0 **11.** $-\frac{13}{6}$ **13.** $\frac{9}{10}$ **15.** -76 **17.** $\frac{21}{4}$
19. Does not exist **21.** $\frac{1}{3}\sin 1$ **23.** 0
25. $\frac{1}{2}\ln(x^2 + 1) + C$ **27.** $\sqrt{x^2 + 4x} + C$
29. $[1/(2\pi)]\sin^2\pi t + C$ **31.** $2e^{\sqrt{x}} + C$
33. $-\frac{1}{2}[\ln(\cos x)]^2 + C$ **35.** $\frac{1}{4}\ln(1 + x^4) + C$
37. $\ln|1 + \sec\theta| + C$ **39.** $-\frac{3}{5}(1 - x)^{5/3} + \frac{3}{8}(1 - x)^{8/3} + C$
41. $\frac{23}{3}$ **43.** $2\sqrt{1 + \sin x} + C$ **45.** $\frac{64}{5}$ **47.** $\frac{124}{3}$
49. (a) 2 (b) 6 **51.** $F'(x) = x^2/(1 + x^3)$
53. $g'(x) = 4x^3\cos(x^8)$ **55.** $y' = \left(2e^x - e^{\sqrt{x}}\right)/(2x)$
57. $4 \leqslant \int_1^3 \sqrt{x^2 + 3}\,dx \leqslant 4\sqrt{3}$ **63.** 0.2810
65. Number of barrels of oil consumed from Jan. 1, 2015, through Jan. 1, 2020
67. 72,400 **69.** 3 **71.** $c \approx 1.62$
73. $f(x) = e^{2x}(2x - 1)/(1 - e^{-x})$

PROBLEMS PLUS ■ PAGE 433

1. $\pi/2$ **3.** $2k$ **5.** -1 **7.** e^{-2} **9.** $[-1, 2]$
11. (a) $\frac{1}{2}(n - 1)n$
(b) $\frac{1}{2}[\![b]\!](2b - [\![b]\!] - 1) - \frac{1}{2}[\![a]\!](2a - [\![a]\!] - 1)$
17. $y = -\dfrac{2b}{a^2}x^2 + \dfrac{3b}{a}x$ **19.** $2(\sqrt{2} - 1)$

CHAPTER 6

EXERCISES 6.1 ■ PAGE 442

1. (a) $\int_0^2 (2x - x^2)\,dx$ (b) $\frac{4}{3}$
3. (a) $\int_{-1}^1 (e^y - y^2 + 2)\,dy$ (b) $e - (1/e) + \frac{10}{3}$
5. 8 **7.** $\int_0^1 (3^x - 2^x)\,dx$ **9.** $\int_1^2 (-x^2 + 3x - 2)\,dx$
11. $\frac{23}{6}$ **13.** $\ln 2 - \frac{1}{2}$ **15.** $\frac{9}{2}$ **17.** $\frac{8}{3}$ **19.** 72
21. $\frac{32}{3}$ **23.** 4 **25.** 9 **27.** $\frac{1}{2}$ **29.** $6\sqrt{3}$
31. $\frac{13}{5}$ **33.** $(4/\pi) - \frac{1}{2}$ **35.** $\ln 2$
37. (a) 39 (b) 15 **39.** $\frac{1}{6}\ln 2$ **41.** $\frac{5}{2}$
43. $\frac{3}{2}\sqrt{3} - 1$ **45.** 0, 0.896; 0.037
47. −1.11, 1.25, 2.86; 8.38 **49.** 2.80123 **51.** 0.25142
53. $12\sqrt{6} - 9$ **55.** 36 m **57.** 4232 cm²

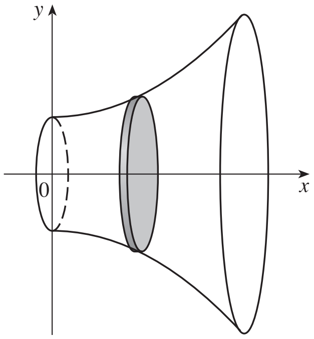
59. (a) Day 12 ($t \approx 11.26$) (b) Day 18 ($t \approx 17.18$)
(c) 706 (cells/mL) · days

61. (a) Car A (b) The distance by which car A is ahead of car B after 1 minute (c) Car A (d) $t \approx 2.2$ min

63. $\frac{24}{5}\sqrt{3}$ **65.** $4^{2/3}$ **67.** ± 6 **69.** $\frac{32}{27}$

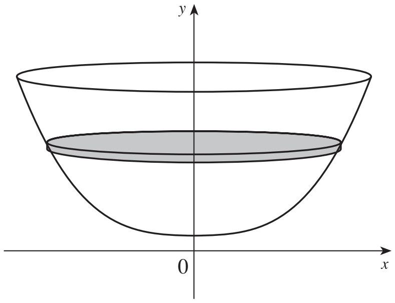
EXERCISES 6.2 ■ PAGE 456

1. (a)



(b) $\int_0^3 \pi(x^4 + 10x^2 + 25)\,dx$ (c) $1068\pi/5$

3. (a)

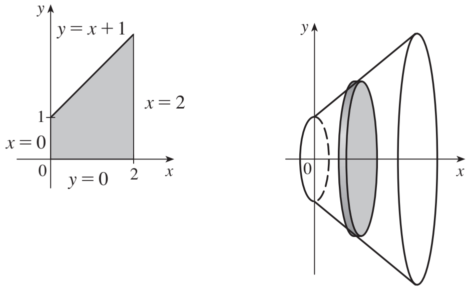


(b) $\int_1^9 \pi(y - 1)^{2/3}\,dy$ (c) $96\pi/5$

5. $\int_1^3 \pi(\ln x)^2\,dx$ **7.** $\int_0^2 \pi(8y - y^4)\,dy$

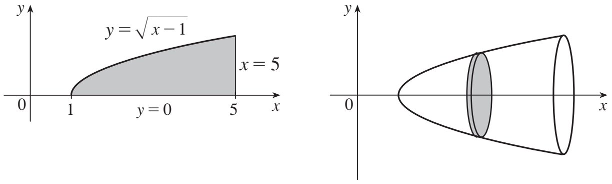
9. $\int_0^\pi \pi[(2 + \sin x)^2 - 4]\,dx$

11. $26\pi/3$



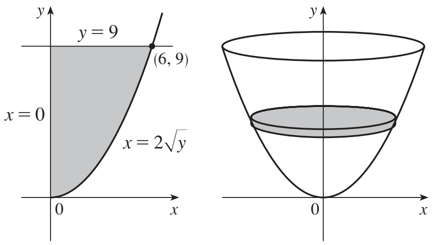
y
y = x + 1
x = 2
1
x = 0
0
y = 0
2
x
y
0
x

13. 8π



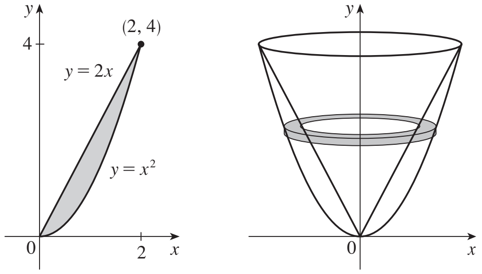
y
y = √(x − 1)
x = 5
0
1
y = 0
5
x
y
0
x

15. 162π



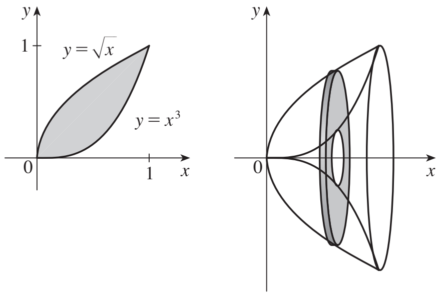
y
y = 9
(6, 9)
x = 0
x = 2√y
0
x
y
0
x

17. $8\pi/3$



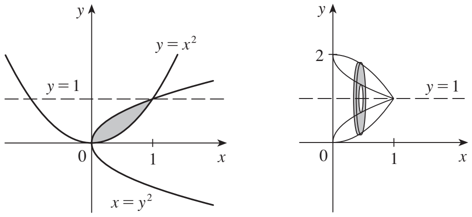
y
(2, 4)
4
y = 2x
y = x²
0
2
x
y
0
x

19. $5\pi/14$



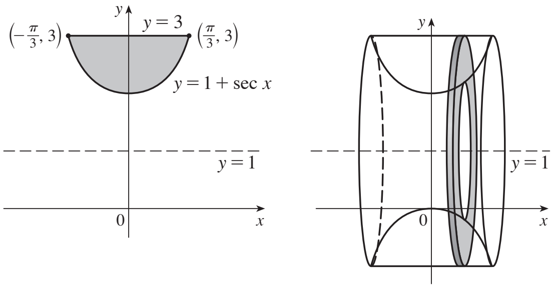
y
1
y = √x
y = x³
0
1
x
y
0
x

21. $11\pi/30$



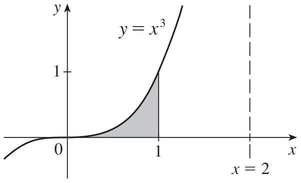
y
y = x²
y = 1
0
1
x
x = y²
y
2
y = 1
0
1
x

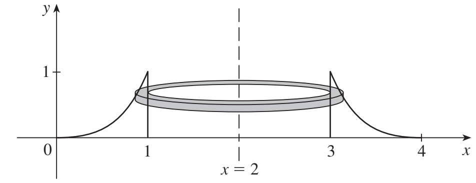
23. $2\pi\left(\frac{4}{3}\pi - \sqrt{3}\right)$



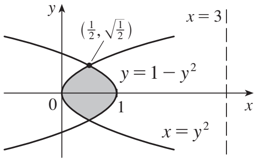
y
y = 3
(−π/3, 3)
(π/3, 3)
y = 1 + sec x
y = 1
0
x
y
y = 1
0
x

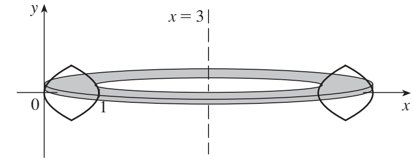
25. $3\pi/5$





27. $10\sqrt{2}\,\pi/3$

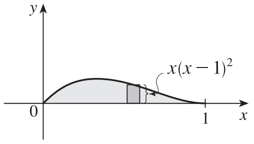


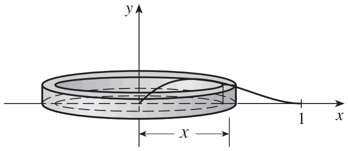


29. $\pi/3$ **31.** $\pi/3$ **33.** $\pi/3$
35. $13\pi/45$ **37.** $\pi/3$ **39.** $17\pi/45$
41. (a) $2\pi \int_0^1 e^{-2x^2}\,dx \approx 3.75825$
(b) $2\pi \int_0^1 \left(e^{-2x^2} + 2e^{-x^2}\right) dx \approx 13.14312$
43. (a) $2\pi \int_0^2 8\sqrt{1 - x^2/4}\ dx \approx 78.95684$
(b) $2\pi \int_0^1 8\sqrt{4 - 4y^2}\ dy \approx 78.95684$
45. $-4.091, -1.467, 1.091; 89.023$ **47.** $\frac{11}{8}\pi^2$
49. Solid obtained by rotating the region $0 \leqslant x \leqslant \pi/2$, $0 \leqslant y \leqslant \sin x$ about the x-axis
51. Solid obtained by rotating the region $0 \leqslant x \leqslant 1$, $x^3 \leqslant y \leqslant x^2$ about the x-axis
53. Solid obtained by rotating the region $0 \leqslant y \leqslant 4$, $0 \leqslant x \leqslant \sqrt{y}$ about the y-axis
55. 1110 cm^3 **57.** (a) 196 (b) 838
59. $\frac{1}{3}\pi r^2 h$ **61.** $\pi h^2\left(r - \frac{1}{3}h\right)$ **63.** $\frac{2}{3}b^2 h$
65. 10 cm^3 **67.** 24 **69.** $\frac{1}{3}$ **71.** $\frac{8}{15}$ **73.** $4\pi/15$
75. (a) $8\pi R \int_0^r \sqrt{r^2 - y^2}\ dy$ (b) $2\pi^2 r^2 R$
77. $\int_0^4 \frac{2}{\sqrt{3}} y\sqrt{16 - y^2}\,dy = \frac{128}{3\sqrt{3}}$ **81.** $\frac{5}{12}\pi r^3$
83. $8 \int_0^r \sqrt{R^2 - y^2}\sqrt{r^2 - y^2}\ dy$
87. (a) $93\pi/5$ (d) $\sqrt[3]{25{,}000/(93\pi)} \approx 4.41$

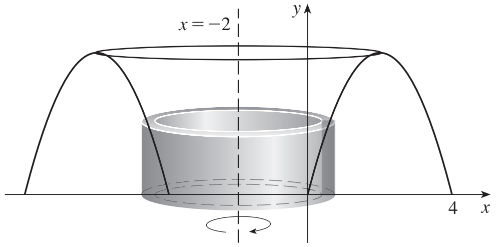
EXERCISES 6.3 ■ PAGE 464

1. Circumference $= 2\pi x$, height $= x(x - 1)^2$; $\pi/15$





3. (a) $\int_0^{\sqrt{\pi/2}} 2\pi x \cos(x^2)\,dx$ (b) π **5.** $\int_1^2 2\pi x \ln x\,dx$
7. $\int_0^{\pi/2} 2\pi(3 - y)\sin y\,dy$ **9.** $128\pi/5$ **11.** 6π
13. $\frac{2}{3}\pi\left(27 - 5\sqrt{5}\right)$ **15.** 4π **17.** 192π **19.** $16\pi/3$
21. $384\pi/5$
23. (a)



(b) $\int_0^4 2\pi(x + 2)(4x - x^2)\,dx$ (c) $256\pi/3$
25. $264\pi/5$ **27.** $8\pi/3$ **29.** $13\pi/3$
31. (a) $2\pi \int_0^2 x^2 e^{-x}\,dx$ (b) 4.06300
33. (a) $4\pi \int_{-\pi/2}^{\pi/2} (\pi - x)\cos^4 x\,dx$ (b) 46.50942
35. (a) $\int_0^{\pi} 2\pi(4 - y)\sqrt{\sin y}\ dy$ (b) 36.57476 **37.** 3.68
39. Solid obtained by rotating the region $0 \leqslant y \leqslant x^4$, $0 \leqslant x \leqslant 3$ about the y-axis
41. Solid obtained (using shells) by rotating the region $0 \leqslant x \leqslant 1/y^2$, $1 \leqslant y \leqslant 4$ about the line $y = -2$
43. 0, 2.175; 14.450 **45.** $\frac{1}{32}\pi^3$
47. (a) $\int_0^1 2\pi x\left(\frac{1}{1 + x^2} - \frac{x}{2}\right)dx$ (b) $\pi\left(\ln 2 - \frac{1}{3}\right)$
49. (a) $\int_0^{\pi} \pi \sin x\,dx$ (b) 2π
51. (a) $\int_0^{1/2} 2\pi(x + 2)(x^2 - x^3)\,dx$ (b) $59\pi/480$
53. 8π **55.** $4\sqrt{3}\,\pi$ **57.** $4\pi/3$
59. $117\pi/5$ **61.** $\frac{4}{3}\pi r^3$ **63.** $\frac{1}{3}\pi r^2 h$

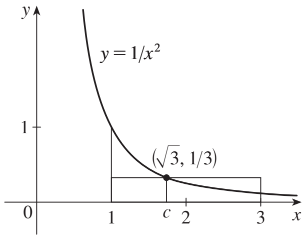
EXERCISES 6.4 ■ PAGE 470

1. 980 J **3.** 4.5 J **5.** 180 J **7.** $\frac{81}{16}$ J
9. (a) $\frac{25}{24} \approx 1.04$ J (b) 10.8 cm **11.** $W_2 = 3W_1$
13. (a) $\frac{6615}{8}$ J (b) ≈ 620 J **15.** 845,250 J
17. 73.5 J **19.** ≈ 3857 J **21.** 2450 J
23. $\approx 1.06 \times 10^6$ J **25.** $\approx 176{,}000$ J
27. ≈ 2.0 m **33.** ≈ 32.14 m/s
35. (a) $Gm_1m_2\left(\frac{1}{a} - \frac{1}{b}\right)$ (b) $\approx 8.50 \times 10^9$ J

EXERCISES 6.5 ■ PAGE 475

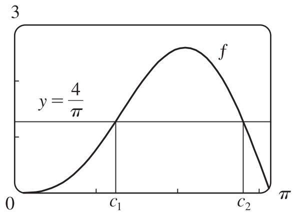
1. 7 **3.** $6/\pi$ **5.** $\frac{9}{2}\tan^{-1}2$ **7.** $2/(5\pi)$

9. (a) $\frac{1}{3}$ (b) $\sqrt{3}$ (c)



11. (a) $4/\pi$ (b) $\approx 1.24, 2.81$

(c)



15. $\frac{9}{8}$ **17.** $(10 + 8/\pi)$°C ≈ 12.5°C **19.** 6 kg/m

21. About 4056 million (or 4 billion) people

23. $5/(4\pi) \approx 0.40$ L

CHAPTER 6 REVIEW ■ PAGE 479

True-False Quiz

1. False **3.** False **5.** True **7.** False **9.** True

11. True

Exercises

1. $\frac{64}{3}$ **3.** $\frac{7}{12}$ **5.** $\frac{4}{3} + 4/\pi$ **7.** $64\pi/15$ **9.** $1656\pi/5$

11. $\frac{4}{3}\pi(2ah + h^2)^{3/2}$ **13.** $\int_{-\pi/3}^{\pi/3} 2\pi(\pi/2 - x)(\cos^2 x - \frac{1}{4})\,dx$

15. $189\pi/5$ **17.** (a) $2\pi/15$ (b) $\pi/6$ (c) $8\pi/15$

19. (a) 0.38 (b) 0.87

21. Solid obtained by rotating the region $0 \leqslant y \leqslant \cos x$, $0 \leqslant x \leqslant \pi/2$ about the y-axis

23. Solid obtained by rotating the region $0 \leqslant y \leqslant 2 - \sin x$, $0 \leqslant x \leqslant \pi$ about the x-axis

25. 36 **27.** $\frac{125}{3}\sqrt{3}$ m^3 **29.** 3.2 J

31. (a) 10,640 J (b) 0.7 m

33. $4/\pi$ **35.** (a) No (b) Yes (c) No (d) Yes

PROBLEMS PLUS ■ PAGE 481

1. $f(x) = \sqrt{2x/\pi}$ **3.** $y = \frac{32}{9}x^2$ **7.** $2/\sqrt{5}$

9. (a) $V = \int_0^h \pi[f(y)]^2\,dy$

(c) $f(y) = \sqrt{kA/(\pi C)}\,y^{1/4}$. Advantage: the markings on the container are equally spaced.

11. $b = 2a$ **13.** $B = 16A$

CHAPTER 7

EXERCISES 7.1 ■ PAGE 490

1. $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$ **3.** $\frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x + C$

5. $\frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + C$ **7.** $-\frac{1}{10}x\cos 10x + \frac{1}{100}\sin 10x + C$

9. $\frac{1}{2}w^2\ln w - \frac{1}{4}w^2 + C$

11. $(x^2 + 2x)\sin x + (2x + 2)\cos x - 2\sin x + C$

13. $x\cos^{-1}x - \sqrt{1 - x^2} + C$ **15.** $\frac{1}{5}t^5\ln t - \frac{1}{25}t^5 + C$

17. $-t\cot t + \ln|\sin t| + C$

19. $x(\ln x)^2 - 2x\ln x + 2x + C$

21. $\frac{1}{10}e^{3x}\sin x + \frac{3}{10}e^{3x}\cos x + C$

23. $\frac{1}{13}e^{2\theta}(2\sin 3\theta - 3\cos 3\theta) + C$

25. $z^3e^z - 3z^2e^z + 6ze^z - 6e^z + C$

27. $\frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{11}{27}e^{3x} + C$ **29.** $\dfrac{3}{\ln 3} - \dfrac{2}{(\ln 3)^2}$

31. $2\cosh 2 - \sinh 2$ **33.** $\frac{4}{5} - \frac{1}{5}\ln 5$ **35.** $-\pi/4$

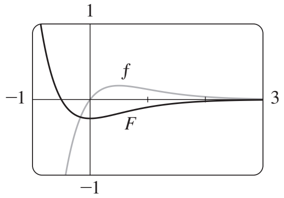
37. $2e^{-1} - 6e^{-5}$ **39.** $\frac{1}{2}\ln 2 - \frac{1}{2}$

41. $-\frac{1}{2}(1 + \cosh\pi) = -\frac{1}{4}(2 + e^{\pi} + e^{-\pi})$

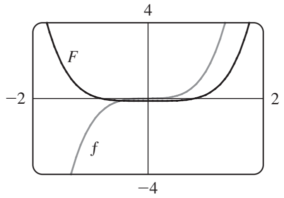
43. $2(\sqrt{x} - 1)e^{\sqrt{x}} + C$ **45.** $-\frac{1}{2} - \pi/4$

47. $\frac{1}{2}(x^2 - 1)\ln(1 + x) - \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{4} + C$

49. $-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$



51. $\frac{1}{3}x^2(1 + x^2)^{3/2} - \frac{2}{15}(1 + x^2)^{5/2} + C$



53. (b) $-\frac{1}{4}\cos x\sin^3 x + \frac{3}{8}x - \frac{3}{16}\sin 2x + C$

55. (b) $\frac{2}{3}, \frac{8}{15}$

61. $x[(\ln x)^3 - 3(\ln x)^2 + 6\ln x - 6] + C$

63. $\frac{16}{3}\ln 2 - \frac{29}{9}$ **65.** $-1.75119, 1.17210; 3.99926$

67. $4 - 8/\pi$ **69.** $2\pi e$

71. (a) $2\pi(2\ln 2 - \frac{3}{4})$ (b) $2\pi[(\ln 2)^2 - 2\ln 2 + 1]$

73. $xS(x) + \dfrac{1}{\pi}\cos(\frac{1}{2}\pi x^2) + C$

75. $2 - e^{-t}(t^2 + 2t + 2)$ m **77.** 2

79. (b) $-\dfrac{\ln x}{x} - \dfrac{1}{x} + C$

EXERCISES 7.2 ■ PAGE 498

1. $\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$ **3.** $\frac{1}{210}$

5. $-\frac{1}{14}\cos^7(2t) + \frac{1}{5}\cos^5(2t) - \frac{1}{6}\cos^3(2t) + C$

7. $\pi/4$ **9.** $3\pi/8$ **11.** $\pi/16$

13. $\frac{2}{7}(\cos\theta)^{7/2} - \frac{2}{3}(\cos\theta)^{3/2} + C$ **15.** $\frac{1}{4}\sec^4 x + C$

17. $\ln|\sin x| - \frac{1}{2}\sin^2 x + C$ **19.** $\frac{1}{2}\sin^4 x + C$

21. $\frac{1}{3}\sec^3 x + C$ **23.** $\tan x - x + C$

25. $\frac{1}{9}\tan^9 x + \frac{2}{7}\tan^7 x + \frac{1}{5}\tan^5 x + C$

27. $\frac{1}{3}\sec^3 x - \sec x + C$ **29.** $\frac{1}{8}\tan^8 x + \frac{1}{3}\tan^6 x + \frac{1}{4}\tan^4 x + C$

31. $\frac{1}{4}\sec^4 x - \tan^2 x + \ln|\sec x| + C$ **33.** $\frac{1}{2}\sin 2x + C$

35. $-\frac{1}{4} - \ln\left(\sqrt{2}/2\right)$ **37.** $\sqrt{3} - \frac{1}{3}\pi$

39. $\frac{22}{105}\sqrt{2} - \frac{8}{105}$ **41.** $\ln|\csc x - \cot x| + C$

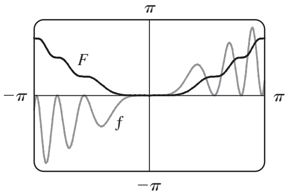
43. $-\frac{1}{6}\cos 3x - \frac{1}{26}\cos 13x + C$ **45.** $\frac{1}{15}$

47. $-1/(2t) + \frac{1}{4}\sin(2/t) + C$ **49.** $\frac{1}{2}\sqrt{2}$

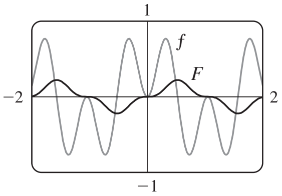
51. $\frac{1}{4}t^2 - \frac{1}{4}t\sin 2t - \frac{1}{8}\cos 2t + C$

53. $x\tan x - \ln|\sec x| - \frac{1}{2}x^2 + C$ **55.** $\csc x + \cot x + C$

57. $\frac{1}{4}x^2 - \frac{1}{4}\sin(x^2)\cos(x^2) + C$



59. $\frac{1}{6}\sin 3x - \frac{1}{18}\sin 9x + C$



61. $\frac{1}{8}\left(\sqrt{2} - 7I\right)$ **63.** 0 **65.** $\frac{1}{2}\pi - \frac{4}{3}$ **67.** 0

69. $\pi^2/4$ **71.** $\pi\left(2\sqrt{2} - \frac{5}{2}\right)$ **73.** $s = (1 - \cos^3\omega t)/(3\omega)$

EXERCISES 7.3 ■ PAGE 505

1. (a) $x = \tan\theta$ (b) $\int \tan^3\theta\sec\theta\,d\theta$

3. (a) $x = \sqrt{2}\sec\theta$ (b) $\int 2\sec^3\theta\,d\theta$

5. $-\sqrt{1 - x^2} + \frac{1}{3}(1 - x^2)^{3/2} + C$

7. $\sqrt{4x^2 - 25} - 5\sec^{-1}\left(\frac{2}{5}x\right) + C$

9. $\frac{1}{15}(16 + x^2)^{3/2}(3x^2 - 32) + C$

11. $\dfrac{1}{3}\dfrac{(x^2 - 1)^{3/2}}{x^3} + C$ **13.** $\dfrac{1}{\sqrt{2}\,a^2}$

15. $\frac{2}{3}\sqrt{3} - \frac{3}{4}\sqrt{2}$ **17.** $\frac{1}{12}$

19. $\frac{1}{6}\sec^{-1}(x/3) - \sqrt{x^2 - 9}/(2x^2) + C$

21. $\frac{1}{16}\pi a^4$ **23.** $\sqrt{x^2 - 7} + C$

25. $\ln\left|\left(\sqrt{1 + x^2} - 1\right)/x\right| + \sqrt{1 + x^2} + C$ **27.** $\frac{9}{500}\pi$

29. $\ln\left|\sqrt{x^2 + 2x + 5} + x + 1\right| + C$

31. $4\sin^{-1}\left(\dfrac{x - 1}{2}\right) + \frac{1}{4}(x - 1)^3\sqrt{3 + 2x - x^2}$
$- \frac{2}{3}(3 + 2x - x^2)^{3/2} + C$

33. $\frac{1}{2}(x + 1)\sqrt{x^2 + 2x} - \frac{1}{2}\ln\left|x + 1 + \sqrt{x^2 + 2x}\right| + C$

35. $\frac{1}{4}\sin^{-1}(x^2) + \frac{1}{4}x^2\sqrt{1 - x^4} + C$

39. $\frac{1}{6}\left(\sqrt{48} - \sec^{-1}7\right)$ **43.** $\frac{3}{8}\pi^2 + \frac{3}{4}\pi$

47. $2\pi^2 Rr^2$ **49.** $r\sqrt{R^2 - r^2} + \pi r^2/2 - R^2\arcsin(r/R)$

EXERCISES 7.4 ■ PAGE 515

1. (a) $\dfrac{A}{x - 3} + \dfrac{B}{x + 5}$ (b) $\dfrac{A}{x - 2} + \dfrac{B}{(x - 2)^2} + \dfrac{Cx + D}{x^2 + 2}$

3. (a) $\dfrac{A}{x} + \dfrac{B}{x - 1} + \dfrac{C}{x - 2}$

(b) $\dfrac{A}{x} + \dfrac{B}{2x - 1} + \dfrac{C}{(2x - 1)^2} + \dfrac{Dx + E}{x^2 + 3} + \dfrac{Fx + G}{(x^2 + 3)^2}$

5. (a) $\dfrac{A}{x} + \dfrac{B}{x - 1} + \dfrac{Cx + D}{x^2 + 1} + \dfrac{Ex + F}{(x^2 + 1)^2}$

(b) $1 + \dfrac{A}{x - 2} + \dfrac{B}{x + 3}$ **7.** $\ln|x - 1| - \ln|x + 4| + C$

9. $\frac{1}{2}\ln|2x + 1| + 2\ln|x - 1| + C$ **11.** $2\ln\frac{3}{2}$

13. $-\dfrac{1}{a}\ln|x| + \dfrac{1}{a}\ln|x - a| + C$

15. $\frac{1}{2}x^2 + x + \ln|x - 1| + C$

17. $\frac{27}{5}\ln 2 - \frac{9}{5}\ln 3$ $\left(\text{or } \frac{9}{5}\ln\frac{8}{3}\right)$

19. $\frac{1}{2} - 5\ln 2 + 3\ln 3$ $\left(\text{or } \frac{1}{2} + \ln\frac{27}{32}\right)$

21. $\dfrac{1}{4}\left[\ln|t + 1| - \dfrac{1}{t + 1} - \ln|t - 1| - \dfrac{1}{t - 1}\right] + C$

23. $\ln|x - 1| - \frac{1}{2}\ln(x^2 + 9) - \frac{1}{3}\tan^{-1}(x/3) + C$

25. $\frac{5}{2} - \ln 2 - \ln 3$ $\left(\text{or } \frac{5}{2} - \ln 6\right)$

27. $-2\ln|x + 1| + \ln(x^2 + 1) + 2\tan^{-1}x + C$

29. $\frac{1}{2}\ln(x^2 + 1) + \tan^{-1}x - \frac{1}{2}\tan^{-1}(x/2) + C$

31. $\frac{1}{2}\ln(x^2 + 2x + 5) + \frac{3}{2}\tan^{-1}\left(\dfrac{x + 1}{2}\right) + C$

33. $\frac{1}{3}\ln|x - 1| - \frac{1}{6}\ln(x^2 + x + 1) - \dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x + 1}{\sqrt{3}} + C$

35. $\frac{1}{4}\ln\frac{8}{3}$

37. $2\ln|x| + \frac{3}{2}\ln(x^2 + 1) + \frac{1}{2}\tan^{-1}x + \dfrac{x}{2(x^2 + 1)} + C$

39. $\frac{7}{8}\sqrt{2}\tan^{-1}\left(\dfrac{x - 2}{\sqrt{2}}\right) + \dfrac{3x - 8}{4(x^2 - 4x + 6)} + C$

41. $2\tan^{-1}\sqrt{x - 1} + C$

43. $-2\ln\sqrt{x} - \dfrac{2}{\sqrt{x}} + 2\ln\left(\sqrt{x} + 1\right) + C$

45. $\frac{3}{10}(x^2+1)^{5/3} - \frac{3}{4}(x^2+1)^{2/3} + C$

47. $2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln|\sqrt[6]{x} - 1| + C$

49. $4\ln|\sqrt{x} - 2| - 2\ln|\sqrt{x} - 1| + C$

51. $\ln\dfrac{(e^x+2)^2}{e^x+1} + C$

53. $\ln|\tan t + 1| - \ln|\tan t + 2| + C$

55. $x - \ln(e^x + 1) + C$

57. $\left(x - \frac{1}{2}\right)\ln(x^2 - x + 2) - 2x + \sqrt{7}\tan^{-1}\left(\dfrac{2x-1}{\sqrt{7}}\right) + C$

59. $-\frac{1}{2}\ln 3 \approx -0.55$

61. $\frac{1}{2}\ln\left|\dfrac{x-2}{x}\right| + C$ **65.** $\frac{1}{5}\ln\left|\dfrac{2\tan(x/2) - 1}{\tan(x/2) + 2}\right| + C$

67. $4\ln\frac{2}{3} + 2$ **69.** $-1 + \frac{11}{3}\ln 2$

71. $t = \ln\dfrac{10{,}000}{P} + 11\ln\dfrac{P - 9000}{1000}$

73. (a) $\dfrac{24{,}110}{4879}\dfrac{1}{5x+2} - \dfrac{668}{323}\dfrac{1}{2x+1} - \dfrac{9438}{80{,}155}\dfrac{1}{3x-7} + \dfrac{1}{260{,}015}\dfrac{22{,}098x + 48{,}935}{x^2+x+5}$

(b) $\dfrac{4822}{4879}\ln|5x+2| - \dfrac{334}{323}\ln|2x+1| - \dfrac{3146}{80{,}155}\ln|3x-7| + \dfrac{11{,}049}{260{,}015}\ln(x^2+x+5) + \dfrac{75{,}772}{260{,}015\sqrt{19}}\tan^{-1}\dfrac{2x+1}{\sqrt{19}} + C$

The CAS omits the absolute value signs and the constant of integration.

77. $\dfrac{1}{a^n(x-a)} - \dfrac{1}{a^n x} - \dfrac{1}{a^{n-1}x^2} - \cdots - \dfrac{1}{ax^n}$

EXERCISES 7.5 ■ PAGE 521

1. (a) $\frac{1}{2}\ln(1+x^2) + C$ (b) $\tan^{-1}x + C$
(c) $\frac{1}{2}\ln|1+x| - \frac{1}{2}\ln|1-x| + C$

3. (a) $\frac{1}{2}(\ln x)^2 + C$ (b) $x\ln(2x) - x + C$
(c) $\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$

5. (a) $\frac{1}{2}\ln|x-3| - \frac{1}{2}\ln|x-1| + C$ (b) $-\dfrac{1}{x-2} + C$
(c) $\tan^{-1}(x-2) + C$

7. (a) $\frac{1}{3}e^{x^3} + C$ (b) $e^x(x^2 - 2x + 2) + C$
(c) $\frac{1}{2}e^{x^2}(x^2 - 1) + C$

9. $-\ln(1 - \sin x) + C$ **11.** $\frac{32}{3}\ln 2 - \frac{28}{9}$

13. $\ln y\,[\ln(\ln y) - 1] + C$ **15.** $\frac{1}{6}\tan^{-1}\left(\frac{1}{3}x^2\right) + C$

17. $\frac{4}{5}\ln 2 + \frac{1}{5}\ln 3$ $\left(\text{or } \frac{1}{5}\ln 48\right)$ **19.** $\frac{1}{2}\sec^{-1}x + \dfrac{\sqrt{x^2-1}}{2x^2} + C$

21. $-\frac{1}{4}\cos^4 x + C$ **23.** $x\sec x - \ln|\sec x + \tan x| + C$

25. $\frac{1}{4}\pi^2$ **27.** $e^{e^x} + C$ **29.** $(x+1)\arctan\sqrt{x} - \sqrt{x} + \mathrm{C}$

31. $\frac{4097}{45}$ **33.** $4 - \ln 4$ **35.** $x - \ln(1 + e^x) + C$

37. $x\ln\left(x + \sqrt{x^2-1}\right) - \sqrt{x^2-1} + C$

39. $\sin^{-1}x - \sqrt{1-x^2} + C$

41. $2\sin^{-1}\left(\dfrac{x+1}{2}\right) + \dfrac{x+1}{2}\sqrt{3 - 2x - x^2} + C$

43. 0 **45.** $\frac{1}{4}$ **47.** $\ln|\sec\theta - 1| - \ln|\sec\theta| + C$

49. $\theta\tan\theta - \frac{1}{2}\theta^2 - \ln|\sec\theta| + C$ **51.** $\frac{2}{3}\tan^{-1}(x^{3/2}) + C$

53. $\frac{2}{3}x^{3/2} - x + 2\sqrt{x} - 2\ln(1 + \sqrt{x}) + C$

55. $\ln|x-1| - 3(x-1)^{-1} - \frac{3}{2}(x-1)^{-2} - \frac{1}{3}(x-1)^{-3} + C$

57. $\ln\left|\dfrac{\sqrt{4x+1} - 1}{\sqrt{4x+1} + 1}\right| + C$

59. $-\ln\left|\dfrac{\sqrt{4x^2+1} + 1}{2x}\right| + C$

61. $\dfrac{1}{m}x^2\cosh mx - \dfrac{2}{m^2}x\sinh mx + \dfrac{2}{m^3}\cosh mx + C$

63. $2\ln\sqrt{x} - 2\ln\left(1 + \sqrt{x}\right) + C$

65. $\frac{3}{7}(x+c)^{7/3} - \frac{3}{4}c(x+c)^{4/3} + C$

67. $\dfrac{1}{32}\ln\left|\dfrac{x-2}{x+2}\right| - \dfrac{1}{16}\tan^{-1}\left(\dfrac{x}{2}\right) + C$

69. $\csc\theta - \cot\theta + C$ or $\tan(\theta/2) + C$

71. $2\left(x - 2\sqrt{x} + 2\right)e^{\sqrt{x}} + C$

73. $-\tan^{-1}(\cos^2 x) + C$ **75.** $\frac{2}{3}[(x+1)^{3/2} - x^{3/2}] + C$

77. $\sqrt{2} - 2/\sqrt{3} + \ln\left(2 + \sqrt{3}\right) - \ln\left(1 + \sqrt{2}\right)$

79. $e^x - \ln(1 + e^x) + C$

81. $-\sqrt{1-x^2} + \frac{1}{2}(\arcsin x)^2 + C$ **83.** $\ln|\ln x - 1| + C$

85. $2(x-2)\sqrt{1+e^x} + 2\ln\dfrac{\sqrt{1+e^x} + 1}{\sqrt{1+e^x} - 1} + C$

87. $\frac{1}{3}x\sin^3 x + \frac{1}{3}\cos x - \frac{1}{9}\cos^3 x + C$

89. $2\sqrt{1 + \sin x} + C$ **91.** $2\sqrt{2}$

93. $\left(3 - \sqrt{3}\right)/2$ or $1 - \sqrt{1 - \left(\sqrt{3}/2\right)}$ **95.** $xe^{x^2} + C$

EXERCISES 7.6 ■ PAGE 527

1. $-\frac{5}{21}$ **3.** $\frac{1}{2}x^2\sin^{-1}(x^2) + \frac{1}{2}\sqrt{1-x^4} + C$

5. $\frac{1}{4}y^2\sqrt{4+y^4} - \ln\left(y^2 + \sqrt{4+y^4}\right) + C$

7. $\dfrac{\pi}{8}\arctan\dfrac{\pi}{4} - \frac{1}{4}\ln\left(1 + \frac{1}{16}\pi^2\right)$ **9.** $\frac{1}{6}\ln\left|\dfrac{\sin x - 3}{\sin x + 3}\right| + C$

11. $-\dfrac{\sqrt{9x^2+4}}{x} + 3\ln\left(3x + \sqrt{9x^2+4}\right) + C$

13. $5\pi/16$ **15.** $2\sqrt{x}\arctan\sqrt{x} - \ln(1+x) + C$

17. $-\ln|\sinh(1/y)| + C$

19. $\dfrac{2y-1}{8}\sqrt{6 + 4y - 4y^2} + \dfrac{7}{8}\sin^{-1}\left(\dfrac{2y-1}{\sqrt{7}}\right) - \frac{1}{12}(6 + 4y - 4y^2)^{3/2} + C$

21. $\frac{1}{9}\sin^3 x\,[3\ln(\sin x) - 1] + C$

23. $-\ln\left(\cos^2\theta + \sqrt{\cos^4\theta + 4}\right) + C$

25. $\frac{1}{8}e^{2x}\left(4x^3 - 6x^2 + 6x - 3\right) + C$

27. $\frac{1}{15}\sin y\left(3\cos^4 y + 4\cos^2 y + 8\right) + C$

29. $-\frac{1}{2}x^{-2}\cos^{-1}(x^{-2}) + \frac{1}{2}\sqrt{1 - x^{-4}} + C$

31. $\sqrt{e^{2x} - 1} - \cos^{-1}(e^{-x}) + C$

33. $\frac{1}{5}\ln\left|x^5 + \sqrt{x^{10} - 2}\right| + C$ **35.** $\frac{3}{8}\pi^2$

39. $\frac{1}{3}\tan x\sec^2 x + \frac{2}{3}\tan x + C$

41. $\frac{1}{4}x(x^2 + 2)\sqrt{x^2 + 4} - 2\ln\left(\sqrt{x^2 + 4} + x\right) + C$

43. $\frac{1}{4}\cos^3 x\sin x + \frac{3}{8}x + \frac{3}{8}\sin x\cos x + C$

45. $-\ln|\cos x| - \frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C$

47. (a) $-\ln\left|\dfrac{1 + \sqrt{1 - x^2}}{x}\right| + C$;

both have domain $(-1, 0) \cup (0, 1)$

EXERCISES 7.7 ■ PAGE 539

1. (a) $L_2 = 6$, $R_2 = 12$, $M_2 \approx 9.6$
(b) L_2 is an underestimate, R_2 and M_2 are overestimates.
(c) $T_2 = 9 < I$ (d) $L_n < T_n < I < M_n < R_n$

3. (a) $T_4 \approx 0.895759$ (underestimate)
(b) $M_4 \approx 0.908907$ (overestimate);
$T_4 < I < M_4$

5. (a) $M_6 \approx 3.177769$, $E_M \approx -0.036176$
(b) $S_6 \approx 3.142949$, $E_S \approx -0.001356$

7. (a) 1.116993 (b) 1.108667 (c) 1.111363

9. (a) 1.777722 (b) 0.784958 (c) 0.780895

11. (a) 10.185560 (b) 10.208618 (c) 10.201790

13. (a) −2.364034 (b) −2.310690 (c) −2.346520

15. (a) 0.243747 (b) 0.243748 (c) 0.243751

17. (a) 8.814278 (b) 8.799212 (c) 8.804229

19. (a) $T_8 \approx 0.902333$, $M_8 \approx 0.905620$
(b) $|E_T| \leqslant 0.0078$, $|E_M| \leqslant 0.0039$
(c) $n = 71$ for T_n, $n = 50$ for M_n

21. (a) $T_{10} \approx 1.983524$, $E_T \approx 0.016476$;
$M_{10} \approx 2.008248$, $E_M \approx -0.008248$;
$S_{10} \approx 2.000110$, $E_S \approx -0.000110$
(b) $|E_T| \leqslant 0.025839$, $|E_M| \leqslant 0.012919$, $|E_S| \leqslant 0.000170$
(c) $n = 509$ for T_n, $n = 360$ for M_n, $n = 22$ for S_n

23. (a) 2.8 (b) 7.954926518 (c) 0.2894
(d) 7.954926521 (e) Actual error is much smaller.
(f) 10.9 (g) 7.953789422 (h) 0.0593
(i) Actual error is smaller. (j) $n \geqslant 50$

25.

n	L_n	R_n	T_n	M_n
5	0.742943	1.286599	1.014771	0.992621
10	0.867782	1.139610	1.003696	0.998152
20	0.932967	1.068881	1.000924	0.999538

n	E_L	E_R	E_T	E_M
5	0.257057	−0.286599	−0.014771	0.007379
10	0.132218	−0.139610	−0.003696	0.001848
20	0.067033	−0.068881	−0.000924	0.000462

Observations are the same as those following Example 1.

27.

n	T_n	M_n	S_n
6	6.695473	6.252572	6.403292
12	6.474023	6.363008	6.400206

n	E_T	E_M	E_S
6	−0.295473	0.147428	−0.003292
12	−0.074023	0.036992	−0.000206

Observations are the same as those following Example 1.

29. (a) 19 (b) 18.6 (c) $18.\overline{6}$

31. (a) 14.4 (b) 0.5

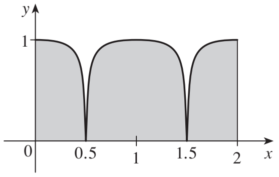
33. 21.6 Degrees Celsius **35.** 18.8 m/s

37. 10,177 megawatt-hours

39. (a) 190 (b) 828

41. 28 **43.** 59.4

45.



EXERCISES 7.8 ■ PAGE 549

Abbreviations: C, convergent; D, divergent

1. (a), (c) Infinite discontinuity (b), (d) Infinite interval

3. $\frac{1}{2} - 1/(2t^2)$; 0.495, 0.49995, 0.4999995; 0.5

5. 1 **7.** $\frac{1}{2}$ **9.** D **11.** 2 **13.** $-\frac{1}{4}$ **15.** $\frac{11}{6}$

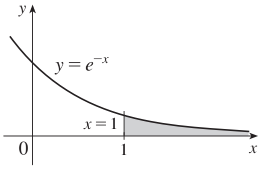
17. $\frac{1}{2}$ **19.** 0 **21.** D **23.** D **25.** ln 2

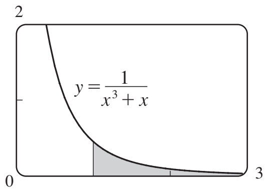
27. $-\frac{1}{4}$ **29.** D **31.** $-\pi/8$ **33.** 2

35. D **37.** $\frac{32}{3}$ **39.** D **41.** $\frac{9}{2}$ **43.** D **45.** $-\frac{1}{4}$

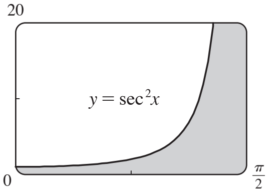
47. $-2/e$

49. $1/e$ **51.** $\frac{1}{2}\ln 2$





53. Infinite area

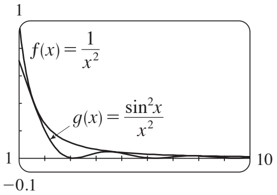


55. (a)

t	$\int_1^t [(\sin^2 x)/x^2]\,dx$
2	0.447453
5	0.577101
10	0.621306
100	0.668479
1,000	0.672957
10,000	0.673407

It appears that the integral is convergent.

(c)

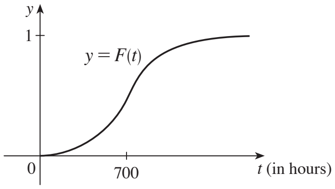


57. C **59.** D **61.** D **63.** D **65.** D **67.** π

69. $p < 1,\ 1/(1 - p)$ **71.** $p > -1,\ -1/(p + 1)^2$

75. π **77.** $\sqrt{2GM/R}$

79. (a)



(b) The rate at which the fraction $F(t)$ increases as t increases

(c) 1; all bulbs burn out eventually

81. $\gamma = \dfrac{cN}{\lambda(k + \lambda)}$ **83.** 1000

85. (a) $F(s) = 1/s,\ s > 0$ (b) $F(s) = 1/(s - 1),\ s > 1$
(c) $F(s) = 1/s^2,\ s > 0$

91. $C = 1$; ln 2 **93.** No

CHAPTER 7 REVIEW ■ PAGE 553

True-False Quiz

1. True **3.** False **5.** False **7.** False
9. False **11.** True **13.** (a) True (b) False
15. False **17.** False

Exercises

1. $\frac{7}{2} + \ln 2$ **3.** $e^{\sin x} + C$ **5.** $\ln|2t + 1| - \ln|t + 1| + C$

7. $\frac{2}{15}$ **9.** $-\cos(\ln t) + C$

11. $\frac{1}{4}x^2[2(\ln x)^2 - 2\ln x + 1] + C$ **13.** $\sqrt{3} - \frac{1}{3}\pi$

15. $3e^{\sqrt[3]{x}}(x^{2/3} - 2x^{1/3} + 2) + C$

17. $\frac{1}{6}[2x^3 \tan^{-1}x - x^2 + \ln(1 + x^2)] + C$

19. $-\frac{1}{2}\ln|x| + \frac{3}{2}\ln|x + 2| + C$

21. $x \sinh x - \cosh x + C$

23. $\ln\left|x - 2 + \sqrt{x^2 - 4x}\right| + C$

25. $\frac{1}{18}\ln(9x^2 + 6x + 5) + \frac{1}{9}\tan^{-1}\left[\frac{1}{2}(3x + 1)\right] + C$

27. $\sqrt{2} + \ln(\sqrt{2} + 1)$ **29.** $\ln\left|\dfrac{\sqrt{x^2 + 1} - 1}{x}\right| + C$

31. $-\cos\left(\sqrt{1 + x^2}\right) + C$

33. $\frac{3}{2}\ln(x^2 + 1) - 3\tan^{-1}x + \sqrt{2}\tan^{-1}\left(x/\sqrt{2}\right) + C$

35. $\frac{2}{5}$ **37.** 0 **39.** $6 - \frac{3}{2}\pi$

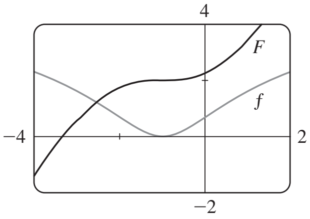
41. $\dfrac{x}{\sqrt{4 - x^2}} - \sin^{-1}\left(\dfrac{x}{2}\right) + C$

43. $4\sqrt{1 + \sqrt{x}} + C$ **45.** $\frac{1}{2}\sin 2x - \frac{1}{8}\cos 4x + C$

47. $\frac{1}{8}e - \frac{1}{4}$ **49.** $\tan^{-1}\left(\frac{1}{2}\sqrt{e^x - 4}\right) + C$ **51.** $\frac{1}{36}$

53. D **55.** $4\ln 4 - 8$ **57.** $-\frac{4}{3}$ **59.** $\pi/4$

61. $(x + 1)\ln(x^2 + 2x + 2) + 2\arctan(x + 1) - 2x + C$



63. 0

65. $\frac{1}{4}(2x - 1)\sqrt{4x^2 - 4x - 3}$
$- \ln\left|2x - 1 + \sqrt{4x^2 - 4x - 3}\right| + C$

67. $\frac{1}{2}\sin x\sqrt{4 + \sin^2 x} + 2\ln\left(\sin x + \sqrt{4 + \sin^2 x}\right) + C$

71. No

73. (a) 1.925444 (b) 1.920915 (c) 1.922470

75. (a) 0.01348, $n \geqslant 368$ (b) 0.00674, $n \geqslant 260$

77. 13.7 km

79. (a) 3.8 (b) 1.786721, 0.000646 (c) $n \geqslant 30$

81. (a) D (b) C

83. 2 **85.** $\frac{3}{16}\pi^2$

PROBLEMS PLUS ■ PAGE 557

1. About 4.7 centimeters from the center **3.** 0

9. $f(\pi) = -\pi/2$ **13.** $(b^b a^{-a})^{1/(b-a)}e^{-1}$ **15.** $\frac{1}{8}\pi - \frac{1}{12}$

17. $2 - \sin^{-1}(2/\sqrt{5})$

CHAPTER 8

EXERCISES 8.1 ■ PAGE 565

1. $4\sqrt{5}$ **3.** $\int_0^2 \sqrt{1 + 9x^4}\,dx$ **5.** $\int_1^4 \sqrt{1 + \left(1 - \dfrac{1}{x}\right)^2}\,dx$

7. $\int_0^{\pi/2} \sqrt{1 + \cos^2 y}\,dy$ **9.** $2\sqrt{3} - \frac{2}{3}$ **11.** $\frac{5}{3}$ **13.** $\frac{59}{24}$

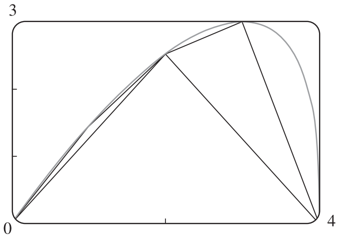
15. $\frac{1}{2}\left[\ln\left(1\sqrt{3}\right) - \ln\left(\sqrt{2} - 1\right)\right]$ **17.** $\ln\left(\sqrt{2} + 1\right)$

19. $\frac{32}{3}$ **21.** $\frac{3}{4} + \frac{1}{2}\ln 2$ **23.** $\ln 3 - \frac{1}{2}$

25. $\sqrt{2} + \ln\left(1 + \sqrt{2}\right)$ **27.** 10.0556 **29.** 3.0609

31. 1.0054 **33.** 15.498085; 15.374568

35. (a), (b)



$L_1 = 4$, $L_2 \approx 6.43$, $L_4 \approx 7.50$

(c) $\int_0^4 \sqrt{1 + [4(3 - x)/(3(4 - x)^{2/3})]^2}\,dx$ (d) 7.7988

37. $\sqrt{1 + e^4} - \ln\left(1 + \sqrt{1 + e^4}\right) + 2 - \sqrt{2} + \ln\left(1 + \sqrt{2}\right)$

39. 6 **41.** $s(x) = \frac{2}{27}\left[(1 + 9x)^{3/2} - 10\sqrt{10}\right]$

43. $s(x) = 2\sqrt{2}\left(\sqrt{1 + x} - 1\right)$ **45.** 209.1 m

47. 62.55 cm **49.** ≈7.42 m above the ground **53.** 12.4

EXERCISES 8.2 ■ PAGE 573

1. (a) $\int_1^8 2\pi\sqrt[3]{x}\sqrt{1 + \frac{1}{9}x^{-4/3}}\,dx$ (b) $\int_1^2 2\pi y\sqrt{1 + 9y^4}\,dy$

3. (a) $\int_0^{\ln 3} \pi(e^x - 1)\sqrt{1 + \frac{1}{4}e^{2x}}\,dx$

(b) $\displaystyle\int_0^1 2\pi y\sqrt{1 + \frac{4}{(2y + 1)^2}}\,dy$

5. (a) $\displaystyle\int_1^8 2\pi x\sqrt{1 + \frac{16}{x^4}}\,dx$ (b) $\displaystyle\int_{1/2}^4 \frac{8\pi}{y}\sqrt{1 + \frac{16}{y^4}}\,dy$

7. (a) $\int_0^{\pi/2} 2\pi x\sqrt{1 + \cos^2 x}\,dx$

(b) $\displaystyle\int_1^2 2\pi\sin^{-1}(y - 1)\sqrt{1 + \frac{1}{2y - y^2}}\,dy$

9. $\frac{1}{27}\pi\left(145\sqrt{145} - 1\right)$ **11.** $\frac{1}{6}\pi\left(17\sqrt{17} - 5\sqrt{5}\right)$

13. $\pi\sqrt{5} + 4\pi\ln\left(\dfrac{1 + \sqrt{5}}{2}\right)$ **15.** $\frac{21}{2}\pi$ **17.** $\frac{3712}{15}\pi$

19. πa^2 **21.** $\int_{-1}^1 2\pi e^{-x^2}\sqrt{1 + 4x^2 e^{-2x^2}}\,dx$; 11.0753

23. $\int_0^1 2\pi(y + y^3)\sqrt{1 + (1 + 3y^2)^2}\,dy$; 13.5134

25. $\int_1^4 2\pi y\sqrt{1 + [2y + (1/y)]^2}\,dy$; 286.9239

27. $\frac{1}{4}\pi\left[4\ln\left(\sqrt{17} + 4\right) - 4\ln\left(\sqrt{2} + 1\right) - \sqrt{17} + 4\sqrt{2}\right]$

29. $\frac{1}{6}\pi\left[\ln\left(\sqrt{10} + 3\right) + 3\sqrt{10}\right]$ **31.** 1,230,507

35. (a) $\frac{1}{3}\pi a^2$ (b) $\frac{56}{45}\pi\sqrt{3}\,a^2$

37. (a) $2\pi\left[b^2 + \dfrac{a^2 b\sin^{-1}\left(\sqrt{a^2 - b^2}/a\right)}{\sqrt{a^2 - b^2}}\right]$

(b) $2\pi a^2 + \dfrac{2\pi ab^2}{\sqrt{a^2 - b^2}}\ln\dfrac{a + \sqrt{a^2 - b^2}}{b}$

39. (a) $\int_a^b 2\pi[c - f(x)]\sqrt{1 + [f'(x)]^2}\,dx$

(b) $\int_0^4 2\pi\left(4 - \sqrt{x}\right)\sqrt{1 + 1/(4x)}\,dx \approx 80.6095$

41. $4\pi^2 r^2$ **45.** Both equal $\pi\int_a^b \left(e^{x/2} + e^{-x/2}\right)^2 dx$.

EXERCISES 8.3 ■ PAGE 584

1. (a) 915.5 kg/m^2 (b) 8340 N (c) 2502 N

3. 31136 N **5.** $\approx 2.36 \times 10^7$ N **7.** 470,400 N

9. 1793 kg **11.** $\frac{2}{3}\delta a h^2$ **13.** ≈9450 N

15. (a) ≈314 N (b) ≈353 N

17. (a) 4.9×10^4 N (shallow end), $\approx$ $\times\ 10^5$ N (deep end), and $\approx 4.2 \times 10^5$ N (one of the sides)

(b) 3.9×10^6 N (bottom of the pool) **19.** 8372 kg **21.** 330; 22

23. 23; −20; (−1, 1.15) **25.** $\left(\frac{2}{3}, \frac{4}{3}\right)$ **27.** $\left(\frac{3}{2}, \frac{3}{5}\right)$

29. $\left(\frac{9}{20}, \frac{9}{20}\right)$ **31.** $\left(\pi - \frac{3}{2}\sqrt{3}, \frac{3}{8}\sqrt{3}\right)$ **33.** $\left(\frac{8}{5}, -\frac{1}{2}\right)$

35. $\left(\dfrac{28}{3(\pi + 2)}, \dfrac{10}{3(\pi + 2)}\right)$ **37.** $\left(-\frac{1}{5}, -\frac{12}{35}\right)$

41. $\left(0, \frac{1}{12}\right)$ **45.** $\frac{1}{3}\pi r^2 h$ **47.** $\left(\dfrac{8}{\pi}, \dfrac{8}{\pi}\right)$

49. $4\pi^2 rR$

EXERCISES 8.4 ■ PAGE 590

1. \$21,104 **3.** \$140,000; \$60,000 **5.** \$11,332.78

7. $p = 25 - \frac{1}{30}x$; \$1500 **9.** \$6.67 **11.** \$55,735

13. (a) 3800 (b) \$324,900

15. $\frac{2}{3}\left(16\sqrt{2} - 8\right) \approx$ \$9.75 million

17. \$65,230.48 **19.** $\dfrac{(1 - k)(b^{2-k} - a^{2-k})}{(2 - k)(b^{1-k} - a^{1-k})}$

21. $\approx 1.19 \times 10^{-4}$ cm^3/s **23.** ≈6.59 L/min

25. 5.77 L/min

EXERCISES 8.5 ■ PAGE 598

1. (a) The probability that a randomly chosen tire will have a lifetime between 50,000 and 65,000 kilometers

(b) The probability that a randomly chosen tire will have a lifetime of at least 40,000 kilometers

3. (a) $f(x) \geqslant 0$ for all x and $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (b) $\frac{17}{81}$

5. (a) $1/\pi$ (b) $\frac{1}{2}$

7. (a) $f(x) \geqslant 0$ for all x and $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (b) 5

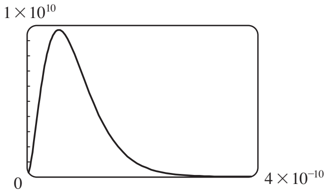
11. (a) ≈0.465 (b) ≈0.153 (c) About 4.8 s

13. (a) $\frac{19}{32}$ (b) 40 min **15.** ≈36%

17. (a) 0.0668 (b) ≈5.21% **19.** ≈0.9545

21. (b) 0; a_0

(c)



(d) $1 - 41e^{-8} \approx 0.986$ (e) $\frac{3}{2}a_0$

CHAPTER 8 REVIEW ■ PAGE 600

True-False Quiz

1. True **3.** False **5.** True **7.** True

Exercises

1. $\frac{1}{54}(109\sqrt{109} - 1)$ **3.** $\frac{53}{6}$

5. (a) 3.5121 (b) 22.1391 (c) 29.8522

7. 3.8202 **9.** $\frac{124}{5}$ **11.** 6533 N **13.** $\left(\frac{4}{3}, \frac{4}{3}\right)$

15. $\left(\frac{8}{5}, 1\right)$ **17.** $2\pi^2$ **19.** \$7166.67

21. (a) $f(x) \geqslant 0$ for all x and $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(b) ≈ 0.3455 (c) 5; yes

23. (a) $1 - e^{-3/8} \approx 0.313$ (b) $e^{-5/4} \approx 0.287$

(c) $8 \ln 2 \approx 5.55$ min

PROBLEMS PLUS ■ PAGE 602

1. $\frac{2}{3}\pi - \frac{1}{2}\sqrt{3}$

3. (a) $2\pi r(r \pm d)$ (b) $\approx 8.69 \times 10^6$ km²

(c) $\approx 2.03 \times 10^8$ km²

5. (a) $P(z) = P_0 + g \int_0^z \rho(x)\,dx$

(b) $(P_0 - \rho_0 g H)(\pi r^2) + \rho_0 g H e^{L/H} \int_{-r}^{r} e^{x/H} \cdot 2\sqrt{r^2 - x^2}\,dx$

7. Height $\sqrt{2}\,b$, volume $\left(\frac{28}{27}\sqrt{6} - 2\right)\pi b^3$ **9.** 0.14 m

11. $2/\pi$; $1/\pi$ **13.** $(0, -1)$

CHAPTER 9

EXERCISES 9.1 ■ PAGE 610

1. $dr/dt = k/r$ **3.** $dv/dt = k(M - v)$

5. $dy/dt = k(N - y)$ **7.** Yes **9.** No **11.** Yes

15. (a) $\frac{1}{2}, -1$ **17.** (d)

19. (a) It must be either 0 or decreasing

(c) $y = 0$ (d) $y = 1/(x + 2)$

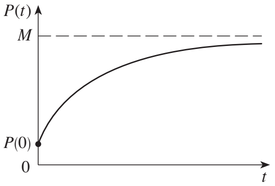
21. (a) $0 < P < 4200$ (b) $P > 4200$

(c) $P = 0$, $P = 4200$

25. (a) III (b) I (c) IV (d) II

27. (a) At the beginning; stays positive, but decreases

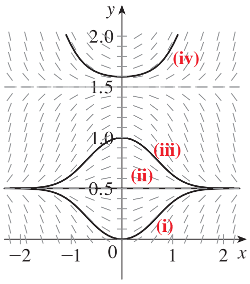
(c)



29. It approaches 0 as c approaches c_s.

EXERCISES 9.2 ■ PAGE 619

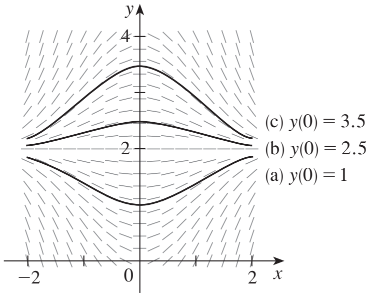
1. (a)



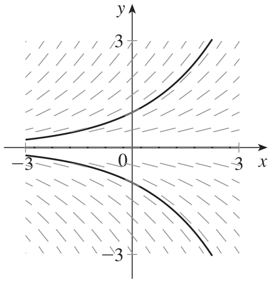
(b) $y = 0.5$, $y = 1.5$

3. III **5.** IV

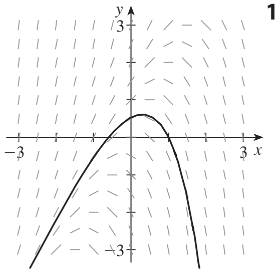
7.



9.



11.



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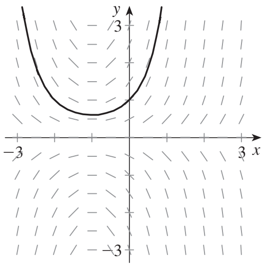

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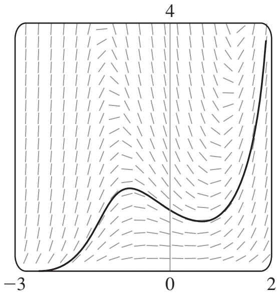
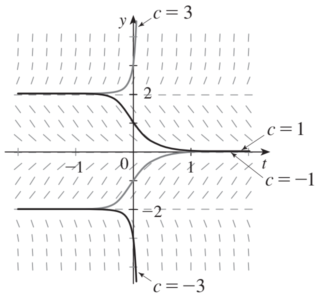


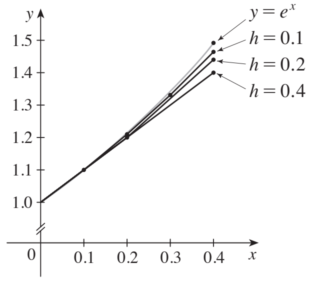
17.



$-2 \leqslant c \leqslant 2$; $-2, 0, 2$

19. (a) (i) 1.4 (ii) 1.44 (iii) 1.4641

(b)



Underestimates

(c) (i) 0.0918 (ii) 0.0518 (iii) 0.0277

It appears that the error is also halved (approximately).

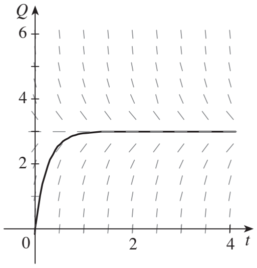
21. $-1, -3, -6.5, -12.25$ **23.** 1.7616

25. (a) (i) 3 (ii) 2.3928 (iii) 2.3701 (iv) 2.3681

(c) (i) -0.6321 (ii) -0.0249 (iii) -0.0022 (iv) -0.0002

It appears that the error is also divided by 10 (approximately).

27. (a), (d)



(b) 3

(c) Yes, $Q = 3$

(e) 2.77 C

EXERCISES 9.3 ■ PAGE 626

1. $y = -1/(x^3 + C)$, $y = 0$ **3.** $y = \left(\frac{1}{4}x^2 + C\right)^2$, $y = 0$

5. $y = \pm\sqrt{x^2 + 2\ln|x| + C}$

7. $e^y - y = 2x + \sin x + C$ **9.** $p = Ke^{(t^3/3)-t} - 1$

11. $\theta \sin\theta + \cos\theta = -\frac{1}{2}e^{-t^2} + C$

13. $y = -\ln\left(1 - \frac{1}{2}x^2\right)$ **15.** $A = b^3 e^{b \sin br}$

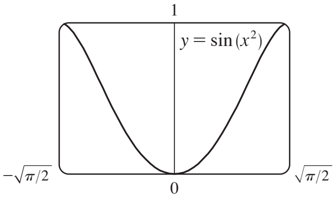
17. $u = -\sqrt{t^2 + \tan t + 25}$

19. $\frac{1}{2}y^2 + \frac{1}{3}(3 + y^2)^{3/2} = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + \frac{41}{12}$

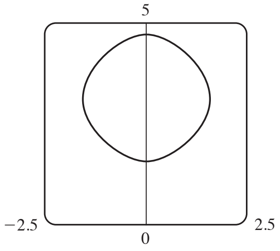
21. $y = \sqrt{x^2 + 4}$ **23.** $y = Ke^x - x - 1$

25. (a) $\sin^{-1} y = x^2 + C$

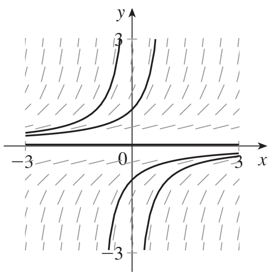
(b) $y = \sin(x^2)$, $-\sqrt{\pi/2} \leqslant x \leqslant \sqrt{\pi/2}$ (c) No



27. $\cos y = \cos x - 1$

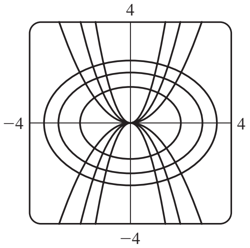


29. (a), (c)

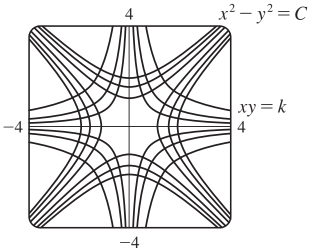


(b) $y = \dfrac{1}{K - x}$

31. $y = Cx^2$



33. $x^2 - y^2 = C$



35. $y = 1 + e^{2-(x^2/2)}$ **37.** $y = \left(\frac{1}{2}x^2 + 2\right)^2$

39. $Q(t) = 3 - 3e^{-4t}$; 3 **41.** $P(t) = M - Me^{-kt}$; M

43. (a) $x = a - \dfrac{4}{(kt + 2/\sqrt{a})^2}$

(b) $t = \dfrac{2}{k\sqrt{a-b}}\left(\tan^{-1}\sqrt{\dfrac{b}{a-b}} - \tan^{-1}\sqrt{\dfrac{b-x}{a-b}}\right)$

45. (a) $C(t) = (C_0 - r/k)e^{-kt} + r/k$ (b) r/k; the concentration approaches r/k regardless of the value of C_0.

47. (a) $15e^{-t/100}$ kg (b) $15e^{-0.2} \approx 12.3$ kg

49. About 4.9% **51.** g/k

53. (a) $L_1 = KL_2^k$ (b) $B = KV^{0.0794}$

55. (a) $dA/dt = k\sqrt{A}\,(M - A)$

(b) $A(t) = M\left(\dfrac{Ce^{\sqrt{M}\,kt} - 1}{Ce^{\sqrt{M}\,kt} + 1}\right)^2$, where $C = \dfrac{\sqrt{M} + \sqrt{A_0}}{\sqrt{M} - \sqrt{A_0}}$ and $A_0 = A(0)$

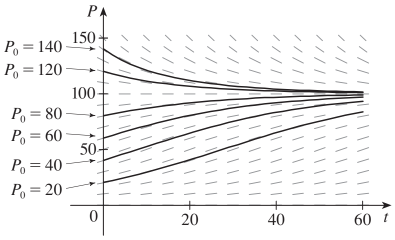
57. (b) $v_e = \sqrt{2gR}$ (c) $v_e \approx 11{,}173$ m/s ≈ 11.2 km/s

EXERCISES 9.4 ■ PAGE 638

1. (a) 1200; 0.04 (b) $P(t) = \dfrac{1200}{1 + 19e^{-0.04t}}$ (c) ≈87

3. (a) 100; 0.05 (b) Where P is close to 0 or 100; on the line $P = 50$; $0 < P_0 < 100$; $P_0 > 100$

(c)



Solutions approach 100; some increase and some decrease, some have an inflection point but others don't; solutions with $P_0 = 20$ and $P_0 = 40$ have inflection points at $P = 50$.

(d) $P = 0$, $P = 100$; other solutions move away from $P = 0$ and toward $P = 100$.

5. (a) $\approx 3.23 \times 10^7$ kg (b) ≈1.55 years **7.** 9000

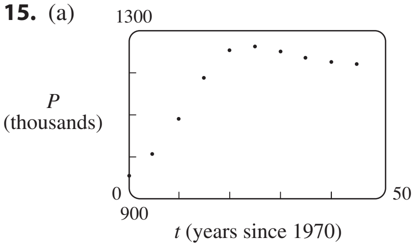
9. (a) $\dfrac{dP}{dt} = \dfrac{1}{305}P\left(1 - \dfrac{P}{20}\right)$ (b) 6.24 billion

(c) 7.57 billion; 13.87 billion

11. (a) $\dfrac{dy}{dt} = ky(1 - y)$ (b) $y = \dfrac{y_0}{y_0 + (1 - y_0)e^{-kt}}$

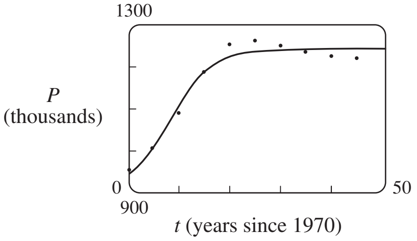
(c) 3:36 PM

15. (a)



(b) $f(t) = \dfrac{345.5899}{1 + 7.9977e^{-0.2482t}}$

(c) $P(t) = 900 + \dfrac{345.5899}{1 + 7.9977e^{-0.2482t}}$



(d) Population approaches 1.246 million

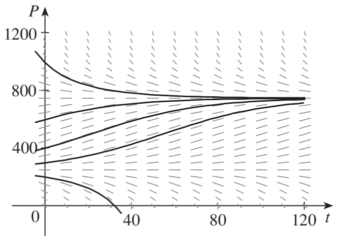
17. (a) $P(t) = \dfrac{m}{k} + \left(P_0 - \dfrac{m}{k}\right)e^{kt}$ (b) $m < kP_0$

(c) $m = kP_0$, $m > kP_0$ (d) Declining

19. (a) Fish are caught at a rate of 15 per week.

(b) See part (d). (c) $P = 250$, $P = 750$

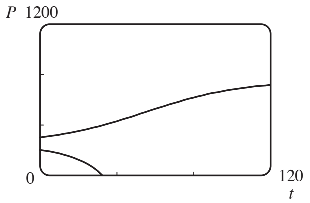
(d)



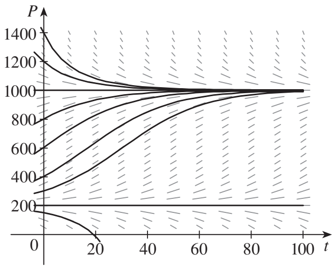
$0 < P_0 < 250$: $P \to 0$;
$P_0 = 250$: $P \to 250$;
$P_0 > 250$: $P \to 750$

(e) $P(t) = \dfrac{250 - 750ke^{t/25}}{1 - ke^{t/25}}$

where $k = \frac{1}{11}, -\frac{1}{9}$



21. (b)



$0 < P_0 < 200$: $P \longrightarrow 0$;
$P_0 = 200$: $P \longrightarrow 200$;
$P_0 > 200$: $P \longrightarrow 1000$;
$P = 200$,
$P = 1000$

(c) $P(t) = \dfrac{m(M - P_0) + M(P_0 - m)e^{(M-m)(k/M)t}}{M - P_0 + (P_0 - m)e^{(M-m)(k/M)t}}$

23. (a) $P(t) = P_0 e^{(k/r)[\sin(rt - \phi) + \sin\phi]}$ (b) Does not exist

EXERCISES 9.5 ■ PAGE 646

1. No **3.** Yes; $\dfrac{du}{dt} - \dfrac{e^t}{\sqrt{t}}u = -\sqrt{t}$ **5.** $y = 1 + Ce^{-x}$

7. $y = x - 1 + Ce^{-x}$ **9.** $y = \frac{2}{3}\sqrt{x} + C/x$

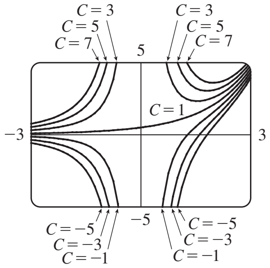
11. $y = x^2(\ln x + C)$ **13.** $y = \frac{1}{3}t^{-3}(1 + t^2)^{3/2} + Ct^{-3}$

15. $y = e^{-\sin x}\int xe^{\sin x}\,dx + Ce^{-\sin x}$ **17.** $y = x^2 + 3/x$

19. $y = \dfrac{1}{x}\ln x - \dfrac{1}{x} + \dfrac{3}{x^2}$ **21.** $u = -t^2 + t^3$

23. $y = -x\cos x - x$

25. $y = \dfrac{(x-1)e^x + C}{x^2}$

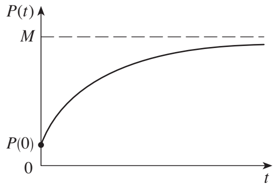


29. $y = \pm\left(Cx^4 + \dfrac{2}{5x}\right)^{-1/2}$

31. (a) $I(t) = 4 - 4e^{-5t}$ (b) $4 - 4e^{-1/2} \approx 1.57$ A

33. $Q(t) = 3(1 - e^{-4t})$, $I(t) = 12e^{-4t}$

35. $P(t) = M + Ce^{-kt}$



37. $y = \frac{2}{5}(100 + 2t) - 40{,}000(100 + 2t)^{-3/2}$; 0.2275 kg/L

39. (b) mg/c (c) $(mg/c)[t + (m/c)e^{-ct/m}] - m^2g/c^2$

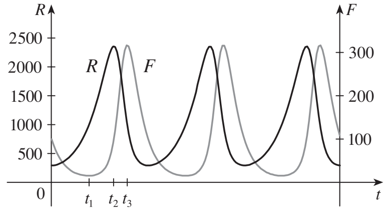
41. (b) $P(t) = \dfrac{M}{1 + MCe^{-kt}}$

EXERCISES 9.6 ■ PAGE 653

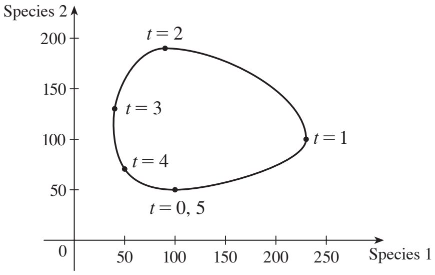
1. (a) x = predators, y = prey; growth is restricted only by predators, which feed only on prey.
(b) x = prey, y = predators; growth is restricted by carrying capacity and by predators, which feed only on prey.

3. (a) Competition
(b) (i) $x = 0$, $y = 0$: zero populations
(ii) $x = 0$, $y = 400$: In the absence of an x-population, the y-population stabilizes at 400.
(iii) $x = 125$, $y = 0$: In the absence of a y-population, the x-population stabilizes at 125.
(iv) $x = 50$, $y = 300$: Both populations are stable.

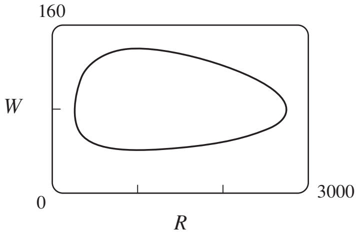
5. (a) The rabbit population starts at about 300, increases to 2400, then decreases back to 300. The fox population starts at 100, decreases to about 20, increases to about 315, decreases to 100, and the cycle starts again.
(b)



7.

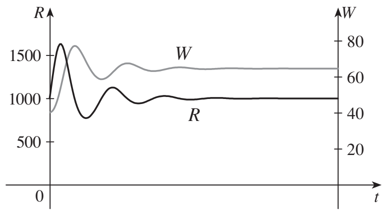


9. (b)



11. (a) Population stabilizes at 5000.
(b) (i) $W = 0$, $R = 0$: Zero populations
(ii) $W = 0$, $R = 5000$: In the absence of wolves, the rabbit population is always 5000.
(iii) $W = 64$, $R = 1000$: Both populations are stable.
(c) The populations stabilize at 1000 rabbits and 64 wolves.

(d)



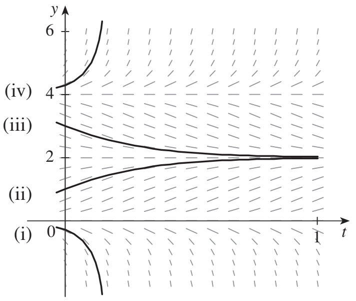
CHAPTER 9 REVIEW ■ PAGE 656

True-False Quiz

1. True **3.** False **5.** True **7.** False **9.** True

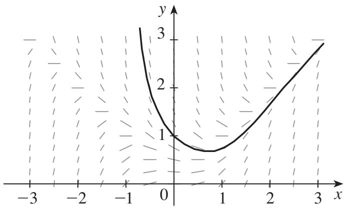
Exercises

1. (a)



(b) $0 \leqslant c \leqslant 4$; $y = 0$, $y = 2$, $y = 4$

3. (a) $y(0.3) \approx 0.8$



(b) 0.75676

(c) $y = x$ and $y = -x$; there is a loc max or loc min.

5. $y = \left(\frac{1}{2}x^2 + C\right)e^{-\sin x}$

7. $y = \pm\sqrt{\ln(x^2 + 2x^{3/2} + C)}$

9. $r(t) = 5e^{t - t^2}$ **11.** $y = \frac{1}{2}x(\ln x)^2 + 2x$

13. $x = C - \frac{1}{2}y^2$

15. (a) $P(t) = \dfrac{2000}{1 + 19e^{-0.1t}}$; ≈ 560

(b) $t = -10 \ln \frac{2}{57} \approx 33.5$

17. (a) $L(t) = L_\infty - [L_\infty - L(0)]e^{-kt}$

(b) $L(t) = 53 - 43e^{-0.2t}$

19. 15 days **21.** $k \ln h + h = (-R/V)t + C$

23. (a) Stabilizes at 200,000

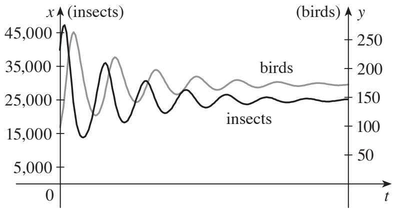
(b) (i) $x = 0$, $y = 0$: Zero populations

(ii) $x = 200{,}000$, $y = 0$: In the absence of birds, the insect population is always 200,000.

(iii) $x = 25{,}000$, $y = 175$: Both populations are stable.

(c) The populations stabilize at 25,000 insects and 175 birds.

(d)



PROBLEMS PLUS ■ PAGE 659

1. $f(x) = \pm 10e^x$ **5.** $y = x^{1/n}$ **7.** 20°C

9. (b) $f(x) = \dfrac{x^2 - L^2}{4L} - \dfrac{L}{2}\ln\left(\dfrac{x}{L}\right)$ (c) No

11. (a) 9.5 h (b) $2{,}700\pi \approx 8482\ \text{m}^2$; $471\ \text{m}^2/\text{h}$ (c) 5.5 h

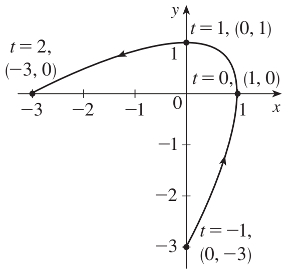
13. $x^2 + (y - 6)^2 = 25$ **15.** $y = K/x$, $K \neq 0$

CHAPTER 10

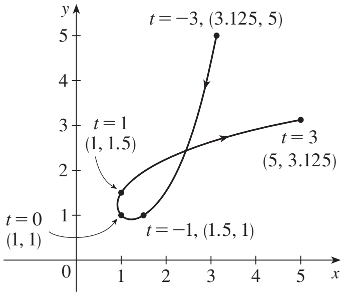
EXERCISES 10.1 ■ PAGE 668

1. $(2, \frac{1}{3})$, $(0, 1)$, $(0, 3)$, $(2, 9)$, $(6, 27)$

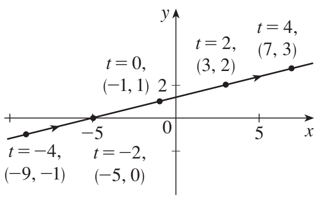
3.



5.

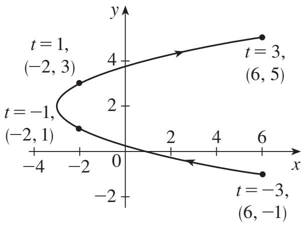


7. (a)



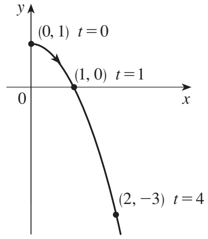
(b) $y = \frac{1}{4}x + \frac{5}{4}$

9. (a)



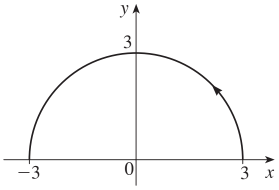
(b) $x = y^2 - 4y + 1$, $-1 \leqslant y \leqslant 5$

11. (a)

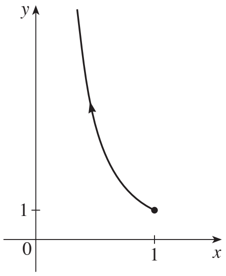


(b) $y = 1 - x^2, x \geqslant 0$

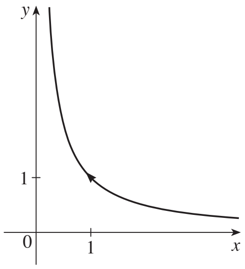
13. (a) $x^2 + y^2 = 9, y \geqslant 0$ (b)



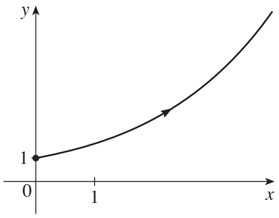
15. (a) $y = 1/x^2, 0 < x \leqslant 1$ (b)



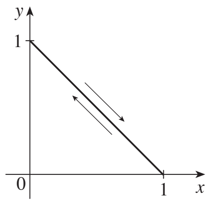
17. (a) $y = 1/x, x > 0$ (b)



19. (a) $y = e^{x/2}, x \geqslant 0$ (b)



21. (a) $x + y = 1, 0 \leqslant x \leqslant 1$ (b)



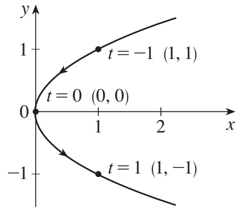
23. 2π seconds; clockwise

25. Moves counterclockwise along the circle $(x - 5)^2 + (y - 3)^2 = 4$ from (3, 3) to (7, 3)

27. Moves 3 times clockwise around the ellipse $(x^2/25) + (y^2/4) = 1$, starting and ending at $(0, -2)$

29. It is contained in the rectangle described by $1 \leqslant x \leqslant 4$ and $2 \leqslant y \leqslant 3$.

31.



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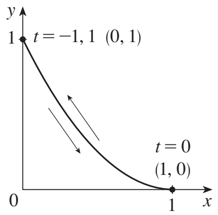
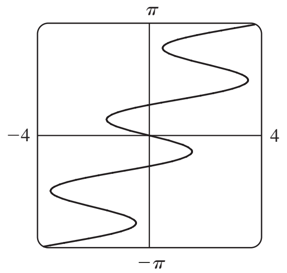


35.

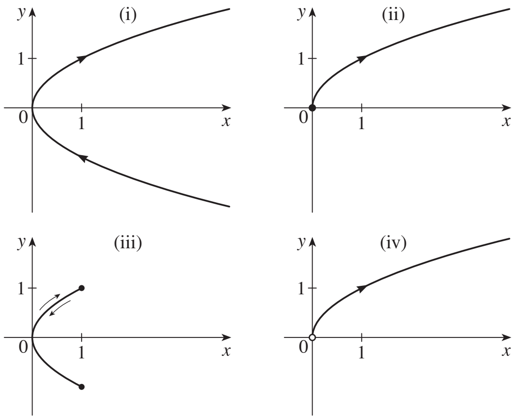


37. (b) $x = -2 + 5t, y = 7 - 8t, 0 \leqslant t \leqslant 1$

39. One option: $x = 5\sin(t/2)$, $y = 5\cos(t/2)$ where t is time in seconds

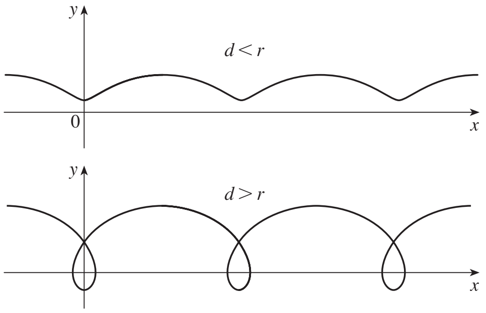
41. (a) $x = 2\cos t, y = 1 - 2\sin t, 0 \leqslant t \leqslant 2\pi$
(b) $x = 2\cos t, y = 1 + 2\sin t, 0 \leqslant t \leqslant 6\pi$
(c) $x = 2\cos t, y = 1 + 2\sin t, \pi/2 \leqslant t \leqslant 3\pi/2$

45. (b)



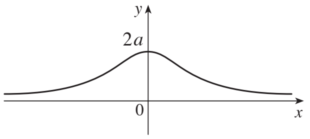
47. The curve $y = x^{2/3}$ is generated in (a). In (b), only the portion with $x \geqslant 0$ is generated, and in (c) we get only the portion with $x > 0$.

49.



51. $x = a\cos\theta$, $y = b\sin\theta$; $(x^2/a^2) + (y^2/b^2) = 1$, ellipse

53.

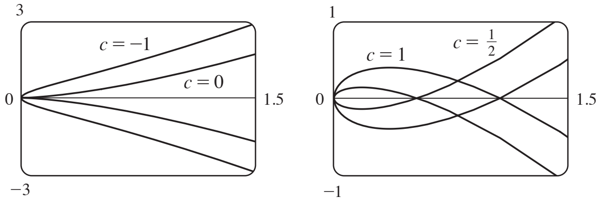


55. (a) No (b) Yes; (6, 11) when $t = 1$

57. (a) (0, 0); $t = 1$, $t = -1$

(b) $(-1, -1)$; $t = \dfrac{1 + \sqrt{5}}{2}$, $t = \dfrac{1 - \sqrt{5}}{2}$

59. For $c = 0$, there is a cusp; for $c > 0$, there is a loop whose size increases as c increases.



61. The curves roughly follow the line $y = x$ and start having loops when a is between 1.4 and 1.6 (more precisely, when $a > \sqrt{2}$); the loops increase in size as a increases.

63. As n increases, the number of oscillations increases; a and b determine the width and height.

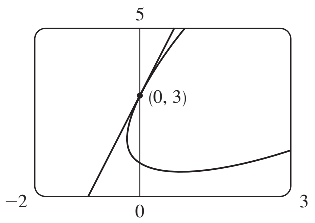
EXERCISES 10.2 ■ PAGE 679

1. $6t^2 + 3, 4 - 10t, \dfrac{4 - 10t}{6t^2 + 3}$

3. $e^t(t + 1), 1 + \cos t, \dfrac{1 + \cos t}{e^t(t + 1)}$ **5.** $\ln 2 - \frac{1}{4}$

7. $y = -x$ **9.** $y = \frac{1}{2}x + \frac{3}{2}$ **11.** $y = -x + \frac{5}{4}$

13. $y = 3x + 3$



15. $\dfrac{2t + 1}{2t}, -\dfrac{1}{4t^3}, t < 0$

17. $e^{-2t}(1 - t), e^{-3t}(2t - 3), \; t > \frac{3}{2}$

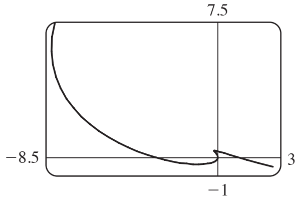
19. $\dfrac{t + 1}{t - 1}, \dfrac{-2t}{(t - 1)^3}, 0 < t < 1$

21. Horizontal at $(0, -3)$, vertical at $(\pm 2, -2)$

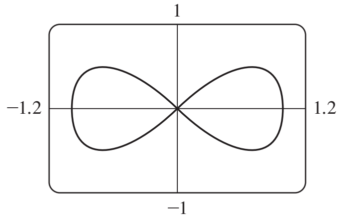
23. Horizontal at $(\frac{1}{2}, -1)$ and $(-\frac{1}{2}, 1)$, no vertical

25. (0.6, 2); $(5 \cdot 6^{-6/5}, e^{6^{-1/5}})$

27.



29. $y = x$, $y = -x$



31. (a) $d\sin\theta/(r - d\cos\theta)$ **33.** (4, 0) **35.** $\frac{24}{5}$

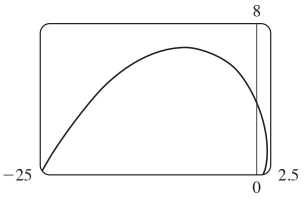
37. $\frac{4}{3}$ **39.** πab **41.** $2\pi r^2 + \pi d^2$

43. $\int_{-1}^{3} \sqrt{(6t - 3t^2)^2 + (2t - 2)^2}\, dt \approx 15.2092$

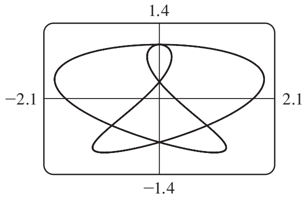
45. $\int_{0}^{4\pi} \sqrt{5 - 4\cos t}\, dt \approx 26.7298$ **47.** $\frac{2}{3}(10\sqrt{10} - 1)$

49. $\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(1 + \sqrt{2})$

51. $\sqrt{2}\,(e^{\pi} - 1)$



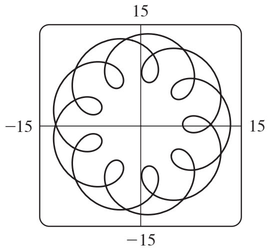
53. 16.7102



55. $6\sqrt{2}, \sqrt{2}$ **57.** $\sqrt{293} \approx 17.12$ m/s

59. $\sqrt{5}\,e \approx 6.08$ m/s **61.** (a) v_0 m/s (b) $v_0 \cos\alpha$ m/s

63. (a)



$t \in [0, 4\pi]$

(b) 294

65. $\frac{3}{8}\pi a^2$ **67.** $\int_0^{\pi/2} 2\pi t \cos t\,\sqrt{t^2+1}\,dt \approx 4.7394$

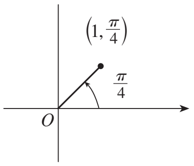
69. $\int_0^1 2\pi e^{-t}\sqrt{1+2e^t+e^{2t}+e^{-2t}}\,dt \approx 10.6705$

71. $\frac{2}{1215}\pi(247\sqrt{13}+64)$ **73.** $\frac{6}{5}\pi a^2$

75. $\frac{24}{5}\pi(949\sqrt{26}+1)$ **81.** $\frac{1}{4}$

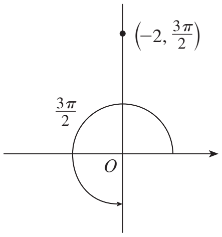
EXERCISES 10.3 ■ PAGE 692

1. (a)



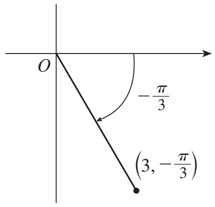
$(1, 9\pi/4), (-1, 5\pi/4)$

(b)



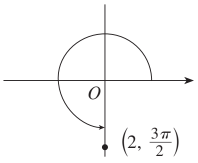
$(2, \pi/2), (-2, 7\pi/2)$

(c)



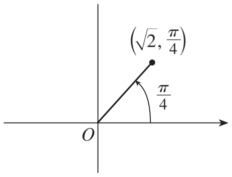
$(3, 5\pi/3), (-3, 2\pi/3)$

3. (a)



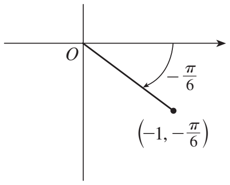
$(0, -2)$

(b)



$(1, 1)$

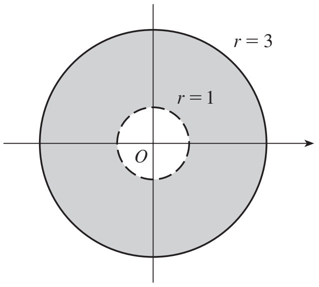
(c)



$(-\sqrt{3}/2, 1/2)$

5. (a) (i) $(4\sqrt{2}, 3\pi/4)$ (ii) $(-4\sqrt{2}, 7\pi/4)$
(b) (i) $(6, \pi/3)$ (ii) $(-6, 4\pi/3)$

7.



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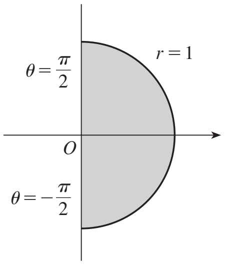

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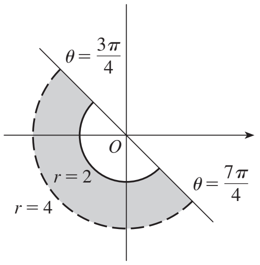

θ = 3π/4
O
r = 2
θ = 7π/4
r = 4

13. $2\sqrt{7}$ **15.** $x^2 + y^2 = 5$; circle, center O, radius $\sqrt{5}$

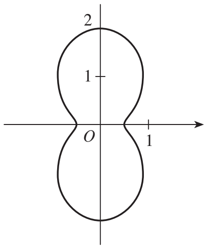
17. $x^2 + y^2 = 5x$; circle, center $(5/2, 0)$, radius $5/2$

19. $x^2 - y^2 = 1$; hyperbola, center O, foci on x-axis

21. $r = \sqrt{7}$ **23.** $\theta = \pi/3$ **25.** $r = 4\sin\theta$

27. (a) $\theta = \pi/6$ (b) $x = 3$

29.



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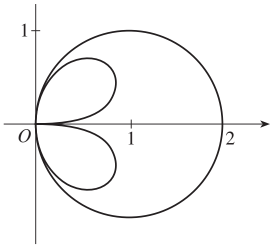

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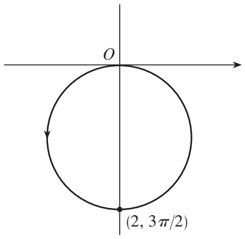

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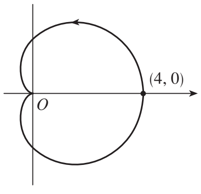

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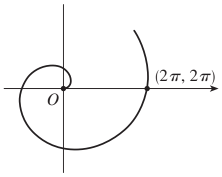

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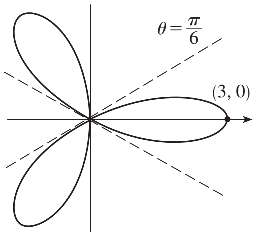

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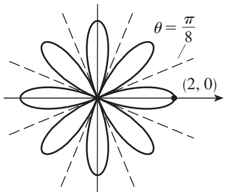

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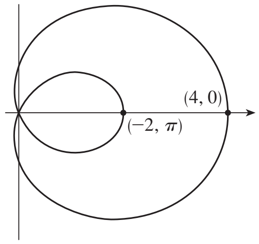

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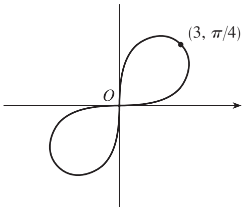

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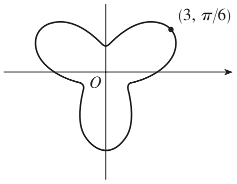

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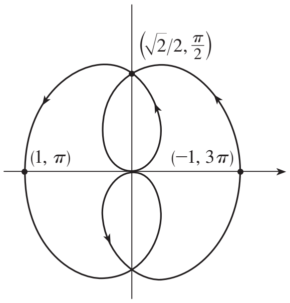

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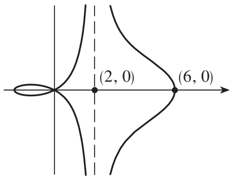

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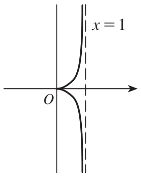
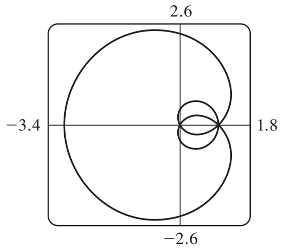


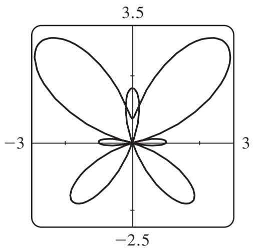
55. (a) For $c < -1$, the inner loop begins at $\theta = \sin^{-1}(-1/c)$ and ends at $\theta = \pi - \sin^{-1}(-1/c)$; for $c > 1$, it begins at $\theta = \pi + \sin^{-1}(1/c)$ and ends at $\theta = 2\pi - \sin^{-1}(1/c)$.

57. Center $(b/2, a/2)$, radius $\sqrt{a^2 + b^2}/2$

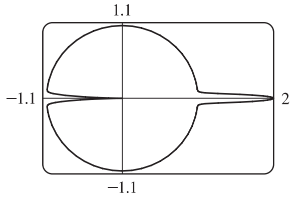
59.



61.



63.



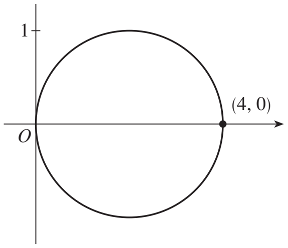
65. By counterclockwise rotation through angle $\pi/6$, $\pi/3$, or α about the origin

67. For $c = 0$, the curve is a circle. As c increases, the left side gets flatter, then has a dimple for $0.5 < c < 1$, a cusp for $c = 1$, and a loop for $c > 1$.

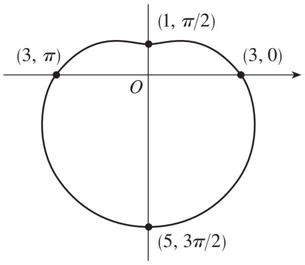
EXERCISES 10.4 ■ PAGE 699

1. $\pi^2/8$ **3.** $\pi/2$ **5.** $\frac{1}{2}$ **7.** $\frac{41}{4}\pi$

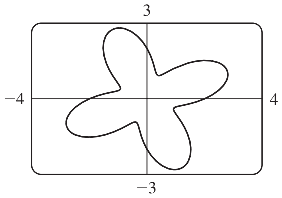
9. 4π



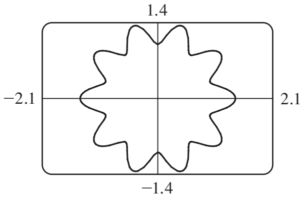
11. 11π



13. $\frac{9}{2}\pi$



15. $\frac{3}{2}\pi$



17. $\frac{4}{3}\pi$ **19.** $\frac{1}{16}\pi$ **21.** $\pi - \frac{3}{2}\sqrt{3}$ **23.** $\frac{4}{3}\pi + 2\sqrt{3}$

25. $4\sqrt{3} - \frac{4}{3}\pi$ **27.** π **29.** $\frac{9}{8}\pi - \frac{9}{4}$ **31.** $\frac{1}{2}\pi - 1$

33. $-\sqrt{3} + 2 + \frac{1}{3}\pi$ **35.** $\frac{1}{4}(\pi + 3\sqrt{3})$

37. $\left(\frac{1}{2}, \pi/6\right), \left(\frac{1}{2}, 5\pi/6\right)$, and the pole

39. $(1, \theta)$ where $\theta = \pi/12, 5\pi/12, 13\pi/12, 17\pi/12$ and $(-1, \theta)$ where $\theta = 7\pi/12, 11\pi/12, 19\pi/12, 23\pi/12$

41. $(1, \pi/6), (1, 5\pi/6), (1, 7\pi/6), (1, 11\pi/6)$

43. $21\pi/2$ **45.** $\pi/8$

47. Intersection at $\theta \approx 0.89, 2.25$; area ≈ 3.46

49. 2π **51.** $\frac{8}{3}[(\pi^2 + 1)^{3/2} - 1]$ **53.** $6\sqrt{2} + 12$

55. $\frac{16}{3}$ **57.** $\int_{\pi}^{4\pi} \sqrt{\cos^2(\theta/5) + \frac{1}{25}\sin^2(\theta/5)}\, d\theta$

59. 2.4221 **61.** 8.0091 **63.** $1/\sqrt{3}$

65. $-\pi$ **67.** 1

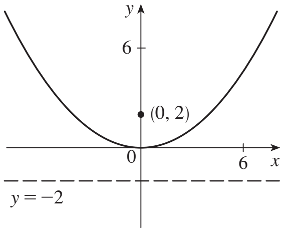
69. Horizontal at (0, 0) [the pole], $(1, \pi/2)$; vertical at $(1/\sqrt{2}, \pi/4), (1/\sqrt{2}, 3\pi/4)$

71. Horizontal at $\left(\frac{3}{2}, \pi/3\right)$, $(0, \pi)$ [the pole], and $\left(\frac{3}{2}, 5\pi/3\right)$; vertical at (2, 0), $\left(\frac{1}{2}, 2\pi/3\right), \left(\frac{1}{2}, 4\pi/3\right)$

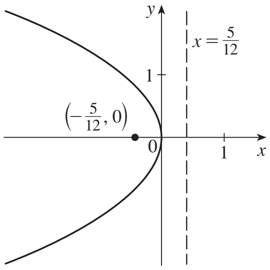
75. (b) $2\pi\left(2 - \sqrt{2}\right)$

EXERCISES 10.5 ■ PAGE 708

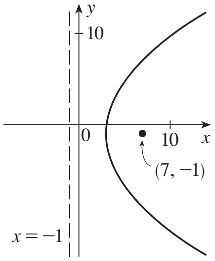
1. $(0, 0), (0, 2), y = -2$



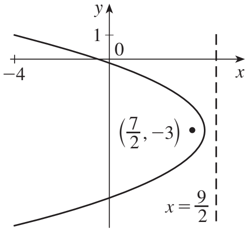
3. $(0, 0), \left(-\frac{5}{12}, 0\right), x = \frac{5}{12}$



5. $(3, -1), (7, -1), x = -1$

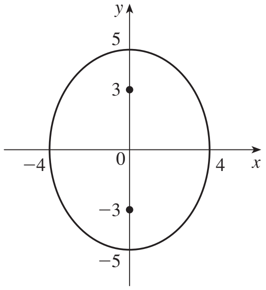


7. $(4, -3), \left(\frac{7}{2}, -3\right), x = \frac{9}{2}$

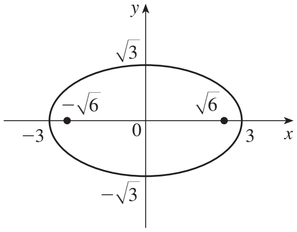


9. $x = -y^2$, focus $\left(-\frac{1}{4}, 0\right)$, directrix $x = \frac{1}{4}$

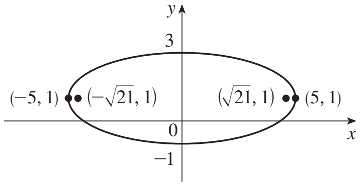
11. $(0, \pm 5), (0, \pm 3)$



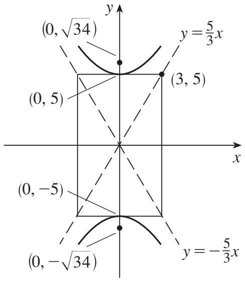
13. $(\pm 3, 0), \left(\pm\sqrt{6}, 0\right)$



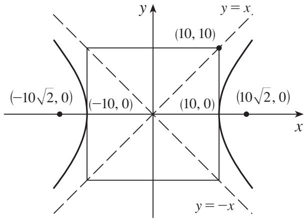
15. $(\pm 5, 1), \left(\pm\sqrt{21}, 1\right)$ **17.** $\frac{x^2}{4} + \frac{y^2}{9} = 1$, foci $\left(0, \pm\sqrt{5}\right)$



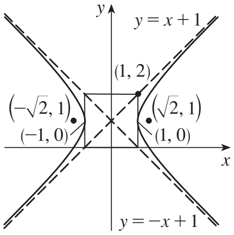
19. $(0, \pm 5), \left(0, \pm\sqrt{34}\right), y = \pm\frac{5}{3}x$



21. $(\pm 10, 0), \left(\pm 10\sqrt{2}, 0\right), y = \pm x$



23. $(\pm 1, 1), \left(\pm\sqrt{2}, 1\right), y - 1 = \pm x$



25. $\frac{x^2}{9} - \frac{y^2}{9} = 1; \left(\pm 3\sqrt{2}, 0\right), y = \pm x$

27. Hyperbola, $(\pm 1, 0), \left(\pm\sqrt{5}, 0\right)$

29. Ellipse, $\left(\pm\sqrt{2}, 1\right), (\pm 1, 1)$

31. Parabola, $(1, -2), \left(1, -\frac{11}{6}\right)$

33. $y^2 = 4x$ **35.** $y^2 = -12(x + 1)$

37. $(y + 1)^2 = -\frac{1}{2}(x - 3)$

39. $\frac{x^2}{25} + \frac{y^2}{21} = 1$ **41.** $\frac{x^2}{12} + \frac{(y - 4)^2}{16} = 1$

43. $\frac{(x + 1)^2}{12} + \frac{(y - 4)^2}{16} = 1$ **45.** $\frac{x^2}{9} - \frac{y^2}{16} = 1$

47. $\frac{(y - 1)^2}{25} - \frac{(x + 3)^2}{39} = 1$ **49.** $\frac{x^2}{9} - \frac{y^2}{36} = 1$

51. $\dfrac{x^2}{3{,}763{,}600} + \dfrac{y^2}{3{,}753{,}196} = 1$

53. (a) $\dfrac{1.30x^2}{10{,}000} + \dfrac{5.83y^2}{100{,}000} = 1$ (b) ≈ 399 km

57. (a) Ellipse (b) Hyperbola (c) No curve

61. 15.9

63. $\dfrac{b^2c}{a} + ab\ln\left(\dfrac{a}{b+c}\right)$ where $c^2 = a^2 + b^2$

65. $(0, 4/\pi)$ **69.** $\dfrac{x^2}{16} + \dfrac{y^2}{15} = 1$

EXERCISES 10.6 ■ PAGE 717

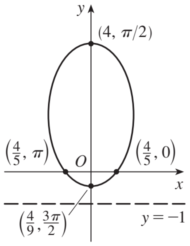
1. $r = \dfrac{2}{1 + \cos\theta}$ **3.** $r = \dfrac{8}{1 - 2\sin\theta}$

5. $r = \dfrac{10}{3 - 2\cos\theta}$ **7.** $r = \dfrac{6}{1 + \sin\theta}$

9. VI **11.** II **13.** IV

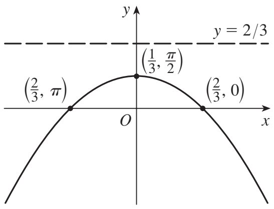
15. (a) $\frac{4}{5}$ (b) Ellipse (c) $y = -1$

(d)



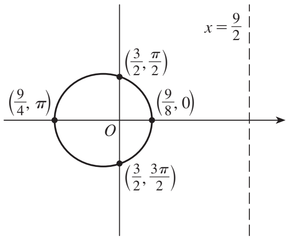
17. (a) 1 (b) Parabola (c) $y = \frac{2}{3}$

(d)



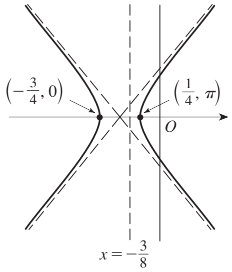
19. (a) $\frac{1}{3}$ (b) Ellipse (c) $x = \frac{9}{2}$

(d)

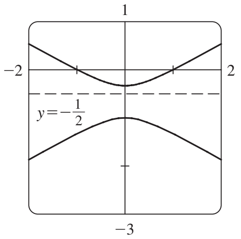


21. (a) 2 (b) Hyperbola (c) $x = -\frac{3}{8}$

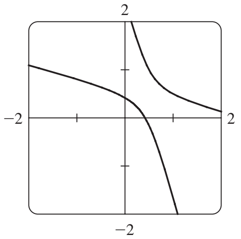
(d)



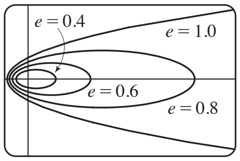
23. (a) $2, y = -\frac{1}{2}$



(b) $r = \dfrac{1}{1 - 2\sin(\theta - 3\pi/4)}$



25. The ellipse is nearly circular when e is close to 0 and becomes more elongated as $e \to 1^-$. At $e = 1$, the curve becomes a parabola.



31. $r = \dfrac{2.26 \times 10^8}{1 + 0.093\cos\theta}$ **33.** $r = \dfrac{1.07}{1 + 0.97\cos\theta}$; 35.64 AU

35. 7.0×10^7 km **37.** 3.6×10^8 km

CHAPTER 10 REVIEW ■ PAGE 719

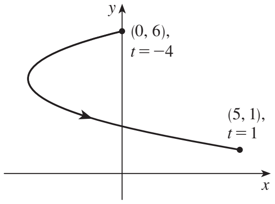
True-False Quiz

1. False **3.** False **5.** False **7.** True **9.** True

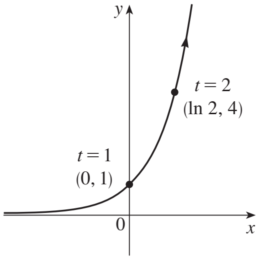
11. True

Exercises

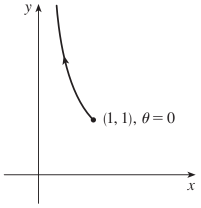
1. $x = y^2 - 8y + 12$, $1 \leqslant y \leqslant 6$



3. $y = e^{2x}$

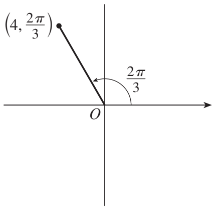


5. $y = 1/x$, $0 < x \leqslant 1$



7. $x = t$, $y = \sqrt{t}$; $x = t^4$, $y = t^2$; $x = \tan^2 t$, $y = \tan t$, $0 \leqslant t < \pi/2$

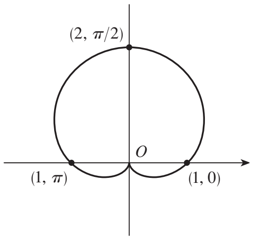
9. (a)



$\left(-2, 2\sqrt{3}\right)$

(b) $\left(3\sqrt{2}, 3\pi/4\right)$, $\left(-3\sqrt{2}, 7\pi/4\right)$

11.



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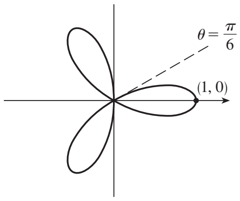

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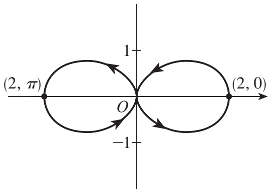
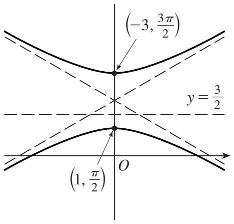
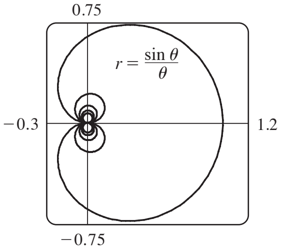


17.



19. $r = \dfrac{2}{\cos\theta + \sin\theta}$

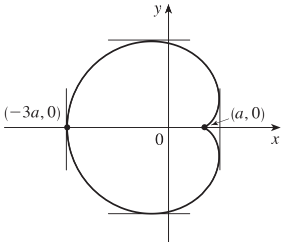
21.



23. 2 **25.** -1 **27.** $\dfrac{1 + \sin t}{1 + \cos t}, \dfrac{1 + \cos t + \sin t}{(1 + \cos t)^3}$

29. $\left(\frac{11}{8}, \frac{3}{4}\right)$

31. Vertical tangent at $\left(\frac{3}{2}a, \pm\frac{1}{2}\sqrt{3}\,a\right)$, $(-3a, 0)$; horizontal tangent at $(a, 0)$, $\left(-\frac{1}{2}a, \pm\frac{3}{2}\sqrt{3}\,a\right)$



33. 18 **35.** $(2, \pm\pi/3)$ **37.** $\frac{1}{2}(\pi - 1)$

39. $2\left(5\sqrt{5} - 1\right)$

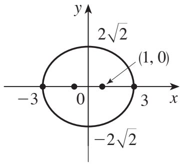
41. $\dfrac{2\sqrt{\pi^2 + 1} - \sqrt{4\pi^2 + 1}}{2\pi} + \ln\left(\dfrac{2\pi + \sqrt{4\pi^2 + 1}}{\pi + \sqrt{\pi^2 + 1}}\right)$

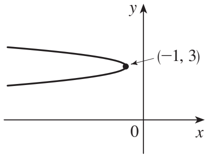
43. (a) $\sqrt{90} \approx 9.49$ m/s (b) $\frac{1}{24}\left(65\sqrt{65} - 1\right) \approx 21.79$ m/s

45. $471{,}295\pi/1024$

47. All curves have the vertical asymptote $x = 1$. For $c < -1$, the curve bulges to the right; at $c = -1$, the curve is the line $x = 1$; and for $-1 < c < 0$, it bulges to the left. At $c = 0$ there is a cusp at $(0, 0)$ and for $c > 0$, there is a loop.

49. $(\pm1, 0)$, $(\pm3, 0)$ **51.** $\left(-\frac{25}{24}, 3\right)$, $(-1, 3)$





53. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ **55.** $\dfrac{y^2}{72/5} - \dfrac{x^2}{8/5} = 1$

57. $\dfrac{x^2}{25} + \dfrac{(8y - 399)^2}{160{,}801} = 1$ **59.** $r = \dfrac{4}{3 + \cos\theta}$

PROBLEMS PLUS ■ PAGE 722

1. $\frac{2}{3}\pi + 2 - 2\sqrt{3}$ **3.** $\left[-\frac{3}{4}\sqrt{3}, \frac{3}{4}\sqrt{3}\right] \times [-1, 2]$

CHAPTER 11

EXERCISES 11.1 ■ PAGE 735

Abbreviations: C, convergent; D, divergent

1. (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.
(b) The terms a_n approach 8 as n becomes large.
(c) The terms a_n become large as n becomes large.

3. 0, 7, 26, 63, 124 **5.** 6, 11, 20, 37, 70 **7.** $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}$.

9. $-1, 1, -1, 1, -1$ **11.** $-1, \frac{2}{3}, -\frac{1}{3}, \frac{2}{15}, -\frac{2}{45}$

13. 1, 3, 7, 15, 31 **15.** $2, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}$ **17.** $a_n = 1/(2n)$

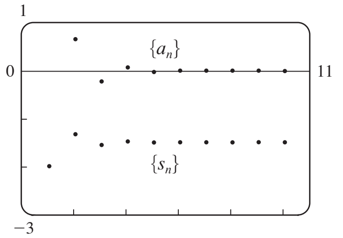
19. $a_n = -3\left(-\frac{2}{3}\right)^{n-1}$ **21.** $a_n = (-1)^{n+1} \dfrac{n^2}{n + 1}$

23. 0.4286, 0.4615, 0.4737, 0.4800, 0.4839, 0.4865, 0.4884, 0.4898, 0.4909, 0.4918; yes; $\frac{1}{2}$
25. 0.5000, 1.2500, 0.8750, 1.0625, 0.9688, 1.0156, 0.9922, 1.0039, 0.9980, 1.0010; yes; 1 **27.** 0 **29.** 2
31. D **33.** 0 **35.** 1 **37.** 2 **39.** D
41. 0 **43.** 0 **45.** D **47.** 0 **49.** 0
51. 1 **53.** e^2 **55.** ln 2 **57.** $\pi/2$ **59.** D
61. D **63.** D **65.** $\pi/4$ **67.** D **69.** 0
71. (a) 1060, 1123.60, 1191.02, 1262.48, 1338.23 (b) D
73. (b) 5734 **75.** $-1 < r < 1$
77. Convergent by the Monotonic Sequence Theorem; $5 \leqslant L < 8$
79. Decreasing; yes **81.** Not monotonic; no
83. Increasing; yes

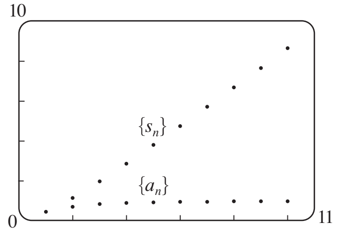
85. 2 **87.** $\frac{1}{2}\left(3 + \sqrt{5}\right)$ **89.** (b) $\frac{1}{2}\left(1 + \sqrt{5}\right)$
91. (a) 0 (b) 9, 11

EXERCISES 11.2 ■ PAGE 747

1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.
(b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.
3. 2
5. 1, 1.125, 1.1620, 1.1777, 1.1857, 1.1903, 1.1932, 1.1952; C
7. 0.8415, 1.7508, 1.8919, 1.1351, 0.1762, −0.1033, 0.5537, 1.5431; D
9. 0.5, 0.55, 0.5611, 0.5648, 0.5663, 0.5671, 0.5675, 0.5677; C
11. −2, −1.33333, −1.55556, −1.48148, −1.50617, −1.49794, −1.50069, −1.49977, −1.50008, −1.49997; convergent, sum = −1.5



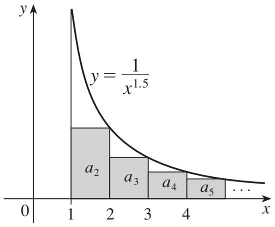
13. 0.44721, 1.15432, 1.98637, 2.88080, 3.80927, 4.75796, 5.71948, 6.68962, 7.66581, 8.64639; divergent



15. (a) Yes (b) No **17.** $-\frac{3}{2}$ **19.** $\frac{11}{6}$
21. $e - 1$ **23.** D **25.** $\frac{25}{3}$ **27.** $\frac{400}{9}$ **29.** $\frac{1}{7}$
31. D **33.** D **35.** $\frac{2}{3}$ **37.** D **39.** 9
41. D **43.** $\dfrac{\sin 100}{1 - \sin 100} \approx -0.336$ **45.** D
47. D **49.** $e/(e - 1)$
51. (b) 1 (c) 2 (d) All rational numbers with a terminating decimal representation, except 0
53. $\frac{8}{9}$ **55.** $\frac{838}{333}$ **57.** 45,679/37,000
59. $-\dfrac{1}{5} < x < \dfrac{1}{5}$; $\dfrac{-5x}{1 + 5x}$
61. $-1 < x < 5$; $\dfrac{3}{5 - x}$
63. $x > 2$ or $x < -2$; $\dfrac{x}{x - 2}$ **65.** $x < 0$; $\dfrac{1}{1 - e^x}$
67. 1 **69.** $a_1 = 0$, $a_n = \dfrac{2}{n(n + 1)}$ for $n > 1$, sum = 1
71. (a) 125 mg; 131.25 mg
(b) $Q_{n+1} = 100 + 0.25Q_n$ (c) $133.\overline{3}$ mg
73. (a) 157.875 mg; $\frac{3000}{19}(1 - 0.05^n)$ (b) $\frac{3000}{19} \approx 157.895$ mg
75. (a) $S_n = \dfrac{D(1 - c^n)}{1 - c}$ (b) 5 **77.** $\frac{1}{2}\left(\sqrt{3} - 1\right)$
83. $\dfrac{1}{n(n + 1)}$ **85.** The series is divergent.
91. $\{s_n\}$ is bounded and increasing.
93. (a) $0, \frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9}, 1$
95. (a) $\frac{1}{2}, \frac{5}{6}, \frac{23}{24}, \frac{119}{120}$; $\dfrac{(n + 1)! - 1}{(n + 1)!}$ (c) 1

EXERCISES 11.3 ■ PAGE 758

1. C



3. C **5.** D **7.** D **9.** C **11.** C **13.** C

15. D **17.** C **19.** C **21.** D **23.** D **25.** C
27. C **29.** f is neither positive nor decreasing.
31. $p > 1$ **33.** $p < -1$ **35.** $(1, \infty)$
37. (a) $\frac{9}{10}\pi^4$ (b) $\frac{1}{90}\pi^4 - \frac{17}{16}$
39. (a) 1.54977, error $\leqslant 0.1$ (b) 1.64522, error $\leqslant 0.005$
(c) 1.64522 compared to 1.64493 (d) $n > 1000$
41. 0.00145 **47.** $b < 1/e$

EXERCISES 11.4 ■ PAGE 764

1. (a) Nothing (b) C **5.** (c) **7.** C **9.** D
11. C **13.** D **15.** C **17.** C **19.** D
21. D **23.** C **25.** D **27.** C **29.** D
31. C **33.** C **35.** C **37.** D **39.** C
41. 0.1993, error $< 2.5 \times 10^{-5}$
43. 0.0739, error $< 6.4 \times 10^{-8}$
53. Yes **55.** (a) False (b) False (c) True

EXERCISES 11.5 ■ PAGE 772

Abbreviations: AC, absolutely convergent;
CC, conditionally convergent
1. (a) A series whose terms are alternately positive and negative (b) $0 < b_{n+1} \leqslant b_n$ and $\lim_{n\to\infty} b_n = 0$, where $b_n = |a_n|$ (c) $|R_n| \leqslant b_{n+1}$
3. D **5.** C **7.** D **9.** C **11.** C **13.** D
15. C **17.** C **19.** C
21. (a) The series Σa_n is absolutely convergent if $\Sigma|a_n|$ converges. (b) The series Σa_n is conditionally convergent if Σa_n converges but $\Sigma|a_n|$ diverges. (c) It converges absolutely.
23. CC **25.** CC **27.** AC **29.** AC **31.** CC
33. CC **35.** -0.5507 **37.** 5 **39.** 5
41. -0.4597 **43.** -0.1050
45. An underestimate **47.** p is not a negative integer.
49. $\{b_n\}$ is not decreasing. **53.** (b) $\sum_{n=2}^{\infty} \dfrac{(-1)^n}{n \ln n}; \sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n}$

EXERCISES 11.6 ■ PAGE 778

1. (a) D (b) C (c) May converge or diverge
3. AC **5.** D **7.** AC **9.** AC **11.** D
13. AC **15.** AC **17.** AC **19.** D **21.** AC
23. AC **25.** D **27.** CC **29.** AC **31.** D
33. AC **35.** D **37.** AC **39.** (a) and (d)
43. (a) $\frac{661}{960} \approx 0.68854$, error < 0.00521
(b) $n \geqslant 11$, 0.693109

EXERCISES 11.7 ■ PAGE 781

1. (a) C (b) C **3.** (a) C (b) D
5. (a) D (b) C **7.** (a) C (b) D
9. D **11.** CC **13.** D **15.** D **17.** C **19.** C
21. C **23.** C **25.** C **27.** C **29.** D **31.** D
33. D **35.** C **37.** C **39.** C **41.** D
43. C **45.** D **47.** C

EXERCISES 11.8 ■ PAGE 786

1. A series of the form $\Sigma_{n=0}^{\infty} c_n(x-a)^n$, where x is a variable and a and the c_n's are constants
3. 1, $[-1, 1)$ **5.** 1, $(-1, 1)$ **7.** 5, $(-5, 5)$
9. 3, $[-3, 3)$ **11.** 1, $[-1, 1)$ **13.** ∞, $(-\infty, \infty)$
15. 4, $[-4, 4]$ **17.** $\frac{1}{4}$, $\left(-\frac{1}{4}, \frac{1}{4}\right]$ **19.** 2, $[-2, 2)$
21. 1, $[1, 3]$ **23.** 2, $[-4, 0)$ **25.** ∞, $(-\infty, \infty)$
27. 1, $[-1, 1)$ **29.** b, $(a - b, a + b)$ **31.** 0, $\left\{\frac{1}{2}\right\}$
33. $\frac{1}{5}$, $\left[\frac{3}{5}, 1\right]$ **35.** ∞, $(-\infty, \infty)$ **37.** (a) Yes (b) No
39. k^k **41.** No **45.** 2

EXERCISES 11.9 ■ PAGE 793

1. 10 **3.** $\sum_{n=0}^{\infty} (-1)^n x^n$, $(-1, 1)$ **5.** $\sum_{n=0}^{\infty} x^{2n}$, $(-1, 1)$

7. $2\sum_{n=0}^{\infty} \dfrac{1}{3^{n+1}} x^n$, $(-3, 3)$ **9.** $\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{4n+2}}{2^{4n+4}}$, $(-2, 2)$

11. $-\dfrac{1}{2} - \sum_{n=1}^{\infty} \dfrac{(-1)^n 3x^n}{2^{n+1}}$, $(-2, 2)$

13. $\sum_{n=0}^{\infty} \left(-1 - \dfrac{1}{3^{n+1}}\right) x^n$, $(-1, 1)$

15. (a) $\sum_{n=0}^{\infty} (-1)^n (n+1) x^n$, $R = 1$

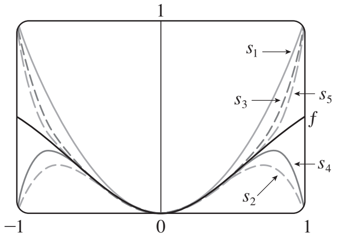
(b) $\dfrac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^n$, $R = 1$

(c) $\dfrac{1}{2} \sum_{n=2}^{\infty} (-1)^n n(n-1) x^n$, $R = 1$

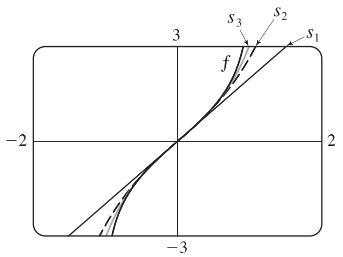
17. $\sum_{n=0}^{\infty} (-1)^n 4^n (n+1) x^{n+1}$, $R = \frac{1}{4}$

19. $\sum_{n=0}^{\infty} (2n+1) x^n$, $R = 1$ **21.** $\ln 5 - \sum_{n=1}^{\infty} \dfrac{x^n}{n5^n}$, $R = 5$

23. $\sum_{n=0}^{\infty} (-1)^n x^{2n+2}$, $R = 1$



25. $\sum_{n=0}^{\infty} \dfrac{2x^{2n+1}}{2n+1}$, $R = 1$



27. $C + \sum_{n=0}^{\infty} \dfrac{t^{8n+2}}{8n+2}$, $R = 1$

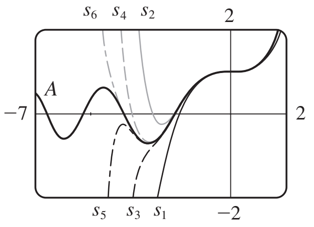
29. $C + \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^{n+3}}{n(n+3)}, R = 1$

31. 0.044522 **33.** 0.000395 **35.** 0.19740

39. (b) 0.920

41. (a) $(-\infty, \infty)$

(b), (c)



43. $(-1, 1), f(x) = (1 + 2x)/(1 - x^2)$

45. $[-1, 1], [-1, 1), (-1, 1)$ **47.** $\sum_{n=1}^{\infty} n^2 x^n, R = 1$

EXERCISES 11.10 ■ PAGE 808

1. $b_8 = f^{(8)}(5)/8!$ **3.** $\sum_{n=0}^{\infty} (n+1)x^n, R = 1$

5. $x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$

7. $2 + \frac{1}{12}(x - 8) - \frac{1}{288}(x - 8)^2 + \frac{5}{20{,}736}(x - 8)^3$

9. $\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\left(x - \dfrac{\pi}{6}\right) - \dfrac{1}{4}\left(x - \dfrac{\pi}{6}\right)^2 - \dfrac{\sqrt{3}}{12}\left(x - \dfrac{\pi}{6}\right)^3$

11. $\sum_{n=0}^{\infty} (n+1)x^n, R = 1$ **13.** $\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}, R = \infty$

15. $3 - 3x^2 + 2x^4, R = \infty$ **17.** $\sum_{n=0}^{\infty} \dfrac{(\ln 2)^n}{n!} x^n, R = \infty$

19. $\sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{(2n+1)!}, R = \infty$

21. $50 + 105(x - 2) + 92(x - 2)^2 + 42(x - 2)^3 + 10(x - 2)^4 + (x - 2)^5, R = \infty$

23. $\ln 2 + \sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{1}{n2^n}(x - 2)^n, R = 2$

25. $\sum_{n=0}^{\infty} \dfrac{2^n e^6}{n!}(x - 3)^n, R = \infty$

27. $\sum_{n=0}^{\infty} \dfrac{(-1)^{n+1}}{(2n+1)!}(x - \pi)^{2n+1}, R = \infty$

29. $\sum_{n=0}^{\infty} (-1)^n \dfrac{2^{2n+1}}{(2n+1)!}(x - \pi)^{2n+1}, R = \infty$

35. $1 - \dfrac{1}{4}x - \sum_{n=2}^{\infty} \dfrac{3 \cdot 7 \cdot \cdots \cdot (4n - 5)}{4^n \cdot n!} x^n, R = 1$

37. $\sum_{n=0}^{\infty} (-1)^n \dfrac{(n+1)(n+2)}{2^{n+4}} x^n, R = 2$

39. $\sum_{n=0}^{\infty} (-1)^n \dfrac{1}{2n+1} x^{4n+2}, R = 1$

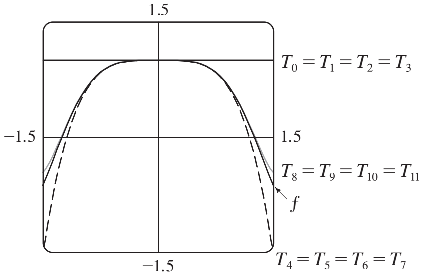
41. $\sum_{n=0}^{\infty} (-1)^n \dfrac{2^{2n}}{(2n)!} x^{2n+1}, R = \infty$

43. $\sum_{n=0}^{\infty} (-1)^n \dfrac{1}{2^{2n}(2n)!} x^{4n+1}, R = \infty$

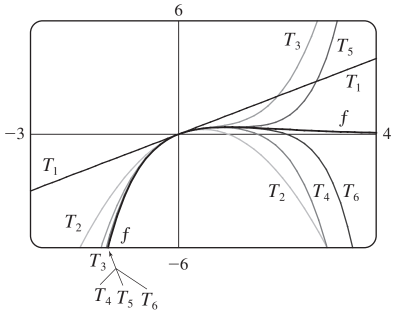
45. $\dfrac{1}{2}x + \sum_{n=1}^{\infty} (-1)^n \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{n!\,2^{3n+1}} x^{2n+1}, R = 2$

47. $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{2^{2n-1}}{(2n)!} x^{2n}, R = \infty$

51. $\sum_{n=0}^{\infty} (-1)^n \dfrac{1}{(2n)!} x^{4n}, R = \infty$



53. $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{(n-1)!} x^n, R = \infty$



55. 0.99619

57. (a) $1 + \sum_{n=1}^{\infty} \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{2^n n!} x^{2n}$

(b) $x + \sum_{n=1}^{\infty} \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{(2n+1)2^n n!} x^{2n+1}$

59. $C + \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \dfrac{x^{3n+1}}{3n+1}, R = 1$

61. $C + \sum_{n=1}^{\infty} (-1)^n \dfrac{1}{2n(2n)!} x^{2n}, R = \infty$

63. 0.0059 **65.** 0.40102 **67.** $\frac{1}{2}$ **69.** $\frac{1}{120}$ **71.** $\frac{3}{5}$

73. $1 - \frac{3}{2}x^2 + \frac{25}{24}x^4$ **75.** $1 + \frac{1}{6}x^2 + \frac{7}{360}x^4$

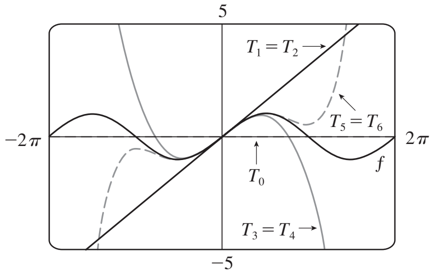
77. $x - \frac{2}{3}x^4 + \frac{23}{45}x^6$ **79.** e^{-x^4} **81.** $\tan^{-1}(x/2)$

83. $1/e$ **85.** $\ln \frac{8}{5}$ **87.** $1/\sqrt{2}$ **89.** $e^3 - 1$

93. $\dfrac{203!}{101!}$

EXERCISES 11.11 ■ PAGE 818

1. (a) $T_0(x) = 0$, $T_1(x) = T_2(x) = x$, $T_3(x) = T_4(x) = x - \frac{1}{6}x^3$, $T_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

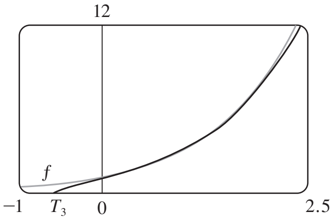


(b)

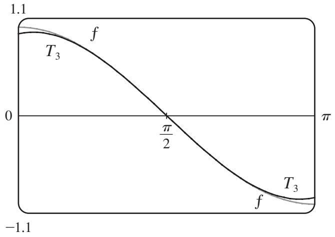
x	f	T_0	$T_1 = T_2$	$T_3 = T_4$	T_5
$\pi/4$	0.7071	0	0.7854	0.7047	0.7071
$\pi/2$	1	0	1.5708	0.9248	1.0045
π	0	0	3.1416	−2.0261	0.5240

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

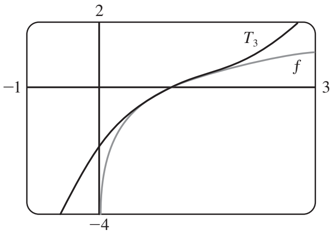
3. $e + e(x - 1) + \frac{1}{2}e(x - 1)^2 + \frac{1}{6}e(x - 1)^3$



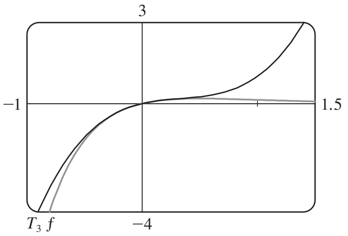
5. $-\left(x - \dfrac{\pi}{2}\right) + \dfrac{1}{6}\left(x - \dfrac{\pi}{2}\right)^3$



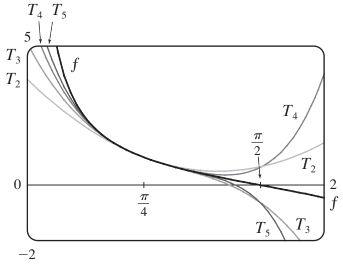
7. $(x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3$



9. $x - 2x^2 + 2x^3$



11. $T_5(x) = 1 - 2\left(x - \dfrac{\pi}{4}\right) + 2\left(x - \dfrac{\pi}{4}\right)^2 - \dfrac{8}{3}\left(x - \dfrac{\pi}{4}\right)^3 + \dfrac{10}{3}\left(x - \dfrac{\pi}{4}\right)^4 - \dfrac{64}{15}\left(x - \dfrac{\pi}{4}\right)^5$



13. (a) $1 - (x - 1) + (x - 1)^2$ (b) 0.112 453

15. (a) $1 + \frac{2}{3}(x - 1) - \frac{1}{9}(x - 1)^2 + \frac{4}{81}(x - 1)^3$ (b) 0.000 097

17. (a) $1 + \frac{1}{2}x^2$ (b) 0.001 447

19. (a) $1 + x^2$ (b) 0.000 053

21. (a) $x^2 - \frac{1}{6}x^4$ (b) 0.041 667

23. 0.17365 **25.** Four **27.** $-1.037 < x < 1.037$

29. $-0.86 < x < 0.86$ **31.** 21 m, no

37. (c) Corrections differ by about 8×10^{-9} km.

CHAPTER 11 REVIEW ■ PAGE 822

True-False Quiz

1. False **3.** True **5.** False **7.** False **9.** False
11. True **13.** True **15.** False **17.** True
19. True **21.** True

Exercises

1. $\frac{1}{2}$ **3.** D **5.** 0 **7.** e^{12} **9.** 2 **11.** C
13. C **15.** D **17.** C **19.** C **21.** C **23.** CC
25. AC **27.** $\frac{1}{11}$ **29.** $\pi/4$ **31.** e^{-e} **35.** 0.9721
37. 0.189 762 24, error $< 6.4 \times 10^{-7}$
41. 4, $[-6, 2)$ **43.** 0.5, $[2.5, 3.5)$

45. $\dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty}(-1)^n\left[\dfrac{1}{(2n)!}\left(x - \dfrac{\pi}{6}\right)^{2n} + \dfrac{\sqrt{3}}{(2n + 1)!}\left(x - \dfrac{\pi}{6}\right)^{2n+1}\right]$

47. $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{n+2}$, $R = 1$ **49.** $\ln 4 - \displaystyle\sum_{n=1}^{\infty}\dfrac{x^n}{n4^n}$, $R = 4$

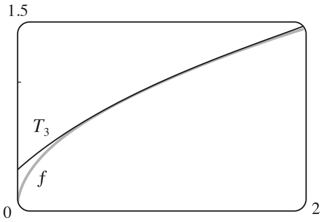
51. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+4}}{(2n+1)!}, R = \infty$

53. $\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{n!\,2^{6n+1}} x^n, R = 16$

55. $C + \ln|x| + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$

57. (a) $1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$

(b)



(c) 0.000 006

59. $-\frac{1}{6}$

PROBLEMS PLUS ■ PAGE 825

1. (b) 0 if $x = 0$, $(1/x) - \cot x$ if $x \neq k\pi$, k an integer

3. (a) $s_n = 3 \cdot 4^n$, $l_n = 1/3^n$, $p_n = 4^n/3^{n-1}$ (c) $\frac{2}{5}\sqrt{3}$

7. $\frac{3\pi}{4}$ **9.** $(-1, 1)$, $\frac{x^3 + 4x^2 + x}{(1-x)^4}$ **11.** $\ln \frac{1}{2}$

15. (a) $\frac{250}{101}\pi(e^{-(n-1)\pi/5} - e^{-n\pi/5})$ (b) $\frac{250}{101}\pi$

17. $\frac{\pi}{2\sqrt{3}} - 1$

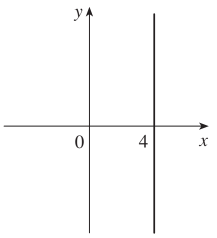
19. $-\left(\frac{\pi}{2} - \pi k\right)^2$, where k is a positive integer

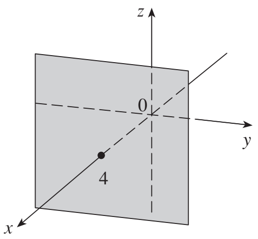
CHAPTER 12

EXERCISES 12.1 ■ PAGE 835

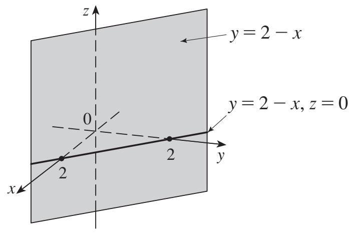
1. $(4, 0, -3)$ **3.** C; A

5. A line parallel to the y-axis and 4 units to the right of it; a vertical plane parallel to the yz-plane and 4 units in front of it.





7. A vertical plane that intersects the xy-plane in the line $y = 2 - x$, $z = 0$



9. 6

11. $|PQ| = 6$, $|QR| = 2\sqrt{10}$, $|RP| = 6$; isosceles triangle

13. (a) No (b) Yes

15. $(x+3)^2 + (y-2)^2 + (z-5)^2 = 16$; $(y-2)^2 + (z-5)^2 = 7$, $x = 0$ (a circle)

17. $(x-3)^2 + (y-8)^2 + (z-1)^2 = 30$

19. $(-4, 0, 1)$, 5 **21.** $\left(\frac{1}{2}, -1, 0\right)$, $\sqrt{3}/2$

25. (a) $(x+1)^2 + (y-4)^2 + (z-5)^2 = 25$
(b) $(x+1)^2 + (y-4)^2 + (z-5)^2 = 1$
(c) $(x+1)^2 + (y-4)^2 + (z-5)^2 = 16$

27. A horizontal plane 2 units below the xy-plane

29. A half-space consisting of all points on or to the right of the plane $y = 1$

31. All points on or between the vertical planes $x = -1$ and $x = 2$

33. All points on a circle with radius 2 and center on the z-axis that is contained in the plane $z = -1$

35. All points on or inside a circular cylinder of radius 5 with axis the x-axis

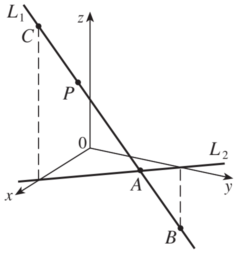
37. All points on a sphere with radius 2 and center (0, 0, 0)

39. All points on or between spheres with radii 1 and $\sqrt{5}$ and centers (0, 0, 0)

41. All points on or inside a cube with edges along the coordinate axes and opposite vertices at the origin and (3, 3, 3)

43. $0 < x < 5$ **45.** $r^2 < x^2 + y^2 + z^2 < R^2$

47. (a) (2, 1, 4) (b)



49. $14x - 6y - 10z = 9$; a plane perpendicular to AB

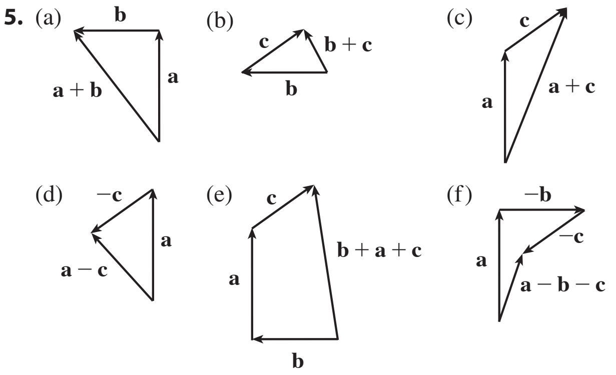
51. $2\sqrt{3} - 3$

EXERCISES 12.2 ■ PAGE 843

1. (a) Scalar (b) Vector (c) Vector (d) Scalar

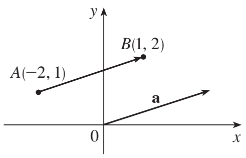
3. $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{DA} = \overrightarrow{CB}$, $\overrightarrow{DE} = \overrightarrow{EB}$, $\overrightarrow{EA} = \overrightarrow{CE}$

5.

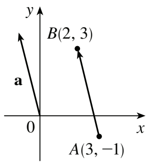


7. $\mathbf{c} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$, $\mathbf{d} = \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}$

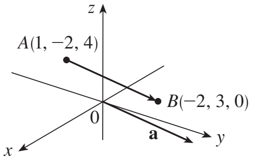
9. $\mathbf{a} = \langle 3, 1 \rangle$



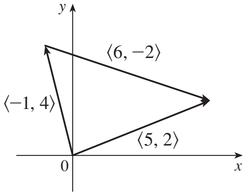
11. $\mathbf{a} = \langle -1, 4 \rangle$



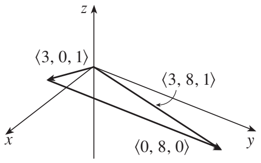
13. $\mathbf{a} = \langle -3, 5, -4 \rangle$



15. $\langle 5, 2 \rangle$



17. $\langle 3, 8, 1 \rangle$



19. $\langle 6, 3 \rangle$, $\langle 6, 14 \rangle$, 5, 13

21. $6\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$, $20\mathbf{i} - 12\mathbf{j}$, $\sqrt{29}$, 7

23. $\left\langle \dfrac{3}{\sqrt{10}}, -\dfrac{1}{\sqrt{10}} \right\rangle$ **25.** $\frac{8}{9}\mathbf{i} - \frac{1}{9}\mathbf{j} + \frac{4}{9}\mathbf{k}$ **27.** $60°$

29. $\langle -2\sqrt{3}, 2 \rangle$ **31.** ≈ 15.32 m/s, ≈ 12.86 m/s

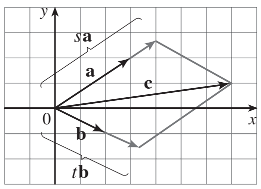
33. $100\sqrt{7} \approx 264.6$ N, $\approx 139.1°$

35. $\approx -177.39\,\mathbf{i} + 211.41\,\mathbf{j}$, $\approx 177.39\,\mathbf{i} + 138.59\,\mathbf{j}$; ≈ 275.97 N, ≈ 225.11 N

37. ≈ 26.1 N **39.** $\approx$N 41.6° W, ≈ 237.3 km/h

41. $\pm(\mathbf{i} + 4\mathbf{j})/\sqrt{17}$ **43.** $\mathbf{0}$

45. (a), (b)



(d) $s = \frac{9}{7}$, $t = \frac{11}{7}$

47. A sphere with radius 1, centered at (x_0, y_0, z_0)

EXERCISES 12.3 ■ PAGE 852

1. (b), (c), (d) are meaningful **3.** -3.6 **5.** 19 **7.** 1

9. $14\sqrt{3}$ **11.** $\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}$, $\mathbf{u} \cdot \mathbf{w} = -\frac{1}{2}$

15. $\cos^{-1}\left(\dfrac{17}{13\sqrt{2}}\right) \approx 22°$ **17.** $\cos^{-1}\left(-\frac{5}{6}\right) \approx 146°$

19. $\cos^{-1}\left(\dfrac{-2}{3\sqrt{70}}\right) \approx 95°$ **21.** 48°, 75°, 57°

23. (a) Orthogonal (b) Neither
(c) Parallel (d) Orthogonal

25. Yes **27.** $(\mathbf{i} - \mathbf{j} - \mathbf{k})/\sqrt{3}$ $\left[\text{or } (-\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}\right]$

29. $\approx 36.9°$ **31.** 0° at (0, 0), $\approx 8.1°$ at (1, 1)

33. $\frac{4}{9}, \frac{1}{9}, \frac{8}{9}$; 63.6°, 83.6°, 27.3°

35. $3/\sqrt{14}$, $-1/\sqrt{14}$, $-2/\sqrt{14}$; 36.7°, 105.5°, 122.3°

37. $1/\sqrt{3}$, $1/\sqrt{3}$, $1/\sqrt{3}$; 54.7°, 54.7°, 54.7° **39.** 4, $\left\langle -\frac{20}{13}, \frac{48}{13} \right\rangle$

41. $\frac{1}{9}$, $\left\langle \frac{4}{81}, \frac{7}{81}, -\frac{4}{81} \right\rangle$ **43.** $-7/\sqrt{19}$, $-\frac{21}{19}\mathbf{i} + \frac{21}{19}\mathbf{j} - \frac{7}{19}\mathbf{k}$

47. $\left\langle 0, 0, -2\sqrt{10} \right\rangle$ or any vector of the form $\left\langle s, t, 3s - 2\sqrt{10} \right\rangle$, $s, t \in \mathbb{R}$

49. 144 J **51.** $2400\cos(40°) \approx 1839$ J

53. $\frac{13}{5}$ **55.** $\approx 54.7°$

EXERCISES 12.4 ■ PAGE 861

1. $15\,\mathbf{i} - 10\,\mathbf{j} - 3\,\mathbf{k}$ **3.** $14\,\mathbf{i} + 4\,\mathbf{j} + 2\,\mathbf{k}$

5. $-\frac{3}{2}\mathbf{i} + \frac{7}{4}\mathbf{j} + \frac{2}{3}\mathbf{k}$

7. $(3t^3 - 2t^2)\,\mathbf{i} + (t^2 - 3t^4)\,\mathbf{j} + (2t^4 - t^3)\,\mathbf{k}$

9. $\mathbf{0}$ **11.** $\mathbf{i} + \mathbf{j} + \mathbf{k}$

13. (a) Scalar (b) Meaningless (c) Vector
(d) Meaningless (e) Meaningless (f) Scalar

15. 6; into the page **17.** $\langle -7, 10, 8 \rangle$, $\langle 7, -10, -8 \rangle$

19. $\left\langle -\dfrac{1}{3\sqrt{3}}, -\dfrac{1}{3\sqrt{3}}, \dfrac{5}{3\sqrt{3}} \right\rangle$, $\left\langle \dfrac{1}{3\sqrt{3}}, \dfrac{1}{3\sqrt{3}}, -\dfrac{5}{3\sqrt{3}} \right\rangle$

27. 20 **29.** (a) $\langle -10, 11, 3 \rangle$ (b) $\frac{1}{2}\sqrt{230}$

31. (a) $\langle 12, -1, 17 \rangle$ (b) $\frac{1}{2}\sqrt{434}$

33. 9 **35.** 16 **39.** $10.8 \sin 80° \approx 10.6$ N·m

41. ≈ 417 N **43.** 60°

45. (b) $\sqrt{97/3}$ **53.** (a) No (b) No (c) Yes

EXERCISES 12.5 ■ PAGE 872

1. (a) True (b) False (c) True (d) False
(e) False (f) True (g) False (h) True (i) True
(j) False (k) True

3. $\mathbf{r} = (-\mathbf{i} + 8\,\mathbf{j} + 7\,\mathbf{k}) + t\left(\frac{1}{2}\,\mathbf{i} + \frac{1}{3}\,\mathbf{j} + \frac{1}{4}\,\mathbf{k}\right)$;
$x = -1 + \frac{1}{2}t,\ y = 8 + \frac{1}{3}t,\ z = 7 + \frac{1}{4}t$

5. $\mathbf{r} = (5\,\mathbf{i} + 7\,\mathbf{j} + \mathbf{k}) + t(3\,\mathbf{i} - 2\,\mathbf{j} + 2\,\mathbf{k})$;
$x = 5 + 3t,\ y = 7 - 2t,\ z = 1 + 2t$

7. $x = 8t,\ y = -t,\ z = 3t;\ x/8 = -y = z/3$

9. $x = 12 - 19t,\ y = 9,\ z = -13 + 24t$;
$(x - 12)/(-19) = (z + 13)/24,\ y = 9$

11. $x = -6 + 2t,\ y = 2 + 3t,\ z = 3 + t$;
$(x + 6)/2 = (y - 2)/3 = z - 3$

13. Yes

15. (a) $(x - 1)/(-1) = (y + 5)/2 = (z - 6)/(-3)$
(b) $(-1, -1, 0),\ \left(-\frac{3}{2}, 0, -\frac{3}{2}\right),\ (0, -3, 3)$

17. $\mathbf{r}(t) = (6\mathbf{i} - \mathbf{j} + 9\mathbf{k}) + t(\mathbf{i} + 7\mathbf{j} - 9\mathbf{k}),\ 0 \leqslant t \leqslant 1$

19. Skew **21.** $(4, -1, -5)$ **23.** $5x + 4y + 6z = 29$

25. $-x + 2y + 3z = 3$

27. $4x - y + 5z = -4$

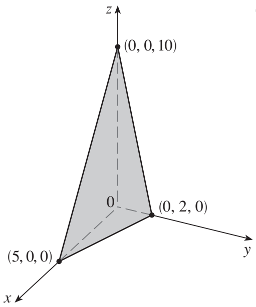
29. $2x - y + 3z = -0.2$ or $10x - 5y + 15z = -1$

31. $x + y + z = 2$ **33.** $5x - 3y - 8z = -9$

35. $8x + y - 2z = 31$ **37.** $x - 2y - z = -3$

39. $3x - 8y - z = -38$

41.



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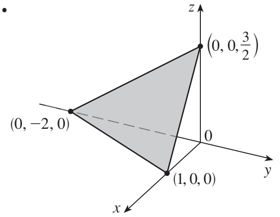


45. $(-2, 6, 3)$ **47.** $\left(\frac{2}{5}, 4, 0\right)$ **49.** $1, 0, -1$

51. Perpendicular **53.** Neither, $\cos^{-1}\left(-\dfrac{1}{\sqrt{6}}\right) \approx 114.1°$

55. Parallel

57. (a) $x = 1,\ y = -t,\ z = t$ (b) $\cos^{-1}\left(\dfrac{5}{3\sqrt{3}}\right) \approx 15.8°$

59. $x = 1,\ y - 2 = -z$ **61.** $x + 2y + z = 5$

63. $(x/a) + (y/b) + (z/c) = 1$

65. $x = 3t,\ y = 1 - t,\ z = 2 - 2t$

67. P_2 and P_3 are parallel, P_1 and P_4 are identical

69. $\sqrt{61/14}$ **71.** $\frac{18}{7}$ **73.** $5/\left(2\sqrt{14}\right)$

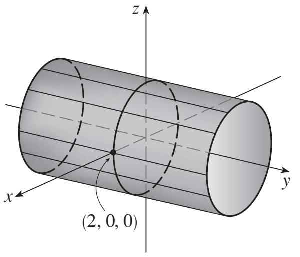
77. $1/\sqrt{6}$ **79.** $13/\sqrt{69}$

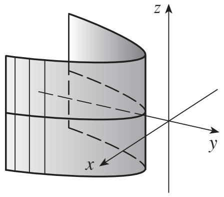
81. (a) $x = 325 + 440t,\ y = 810 - 135t,\ z = 561 + 38t$,
$0 \leqslant t \leqslant 1$ (b) No

EXERCISES 12.6 ■ PAGE 881

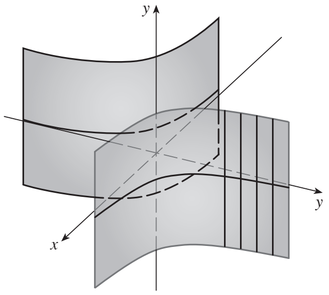
1. (a) Parabola
(b) Parabolic cylinder with rulings parallel to the z-axis
(c) Parabolic cylinder with rulings parallel to the x-axis

3. Circular cylinder of radius 2 **5.** Parabolic cylinder





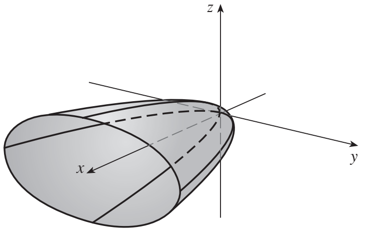
7. Hyperbolic cylinder



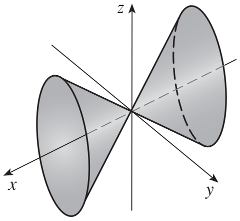
9. $z = \cos x$

11. (a) $x = k,\ y^2 - z^2 = 1 - k^2$, hyperbola $(k \neq \pm 1)$;
$y = k,\ x^2 - z^2 = 1 - k^2$, hyperbola $(k \neq \pm 1)$;
$z = k,\ x^2 + y^2 = 1 + k^2$, circle
(b) The hyperboloid is rotated so that its axis is the y-axis.
(c) The hyperboloid is shifted one unit in the negative y-direction.

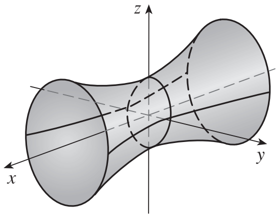
13. Elliptic paraboloid with axis the x-axis



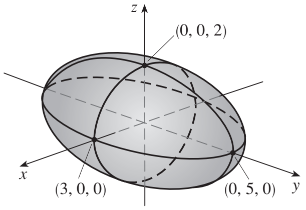
15. Elliptic cone with axis the x-axis



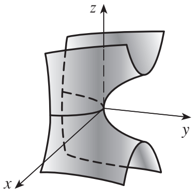
17. Hyperboloid of one sheet with axis the *x*-axis



19. Ellipsoid

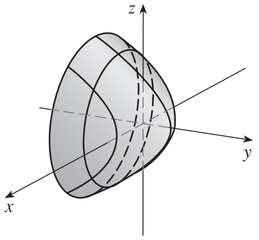


21. Hyperbolic paraboloid



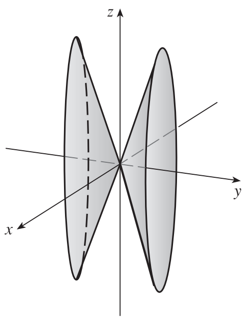
23. VII **25.** II **27.** VI **29.** VIII

31. Circular paraboloid



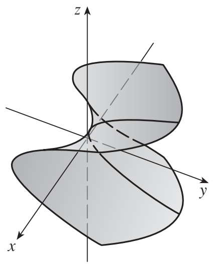
33. $y^2 = x^2 + \dfrac{z^2}{9}$

Elliptic cone with axis the *y*-axis



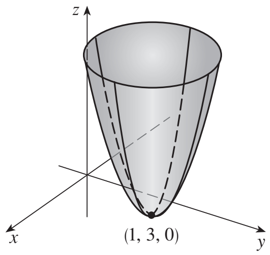
35. $y = z^2 - \dfrac{x^2}{2}$

Hyperbolic paraboloid



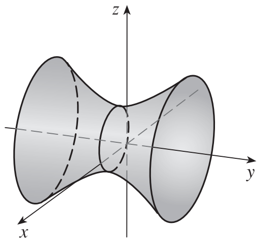
37. $z = (x - 1)^2 + (y - 3)^2$

Circular paraboloid with vertex (1, 3, 0) and axis the vertical line $x = 1$, $y = 3$



39. $\dfrac{(x - 2)^2}{5} - \dfrac{y^2}{5} + \dfrac{(z - 1)^2}{5} = 1$

Hyperboloid of one sheet with center (2, 0, 1) and axis the horizontal line $x = 2$, $z = 1$



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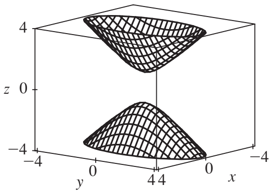

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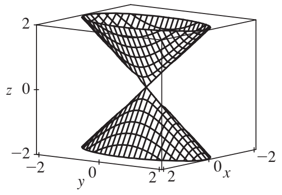

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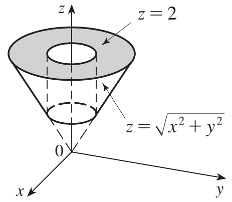
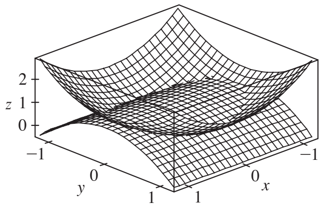


47. $x = y^2 + z^2$ **49.** $-4x = y^2 + z^2$, paraboloid

51. (a) $\dfrac{x^2}{(6378.137)^2} + \dfrac{y^2}{(6378.137)^2} + \dfrac{z^2}{(6356.523)^2} = 1$

(b) Circle (c) Ellipse

55.



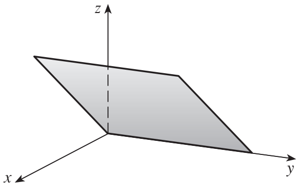
CHAPTER 12 REVIEW ■ PAGE 884

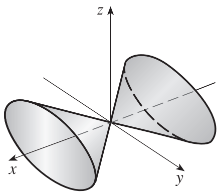
True-False Quiz

1. False **3.** False **5.** True **7.** True **9.** True
11. True **13.** True **15.** False **17.** False
19. False **21.** True

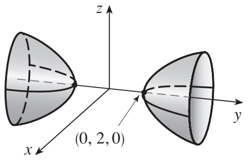
Exercises

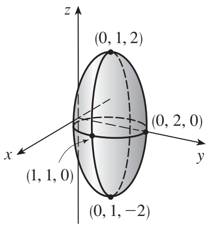
1. (a) $(x + 1)^2 + (y - 2)^2 + (z - 1)^2 = 69$
(b) $(y - 2)^2 + (z - 1)^2 = 68,\ x = 0$
(c) Center $(4, -1, -3)$, radius 5
3. $\mathbf{u} \cdot \mathbf{v} = 3\sqrt{2}$; $|\mathbf{u} \times \mathbf{v}| = 3\sqrt{2}$; out of the page
5. $-2, -4$ **7.** (a) 2 (b) -2 (c) -2 (d) 0
9. $\cos^{-1}\left(\frac{1}{3}\right) \approx 71°$ **11.** (a) $\langle 4, -3, 4\rangle$ (b) $\sqrt{41}/2$
13. ≈ 166 N, ≈ 114 N
15. $x = 4 - 3t,\ y = -1 + 2t,\ z = 2 + 3t$
17. $x = -2 + 2t,\ y = 2 - t,\ z = 4 + 5t$
19. $-4x + 3y + z = -14$ **21.** $(1, 4, 4)$ **23.** Skew
25. $x + y + z = 4$ **27.** $22/\sqrt{26}$
29. Plane **31.** Cone





33. Hyperboloid of two sheets **35.** Ellipsoid





37. $4x^2 + y^2 + z^2 = 16$

PROBLEMS PLUS ■ PAGE 887

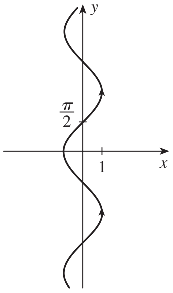
1. $\left(\sqrt{3} - \frac{3}{2}\right)$ m
3. (a) $(x + 1)/(-2c) = (y - c)/(c^2 - 1) = (z - c)/(c^2 + 1)$
(b) $x^2 + y^2 = t^2 + 1,\ z = t$ (c) $4\pi/3$
5. 20

CHAPTER 13

EXERCISES 13.1 ■ PAGE 895

1. $(-1, 3)$ **3.** $\mathbf{i} + \mathbf{j} + \mathbf{k}$ **5.** $\langle -1, \pi/2, 0\rangle$

7.



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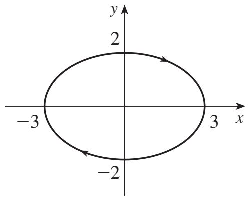

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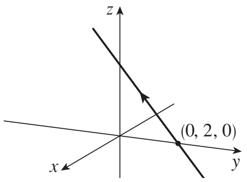

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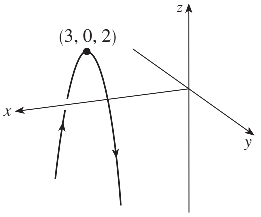

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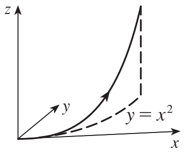

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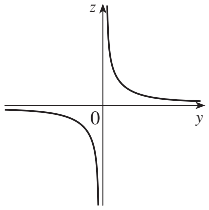

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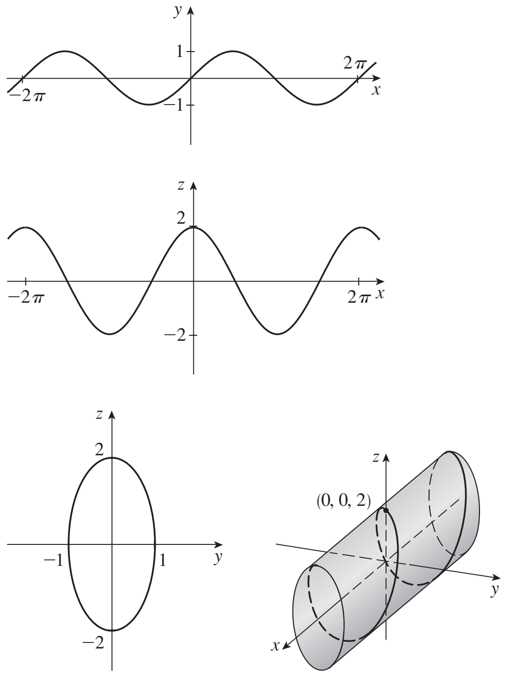
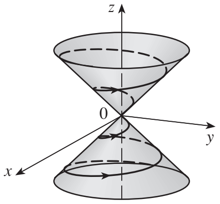


21. $\langle -2 + 7t, 1 + t, -3t \rangle$, $0 \leqslant t \leqslant 1$;
$x = -2 + 7t$, $y = 1 + t$, $z = -3t$, $0 \leqslant t \leqslant 1$

23. $\langle 3.5 - 1.7t, -1.4 + 1.7t, 2.1 \rangle$, $0 \leqslant t \leqslant 1$;
$x = 3.5 - 1.7t$, $y = -1.4 + 1.7t$, $z = 2.1$, $0 \leqslant t \leqslant 1$

25. II **27.** V **29.** IV **31.** $y = 4$ **33.** $z = -y$

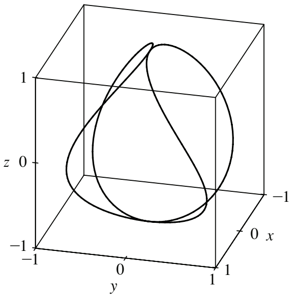
35.



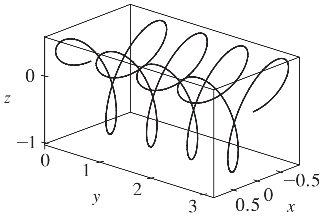
37. $y = e^{x/2}$, $z = e^x$, $z = y^2$

39. (0, 0, 0), (1, 0, 1)

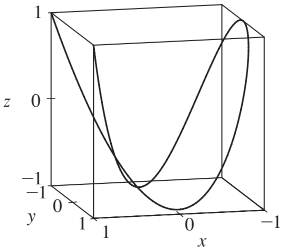
41.



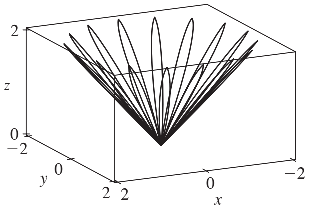
43.



45.



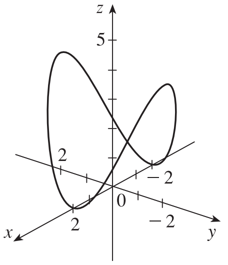
47.



51. $\mathbf{r}(t) = t\,\mathbf{i} + \frac{1}{2}(t^2 - 1)\,\mathbf{j} + \frac{1}{2}(t^2 + 1)\,\mathbf{k}$

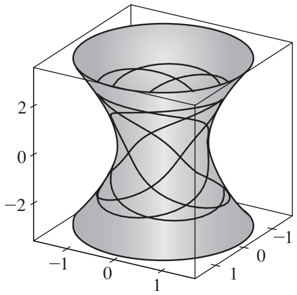
53. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos 2t\,\mathbf{k}$, $0 \leqslant t \leqslant 2\pi$

55. $x = 2\cos t$, $y = 2\sin t$, $z = 4\cos^2 t$, $0 \leqslant t \leqslant 2\pi$



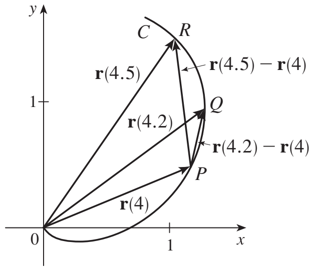
57. Yes

59. (a)

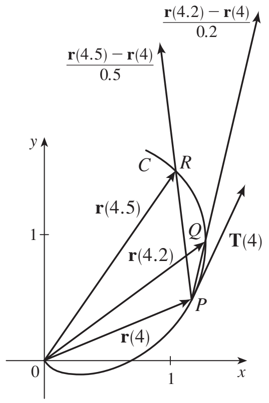


EXERCISES 13.2 ■ PAGE 902

1. (a)

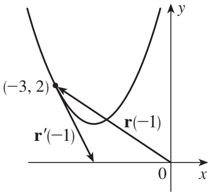

y
C
R
r(4.5)
r(4.5) − r(4)
1
Q
r(4.2)
r(4.2) − r(4)
P
r(4)
0
1
x

(b), (d)


r(4.2) − r(4) / 0.2
r(4.5) − r(4) / 0.5
y
C
R
r(4.5)
1
Q
T(4)
r(4.2)
P
r(4)
0
1
x

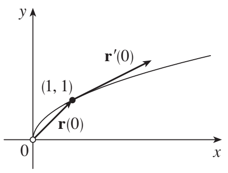
(c) $\mathbf{r}'(4) = \lim_{h \to 0} \dfrac{\mathbf{r}(4 + h) - \mathbf{r}(4)}{h}$; $\mathbf{T}(4) = \dfrac{\mathbf{r}'(4)}{|\mathbf{r}'(4)|}$

3. (a), (c)



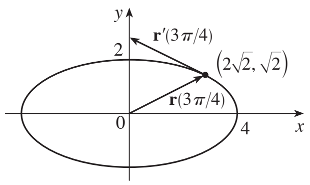
(b) $\mathbf{r}'(t) = \langle 1, 2t \rangle$

5. (a), (c)



(b) $\mathbf{r}'(t) = 2e^{2t}\,\mathbf{i} + e^t\,\mathbf{j}$

7. (a), (c)



(b) $\mathbf{r}'(t) = 4\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$

9. $\mathbf{r}'(t) = \left\langle \dfrac{1}{2\sqrt{t-2}}, 0, -\dfrac{2}{t^3} \right\rangle$

11. $\mathbf{r}'(t) = 2t\,\mathbf{i} - 2t\sin(t^2)\,\mathbf{j} + 2\sin t\cos t\,\mathbf{k}$

13. $\mathbf{r}'(t) = (t\cos t + \sin t)\,\mathbf{i} + e^t(\cos t - \sin t)\,\mathbf{j} + (\cos^2 t - \sin^2 t)\,\mathbf{k}$

15. $\mathbf{r}'(t) = \mathbf{b} + 2t\mathbf{c}$ **17.** $\left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \right\rangle$ **19.** $\frac{3}{5}\,\mathbf{j} + \frac{4}{5}\,\mathbf{k}$

21. $\langle 3/\sqrt{34}, 3/\sqrt{34}, -4/\sqrt{34} \rangle$

23. $\langle 4t^3, 1, 2t \rangle$, $\langle 4/\sqrt{21}, 1/\sqrt{21}, 2/\sqrt{21} \rangle$, $\langle 12t^2, 0, 2 \rangle$, $\langle 2, 16t^3, -12t^2 \rangle$

25. $x = 2 + 2t$, $y = 4 + 2t$, $z = 1 + t$

27. $x = 1 - t$, $y = t$, $z = 1 - t$

29. $\mathbf{r}(t) = (3 - 4t)\,\mathbf{i} + (4 + 3t)\,\mathbf{j} + (2 - 6t)\,\mathbf{k}$

31. $x = t$, $y = 1 - t$, $z = 2t$

33. $x = -\pi - t$, $y = \pi + t$, $z = -\pi t$

35. 66° **37.** $2\,\mathbf{i} - 4\,\mathbf{j} + 32\,\mathbf{k}$

39. $(\ln 2)\,\mathbf{i} + (\pi/4)\,\mathbf{j} + \frac{1}{2}\ln 2\,\mathbf{k}$

41. $\tan^{-1} t\,\mathbf{i} + \frac{1}{2}e^{t^2}\mathbf{j} + \frac{2}{3}t^{3/2}\,\mathbf{k} + \mathbf{C}$

43. $t^2\mathbf{i} + t^3\mathbf{j} + \left(\frac{2}{3}t^{3/2} - \frac{2}{3}\right)\mathbf{k}$

49. $2t\cos t + 2\sin t - 2\cos t\sin t$ **51.** 35

EXERCISES 13.3 ■ PAGE 913

1. (a) $2\sqrt{21}$ **3.** $10\sqrt{10}$ **5.** $e - e^{-1}$ **7.** $\frac{1}{27}(13^{3/2} - 8)$

9. 18.6833 **11.** 10.3311 **13.** 42

15. (a) $s(t) = \sqrt{26}\,(t - 1)$;

$\mathbf{r}(t(s)) = \left(4 - \dfrac{s}{\sqrt{26}}\right)\mathbf{i} + \left(\dfrac{4s}{\sqrt{26}} + 1\right)\mathbf{j} + \left(\dfrac{3s}{\sqrt{26}} + 3\right)\mathbf{k}$

(b) $\left(4 - \dfrac{4}{\sqrt{26}}, \dfrac{16}{\sqrt{26}} + 1, \dfrac{12}{\sqrt{26}} + 3\right)$

17. $(3\sin 1, 4, 3\cos 1)$

19. (a) $\dfrac{1}{\sqrt{5}}\langle 2, \sin t, \cos t \rangle$, $\langle 0, \cos t, -\sin t \rangle$ (b) $1/(5t)$

21. (a) $\dfrac{1}{\sqrt{1 + 4t^2}}\langle 1, 2t, 0 \rangle$, $\dfrac{1}{\sqrt{1 + 4t^2}}\langle -2t, 1, 0 \rangle$

(b) $2/(1 + 4t^2)^{3/2}$

23. (a) $\dfrac{1}{\sqrt{1 + 5t^2}}\langle 1, t, 2t \rangle$, $\dfrac{1}{\sqrt{5 + 25t^2}}\langle -5t, 1, 2 \rangle$

(b) $\sqrt{5}/(1 + 5t^2)^{3/2}$

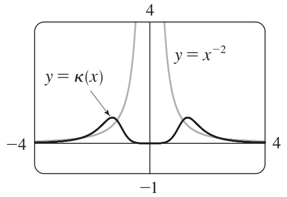
25. $6t^2/(9t^4 + 4t^2)^{3/2}$ **27.** $\dfrac{\sqrt{6}}{2(3t^2 + 1)^2}$

29. $\frac{1}{7}\sqrt{19/14}$ **31.** $12x^2/(1 + 16x^6)^{3/2}$

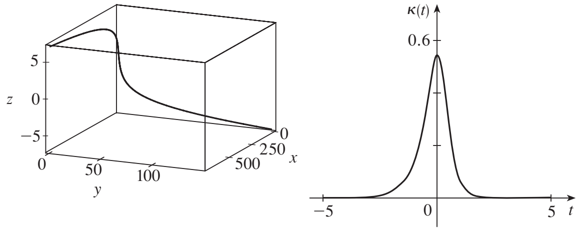
33. $e^x|x + 2|/[1 + (xe^x + e^x)^2]^{3/2}$

35. $\left(-\frac{1}{2}\ln 2, 1/\sqrt{2}\right)$; approaches 0 **37.** (a) P (b) 1.3, 0.7

39.

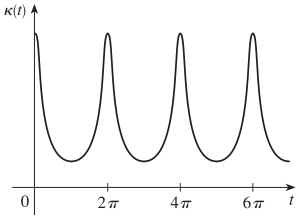


41.



43. a is $y = f(x)$, b is $y = \kappa(x)$

45. $\kappa(t) = \dfrac{6\sqrt{4\cos^2 t - 12\cos t + 13}}{(17 - 12\cos t)^{3/2}}$



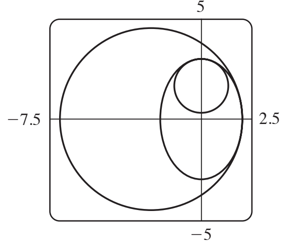
largest at integer multiples of 2π

47. $6t^2/(4t^2 + 9t^4)^{3/2}$

49. $1/(\sqrt{2}e^t)$ **51.** $\left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$, $\left\langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle$, $\left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$

53. $x - 2z = -4\pi$, $2x + z = 2\pi$

55. $\left(x + \frac{5}{2}\right)^2 + y^2 = \frac{81}{4}$, $x^2 + \left(y - \frac{5}{3}\right)^2 = \frac{16}{9}$



57. $(-1, -3, 1)$

59. $2x + y + 4z = 7$, $6x - 8y - z = -3$ **67.** 0

69. $-2/(e^{2t} + e^{-2t} + 4)$, $-\frac{1}{3}$

75. (b) $\mathbf{r}_e(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j} + t\,\mathbf{k}$

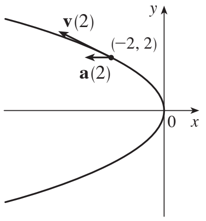
(c) $\mathbf{r}_e(t) = -4t^3\,\mathbf{i} + \left(3t^2 + \frac{1}{2}\right)\mathbf{j}$ or $y_e = \frac{1}{2} + 3(x/4)^{2/3}$

77. 2.07×10^{10} Å ≈ 2 m

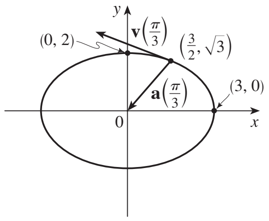
EXERCISES 13.4 ■ PAGE 923

1. (a) $1.8\mathbf{i} - 3.8\mathbf{j} - 0.7\mathbf{k}$, $2.0\mathbf{i} - 2.4\mathbf{j} - 0.6\mathbf{k}$, $2.8\mathbf{i} + 1.8\mathbf{j} - 0.3\mathbf{k}$, $2.8\mathbf{i} + 0.8\mathbf{j} - 0.4\mathbf{k}$
(b) $2.4\mathbf{i} - 0.8\mathbf{j} - 0.5\mathbf{k}$, 2.58

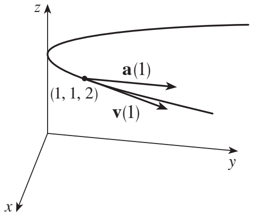
3. $\mathbf{v}(t) = \langle -t, 1\rangle$
$\mathbf{a}(t) = \langle -1, 0\rangle$
$|\mathbf{v}(t)| = \sqrt{t^2 + 1}$



5. $\mathbf{v}(t) = -3\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$
$\mathbf{a}(t) = -3\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}$
$|\mathbf{v}(t)| = \sqrt{5\sin^2 t + 4}$



7. $\mathbf{v}(t) = \mathbf{i} + 2t\,\mathbf{j}$
$\mathbf{a}(t) = 2\,\mathbf{j}$
$|\mathbf{v}(t)| = \sqrt{1 + 4t^2}$



9. $\langle 2t + 1, 2t - 1, 3t^2\rangle$, $\langle 2, 2, 6t\rangle$, $\sqrt{9t^4 + 8t^2 + 2}$

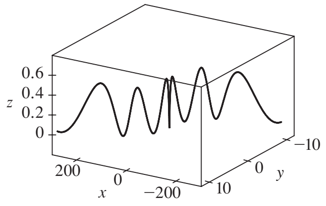
11. $\sqrt{2}\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k}$, $e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}$, $e^t + e^{-t}$

13. $e^t[(\cos t - \sin t)\mathbf{i} + (\sin t + \cos t)\mathbf{j} + (t + 1)\mathbf{k}]$, $e^t[-2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + (t + 2)\mathbf{k}]$, $e^t\sqrt{t^2 + 2t + 3}$

15. $\mathbf{v}(t) = (2t + 3)\,\mathbf{i} - \mathbf{j} + t^2\,\mathbf{k}$,
$\mathbf{r}(t) = (t^2 + 3t)\,\mathbf{i} + (1 - t)\,\mathbf{j} + \left(\frac{1}{3}t^3 + 1\right)\mathbf{k}$

17. (a) $\mathbf{r}(t) = \left(\frac{1}{3}t^3 + t\right)\mathbf{i} + (t - \sin t + 1)\,\mathbf{j} + \left(\frac{1}{4} - \frac{1}{4}\cos 2t\right)\mathbf{k}$

(b)



19. $t = 4$

21. $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{5}{2}t^2\,\mathbf{k}$, $|\mathbf{v}(t)| = \sqrt{25t^2 + 2}$

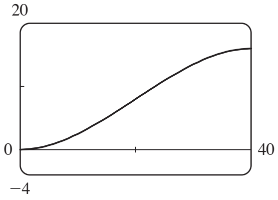
23. (a) ≈3535 m (b) ≈1531 m (c) 200 m/s

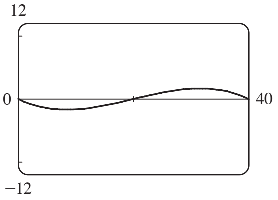
25. ≈30 m/s **27.** ≈ 198 m/s

29. $13.0° < \theta < 36.0°$, $55.4° < \theta < 85.5°$

31. $(250, -50, 0)$; $10\sqrt{93} \approx 96.4$ m/s

33. (a) 16 m (b) ≈23.6° upstream





35. The path is contained in a circle that lies in a plane perpendicular to **c** with center on a line through the origin in the direction of **c**.

37. $\dfrac{4 + 18t^2}{\sqrt{4 + 9t^2}}$, $\dfrac{6t}{\sqrt{4 + 9t^2}}$ **39.** 0, 1

41. $\dfrac{7}{\sqrt{30}}$, $\sqrt{\dfrac{131}{30}}$

43. 4.5 cm/s^2, 9.0 cm/s^2 **45.** $t = 1$

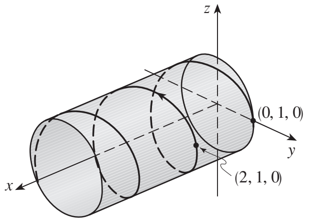
CHAPTER 13 REVIEW ■ PAGE 927

True-False Quiz

1. True **3.** False **5.** False **7.** False
9. True **11.** False **13.** True **15.** True

Exercises

1. (a)



(b) $\mathbf{r}'(t) = \mathbf{i} - \pi\sin\pi t\,\mathbf{j} + \pi\cos\pi t\,\mathbf{k}$,
$\mathbf{r}''(t) = -\pi^2\cos\pi t\,\mathbf{j} - \pi^2\sin\pi t\,\mathbf{k}$

3. $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + (5 - 4\cos t)\mathbf{k}$, $0 \leqslant t \leqslant 2\pi$

5. $\frac{1}{3}\mathbf{i} - (2/\pi^2)\mathbf{j} + (2/\pi)\mathbf{k}$ **7.** 86.631 **9.** 90°

11. (a) $\dfrac{1}{\sqrt{13}}\langle 3\sin t, -3\cos t, 2\rangle$ (b) $\langle \cos t, \sin t, 0\rangle$

(c) $\dfrac{1}{\sqrt{13}}\langle -2\sin t, 2\cos t, 3\rangle$

(d) $\dfrac{3}{13\sin t\cos t}$ or $\dfrac{3}{13}\sec t\csc t$

(e) $\dfrac{2}{13\sin t\cos t}$ or $\dfrac{2}{13}\sec t\csc t$

13. $12/17^{3/2}$ **15.** $x - 2y + 2\pi = 0$

17. $\mathbf{v}(t) = (1 + \ln t)\mathbf{i} + \mathbf{j} - e^{-t}\mathbf{k}$,
$|\mathbf{v}(t)| = \sqrt{2 + 2\ln t + (\ln t)^2 + e^{-2t}}$, $\mathbf{a}(t) = (1/t)\mathbf{i} + e^{-t}\mathbf{k}$

19. $\mathbf{r}(t) = (t^3 + t)\,\mathbf{i} + (t^4 - t)\,\mathbf{j} + \left(3t - t^3\right)\mathbf{k}$

21. ≈37.3°, ≈157.4 m

23. (c) $-2e^{-t}\,\mathbf{v}_d + e^{-t}\,\mathbf{R}$

PROBLEMS PLUS ■ PAGE 930

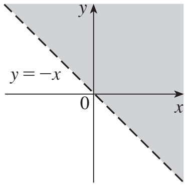
1. (a) $\mathbf{v} = \omega R(-\sin \omega t\,\mathbf{i} + \cos \omega t\,\mathbf{j})$ (c) $\mathbf{a} = -\omega^2\mathbf{r}$
3. (a) 90°, $v_0^2/(2g)$
5. (a) $\approx$0.25 m to the right of the table's edge, $\approx$ 4.9 m/s
(b) $\approx$ 5.9° (c) $\approx$ 0.56 m to the right of the table's edge
7. 56°
9. $(a_2b_3 - a_3b_2)(x - c_1) + (a_3b_1 - a_1b_3)(y - c_2) + (a_1b_2 - a_2b_1)(z - c_3) = 0$

CHAPTER 14

EXERCISES 14.1 ■ PAGE 946

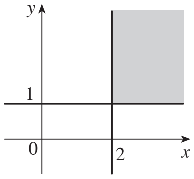
1. (a) $-\frac{3}{7}$ (b) $\frac{4}{5}$ (c) $\dfrac{(x+h)^2y}{2(x+h) - y^2}$ (d) $\dfrac{x^2}{2 - x}$

3. (a) 9 ln 4 (b) $\{(x, y) \mid y > -x\}$

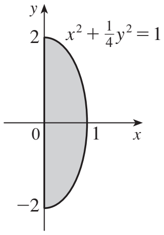


(c) $\mathbb{R}$
5. (a) 1 (b) $\{(x, y, z) \mid z \leqslant x/2, y \leqslant 0\}$, the points on or below the plane $z = x/2$ that are to the right of the xz-plane

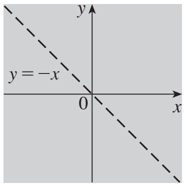
7. $\{(x, y) \mid x \geqslant 2, y \geqslant 1\}$



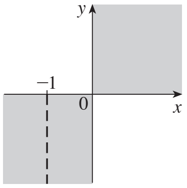
9. $\left\{(x, y) \mid x^2 + \frac{1}{4}y^2 \leqslant 1, x \geqslant 0\right\}$



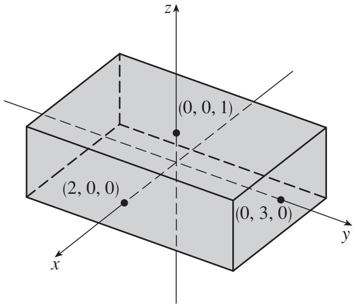
11. $\{(x, y) \mid y \neq -x\}$



13. $\{(x, y) \mid xy \geqslant 0, x \neq -1\}$



15. $\{(x, y, z) \mid -2 \leqslant x \leqslant 2, -3 \leqslant y \leqslant 3, -1 \leqslant z \leqslant 1\}$



17. (a) $\approx$ 1.90 m^2; the surface area of a person 178 cm tall who weighs 73 kg is approximately 1.90 square meters.

19. (a) −27; a temperature of −15°C with wind blowing at 40 km/h feels equivalent to about −27°C without wind.

(b) When the temperature is −20°C, what wind speed gives a wind chill of −30°C? 20 km/h

(c) With a wind speed of 20 km/h, what temperature gives a wind chill of −49°C? −35°C

(d) A function of wind speed that gives wind-chill values when the temperature is −5°C

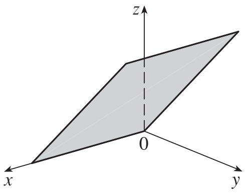
(e) A function of temperature that gives wind-chill values when the wind speed is 50 km/h

21. (a) 2.4; a 40 km/h wind blowing in the open sea for 15 h will create waves about 2.4 m high.

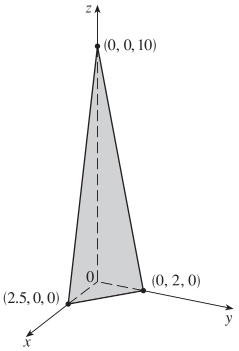
(b) $f(30, t)$ is a function of t giving the wave heights produced by 30 km/h winds blowing for t hours.

(c) $f(v, 30)$ is a function of v giving the wave heights produced by winds of speed v blowing for 30 hours.

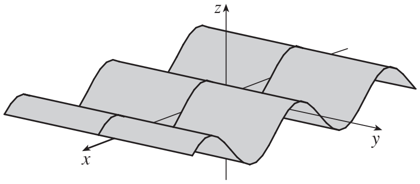
23. $z = y$, plane through the x-axis



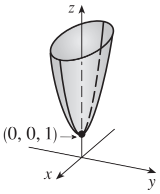
25. $4x + 5y + z = 10$, plane


z
(0, 0, 10)
0
(2.5, 0, 0)
(0, 2, 0)
y
x

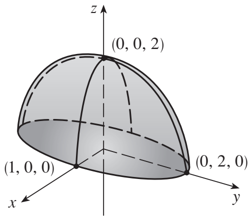
27. $z = \sin x$, cylinder



29. $z = x^2 + 4y^2 + 1$, elliptic paraboloid

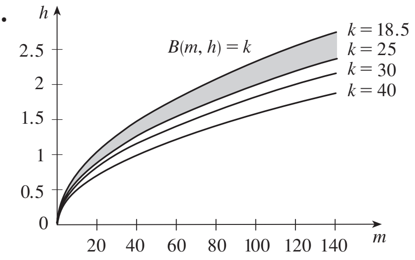


31. $z = \sqrt{4 - 4x^2 - y^2}$, top half of ellipsoid


z
(0, 0, 2)
(1, 0, 0)
(0, 2, 0)
x
y

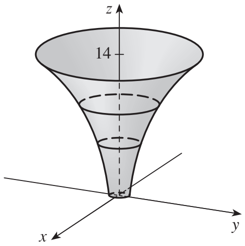
33. $\approx 56, \approx 35$ **35.** 11°C, 19.5°C **37.** Steep; nearly flat

39.



No

41.



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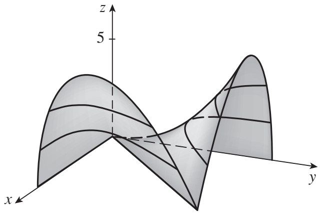
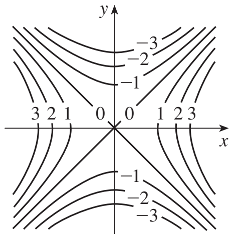
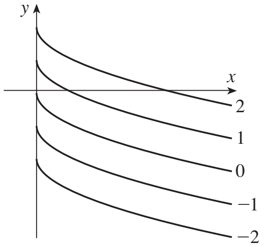


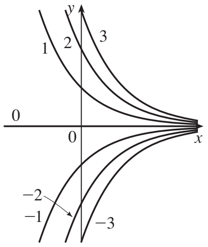
45. $x^2 - y^2 = k$



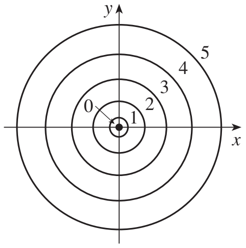
47. $y = -\sqrt{x} + k$



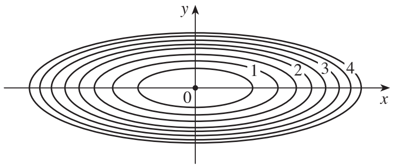
49. $y = ke^{-x}$

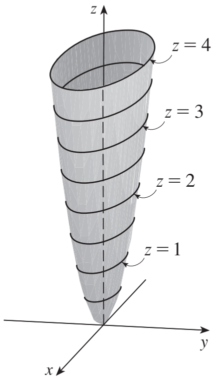


51. $x^2 + y^2 = k^3$ $(k \geqslant 0)$

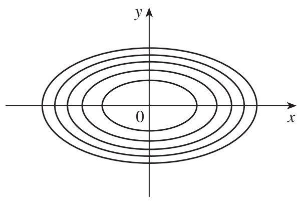


53. $x^2 + 9y^2 = k$




z
z = 4
z = 3
z = 2
z = 1
x
y

55.



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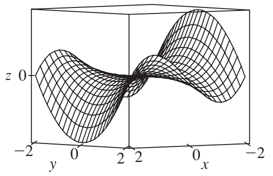
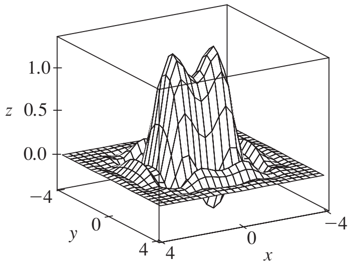
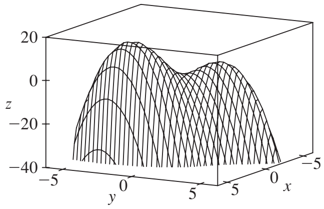


59.



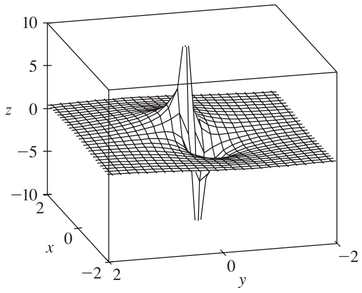
61. (a) C (b) II **63.** (a) F (b) I
65. (a) B (b) VI **67.** Family of parallel planes
69. $k = 0$: cone with axis the z-axis;
$k > 0$: family of hyperboloids of one sheet with axis the z-axis;
$k < 0$: family of hyperboloids of two sheets with axis the z-axis
71. (a) Shift the graph of f upward 2 units
(b) Stretch the graph of f vertically by a factor of 2
(c) Reflect the graph of f about the xy-plane
(d) Reflect the graph of f about the xy-plane and then shift it upward 2 units

73.



f appears to have a maximum value of about 15. There are two local maximum points but no local minimum point.

75.



The function values approach 0 as x, y become large; as (x, y) approaches the origin, f approaches $\pm\infty$ or 0, depending on the direction of approach.

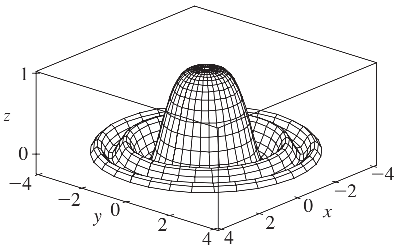
77. If $c = 0$, the graph is a cylindrical surface. For $c > 0$, the level curves are ellipses. The graph curves upward as we leave the origin, and the steepness increases as c increases. For $c < 0$, the level curves are hyperbolas. The graph curves upward in the y-direction and downward, approaching the xy-plane, in the x-direction giving a saddle-shaped appearance near (0, 0, 1).

79. $c = -2, 0, 2$ **81.** (b) $y = 0.75x + 0.01$

EXERCISES 14.2 ■ PAGE 960

1. Nothing; if f is continuous, then $f(3, 1) = 6$ **3.** $-\frac{5}{2}$
5. 56 **7.** −6 **9.** $\pi/2$ **11.** $-\frac{1}{2}$ **19.** 125
21. 0 **23.** Does not exist **25.** 2 **27.** −2
29. Does not exist **31.** 0 **33.** 0
35. The graph shows that the function approaches different numbers along different lines.
37. $h(x, y) = (2x + 3y - 6)^2 + \sqrt{2x + 3y - 6}$; $\{(x, y) \mid 2x + 3y \geqslant 6\}$
39. Along the line $y = x$ **41.** $\mathbb{R}^2$
43. $\{(x, y) \mid x^2 + y^2 \neq 1\}$ **45.** $\{(x, y) \mid x^2 + y^2 \leqslant 1, x \geqslant 0\}$
47. $\{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant 1\}$
49. $\{(x, y) \mid (x, y) \neq (0, 0)\}$ **51.** 0 **53.** −1

55.



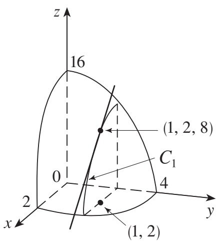
EXERCISES 14.3 ■ PAGE 969

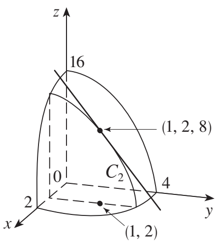
1. $f_T(34, 75) \approx 2$°C; for a temperature of 34°C and relative humidity of 60%, the apparent temperature rises by 2°C for each degree the actual temperature increases. $f_H(34, 75) \approx 0.3$°C; for a temperature of 34°C and relative humidity of 60%, the apparent temperature rises by 0.3°C for each percent that the relative humidity increases.

3. (a) The rate of change of temperature as longitude varies, with latitude and time fixed; the rate of change as only latitude varies; the rate of change as only time varies
(b) Positive, negative, positive

5. (a) Negative (b) Negative

7. $f_x(1, 2) = -8 =$ slope of C_1, $f_y(1, 2) = -4 =$ slope of C_2





9. $f_x(x, y) = 4x^3 + 5y^3$, $f_y(x, y) = 15xy^2$

11. $g_x(x, y) = 3x^2 \sin y$, $g_y(x, y) = x^3 \cos y$

13. $\dfrac{\partial z}{\partial x} = \dfrac{1}{x + t^2}$, $\dfrac{\partial z}{\partial t} = \dfrac{2t}{x + t^2}$

15. $f_x(x, y) = y^2 e^{xy}$, $f_y(x, y) = e^{xy} + xye^{xy}$

17. $g_x(x, y) = 5y(1 + 2xy)(x + x^2y)^4$,
$g_y(x, y) = 5x^2y(x + x^2y)^4 + (x + x^2y)^5$

19. $f_x(x, y) = \dfrac{(ad - bc)y}{(cx + dy)^2}$, $f_y(x, y) = \dfrac{(bc - ad)x}{(cx + dy)^2}$

21. $g_u(u, v) = 10uv(u^2v - v^3)^4$,
$g_v(u, v) = 5(u^2 - 3v^2)(u^2v - v^3)^4$

23. $R_p(p, q) = \dfrac{q^2}{1 + p^2q^4}$, $R_q(p, q) = \dfrac{2pq}{1 + p^2q^4}$

25. $F_x(x, y) = \cos(e^x)$, $F_y(x, y) = -\cos(e^y)$

27. $f_x = 3x^2yz^2$, $f_y = x^3z^2 + 2z$, $f_z = 2x^3yz + 2y$

29. $\partial w/\partial x = 1/(x + 2y + 3z)$, $\partial w/\partial y = 2/(x + 2y + 3z)$,
$\partial w/\partial z = 3/(x + 2y + 3z)$

31. $\partial p/\partial t = 2t^3/\sqrt{t^4 + u^2 \cos v}$,
$\partial p/\partial u = u \cos v/\sqrt{t^4 + u^2 \cos v}$,
$\partial p/\partial v = -u^2 \sin v/\left(2\sqrt{t^4 + u^2 \cos v}\,\right)$

33. $h_x = 2xy \cos(z/t)$, $h_y = x^2 \cos(z/t)$,
$h_z = (-x^2y/t) \sin(z/t)$, $h_t = (x^2yz/t^2) \sin(z/t)$

35. $\partial u/\partial x_i = x_i/\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

37. 1 **39.** $\frac{1}{6}$ **41.** $\dfrac{\partial z}{\partial x} = -\dfrac{x}{3z}$, $\dfrac{\partial z}{\partial y} = -\dfrac{2y}{3z}$

43. $\dfrac{\partial z}{\partial x} = \dfrac{yz}{e^z - xy}$, $\dfrac{\partial z}{\partial y} = \dfrac{xz}{e^z - xy}$

45. (a) $f'(x), g'(y)$ (b) $f'(x + y), f'(x + y)$

47. $f_{xx} = 12x^2y - 12xy^2$, $f_{xy} = 4x^3 - 12x^2y = f_{yx}$, $f_{yy} = -4x^3$

49. $z_{xx} = \dfrac{8y}{(2x + 3y)^3}$, $z_{xy} = \dfrac{6y - 4x}{(2x + 3y)^3} = z_{yx}$,
$z_{yy} = -\dfrac{12x}{(2x + 3y)^3}$

51. $v_{ss} = 2 \cos(s^2 - t^2) - 4s^2 \sin(s^2 - t^2)$,
$v_{st} = 4st \sin(s^2 - t^2) = v_{ts}$,
$v_{tt} = -2 \cos(s^2 - t^2) - 4t^2 \sin(s^2 - t^2)$

57. $24xy^2 - 6y$, $24x^2y - 6x$

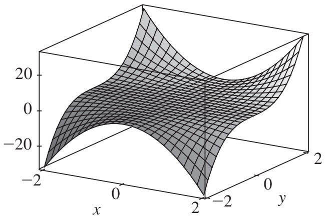
59. $(2x^2y^2z^5 + 6xyz^3 + 2z)e^{xyz^2}$

61. $\frac{3}{4}v(u + v^2)^{-5/2}$ **63.** $4/(y + 2z)^3$, 0

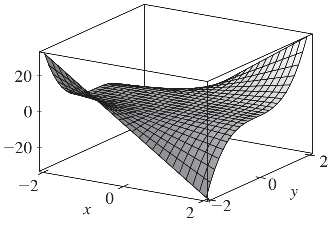
65. $f_x(x, y) = y^2 - 3x^2y$, $f_y(x, y) = 2xy - x^3$

67. $6yz^2$ **69.** $c = f$, $b = f_x$, $a = f_y$

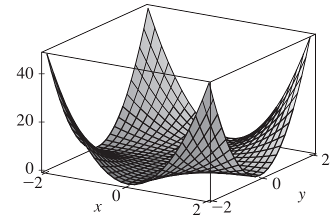
71.



$f(x, y) = x^2y^3$



$f_x(x, y) = 2xy^3$



$f_y(x, y) = 3x^2y^2$

73. ≈ 12.2, ≈ 16.8, ≈ 23.25 **83.** R^2/R_1^2

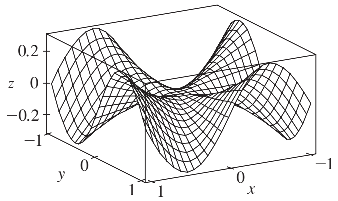
85. $\dfrac{\partial T}{\partial P} = \dfrac{V - nb}{nR}$, $\dfrac{\partial P}{\partial V} = \dfrac{2n^2a}{V^3} - \dfrac{nRT}{(V - nb)^2}$

87. (a) ≈ 0.0035; for a person 178 cm tall who weighs 73 kg, an increase in weight causes the surface area to increase at a rate of about 0.0035 m^2/kg. (b) ≈ 0.0145; for a person 178 cm tall who weighs 73 kg, an increase in height (with no change in weight) causes the surface area to increase at a rate of about 0.0145 m^2/kg of height.

89. $\partial P/\partial v = 3Av^2 - \dfrac{B(mg/x)^2}{v^2}$ is the rate of change of the power needed during flapping mode with respect to the bird's velocity when the mass and fraction of flapping time remain constant; $\partial P/\partial x = -\dfrac{2Bm^2g^2}{x^3v}$ is the rate at which the power changes when only the fraction of time spent in flapping mode varies; $\partial P/\partial m = \dfrac{2Bmg^2}{x^2v}$ is the rate of change of the power when only the mass varies.

93. $x = 1 + t,\ y = 2,\ z = 2 - 2t$ **95.** No **99.** -2

101. (a)



(b) $f_x(x, y) = \dfrac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2},\ f_y(x, y) = \dfrac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2}$

(c) 0, 0 (e) No, because f_{xy} and f_{yx} are not continuous.

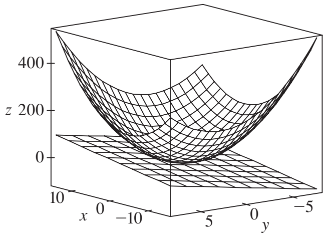
EXERCISES 14.4 ■ PAGE 981

1. $z = -4x - 4y + 24$ **3.** $z = 4x - y - 6$

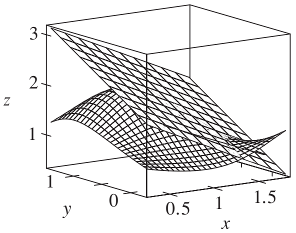
5. $z = x - y + 1$ **7.** $z = -2x - y - 3$

9. $x + y + z = 0$

11.



13.



15. $12x - 16y + 32$ **17.** $6x + 4y - 23$

19. $2x + y - 1$ **21.** $2x + 2y + \pi - 4$ **25.** 6.3

27. $\frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z$; 6.9914 **29.** $2T + 0.3H - 40.5$; 44.4°C

31. $dm = 5p^4q^3\,dp + 3p^5q^2\,dq$

33. $dz = -2e^{-2x}\cos 2\pi t\,dx - 2\pi e^{-2x}\sin 2\pi t\,dt$

35. $dH = 2xy^4\,dx + (4x^2y^3 + 3y^2z^5)\,dy + 5y^3z^4\,dz$

37. $dR = \beta^2\cos\gamma\,d\alpha + 2\alpha\beta\cos\gamma\,d\beta - \alpha\beta^2\sin\gamma\,d\gamma$

39. $\Delta z = 0.9225,\ dz = 0.9$ **41.** 5.4 cm^2 **43.** 16 cm^3

45. (a) $5.89\pi\varepsilon$ m^3 (b) ≈ 0.0015 m ≈ 0.15 cm

47. $\approx -0.0165mg$; decrease **49.** $\frac{1}{17} \approx 0.059\ \Omega$

51. (a) $0.8264m - 34.56h + 38.02$ (b) 18.801

EXERCISES 14.5 ■ PAGE 991

1. $36t^3 + 15t^4$ **3.** $2t(y^3 - 2xy + 3xy^2 - x^2)$

5. $\dfrac{1}{2\sqrt{t}}\cos x\cos y + \dfrac{1}{t^2}\sin x\sin y$

7. $e^{y/z}[2t - (x/z) - (2xy/z^2)]$

9. $\partial z/\partial s = 10s + 14t,\ \partial z/\partial t = 14s + 20t$

11. $\partial z/\partial s = 5(x - y)^4(2st - t^2),\ \partial z/\partial t = 5(x - y)^4(s^2 - 2st)$

13. $\dfrac{\partial z}{\partial s} = \dfrac{3\sin t - 2t\sin s}{3x + 2y},\ \dfrac{\partial z}{\partial t} = \dfrac{3s\cos t + 2\cos s}{3x + 2y}$

15. $\dfrac{\partial z}{\partial s} = -\dfrac{t\sin\theta}{r^2} + \dfrac{2s\cos\theta}{r},\ \dfrac{\partial z}{\partial t} = -\dfrac{s\sin\theta}{r^2} + \dfrac{2t\cos\theta}{r}$

17. 42 **19.** 7, 2

21. $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r},\ \dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s},$

$\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial t}$

23. $\dfrac{\partial T}{\partial x} = \dfrac{\partial T}{\partial p}\dfrac{\partial p}{\partial x} + \dfrac{\partial T}{\partial q}\dfrac{\partial q}{\partial x} + \dfrac{\partial T}{\partial r}\dfrac{\partial r}{\partial x},$

$\dfrac{\partial T}{\partial y} = \dfrac{\partial T}{\partial p}\dfrac{\partial p}{\partial y} + \dfrac{\partial T}{\partial q}\dfrac{\partial q}{\partial y} + \dfrac{\partial T}{\partial r}\dfrac{\partial r}{\partial y},$

$\dfrac{\partial T}{\partial z} = \dfrac{\partial T}{\partial p}\dfrac{\partial p}{\partial z} + \dfrac{\partial T}{\partial q}\dfrac{\partial q}{\partial z} + \dfrac{\partial T}{\partial r}\dfrac{\partial r}{\partial z}$

25. 1582, 3164, −700 **27.** $2\pi,\ -2\pi$

29. $\frac{5}{144}, -\frac{5}{96}, \frac{5}{144}$ **31.** $\dfrac{2x + y\sin x}{\cos x - 2y}$

33. $\dfrac{1 + x^4y^2 + y^2 + x^4y^4 - 2xy}{x^2 - 2xy - 2x^5y^3}$

35. $-\dfrac{x}{3z}, -\dfrac{2y}{3z}$ **37.** $\dfrac{yz}{e^z - xy}, \dfrac{xz}{e^z - xy}$

39. 2°C/s **41.** ≈ -0.33 m/s per minute

43. (a) 6 m^3/s (b) 10 m^2/s (c) 0 m/s

45. ≈ -0.27 L/s **47.** $-1/(12\sqrt{3})$ rad/s

49. (a) $\partial z/\partial r = (\partial z/\partial x)\cos\theta + (\partial z/\partial y)\sin\theta$, $\partial z/\partial\theta = -(\partial z/\partial x)\,r\sin\theta + (\partial z/\partial y)\,r\cos\theta$

53. $4rs\dfrac{\partial^2 z}{\partial x^2} + (4r^2 + 4s^2)\dfrac{\partial^2 z}{\partial x\,\partial y} + 4rs\dfrac{\partial^2 z}{\partial y^2} + 2\dfrac{\partial z}{\partial y}$

EXERCISES 14.6 ■ PAGE 1005

1. ≈ -0.08 mb/km **3.** ≈ 0.778 **5.** $\sqrt{2}/2$

7. $5\sqrt{2}/74$ **9.** (a) $\nabla f(x, y) = (1/y)\mathbf{i} - (x/y^2)\mathbf{j}$

(b) $\mathbf{i} - 2\mathbf{j}$ (c) -1

11. (a) $\langle 2xyz - yz^3, x^2z - xz^3, x^2y - 3xyz^2\rangle$

(b) $\langle -3, 2, 2\rangle$ (c) $\frac{2}{5}$

13. $\dfrac{4 - 3\sqrt{3}}{10}$ **15.** $7/(2\sqrt{5})$ **17.** 1 **19.** $\frac{23}{42}$

21. $-\frac{56}{5}$ **23.** $\frac{2}{5}$ **25.** $-\frac{18}{7}$ **27.** $20\sqrt{10}, \langle 20, -60\rangle$

29. $1, \langle 0, 1\rangle$ **31.** $\frac{3}{4}, \langle 1, -2, -2\rangle$

33. (b) $\langle -12, 92\rangle, -4\sqrt{538}$

35. All points on the line $y = x + 1$ **37.** (a) $-40/(3\sqrt{3})$

39. (a) $32/\sqrt{3}$ (b) $\langle 38, 6, 12\rangle$ (c) $2\sqrt{406}$

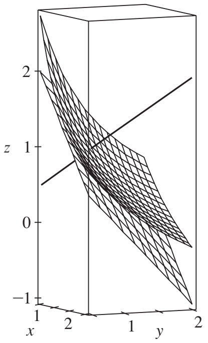
41. $\frac{327}{13}$ **45.** $\frac{774}{25}$

47. (a) $x + y + z = 11$ (b) $x - 3 = y - 3 = z - 5$

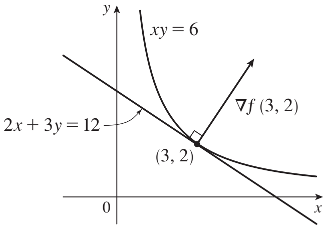
49. (a) $x + 2y + 6z = 12$ (b) $x - 2 = \dfrac{y - 2}{2} = \dfrac{z - 1}{6}$

51. (a) $x + y + z = 1$ (b) $x = y = z - 1$

53.



55. $\langle 2, 3\rangle$, $2x + 3y = 12$



61. No **65.** $\left(-\frac{5}{4}, -\frac{5}{4}, \frac{25}{8}\right)$

69. $x = -1 - 10t, y = 1 - 16t, z = 2 - 12t$

71. $(-1, 0, 1)$; $\approx 7.8°$

75. If $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$, then $af_x + bf_y$ and $cf_x + df_y$ are known, so we solve linear equations for f_x and f_y.

EXERCISES 14.7 ■ PAGE 1016

1. (a) f has a local minimum at (1, 1).
(b) f has a saddle point at (1, 1).

3. Local minimum at (1, 1), saddle point at (0, 0)

5. Minimum $f\left(\frac{1}{3}, -\frac{2}{3}\right) = -\frac{1}{3}$

7. Minima $f(-2, -1) = -3$, $f(8, 4) = -128$, saddle point at (0, 0)

9. Saddle points at (1, 1), (−1, −1)

11. Maximum $f(1, 4) = 14$

13. Maximum $f(-1, 0) = 2$, minimum $f(1, 0) = -2$, saddle points at $(0, \pm 1)$

15. Maximum $f(0, -1) = 2$, minima $f(\pm 1, 1) = -3$, saddle points at (0, 1), $(\pm 1, -1)$

17. Maximum $f\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{27}$, saddle points at (0, 0), (1, 0), (0, 1)

19. None

21. Minima $f(0, 1) = f(\pi, -1) = f(2\pi, 1) = -1$, saddle points at $(\pi/2, 0)$, $(3\pi/2, 0)$

25. Minima $f(1, \pm 1) = f(-1, \pm 1) = 3$

27. Maximum $f(\pi/3, \pi/3) = 3\sqrt{3}/2$, minimum $f(5\pi/3, 5\pi/3) = -3\sqrt{3}/2$, saddle point at (π, π)

29. Minima $f(0, -0.794) \approx -1.191$, $f(\pm 1.592, 1.267) \approx -1.310$, saddle points $(\pm 0.720, 0.259)$, lowest points $(\pm 1.592, 1.267, -1.310)$

31. Maximum $f(0.170, -1.215) \approx 3.197$, minima $f(-1.301, 0.549) \approx -3.145$, $f(1.131, 0.549) \approx -0.701$, saddle points $(-1.301, -1.215)$, $(0.170, 0.549)$, $(1.131, -1.215)$, no highest or lowest point

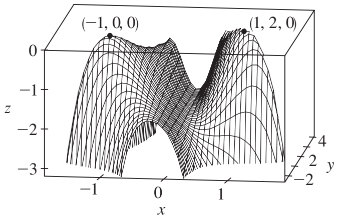
33. Maximum $f(0, \pm 2) = 4$, minimum $f(1, 0) = -1$

35. Maximum $f(\pm 1, 1) = 7$, minimum $f(0, 0) = 4$

37. Maximum $f(0, 3) = f(2, 3) = 7$, minimum $f(1, 1) = -2$

39. Maximum $f(1, 0) = 2$, minimum $f(-1, 0) = -2$

41.



43. $2/\sqrt{3}$ **45.** $\left(2, 1, \sqrt{5}\right)$, $\left(2, 1, -\sqrt{5}\right)$ **47.** $\frac{100}{3}, \frac{100}{3}, \frac{100}{3}$

49. $8r^3/\left(3\sqrt{3}\right)$ **51.** $\frac{4}{3}$ **53.** Cube, edge length $c/12$

55. Square base of side 40 cm, height 20 cm **57.** $L^3/\left(3\sqrt{3}\right)$

59. (a) $H = -p_1 \ln p_1 - p_2 \ln p_2 - (1 - p_1 - p_2)\ln(1 - p_1 - p_2)$
(b) $\{(p_1, p_2) \mid 0 < p_1 < 1, p_2 < 1 - p_1\}$
(c) $\ln 3$; $p_1 = p_2 = p_3 = \frac{1}{3}$

EXERCISES 14.8 ■ PAGE 1026

1. $\approx 59, 30$

3. Maximum $f(\pm 1, 0) = 1$, minimum $f(0, \pm 1) = -1$

5. Maximum $f(1, 2) = f(-1, -2) = 2$, minimum $f(1, -2) = f(-1, 2) = -2$

7. Maximum $f\left(1/\sqrt{2}, \pm 1/\sqrt{2}\right) = f\left(-1/\sqrt{2}, \pm 1/\sqrt{2}\right) = 4$, minimum $f(\pm 1, 0) = 2$

9. Maximum $f(2, 2, 1) = 9$, minimum $f(-2, -2, -1) = -9$

11. Maximum $f\left(1, \pm\sqrt{2}, 1\right) = f\left(-1, \pm\sqrt{2}, -1\right) = 2$, minimum $f\left(1, \pm\sqrt{2}, -1\right) = f\left(-1, \pm\sqrt{2}, 1\right) = -2$

13. Maximum $\sqrt{3}$, minimum 1

15. Maximum $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = 2$, minimum $f\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = -2$

17. 10, 10

19. 25 m by 25 m

21. $\left(-\frac{6}{5}, \frac{3}{5}\right)$

23. Minimum $f(1, 1) = f(-1, -1) = 2$

25. Maximum $f(2, 2) = e^4$

27. Maximum $f\left(3/\sqrt{2}, -3/\sqrt{2}\right) = 9 + 12\sqrt{2}$, minimum $f(-2, 2) = -8$

29. Maximum $f\left(\pm 1/\sqrt{2}, \mp 1/\left(2\sqrt{2}\right)\right) = e^{1/4}$, minimum $f\left(\pm 1/\sqrt{2}, \pm 1/\left(2\sqrt{2}\right)\right) = e^{-1/4}$

31. Maximum $f\left(0, 1, \sqrt{2}\right) = 1 + \sqrt{2}$, minimum $f\left(0, 1, -\sqrt{2}\right) = 1 - \sqrt{2}$

33. Maximum $\frac{3}{2}$, minimum $\frac{1}{2}$

41–53. See Exercises 43–57 in Section 14.7.

57. Nearest $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, farthest $(-1, -1, 2)$

59. Maximum ≈ 9.7938, minimum ≈ -5.3506

61. Maximum $f\left(\pm\sqrt{3}, 3\right) = 18$, minimum $f(0, 0) = 0$

63. (a) c/n (b) When $x_1 = x_2 = \cdots = x_n$

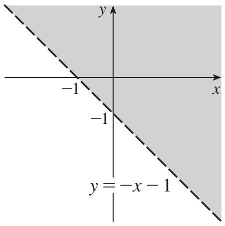
CHAPTER 14 REVIEW ■ PAGE 1031

True-False Quiz

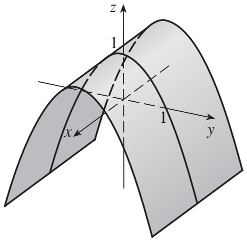
1. True **3.** False **5.** False **7.** True **9.** False
11. True

Exercises

1. $\{(x, y) \mid y > -x - 1\}$



3.



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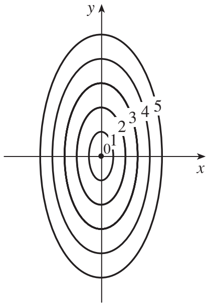

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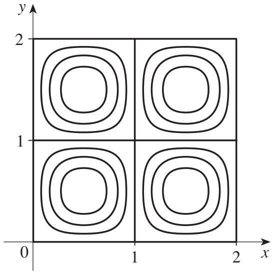


9. $\frac{2}{3}$

11. (a) $\approx 3.5°\text{C/m}, -3.0°\text{C/m}$
(b) $\approx 0.35°\text{C/m}$ by Equation 14.6.9 (Definition 14.6.2 gives $\approx 1.1°\text{C/m}$.)
(c) -0.25

13. $f_x = 32xy(5y^3 + 2x^2y)^7,\ f_y = (16x^2 + 120y^2)(5y^3 + 2x^2y)^7$

15. $F_\alpha = \dfrac{2\alpha^3}{\alpha^2 + \beta^2} + 2\alpha \ln(\alpha^2 + \beta^2),\ F_\beta = \dfrac{2\alpha^2\beta}{\alpha^2 + \beta^2}$

17. $S_u = \arctan\left(v\sqrt{w}\right),\ S_v = \dfrac{u\sqrt{w}}{1 + v^2w},\ S_w = \dfrac{uv}{2\sqrt{w}\,(1 + v^2w)}$

19. $f_{xx} = 24x,\ f_{xy} = -2y = f_{yx},\ f_{yy} = -2x$

21. $f_{xx} = k(k-1)x^{k-2}y^lz^m,\ f_{xy} = klx^{k-1}y^{l-1}z^m = f_{yx},$
$f_{xz} = kmx^{k-1}y^lz^{m-1} = f_{zx},\ f_{yy} = l(l-1)x^ky^{l-2}z^m,$
$f_{yz} = lmx^ky^{l-1}z^{m-1} = f_{zy},\ f_{zz} = m(m-1)x^ky^lz^{m-2}$

25. (a) $z = 8x + 4y + 1$
(b) $x = 1 + 8t,\ y = -2 + 4t,\ z = 1 - t$

27. (a) $2x - 2y - 3z = 3$
(b) $x = 2 + 4t,\ y = -1 - 4t,\ z = 1 - 6t$

29. (a) $x + 2y + 5z = 0$
(b) $x = 2 + t,\ y = -1 + 2t,\ z = 5t$

31. $\left(2, \frac{1}{2}, -1\right), \left(-2, -\frac{1}{2}, 1\right)$

33. $60x + \frac{24}{5}y + \frac{32}{5}z - 120$; 38.656

35. $2xy^3(1 + 6p) + 3x^2y^2(pe^p + e^p) + 4z^3(p\cos p + \sin p)$

37. $-47, 108$

43. $\langle 2xe^{yz^2}, x^2z^2e^{yz^2}, 2x^2yze^{yz^2}\rangle$ **45.** $-\frac{4}{5}$

47. $\sqrt{145}/2, \left\langle 4, \frac{9}{2}\right\rangle$ **49.** ≈ 0.72 km/h

51. Minimum $f(-4, 1) = -11$

53. Maximum $f(1, 1) = 1$; saddle points at (0, 0), (0, 3), (3, 0)

55. Maximum $f(1, 2) = 4$, minimum $f(2, 4) = -64$

57. Maximum $f(-1, 0) = 2$, minima $f(1, \pm 1) = -3$, saddle points at $(-1, \pm 1), (1, 0)$

59. Maximum $f\left(\pm\sqrt{2/3}, 1/\sqrt{3}\right) = 2/\left(3\sqrt{3}\right)$, minimum $f\left(\pm\sqrt{2/3}, -1/\sqrt{3}\right) = -2/\left(3\sqrt{3}\right)$

61. Maximum 1, minimum -1

63. $\left(\pm 3^{-1/4}, 3^{-1/4}\sqrt{2}, \pm 3^{1/4}\right), \left(\pm 3^{-1/4}, -3^{-1/4}\sqrt{2}, \pm 3^{1/4}\right)$

65. $P\left(2 - \sqrt{3}\right), P\left(3 - \sqrt{3}\right)/6, P\left(2\sqrt{3} - 3\right)/3$

PROBLEMS PLUS ■ PAGE 1035

1. $L^2W^2, \frac{1}{4}L^2W^2$ **3.** (a) $x = w/3$, base $= w/3$ (b) Yes

7. $\sqrt{3/2}, 3/\sqrt{2}$

CHAPTER 15

EXERCISES 15.1 ■ PAGE 1049

1. (a) 288 (b) 144 **3.** (a) 0.990 (b) 1.151

5. $U < V < L$ **7.** (a) ≈ 248 (b) ≈ 15.5

9. $24\sqrt{2}$ **11.** 3 **13.** $2 + 8y^2, 3x + 27x^2$

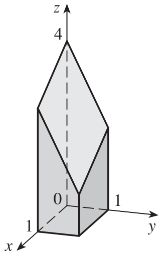
15. 222 **17.** $\frac{5}{2} - e^{-1}$ **19.** 18

21. $\frac{15}{2}\ln 2 + \frac{3}{2}\ln 4$ or $\frac{21}{2}\ln 2$ **23.** 6

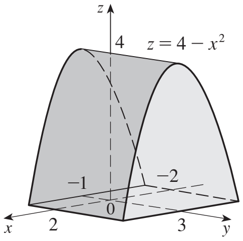
25. $\frac{31}{30}$ **27.** 2 **29.** $9\ln 2$

31. $\frac{1}{2}\left(\sqrt{3} - 1\right) - \frac{1}{12}\pi$ **33.** $\frac{1}{2}e^{-6} + \frac{5}{2}$

35.



37.

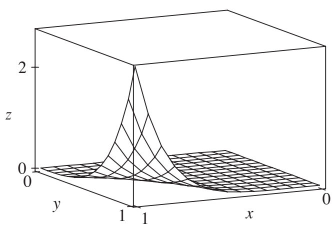


39. (a) $\int_0^2\int_0^2 xy\,dx\,dy$ (b) 4

41. (a) $\int_1^2\int_0^1 (1 + ye^{xy})\,dx\,dy$ (b) $e^2 - e$

43. 51 **45.** $\frac{166}{27}$ **47.** $\frac{8}{3}$ **49.** $\frac{64}{3}$

51. $21e - 57$



53. $\frac{5}{6}$ **55.** 0

57. Fubini's Theorem does not apply. The integrand has an infinite discontinuity at the origin.

EXERCISES 15.2 ■ PAGE 1059

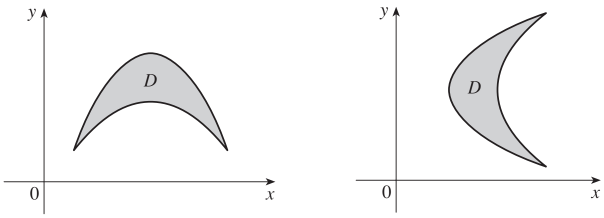
1. $\frac{868}{3}$ **3.** $\frac{1}{6}(e - 1)$ **5.** $\frac{1}{3}\sin 1$

7. (a) $\int_0^2 \int_x^{3x-x^2} 2y\,dy\,dx$ (b) $\frac{56}{15}$

9. (a) $\int_0^2 \int_{y^2}^{y+2} xy\,dx\,dy$ (b) 6

11. $\frac{1}{4}\ln 17$ **13.** $\frac{1}{2}(1 - e^{-9})$

15. (a) (b)



17. Type I: $D = \{(x, y) \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x\}$, type II: $D = \{(x, y) \mid 0 \leqslant y \leqslant 1, y \leqslant x \leqslant 1\}$; $\frac{1}{3}$

19. $\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} y\,dy\,dx + \int_1^4 \int_{x-2}^{\sqrt{x}} y\,dy\,dx = \int_{-1}^2 \int_{y^2}^{y+2} y\,dx\,dy = \frac{9}{4}$

21. $\int_0^1 \int_0^{\cos^{-1}y} \sin^2 x\,dx\,dy = \int_0^{\pi/2} \int_0^{\cos x} \sin^2 x\,dy\,dx = \frac{1}{3}$

23. $\frac{1}{2}(1 - \cos 1)$ **25.** $\frac{11}{3}$ **27.** 0

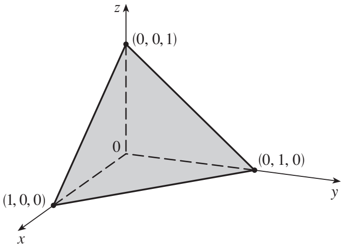
29. (a) $\int_0^1 \int_0^y (1 + xy)\,dx\,dy$ (b) $\frac{5}{8}$ **31.** $\frac{3}{4}$

33. $\frac{31}{8}$ **35.** $\frac{16}{3}$ **37.** $\frac{128}{15}$ **39.** $\frac{1}{3}$

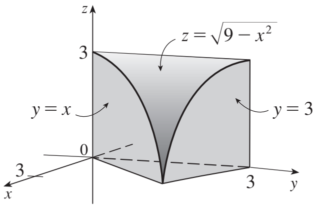
41. 0, 1.213; 0.713 **43.** $\frac{64}{3}$

45. $\dfrac{10}{3\sqrt{2}}$ or $\dfrac{5\sqrt{2}}{3}$

47.

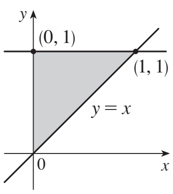


49.



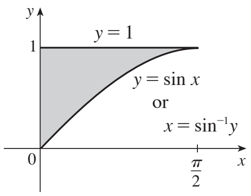
51. 13,984,735,616/14,549,535 **53.** $\pi/2$

55.



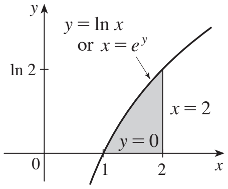
$\int_0^1 \int_x^1 f(x, y)\,dy\,dx$

57.



$\int_0^1 \int_0^{\sin^{-1}y} f(x, y)\,dx\,dy$

59.



$\int_0^{\ln 2} \int_{e^y}^2 f(x, y)\,dx\,dy$

61. $\frac{1}{6}(e^9 - 1)$ **63.** $\frac{2}{9}\left(2\sqrt{2} - 1\right)$

65. $\frac{1}{3}\left(2\sqrt{2} - 1\right)$ **67.** 1

69. $\dfrac{\sqrt{3}}{2}\pi \leqslant \iint_S \sqrt{4 - x^2y^2}\,dA \leqslant \pi$

71. $\frac{3}{4}$ **75.** 9π **77.** $a^2b + \frac{3}{2}ab^2$ **79.** $\pi a^2 b$

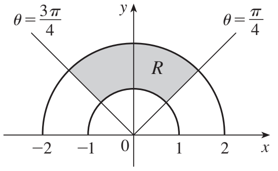
EXERCISES 15.3 ■ PAGE 1067

1. $\int_0^{3\pi/2} \int_0^4 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

3. $\int_0^{\pi} \int_1^3 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

5. $\int_0^1 \int_{2y-2}^{2-2y} f(x, y)\,dx\,dy$

7.



$3\pi/4$

9. $\frac{1250}{3}$ **11.** $(\pi/4)(\cos 1 - \cos 9)$

13. $(\pi/2)(1 - e^{-4})$ **15.** $\frac{3}{64}\pi^2$

17. $\dfrac{3\pi}{2} - 4$ **19.** $\dfrac{3\pi}{8} + \dfrac{1}{4}$ **21.** $\pi/12$

23. (a) $\int_0^{\pi/2} \int_0^2 (r + r^3\cos\theta\sin\theta)\,dr\,d\theta$ (b) $\pi + 2$

25. (a) $\int_0^{3\pi/2} \int_0^3 r^2\sin\theta\,dr\,d\theta$ (b) 9

27. (a) $\int_0^{\pi/2} \int_0^{\sin\theta} r^2\cos\theta\,dr\,d\theta$ (b) $\frac{1}{12}$

29. $\frac{625}{2}\pi$ **31.** 4π **33.** $\frac{4}{3}\pi a^3$

35. $(\pi/3)\left(2 - \sqrt{2}\right)$ **37.** $(8\pi/3)\left(64 - 24\sqrt{3}\right)$

39. $(\pi/4)(1 - e^{-4})$ **41.** $\frac{1}{120}$ **43.** 4.5951

45. $38\pi\ \text{m}^3$ **47.** $2/(a+b)$ **49.** $\frac{15}{16}$

51. (a) $\sqrt{\pi}/4$ (b) $\sqrt{\pi}/2$

EXERCISES 15.4 ■ PAGE 1078

1. 285 C **3.** $\left(\frac{3}{4}, \frac{1}{2}\right)$ **5.** $42k, \left(2, \frac{85}{28}\right)$ **7.** $6, \left(\frac{3}{4}, \frac{3}{2}\right)$

9. $\frac{8}{15}k, \left(0, \frac{4}{7}\right)$ **11.** $\frac{1}{8}(1 - 3e^{-2}), \left(\dfrac{e^2 - 5}{e^2 - 3}, \dfrac{8(e^3 - 4)}{27(e^3 - 3e)}\right)$

13. $\left(\frac{3}{8}, 3\pi/16\right)$ **15.** $(0, 45/(14\pi))$

17. $(2a/5, 2a/5)$ if vertex is $(0, 0)$ and sides are along positive axes

19. $409.2k, 182k, 591.2k$

21. $7ka^6/180, 7ka^6/180, 7ka^6/90$ if vertex is $(0, 0)$ and sides are along positive axes

23. $\rho bh^3/3, \rho b^3h/3; b/\sqrt{3}, h/\sqrt{3}$

25. $\rho a^4\pi/16, \rho a^4\pi/16; a/2, a/2$

27. $m = 3\pi/64, (\bar{x}, \bar{y}) = \left(\dfrac{16384\sqrt{2}}{10395\pi}, 0\right),$

$I_x = \dfrac{5\pi}{384} - \dfrac{4}{105}, I_y = \dfrac{5\pi}{384} + \dfrac{4}{105}, I_0 = \dfrac{5\pi}{192}$

29. (a) $\frac{1}{2}$ (b) 0.375 (c) $\frac{5}{48} \approx 0.1042$

31. (b) (i) $e^{-0.2} \approx 0.8187$

(ii) $1 + e^{-1.8} - e^{-0.8} - e^{-1} \approx 0.3481$ (c) 2, 5

33. (a) ≈ 0.500 (b) ≈ 0.632

35. (a) $\iint_D k\left[1 - \frac{1}{20}\sqrt{(x - x_0)^2 + (y - y_0)^2}\right] dA$, where D is the disk with radius 10 km centered at the center of the city

(b) $200\pi k/3 \approx 209k, 200\left(\pi/2 - \frac{8}{9}\right)k \approx 136k$; on the edge

EXERCISES 15.5 ■ PAGE 1081

1. $\frac{13}{3}\sqrt{2}$ **3.** $12\sqrt{35}$ **5.** $3\sqrt{14}$

7. $(\pi/6)\left(13\sqrt{13} - 1\right)$ **9.** $(\pi/6)\left(17\sqrt{17} - 5\sqrt{5}\right)$

11. $(2\pi/3)\left(2\sqrt{2} - 1\right)$ **13.** $a^2(\pi - 2)$ **15.** 3.6258

17. (a) ≈ 1.83 (b) ≈ 1.8616

19. $\frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\left[\left(11\sqrt{5} + 3\sqrt{70}\right)/\left(3\sqrt{5} + \sqrt{70}\right)\right]$

21. 3.3213 **25.** $(\pi/6)\left(101\sqrt{101} - 1\right)$

EXERCISES 15.6 ■ PAGE 1092

1. $\frac{27}{4}$ **3.** $\frac{16}{15}$ **5.** $\frac{5}{3}$ **7.** $3\ln 3 + 3$

9. (a) $\int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{2-z} x\, dy\, dz\, dx$ (b) 0

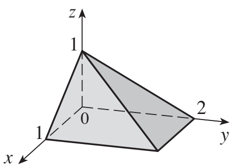
11. (a) $\int_{0}^{2}\int_{0}^{2-x}\int_{0}^{x^2} (x + y)\, dy\, dz\, dx$ (b) $\frac{8}{3}$

13. $\frac{27}{2}$ **15.** $\pi/8 - \frac{1}{3}$ **17.** $\frac{65}{28}$

19. $\frac{8}{15}$ **21.** $16\pi/3$ **23.** $\frac{16}{3}$ **25.** $\frac{8}{15}$

27. (a) $\int_{0}^{1}\int_{0}^{x}\int_{0}^{\sqrt{1-y^2}} dz\, dy\, dx$ (b) $\frac{1}{4}\pi - \frac{1}{3}$

29. ≈ 0.985 **31.**



33. $\int_{-2}^{2}\int_{0}^{4-x^2}\int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x, y, z)\, dz\, dy\, dx$

$= \int_{0}^{4}\int_{-\sqrt{4-y}}^{\sqrt{4-y}}\int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x, y, z)\, dz\, dx\, dy$

$= \int_{-1}^{1}\int_{0}^{4-4z^2}\int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x, y, z)\, dx\, dy\, dz$

$= \int_{0}^{4}\int_{-\sqrt{4-y}/2}^{\sqrt{4-y}/2}\int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x, y, z)\, dx\, dz\, dy$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}\int_{0}^{4-x^2-4z^2} f(x, y, z)\, dy\, dz\, dx$

$= \int_{-1}^{1}\int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}}\int_{0}^{4-x^2-4z^2} f(x, y, z)\, dy\, dx\, dz$

35. $\int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{2-y/2} f(x, y, z)\, dz\, dy\, dx$

$= \int_{0}^{4}\int_{-\sqrt{y}}^{\sqrt{y}}\int_{0}^{2-y/2} f(x, y, z)\, dz\, dx\, dy$

$= \int_{0}^{2}\int_{0}^{4-2z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\, dx\, dy\, dz$

$= \int_{0}^{4}\int_{0}^{2-y/2}\int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\, dx\, dz\, dy$

$= \int_{-2}^{2}\int_{0}^{2-x^2/2}\int_{x^2}^{4-2z} f(x, y, z)\, dy\, dz\, dx$

$= \int_{0}^{2}\int_{-\sqrt{4-2z}}^{\sqrt{4-2z}}\int_{x^2}^{4-2z} f(x, y, z)\, dy\, dx\, dz$

37. $\int_{0}^{1}\int_{\sqrt{x}}^{1}\int_{0}^{1-y} f(x, y, z)\, dz\, dy\, dx = \int_{0}^{1}\int_{0}^{y^2}\int_{0}^{1-y} f(x, y, z)\, dz\, dx\, dy$

$= \int_{0}^{1}\int_{0}^{1-z}\int_{0}^{y^2} f(x, y, z)\, dx\, dy\, dz = \int_{0}^{1}\int_{0}^{1-y}\int_{0}^{y^2} f(x, y, z)\, dx\, dz\, dy$

$= \int_{0}^{1}\int_{0}^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f(x, y, z)\, dy\, dz\, dx = \int_{0}^{1}\int_{0}^{(1-z)^2}\int_{\sqrt{x}}^{1-z} f(x, y, z)\, dy\, dx\, dz$

39. $\int_{0}^{1}\int_{y}^{1}\int_{0}^{y} f(x, y, z)\, dz\, dx\, dy = \int_{0}^{1}\int_{0}^{x}\int_{0}^{y} f(x, y, z)\, dz\, dy\, dx$

$= \int_{0}^{1}\int_{z}^{1}\int_{y}^{1} f(x, y, z)\, dx\, dy\, dz = \int_{0}^{1}\int_{0}^{y}\int_{y}^{1} f(x, y, z)\, dx\, dz\, dy$

$= \int_{0}^{1}\int_{0}^{x}\int_{z}^{x} f(x, y, z)\, dy\, dz\, dx = \int_{0}^{1}\int_{z}^{1}\int_{z}^{x} f(x, y, z)\, dy\, dx\, dz$

41. 64π **43.** $\frac{3}{2}\pi, \left(0, 0, \frac{1}{3}\right)$

45. $a^5, (7a/12, 7a/12, 7a/12)$

47. $I_x = I_y = I_z = \frac{2}{3}kL^5$ **49.** $\frac{1}{2}\pi kha^4$

51. (a) $m = \int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} \sqrt{x^2 + y^2}\, dz\, dy\, dx$

(b) $(\bar{x}, \bar{y}, \bar{z})$, where

$\bar{x} = (1/m)\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} x\sqrt{x^2 + y^2}\, dz\, dy\, dx,$

$\bar{y} = (1/m)\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} y\sqrt{x^2 + y^2}\, dz\, dy\, dx,$

and $\bar{z} = (1/m)\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} z\sqrt{x^2 + y^2}\, dz\, dy\, dx$

(c) $\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} (x^2 + y^2)^{3/2}\, dz\, dy\, dx$

53. (a) $\frac{3}{32}\pi + \frac{11}{24}$

(b) $\left(\dfrac{28}{9\pi + 44}, \dfrac{30\pi + 128}{45\pi + 220}, \dfrac{45\pi + 208}{135\pi + 660}\right)$

(c) $\frac{1}{240}(68 + 15\pi)$

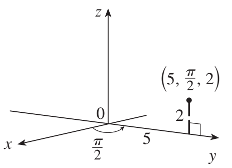
55. (a) $\frac{1}{8}$ (b) $\frac{1}{64}$ (c) $\frac{1}{5760}$ **57.** $L^3/8$

59. (a) The region bounded by the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$

(b) $4\sqrt{6}\pi/45$

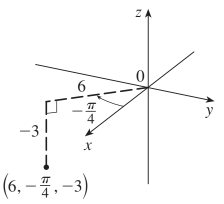
EXERCISES 15.7 ■ PAGE 1100

1. (a)



$(0, 5, 2)$

(b)



$\left(3\sqrt{2}, -3\sqrt{2}, -3\right)$

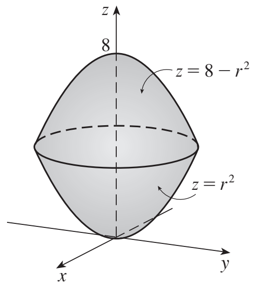
3. (a) $(4\sqrt{2}, \pi/4, -3)$ (b) $(10, -\pi/6, \sqrt{3})$

5. Circular cylinder with radius 2 and axis the z-axis

7. Sphere, radius 2, centered at the origin

9. (a) $z^2 = 1 + r\cos\theta - r^2$ (b) $z = r^2 \cos 2\theta$

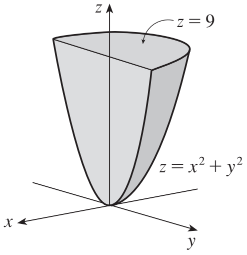
11.



13. Cylindrical coordinates: $6 \leqslant r \leqslant 7$, $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant z \leqslant 20$

15. (a) $\int_0^{\pi} \int_0^1 \int_0^{2-r^2} r^3 \, dz \, dr \, d\theta$ (b) $\pi/3$

17.



$\frac{81}{4}\pi$

19. 384π **21.** $\frac{8}{3}\pi + \frac{128}{15}$ **23.** $2\pi/5$ **25.** $\frac{4}{3}\pi(\sqrt{2} - 1)$

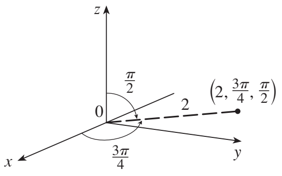
27. (a) $\frac{512}{3}\pi$ (b) $(0, 0, \frac{23}{2})$

29. $\pi K a^2/8$, $(0, 0, 2a/3)$ **31.** 0

33. (a) $\iiint_C h(P)g(P)\, dV$, where C is the cone
(b) $\approx 4.4 \times 10^{18}$ J

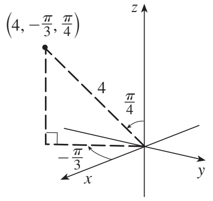
EXERCISES 15.8 ■ PAGE 1106

1. (a)



$(-\sqrt{2}, \sqrt{2}, 0)$

(b)



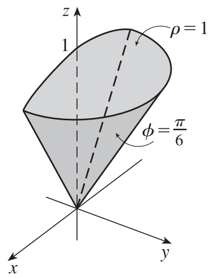
$(\sqrt{2}, -\sqrt{6}, 2\sqrt{2})$

3. (a) $(3\sqrt{2}, \pi/4, \pi/2)$ (b) $(4, -\pi/3, \pi/6)$

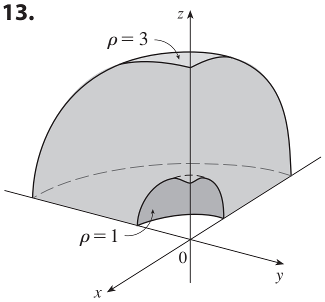
5. Bottom half of a cone **7.** Horizontal plane

9. (a) $\rho = 3$ (b) $\rho^2(\sin^2\phi \cos 2\theta - \cos^2\phi) = 1$

11.

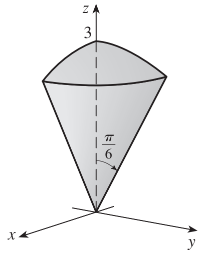


13.



15. $\pi/4 \leqslant \phi \leqslant \pi/2$, $0 \leqslant \rho \leqslant 4\cos\phi$

17.



$(9\pi/4)(2 - \sqrt{3})$

19. $\int_0^{\pi/2} \int_0^3 \int_0^2 f(r\cos\theta, r\sin\theta, z)\, r \, dz \, dr \, d\theta$

21. (a) $\int_{\pi/2}^{\pi} \int_{\pi/2}^{3\pi/2} \int_2^3 \rho^3 \sin\phi \, d\rho \, d\theta \, d\phi$ (b) $\frac{65}{4}\pi$

23. $312{,}500\pi/7$ **25.** $1688\pi/15$ **27.** $\pi/8$

29. $(\sqrt{3} - 1)\pi a^3/3$ **31.** (a) 10π (b) $(0, 0, 2.1)$

33. (a) $(0, 0, \frac{7}{12})$ (b) $11K\pi/960$

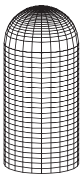
35. (a) $(0, 0, \frac{3}{8}a)$ (b) $4K\pi a^5/15$ (K is the density)

37. $\frac{1}{3}\pi(2 - \sqrt{2})$, $(0, 0, 3/[8(2 - \sqrt{2})])$

39. (a) $\pi K a^4 h/2$ (K is the density) (b) $\pi K a^2 h(3a^2 + 4h^2)/12$

41. $5\pi/6$ **43.** $(4\sqrt{2} - 5)/15$ **45.** $4096\pi/21$

47.



49. $136\pi/99$

EXERCISES 15.9 ■ PAGE 1116

1. (a) VI (b) I (c) IV (d) V (e) III (f) II

3. The parallelogram with vertices (0, 0), (6, 3), (12, 1), (6, −2)

5. The region bounded by the line $y = 1$, the y-axis, and $y = \sqrt{x}$

7. $x = \frac{1}{3}(v - u)$, $y = \frac{1}{3}(u + 2v)$ is one possible transformation, where $S = \{(u, v) \mid -1 \leqslant u \leqslant 1, 1 \leqslant v \leqslant 3\}$

9. $x = u\cos v$, $y = u \sin v$ is one possible transformation, where $S = \{(u, v) \mid 1 \leqslant u \leqslant \sqrt{2}, 0 \leqslant v \leqslant \pi/2\}$

11. -6 **13.** s **15.** $2uvw$

17. -3 **19.** 6π **21.** $2 \ln 3$

23. (a) $\frac{4}{3}\pi abc$ (b) $1.083 \times 10^{12}\text{ km}^3$
(c) $\frac{4}{15}\pi(a^2 + b^2)abck$

25. $\frac{8}{5} \ln 8$ **27** $\frac{3}{2} \sin 1$ **29.** $e - e^{-1}$

CHAPTER 15 REVIEW ■ PAGE 1118

True-False Quiz

1. True **3.** True **5.** True **7.** True **9.** False

Exercises

1. ≈ 64.0 **3.** $4e^2 - 4e + 3$ **5.** $\frac{1}{2} \sin 1$ **7.** $\frac{2}{3}$

9. $\int_0^{\pi} \int_2^4 f(r \cos \theta, r \sin \theta)\, r\, dr\, d\theta$

11. $(\sqrt{3}, 3, 2), (4, \pi/3, \pi/3)$

13. $(2\sqrt{2}, 2\sqrt{2}, 4\sqrt{3}), (4, \pi/4, 4\sqrt{3})$

15. (a) $r^2 + z^2 = 4, \rho = 2$ (b) $r = 2, \rho \sin \phi = 2$

17. The region inside the loop of the four-leaved rose $r = \sin 2\theta$ in the first quadrant

19. $\frac{1}{2} \sin 1$ **21.** $\frac{1}{2}e^6 - \frac{7}{2}$ **23.** $\frac{1}{4} \ln 2$ **25.** 8

27. $81\pi/5$ **29.** $\frac{81}{2}$ **31.** $\pi/96$ **33.** $\frac{64}{15}$

35. 176 **37.** $\frac{2}{3}$ **39.** $2ma^3/9$

41. (a) $\frac{1}{4}$ (b) $\left(\frac{1}{3}, \frac{8}{15}\right)$
(c) $I_x = \frac{1}{12}, I_y = \frac{1}{24}; \bar{\bar{y}} = 1/\sqrt{3}, \bar{\bar{x}} = 1/\sqrt{6}$

43. (a) $(0, 0, h/4)$ (b) $\pi a^5 h/15$

45. $\ln(\sqrt{2} + \sqrt{3}) + \sqrt{2}/3$ **47.** $\frac{486}{5}$ **49.** 0.0512

51. (a) $\frac{1}{15}$ (b) $\frac{1}{3}$ (c) $\frac{1}{45}$

53. $\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\, dx\, dy\, dz$ **55.** $-\ln 2$ **57.** 0

PROBLEMS PLUS ■ PAGE 1121

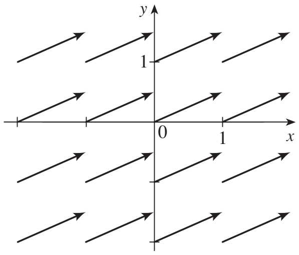
1. 30 **3.** $\frac{1}{2} \sin 1$ **7.** (b) 0.90

13. $abc\pi\left(\dfrac{2}{3} - \dfrac{8}{9\sqrt{3}}\right)$

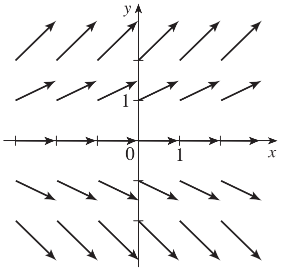
CHAPTER 16

EXERCISES 16.1 ■ PAGE 1129

1.



3.



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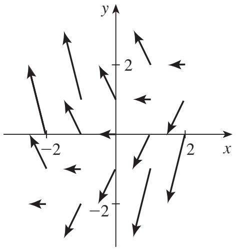

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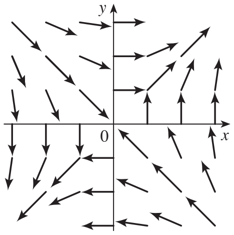

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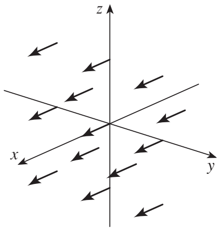
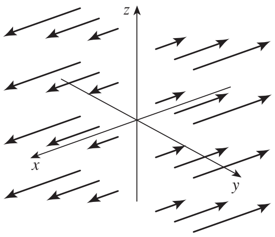
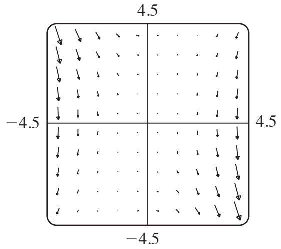


11.



13. IV **15.** I **17.** III **19.** IV **21.** III

23.

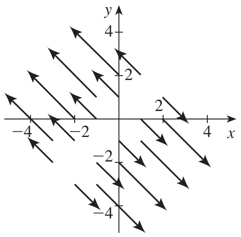


The line $y = 2x$

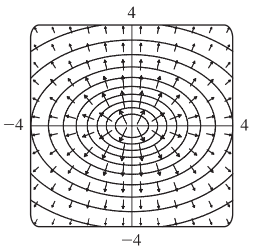
25. $\nabla f(x, y) = y^2 \cos(xy)\,\mathbf{i} + [xy \cos(xy) + \sin(xy)]\,\mathbf{j}$

27. $$\nabla f(x, y, z) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{k}$$

29. $\nabla f(x, y) = (x - y)\,\mathbf{i} + (y - x)\,\mathbf{j}$

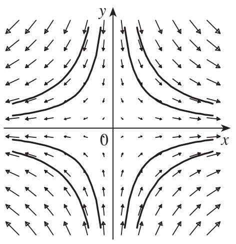


31. III **33.** II **35.**



37. (2.04, 1.03)

39. (a)



(b) $y = 1/x, x > 0$

EXERCISES 16.2 ■ PAGE 1141

1. $\frac{4}{3}(10^{3/2} - 1)$ **3.** 1638.4 **5.** $\frac{1}{3}\pi^6 + 2\pi$ **7.** $\frac{5}{2}$

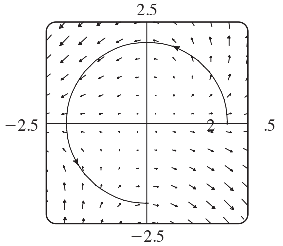
9. $\sqrt{2}/3$ **11.** $\frac{1}{12}\sqrt{14}(e^6 - 1)$ **13.** $\frac{2}{5}(e - 1)$

15. $\pi/2 - \frac{1}{6}\sqrt{2}$ **17.** $\frac{35}{3}$

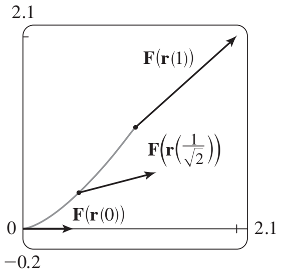
19. (a) Positive (b) Negative **21.** $\frac{1}{20}$

23. $\frac{6}{5} - \cos 1 - \sin 1$ **25.** 0.5424 **27.** 94.8231

29. $3\pi + \frac{2}{3}$



31. (a) $\frac{11}{8} - 1/e$ (b)



33. $\frac{172{,}704}{5{,}632{,}705}\sqrt{2}(1 - e^{-14\pi})$ **35.** $2\pi k, (4/\pi, 0)$

37. (a) $\bar{x} = (1/m)\int_C x\rho(x, y, z)\, ds$,
$\bar{y} = (1/m)\int_C y\rho(x, y, z)\, ds$,
$\bar{z} = (1/m)\int_C z\rho(x, y, z)\, ds$, where $m = \int_C \rho(x, y, z)\, ds$
(b) $(0, 0, 3\pi)$

39. $I_x = k\left(\frac{1}{2}\pi - \frac{4}{3}\right), I_y = k\left(\frac{1}{2}\pi - \frac{2}{3}\right)$ **41.** $2\pi^2$ **43.** $\frac{7}{3}$

45. (a) $2ma\,\mathbf{i} + 6mbt\,\mathbf{j}, 0 \leqslant t \leqslant 1$ (b) $2ma^2 + \frac{9}{2}mb^2$

47. $\approx 2.26 \times 10^4$ J **49.** (b) Yes **53.** ≈ 22 J

EXERCISES 16.3 ■ PAGE 1151

1. 40 **3.** Not conservative

5. $f(x, y) = ye^{xy} + K$ **7.** $f(x, y) = ye^x + x \sin y + K$

9. $f(x, y) = y^2 \sin x + x \cos y + K$

11. (b) 16 **13.** (a) 16 (b) $f(x, y) = x^3 + xy^2 + K$

15. (a) $f(x, y) = e^{xy} + K$ (b) $e^2 - 1$

17. (a) $f(x, y) = x^2 + 2y^2$ (b) -21

19. (a) $f(x, y) = \frac{1}{3}x^3y^3$ (b) -9

21. (a) $f(x, y, z) = x^2y + y^2z$ (b) 30

23. (a) $f(x, y, z) = ye^{xz}$ (b) 4 **25.** $4/e$

27. It doesn't matter which curve is chosen.

29. $\frac{31}{4}$ **31.** No **33.** Conservative

37. (a) Yes (b) Yes (c) Yes

39. (a) No (b) Yes (c) Yes

EXERCISES 16.4 ■ PAGE 1159

1. 120 **3.** $\frac{2}{3}$ **5.** $4(e^3 - 1)$ **7.** $-\frac{9}{5}$ **9.** $\frac{1}{3}$

11. -24π **13.** 14 **15.** $-\frac{16}{3}$ **17.** 4π

19. $\frac{1}{15}\pi^4 - \frac{4144}{1125}\pi^2 + \frac{7{,}578{,}368}{253{,}125} \approx 0.0779$

21. $-\frac{1}{12}$ **23.** 3π **25.** (c) $\frac{9}{2}$

27. $(4a/3\pi, 4a/3\pi)$ if the region is the portion of the disk $x^2 + y^2 = a^2$ in the first quadrant

31. 0

EXERCISES 16.5 ■ PAGE 1168

1. (a) $\mathbf{0}$ (b) $y^2z^2 + x^2z^2 + x^2y^2$

3. (a) $ze^x\,\mathbf{i} + (xye^z - yze^x)\,\mathbf{j} - xe^z\,\mathbf{k}$ (b) $y(e^z + e^x)$

5. (a) $-\dfrac{\sqrt{z}}{(1 + y)^2}\,\mathbf{i} - \dfrac{\sqrt{x}}{(1 + z)^2}\,\mathbf{j} - \dfrac{\sqrt{y}}{(1 + x)^2}\,\mathbf{k}$

(b) $\dfrac{1}{2\sqrt{x}\,(1 + z)} + \dfrac{1}{2\sqrt{y}\,(1 + x)} + \dfrac{1}{2\sqrt{z}\,(1 + y)}$

7. (a) $\langle -e^y \cos z, -e^z \cos x, -e^x \cos y\rangle$
(b) $e^x \sin y + e^y \sin z + e^z \sin x$

9. (a) Negative (b) curl $\mathbf{F} = \mathbf{0}$

11. (a) Zero (b) curl **F** points in the negative z-direction.

15. $f(x, y, z) = x^2y^3z^2 + K$

17. $f(x, y, z) = x \ln y + y \ln z + K$

19. Not conservative **21.** No

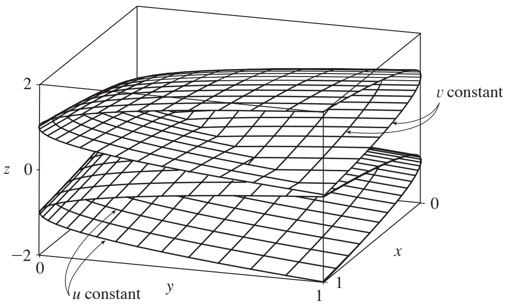
EXERCISES 16.6 ■ PAGE 1180

1. P: yes; Q: no

3. Plane through (0, 3, 1) containing vectors ⟨1, 0, 4⟩, ⟨1, −1, 5⟩

5. Circular cone with axis the z-axis

7.



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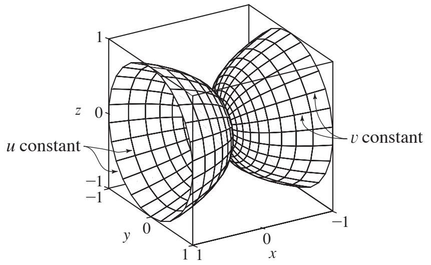

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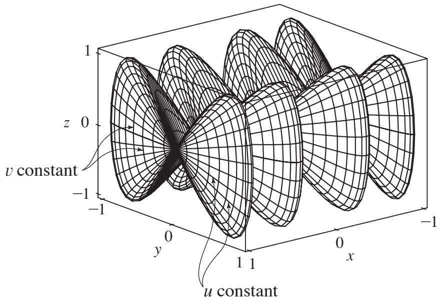


13. IV **15.** I **17.** III

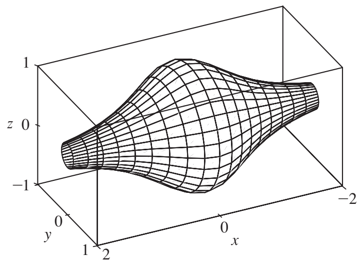
19. $x = u, y = v - u, z = -v$

21. $y = y, z = z, x = \sqrt{1 + y^2 + \frac{1}{4}z^2}$

23. $x = 2\sin\phi\cos\theta, y = 2\sin\phi\sin\theta,$
$z = 2\cos\phi, 0 \leqslant \phi \leqslant \pi/4, 0 \leqslant \theta \leqslant 2\pi$
$\left[\text{or } x = x, y = y, z = \sqrt{4 - x^2 - y^2}, x^2 + y^2 \leqslant 2\right]$

25. $x = 6\sin\phi\cos\theta, y = 6\sin\phi\sin\theta, z = 6\cos\phi,$
$\pi/6 \leqslant \phi \leqslant \pi/2, 0 \leqslant \theta \leqslant 2\pi$

29. $x = x, y = \dfrac{1}{1+x^2}\cos\theta, y = \dfrac{1}{1+x^2}\sin\theta,$
$-2 \leqslant x \leqslant 2, 0 \leqslant \theta \leqslant 2\pi$



31. (a) Direction reverses (b) Number of coils doubles

33. $3x - y + 3z = 3$ **35.** $\dfrac{\sqrt{3}}{2}x - \dfrac{1}{2}y + z = \dfrac{\pi}{3}$

37. $-x + 2z = 1$ **39.** $3\sqrt{14}$ **41.** $\sqrt{14}\,\pi$

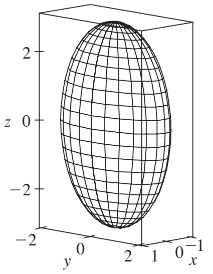
43. $\frac{4}{15}(3^{5/2} - 2^{7/2} + 1)$ **45.** $(2\pi/3)(2\sqrt{2} - 1)$

47. $(\pi/6)(65^{3/2} - 1)$ **49.** 4 **51.** $\pi R^2 \leqslant A(S) \leqslant \sqrt{3}\,\pi R^2$

53. 3.5618 **55.** (a) ≈ 24.2055 (b) 24.2476

57. $\frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\left[(11\sqrt{5} + 3\sqrt{70})/(3\sqrt{5} + \sqrt{70})\right]$

59. (b)



(c) $\int_0^{2\pi}\int_0^{\pi}\sqrt{36\sin^4 u\cos^2 v + 9\sin^4 u\sin^2 v + 4\cos^2 u\sin^2 u}\,du\,dv$

61. 4π **63.** $2a^2(\pi - 2)$

EXERCISES 16.7 ■ PAGE 1192

1. ≈ -6.93 **3.** 900π **5.** $11\sqrt{14}$ **7.** $\frac{2}{3}(2\sqrt{2} - 1)$

9. $171\sqrt{14}$ **11.** $\sqrt{21}/3$ **13.** $(\pi/120)(25\sqrt{5} + 1)$

15. $\frac{7}{4}\sqrt{21} - \frac{17}{12}\sqrt{17}$ **17.** 16π **19.** 0 **21.** 4

23. $\frac{713}{180}$ **25.** $\frac{8}{3}\pi$ **27.** 0 **29.** 48 **31.** $2\pi + \frac{8}{3}$

33. 4.5822 **35.** 3.4895

37. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D [P(\partial h/\partial x) - Q + R(\partial h/\partial z)]\,dA$, where D = projection of S onto xz-plane

39. $(0, 0, a/2)$

41. (a) $I_z = \iint_S (x^2 + y^2)\rho(x, y, z)\,dS$ (b) $4329\sqrt{2}\,\pi/5$

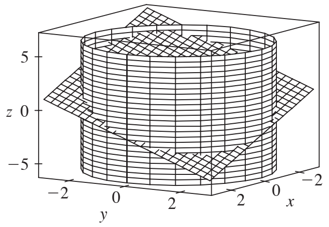
43. 0 kg/s **45.** $\frac{8}{3}\pi a^3\varepsilon_0$ **47.** 1248π

EXERCISES 16.8 ■ PAGE 1199

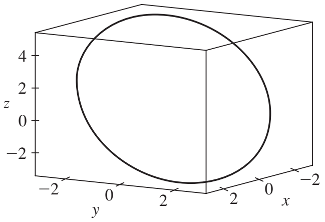
3. 16π **5.** 0 **7.** -1 **9.** $-\frac{17}{20}$

11. 8π **13.** $\pi/2$

15. (a) $81\pi/2$ (b)



(c) $x = 3\cos t$, $y = 3\sin t$,
$z = 1 - 3(\cos t + \sin t)$,
$0 \leqslant t \leqslant 2\pi$



17. -32π **19.** $-\pi$ **21.** 3

EXERCISES 16.9 ■ PAGE 1206

1. $\frac{9}{2}$ **3.** $256\pi/3$ **5.** $\frac{9}{2}$ **7.** $9\pi/2$ **9.** 0
11. π **13.** 16 **15.** $\frac{1}{24}abc(a + 4)$ **17.** 2π
19. $13\pi/20$ **21.** Negative at P_1, positive at P_2
23. div $\mathbf{F} > 0$ in quadrants I, II; div $\mathbf{F} < 0$ in quadrants III, IV

CHAPTER 16 REVIEW ■ PAGE 1209

True-False Quiz

1. False **3.** True **5.** False **7.** False
9. True **11.** True **13.** False

Exercises

1. (a) Negative (b) Positive **3.** $6\sqrt{10}$ **5.** $\frac{4}{15}$
7. $\frac{110}{3}$ **9.** $\frac{11}{12} - 4/e$ **11.** $f(x, y) = e^y + xe^{xy} + K$
13. 0 **15.** 0 **17.** -8π **25.** $\frac{1}{6}\left(27 - 5\sqrt{5}\right)$
27. $(\pi/60)\left(391\sqrt{17} + 1\right)$ **29.** $-64\pi/3$ **31.** 0
33. $-\frac{1}{2}$ **35.** 4π **37.** -4 **39.** 21

PROBLEMS PLUS ■ PAGE 1213

7. (d) $\dfrac{4\sqrt{2}\pi^2}{25}$ (e) $2\pi^2 r^2 R$

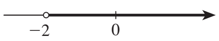
APPENDIXES

EXERCISES A ■ PAGE A9

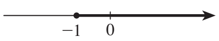
1. 18 **3.** π **5.** $5 - \sqrt{5}$ **7.** $2 - x$

9. $|x + 1| = \begin{cases} x + 1 & \text{for } x \geqslant -1 \\ -x - 1 & \text{for } x < -1 \end{cases}$ **11.** $x^2 + 1$

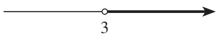
13. $(-2, \infty)$



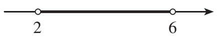
15. $[-1, \infty)$



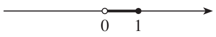
17. $(3, \infty)$



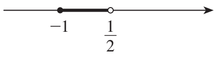
19. $(2, 6)$



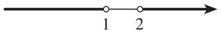
21. $(0, 1]$



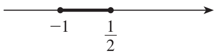
23. $\left[-1, \frac{1}{2}\right)$



25. $(-\infty, 1) \cup (2, \infty)$

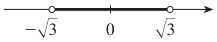


27. $\left[-1, \frac{1}{2}\right]$

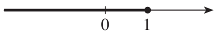


29. $(-\infty, \infty)$

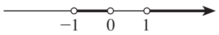
31. $\left(-\sqrt{3}, \sqrt{3}\right)$



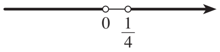
33. $(-\infty, 1]$



35. $(-1, 0) \cup (1, \infty)$



37. $(-\infty, 0) \cup \left(\frac{1}{4}, \infty\right)$

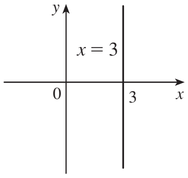


39. $10 \leqslant C \leqslant 35$ **41.** (a) $T = 20 - 10h$, $0 \leqslant h \leqslant 12$
(b) $-30\,°\text{C} \leqslant T \leqslant 20\,°\text{C}$ **43.** $\pm\frac{3}{2}$ **45.** $2, -\frac{4}{3}$
47. $(-3, 3)$ **49.** $(3, 5)$ **51.** $(-\infty, -7] \cup [-3, \infty)$
53. $[1.3, 1.7]$ **55.** $[-4, -1] \cup [1, 4]$
57. $x \geqslant (a + b)c/(ab)$ **59.** $x > (c - b)/a$

EXERCISES B ■ PAGE A15

1. 5 **3.** $\sqrt{74}$ **5.** $2\sqrt{37}$ **7.** 2 **9.** $-\frac{9}{2}$

17.



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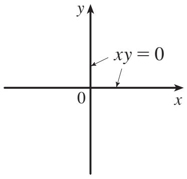
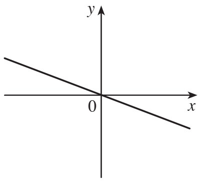
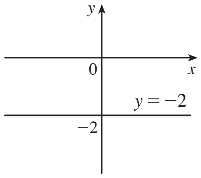
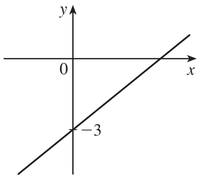


21. $y = 6x - 15$ **23.** $2x - 3y + 19 = 0$
25. $5x + y = 11$ **27.** $y = 3x - 2$ **29.** $y = 3x - 3$
31. $y = 5$ **33.** $x + 2y + 11 = 0$ **35.** $5x - 2y + 1 = 0$
37. $m = -\frac{1}{3}$, $b = 0$ **39.** $m = 0$, $b = -2$ **41.** $m = \frac{3}{4}$, $b = -3$







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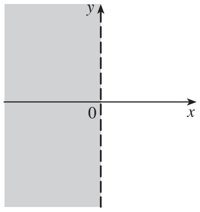

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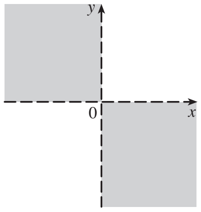

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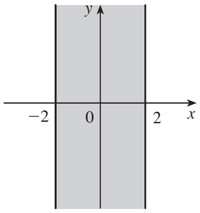

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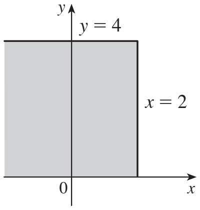

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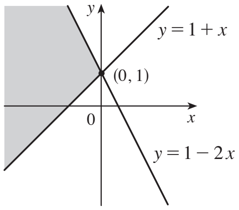


53. $(0, -4)$ **55.** (a) $(4, 9)$ (b) $(3.5, -3)$ **57.** $(1, -2)$

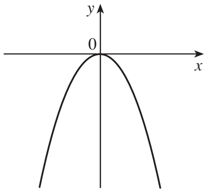
59. $y = x - 3$ **61.** (b) $4x - 3y - 24 = 0$

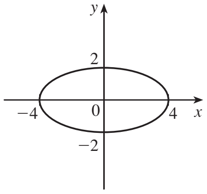
EXERCISES C ■ PAGE A23

1. $(x - 3)^2 + (y + 1)^2 = 25$ **3.** $x^2 + y^2 = 65$

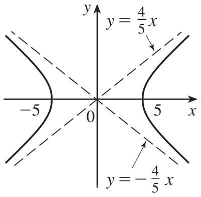
5. $(2, -5), 4$ **7.** $\left(-\frac{1}{2}, 0\right), \frac{1}{2}$ **9.** $\left(\frac{1}{4}, -\frac{1}{4}\right), \sqrt{10}/4$

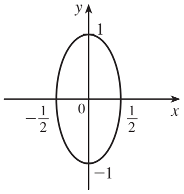
11. Parabola **13.** Ellipse



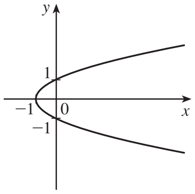


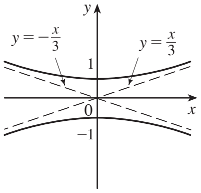
15. Hyperbola **17.** Ellipse



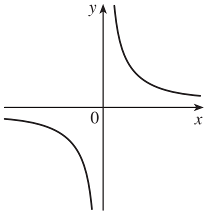


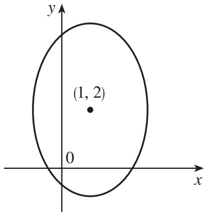
19. Parabola **21.** Hyperbola



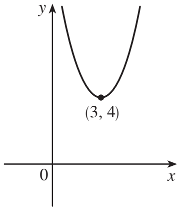


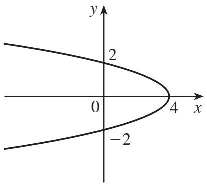
23. Hyperbola **25.** Ellipse



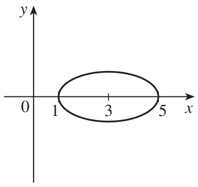


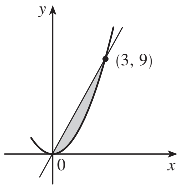
27. Parabola **29.** Parabola





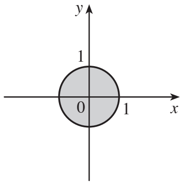
31. Ellipse **33.**

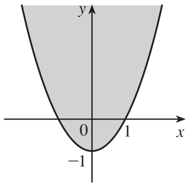




35. $y = x^2 - 2x$

37. **39.**



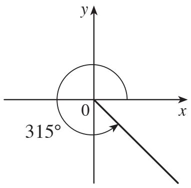


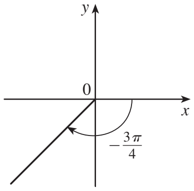
EXERCISES D ■ PAGE A33

1. $7\pi/6$ **3.** $\pi/20$ **5.** 5π **7.** 720° **9.** 75°

11. -67.5° **13.** 3π cm **15.** $\frac{2}{3}$ rad $= (120/\pi)^{\circ}$

17. **19.**





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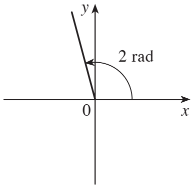


23. $\sin(3\pi/4) = 1/\sqrt{2}$, $\cos(3\pi/4) = -1/\sqrt{2}$, $\tan(3\pi/4) = -1$, $\csc(3\pi/4) = \sqrt{2}$, $\sec(3\pi/4) = -\sqrt{2}$, $\cot(3\pi/4) = -1$

25. $\sin(9\pi/2) = 1$, $\cos(9\pi/2) = 0$, $\csc(9\pi/2) = 1$, $\cot(9\pi/2) = 0$, $\tan(9\pi/2)$ and $\sec(9\pi/2)$ undefined

27. $\sin(5\pi/6) = \frac{1}{2}$, $\cos(5\pi/6) = -\sqrt{3}/2$, $\tan(5\pi/6) = -1/\sqrt{3}$, $\csc(5\pi/6) = 2$, $\sec(5\pi/6) = -2/\sqrt{3}$, $\cot(5\pi/6) = -\sqrt{3}$

29. $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\csc\theta = \frac{5}{3}$, $\sec\theta = \frac{5}{4}$, $\cot\theta = \frac{4}{3}$

31. $\sin\phi = \sqrt{5}/3$, $\cos\phi = -\frac{2}{3}$, $\tan\phi = -\sqrt{5}/2$, $\csc\phi = 3/\sqrt{5}$, $\cot\phi = -2/\sqrt{5}$

33. $\sin\beta = -1/\sqrt{10}$, $\cos\beta = -3/\sqrt{10}$, $\tan\beta = \frac{1}{3}$, $\csc\beta = -\sqrt{10}$, $\sec\beta = -\sqrt{10}/3$

35. 5.73576 cm **37.** 24.62147 cm

59. $\frac{1}{15}(4 + 6\sqrt{2})$ **61.** $\frac{1}{15}(3 + 8\sqrt{2})$

63. $\frac{24}{25}$ **65.** $\pi/3, 5\pi/3$

67. $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ **69.** $\pi/6, \pi/2, 5\pi/6, 3\pi/2$

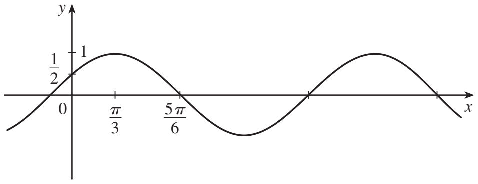
71. $0, \pi, 2\pi$ **73.** $0 \leqslant x \leqslant \pi/6$ and $5\pi/6 \leqslant x \leqslant 2\pi$

75. $0 \leqslant x < \pi/4$, $3\pi/4 < x < 5\pi/4$, $7\pi/4 < x \leqslant 2\pi$

77. $\angle C = 62°$, $a \approx 199.55$, $b \approx 241.52$

79. $\approx$1355 m **81.** 14.34457 cm^2

83.



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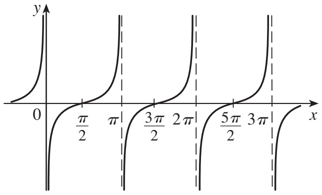

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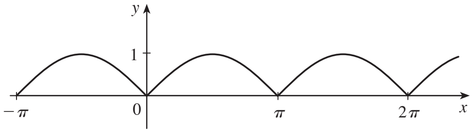


EXERCISES E ■ PAGE A40

1. $\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5}$ **3.** $3^4 + 3^5 + 3^6$

5. $-1 + \frac{1}{3} + \frac{3}{5} + \frac{5}{7} + \frac{7}{9}$ **7.** $1^{10} + 2^{10} + 3^{10} + \cdots + n^{10}$

9. $1 - 1 + 1 - 1 + \cdots + (-1)^{n-1}$ **11.** $\sum_{i=1}^{10} i$

13. $\sum_{i=1}^{19} \frac{i}{i+1}$ **15.** $\sum_{i=1}^{n} 2i$ **17.** $\sum_{i=0}^{5} 2^i$ **19.** $\sum_{i=1}^{n} x^i$

21. 80 **23.** 3276 **25.** 0 **27.** 61 **29.** $n(n + 1)$

31. $n(n^2 + 6n + 17)/3$ **33.** $n(n^2 + 6n + 11)/3$

35. $n(n^3 + 2n^2 - n - 10)/4$

41. (a) n^4 (b) $5^{100} - 1$ (c) $\frac{97}{300}$ (d) $a_n - a_0$

43. $\frac{1}{3}$ **45.** 14 **49.** $2^{n+1} + n^2 + n - 2$

EXERCISES G ■ PAGE 000

1. (b) 0.405

Index

RP denotes Reference Page numbers at the front and back of the book.

Abel, Niels, 213
absolute maximum and minimum, 280, 284
absolute maximum and minimum values, 280, 284, 1008, 1014, 1015
absolute value, 14, 126, A6
absolute value function, 14
absolutely convergent series, 769
acceleration of a particle, 916
  components of, 919
  as a vector, 916
acceleration as a rate of change, 160, 226
adaptive numerical integration, 538
addition formulas for sine and cosine, A29
addition of vectors, 836, 840
Airy, Sir George, 794
Airy function, 794
algebraic function, 29
algebraic vector, 839
allometric growth, 628
alternating harmonic series, 767, 770
alternating series, 765, 768, 769
Alternating Series Estimation Theorem, 768, 769
Alternating Series Test, 766, 768, 780
Ampère's Law, 1144
analytic geometry, A11
angle, A24
  between curves, 277
  of deviation, 289
  negative, A25
  between planes, 869
  positive, A25
  standard position, A25
  between vectors, 848, 849
angular momentum, 925
angular speed, 918
antiderivative, 356, 357
antiderivatives, graphing, 360
antidifferentiation formulas, 358
aphelion, 716
apolune, 709
application(s)
  areas between curves, 441
  business and economics, 341, 937, 943, 973, 1027
  economics and biology, 587
  engineering, 576
  exponential functions, 48
  integration, 435, 559
  physics, 256, 815, 576
  Taylor polynomials, 811
approach path of an aircraft, 209
approximate integration, 529
approximating cylinder, 449
approximating functions by polynomials, 811
approximating surface, 568
approximation
  by differentials, 254
  to *e*, 180, 390
  linear, 254, 976, 980
  by the Midpoint Rule, 391, 530
  by Newton's method, 351
  by an *n*th-degree Taylor polynomial, 258
  polynomial, 260
  quadratic, 260
  by Riemann sums, 386
  by Simpson's Rule, 534, 535
  tangent line, 254
  by Taylor polynomials, 811, 812
  by Taylor's inequality, 812, 814
  by the Trapezoidal Rule, 531
Archimedes, 97, 418
Archimedes' Principle, 481, 1207
arc length
  of a curve, 560
  of a parametric curve, 676
  of a polar curve, 697
  of a space curve, 904
arc length contest, 567
arc length formula, 561
arc length formula for a space curve, 904
arc length function, 563, 905
arc length function for a parametric curve, 678
arcsine function, 62
area, 2, 377
  of a circle, 502
  under a curve, 372, 377, 385
  between curves, 435, 436, 438, 439
  of an ellipse, 501
  by exhaustion, 2, 97
  by Green's Theorem, 1156, 1157
  enclosed by a parametric curve, 675
  of a plane region, 1156
  in polar coordinates, 661, 694, 1065
  of a region, 694
  of a sector of a circle, 694
  surface (*see* surface area)
  of a surface of a revolution, 567, 575
area function, 398
area problem, 2, 372
arithmetic-geometric mean, 737
arrow diagram, 9
astroid, 215, 673
asymptote(s), 321, 322
  in graphing, 321
  horizontal, 127, 128, 321
  of a hyperbola, 706, A20
  slant, 321, 326
  vertical, 89 90, 321
asymptotic curve, 329
autonomous differential equation, 616
average blood alcohol concentration (BAC), 208
average cost function, 346
average productivity, 238
average rate of change, 146, 225
average speed of molecules, 551
average value of a function, 473, 594, 1047
  over a solid region, 1095
average velocity, 4, 80, 143, 226
axes, coordinate, 830, A11
axes of an ellipse, A19
axis of a parabola, 703

bacterial growth, 631, 636
Barrow, Isaac, 3, 97, 152, 399, 404, 418
base of a cylinder, 447
base of a logarithm, 57, A51
  change of, 60

baseball and calculus, 476
basis vectors, 841
Bernoulli, James, 621, 646
Bernoulli, John, 311, 319, 621, 796, 800
Bernoulli differential equation, 646
Bessel, Friedrich, 791
Bessel function, 216, 791
Bézier, Pierre, 684
Bézier curves, 666, 684
binomial coefficients, 803
binomial series, 803
  discovery by Newton, 811
binomial theorem, 175, 526, 811, RP1
binormal vector, 909, 911
biology, rates of change in, 230
bird, minimizing energy of, 350
blackbody radiation, 820
blood flow, 232, 348, 588
body mass index (BMI), 949, 965
Bohr radius, 599
boundary curve
  in double integrals, 1052
  positively oriented, 1154, 1195
bounded sequence, 732, 733
bounded set, 1014
Boyle's Law, 236, 253
brachistochrone problem, 667
Brahe, Tycho, 921
branches of a hyperbola, 706, A20
Buffon's needle problem, 604
bullet-nose curve, 206

C^1 transformation, 1109
cable (hanging), 262
calculator, graphing with 329, 690
calculus, 8
  differential, 4
  integral, 3
  invention of, 8
calculus of rainbows, 289
cancellation equations
  for inverse functions, 56
  for inverse trigonometric functions, 58, 62
  for logarithms, 57
can, minimizing manufacturing cost of, 349
Cantor, Georg, 750
Cantor set, 750
capital formation, 591
cardiac output, 589
cardioid, 215, 688
carrying capacity, 238, 301, 318, 607, 633
Cartesian coordinate system, 684, 688, A11
  relationship to polar coordinates, 686
Cartesian plane, A11
Cassini, Giovanni, 694
catenary, 262, 566, 846
Cauchy, Augustin-Louis, 109, 1044, 1111, A47
Cauchy principal value of an integral, 551
Cauchy-Schwarz Inequality, 854, 1028
Cauchy's Mean Value Theorem, A47
Cavalieri, 535
Cavalieri's principle, 459
center of curvature, 910
center of gravity, 578. *See also* center of mass
center of mass, 558, 578, 1070, 1071, 1134
  of a lamina, 1071
  of a plate, 581
  of a solid, 1091
  of a surface, 1184
  of a wire, 1134
centripetal acceleration, 930
centripetal force, 930
centroid, 580
  of a curve, 586
  of a plane region, 562
  of a solid, 1091
Chain Rule, 199, 200, 201, 205
  for several variables, 985, 987, 988
change of base, formula for, 60
change of variable(s)
  in a double integral, 1109, 1112
  in integration, 419
  in a triple integral, 1097, 1098, 1104, 1114, 1115
chaotic behavior of a sequence, 738
charge, electric, 228, 1070, 1091
charge density, 1070, 1091
chemical reaction, 229
circle, A16
  area of, 502
  equation of, A17
  osculating, 910
circle of curvature, 910
circular cylinder, 447
circular paraboloid, 880
circulation of a velocity field, 1198
cissoid of Diocles, 215, 671, 693
Clairaut, Alexis, 967
Clairaut's Theorem, 967, A2
Clarke, Author C., 926
Clarke geosynchronous orbit, 926
clipping planes, 874
closed curve, 1146
closed interval, A3
Closed Interval Method, 282
  for a function of two variables, 1014, 1015
closed set, 1014
closed surface, 1188
Cobb, Charles, 936
Cobb-Douglas production function, 937, 943, 973, 1027
  graph of, 938
  level curves for, 943
cochleoid, 720
coefficient(s)
  binomial, 803
  of friction, 198, 288
  of inequality, 445
  leading, 25
  of a polynomial, 25
  of a power series, 782
  of a series, 782
  of static friction, 887
coffee cups as surfaces of revolution, 587
collision and intersection
  of particles, 671
  of particles in space, 897
collision point, 671
combinations of functions, 40
comets, orbits of, 718
common polar curves, 691
common ratio of a geometric series, 742
comparison properties of the integral, 393
comparison test for improper integrals, 548
Comparison Test for series, 760, 779
Comparison Theorem for integrals, 548
Completeness Axiom, 734
component of **b** along **a**, 851
component function, 890, 1125
components of acceleration, 919
components of a vector, 838, 851, 919
composition of functions, 41, 200
  continuity of, 121, 958
  derivative of, 201
compound interest, 241, 317
compressibility, 230
concavity, 300
Concavity Test, 300, A47
concentration, 229, 289
conchoid, 668, 693
conditionally convergent series, 769, 770, 774
conductivity (of a substance), 1192
conductivity, thermal, 629
cone, 702, 879
  parametrization of, 1174

conic section, 702, 711
  directrix, 703, 711
  eccentricity, 711
  focus, 703, 704, 711
  in polar coordinates, 711
  polar equation, 713
  shifted, 707, A21
  unified description, 711
  vertex (vertices), 703
conics, 702
connected region, 1113
conservation of energy, 1150
conservative vector field, 1129, 1147, 1148, 1163
constant force, 852
constant function, 174
Constant Multiple Law of limits, 95, 728
Constant Multiple Rule, 177
constant spring, 468
constraint, 1020, 1025
consumer surplus, 587
continued fraction expansion, 738
continuity
  of a function, 115
  of a function of three variables, 959
  of a function of two variables, 957
  on an interval, 118
  from the left or right, 117
  of a vector function, 891
continuous function, 98, 115
  integration of, 520
continuous random variable, 592
contour curves, 939
contour map, 939, 940
convergence
  absolute, 774
  conditional, 774
  of an improper integral, 543, 546
  interval of, 783
  radius of, 783
  of a sequence, 726
  of a series, 740
convergent improper integral, 546, 549
convergent sequence, 726, 740
convergent series, 740
  properties of, 728, 746
conversion of coordinates
  cylindrical to rectangular, 1096
  rectangular to cylindrical, 1096
  rectangular to spherical, 1102
  spherical to rectangular, 1102
cooling tower, hyperbolic, 881
coordinate axes, 830, A11
coordinate planes, 830
coordinate system, A2
  Cartesian, A11
  cylindrical, 1096
  polar, 684
  rectangular, A11
  spherical, 1102
  three-dimensional rectangular, 830, 831
coplanar vectors, 860
Coriolis acceleration, 929
corner reflector, 846
Cornu's spiral, 682
cosine function, A26
  derivative of, 194
  graph of, 30, A33
  power series for, 800, 802
cost, marginal, 233
cost function, 233, 341
Coulomb's constant, 819
Coulomb's Law, 328
critical number, 282, 284
critical point(s), 1009, 1019
cross product, 855
  direction of, 857
  geometric characterization of, 858
  length of, 857
  magnitude of, 858
  properties of, 857, 859
cross-section, 447
cross-section of a surface, 875
cubic function, 26
curl of a vector field, 1161, 1162
current, 228
curvature, 683, 906, 907, 908, 911
curvature of a plane parametric curve, 914
curve(s), 664
  angle between, 277
  area between, 436
  asymptotic, 328
  Bézier, 666, 684
  boundary (*see* boundary curve)
  bullet-nose, 206
  cissoid of Diocles, 693
  closed, 1146
  contour, 939
  Cornu's spiral, 682
  demand, 587
  devil's, 213
  dog saddle, 949
  Ebbinghaus forgetting curve, 239
  epicycloid, 673
  equipotential, 949
  families of implicit, 217
  grid, 1172
  helix, 892, 900
  learning, 234, 612, 647
  length of, 560, 904
  level, 939
  longbow, 670
  monkey saddle, 949
  orientation of, 1136, 1154
  orthogonal, 216
  ovals of Cassini, 694
  parametric, 662
  piecewise-smooth, 1133
  polar, 691
  serpentine, 190
  simple, 1147
  smooth, 560, 906
  space, 891
  strophoid, 700, 721
  swallowtail catastrophe, 672
  toroidal spiral, 893
  trochoid, 670
  twisted cubic, 894
  witch of Maria Agnesi, 671
curve fitting, 23
curve-sketching 303,
  procedure for, 320
cusp, 673
cycloid, 666
cylinder, 4, 832, 875
  parabolic, 875
  parametrization of, 1173
  rulings of, 875
cylinders, intersection of, 1101
cylindrical coordinate system, 1096
  conversion equations for, 1096
  triple integrals in, 1097
cylindrical coordinates, 1096
cylindrical shell, 460, 463
cylindrical shells, method of, 460

decay, exponential, 239
decay, law of natural, 239
decay, radioactive, 241
decreasing function, 16, 17
decreasing sequence, 732
definite integral, 384
  evaluating, 387
  properties of, 391
  review of, 1038
  Substitution Rule for, 423
  of a vector function, 901
definite integration
  by parts, 486, 487, 488, 489
  by substitution, 423
degree of a polynomial, 25
del (∇), 997, 999
delta (Δ) notation, 146
demand curve, 341, 587
demand function, 341, 587

density
  of a lamina, 1069
  linear, 228, 413
  liquid, 576
  mass vs. weight, 576
  of a solid, 1091
dependent variable, 9, 934, 987
derivative(s), 140, 144
  of a composite function, 200
  of a constant function, 174
  directional (*see* directional derivative)
  domain of, 153
  of an even function, 165
  of exponential functions, 174, 204, A56
  as a function, 153
  higher, 159
  higher partial, 966
  of hyperbolic functions, 261, 263
  of an increasing or decreasing function, 297
  of an integral, 399
  of an inverse function, 217
  of an inverse hyperbolic function, 265
  of inverse trigonometric functions, 217, 220, 223
  left-hand, 165
  of logarithmic functions, 217, A57
  of natural exponential function, 180
  normal, 1169
  notation, 156, 963
  of an odd function, 165
  partial, 961
  of a polynomial, 174
  of a power function, 174
  of a power series, 789
  of a product, 185, 186
  of a quotient, 187
  as a rate of change, 140
  right-hand, 165
  second, 159, 900
  second directional, 1007
  second partial, 967
  and the shape of a graph, 296
  as the slope of a tangent, 140, 147
  third, 160
  of trigonometric functions, 191, 194
  of a vector function, 898, 900
Descartes, René, A11
descent of aircraft, determining start of, 209
determinant, 855, 856
deviation, angle of, 289
devil's curve, 215
Difference Law of Limits, 95, 728
difference quotient, 10
Difference Rule, 178
difference of vectors, 838
differentiable function, 156
  failure to be, 158
  of two variables, 977
differential, 254, 256, 420, 979, 981
differential calculus, 4
differential equation, 183, 239, 359, 605, 606, 608, 609
  autonomous, 616
  Bernoulli, 646
  family of solutions, 606, 607, 610
  first-order, 608
  general solution of, 610
  linear, 642
  logistic, 633
  modeling population growth with, 606
  order of, 608
  partial, 968
  second-order, 608
  separable, 621
  solution of, 608
differentiation, 156, 173
  formulas for, 189, RP5
  formulas for vector functions, 900
  implicit, 209, 210, 966, 990
  as an inverse process of integration, 405
  logarithmic, 220
  partial, 961, 962, 963, 966
  of a power series, 788
  term-by-term, 788
  of a vector function, 898, 900
differentiation operator, 156
diffusion equation, 972
Direct Comparison Test, 760
Direct Substitution Property, 98
directed line segment, 836, 839
direction angles, 850
direction cosines, 850
direction field, 612, 613
direction of most rapid decrease, 1006
direction numbers, 866
directional derivative, 994, 995, 996, 999
  maximum value of, 1000
  second, 1007
  of a temperature function, 994, 996
directrix, 703, 711
discharge (flux), 589
discontinuity, 115, 116
discontinuous function, 115
discontinuous integrand, 546
disk method for approximating volume, 449
disks and washers vs. cylindrical shells, computing volume by, 463
dispersion, 289
displacement, 143, 413
displacement vector, 836, 837, 852
distance
  between parallel planes, 871, 874
  between point and line in space, 854, 863
  between point and plane, 863, 870, 871
  between points in a plane, A11
  between points in space, 833
  between real numbers, A7
  between skew lines, 871
distance formula, A12
  in three dimensions, 833
distance problem, 379
distinct linear factors, 508
divergence
  of the harmonic series, 750
  of an improper integral, 543, 546
  of an infinite series, 740
  of a sequence, 726
  of a vector field, 1165
Divergence, Test for, 744, 779
Divergence Theorem, 1201, 1208
divergent improper integral, 543, 546
divergent sequence, 726
divergent series, 740
division of power series, 807
DNA, helical shape of, 892
dog saddle, 949
domain, determining in curve sketching, 320, 322
domain convention, 13
domain of a function, 8
  of three variables, 944
  of two variables, 934
  of a vector function, 890
domain sketching, 934
doomsday equation, 640
Doppler effect, 993
dot product, 847
  in component form, 847
  properties of, 848
  in vector form, 848
double-angle formulas, A29
double helix, 892
double integral(s), 1038, 1040
  applications of, 1069
  change of variables in, 1063, 1109, 1112
  over general regions, 1051
  Midpoint Rule for, 1042
  in polar coordinates, 1063, 1064, 1065
  properties of, 1058, 1059

  over rectangles, 1038
  volumes and, 1038
double Riemann sum, 1041
Douglas, Paul, 936
drug response curve, 308
dye dilution method, 589

e (the number), 50, 180, A54
  as a limit, 221
  as a sum of an infinite series, 800
Ebbinghaus, Hermann, 239
Ebbinghaus forgetting curve, 239
eccentricity, 711
economic applications, rates of change in, 233
Einstein's theory of special relativity, 815
electric charge, 1070, 1091
electric circuit, 620, 623, 644
electric current to a pulse laser, 79
electric field (force per unit charge), 1128
electric field **E**, 895, 1204
electric flux, 1191, 1204
electric force, 1128
element of a set, A3
elementary function, integrability of, 521
elimination of a parameter, 663
ellipse, 215, 704, 711, A19
  area, 501
  directrix, 711
  eccentricity, 711
  foci, 704, 711
  major axis, 705, 716
  minor axis, 676
  polar equation, 713, 716
  reflection property, 705, 710
  rotated, 216
  vertices, 705
ellipsoid, 877, 879
elliptic paraboloid, 879
  parametrization of, 1174
elusive limit, evaluation of, 810
empirical model, 23
end behavior of a function, 139
endpoint extreme values, 281
energy
  conservation of, 1150
  kinetic, 477, 1151
  potential, 1151
epicycloid, 672, 673
epistola posterior, 811
epistola prior, 811
epitrochoid, 682, 722
equation(s)
  cancellation, 56
  of a circle, A17
  of degree *n*, 213
  diffusion, 972
  of an ellipse, 705, 713, 716, A19
  of a graph, A16, A17
  heat conduction, 972
  of a hyperbola, 706, 707, 713, A20
  integral, 627
  Laplace's, 968, 1112
  of a line, A12, A13, A14, A16
  of a line in space, 865
  of a line through two points, 866
  linear, A14
  linear, of a plane, 868
  logistic difference, 738
  logistic differential, 607
  Lotka-Volterra, 649
  *n*th-degree, 213
  of a parabola, 703, 713, A18
  parametric, 662, 865, 891
  of a plane, 868
  of a plane through three points, 869
  point-slope, A12
  polar, 687, 713
  predator-prey, 649
  second-degree, A16
  of a space curve, 891
  of a sphere, 833
  slope-intercept, A13
  symmetric, 866
  two-intercept form, A16
  van der Waals, 972
  vector, of a line, 865, 867
  wave, 968
equilateral hyperbola, A21
equilibrium point, 651
equilibrium solution, 607, 650
equipotential curves, 949
equivalent vectors, 836
error
  in approximate integration, 532, 533
  percentage, 258
  relative, 258
  in Taylor approximation, 812
error bounds, 531, 533
error estimate
  for alternating series, 768
  for the Midpoint Rule, 532, 533
  for Simpson's Rule, 537
  for the Trapezoidal Rule, 532, 533
error function, 408
error tolerance, 106
escape velocity, 551
estimate of the sum of a series, 755, 763, 768, 775
  of an alternating series, 768, 775
Euclid, 97
Eudoxus, 2, 97, 418
Euler, Leonhard, 52, 279, 618, 752, 759, 1121
Euler's method, 612, 616, 617
evaluating definite integrals, 387
even function, 16, 320
expected values, 1076, 1077
exponential decay, 239
exponential function(s), 31, 48, 179, 204, A53, A55, RP4
  with base *b*, A55
  derivative, of 174, 201, A55
  graphs of, 45
  integration of, 389, 419, 420, 805, 806
  limits of, 127, A54
  power series for, 796
  properties of, A54, A56
exponential graph, 44
exponential growth, 239, 636
exponents, laws of, 47, A54, A56
extended product rule for three functions, 191
extrapolation, 25
extreme value, 280
Extreme Value Theorem, 281, 1014

family
  of epicycloids and hypocycloids, 672, 673
  of exponential functions, 45
  of functions, 27, 333, 334
  of implicit curves, 217
  of parametric curves, 667
  of polar curves, 694
  of solutions, 606, 607
fat circles, 214, 567
Fermat, Pierre, 3, 152, 282, 418, A11
Fermat's Principle, 347
Fermat's Theorem, 282, 283
Fibonacci, 725, 737
Fibonacci sequence, 725, 737
field
  conservative, 1129, 1147, 1148, 1163
  electric (force per unit charge), 1128
  force, 1124, 1128
  gradient, 1128
  gravitational, 1128, 1145
  incompressible, 1166
  irrotational, 1164
  scalar, 1125
  vector (*see* vector field)
  velocity, 1124, 1164
First Derivative Test, 298
  for Absolute Extreme Values, 339
first-degree Taylor polynomial, 1019
first octant, 830

first-order linear differential equation, 608, 641, 642
first-order optics, 817
fixed point of a function, 172, 296
floor function, 101
flow lines, 1131
fluid flow, 1127, 1164, 1165, 1166, 1198, 1205
flux, 589, 1189, 1191
flux integral, 1189
FM synthesis, 333
foci, 704, 706
focus, 704, 711
  of a conic section, 711
  of an ellipse, 704, 705, 711
  of a hyperbola, 706, 711
  of a parabola, 703, 711
folium of Descartes, 210, 721
force, 467
  centripetal, 930
  constant, 852
  exerted by fluid, 577
  resultant, 842
  torque, 861, 925
force field, 1124, 1128
Fourier, Joseph, 234
Fourier series, finite, 500
four-leaved rose, 689
fractions (partial), 507, 508
Frenet-Serret formulas, 915
Fresnel, Augustin, 402
Fresnel function, 402
friction, coefficient of, 198
frustum, 457, 458
Fubini, Guido, 1044
Fubini's Theorem
  for double integrals, 1044
  for triple integrals, 1083
function(s), 8
  absolute value, 14
  Airy, 794
  algebraic, 29
  arc length, 560, 905
  arcsine, 62
  area, 398
  arrow diagram of, 9
  average cost, 346
  average value of, 473, 594, 1047, 1095
  Bessel, 216, 791
  Cobb-Douglas production function, 937, 938, 943, 973, 1027
  combinations of, 40
  component, 890, 1125
  composite, 41, 199, 958
  constant, 174
  continuity of, 115, 891, 957, 959
  continuous, 891, 957, 959
  continuous, integration of, 520
  continuous properties, 117
  cost, 233, 341
  cubic, 26
  decreasing, 16
  demand, 341, 587
  derivative of, 144, 153
  differentiability of, 156, 977
  discontinuous, 115
  domain of, 8, 890, 934, 944
  elementary, 521
  empirical, 23
  end behavior of, 139
  epicycloids, 672
  error, 408
  escape velocity, 629
  even, 16, 165, 320
  exponential, 31, 45, 48, 51, 179, A53, A55, RP4
  extreme values of, 280
  family of, 27, 333, 334
  fixed point of, 172, 296
  Fresnel, 402, 408
  Gompertz, 638, 641
  gradient of, 997, 999
  graph of, 9, 937
  greatest integer, 101
  harmonic, 968, 1169
  Heaviside, 45
  homogeneous, 993
  hyperbolic, 263
  implicit, 209
  increasing, 16, 17
  integrable, 1040
  inverse, 54, 55
  inverse cosine, 63
  inverse hyperbolic, 264
  inverse sine, 62
  inverse tangent, 63
  inverse trigonometric, 61, 64, 222, 223
  joint density, 1074, 1091
  limit of, 83, 105, 951, 952, 959
  linear, 22, 937
  logarithmic, 31, 57, A51, A56
  machine diagram of, 11
  marginal cost, 233, 341, 413
  marginal profit, 341
  marginal revenue, 341
  maximum and minimum values of, 280, 1008
  of *n* variables, 945
  natural exponential, 51, 180, A53
  natural logarithmic, 59, A51
  nondifferentiable, 158
  normal density, 308
  odd, 16, 165, 320
  one-to-one, 54
  periodic, 321
  piecewise defined, 14
  polynomial, 25
  polynomial, of two variables, 955
  position, 142
  potential, 1129
  power, 27, 174
  power series representation for, 791
  probability density, 592, 1074
  profit, 336
  quadratic, 25
  ramp, 45
  range of, 8
  range, of two variables 934
  rational, 29, 518
  rational, of two variables 955
  reciprocal, 28
  reflected, 37
  representation as a power series, 787
  representation by a Taylor series, 797
  representations of, 8, 10
  revenue, 341
  root, 27
  rules for defining, 13
  of several variables, 934, 944, 945, 966
  shifted, 37
  signum, 103
  sine integral, 408
  smooth, 560
  step, 16
  stretched, 37
  symmetric, 424
  tabular, 11
  of three variables, 944, 966
  transcendental, 30
  transformation of, 36
  translation of, 37
  trigonometric, 30, A26
  of two variables, 934
  value of, 8, 9
  vector, 890
function notation, 8
Fundamental Theorem of Calculus, 399, 405
  for double integrals, 1154
  for line integrals, 1144
  Part 1, 399, 400
  Part 2, 402, 403
  summary of higher-dimensional versions, 1208
  for surface integrals, 1195
  for vector functions, 902
future value of income, 591

G (gravitational constant), 236, 472
Gabriel's horn, 551, 574
Galileo, 667, 676, 703
Galois, Evariste, 213
Gause, G. F., 636
Gauss, Karl Friedrich, 817, 1201, A37
Gaussian optics, 817
Gauss's Law, 1191, 1205
Gauss's Theorem, 1201
general region, integration over, 1052
geometric series, 742, 779
geometric vector, 836, 839
geometry of a tetrahedron, 864
geosynchronous orbit, 926
Gibbs, Josiah Willard, 842
Gini, Corrado, 445
Gini coefficient, 445
Gini index, 445
global maximum and minimum, 280
Gompertz function, 638, 641
grad *f*, 997, 999
gradient, 997, 999
gradient, velocity, 732
gradient vector, 997, 998, 999, 1004
gradient vector field, 1004, 1128
graph(s)
  of an equation, A16, A17
  of equations in three dimensions, 831, 832
  of exponential functions, 46, 179, RP4
  of a function, 9
  of a function of two variables, 937
  of logarithmic functions, 61, 62
  of a parametric curve, 662
  of a parametric surface, 1184
  polar, 687, 691
  of power functions, 27, RP3
  of a sequence, 730
  of a surface, 1184
  of trigonometric functions, 30, A32, RP2
gravitation law, 236, 472
gravitational acceleration, 467
gravitational field, 1128, 1145
great circle, 1107
greatest integer function, 101
Green, George, 1155
Green's identities, 1169
Green's Theorem, 1154, 1208
  extended versions, 1157
  for a union of simple regions, 1157, 1158
  vector forms, 1167
Gregory, James, 200, 497, 535, 790, 796
Gregory's series, 790
grid curves, 1172
ground state, 599
growth, exponential, 239
growth, law of natural, 239, 631
growth, population, 240
growth rate, 239, 413
  relative, 240, 632
guidelines for curve sketching, 320
guidelines for integration, 517

half-angle formulas, A30
half-life, 48, 241
half-space, 944
Halley, Edmund, 931
Hamilton, Sir William Rowan, 855
hare-lynx system, 653
harmonic function, 968, 1169
harmonic series, 744, 754
harmonic series, alternating, 767
heat conduction equation, 972
heat conductivity, 1192
heat equation, 972
heat flow, 1192
heat index, 961
Heaviside, Oliver, 86
Heaviside function, 45
helix, 892, 900
hidden line rendering, 875
higher derivatives, 159
higher partial derivatives, 967
homogeneous function, 993
Hooke's Law, 468, 607
horizontal asymptote, 127, 321
horizontal line, equation of, A13
Horizontal Line Test, 54
horizontal plane, 831
horizontal shift of a function, 37
Hubble space telescope, 287
humidex, 947, 961
Huygens, Christiaan, 667
hydrostatic pressure and force, 576
hydro-turbine optimization, 1030
hyperbola, 215, 706, 711, A20
  asymptotes, 706, A20
  branches, 706, A20
  directrix, 711
  eccentricity, 711
  equation, 706, 707, 713, A20
  equilateral, A21
  foci, 706 711
  polar equation, 713
  reflection property, 710
  vertices, 706
hyperbolic cosine, 261
hyperbolic function(s), 261
  derivatives of, 263
  inverse, 264, 265
hyperbolic identities, 262
hyperbolic paraboloid, 878, 879
hyperbolic sine, 261
hyperbolic substitution, 503, 504
hyperboloid, 878, 879, 880, 881
hypersphere, volume of, 1095
hypervolume, 1089
hypocycloid, 672

i (standard basis vector), 841
I/D Test, 297
ideal gas law, 232, 972
identities
  hyperbolic, 262
  product, for trigonometric integrals, 498
  trigonometric, 500, A28
image of a point, 1109
image of a region, 1109
implicit differentiation, 209, 210, 966, 990
implicit function, 209, 210
Implicit Function Theorem, 990, 991
improper integral, 543, 546
  comparison test for, 548
  convergence or divergence of, 544, 546
impulse of a force, 477
incompressible velocity field, 1166
Increasing/Decreasing Test, 321
increasing function, 16
increasing sequence, 732
increment, 146, 981
indefinite integral(s), 409
  Substitution Rule for, 419
  table of, 410
independence of path, 1145, 1146
independent random variable, 1076
independent variable, 9, 934, 987
indeterminate difference, 314
indeterminate forms of limits, 309, 313, 314
indeterminate powers, 315
indeterminate product, 313
index of summation, A36
inequalities, rules for, A4
inequality, coefficient of, 445
inertia, moment of. *See* moment of inertia
infinite discontinuity, 116
infinite interval, 542, 543
infinite limit, 89, 112, 133
  of a sequence, 728
infinite sequence. *See* sequence
infinite series. *See* series
inflection point, 301, 321
initial condition, 610
initial point
  of a parametric curve, 663
  of a vector, 836

initial value of a growth function, 240
initial-value problem, 610, 632
inner product, 847
input of a function machine, 9
instantaneous rate of change, 80, 146, 225
instantaneous rate of growth, 230
instantaneous rate of reaction, 229
instantaneous velocity, 81, 143, 226
integer, A2
integrable function, 384, 1040
integral(s)
  approximations to, 390
  change of variables in, 419, 1064, 1097, 1103
  comparison properties of, 393
  conversion to cylindrical coordinates, 1097, 1098
  conversion to polar coordinates, 1064
  conversion to spherical coordinates, 1103
  definite, 384
  definite, review of, 1038
  derivative of, 400
  double (*see* double integral(s))
  equation, 627
  evaluation of, 387
  improper, 542
  indefinite, 409, 486
  iterated, 1043
  line (*see* line integral)
  patterns in, 528
  properties of, 391
  surface (*see* surface integral)
  of symmetric functions, 424
  table of, 485, 517, 523, RP6–10
  trigonometric, 493
  triple (*see* triple integral(s))
  units for, 415
integral calculus, 3
integral equation, 627
integral sign, 384
Integral Test, 753, 780
  proof of, 757
integrand, 384, 493
  discontinuous, 546
integrating factor, 643
integration, 384, 405
  approximate, 529
  of exponential functions, 389, 422
  formulas, 485, 517, RP6–10
  indefinite, 409
  limits of, 384
  numerical, 538
  partial, 1043
  by partial fractions, 507
  by parts, 486, 487, 488, 489, 518
  of a power series, 788
  of powers of secant and tangents, 495
  of powers of sine and cosine, 493
  of rational functions, 507
  by a rationalizing substitution, 514
  with respect to *x*, 436
  with respect to *y*, 439
  reversing order of, 1043
  over a solid, 1084
  substitution in, 419
  with tables, 523
  techniques of, 485
  with technology, 523, 525
  term-by-term, 788
  of trigonometric functions, 493
  of a vector function, 901
intercepts, 320, 322, A19
interest compounded continuously, 243
Intermediate Value Theorem, 122
intermediate variable, 987
interpolation, 25
interpretations of a derivative, 234
intersection
  of objects in space, 897
  of planes, 869
  of polar graphs, area of, 696
  of three cylinders, 1101
intersection and collision of particles, 671
intersection of sets, A3
interval, A3
interval of convergence, 783
intervals of increase or decrease, 322
invention of calculus, 8
  Newton and Leibniz, priority dispute between, 418
inverse cosine function, 63
inverse function(s), 54, 55
inverse hyperbolic functions, 264
inverse processes, differentiation and integration as, 405
inverse sine function, 62
inverse square field, 1153
inverse square law, 28
inverse tangent function, 63
inverse transformation, 1110
inverse trigonometric functions, 61, 62, 217, 222, 223
involute, 684
irrational number, A2
irreducible quadratic factor, 511
  repeated, 513
irrotational vector field, 1164
isobar, 940
isothermal, 940
isothermal compressibility, 230
iterated integral, 1043

j (standard basis vector), 841
Jacobi, Carl Gustav Jacob, 1111
Jacobian of a transformation, 1111
jerk, 161
joint density function
  of three variables, 1091
  of two variables, 1074
joule, 467
jump discontinuity, 116

k (standard basis vector), 841
kampyle of Eudoxus, 215
Kepler, Johannes, 715, 921, 925
Kepler's Laws, 715, 921, 925
kinetic energy, 477, 1151
Kirchhoff's laws, 614

Lagrange, Joseph-Louis, 293, 294, 1021
Lagrange multiplier, 1020
  with one constraint, 1020, 1021
  with two constraints, 1025
Lagrange's form of the remainder term, 799
lamina, 580
lamina (of variable density)
  center of mass of, 1071
  density at a point on, 1069
  moment about an axis, 1070, 1071
  moment of inertia about an axis, 1072
  radius of gyration about an axis, 1074
laminar flow, law of, 232, 588
Laplace, Pierre, 968
Laplace operator, 1167
Laplace transform, 552
Laplace's equation, 968, 1122, 1167
lattice point, 278
law, inverse square, 28
law of conservation of angular momentum, 925
Law of Conservation of Energy, 1151
law of cooling, 242
law of cosines, A31
law of gravitation, 472
law of laminar flow, 232, 588
law of natural growth or decay, 239, 631
Law of Universal Gravitation, 921, 926
laws of exponents, 47, A53
laws of logarithms, 58, A51
leading coefficient, 25
learning curve, 234, 612, 647
least squares method, 24, 1018

least upper bound, 734
left endpoint approximation, 530
left-hand derivative, 165
left-hand limit, 86, 110
Leibniz, Gottfried Wilhelm, 3, 156, 399, 418, 627, 811
Leibniz formula for π, 790
Leibniz notation, 156, 622
lemniscate, 215
length
  of a curve, 560
  of a line segment, A7, A12
  of a parametric curve, 676
  of a polar curve, 696
  of a space curve, 904
  of a vector, 839
level curves, 939
level surface, 945
  tangent plane to, 1002
l'Hospital, Marquis de, 311
l'Hospital's Rule, 309, 310, 311, 319, A48
  origins of, 319
libration point, 356
limaçon, 690
Limit Comparison Test, 762
Limit Laws, 94, 95, 111, A41
  for functions of two variables, 955
  for sequences, 728
limit(s), 2, 83
  calculating, 94
  e (the number) as, 221
  evaluation of, 97
  of exponential functions, 133, A54
  of a function, 83
  of a function of three variables, 959
  of a function of two variables, 951, 952
  graphical evaluation, 83
  infinite, 89, 112, 132
  at infinity, 127, 128, 130, 132, 134, 137
  of integration, 384
  intuitive definition, 83, 86
  left-hand, 86, 110
  of logarithmic functions, 91, A53
  one-sided, 86, 110
  precise definitions, 105, 106, 110, 113, 134, 135, 137
  properties of, 94
  properties of, for vector functions, 898
  right-hand, 86, 110
  of a sequence, 5, 374, 726, 727
  involving sine and cosine functions, 192, 193, 194
  of a trigonometric function, 195
  of a vector function, 890, 898
limiting value, 81
line integral
  Fundamental Theorem for, 1144
  of the normal component of **F**, 1168
  for a plane curve, 1131, 1132
  with respect to arc length, 1132, 1135, 1137
  with respect to x and y, 1132, 1135
  for a space curve, 1137
  of the tangential component of **F**, 1167
  of vector fields, 1138, 1140, 1141
  work defined as, 1139
line(s) in the plane, 78, A12
  equations of, A12, A13, A14
  horizontal, A13
  normal, 177
  parallel, A14
  perpendicular, A14
  secant, 78, 79
  slope of, A12
  tangent, 78, 79, 141
line(s) in space, 863, 864
  equation of, through two points, 866
  normal, 1002
  parametric equations of, 865
  skew, 867
  symmetric equations of, 866
  tangent, 898
  vector equation of, 865
linear approximation, 254, 976, 980
linear density, 228, 413
linear differential equation, 642
linear equation, A14
linear equation of a plane, 868
linear function, 22, 937
linear model, 22
linear motion of an object, 360
linear regression, 24
linearization, 254, 976
liquid force, 577
Lissajous figure, 665, 672
lithotripsy, 706
local maximum and minimum, 280, 321, 322, 1008
logarithm(s), 31, 57, A56
  laws of, 58, A52
  natural, 59, A51
  notation for, 59
logarithmic differentiation, 217
logarithmic function(s), 31, 57, A51
  with base b, 60, A56
  derivatives of, 220, A52, A57
  graphs of, 58, 63
  limits of, 91, A53
  properties of, 58, A52
logistic difference equation, 738
logistic differential equation, 607, 633
logistic distribution, 598
logistic equation, 318
logistic model, 607, 632
  vs. natural growth model, 636
logistic sequence, 738
longbow curve, 670
LORAN system, 709
Lorenz curve, 445
Lotka-Volterra equations, 649
lower limit of integration, 384
lower sum, 377
LZR Racer, 984

machine diagram of a function, 9
Maclaurin, Colin, 796
Maclaurin series, 795, 796, 802
  table of, 804
magnetic field **B**, 961
magnitude of a vector, 839
major axis of ellipse, 705
marginal cost function, 146, 233, 341, 413
marginal productivity of capital, 973
marginal productivity of labor, 973
marginal profit function, 341
marginal propensity to consume or save, 749
marginal revenue function, 341
market equilibrium, 591
mass
  of a lamina, 1069
  of a solid, 1090
  of a surface, 1184
  of a wire, 1134
mass, center of. *See* center of mass
mathematical induction, 71, 72, 734
  principle of, 71, 72, A38
mathematical model, 11, 21
maximum and minimum values, 280, 1008
mean life of an atom, 551
mean of a probability density function, 595
Mean Value Theorem, 290, 291, 293
  for double integrals, 1062
  for integrals, 474
mean waiting time, 595
median of a probability density function, 596
method of cylindrical shells, 460
method of exhaustion, 2, 97
method of Lagrange multipliers
  with one constraint, 1020, 1021
  with two constraints, 1025
method of least squares, 24, 1018
method of partial fractions, 507

Midpoint Formula, A16
  for points in space, 835
midpoint rule, 390, 530, 531
  for double integrals, 1042
  error in using, 532
  for triple integrals, 1093
minor axis of ellipse, 705
mixing problems, 625
Möbius, August, 1187
Möbius strip, 1181, 1187
modeling
  with differential equations, 606
  motion of a spring, 607
  population growth, 48, 240, 606, 632, 638, 657
model(s), mathematical, 11, 21
  comparison of natural growth vs. logistic, 636
  of electric current, 614
  empirical, 23
  exponential, 31, 45
  Gompertz function, 638, 641
  linear, 22
  logarithmic, 31
  polynomial, 25
  for population growth, 240, 606, 631, 638
  power function, 27
  predator-prey, 649
  rational function, 29
  seasonal-growth, 641
  trigonometric, 30, 31
  von Bertalanffy, 657
moment(s)
  about an axis, 579, 1070
  of a lamina, 580, 1070
  of a mass, 579
  about a plane, 1090
  polar, 1073
  second, 1072
  of a solid, 1090
  of a system of particles, 579
moment of inertia, 1072, 1091
  about axes, 1143
  about the origin, 1073
momentum of an object, 477
monkey saddle, 949
monotonic sequence, 732
Monotonic Sequence Theorem, 733, 752
motion of a projectile, 918
motion in space, 916
movie theater seating, 478
multiple integrals, 1037. *See also* double integral(s); triple integral(s)
multiplication, scalar, 837, 840
multiplication and division of power series, 807
multiplier (Lagrange), 1020, 1021, 1025
multiplier effect, 749
natural exponential function, 51, 179, A53
  derivative of, 177, A55
  graph of, 181
  power series for, 796
  properties of, A54
natural growth law, 239, 631
  vs. logistic model, 636
natural logarithm function, 59, 61, A51
  derivative of, 217, A52
  limits of, A53
  properties of, A52
n-dimensional vector, 840
negative angle, A25
negative of a vector, 838
net area, 385
Net Change Theorem, 412
net investment flow, 591
newton (unit of force), 477
Newton, Sir Isaac, 3, 8, 97, 152, 399, 418, 811, 921, 926
Newton's discovery of the binomial series, 811
Newton's Law of Cooling, 242, 612
Newton's Law of Gravitation, 236, 472, 922, 926, 1127
Newton's method, 351
  for functions of two variables, 1035
Newton's Second Law of Motion, 467, 477, 918, 922, 926
Newton-Raphson method, 351
Nicomedes, 668
nondifferentiable function, 158
nonparallel planes, 870
normal component
  of acceleration, 919, 920
  of **F**, line integral of, 1168
normal derivative, 1169
normal distribution, 597
normal line, 177
normal line to a surface, 1002
normal plane, 910
normal vector, 868, 909
normally distributed random variable, probability density function of, 1077
*n*th term of a sequence, 724
*n*th-degree Taylor polynomial, 798
n-tuple, 840
nuclear reactor, cooling towers of, 881
number
  integer, A2
  irrational, A2
  rational, A2
  real, A2
numerical integration, 529

O (origin), 830
octant, 830
odd function, 16, 320
Ohm's Law, 614
one-sided limits, 86, 110
one-to-one function, 54
one-to-one transformation, 1109
open connected region, 1146
open interval, A3
open interal, differentiability on, 56
optics
  first-order, 817
  Gaussian, 817
  third-order, 817
optimization problems, 280, 336
orbit of a planet, 921
orbital, 599
order of a differential equation, 608
order of integration, reversed, 1045
ordered pair, A10
ordered triple, 830
Oresme, Nicole, 745
orientation of a curve, 1136, 1154
orientation of a surface, 1187
oriented surface, 1187
origin, 830, A2, A10
orthogonal curves, 216
orthogonal projection of a vector, 854
orthogonal surfaces, 1008
orthogonal trajectory, 216, 624
orthogonal vectors, 849
osculating circle, 910
osculating plane, 910
Ostrogradsky, Mikhail, 1201
output of a function rule, 9
ovals of Cassini, 694

Pappus, Theorem of, 583
Pappus of Alexandria, 583
parabola, 703, 711, A18
  axis, 703
  directrix, 703
  equation, 675, 703
  focus, 703, 711
  polar equation, 713
  reflection property, 274
  vertex, 703
parabolic cylinder, 875
paraboloid, 277
  circular, 880

elliptic, 877
  hyperbolic, 878
paradoxes of Zeno, 724
parallel lines, A14
parallel planes, 869
parallel vectors, 838, 858
parallelepiped, 447
  volume of, 860
Parallelogram Identity, 854
Parallelogram Law, 837
Paramecium, 636, 637
parameter, 662, 865, 891
parametric curve, 662, 664, 667
  arc length of, 662
  area under, 675
  slope of tangent line to, 673
parametric equations, 662, 865, 891
  of a line in space, 865
  of a space curve, 891, 905
  of a surface, 1170
  of a trajectory, 919
parametric surface, 1170, 1183
  graph of, 1165, 1172
  smooth, 1176
  surface area of, 1177
  surface integral over, 1183
  tangent plane to, 1175
  given by a vector function, 1171
parametrization of a space curve, 905
  with respect to arc length, 905
  smooth, 906
paraxial rays, 256
Pareto's law of income, 592
partial derivative(s), 961
  of a function of more than three variables, 966
  of a function of more than two variables, 966
  of a function of two variables, 961, 963
  interpretations of, 964
  at maximum and minimum values, 1009
  notations for, 963
  as a rate of change, 962
  with respect to x, 962, 963
  with respect to y, 963
  rules for finding, 963
  second, 967
  as slopes of tangent lines, 964
partial differential equation, 968
partial differentiation, 961, 962, 963, 966
partial fractions, 507, 508
partial integration, 486, 487, 488
  for double integrals, 1043
partial sum of a series, 740
particle, motion of, 916
particular antiderivative, 526
parts, integration by, 486, 487, 488
pascal (unit of pressure), 576
path, 1145
patterns in integrals, 528
pendulum, approximating the period of, 256, 260
percentage error, 258
perihelion, 709
perilune, 681
period, 321
period of a particle, 930
periodic function, 321
perpendicular lines, A14
perpendicular vectors, 849
phase plane, 651
phase portrait, 651
phase trajectory, 651
physics applications, 256, 815
  rates of change in, 226
piecewise defined function, 14
piecewise-smooth curve, 1133
planar curve, 915
Planck's law, 820
plane(s), 868
  angle between, 869
  coordinate, 830
  distance between, 874
  distance from point to, 870, 871
  equation of, 868
  equation of, through three points, 869
  horizontal, 831
  line of intersection, 869
  linear equation of, 868
  normal, 910
  osculating, 910
  parallel, 869
  scalar equation of, 868
  tangent to a surface, 974, 1175
  vector equation of, 868
  vertical, 831
plane, descent of, 209
plane, minimizing energy of, 350
plane region of type I or type II, 1053, 1054
planetary motion, laws of, 715, 921
planimeter, 1157
point of inflection, 301, 321
point(s) in space
  coordinates of, 830
  distance between, 833
  projection of, 831
point-slope equation of a line, A12
Poiseuille, Jean-Louis-Marie, 232
Poiseuille's Law(s), 260, 348, 589, 592, 972
polar axis, 685
polar coordinate system, 684, 685, 1062
  arc length in, 697
  area in, 694
  calculus in, 694
  conic sections in, 711
  conversion of double integral to, 1063, 1064, 1065
  conversion equations for Cartesian coordinates, 686
  relationship to Cartesian coordinates, 686
  tangents in, 698
polar curve, 687, 694
  arc length of, 697
  of a conic, 713, 923
  graph of, 687
  graphing with technology, 690
  parametric equations for, 697
  polar equation(s), 686
  symmetry in, 689
  table of, 691
  tangent line to, 698
polar equation(s), 686
  of a conic, 713, 923
  graph of, 687
polar graph, 687
  using technology, 690
polar moment of inertia, 1073
polar rectangle, 1063
polar region, area of, 694
pole, 685
polynomial, 25
polynomial approximations, 260
polynomial function, 25
  of two variables, 955
population growth, 48, 240, 606
  of bacteria, 631, 636
  of insects, 516
  models, 631
  world, 49
position function, 142
position vector, 839
positive angle, A25
positive orientation
  of a boundary curve, 1195
  of a closed curve, 1154
  of a surface, 1188
potential, 555
potential energy, 1151
potential function, 1129
pound (unit of force), 467
power, 148
power function(s), 27
  derivative of, 174
Power Law of Limits, 96, 729

Power Rule, 175, 176, 200, 220
power series, 781, 782, 789
  coefficients of, 782
  for cosine and sine, 801
  differentiation of, 788
  division of, 807
  for exponential function, 800
  integration of, 788
  interval of convergence, 783
  multiplication of, 807
  radius of convergence, 783
  representations of functions as, 787
predator-prey equation, 649
predator-prey model, 238
predator-prey systems, 649
present value of income, 591
pressure exerted by a fluid, 576
prime notation, 144, 178
principal unit normal vector, 909
principle of mathematical induction, 71, 72, 734, A38
probability, 592, 1074
probability density function, 592, 593, 1074
problem-solving principles, 70, 825
  carrying out a plan, 71
  introducing something extra, 70, 419
  looking back on a solution, 71, 249
  recognizing patterns, 70
  recognizing something familiar, 70, 825
  taking cases, 70, 73, 291
  thinking of a plan, 70
  understanding the problem, 7
  uses of, 171, 363, 419, 432
  using analogy, 70
problem-solving strategy, 249
producer surplus, 591
product
  cross, 855 (*see also* cross product)
  dot, 847 (*see also* dot product)
  scalar, 847
  scalar triple, 860
  triple, 860
product identities, A29
product identities for trigonometric integrals, 498
Product Law of Limits, 95, 728
Product Rule, 185, 186
  extended to three functions, 191
profit function, 341
projectile, path of, 672, 918
projectile motion, 918
  parametric equations for, 919
projection, 831, 851
  orthogonal, 854
proof of the Chain Rule, 205
properties of continuous functions, 117
properties of convergent series, 746
properties of a definite integral, 391
properties of limits, 94
p-series, 754, 779
Pyramid, Great, of Khufu, 472
Pythagorean Theorem, 501
  three-dimensional version of, 864

quadrant, A11
quadratic approximation, 260, 1019
quadratic factor, 511
quadratic function, 25
quadratic model, 648
quadric surface(s), 875, 876
  cone, 879
  ellipsoid, 877, 879
  elliptic paraboloid, 877, 879
  hyperbolic paraboloid, 878, 879
  hyperboloid, 878, 879
  paraboloid, 877
  standard form of equation for, 876
  table of graphs, 879
quaternion, 843
quotient, symmetric difference, 152
Quotient Law of Limits, 95, 728
Quotient Rule, 185, 187, 188

radian measure, 30, 85, A24
radiation from stars, 820
radioactive decay, 241
radiocarbon dating, 246
radius of convergence
  of a Maclaurin series, 804
  of a power series, 783
radius of gyration of a lamina, 1074
rainbow, formation and location of, 289
rainbow angle, 289
ramp function, 45
range of a function, 8
  of two variables, 934
rate of change
  average, 146, 225
  derivative as, 140
  instantaneous, 81, 146, 225
  interpretations of, 234
  in natural science, 225
  in social sciences, s225
rate of growth, 230, 413
rate of reaction, 148, 229, 413
rates, related, 247
ratio, common, of a geometric series, 742
Ratio Test, 774, 780
rational function, 29, 518
  continuity of, 118
  integration of, 507
  of two variables, 955
rational number, A2
rationalizing substitution for integration, 514
Rayleigh-Jeans law, 820
real line, A3
real number, A2
rearrangement of a series, 771, 772
reciprocal function, 28
Reciprocal Rule, 191
rectangular coordinate system, A11
rectangular coordinate system, three-dimensional, 831
  conversion to cylindrical coordinates, 1096
  conversion to spherical coordinates, 1102
rectangular parallelepiped, 447
rectifying plane, 915
recurrence relation, 734
red blood cell loss during surgery, 247
reduction formula, 489, 490, 494
reflecting a function, 37
reflection property
  of conics, 710
  of an ellipse, 705
  of a hyperbola, 711
  of a parabola, 274, 275
region
  connected, 1146
  under a graph, 372, 377
  open, 1146
  plane (of type I or II), 1053, 1054
  simple plane, 1155
  simple solid, 1201
  simply-connected, 1148
  solid (of type 1, 2, or 3), 1084, 1086
  between two graphs, 436
regression, linear, 24
related rates, 247
relationship between polar and Cartesian coordinates, 686
relative error, 258
relative growth rate, 240, 632
remainder, 755
remainder estimates
  for the Alternating Series, 768
  for the Integral Test, 755
remainder of the Taylor series, 798
removable discontinuity, 116
repeated irreducible quadratic factor, 513
repeated linear factors, 510

representation(s) of a function, 10, 11
  using geometric series, 787
  as a power series, 787
resultant force, 842
revenue function, 341
reversing order of integration, 1043
revolution, solid of, 449
revolution, surface of, 567
Riemann, Georg Bernhard, 385
Riemann sum(s), 385, 530
  double, 1041
  triple, 1083
right circular cylinder, 446
right-hand derivative, 165
right-hand limit, 86, 110
right-hand rule, 830, 857
Roberval, Gilles de, 404, 676
rocket equation, 492
rocket stages, determining optimal masses for, 1028
Rolle, Michel, 290
roller coaster, design of, 184
roller derby, 1108
Rolle's Theorem, 290
root function, 27
Root Law of Limits, 96
Root Test, 776, 777, 780
ruled surface, 882, 883
ruling of a surface, 876
rumor, rate of spread, 234

saddle point, 1010
sample point, 377, 384, 1039
satellite dish, parabolic, 881
scalar, 837
scalar equation of a plane, 868
scalar field, 1125
scalar multiple of a vector, 837, 840
scalar product, 847
scalar projection, 851
scalar triple product, 860
  geometric characterization of, 860
scatter plot, 11
sea ice, 629
seasonal-growth model, 620
secant function, A33
  derivative of, 194
  graph of, A33
secant line, 3, 78, 79, 81
secant vector, 898
second derivative, 159
  of an implicit function, 213
  of a vector function, 900
second-degree Taylor polynomial, 1019
Second Derivative Test, 302
Second Derivatives Test, 1010, 1015
second directional derivative, 1007
second moment of inertia, 1072
second partial derivative, 967
second-order differential equation, 608
Second Theorem of Pappas, 586
sector of a circle, area of, 694
sensitivity, 238
separable differential equation, 621
sequence, 5, 724
  bounded, 732, 733
  convergent, 726
  decreasing, 732
  divergent, 726
  Fibonacci, 725
  graph of, 730
  increasing, 732
  limit of, 5, 368, 726, 727
  logistic, 738
  monotonic, 732
  of partial sums, 740
  Squeeze Theorem for, 729
  term of, 724
series, 6, 738
  absolutely convergent, 774
  alternating, 765, 780
  alternating harmonic, 77å0, 773
  binomial, 803
  coefficients of, 782
  Comparison Test for, 779
  conditionally convergent, 774
  convergent, 740, 760
  divergent, 740, 760
  geometric, 742, 779
  Gregory's, 790
  harmonic, 744, 754
  infinite, 739
  Maclaurin, 795, 796, 802
  p-, 754, 779
  partial sum of, 740
  power, 781. 782, 789
  rearrangement of, 771
  strategy for testing, 779
  sum of, 740
  Taylor, 795, 796, 802
  term of, 739
  trigonometric, 782
serpentine (curve), 190
set, bounded or closed, 1014
set notation, A3
Shannon index, 1018
shell method for approximating volume, 460
shift of a function, 37
shifted conic, 709, A21
shifted logistic model, 640
Sierpinski carpet, 751
sigma notation, 378, A36
signum function, 103
simple curve, 1147
simple harmonic motion, 207
simple plane region, 1155
simple solid region, 1201
simply-connected region, 1148
Simpson, Thomas, 535, 1035
Simpson's Rule, 534, 535, 542
  error bounds for, 537
sine function, A26
  derivative of, 193, 194
  graph of, 30, A33
  power series for, 801
sine integral function, 408
sink, 1205
sketching curves, guidelines for, 320
skew lines, 867
slant asymptote, 316, 326
slope, A12
  of a curve, 141
slope field, 613
slope-intercept equation of a line, A13
smooth curve, 560, 906
smooth function, 560
smooth parametrization of a space curve, 906
smooth surface, 1176
Snell's law, 347
snowflake curve, 826
solid, 466
solid, volume of, 446, 447, 1040
solid angle, 1213
solid region (of type 1, 2, or 3), 1084, 1086
solid of revolution, 449
  rotated on a slant, 575
  volume of, 454, 461, 575
solution curve, 613
solution of a differential equation, 608
solution of predator-prey equations, 649
source, 1205
space, three-dimensional, 830
space curve, 891
  arc length of, 904
  graph of, 893
  parametrization of, 893
speed of a particle, 147, 678, 916
Speedo LZR racer, 984
sphere, 833
  equation of, 834
  flux across, 1190
  graph of, 1173
  parametrization of, 1173
  surface area of, 1178

spherical coordinate system, 1102
  conversion equations for, 1102
  triple integrals in, 1103
spherical wedge, 1103
spherical zones, 603
spring constant, 468, 607
Squeeze Theorem, 101, A44
  for sequences, 729
standard basis vectors, 841
  properties of, 859
standard deviation, 579
standard position of an angle, A25
stationary points, 1009
stellar stereography, 551
step function, 16
Stiles-Crawford effect, 318
Stokes, Sir George, 1195
Stokes' Theorem, 1195, 1208
strategy
  for integration, 517, 518
  for optimization problems, 336, 337, 338
  for problem solving, 70, 249
  for related rates, 247
  for testing series, 779
  for trigonometric integrals, 495, 496
streamlines, 1131
stretching of a function, 37
strophoid, 700, 721
Substitution Rule, 419, 420, 423
  for definite integrals, 423
  for indefinite integrals, 420
substitution, trigonometric, 500
substitution, Weierstrass, 516
subtraction formulas for sine and cosine, A29
subtraction of vectors, 838, 840
sum, 377
  of a geometric series, 742
  of an infinite series, 740
  lower, 377
  of partial fractions, 508
  Riemann, 386
  telescoping, 741
  upper, 377
  of vectors, 836, 837
Sum Law of limits, 95, 111
  for sequences, 728
Sum Rule, 178
summation notation, A36
supply function, 591
surface(s), 831
  closed, 1188
  graph of, 1184
  level, 945
  oriented, 1187
  orthogonal, 1008
  parametric (see parametric surface)
  positive orientation of, 1188
  quadric, 876
  smooth, 1176
  traces of, 875
surface area, 569, 679
  of a function of two variables, 1079, 1080
  of a graph of a function, 1178, 1179
  of a parametric surface, 679, 1177
  of a sphere, 1178
surface integral, 1182
  over a parametric surface, 1183
  of a vector field, 1188, 1189
surface of revolution, 567
  parametric representation of, 1175
  surface area of, 569
swallowtail catastrophe curve, 672
symmetric difference quotient, 152
symmetric equations of a line, 866
symmetric functions, integrals of, 424
symmetry, 320, 332, 424
  in polar graphs, 689
symmetry principle, 580

T and T^{-1} transformations, 1109, 1110
table of antidifferentiation formulas, 358
table of differentiation formulas, 189, RP5
table of integrals, 517, 523, RP6–10
  use of, 523
table of trigonometric substitutions, 500
tabular function, 11
tangent function, A26
  derivative of, 193
  graph of, 32, A33
tangent line(s), 140, 141
  to a curve, 3, 78, 141
  early methods of finding, 152
  to a parametric curve, 673, 674
  to a polar curve, 698
  to a space curve, 898
  vertical, 159
tangent line approximation, 254
tangent plane, 974, 975
  to a level surface, 1002
  to a parametric surface, 1175, 1176
  to a surface $z = f(x, y)$, 974, 975
tangent plane approximation, 976
tangent problem, 3, 78, 144
tangent vector, 898, 1176
tangential component of acceleration, 919, 920
tangential component of **F**, line integral of, 1167
tautochrone problem, 667
Taylor, Brook, 796
Taylor polynomial, 261, 798, 812, 1010
  applications of, 811
Taylor remainder term, 799
Taylor series, 795, 796, 802
  obtaining a new series, 805
Taylor's inequality, 798, 812, 814
techniques of integration, summary, 518
technology, graphing with
  function of two variables, 939
  gradient vector field, 1004
  level curves, 944
  parametric curves, 665
  parametric equations, 690
  parametric surface, 1172, 1173, 1176
  polar curve, 690
  space curve, 893, 894 ,895
  vector field, 1126, 1127
technology, pitfalls of using, 88
technology, using, 88, 540, 525, 555, 665, 791
  for integration, 791
telescoping sum, 741
temperature-humidity index, 947
term of a sequence, 724
term of a series, 739
term-by-term differentiation and integration, 788
terminal point
  of a parametric curve, 663
  of a vector, 836
terminal velocity, 628
Test for Divergence, 744
tests for convergence and divergence of series
  Alternating Series Test, 766
  Direct Comparison Test, 760
  Integral Test, 751
  Limit Comparison Test, 762
  Ratio Test, 774
  Root Test, 776, 777
  summary of tests, 779
tetrahedron, 864
thermal conductivity, 629
third derivative, 160
third-order optics, 817
Thomson, William (Lord Kelvin), 1155, 1195
three-dimensional coordinate systems, 830, 831
three-dimensional vector, 839

TNB frame, 909
toroidal spiral, 893
torque, 861, 925
Torricelli, Evangelista, 676
Torricelli's Law, 236
torrid zone, 603
torsion of a space curve, 911, 912, 913
torus, 458, 583, 1182
total differential, 979
total electric charge, 1070, 1091
total fertility rate, 170
total surplus, 591
trace of a surface, 875
trajectories, orthogonal, 216
trajectory, parametric equations for, 919
transcendental function, 30
transfer curve, 929
transform, Laplace, 552
transformation, 1109
  of a function, 36
  inverse, 1110
  Jacobian of, 1111
  one-to-one, 1109
  of a root function, 38
translation of a function, 37
Trapezoidal Rule, 530, 531
  error in, 532
tree diagram, 987
trefoil knot, 893, 897
Triangle Inequality, 111, A8
  for vectors, 854
Triangle Law, 836
trigonometric forms of integrals, 524
trigonometric functions, 30, 518, A26
  derivatives of, 191, 193
  graphs of, 30, 31, A32, A33
  integrals of, 409, 493
  inverse, 61, 222, 223
  limits involving, 195, 196
trigonometric identities, 500, A28
trigonometric integrals, 493
  strategy for evaluating, 495, 496
trigonometric series, 782
trigonometric substitutions, 500, 503, 504
  table of, 500
triple integral(s), 1082, 1083
  applications of, 1089
  change of order of integration in, 1088
  change of variables in, 1114, 1115
  in cylindrical coordinates, 1095, 1097, 1098
  over a general bounded region, 1084
  Midpoint Rule for, 1093
  over a rectangular box, 1082, 1083
  in spherical coordinates, 1102, 1104
  over type I or type II plane region, 1084, 1085
  type 1, 2, or 3 solid region, 1084, 1086
  volume in, 1089
triple product, 860
triple Riemann sum, 1083
trochoid, 670
Tschirnhausen cubic, 215, 444
twisted cubic, 894
type I or type II plane region, 1053, 1054
type 1, 2, or 3 solid region, 1084, 1086

ultraviolet catastrophe, 820
unified description of conics, 711
uniform circular motion, 930
union of sets, A3
unit normal vector, 909, 911
unit tangent vector, 899, 911
unit vector, 842
upper limit of integration, 384
upper sum, 377

value, initial, 240
value of a function, 8
van der Waals equation, 216, 972
variable(s)
  change of, 420
  continuous random, 592
  dependent, 9, 934, 987
  independent, 9, 934, 987
  independent random, 1076
  intermediate, 987
variables, change of. *See* change of variable(s)
vascular branching, 348
vector(s), 836
  acceleration, 916
  addition of, 836, 840
  algebraic, 839
  angle between, 848, 849
  basis, 841
  binormal, 909, 911
  components of, 838
  coplanar, 860
  cross product of, 855
  difference, 838
  displacement, 836, 837, 852
  dot product, 847, 848
  equality of, 836
  geometric representation of, 836, 839
  gradient, 997, 998, 999, 1004
  gravitational force, 1127
  i, **j**, and **k**, 841
  length of, 839
  magnitude of, 839
  multiplication of, 837, 840
  n-dimensional, 840
  normal, 868, 909
  orthogonal, 849
  orthogonal, projection of, 851
  parallel, 838, 858
  perpendicular, 849
  position, 839
  properties of, 840
  representation of, 839, 840
  scalar multiple of, 837, 840
  secant, 898
  standard basis, 841
  subtraction of, 838, 840
  tangent, 898, 1176
  three-dimensional, 839, 840
  triple product, 860
  two-dimensional, 840
  unit, 842
  unit normal, 909, 911
  unit tangent, 899, 911
  velocity, 916
  zero, 836
vector equation
  of a line, 867
  of a plane, 868
vector field, 1124, 1125
  component functions, 1125
  conservative, 1129, 1147, 1148, 1163
  curl of, 1161, 1162
  divergence of, 1165
  electric flux of, 1191, 1204
  flux of, 1189, 1191
  force, 1124, 1128
  gradient, 997, 998, 999, 1128
  gravitational, 1128
  incompressible, 1166
  irrotational, 1164
  line integral of, 1138, 1139
  potential function, 1129
  surface integral of, 1188, 1189
  velocity, 1124, 1164
vector function, 890
  component functions of, 890
  continuity of, 891
  differentiation of, 898, 900
  integration of, 901
  limit of, 890, 898
  second derivative, 900
vector product, 855
  properties of, 857, 859
vector projection, 851
vector triple product, 861
vector-valued function. *See* vector function

velocity, 3, 80, 142, 226, 413
  average, 4, 81, 143, 226
  escape, 551, 629
  instantaneous, 81, 143, 226
velocity field, 1124, 1127
  airflow, 1124
  ocean currents, 1124
  wind patterns, 1124
velocity gradient, 232
velocity problem, 80, 143
velocity vector, 916
velocity vector field, 1124, 1164
Verhulst, Pierre-François, 607
vertex of a parabola, 703
vertical asymptote, 89, 90, 321
vertical line, A13
Vertical Line Test, 13
vertical plate, 577
vertical shift of a graph, 37
vertical tangent line, 159
vertical translation of a graph, 35
vertices of an ellipse, 705
vertices of a hyperbola, 706
vibration of a drumhead, computer model for, 792
visual representations of a function, 8, 10
volume, 446
  by cross-sections, 454, 455, 589
  by cylindrical shells, 460
  definition of, 446, 448
  by disks, 449, 453
  by double integrals, 1038
  of a hypersphere, 1095
  of a parallelepiped, 860
  by polar coordinates, 1065
  of a solid, 446, 1040
  of a solid of revolution, 449, 575
  of a solid on a slant, 575
  by triple integrals, 1089
  by washers, 451, 453
Volterra, Vito, 649
von Bertalanffy model, 657

Wallis, John, 3
Wallis product, 492
washer method, 451
wave equation, 968
wave height as a function of two variables, 947
Weierstrass, Karl, 516
Weierstrass substitution, 516
weight (force), 467
wind patterns in San Francisco Bay area, 1124
wind-chill index, 935, 936
witch of Maria Agnesi, 190, 671
work (force), 467, 468, 852
  defined as a line integral, 1139
Wren, Sir Christopher, 678

x-axis, 830, A10
x-coordinate, 830, A10
x-intercept, A13, A19
X-mean, 1077

y-axis, 830, A10
y-coordinate, 830, A10
y-intercept, A13, A19
Y-mean, 1077

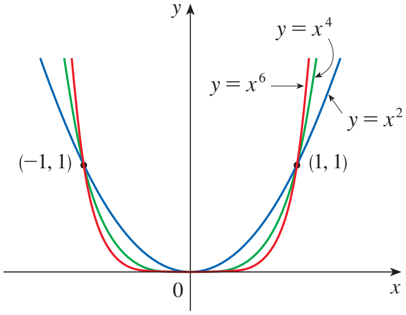
z-axis, 830
z-coordinate, 830
Zeno's paradoxes, 724
zero vector, 836
zone of a sphere, 574

Cut here and keep for reference

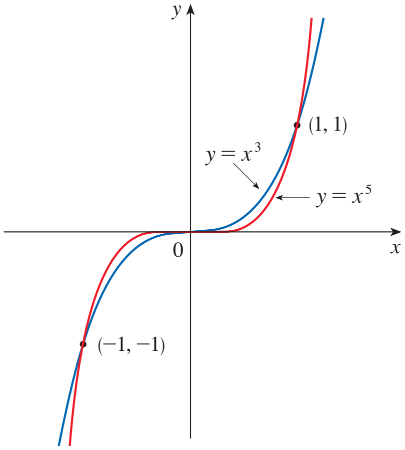
SPECIAL FUNCTIONS

Power Functions $f(x) = x^a$

(i) $f(x) = x^n$, n a positive integer

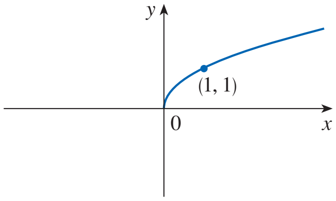


n even

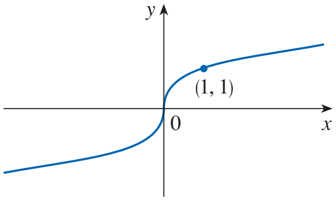


n odd

(ii) $f(x) = x^{1/n} = \sqrt[n]{x}$, n a positive integer

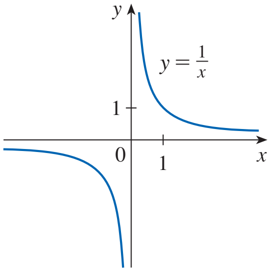


$f(x) = \sqrt{x}$



$f(x) = \sqrt[3]{x}$

(iii) $f(x) = x^{-1} = \dfrac{1}{x}$

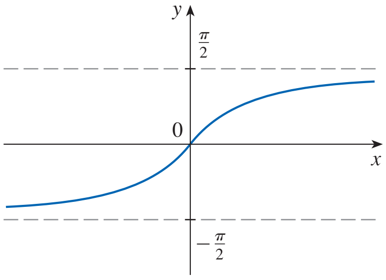


Inverse Trigonometric Functions

$$\arcsin x = \sin^{-1}x = y \iff \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

$$\arccos x = \cos^{-1}x = y \iff \cos y = x \quad \text{and} \quad 0 \leqslant y \leqslant \pi$$

$$\arctan x = \tan^{-1}x = y \iff \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$



$y = \tan^{-1}x = \arctan x$

$$\lim_{x \to -\infty} \tan^{-1}x = -\frac{\pi}{2}$$

$$\lim_{x \to \infty} \tan^{-1}x = \frac{\pi}{2}$$

REFERENCE page 4

SPECIAL FUNCTIONS

Exponential and Logarithmic Functions

$\log_b x = y \iff b^y = x$

$\ln x = \log_e x$, where $\ln e = 1$

$\ln x = y \iff e^y = x$

Cancellation Equations

$\log_b(b^x) = x \qquad b^{\log_b x} = x$

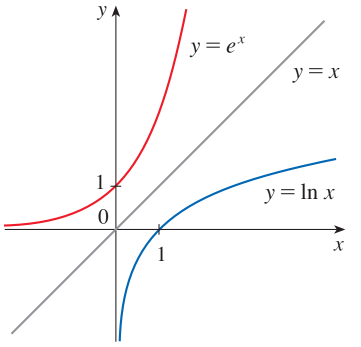
$\ln(e^x) = x \qquad e^{\ln x} = x$

Laws of Logarithms

1. $\log_b(xy) = \log_b x + \log_b y$

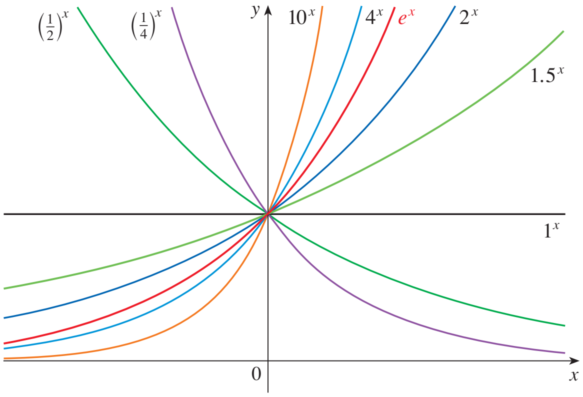
2. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

3. $\log_b(x^r) = r \log_b x$

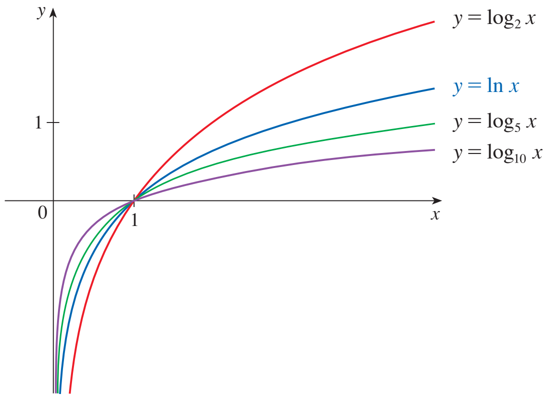


$\displaystyle\lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to\infty} e^x = \infty$

$\displaystyle\lim_{x\to0^+} \ln x = -\infty \qquad \lim_{x\to\infty} \ln x = \infty$



Exponential functions



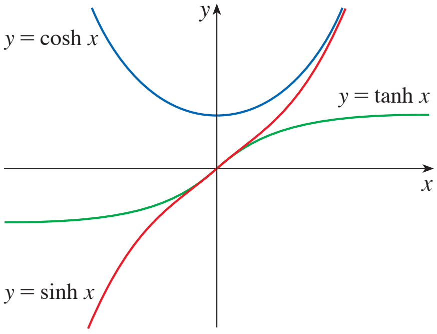
Logarithmic functions

Hyperbolic Functions

$\sinh x = \dfrac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \dfrac{1}{\sinh x}$

$\cosh x = \dfrac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \dfrac{1}{\cosh x}$

$\tanh x = \dfrac{\sinh x}{\cosh x} \qquad \coth x = \dfrac{\cosh x}{\sinh x}$



Inverse Hyperbolic Functions

$y = \sinh^{-1}x \iff \sinh y = x$

$y = \cosh^{-1}x \iff \cosh y = x$ and $y \geqslant 0$

$y = \tanh^{-1}x \iff \tanh y = x$

$\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right)$

$\cosh^{-1}x = \ln\left(x + \sqrt{x^2 - 1}\right)$

$\tanh^{-1}x = \frac{1}{2}\ln\left(\dfrac{1 + x}{1 - x}\right)$

Cut here and keep for reference

DIFFERENTIATION RULES

General Formulas

1. $\dfrac{d}{dx}(c) = 0$

2. $\dfrac{d}{dx}[cf(x)] = cf'(x)$

3. $\dfrac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

4. $\dfrac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

5. $\dfrac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$ (Product Rule)

6. $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$ (Quotient Rule)

7. $\dfrac{d}{dx}f(g(x)) = f'(g(x))g'(x)$ (Chain Rule)

8. $\dfrac{d}{dx}(x^n) = nx^{n-1}$ (Power Rule)

Exponential and Logarithmic Functions

9. $\dfrac{d}{dx}(e^x) = e^x$

10. $\dfrac{d}{dx}(b^x) = b^x \ln b$

11. $\dfrac{d}{dx}\ln|x| = \dfrac{1}{x}$

12. $\dfrac{d}{dx}(\log_b x) = \dfrac{1}{x \ln b}$

Trigonometric Functions

13. $\dfrac{d}{dx}(\sin x) = \cos x$

14. $\dfrac{d}{dx}(\cos x) = -\sin x$

15. $\dfrac{d}{dx}(\tan x) = \sec^2 x$

16. $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

17. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

18. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$

Inverse Trigonometric Functions

19. $\dfrac{d}{dx}(\sin^{-1}x) = \dfrac{1}{\sqrt{1 - x^2}}$

20. $\dfrac{d}{dx}(\cos^{-1}x) = -\dfrac{1}{\sqrt{1 - x^2}}$

21. $\dfrac{d}{dx}(\tan^{-1}x) = \dfrac{1}{1 + x^2}$

22. $\dfrac{d}{dx}(\csc^{-1}x) = -\dfrac{1}{x\sqrt{x^2 - 1}}$

23. $\dfrac{d}{dx}(\sec^{-1}x) = \dfrac{1}{x\sqrt{x^2 - 1}}$

24. $\dfrac{d}{dx}(\cot^{-1}x) = -\dfrac{1}{1 + x^2}$

Hyperbolic Functions

25. $\dfrac{d}{dx}(\sinh x) = \cosh x$

26. $\dfrac{d}{dx}(\cosh x) = \sinh x$

27. $\dfrac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

28. $\dfrac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

29. $\dfrac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

30. $\dfrac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

Inverse Hyperbolic Functions

31. $\dfrac{d}{dx}(\sinh^{-1}x) = \dfrac{1}{\sqrt{1 + x^2}}$

32. $\dfrac{d}{dx}(\cosh^{-1}x) = \dfrac{1}{\sqrt{x^2 - 1}}$

33. $\dfrac{d}{dx}(\tanh^{-1}x) = \dfrac{1}{1 - x^2}$

34. $\dfrac{d}{dx}(\operatorname{csch}^{-1}x) = -\dfrac{1}{|x|\sqrt{x^2 + 1}}$

35. $\dfrac{d}{dx}(\operatorname{sech}^{-1}x) = -\dfrac{1}{x\sqrt{1 - x^2}}$

36. $\dfrac{d}{dx}(\coth^{-1}x) = \dfrac{1}{1 - x^2}$

TABLE OF INTEGRALS

Basic Forms

1. $\displaystyle\int u\,dv = uv - \int v\,du$

2. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$

3. $\displaystyle\int \frac{du}{u} = \ln|u| + C$

4. $\displaystyle\int e^u\,du = e^u + C$

5. $\displaystyle\int b^u\,du = \frac{b^u}{\ln b} + C$

6. $\displaystyle\int \sin u\,du = -\cos u + C$

7. $\displaystyle\int \cos u\,du = \sin u + C$

8. $\displaystyle\int \sec^2 u\,du = \tan u + C$

9. $\displaystyle\int \csc^2 u\,du = -\cot u + C$

10. $\displaystyle\int \sec u\,\tan u\,du = \sec u + C$

11. $\displaystyle\int \csc u\,\cot u\,du = -\csc u + C$

12. $\displaystyle\int \tan u\,du = \ln|\sec u| + C$

13. $\displaystyle\int \cot u\,du = \ln|\sin u| + C$

14. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C$

15. $\displaystyle\int \csc u\,du = \ln|\csc u - \cot u| + C$

16. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C, \quad a > 0$

17. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

18. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$

19. $\displaystyle\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ln\left|\frac{u + a}{u - a}\right| + C$

20. $\displaystyle\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ln\left|\frac{u - a}{u + a}\right| + C$

Forms Involving $\sqrt{a^2 + u^2}$, $a > 0$

21. $\displaystyle\int \sqrt{a^2 + u^2}\,du = \frac{u}{2}\sqrt{a^2 + u^2} + \frac{a^2}{2}\ln\left(u + \sqrt{a^2 + u^2}\right) + C$

22. $\displaystyle\int u^2\sqrt{a^2 + u^2}\,du = \frac{u}{8}(a^2 + 2u^2)\sqrt{a^2 + u^2} - \frac{a^4}{8}\ln\left(u + \sqrt{a^2 + u^2}\right) + C$

23. $\displaystyle\int \frac{\sqrt{a^2 + u^2}}{u}\,du = \sqrt{a^2 + u^2} - a\ln\left|\frac{a + \sqrt{a^2 + u^2}}{u}\right| + C$

24. $\displaystyle\int \frac{\sqrt{a^2 + u^2}}{u^2}\,du = -\frac{\sqrt{a^2 + u^2}}{u} + \ln\left(u + \sqrt{a^2 + u^2}\right) + C$

25. $\displaystyle\int \frac{du}{\sqrt{a^2 + u^2}} = \ln\left(u + \sqrt{a^2 + u^2}\right) + C$

26. $\displaystyle\int \frac{u^2\,du}{\sqrt{a^2 + u^2}} = \frac{u}{2}\sqrt{a^2 + u^2} - \frac{a^2}{2}\ln\left(u + \sqrt{a^2 + u^2}\right) + C$

27. $\displaystyle\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2 + u^2} + a}{u}\right| + C$

28. $\displaystyle\int \frac{du}{u^2\sqrt{a^2 + u^2}} = -\frac{\sqrt{a^2 + u^2}}{a^2 u} + C$

29. $\displaystyle\int \frac{du}{(a^2 + u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2 + u^2}} + C$

TABLE OF INTEGRALS

Forms Involving $\sqrt{a^2 - u^2}$, $a > 0$

30. $\displaystyle\int \sqrt{a^2 - u^2}\, du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

31. $\displaystyle\int u^2\sqrt{a^2 - u^2}\, du = \frac{u}{8}(2u^2 - a^2)\sqrt{a^2 - u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$

32. $\displaystyle\int \frac{\sqrt{a^2 - u^2}}{u}\, du = \sqrt{a^2 - u^2} - a\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

33. $\displaystyle\int \frac{\sqrt{a^2 - u^2}}{u^2}\, du = -\frac{1}{u}\sqrt{a^2 - u^2} - \sin^{-1}\frac{u}{a} + C$

34. $\displaystyle\int \frac{u^2\, du}{\sqrt{a^2 - u^2}} = -\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

35. $\displaystyle\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

36. $\displaystyle\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{1}{a^2 u}\sqrt{a^2 - u^2} + C$

37. $\displaystyle\int (a^2 - u^2)^{3/2}\, du = -\frac{u}{8}(2u^2 - 5a^2)\sqrt{a^2 - u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$

38. $\displaystyle\int \frac{du}{(a^2 - u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2 - u^2}} + C$

Forms Involving $\sqrt{u^2 - a^2}$, $a > 0$

39. $\displaystyle\int \sqrt{u^2 - a^2}\, du = \frac{u}{2}\sqrt{u^2 - a^2} - \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

40. $\displaystyle\int u^2\sqrt{u^2 - a^2}\, du = \frac{u}{8}(2u^2 - a^2)\sqrt{u^2 - a^2} - \frac{a^4}{8}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

41. $\displaystyle\int \frac{\sqrt{u^2 - a^2}}{u}\, du = \sqrt{u^2 - a^2} - a\cos^{-1}\frac{a}{|u|} + C$

42. $\displaystyle\int \frac{\sqrt{u^2 - a^2}}{u^2}\, du = -\frac{\sqrt{u^2 - a^2}}{u} + \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

43. $\displaystyle\int \frac{du}{\sqrt{u^2 - a^2}} = \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

44. $\displaystyle\int \frac{u^2\, du}{\sqrt{u^2 - a^2}} = \frac{u}{2}\sqrt{u^2 - a^2} + \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

45. $\displaystyle\int \frac{du}{u^2\sqrt{u^2 - a^2}} = \frac{\sqrt{u^2 - a^2}}{a^2 u} + C$

46. $\displaystyle\int \frac{du}{(u^2 - a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2 - a^2}} + C$

(continued)

TABLE OF INTEGRALS

Forms Involving $a + bu$

47.
$$\int \frac{u\,du}{a+bu} = \frac{1}{b^2}\left(a + bu - a\ln|a+bu|\right) + C$$

48.
$$\int \frac{u^2\,du}{a+bu} = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$$

49.
$$\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$$

50.
$$\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$$

51.
$$\int \frac{u\,du}{(a+bu)^2} = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$$

52.
$$\int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$$

53.
$$\int \frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\left(a + bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$$

54.
$$\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu - 2a)(a+bu)^{3/2} + C$$

55.
$$\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu - 2a)\sqrt{a+bu} + C$$

56.
$$\int \frac{u^2\,du}{\sqrt{a+bu}} = \frac{2}{15b^3}(8a^2 + 3b^2u^2 - 4abu)\sqrt{a+bu} + C$$

57.
$$\begin{aligned}\int \frac{du}{u\sqrt{a+bu}} &= \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, \quad \text{if } a > 0\\ &= \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bu}{-a}} + C, \quad \text{if } a < 0\end{aligned}$$

58.
$$\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$$

59.
$$\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$$

60.
$$\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$$

61.
$$\int \frac{u^n\,du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}\,du}{\sqrt{a+bu}}$$

62.
$$\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$$

TABLE OF INTEGRALS

Trigonometric Forms

63. $\displaystyle\int \sin^2 u\,du = \tfrac{1}{2}u - \tfrac{1}{4}\sin 2u + C$

64. $\displaystyle\int \cos^2 u\,du = \tfrac{1}{2}u + \tfrac{1}{4}\sin 2u + C$

65. $\displaystyle\int \tan^2 u\,du = \tan u - u + C$

66. $\displaystyle\int \cot^2 u\,du = -\cot u - u + C$

67. $\displaystyle\int \sin^3 u\,du = -\tfrac{1}{3}(2 + \sin^2 u)\cos u + C$

68. $\displaystyle\int \cos^3 u\,du = \tfrac{1}{3}(2 + \cos^2 u)\sin u + C$

69. $\displaystyle\int \tan^3 u\,du = \tfrac{1}{2}\tan^2 u + \ln|\cos u| + C$

70. $\displaystyle\int \cot^3 u\,du = -\tfrac{1}{2}\cot^2 u - \ln|\sin u| + C$

71. $\displaystyle\int \sec^3 u\,du = \tfrac{1}{2}\sec u\tan u + \tfrac{1}{2}\ln|\sec u + \tan u| + C$

72. $\displaystyle\int \csc^3 u\,du = -\tfrac{1}{2}\csc u\cot u + \tfrac{1}{2}\ln|\csc u - \cot u| + C$

73. $\displaystyle\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1}u\cos u + \frac{n-1}{n}\int \sin^{n-2}u\,du$

74. $\displaystyle\int \cos^n u\,du = \frac{1}{n}\cos^{n-1}u\sin u + \frac{n-1}{n}\int \cos^{n-2}u\,du$

75. $\displaystyle\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1}u - \int \tan^{n-2}u\,du$

76. $\displaystyle\int \cot^n u\,du = \frac{-1}{n-1}\cot^{n-1}u - \int \cot^{n-2}u\,du$

77. $\displaystyle\int \sec^n u\,du = \frac{1}{n-1}\tan u\sec^{n-2}u + \frac{n-2}{n-1}\int \sec^{n-2}u\,du$

78. $\displaystyle\int \csc^n u\,du = \frac{-1}{n-1}\cot u\csc^{n-2}u + \frac{n-2}{n-1}\int \csc^{n-2}u\,du$

79. $\displaystyle\int \sin au\sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

80. $\displaystyle\int \cos au\cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

81. $\displaystyle\int \sin au\cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

82. $\displaystyle\int u\sin u\,du = \sin u - u\cos u + C$

83. $\displaystyle\int u\cos u\,du = \cos u + u\sin u + C$

84. $\displaystyle\int u^n\sin u\,du = -u^n\cos u + n\int u^{n-1}\cos u\,du$

85. $\displaystyle\int u^n\cos u\,du = u^n\sin u - n\int u^{n-1}\sin u\,du$

86.
$$\int \sin^n u\cos^m u\,du = -\frac{\sin^{n-1}u\cos^{m+1}u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2}u\cos^m u\,du$$
$$= \frac{\sin^{n+1}u\cos^{m-1}u}{n+m} + \frac{m-1}{n+m}\int \sin^n u\cos^{m-2}u\,du$$

Inverse Trigonometric Forms

87. $\displaystyle\int \sin^{-1}u\,du = u\sin^{-1}u + \sqrt{1-u^2} + C$

88. $\displaystyle\int \cos^{-1}u\,du = u\cos^{-1}u - \sqrt{1-u^2} + C$

89. $\displaystyle\int \tan^{-1}u\,du = u\tan^{-1}u - \tfrac{1}{2}\ln(1+u^2) + C$

90. $\displaystyle\int u\sin^{-1}u\,du = \frac{2u^2-1}{4}\sin^{-1}u + \frac{u\sqrt{1-u^2}}{4} + C$

91. $\displaystyle\int u\cos^{-1}u\,du = \frac{2u^2-1}{4}\cos^{-1}u - \frac{u\sqrt{1-u^2}}{4} + C$

92. $\displaystyle\int u\tan^{-1}u\,du = \frac{u^2+1}{2}\tan^{-1}u - \frac{u}{2} + C$

93. $\displaystyle\int u^n\sin^{-1}u\,du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1}u - \int\frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

94. $\displaystyle\int u^n\cos^{-1}u\,du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1}u + \int\frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

95. $\displaystyle\int u^n\tan^{-1}u\,du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1}u - \int\frac{u^{n+1}\,du}{1+u^2}\right], \quad n \neq -1$

(continued)

TABLE OF INTEGRALS

Exponential and Logarithmic Forms

96. $\int ue^{au}\,du = \frac{1}{a^2}(au - 1)e^{au} + C$

97. $\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$

98. $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2 + b^2}(a\sin bu - b\cos bu) + C$

99. $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2 + b^2}(a\cos bu + b\sin bu) + C$

100. $\int \ln u\,du = u\ln u - u + C$

101. $\int u^n \ln u\,du = \frac{u^{n+1}}{(n + 1)^2}[(n + 1)\ln u - 1] + C$

102. $\int \frac{1}{u\ln u}\,du = \ln|\ln u| + C$

Hyperbolic Forms

103. $\int \sinh u\,du = \cosh u + C$

104. $\int \cosh u\,du = \sinh u + C$

105. $\int \tanh u\,du = \ln\cosh u + C$

106. $\int \coth u\,du = \ln|\sinh u| + C$

107. $\int \operatorname{sech} u\,du = \tan^{-1}|\sinh u| + C$

108. $\int \operatorname{csch} u\,du = \ln\left|\tanh \tfrac{1}{2}u\right| + C$

109. $\int \operatorname{sech}^2 u\,du = \tanh u + C$

110. $\int \operatorname{csch}^2 u\,du = -\coth u + C$

111. $\int \operatorname{sech} u\tanh u\,du = -\operatorname{sech} u + C$

112. $\int \operatorname{csch} u\coth u\,du = -\operatorname{csch} u + C$

Forms Involving $\sqrt{2au - u^2}$, $a > 0$

113. $\int \sqrt{2au - u^2}\,du = \frac{u - a}{2}\sqrt{2au - u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a - u}{a}\right) + C$

114. $\int u\sqrt{2au - u^2}\,du = \frac{2u^2 - au - 3a^2}{6}\sqrt{2au - u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a - u}{a}\right) + C$

115. $\int \frac{\sqrt{2au - u^2}}{u}\,du = \sqrt{2au - u^2} + a\cos^{-1}\left(\frac{a - u}{a}\right) + C$

116. $\int \frac{\sqrt{2au - u^2}}{u^2}\,du = -\frac{2\sqrt{2au - u^2}}{u} - \cos^{-1}\left(\frac{a - u}{a}\right) + C$

117. $\int \frac{du}{\sqrt{2au - u^2}} = \cos^{-1}\left(\frac{a - u}{a}\right) + C$

118. $\int \frac{u\,du}{\sqrt{2au - u^2}} = -\sqrt{2au - u^2} + a\cos^{-1}\left(\frac{a - u}{a}\right) + C$

119. $\int \frac{u^2\,du}{\sqrt{2au - u^2}} = -\frac{(u + 3a)}{2}\sqrt{2au - u^2} + \frac{3a^2}{2}\cos^{-1}\left(\frac{a - u}{a}\right) + C$

120. $\int \frac{du}{u\sqrt{2au - u^2}} = -\frac{\sqrt{2au - u^2}}{au} + C$